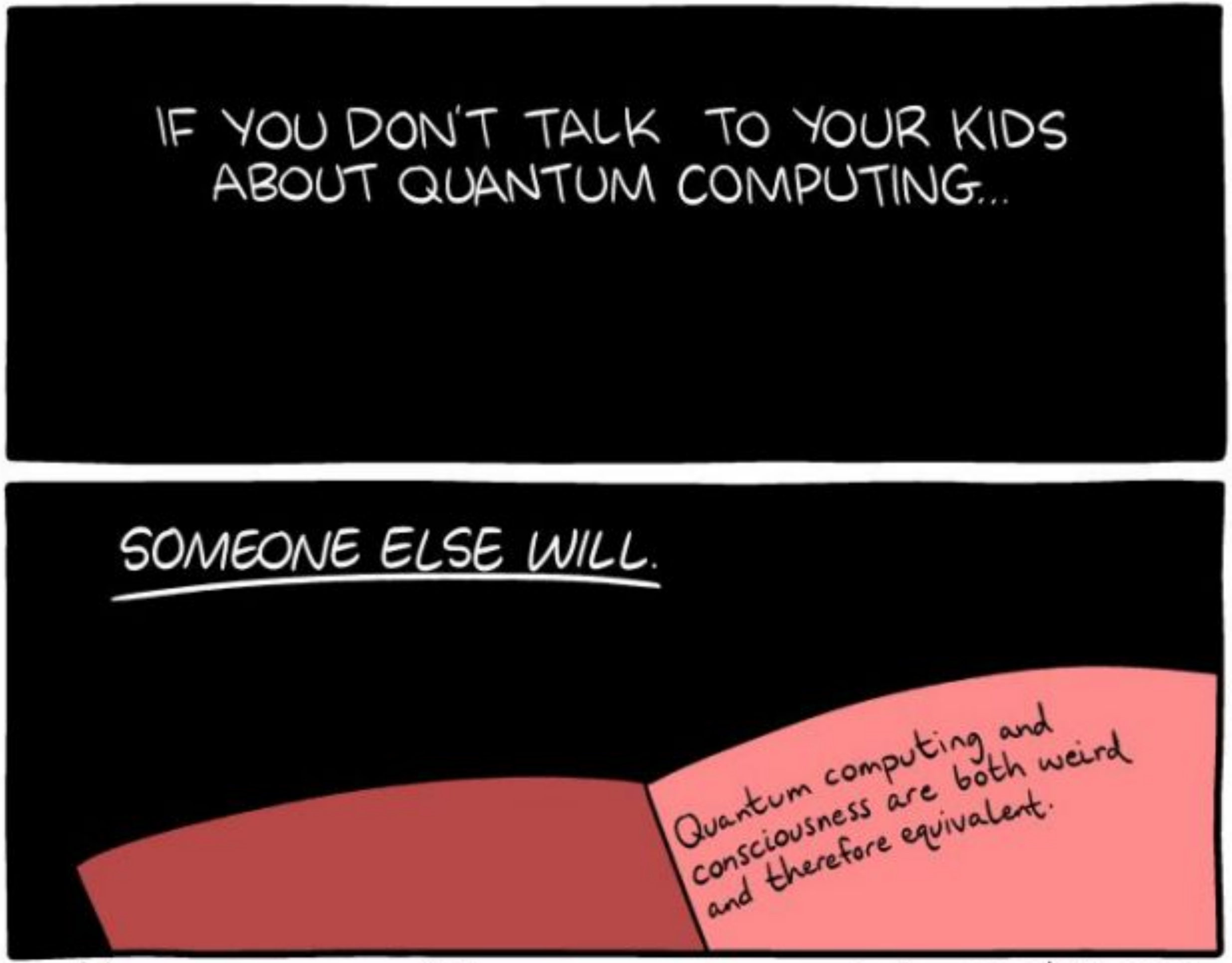
This is CS4084!



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Multi Qubit System

Content

We are covering chapter 4 “Multiple Quantum Bits” from the book Introduction to Classical and Quantum Computing By Thomas G Wong.

4.2

4.3

4.4.1 , 4.4.2 , 4.4.3
Practise book exercise for these sections.

Tensor Product

This is pronounced “zero tensor zero.” $|0\rangle \otimes |0\rangle$

Often, we compress the notation and leave out the tensor product in both writing and speech:

$$|0\rangle|0\rangle \qquad |00\rangle$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Two Qubit

With two qubits, the Z-basis is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

A general state is a superposition of these basis states:

$$c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$$

Kronecker Product

$$|00\rangle = |0\rangle|0\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}1\\0\end{pmatrix}\\0\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

$$|01\rangle = |0\rangle|1\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\0\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}.$$

$$|10\rangle = |1\rangle|0\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}1\\0\end{pmatrix}\\1\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$

$$|11\rangle = |1\rangle|1\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}0\\1\end{pmatrix}\\1\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Probabilistic states

Definition

For a given probabilistic state of (X, Y), we say that X and Y are *independent* if

$$\Pr((X, Y) = (a, b)) = \Pr(X = a)\Pr(Y = b)$$

for all $a \in \Sigma$ and $b \in \Gamma$.

Suppose that a probabilistic state of (X, Y) is expressed as a vector:

$$|\pi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|ab\rangle$$

The systems X and Y are independent if there exist probability vectors

$$|\phi\rangle = \sum_{a\in\Sigma} q_a|a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} r_b|b\rangle$$

such that $p_{ab} = q_a r_b$ for all $a \in \Sigma$ and $b \in \Gamma$.

Probabilistic states

Example

The probabilistic state of a pair of bits (X, Y) represented by the vector

$$|\pi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

is one in which X and Y are independent. The required condition is true for these probability vectors:

$$|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

Probabilistic states

Example

For the probabilistic state

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

of two bits (X, Y), we have that X and Y are not independent.

If they were, we would have numbers q_0, q_1, r_0, r_1 such that

$$\begin{aligned} q_0 r_0 &= \tfrac{1}{2} \\ q_0 r_1 &= 0 \\ q_1 r_0 &= 0 \\ q_1 r_1 &= \tfrac{1}{2} \end{aligned}$$

But if $q_0 r_1 = 0$, then either $q_0 = 0$ or $r_1 = 0$ (or both), contradicting either the first or last equality.

Tensor products of vectors

Definition

$$|\phi\rangle = \sum_{a\in\Sigma} \alpha_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} \beta_b |b\rangle$$

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_a \beta_b |ab\rangle$$

Example

$$|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

$$|\phi\rangle \otimes |\psi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

Two Qubits

$$c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Exercise 4.4. Verify that

$$|1\rangle \otimes |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Exercise 4.5. Consider a two-qubit state

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{i}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}+i}{4}|11\rangle.$$

(a) What is $|\psi\rangle$ as a (column) vector?

Measuring Individual Qubit

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle.$$

What is the probability?

Measuring Individual Qubit

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle.$$

What is the probability if we measure only 1 qubit?

$|0\rangle$ with some probability, and the state collapses to something,

$|1\rangle$ with some probability, and the state collapses to something.

Measuring Individual Qubit

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle.$$

The probability of getting $|0\rangle$ when measuring the left qubit is given by the sum of the norm-squares of the amplitudes of $|00\rangle$ and $|01\rangle$, since those both have the left qubit as $|0\rangle$.

$$\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{3}{4}.$$

Measuring Individual Qubit

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle.$$

What is the probability if we measure only 1 qubit?

$$|0\rangle \text{ with probability } \frac{3}{4}, \text{ and the state collapses to something,}$$
$$|1\rangle \text{ with probability } \frac{1}{4}, \text{ and the state collapses to something.}$$

Measuring Individual Qubit - Left Qubit

State collapses to

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle.$$

$$A\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle\right)$$

$$B\left(\frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle\right)$$

Measuring Individual Qubit - Left Qubit

State collapses to

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle.$$

$|0\rangle$ with probability $\frac{3}{4}$, and the state collapses to $\sqrt{\frac{2}{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle$,

$|1\rangle$ with probability $\frac{1}{4}$, and the state collapses to $\frac{\sqrt{3}}{2}|10\rangle + \frac{1}{2}|11\rangle$.

Sequential Single-Qubit Measurements

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle.$$

If we first measure the left qubit, we get

$$|0\rangle \text{ with probability } \frac{3}{4}, \text{ and the state collapses to } \sqrt{\frac{2}{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle,$$

$$|1\rangle \text{ with probability } \frac{1}{4}, \text{ and the state collapses to } \frac{\sqrt{3}}{2}|10\rangle + \frac{1}{2}|11\rangle.$$

If we measure right qubit now

$$
\begin{aligned}
\text{Prob}(|00\rangle) &= \text{Prob}(\text{first left } |0\rangle)\,\text{Prob}(\text{then right } |0\rangle) = \frac{3}{4}\frac{2}{3} = \frac{1}{2},\\
\text{Prob}(|01\rangle) &= \text{Prob}(\text{first left } |0\rangle)\,\text{Prob}(\text{then right } |1\rangle) = \frac{3}{4}\frac{1}{3} = \frac{1}{4},\\
\text{Prob}(|10\rangle) &= \text{Prob}(\text{first left } |1\rangle)\,\text{Prob}(\text{then right } |0\rangle) = \frac{1}{4}\frac{3}{4} = \frac{3}{16},\\
\text{Prob}(|11\rangle) &= \text{Prob}(\text{first left } |1\rangle)\,\text{Prob}(\text{then right } |1\rangle) = \frac{1}{4}\frac{1}{4} = \frac{1}{16}.
\end{aligned}
$$

Measuring Individual Qubit

Home Activity

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle.$$

What is the probability if we measure only 1 qubit (right one)?

$|0\rangle$ with some probability, and the state collapses to something,

$|1\rangle$ with some probability, and the state collapses to something.

Exercise 4.7. Two qubits are in the state

$$\frac{i}{\sqrt{10}}|00\rangle + \frac{1-2i}{\sqrt{10}}|01\rangle + \frac{e^{i\pi/100}}{\sqrt{10}}|10\rangle + \frac{\sqrt{3}}{\sqrt{10}}|11\rangle.$$

If we measure the qubits in the Z-basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, what are the possible outcomes and with what probabilities?

Exercise 4.8. Normalize the following quantum state:

$$A\left(\frac{1}{2}|00\rangle + i|01\rangle + \sqrt{2}|10\rangle - |11\rangle\right).$$

Exercise 4.8. Normalize the following quantum state:

$$A\left(\frac{1}{2}|00\rangle + i|01\rangle + \sqrt{2}|10\rangle - |11\rangle\right).$$

$$\boxed{|\psi\rangle = \frac{2}{\sqrt{17}}\left(\frac{1}{2}|00\rangle + i|01\rangle + \sqrt{2}|10\rangle - |11\rangle\right)}$$

Measuring 3 qubit state

We can apply these ideas to any number of qubits. For example, if we have three qubits in the state.

$$c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + c_3|011\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + c_7|111\rangle.$$

Measuring 3 qubit state

If we measure the left and middle qubits

$$c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + c_3|011\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + c_7|111\rangle.$$

$$|00\rangle \text{ with probability } |c_0|^2 + |c_1|^2, \text{ collapses to } \frac{c_0|000\rangle + c_1|001\rangle}{\sqrt{|c_0|^2 + |c_1|^2}},$$

$$|01\rangle \text{ with probability } |c_2|^2 + |c_3|^2, \text{ collapses to } \frac{c_2|010\rangle + c_3|011\rangle}{\sqrt{|c_2|^2 + |c_3|^2}},$$

$$|10\rangle \text{ with probability } |c_4|^2 + |c_5|^2, \text{ collapses to } \frac{c_4|100\rangle + c_5|101\rangle}{\sqrt{|c_4|^2 + |c_5|^2}},$$

$$|11\rangle \text{ with probability } |c_6|^2 + |c_7|^2, \text{ collapses to } \frac{c_6|110\rangle + c_7|111\rangle}{\sqrt{|c_6|^2 + |c_7|^2}}.$$

Exercise 4.9. Consider the two-qubit state

$$\frac{1}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle.$$

If you measure only the left qubit, what are the resulting states, and with what probabilities?

Exercise 4.9. Consider the two-qubit state

$$\frac{1}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle.$$

If you measure only the left qubit, what are the resulting states, and with what probabilities?

5/16 – State 0

11/16 – state 1

Exercise 4.10. Consider the three-qubit state

$$\frac{1}{6}|000\rangle + \frac{1}{3\sqrt{2}}|001\rangle + \frac{1}{\sqrt{6}}|010\rangle + \frac{1}{2}|011\rangle + \frac{1}{6}|100\rangle + \frac{1}{3}|101\rangle + \frac{1}{6}|110\rangle + \frac{1}{\sqrt{3}}|111\rangle.$$

If you measure only the left and right qubits, but not the middle qubit, what are the resulting states, and with what probabilities?

Exercise 4.10. Consider the three-qubit state

$$\frac{1}{6}|000\rangle + \frac{1}{3\sqrt{2}}|001\rangle + \frac{1}{\sqrt{6}}|010\rangle + \frac{1}{2}|011\rangle + \frac{1}{6}|100\rangle + \frac{1}{3}|101\rangle + \frac{1}{6}|110\rangle + \frac{1}{\sqrt{3}}|111\rangle.$$

If you measure only the left and right qubits, but not the middle qubit, what are the resulting states, and with what probabilities?

$$P(00) = \left|\frac{1}{6}\right|^2 + \left|\frac{1}{\sqrt{6}}\right|^2 = \frac{1}{36} + \frac{1}{6} = \frac{7}{36}$$

$$P(01) = \frac{1}{18} + \frac{1}{4} = \frac{11}{36}$$

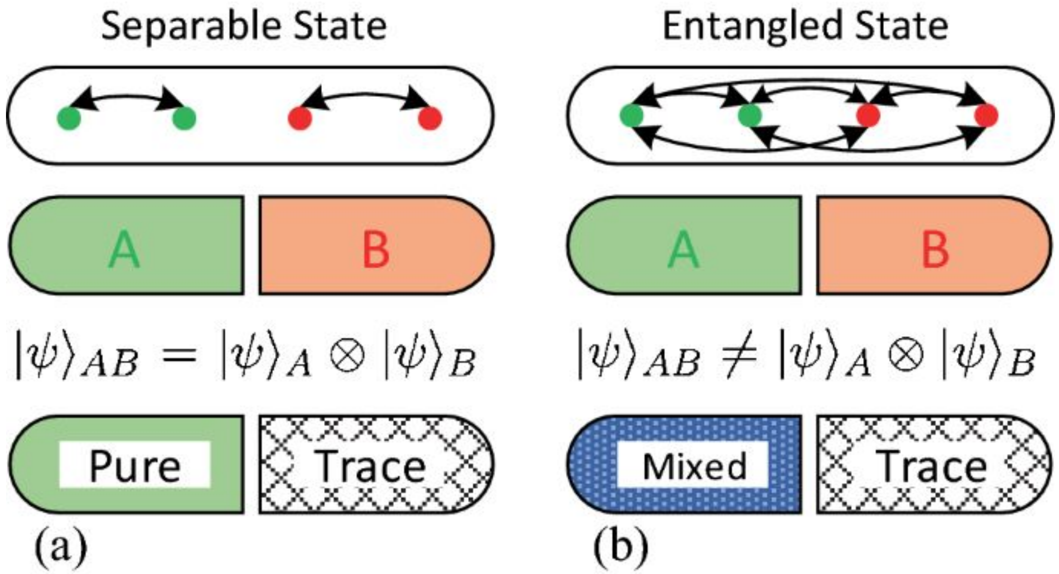
$$P(10) = \frac{1}{36} + \frac{1}{36} = \frac{1}{18} = \frac{2}{36}$$

$$P(11) = \frac{1}{9} + \frac{1}{3} = \frac{4}{9} = \frac{16}{36}$$


ENTANGLEMENT
Natasha Hanacek/NIST

Quantum Entanglement

Spooky action at a distance


Separable State
A
B
$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$
Pure
Trace
(a)
Entangled State
A
B
$|\psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\psi\rangle_B$
Mixed
Trace
(b)

Product States

Some quantum states can be factored into (the tensor product of) individual qubit states. For example,

$$\begin{aligned}\frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right) &= \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)}_{|+\rangle} \otimes \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)}_{|-\rangle} \\ &= |+\rangle \otimes |-\rangle \\ &= |+\rangle|-\rangle.\end{aligned}$$

Product States

$$\frac{1}{2\sqrt{2}}\left(\sqrt{3}|00\rangle - \sqrt{3}|01\rangle + |10\rangle - |11\rangle\right).$$

We want to write this as the product of two single-qubit states,

$$|\psi_1\rangle|\psi_0\rangle,$$

where

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle, \quad |\psi_0\rangle = \alpha_0|0\rangle + \beta_0|1\rangle.$$

$$\begin{aligned}|\psi_1\rangle|\psi_0\rangle &= (\alpha_1|0\rangle + \beta_1|1\rangle)(\alpha_0|0\rangle + \beta_0|1\rangle)\\ &= \alpha_1\alpha_0|00\rangle + \alpha_1\beta_0|01\rangle + \beta_1\alpha_0|10\rangle + \beta_1\beta_0|11\rangle.\end{aligned}$$

Matching up the coefficients with our original state,

$$\alpha_1\alpha_0 = \frac{\sqrt{3}}{2\sqrt{2}}, \quad \alpha_1\beta_0 = \frac{-\sqrt{3}}{2\sqrt{2}}, \quad \beta_1\alpha_0 = \frac{1}{2\sqrt{2}}, \quad \beta_1\beta_0 = \frac{-1}{2\sqrt{2}}.$$

Product States

$$
\begin{aligned}
|\psi_1\rangle|\psi_0\rangle &= (\alpha_1|0\rangle + \beta_1|1\rangle)(\alpha_0|0\rangle + \beta_0|1\rangle) \\
&= \left(\frac{\sqrt{3}}{2\sqrt{2}\alpha_0}|0\rangle + \frac{1}{2\sqrt{2}}\frac{1}{\alpha_0}|1\rangle\right)(\alpha_0|0\rangle - \alpha_0|1\rangle).
\end{aligned}
$$

We see that α_0 cancels, yielding

$$
|\psi_1\rangle|\psi_0\rangle = \left(\frac{\sqrt{3}}{2\sqrt{2}}|0\rangle + \frac{1}{2\sqrt{2}}|1\rangle\right)(|0\rangle - |1\rangle).
$$

Moving the factor of $1/\sqrt{2}$ to the right qubit so that both qubits are normalized,

$$
|\psi_1\rangle|\psi_0\rangle = \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right).
$$

Entangled States

There exist quantum states that cannot be factored into product states. These are called *entangled states*. For example, with two qubits,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

cannot be written as $|\psi_1\rangle|\psi_0\rangle$. As a proof, let us try writing it as a product state using the procedure from the last section:

$$\begin{aligned} |\psi_1\rangle|\psi_0\rangle &= (\alpha_1|0\rangle + \beta_1|1\rangle)(\alpha_0|0\rangle + \beta_0|1\rangle) \\ &= \alpha_1\alpha_0|00\rangle + \alpha_1\beta_0|01\rangle + \beta_1\alpha_0|10\rangle + \beta_1\beta_0|11\rangle. \end{aligned}$$

Matching the coefficients, we get

$$\alpha_1\alpha_0 = \frac{1}{\sqrt{2}}, \quad \alpha_1\beta_0 = 0, \quad \beta_1\alpha_0 = 0, \quad \beta_1\beta_0 = \frac{1}{\sqrt{2}}.$$

Simple Test

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$$(a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$$

$$a = a_0 b_0$$

$$b = a_0 b_1$$

$$c = a_1 b_0$$

$$d = a_1 b_1$$

$$ad = (a_0 b_0)(a_1 b_1)$$

$$bc = (a_0 b_1)(a_1 b_0)$$

- If $ad = bc$ → product state possible
- If $ad \neq bc$ → impossible to factor → entangled

So this equation is the **entanglement test** for 2 qubits.

Exercise 4.11. Are each of the following states a product state or entangled state? If it is a product state, give the factorization.

(a) $\dfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

(b) $\dfrac{1}{\sqrt{2}}(|10\rangle + i|11\rangle)$.

Exercise 4.12. Are each of the following states a product state or entangled state? If it is a product state, give the factorization.

(a) $\frac{1}{4}\left(3|00\rangle - \sqrt{3}|01\rangle + \sqrt{3}|10\rangle - |11\rangle\right)$.

(b) $\frac{1}{\sqrt{3}}|0\rangle|+\rangle + \sqrt{\frac{2}{3}}|1\rangle|-\rangle$.

Two Qubit Gates

CNOT

The CNOT gate or controlled-NOT gate inverts the right qubit if the left qubit is 1

The left qubit is called the control qubit, and the right qubit is called the target qubit.

Control qubit is unchanged by CNOT

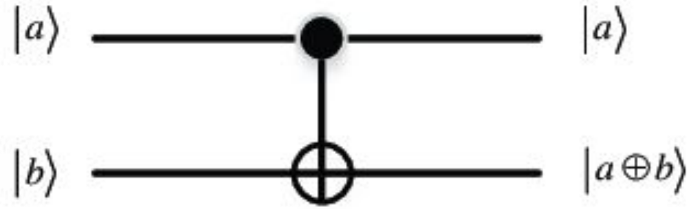
Target qubit becomes the XOR of the inputs

$$
\begin{aligned}
\mathrm{CNOT}|00\rangle &= |00\rangle, \\
\mathrm{CNOT}|01\rangle &= |01\rangle, \\
\mathrm{CNOT}|10\rangle &= |11\rangle, \\
\mathrm{CNOT}|11\rangle &= |10\rangle.
\end{aligned}
$$

CNOT - CX - Controlled X

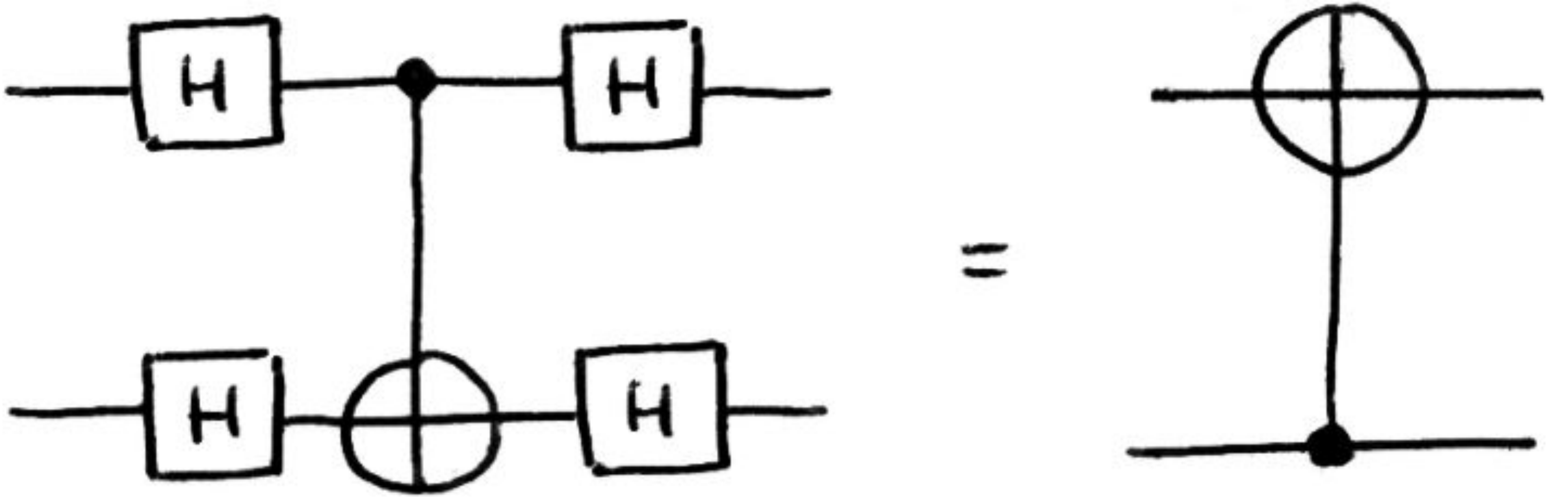
$$\begin{aligned}
\text{CNOT}|00\rangle &= |00\rangle, \\
\text{CNOT}|01\rangle &= |01\rangle, \\
\text{CNOT}|10\rangle &= |11\rangle, \\
\text{CNOT}|11\rangle &= |10\rangle.
\end{aligned}$$

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



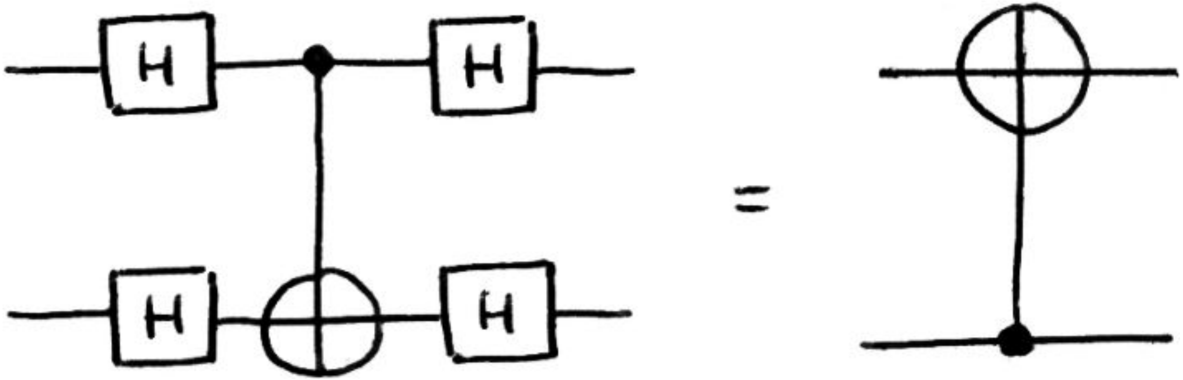
$\text{CNOT}_{ij} = $ CNOT with qubit i as the control and qubit j as the target.

We did the calculations in class



https://cnot.io/quantum_computing/circuit_examples.html

We did the calculations in class



$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H \otimes H = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$(H\otimes H)\,\mathrm{CNOT}\,(H\otimes H)=\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}$$

$$
\begin{aligned}
&= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Bell states

Bell states are quantum states of two qubits that represent simple examples of quantum entanglement.

When one of the two qubits is measured, it takes on a specific value, and the second qubit is forced to also take on a specific value, as the entangled state collapses.

Bell states

Bell states are also known as EPR states or EPR pairs.

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Quantum states

The previous example of a quantum state vector is one of the four *Bell states,* which collectively form the *Bell basis.*

The Bell basis

$$
\begin{aligned}
|\phi^{+}\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
|\phi^{-}\rangle &= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \\
|\psi^{+}\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\
|\psi^{-}\rangle &= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle
\end{aligned}
$$

Quantum states

Here are a couple of well-known examples of quantum state vectors for three-qubits.

GHZ state

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

W state

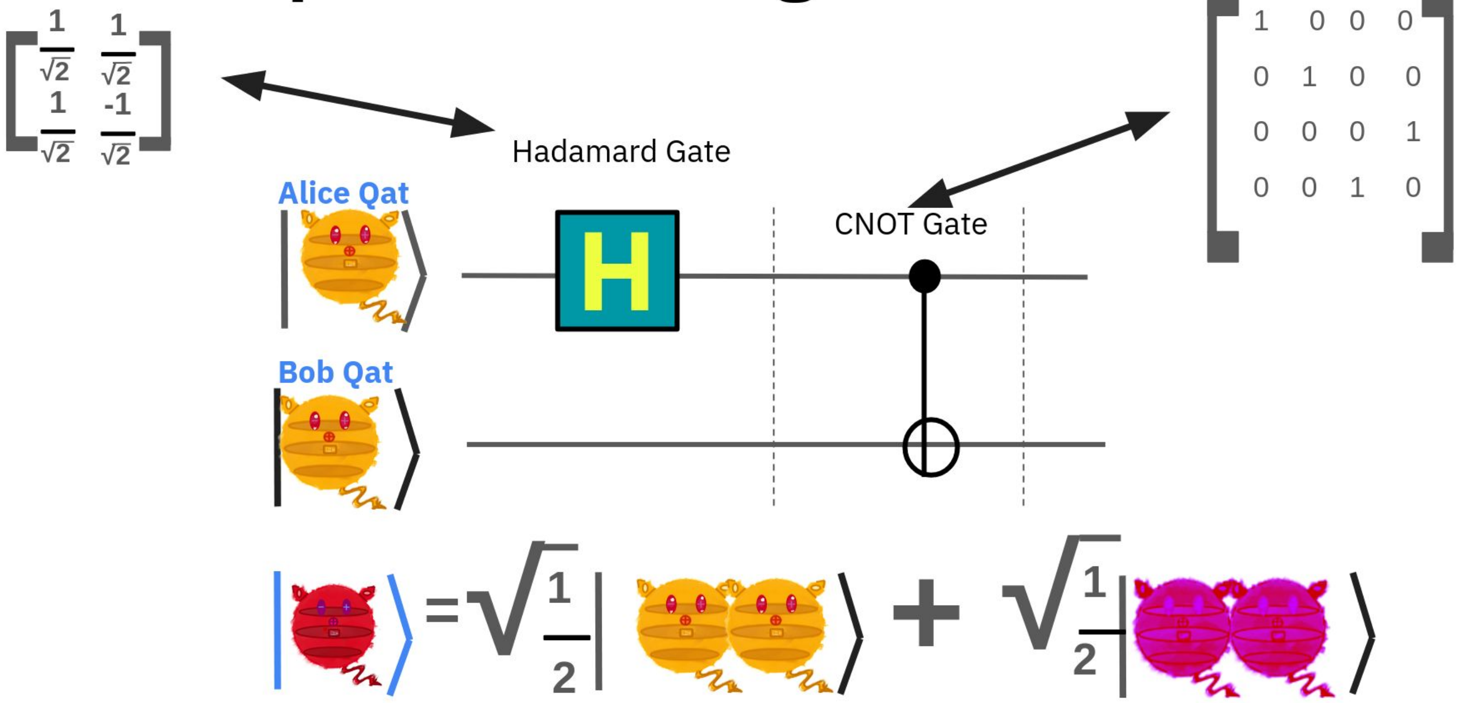
$$\frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$$

Bell states

Bell states are quantum states of two qubits that represent simple examples of quantum entanglement.

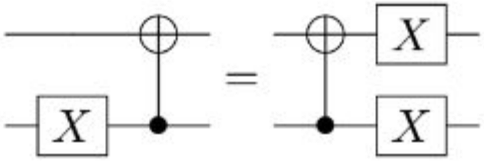
When one of the two qubits is measured, it takes on a specific value, and the second qubit is forced to also take on a specific value, as the entangled state collapses.

quantum entanglement

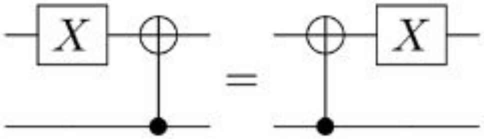


Exercise 4.15. Prove the following circuit identities, such as by finding the matrix representation of each circuit.

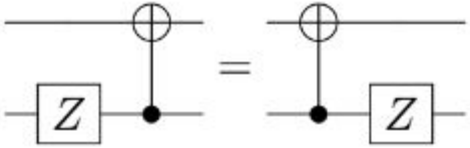
(a) $\text{CNOT}(X \otimes I) = (X \otimes X)\text{CNOT}$.



(b) $\text{CNOT}(I \otimes X) = (I \otimes X)\text{CNOT}$.



(c) $\text{CNOT}(Z \otimes I) = (Z \otimes I)\text{CNOT}$.



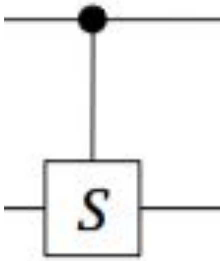
(d) $\text{CNOT}(I \otimes Z) = (Z \otimes Z)\text{CNOT}$.

Controlled Z Gate

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Controlled S Gate

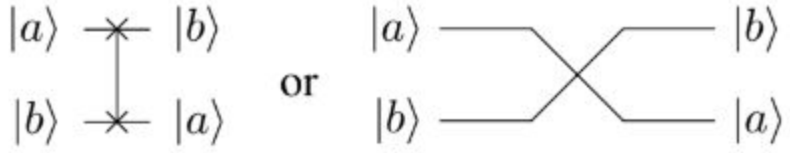
$$CS = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$



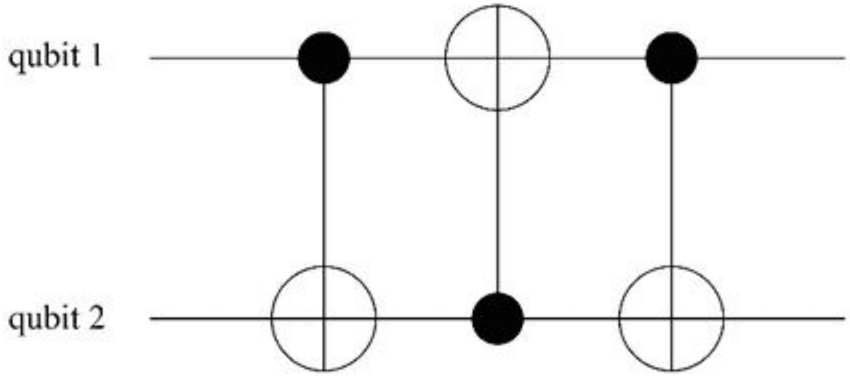
Swap gate

$$\mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} \mathrm{SWAP}|00\rangle &= |00\rangle, \\ \mathrm{SWAP}|01\rangle &= |10\rangle, \\ \mathrm{SWAP}|10\rangle &= |01\rangle, \\ \mathrm{SWAP}|11\rangle &= |11\rangle. \end{aligned}$$

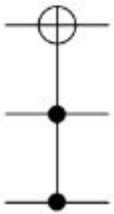


Swap Gate using Cnot Only



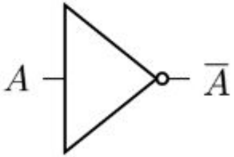
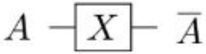
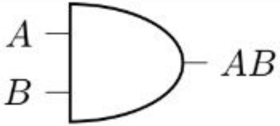
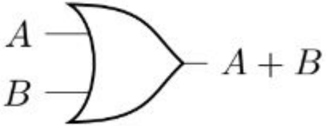
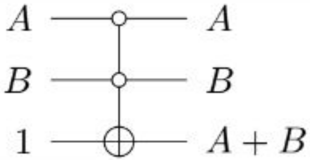
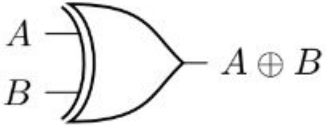
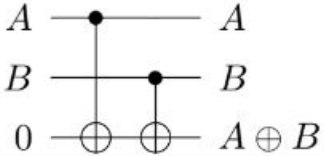
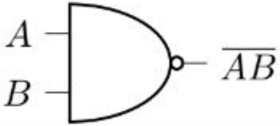
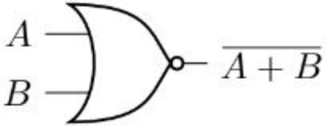
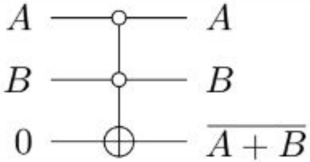
Tofoli gate

$$\text{Toffoli} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

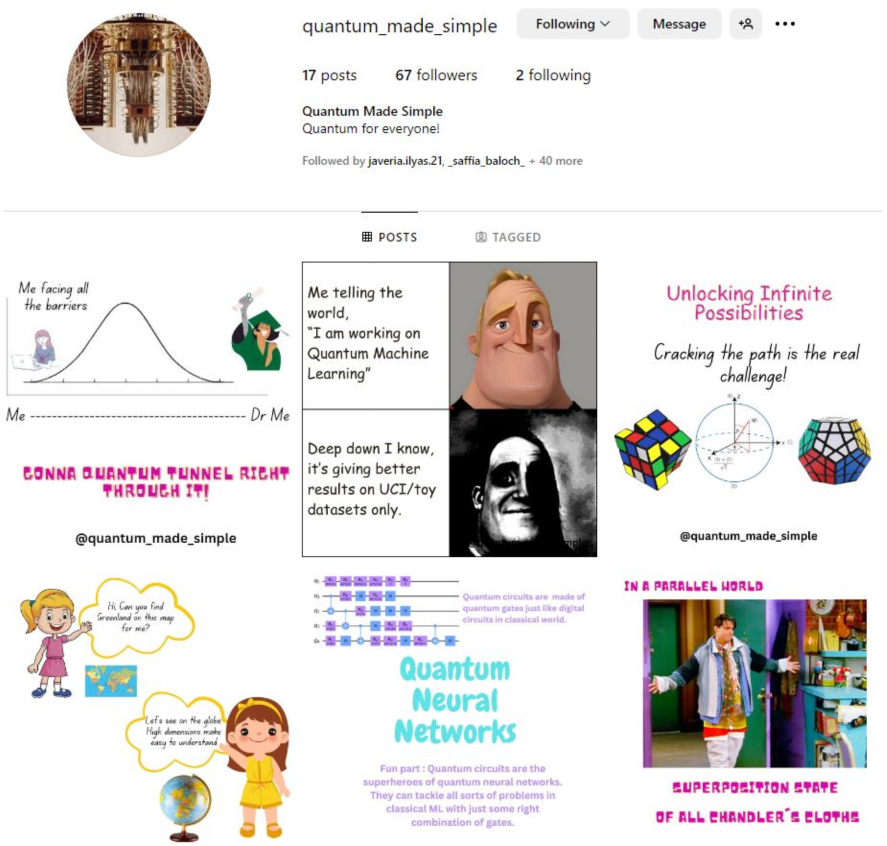


$$\begin{aligned}
\text{Toffoli}|000\rangle &= |000\rangle, \\
\text{Toffoli}|001\rangle &= |001\rangle, \\
\text{Toffoli}|010\rangle &= |010\rangle, \\
\text{Toffoli}|011\rangle &= |011\rangle, \\
\text{Toffoli}|100\rangle &= |100\rangle, \\
\text{Toffoli}|101\rangle &= |101\rangle, \\
\text{Toffoli}|110\rangle &= |111\rangle, \\
\text{Toffoli}|111\rangle &= |110\rangle.
\end{aligned}$$

$$\text{Toffoli}|a\rangle|b\rangle|c\rangle = |a\rangle|b\rangle|ab \oplus c\rangle.$$

Classical		Reversible/Quantum	
NOT		X-Gate	
AND		Toffoli	
OR		anti-Toffoli	
XOR		CNOTs	
NAND		Toffoli	
NOR		anti-Toffoli	

Two Qubit Gates


quantum_made_simple
Following
Message
17 posts
67 followers
2 following
Quantum Made Simple
Quantum for everyone!
Followed by javeria.ilyas.21, _saffia_baloch_ + 40 more
POSTS
TAGGED
Me facing all the barriers
Me -- Dr Me
GONNA QUANTUM TUNNEL RIGHT THROUGH IT!
@quantum_made_simple
Me telling the world,
"I am working on Quantum Machine Learning"
Deep down I know, it's giving better results on UCI/toy datasets only.
Unlocking Infinite Possibilities
Cracking the path is the real challenge!
@quantum_made_simple
Hi, Can you find Greenland on this map for me?
Let's see on the globe. High dimensions make easy to understand
Quantum circuits are made of quantum gates just like digital circuits in classical world.
Quantum Neural Networks
Fun part : Quantum circuits are the superheroes of quantum neural networks. They can tackle all sorts of problems in classical ML with just some right combination of gates.
IN A PARALLEL WORLD
SUPERPOSITION STATE
OF ALL CHANDLER'S CLOTHS

References

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