Circular Cylindrical Coordinate System

Presented by

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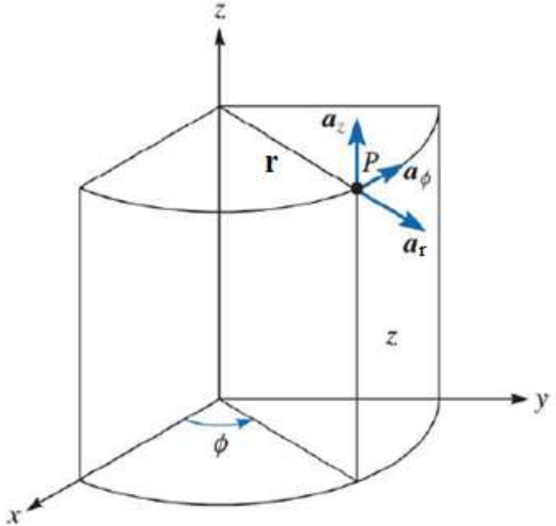
Circular Cylindrical Coordinate System:-

This type of coordinate system is very important whenever we are dealing with problems having cylindrical symmetry.

Take a point "*P*" in this coordinate system.

The position of this point can be represented in the cylindrical coordinate system by three variables.



r = radius of the cylinder passing through the point *"P"* or radial distance from z-axis.

φ = *azimuthal* angle which is measured from the *x*-axis in *x*-*y* plane.

z = same as the Cartesian coordinate system.

Range of coordinates

$$0 \le r \le \infty$$

$$0 \le \varphi \le 2\pi$$

$$-\infty \le z \le \infty$$

So the components of vector $\vec{P}$ will be $\vec{P}_r$, $\vec{P}_\varphi$ and $\vec{P}_z$

Then, the vector can be written as- $\vec{P} = P_r\hat{a}_r + P_\varphi\hat{a}_\varphi + P_z\hat{a}_z$

Here, $\hat{a}_r, \hat{a}_\varphi$ and $\hat{a}_z$ are the unit vectors in "r", "φ" and "z" directions respectively.

Now, any vector, say $\vec{A}$ can be represented in cylindrical coordinate system as

$$\vec{A} = A_r\hat{a}_r + A_\varphi\hat{a}_\varphi + A_z\hat{a}_z$$

Note: $\hat{a}_\varphi$ is not in degree.

The magnitude of vector $\vec{A}$ will be- $\left|\vec{A}\right| = \sqrt{A_r^2 + A_\varphi^2 + A_z^2}$

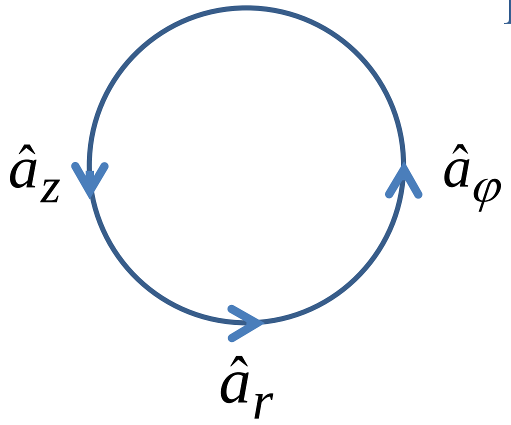
Here, we can see that the unit vectors $\hat{a}_r, \hat{a}_\varphi$ and $\hat{a}_z$ are mutually perpendicular because our coordinate system is orthogonal.

Cross product :-

$$\hat{a}_r \times \hat{a}_\varphi = \hat{a}_z \qquad \hat{a}_r \times \hat{a}_z = -\hat{a}_\varphi$$

$$\hat{a}_\varphi \times \hat{a}_z = \hat{a}_r \quad \text{or} \quad \hat{a}_z \times \hat{a}_\varphi = -\hat{a}_r$$

$$\hat{a}_z \times \hat{a}_r = \hat{a}_\varphi \qquad \hat{a}_\varphi \times \hat{a}_r = -\hat{a}_z$$



Dot product :-

$$\hat{a}_r \bullet \hat{a}_r = \hat{a}_\varphi \bullet \hat{a}_\varphi = \hat{a}_z \bullet \hat{a}_z = 1$$

$$\hat{a}_r \bullet \hat{a}_\varphi = \hat{a}_\varphi \bullet \hat{a}_z = \hat{a}_z \bullet \hat{a}_r = 0$$

Relationship between components of Cartesian and Cylindrical coordinate system

If we divide vector *"r"* into two components along *x*-axis and *y*-axis; then

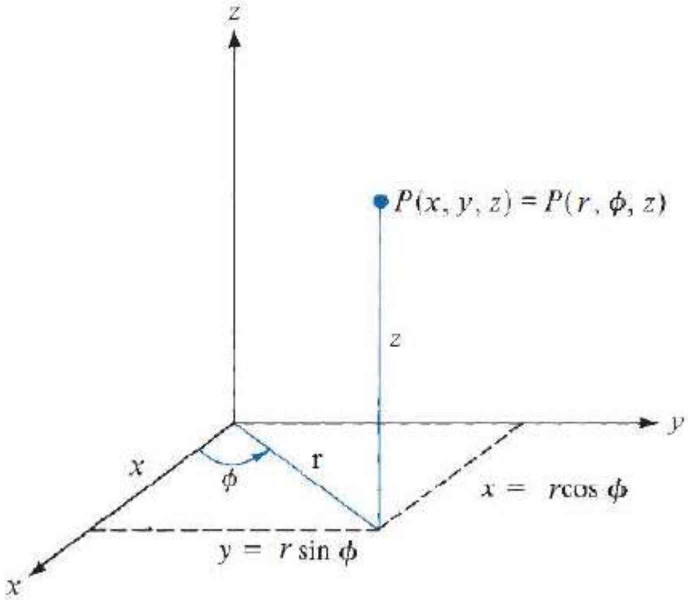
$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

So,

$$\boxed{x = r\cos\varphi\,,\; y = r\sin\varphi \text{ and } z = z}$$

Transformation-Cylindrical to Cartesian coordinate



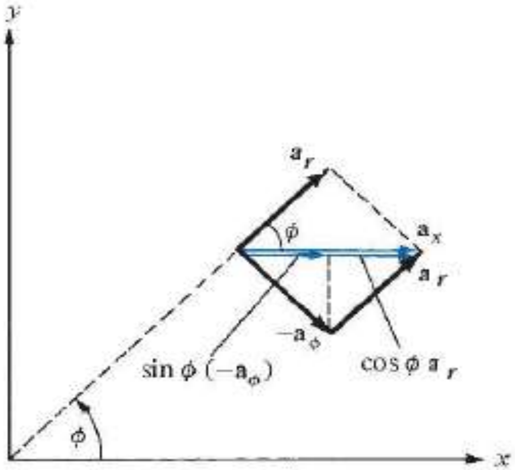
and

$$\boxed{r = \sqrt{x^2 + y^2}\,;\varphi = \tan^{-1}\frac{y}{x} \text{ and } z{=}z}$$

Transformation from Cartesian to Cylindrical coordinate

Relationship between unit vectors

From geometry , we can find the relationship between $(\hat{a}_x, \hat{a}_y, \hat{a}_z)$ and $(\hat{a}_r, \hat{a}_\varphi, \hat{a}_z)$



$$\hat{a}_x = \cos\varphi\hat{a}_r - \sin\varphi\hat{a}_\varphi$$

$$\hat{a}_y = \sin\varphi\hat{a}_r + \cos\varphi\hat{a}_\varphi$$

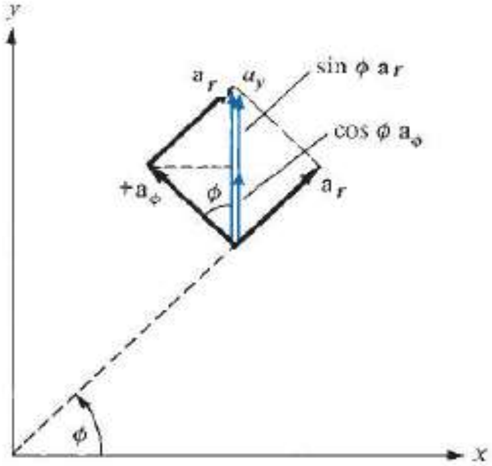
$$\hat{a}_z = \hat{a}_z$$

or,

$$\hat{a}_r = \cos\varphi\hat{a}_x + \sin\varphi\hat{a}_y$$

$$\hat{a}_\varphi = -\sin\varphi\hat{a}_x + \cos\varphi\hat{a}_y$$

$$\hat{a}_z = \hat{a}_z$$



Finally, the relationship between components of Cartesian and cylindrical coordinate systems of vector $\vec{A}$ can be written as:-

$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$$
$$= A_x\left(\cos\varphi\hat{a}_r - \sin\varphi\hat{a}_\varphi\right) + A_y\left(\sin\varphi\hat{a}_r + \cos\varphi\hat{a}_\varphi\right) + A_z\hat{a}_z$$
$$= \left(A_x\cos\varphi + A_y\sin\varphi\right)\hat{a}_r + \left(-A_x\sin\varphi + A_y\cos\varphi\right)\hat{a}_\varphi + A_z\hat{a}_z$$
$$= A_r\hat{a}_r + A_\varphi\hat{a}_\varphi + A_z\hat{a}_z$$

Hence,

$$A_r = \left(A_x\cos\varphi + A_y\sin\varphi\right)$$
$$A_\varphi = \left(-A_x\sin\varphi + A_y\cos\varphi\right)$$
$$A_z = A_z$$

We can write the transformation of the vector as:-

$$\begin{bmatrix} A_r \\ A_\varphi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \text{or,} \quad \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} A_r \\ A_\varphi \\ A_z \end{bmatrix}$$

Thank you