The Routh-Hurwitz Criterion for Stability Analysis

3.1 Condition for Stability

- ✓ We know that if we can factorize the characteristic equation then, from pole locations we can understand the system is stable or not.
- ✓ But it may happen that the characteristic equation is too big or difficult to factorize.
- ✓ In such cases, we can apply two necessary conditions, but not sufficient for the system stability.
- ✓ This is described as follows
- ✓ Consider the system having the characteristic equation

$$a_m S^m + a_{m-1} S^{m-1} + \cdots\cdots + a_2 S^2 + a_1 S + a_0 = 0 \tag{3.1}$$

1. ***All the coefficients of the above equation should have same sign.***
2. ***There should be alternative terms, i.e., there should be no missing term.***

- ✓ The above two conditions are necessary but not sufficient conditions for stability.
- ✓ If the above two conditions are satisfied, there is still no guarantee that the system will be stable.
- ✓ ***Let's understand the stability from the above two necessary conditions***.

Ex. 3.1 Reveal the stability of the system described by the following characteristic equation.

$$S^2 + S - 2 = 0$$

- ➢ Ans.: We can see that all the coefficients are positive except the coefficient of S^0 which is negative. So, according to the criterion 1, the system is unstable.

  This can also be verified by factorizing the equation and finding out the exact root locations.

  We get by factorizing

$$(S+2)(S-1) = 0$$

  We can see that the system has two roots, one root has +ve real part and the other root have –ve real part. So, the system is unstable.

  One point to be noted here that we can test the stability both ways. Since this example has a simple algebraic equation, we can factorize it. But for more complex algebraic equations factorization is quite cumbersome. For such complex algebraic equations we can't easily apply the aforesaid two stability criteria.

Ex. 3.2 Reveal the stability of the system described by the following characteristic equation.

$$S^3 + S^2 = 0$$

- ➢ Ans.: Here S and S^0 terms are missing, and therefore as per the criterion 2, the system is unstable. Hence, we can easily conclude whether the system is stable or not without factorizing. However, we can also cross-check by finding the exact root locations which is given as $S = 0,0,-1$. So, we can conclude that the system is unstable since there are repeated poles at the origin.

Ex. 3.3 Reveal the stability of the system described by the following characteristic equation.

$$S^3 + S^2 + 2S^1 + 8 = 0$$

- ➢ Ans.: According to criteria 1 and 2, the system seems stable. But, factorizing the equation, we get

$$S^3 + S^2 + 2S^1 + 8 = 0$$
$$(S+2)\left(S^2 - S + 4\right) = 0$$

$\left(S^2 - S + 4\right)$ have two roots on the right half of the S-plane, and thus the system is unstable.

- ✓ So, we can observe that although stability criteria are satisfied, the system is still not stable.
- ✓ Hence, the above two stability criteria are not the necessary conditions, but are just sufficient to investigate the stability of any system.
- ✓ To find the stability, we will be using the Routh-Hurwitz Criterion.

3.2 The Routh-Hurwitz Criterion

- ✓ If the characteristic equation satisfies the condition for the stability, we then construct the ***Routh array*** to find the necessary and sufficient conditions for the stability.
- ✓ Generally, for an n^{th}-order polynomial, we need (n+1) rows.
- ✓ The first two rows are filled with the coefficients of the polynomial in a column-wise order.
- ✓ The computation of the array entries is very much like the –ve of the normalized determinant anchored by the first column.
- ✓ The ***Routh criterion*** states that in order to have a stable system, all the coefficients in the first column of the array must be positive definite.
- ✓ If any of the coefficients in the first column is –ve, there is at least one root with +ve real part.
- ✓ The number of sign changes is the number of positive poles present.
- ✓ Consider a system with characteristic equation given by

$$a_m S^m + a_{m-1} S^{m-1} + \cdots\cdots + a_2 S^2 + a_1 S + a_0 = 0$$

The array can be written as

Step-1: Arrange all the coefficients in the two rows shown below

$$\begin{array}{lllllll} 1: & a_m & a_{m-2} & a_{m-4} & \cdots\cdots & a_2 & a_0 \\ 2: & a_{m-1} & a_{m-3} & a_{m-5} & \cdots\cdots & a_3 & a_1 \end{array}$$

Step-2: From rows 1 and 2, construct the 3rd row as shown below.

$$\begin{array}{lllll} 3: & b_1 & b_2 & b_3 & \cdots\cdots \end{array}$$

where, $b_1 = \dfrac{a_{m-1}a_{m-2} - a_m a_{m-3}}{a_{m-1}}$; $\quad b_2 = \dfrac{a_{m-1}a_{m-4} - a_{m-5}a_m}{a_{m-1}}$; $\quad \ldots\ldots\ldots$

Step-3: From rows 2 and 3, construct the 4th row as shown below.

4: $\quad c_1 \quad c_2 \quad \ldots\ldots$

where, $c_1 = \dfrac{b_1 a_{m-3} - b_2 a_{m-1}}{b_1}$; $\quad c_2 = \dfrac{b_1 a_{m-5} - b_3 a_{m-1}}{b_1}$; $\quad \ldots\ldots\ldots$

- ✓ Continue the process to form new rows until we get the last row having only one element.
- ✓ Lets understand the method through examples;

Ex.3.1 Investigate the stability of the system whose characteristic equation is given by

$$S^4 + 2S^3 + 6S^2 + 4S + 1 = 0$$

Sol^n: Let's form the Routh array as follows

$$\begin{array}{llll}
S^4 & 1 & 6 & 1 \\
S^3 & 2 & 4 & 0 \\
S^2 & \dfrac{2\times 6 - 4\times 1}{2} = 4 & \dfrac{2\times 1 - 0\times 1}{2} = 1 & \\
S^1 & \dfrac{4\times 4 - 1\times 2}{4} = 3.5 & 0 & \\
S^0 & \dfrac{3.5\times 1 - 0\times 4}{3.5} = 1 & &
\end{array}$$

All the coefficients in the 1st column are +ve, which indicates there are no roots with +ve real parts. Hence the system is stable.

Ex.3.2 Determine the stability of the system whose characteristic equation is given by

$$3S^4 + 2S^3 + S^2 + 4S + 4 = 0$$

Sol^n: Let's form the Routh array as follows

$$\begin{array}{llll}
S^4 & 3 & 1 & 4 \\
S^3 & 2 & 4 & 0 \\
S^2 & -5 & 4 & \\
S^1 & 5.6 & 0 & \\
S^0 & 4 & &
\end{array}$$

Here, two sign changes occur in the first column (from 2 to -5 and -5 to 5.6). That means there are two roots present in the positive half of the S-plane. Hence, the system is unstable.

3.2.1 Division of a Row

- ✓ The coefficient of any row may be multiplied or divided by a positive number without changing the signs of the first column.
- ✓ The effort of evaluating the coefficients in Routh's array can be reduced by multiplying or dividing any row by a constant.

- ✓ This may result, for example, in reducing the size of the coefficients and therefore simplifying the evaluation of the remaining coefficients.
- ✓ Ex.3.3 Consider the following characteristic equation:

$$S^6 + 3S^5 + 2S^4 + 9S^3 + 5S^2 + 12S + 20 = 0$$

Check the stability with the help of the Routh criterion

- ✓ Sol^n: Let's form the Routh array as follows

S^6	1	2	5	20			
S^5	**1**	**3**	**4**		~~3~~ **1**	~~9~~ **3**	~~12~~ **4**
S^4	-1	1	20				
S^3	**1**	**6**			~~4~~ **1**	~~24~~ **6**	
S^2	7	20					
S^1	**22**				$22/7$		
S^0	20						

Note that the size of the numbers has been reduced by dividing S^5 row by 3 and S^3 row by 4.

The result is unchanged; i.e., there are two changes of sign in the first column and, therefore, there are two roots present in the right half of the s-plane.

Hence, the system is unstable.

3.2.2 Special Cases

- ✓ Occasionally, the following kinds of difficulties may arise causing breakdown of Routh's criterion.

Case-1: When the first element in any row of the Routh's table is zero, but the other elements are not.

- ✓ Ex.3.4 Consider the following characteristic equation:

$$S^3 + S + 2 = 0$$

Determine stability with the help of Routh-Hurwitz criterion

- ✓ Sol^n: Let's form the Routh array as follows

S^3	1	1
S^2	0	2
S^1		
S^0		

- ✓ Here, the 1st element of the second row is zero; hence we can't calculate the remaining coefficients further.
- ✓ To remove this difficulty, replace zero by a small elements called ε .

S^3	1	1
S^2	ε	2
S^1	$\frac{\varepsilon-2}{\varepsilon}=-\frac{2-\varepsilon}{\varepsilon}$	0
S^0	2	

- ✓ As $\varepsilon \to 0$ (very small), $-\frac{2-\varepsilon}{\varepsilon}$ = negative quantity, which indicates that there are two sign changes (ε to –ve quantity and –ve to 2), it means that two roots are present in the right half of the S-plane.
- ✓ Hence, the system is unstable.

<u>Case-2:</u> When all elements in any one row are zero, it indicates that there are symmetrically located roots in the S-plane. This may have residues of the following:

1. Pair of real roots of opposite sign
2. Pair of conjugate roots on the imaginary axis
3. Complex conjugate roots forming quadrants.

Ex.3.5 A system having characteristic equation as:

$$S^6+2S^5+8S^4+12S^3+20S^2+16S+16=0$$

a) Determine whether the system is stable or not.
b) Determine how many roots are present in the imaginary axis.
c) Determine how many roots are present in the negative real axis.

Sol^n:

(a) Let's form the Routh array as follows

S^6	1	8	20	16
S^5	2	12	16	0
S^4	2	12	16	
S^3	0	0	0	
S^2				
S^1				
S^0				

- ✓ Here, all elements of row-4 are zero; thus we can't proceed further and Routh's test breaks down.
- ✓ This situation is overcome by replacing the rows of zeros in Routh's table by a row of coefficients of the polynomial generated by taking the first derivative of the **auxiliary polynomial**.

Auxiliary Polynomial: It is the polynomial whose coefficients are the elements of the row just above the row of zeros in Routh's table.

This polynomial gives the number and location of root pairs of the characteristic equation which are symmetrically located in the S-plane.

$$A(s) = 2S^4 + 12S^2 + 16 = 0$$

or $$\frac{sA(s)}{dS} = 8S^3 + 24S$$

Replacing ‘0’ in row-4 by the coefficients 8, 24, the modified Routh’s table is

S^6	1	8	20	16
S^5	2	12	16	0
S^4	2	12	16	0
S^3	**8**	**24**	**0**	**0**
S^2	6	16	0	0
S^1	2.67	0		
S^0	16			

- ✓ There are no sign changes in the 1st column; hence no roots are present in the +ve half of the S-plane.
- ✓ However, we can’t conclude here that the system is absolutely stable.
- ✓ The presence of ‘0’ in the 4th row of the original Routh table implies that there are poles which are symmetrically located.
- ✓ But as from the above result, there are no roots on the right-half of the S-plane. Thus the only option left is that there are two imaginary poles on the $j\omega$.
- ✓ The exact location of the poles can be found by finding the roots of the auxiliary equation.

(b)The auxiliary equation is : $2S^4 + 12S^2 + 16 = 0$

Putting, $S^2 = x$

We get $$2x^2 + 12x + 16 = 0$$

or, $$x^2 + 6x + 8 = 0$$

$$\Rightarrow (x+2)(x+4) = 0$$

$$\Rightarrow x = -2,\ x = -4$$

where, $x = S^2 = -2 \Rightarrow S = \pm j\sqrt{2}$ and $x = S^2 = -4 \Rightarrow S = \pm 2j$

Therefore, 4 roots are present on the imaginary axis and these roots are at $S = \pm j\sqrt{2}$ and $S = \pm 2j$.

Hence, the system is marginally stable.

(c)The characteristics equation has 6 roots, in which 4 roots are imaginary; hence remaining 2 roots are present on the –ve real axis.