This is CS4084!

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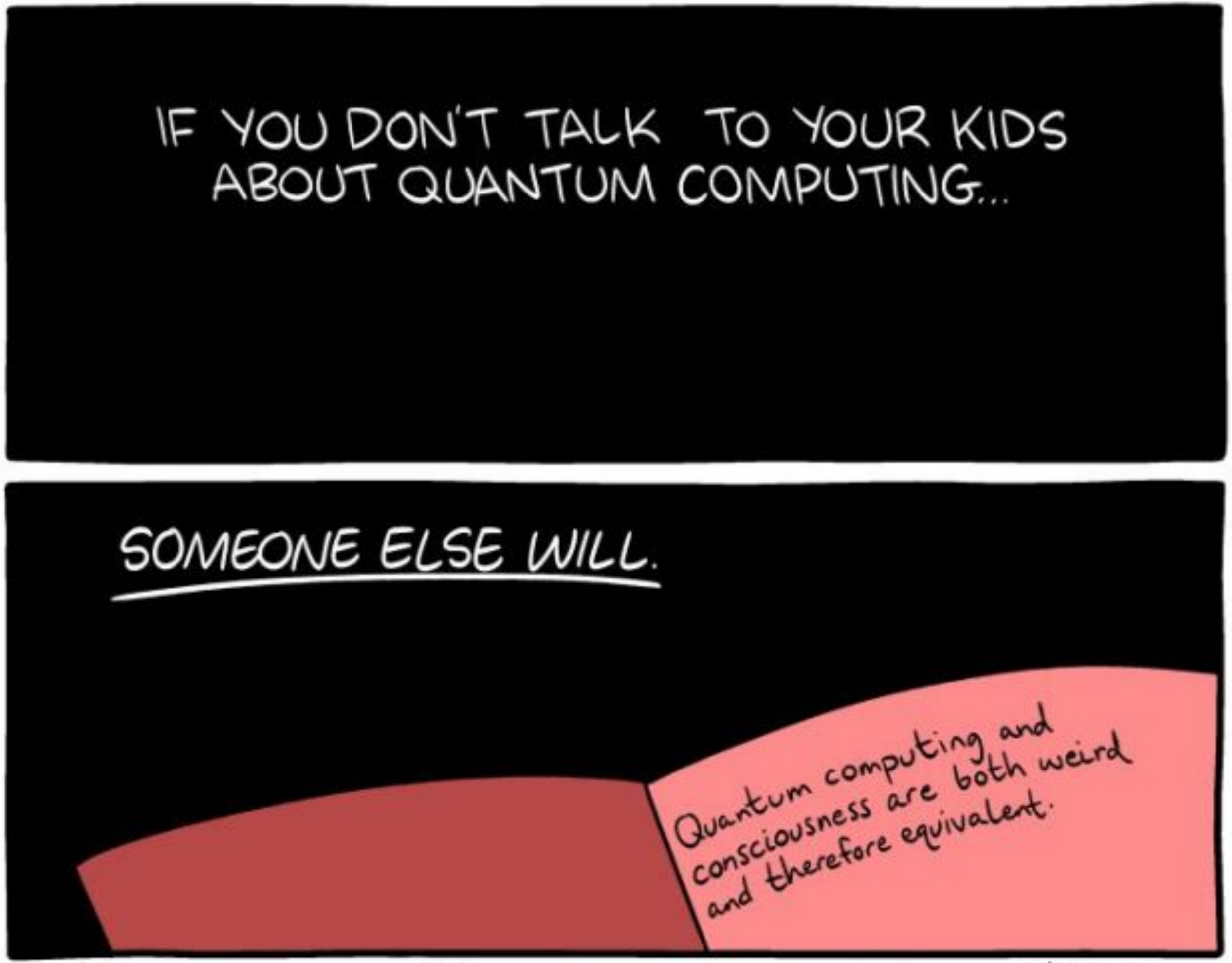

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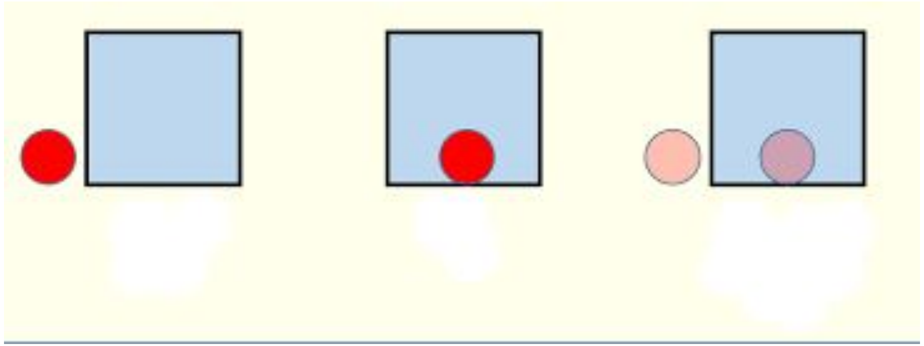
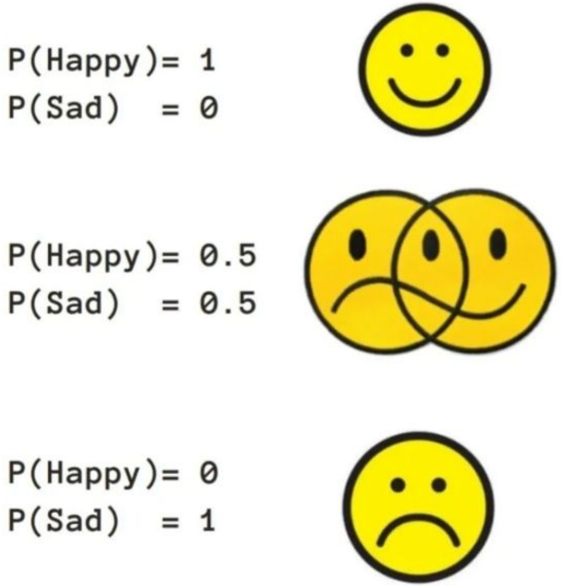
Single Qubit System

Superposition

Superposition is the ability of a quantum system to be in multiple states at the same time until it is measured.





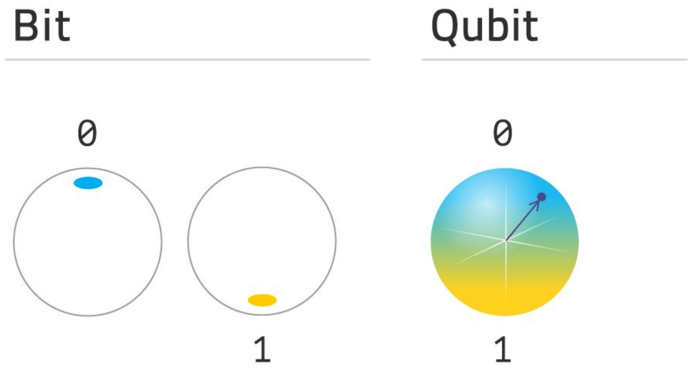
Quantum System

Any system that obeys the laws of quantum mechanics.

- Superposition
- Entanglement (We will study later)
- Quantization

Bit Vs Qubit

Quantum bits or qubits are similar to bits in that there are two measurable states called the 0 & 1.



However, qubits can also be in a superposition state of these 0 and 1 states.

Bit Vs Qubit

A classical bit can take two different values (0 or 1). It is discrete.

A qubit can “take” infinitely many different values.

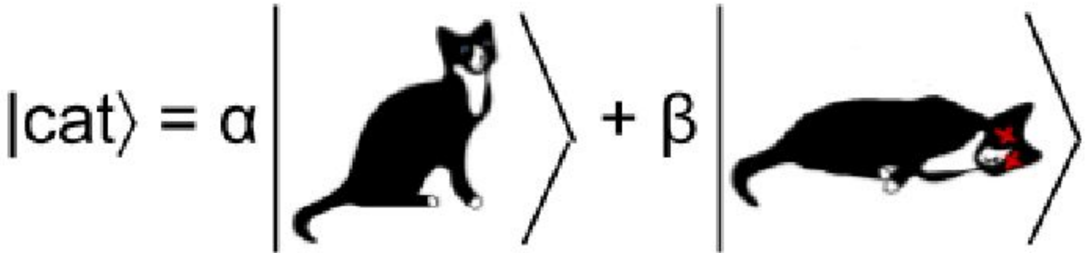
Dirac Bra-Ket Notation

Bra-ket notation is named after the symbols it uses:

“bra” ⟨ and “ket” ⟩

A quantum state is represented by a ket vector = $|\psi\rangle$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



Dirac Bra-Ket Notation

The symbol “|>” denotes a column vector, and is known as a “ket”.

The “bra” (<|) form of a vector is just the conjugate transpose of the original.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

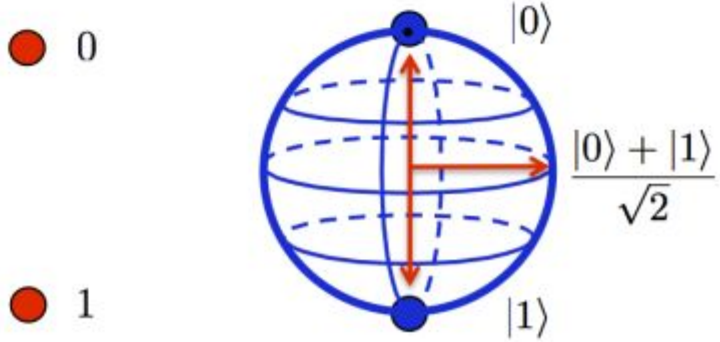
$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A generic qubit is in a **superposition**

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

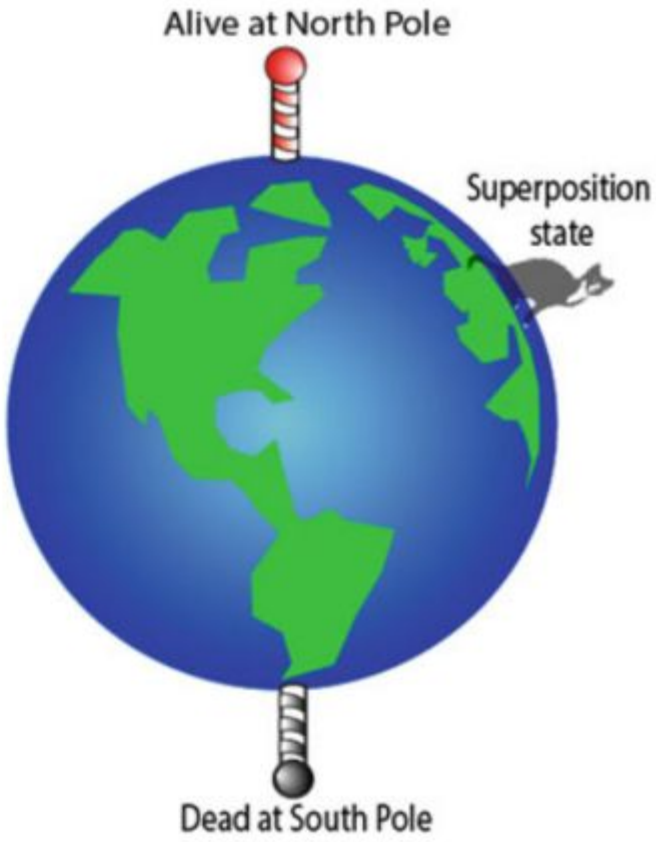
where α and β are **complex numbers** such that

$$|\alpha|^2 + |\beta|^2 = 1$$



Classical Bit **Qubit**

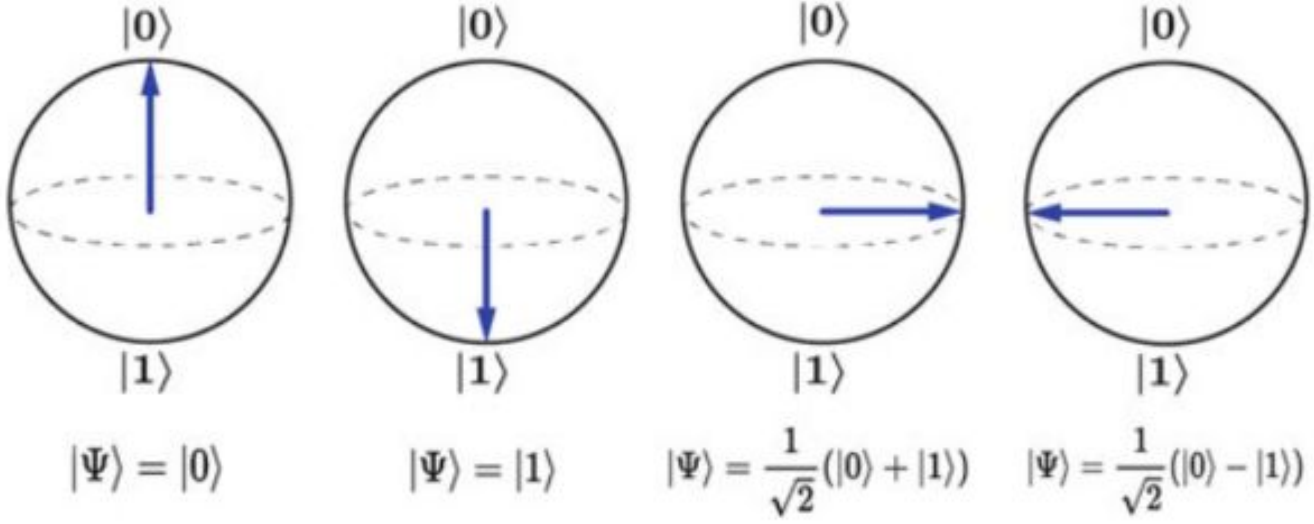
Bloch Sphere



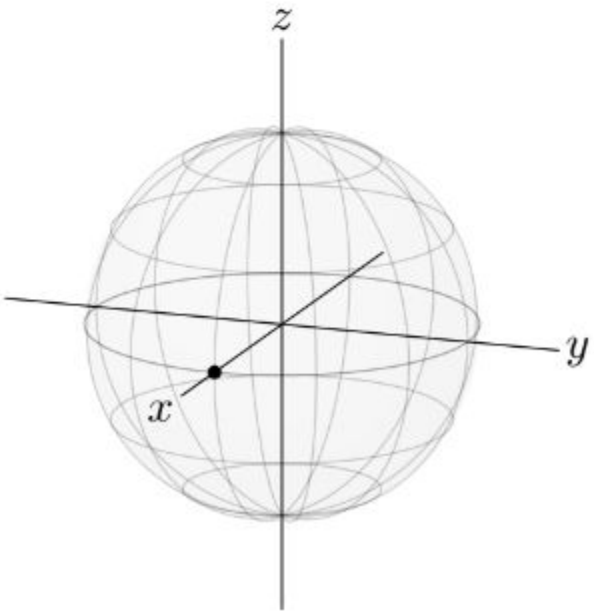
Bloch Sphere

A single qubit can be visualized using the Bloch sphere.

It is a unit sphere which means radius=1



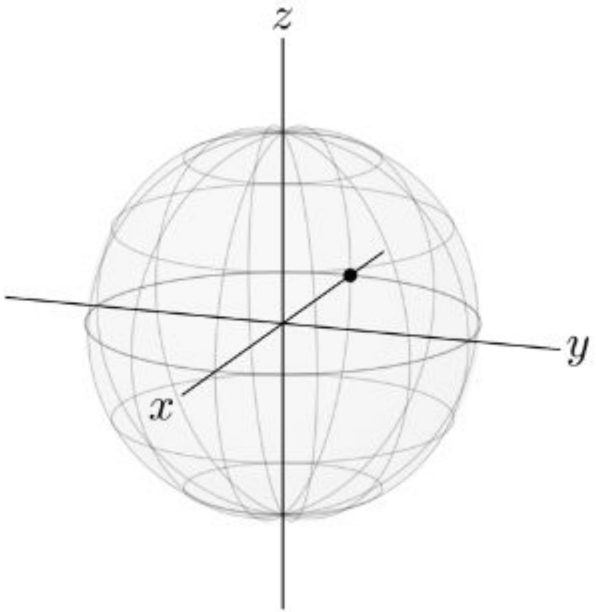
Bloch Sphere



$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right).$$

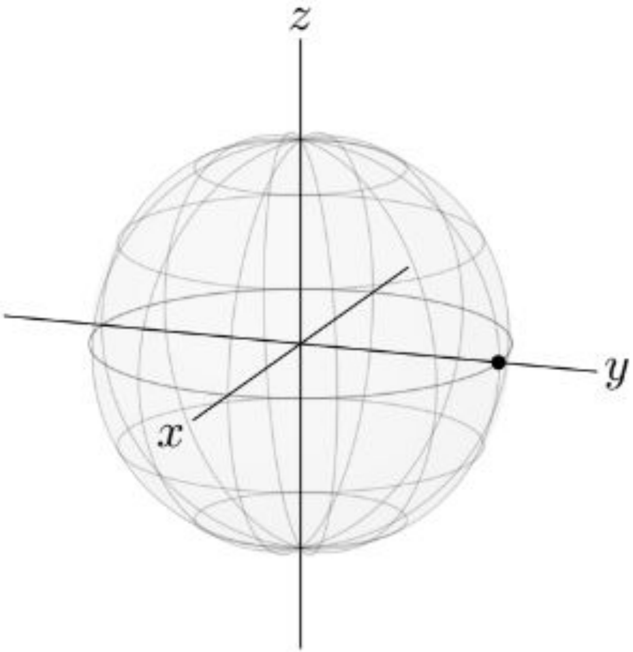
Bloch Sphere

$$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

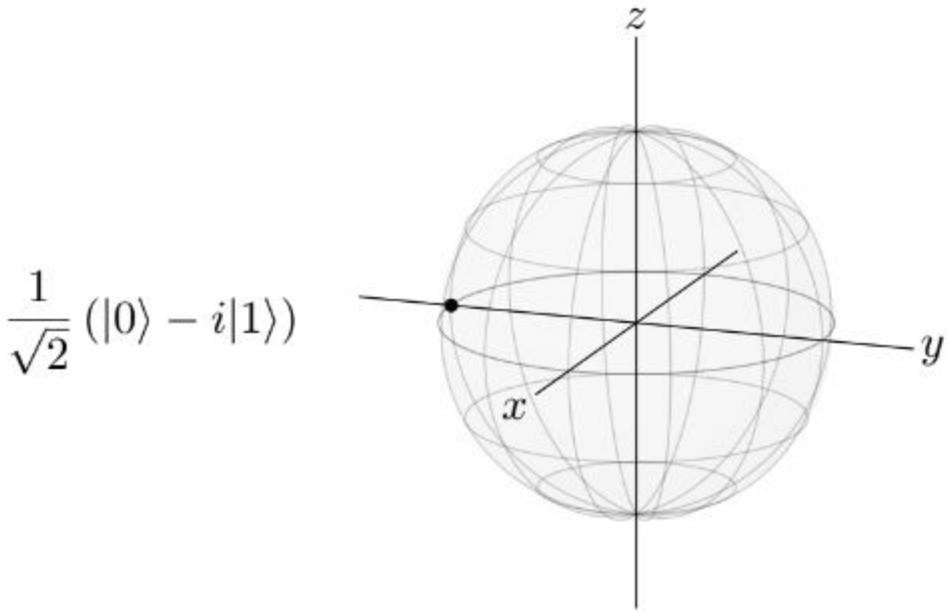


Bloch Sphere

$$\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$



Bloch Sphere



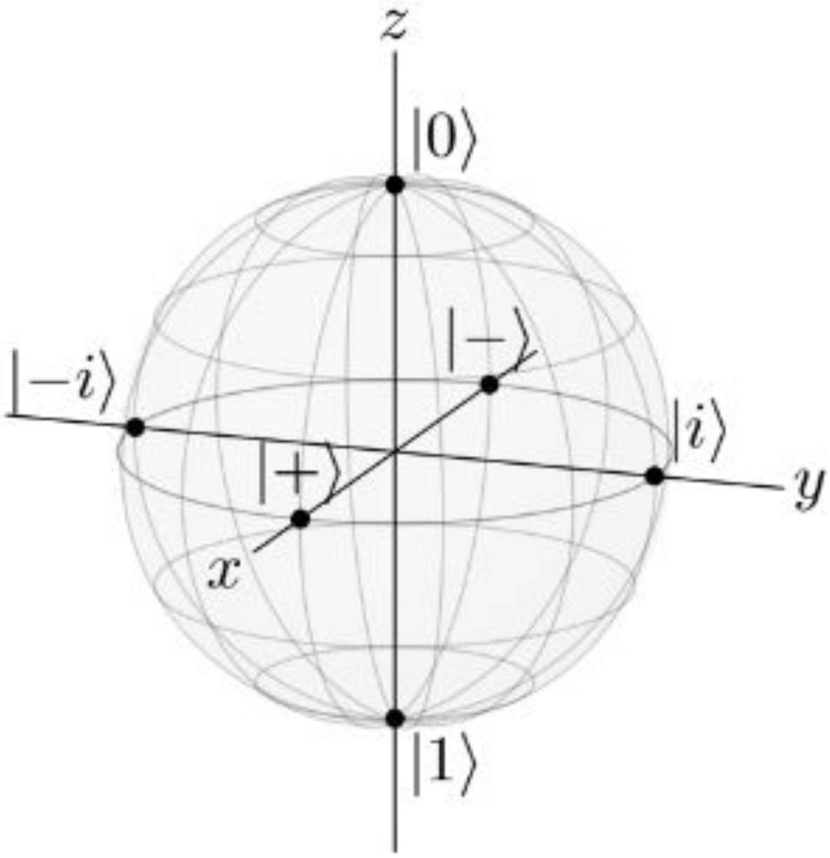
Bloch Sphere

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right),$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right),$$

$$|i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right),$$

$$|-i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right).$$



Review of Complex Numbers

$z = x+iy$ // Cartesian form of a complex number

In quantum computing, it is often useful to write a complex number as its length r times its complex phase $e^{i\theta}$

$z = re^{i\theta}$ // Polar form of a complex number

Note: We are covering chapter 2 from Introduction to classical and quantum computing by Thomas G wong

Review of Complex Numbers

How to convert cartesian to polar?

$$r = \sqrt{x^2 + y^2},$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

$$|z| = \sqrt{zz^*} = \sqrt{(x+iy)(x-iy)}$$
$$= \sqrt{x^2 + y^2}$$

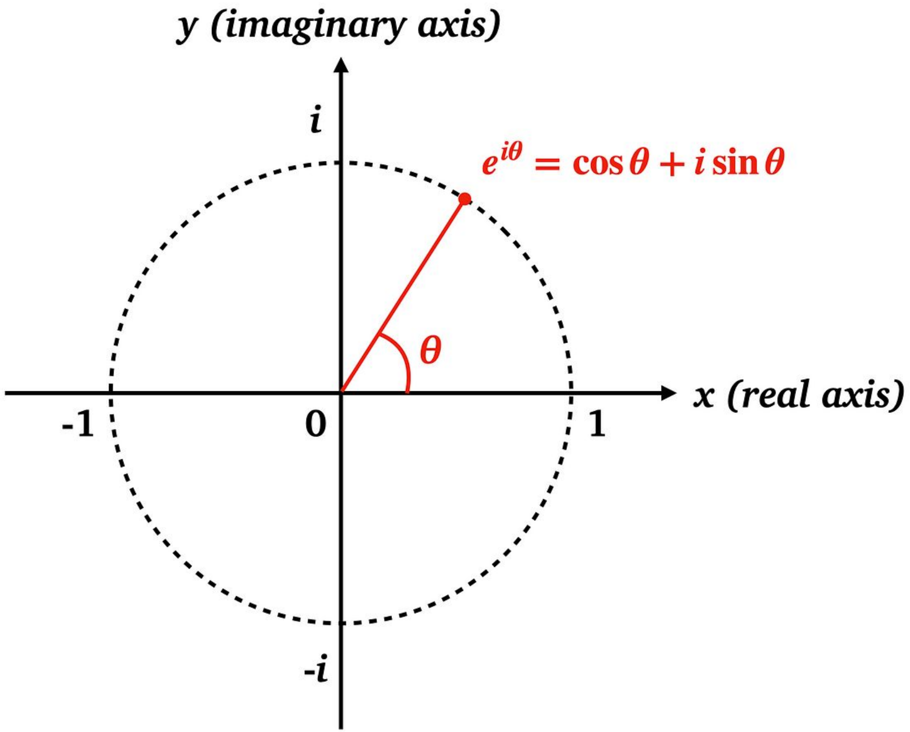
How to convert polar to cartesian?

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) = \underbrace{r\cos\theta}_{x} + i\underbrace{r\sin\theta}_{y}.$$



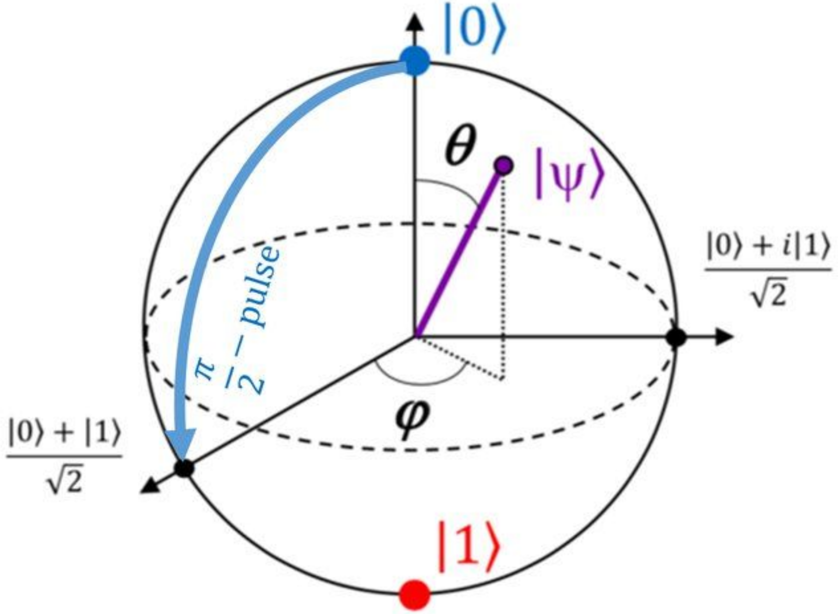
Euler formula is a bridge between trigonometric functions and exponential functions.

Bloch Sphere

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

$$\alpha = \cos\left(\frac{\theta}{2}\right), \quad \beta = e^{i\phi}\sin\left(\frac{\theta}{2}\right)$$

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle$$

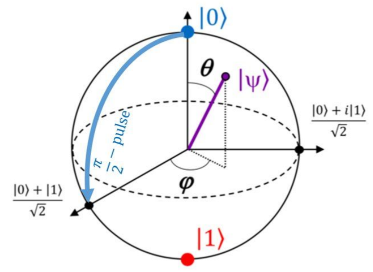


$0 \le \theta \le \pi$, θ is the angle wrt z axis

$0 \le \phi < 2\pi$, ϕ is wrt x axis

α is real and positive, β is complex, and the state is normalized.

Why do we use Complex Plane to explain Qubit?



Qubit works like waves, not switches

Waves need phase, not just size

Real numbers can tell you how big something is

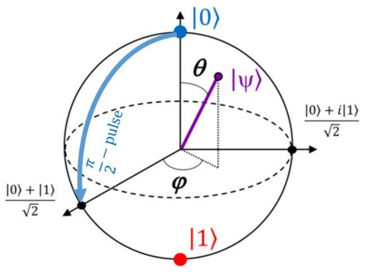
But quantum states also care about phase (the “where are you on the wave?” part)

$$|0\rangle + |1\rangle \qquad \text{vs} \qquad |0\rangle - |1\rangle$$

Both have the same probabilities

But they behave very differently in experiments

Why do we use Complex Plane to explain Qubit?



Real numbers can’t store phase properly

Complex numbers can

r → probability amplitude (how strong)

θ → phase (how rotated)

Quantum computing works because of interference:

Amplitudes add

Amplitudes cancel

Why do we use Complex Plane to explain Qubit?

Qubits are made from quantum objects electrons, photons, ions, superconducting currents, etc

And all of these obey quantum mechanics

Superposition = wave property (a wave being here and there at the same time)

A classical bit can’t do this because it’s like a switch

A qubit can because it’s like a wave pattern

We will discuss this in detail later in Quantum Hardware

Why do we use Complex Plane to explain Qubit?

Measurement kills the wave

Before measurement:

- wave-like
- spread out
- phase matters

After measurement: you see a single outcome (0 or 1)

- looks particle-like

This is why people say:

“Quantum objects act like waves until you look at them.”

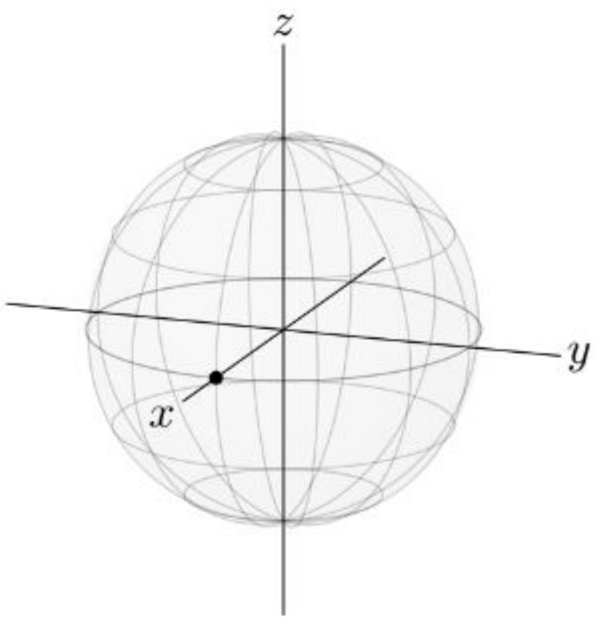
We will discuss this in detail later in Quantum Hardware

Bloch Sphere

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi} sin\frac{\theta}{2}|1\rangle$$

(1,0,0) θ=pi/2, ϕ=0

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

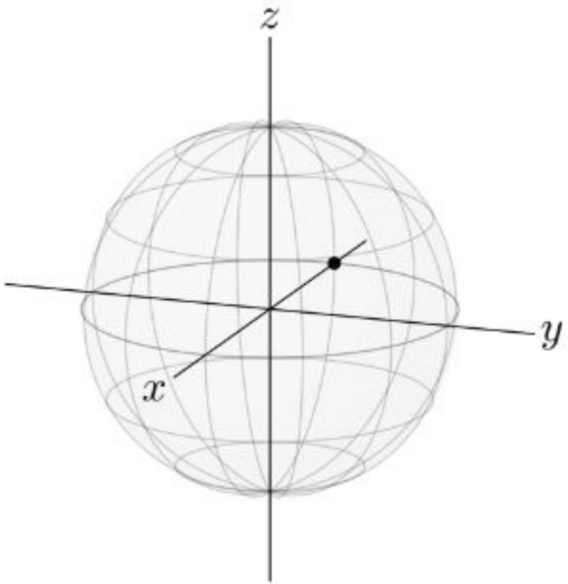


Bloch Sphere

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

(-1,0,0) θ=pi/2, ϕ=pi

$$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

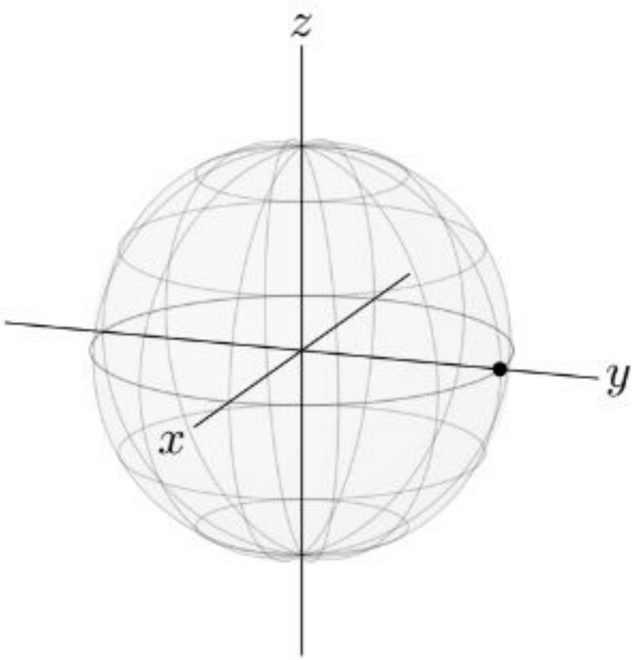


Bloch Sphere

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle$$

(0,1,0) θ=pi/2, ϕ=pi/2

$$\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

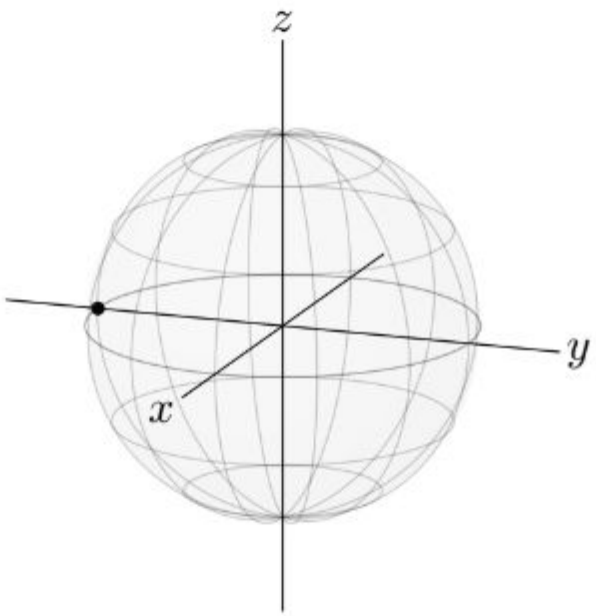


Bloch Sphere

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle$$

(0,-1,0) θ=pi/2, ϕ=3pi/2

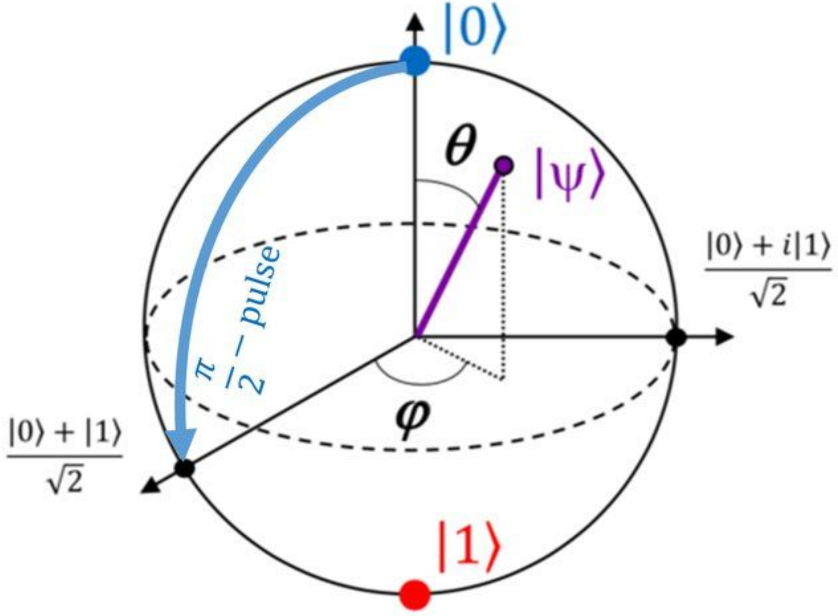
$$\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$



For calculating anywhere on Bloch Sphere

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle$$

θ=? , ϕ=?



For calculating anywhere on Bloch Sphere

Calculations done in class

Example 1

1. The quantum state of a spinning coin can be written as a superposition of heads and tails. Using heads as $|1\rangle$ and tails as $|0\rangle$, the quantum state of the coin is

$$|\text{coin}\rangle = \frac{1}{\sqrt{2}} \left(|1\rangle + |0\rangle \right). \tag{2.3}$$

What is the probability of getting heads?

The amplitude of $|1\rangle$ is $\beta = 1/\sqrt{2}$, so $|\beta|^2 = \left(1/\sqrt{2}\right)^2 = 1/2$. So the probability is 0.5, or 50%.

Example 2

A weighted coin has twice the probability of landing on heads vs. tails. What is the state of the coin in “ket” notation?

Example 2

A weighted coin has twice the probability of landing on heads vs. tails. What is the state of the coin in "ket" notation?

$$P_{\text{heads}} + P_{\text{tails}} = 1 \quad \text{(Normalization Condition)}$$

$$P_{\text{heads}} = 2P_{\text{tails}} \quad \text{(Statement in Example)}$$

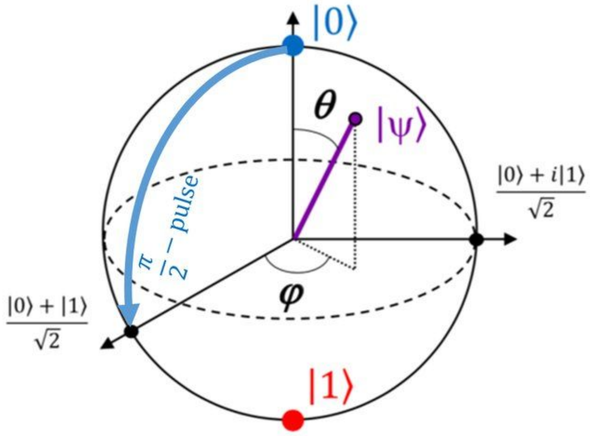
$$\rightarrow P_{\text{tails}} = \frac{1}{3} = \alpha^2$$

$$\rightarrow P_{\text{heads}} = \frac{2}{3} = \beta^2$$

$$\rightarrow \alpha = \sqrt{\frac{1}{3}},\ \beta = \sqrt{\frac{2}{3}}$$

$$\rightarrow |\text{coin}\rangle = \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle.$$

For calculating anywhere on Bloch Sphere



$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

θ=?, ϕ=?

θ is the polar angle (latitude) on the Bloch sphere, ranging from 0 to π

ϕ is the azimuthal angle (longitude), ranging from 0 to 2π

Relative Phase

On the Bloch sphere, the relative phase ϕ \phi determines the qubit’s longitude.

Changing the relative phase ϕ rotates the qubit state around the Z-axis.

The relative phase of a quantum state is a measure of the angle in the complex plane.

Relative Phase

Two superpositions states whose amplitudes have the same magnitudes but that differ in a relative phase represent different states.

Relative phase is a physically important quantity.

Global Phase

Applying a global phase is like rotating the entire carousel by a certain angle. It doesn't change the relative positions of where you are on the carousel; it just shifts everything uniformly.

Global Phase

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

$$|\psi'\rangle = e^{i\phi_{\text{global}}}|\psi\rangle$$

$$|\psi'\rangle = e^{i\phi_{\text{global}}}\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle\right]$$

Global Phase

$$|\psi'\rangle = e^{i\phi_{\text{global}}} \left[\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi}|1\rangle \right]$$

$$|\psi'\rangle = \cos\left(\frac{\theta}{2}\right) e^{i\phi_{\text{global}}}|0\rangle + \sin\left(\frac{\theta}{2}\right) e^{i(\phi+\phi_{\text{global}})}|1\rangle$$

$$\alpha' = \cos\left(\frac{\theta}{2}\right) e^{i\phi_{\text{global}}}$$

$$\beta' = \sin\left(\frac{\theta}{2}\right) e^{i(\phi+\phi_{\text{global}})}$$

Global Phase

Global phases are physically irrelevant.

Just like Upgrading

Global Phase

$$e^{i\phi} = \cos(\phi) + i\sin(\phi)$$

$$|e^{i\phi}| = \sqrt{\cos^2(\phi) + \sin^2(\phi)}$$

$$|e^{i\phi}| = \sqrt{1} = 1$$

Global Phase

$$e^{i\theta}\left(\frac{\sqrt{3}}{2}|0\rangle+\frac{1}{2}|1\rangle\right),$$

Calculate probabilities

Global phases can be dropped/ignored.

States that differ by a global phase are actually the same state; they correspond to the same point on the Bloch sphere.

Exercise 2.6. A qubit is in the state

$$\frac{1+i\sqrt{3}}{3}|0\rangle + \frac{2-i}{3}|1\rangle.$$

If you measure the qubit, what is the probability of getting

(a) $|0\rangle$?

(b) $|1\rangle$?

Exercise 2.7. A qubit is in the state

$$\frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle.$$

Say you measure the qubit and get $|0\rangle$. If you measure the qubit a *second time*, what is the probability of getting

(a) $|0\rangle$?

(b) $|1\rangle$?

Normalization

A quantum state is normalized if its total probability is 1, as it should be.

Sometimes, we must find an overall normalization constant to make this true.

Normalization

$$A\left(\sqrt{2}|0\rangle + i|1\rangle\right).$$

$$\begin{aligned} 1 &= (A\sqrt{2})(A\sqrt{2})^* + (Ai)(Ai)^* \\ &= 2|A|^2 + |A|^2 \\ &= 3|A|^2 \\ |A|^2 &= \frac{1}{3}. \end{aligned}$$

$$A = \frac{1}{\sqrt{3}},$$

$$\frac{1}{\sqrt{3}}\left(\sqrt{2}|0\rangle + i|1\rangle\right).$$

Exercise 2.8. A qubit is in the state

$$\frac{e^{i\pi/8}}{\sqrt{5}}|0\rangle + \beta|1\rangle.$$

What is a possible value of β?

Exercise 2.9. A qubit is in the state

$$A\left(2e^{i\pi/6}|0\rangle - 3|1\rangle\right).$$

(a) Normalize the state (i.e., find A).
(b) If you measure the qubit, what is the probability that you get $|0\rangle$?
(c) If you measure the qubit, what is the probability that you get $|1\rangle$?

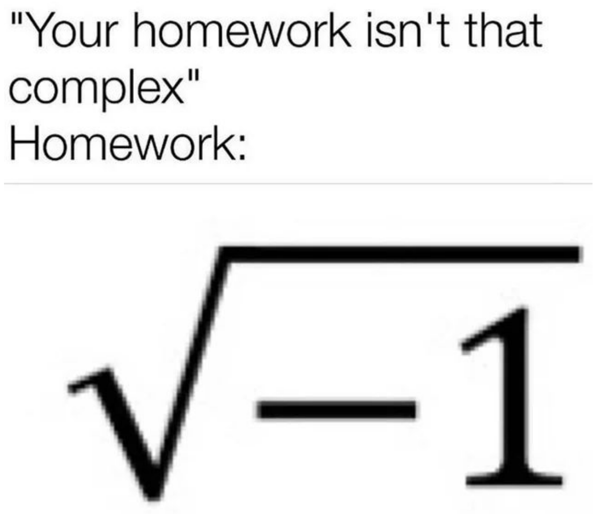
Qubit Rotations

Complex Conjugate

The complex conjugate of a complex number is obtained by changing the sign of its imaginary part.

Z = x + iy

Z’ = x − iy



Hermitian Matrix

A matrix H is called Hermitian if it is equal to its own conjugate transpose (or Hermitian adjoint)

$$H = \begin{pmatrix} 2 & i \\ -i & 3 \end{pmatrix}$$

$$H = \begin{pmatrix} 4 & 1+2i & 3-i \\ 1-2i & 5 & 2+4i \\ 3+i & 2-4i & 6 \end{pmatrix}$$

Unitary Matrix

Changing a qubit's state through a physical action mathematically corresponds to multiplying the qubit vector by some unitary matrix U so that after the operation the state is now

$$|\psi'\rangle = U|\psi\rangle$$

Unitary is a mathematical term which expresses that U can only act on the qubit in such a way that the total probability remains same

Unitary Matrix

A matrix U is unitary if the matrix product of U and its conjugate transpose U† (called U-dagger) multiply to give the identity matrix:

$$U^\dagger U = UU^\dagger = I$$

One fundamental assumption is that each (matrix) operator must be unitary in all mathematical constructions of quantum mechanics

Single Qubit Gates

Analogous to logical gates in classical computing, single-qubit gates act on individual qubits, modifying their quantum states.

PAULI GATES

PHYSICIST WOLFGANG PAULI

When a Pauli gate is applied to a qubit, the state of the qubit is rotated around the corresponding axis of the Bloch sphere

These gates are important for manipulating the phase of a qubit

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli X-Gate

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q_0: ┤ Rx(Θ) ├
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Pauli X-gate perform a rotation of 180 degrees around the X axis

The gate is called NOT gate as it flips the qubit from |1> to |0> and vice versa

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

Pauli Z-Gate

Pauli z-gate perform a rotation of 180 degrees around the z axis

The gate leaves state |0> as such but flips the state |1> to -|1>

The gate is called phase flip

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$$

Pauli Y-Gate

Pauli y-gate perform a rotation of 180 degrees around the y axis

The gate is a combination of bit flip and phase flip

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$$

$$Y|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle$$

Exercise

Prove each of the pauli matrices satisfies the unitary condition

$$X^\dagger X = I, \quad Y^\dagger Y = I, \quad Z^\dagger Z = I$$

Prove that Pauli matrices are Hermitian

Exercise 2.7. A qubit is in the state

$$\frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle.$$

Apply the X gate to this qubit state.

Exercise 2.7. A qubit is in the state

$$\frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle.$$

Apply the Y gate to this qubit state.

Exercise 2.7. A qubit is in the state

$$\frac{1+i\sqrt{3}}{3}|0\rangle + \frac{2-i}{3}|1\rangle.$$

Apply the X gate to this qubit state.

Exercise 2.7. A qubit is in the state

$$\frac{1+i\sqrt{3}}{3}|0\rangle + \frac{2-i}{3}|1\rangle.$$

Apply the Y gate to this qubit state.

Exercise 2.7. A qubit is in the state

$$\frac{1+i\sqrt{3}}{3}|0\rangle + \frac{2-i}{3}|1\rangle.$$

Apply the Z gate to this qubit state.

Qubit unitary operations

1. Pauli operations

Pauli operations are ones represented by the Pauli matrices:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Common alternative notations: $X = \sigma_x$, $Y = \sigma_y$, and $Z = \sigma_z$.

The operation σ_x is also called a *bit flip* (or a NOT operation) and the σ_z operation is called a *phase flip:*

$$\begin{aligned} \sigma_x|0\rangle &= |1\rangle \qquad & \sigma_z|0\rangle &= |0\rangle \\ \sigma_x|1\rangle &= |0\rangle \qquad & \sigma_z|1\rangle &= -|1\rangle \end{aligned}$$

Hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|0\rangle \; -\boxed{H}- \; \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1\rangle \; -\boxed{H}- \; \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Use matrix multiplication to show how applying the Hadamard gate twice to a $|0\rangle$ state qubit recovers its original state.

Use matrix multiplication to show how applying the Hadamard gate twice to a $|0\rangle$ state qubit recovers its original state.

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$H\,H|0\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Qubit unitary operations

2. Hadamard operation

The Hadamard operation is represented by this matrix:

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Checking that H is unitary is a straightforward calculation:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^{\dagger} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Qubit unitary operations

Example 1

$$H\,|0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle$$

$$H\,|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle$$

$$H\,|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$H\,|-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

Qubit unitary operations

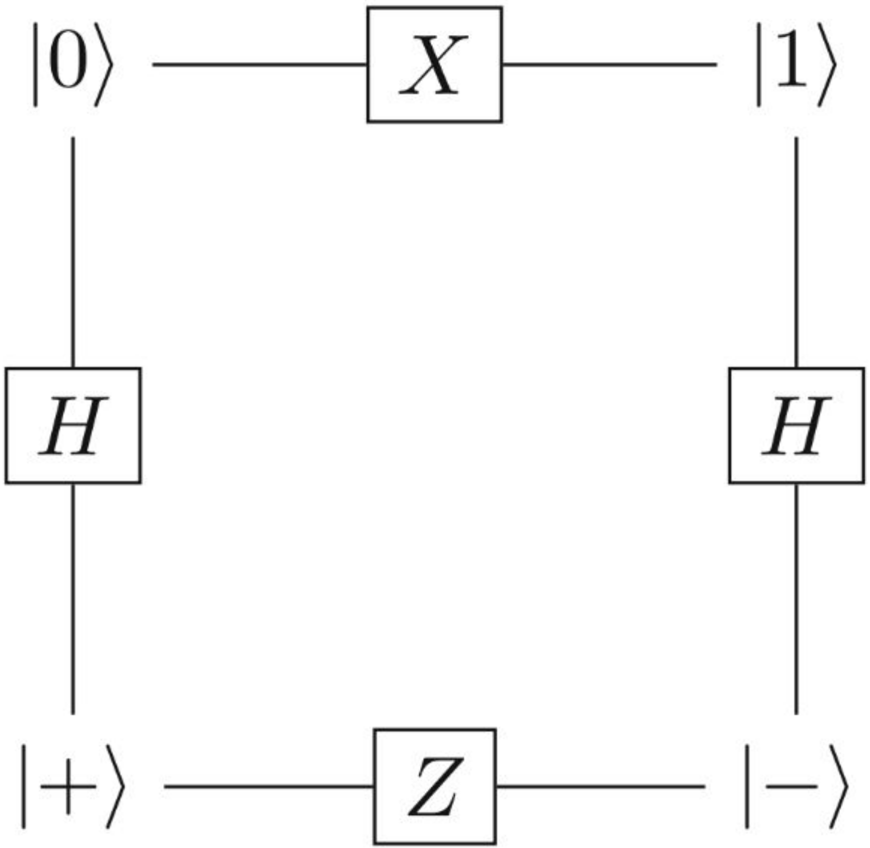
Example 1

$$H|0\rangle = |+\rangle \qquad H|+\rangle = |0\rangle$$

$$H|1\rangle = |-\rangle \qquad H|-\rangle = |1\rangle$$

$$H\left(\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{-1+2i}{3\sqrt{2}} \\ \frac{3+2i}{3\sqrt{2}} \end{pmatrix}$$

$$= \frac{-1+2i}{3\sqrt{2}}|0\rangle + \frac{3+2i}{3\sqrt{2}}|1\rangle$$


$|0\rangle$
X
$|1\rangle$
H
H
$|+\rangle$
Z
$|-\rangle$

6.4.1 Examples

1. A spin right $1/\sqrt{2}(|0\rangle + |1\rangle)$ is sent through a Hadamard gate, creating a superposition of $|+\rangle$ and $|-\rangle$ given by $1/\sqrt{2}(|+\rangle + |-\rangle)$. By performing a basis change, show that this is equivalent to producing a $|0\rangle$ state.

6.4.1 Examples

1. A spin right $1/\sqrt{2}(|0\rangle + |1\rangle)$ is sent through a Hadamard gate, creating a superposition of $|+\rangle$ and $|-\rangle$ given by $1/\sqrt{2}(|+\rangle + |-\rangle)$. By performing a basis change, show that this is equivalent to producing a $|0\rangle$ state.

$$\frac{1}{\sqrt{2}}\Big(|+\rangle + |-\rangle\Big) = \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\Big) + \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\Big), \tag{6.6}$$

$$= \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle, \tag{6.7}$$

$$= |0\rangle. \tag{6.8}$$

Tasks

7. Use matrix multiplication to demonstrate
   (a) The Hadamard gate applied to a $|1\rangle$ state qubit turns it into a $|-\rangle$.
   (b) A second Hadamard gate turns it back into the $|1\rangle$ state.
   (c) The output after applying the Hadamard gate twice to a general state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

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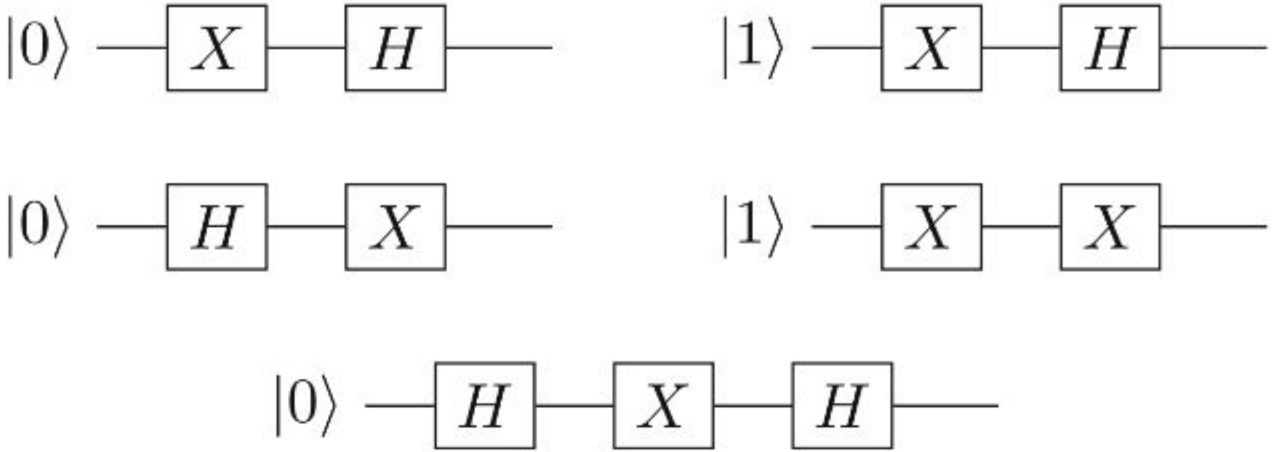


Fig. 6.7 Five quantum circuits for Problem 8

S gate or Phase Gate

This gate is a 90-degree phase shift gate around the z axis that introduces a phase shift of $\pi/2$ radians to the $|1\rangle$ state. Phase gate is the square root of the Z gate (i.e., $S^2 = Z$):

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$S|0\rangle = |0\rangle,$$
$$S|1\rangle = i|1\rangle.$$

T GATE

This gate is a 45-degree phase shift gate that introduces a phase shift of π/4 radians to the |1⟩ state. While leaving the state |0⟩ unchanged.

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$T|0\rangle = |0\rangle,$$
$$T|1\rangle = e^{i\pi/4}|1\rangle.$$

Qubit unitary operations

3. Phase operations

A phase operation is one described by the matrix

$$P_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

for any choice of a real number θ.

The operations

$$S = P_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad T = P_{\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

are important examples.

Qubit unitary operations

Example 2

$$T|0\rangle = |0\rangle \quad \text{and} \quad T|1\rangle = \frac{1+i}{\sqrt{2}}|1\rangle$$

$$\begin{aligned} T\,|+\rangle &= T\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \\ &= \frac{1}{\sqrt{2}}T\,|0\rangle + \frac{1}{\sqrt{2}}T\,|1\rangle \\ &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle \end{aligned}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

Qubit unitary operations

Example 2

$$T\,|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$$

$$\begin{aligned}
HT|+\rangle &= H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}H|0\rangle + \frac{1+i}{2}H|1\rangle \\
&= \frac{1}{\sqrt{2}}|+\rangle + \frac{1+i}{2}|-\rangle \\
&= \left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle\right) + \left(\frac{1+i}{2\sqrt{2}}|0\rangle - \frac{1+i}{2\sqrt{2}}|1\rangle\right) \\
&= \left(\frac{1}{2} + \frac{1+i}{2\sqrt{2}}\right)|0\rangle + \left(\frac{1}{2} - \frac{1+i}{2\sqrt{2}}\right)|1\rangle
\end{aligned}$$

$$H\,|0\rangle = |+\rangle$$
$$H\,|1\rangle = |-\rangle$$

$$X^{1001} = X^{1000}X = \left(X^2\right)^{500}X = I^{500}X = X.$$

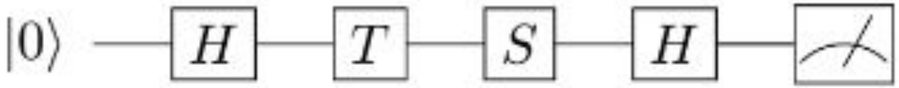
Exercise 2.26. Calculate $Z^{217}X^{101}Y^{50}(\alpha|0\rangle + \beta|1\rangle)$.

Exercise 2.27. Prove that

(a) $XZXZ(\alpha|0\rangle + \beta|1\rangle) = -(\alpha|0\rangle + \beta|1\rangle)$.

(b) $ZXZX(\alpha|0\rangle + \beta|1\rangle) = -(\alpha|0\rangle + \beta|1\rangle)$.

$$HSTH|0\rangle = HST\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$



$$
\begin{aligned}
HSTH|0\rangle &= HST\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \\
&= HS\frac{1}{\sqrt{2}}\left(|0\rangle+e^{i\pi/4}|1\rangle\right) \\
&= H\frac{1}{\sqrt{2}}\left(|0\rangle+e^{i3\pi/4}|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)+e^{i3\pi/4}\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right] \\
&= \frac{1}{2}\left[\left(1+e^{i3\pi/4}\right)|0\rangle+\left(1-e^{i3\pi/4}\right)|1\rangle\right], \qquad (2.9)
\end{aligned}
$$

where in the third line, we used $ie^{i\pi/4} = e^{i\pi/2}e^{i\pi/4} = e^{i3\pi/4}$. On the Bloch sphere, this state is in the southern hemisphere:

Composing unitary operations

Compositions of unitary operations are represented by *matrix multiplication* (similar to the probabilistic setting).

Example: square root of NOT

Applying a Hadamard operation, followed by the phase operation S, followed by another Hadamard operation yields this operation:

$$\mathrm{HSH} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}$$

Applying this unitary operation twice yields a NOT operation:

$$(\mathrm{HSH})^2 = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Rotation Operator

Rotation around the x-axis

$$R_x(\theta) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) \sigma_x$$

$$R_x(\theta) = \cos\left(\frac{\theta}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin\left(\frac{\theta}{2}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$R_x(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i \sin\left(\frac{\theta}{2}\right) \\ -i \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Task

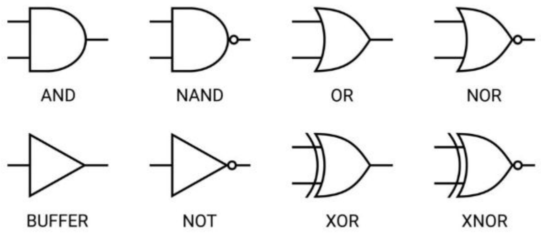
Prove this

$$\sigma_x = iR_x(\ \ 180^\circ) = -iR_x(-180^\circ)$$

Do the same exercise for Ry and Rz.

Rotation Operator

Life was easier back then when you only knew these:



Rotation around the y-axis

$$R_y(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)\sigma_y$$

$$R_y(\theta) = \cos\left(\frac{\theta}{2}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - i\sin\left(\frac{\theta}{2}\right)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{pmatrix}$$

Rotation Operator

$$R_z(\theta)|0\rangle = |0\rangle,$$
$$R_z(\theta)|1\rangle = e^{i\theta}|1\rangle.$$

Rotation around the z-axis

$$R_z(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)\sigma_z$$

$$R_z(\theta) = \cos\left(\frac{\theta}{2}\right)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - i\sin\left(\frac{\theta}{2}\right)\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix}e^{-i\frac{\theta}{2}} & 0\\0 & e^{i\frac{\theta}{2}}\end{pmatrix}$$

Exercise 2.7. A qubit is in the state

$$\frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle.$$

Apply the RX gate to this qubit state with θ=90.

Exercise 2.7. A qubit is in the state

$$\frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle.$$

Apply the RX gate to this qubit state with θ=90. And then apply X gate.

Exercise 2.7. A qubit is in the state

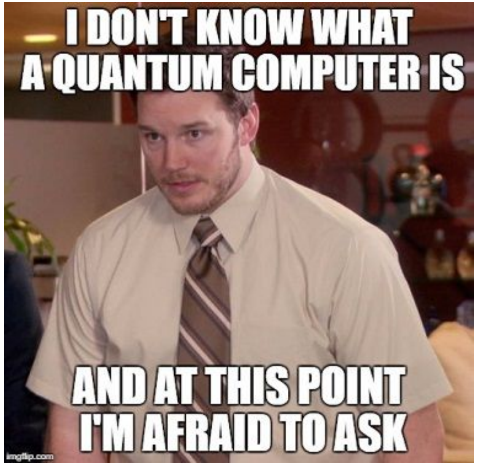
$$\frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle.$$

Apply the RX gate to this qubit state with θ=90. And then apply X gate. And then apply H gate. What will be the final state?

Exercise 2.7. A qubit is in the state

$$\frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle.$$

Apply the RX gate to this qubit state with θ=90. And then apply X , H and Y gate. What will be the final state?



Single Qubit System

Electron as a quantum system

Electron as Qubit

An electron is a prototype for a qubit.

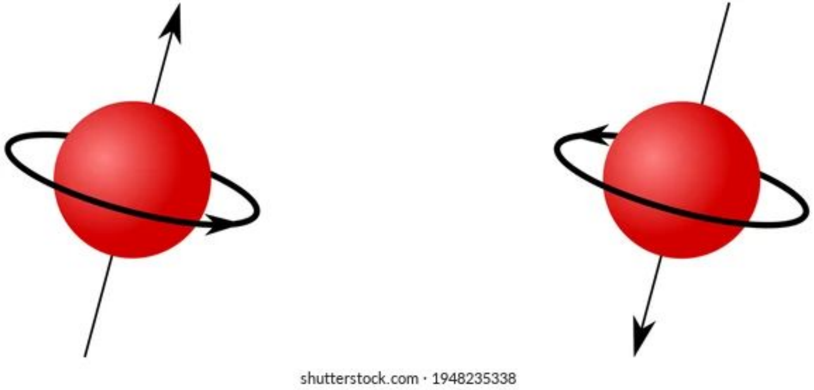
An electron has many measurable properties such as energy, mass, momentum.

But, for the purposes of creating a qubit, we want to focus on a property with only two measurable values. An electron has a two-state property which is called **spin**.

Electron as Qubit

The property was called spin because it can be described mathematically just like orbital momentum (angular momentum), but spin does not actually correspond to the electron physically rotating.

Just like a lot of quantum phenomena, spin can be confusing at first.



Electron Spin

$|\uparrow\rangle$ = spin up → clockwise

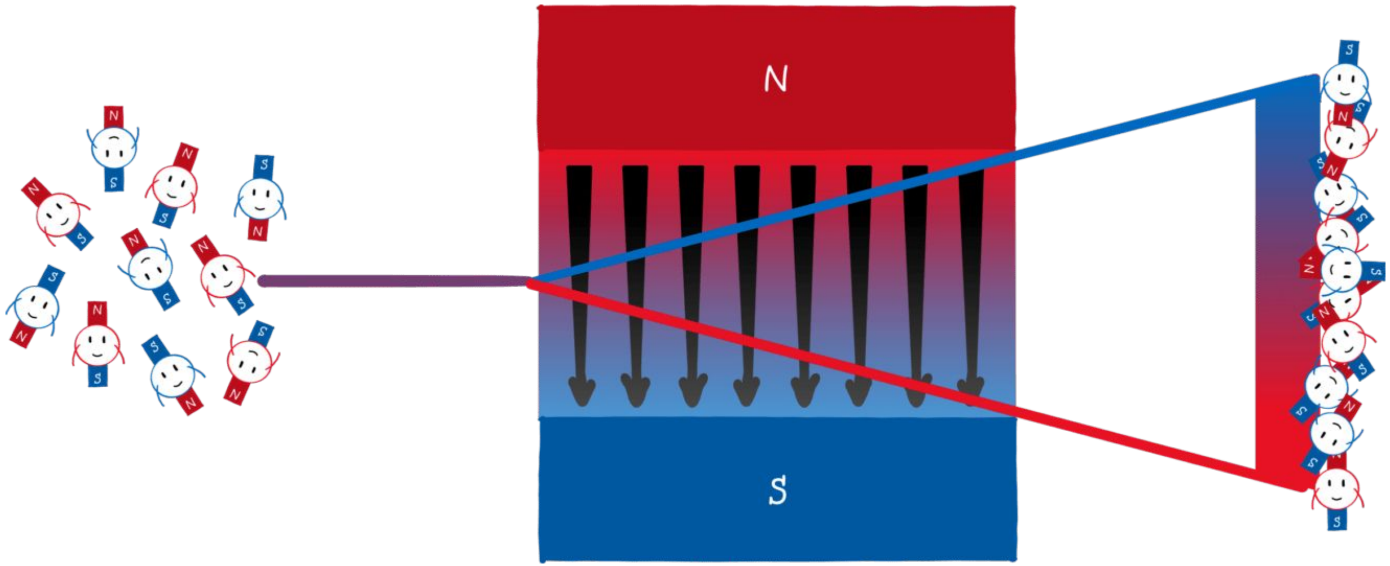
$|\downarrow\rangle$ = spin down → anticlockwise

If our electron — our quantum system — is just left alone then it is said to be in a superposition of both these states, In other words, the electron isn't $|\uparrow\rangle$ or $|\downarrow\rangle$, it's $|\uparrow\rangle$ and $|\downarrow\rangle$.

Stern-Gerlach Experiment

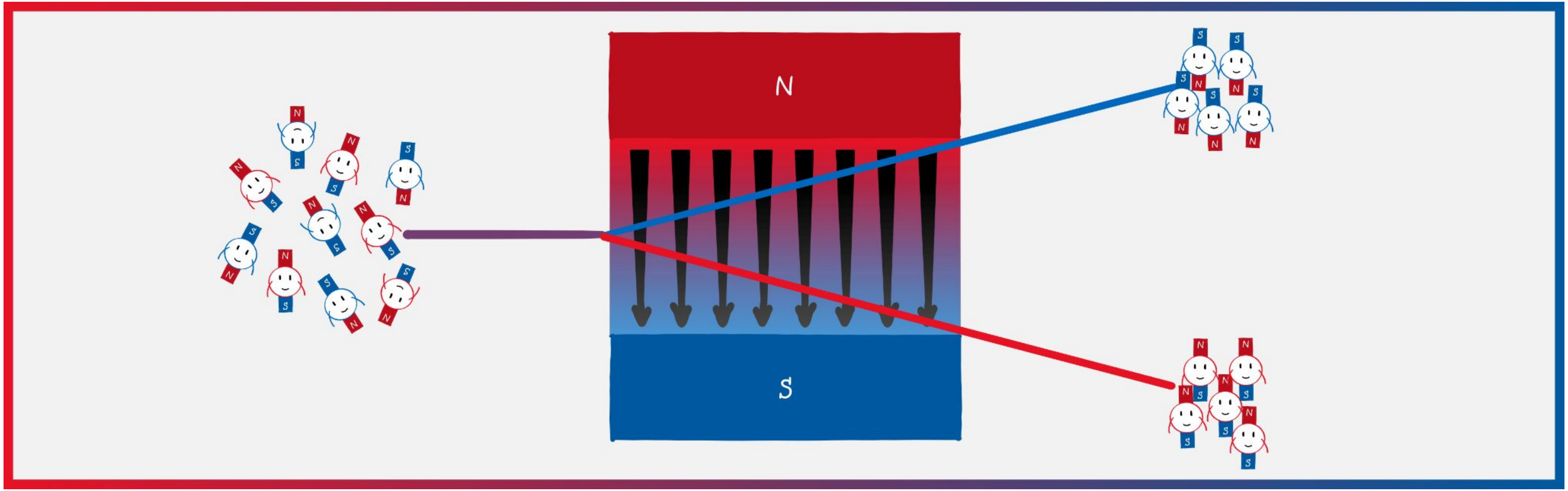
Otto Stern and Walther Gerlach in 1921

Atoms behave like mini-magnets



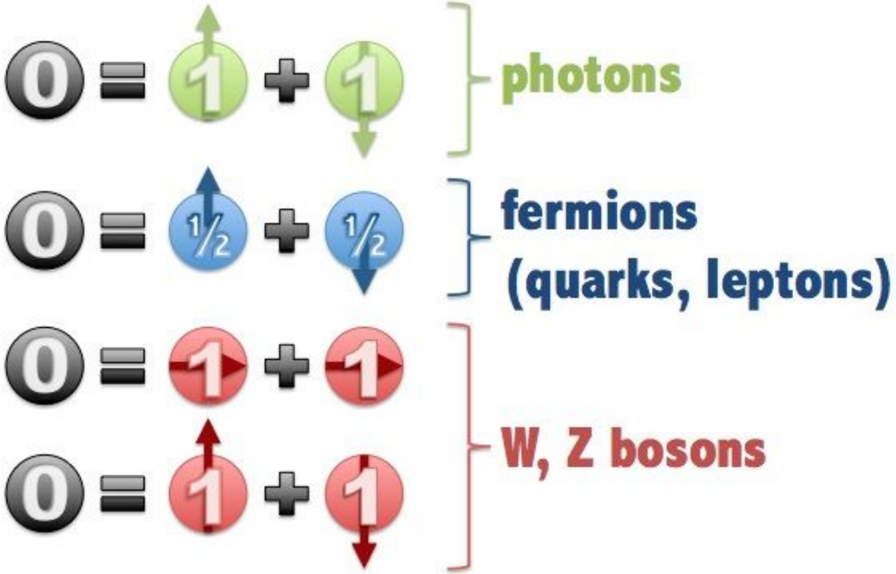
Stern-Gerlach Experiment

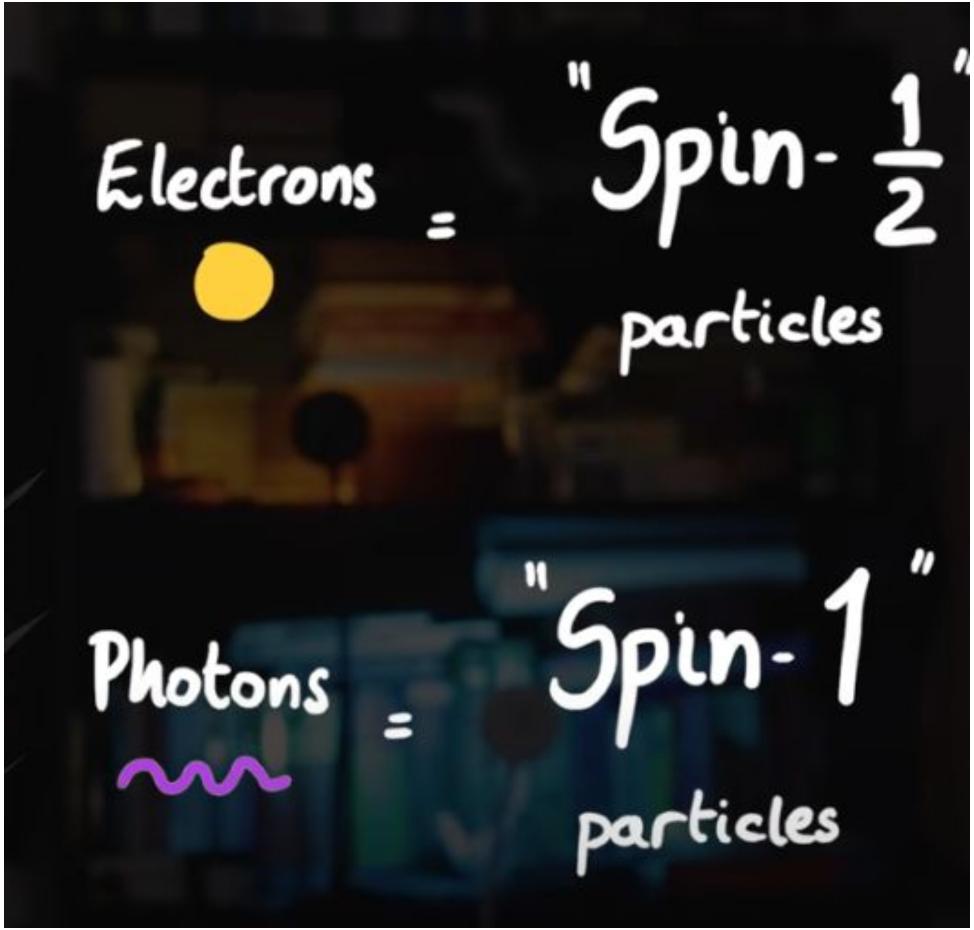
Atoms behave like mini-magnets



Spin -n

n= 0, 1/2 ,1, 3/2 ,2 …..




Electrons = "Spin-1/2" particles
Photons = "Spin-1" particles

If $n = \frac{1}{2}$ (e.g. electron):

Maximum spin $= +\frac{1}{2}\hbar$

Direction of spin: ↑

Magnitude of spin: $\frac{1}{2}\hbar$

Other possible states: $n-1 = \left(-\frac{1}{2}\right)$

Spin $= -\frac{1}{2}\hbar$ ("spin down")

If $n = \frac{3}{2}$:

$\Uparrow \quad +\frac{3}{2}\hbar$

$\uparrow \quad +\frac{1}{2}\hbar$

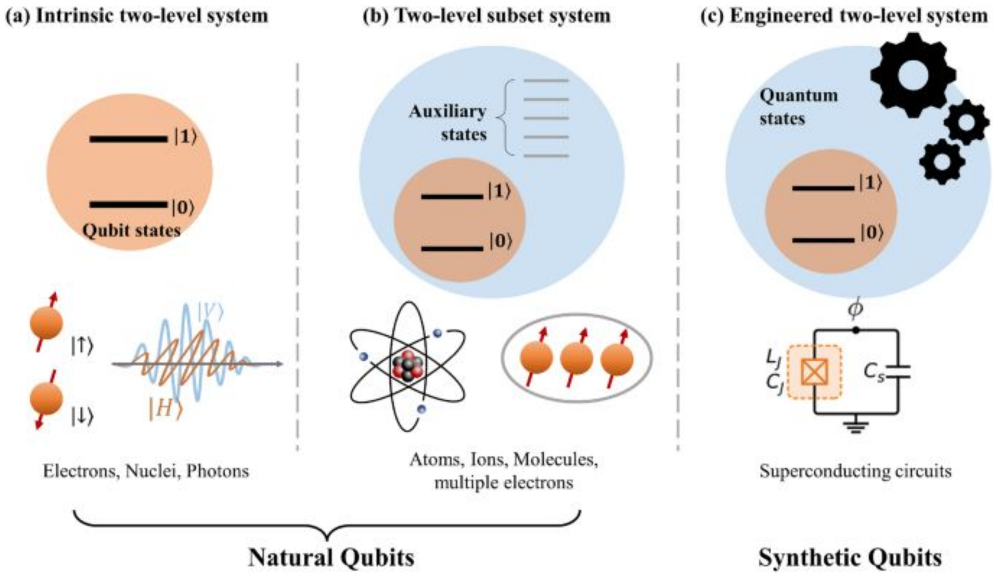
$\downarrow \quad -\frac{1}{2}\hbar$

$\Downarrow \quad -\frac{3}{2}\hbar$

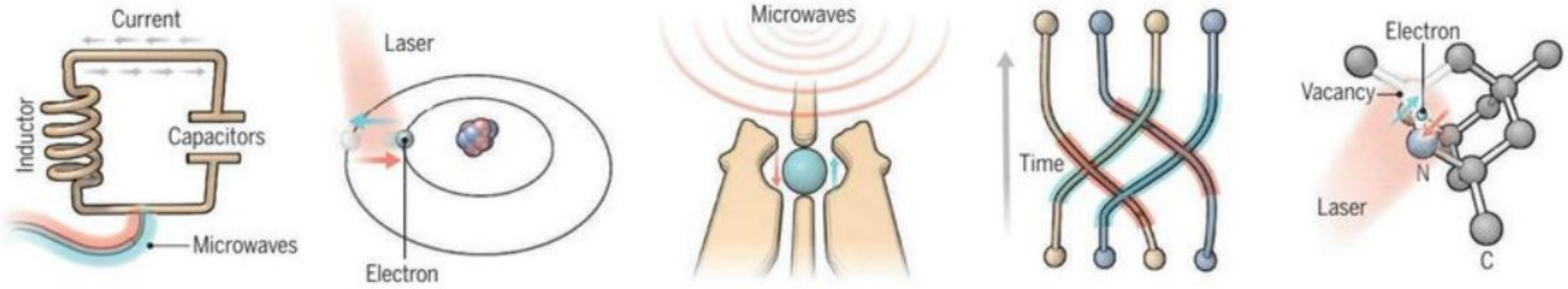
Types of Qubits

There are many kinds of qubits, some occurring naturally and others that are engineered. Some of the most common types include:

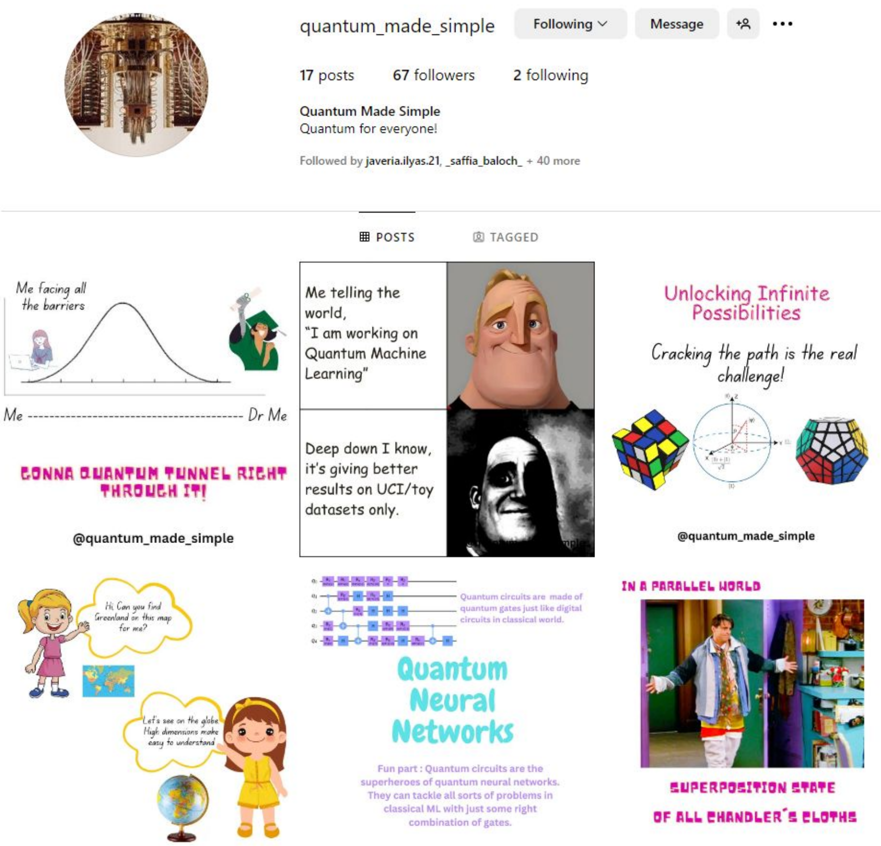
- Spin
- Trapped Atoms and Ions
- Photons
- Superconducting Circuits



What technologies are used to build quantum computers?



	Superconducting loops	**Trapped ions**	**Silicon quantum dots**	**Topological qubits**	**Diamond vacancies**
Company support	Google, IBM, Quantum Circuits	ionQ	Intel	Microsoft, Bell Labs	Quantum Diamond Technologies
Pros	Fast working. Build on existing semiconductor industry.	Very stable. Highest achieved gate fidelities.	Stable. Build on existing semiconductor industry.	Greatly reduce errors.	Can operate at room temperature.
Cons	Collapse easily and must be kept cold.	Slow operation. Many lasers are needed.	Only a few entangled. Must be kept cold.	Existence not yet confirmed.	Difficult to entangle.


quantum_made_simple
Following
Message
17 posts
67 followers
2 following
Quantum Made Simple
Quantum for everyone!
Followed by javeria.ilyas.21, _saffia_baloch_ + 40 more
POSTS
TAGGED
Me facing all the barriers
Me -- Dr Me
GONNA QUANTUM TUNNEL RIGHT THROUGH IT!
@quantum_made_simple
Me telling the world, "I am working on Quantum Machine Learning"
Deep down I know, it's giving better results on UCI/toy datasets only.
Unlocking Infinite Possibilities
Cracking the path is the real challenge!
@quantum_made_simple
Hi, Can you find Greenland on this map for me?
Let's see on the globe. High dimensions make easy to understand
Quantum circuits are made of quantum gates just like digital circuits in classical world.
Quantum Neural Networks
Fun part : Quantum circuits are the superheroes of quantum neural networks. They can tackle all sorts of problems in classical ML with just some right combination of gates.
IN A PARALLEL WORLD
SUPERPOSITION STATE
OF ALL CHANDLER'S CLOTHS

References

https://www.youtube.com/watch?v=UjaAxUO6-Uw

https://physicus-minimus.com/en/the-stern-gerlach-experiment/

https://medium.com/@satoshihgsn/e-i%CF%80-1-0-the-most-beautiful-theorem-in-mathematics-493a7f4e7332

https://javafxpert.github.io/grok-bloch/