Brief Description of Vector Calculus: Part-2

Presented by

Dr. Asim Halder

Dept. of Applied Electronics and Instrumentation Engineering

Haldia Institute of Technology

Vector Differential Operator

Let us consider $f(x)$ is a function of *'x'* then $\frac{df(x)}{dx}$ is the derived of function of $f(x)$.

Here, $\frac{d}{dx}$ is called differential operator for 2-D coordinate system where

$$\frac{d}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Just like that, for 3-D space in Cartesian coordinate system the vector differential operation $\vec{\nabla}$ is represented by-

$$\boxed{\vec{\nabla} = \left[\frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z \right]}$$

The vector differential operator, also known as the gradient operator.

It is not a vector itself, but when it operates on a scalar function we will get vector function.

This operator is useful in defining –

i. The **gradient** of a scalar 'V'; written as ∇V

ii. The **divergence** of a vector '$\vec{A}$'; written as $\nabla \bullet \vec{A}$

iii. The **curl** of a vector '$\vec{A}$'; written as $\nabla \times \vec{A}$

iv. The **laplacian** of a scalar 'V'; written as $\nabla^2 V$

Before going to define the above points, let's see the expression of ∇ in cylindrical and spherical coordinate system.

$$\boxed{\nabla = \left[\frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \varphi}\hat{a}_\varphi + \frac{\partial}{\partial z}\hat{a}_z \right]}$$

Cylindrical Coordinate System

and

$$\boxed{\nabla = \left[\frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\hat{a}_\varphi \right]}$$

Spherical Coordinate System

Hence, the gradient of a scalar '*V*' in three coordinate systems can be represent as-

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \varphi}\hat{a}_\varphi + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\nabla V = \left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{a}_\varphi \right]$$

Definition of Divergence :-

The divergence of $\vec{A}$ at a given point 'P' is the outward flux per unit volume as the volume shrinks about 'P'.

$$div\vec{A} = \nabla \bullet \vec{A} = \lim_{\Delta v \to 0} \frac{\int_s \vec{A} \bullet ds}{\Delta v}$$

Let's see divergence of the vector $\vec{A}$ in those coordinate systems.

$$\nabla \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \Longrightarrow \text{Cartesian}$$

Similarly,

$$\nabla \bullet \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} \Longrightarrow \text{Cylindrical}$$

and $$\nabla \bullet \vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$ ⇨ Spherical

Divergence Theorem :-

The divergence theorem states that the total outward flux of a vector field $\vec{A}$ Through the closed surface 'S' is the same as the volume integral of the divergence $\vec{A}$.

i.e., $$\boxed{\oint_s \vec{A}\cdot ds = \int_v \nabla\cdot\vec{A}dv}$$

Definition of Curl:-

The curl of a vector $\vec{A}$ is an axial (or rotational) vector whose magnitude is the maximum circulation of $\vec{A}$ / unit area as the area tends to zero and whose direction is the normal direction of the area where the area is oriented to make the circulation maximum.

i.e., $$\nabla \times \vec{A} = \left[\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot dl}{\Delta S} \right]_{\max} \hat{a}_n$$

$$\nabla \times \vec{A} = \begin{bmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

or, $$\nabla \times \vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] \hat{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] \hat{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] \hat{a}_z$$

- **For Cylindrical Co-ordinate System**

$$\nabla \times \vec{A} = \frac{1}{r}\begin{bmatrix} \hat{a}_r & r\hat{a}_\varphi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{bmatrix}$$

or, $$\nabla \times \vec{A} = \frac{1}{r}\left[\frac{\partial A_z}{\partial \varphi} - \frac{\partial (rA_\varphi)}{\partial z}\right]\hat{a}_r + \frac{1}{r}\left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]r\hat{a}_\varphi + \frac{1}{r}\left[\frac{\partial (rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial z}\right]\hat{a}_z$$

- **For Spherical Co-ordinate System**

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta}\begin{bmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{bmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial\left(A_\varphi \sin\theta\right)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right]\hat{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial\left(rA_\varphi\right)}{\partial r}\right]\hat{a}_\theta + \frac{1}{r}\left[\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\hat{a}_\varphi$$

Stokes's Theorem :-

It states that the circulation of a vector field $\vec{A}$ around a closed path 'L' is equal to the surface integral of the curl of $\vec{A}$ over the open surface 'S' bounded by 'L', provided $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on 'S' .

$$\boxed{\oint_L \vec{A} \cdot dl = \int_s \left(\nabla \times \vec{A}\right) \cdot ds}$$

Laplacian :-

Laplacian is another operator, like **divergence** and **curl**, which performs both on scalar and vector function.

The Laplacian of a scalar :-

The ***laplacian*** of a scalar function '*V*' is defined as the divergence of the gradient of '*V*'.

The gradient of a scalar *V* is a vector and the divergence of a vector is a scalar.

Hence, the ***laplacian*** of a scalar results in a scalar.

The ***laplacian*** operation has a symbol of- $\nabla^2 V = \nabla \cdot \nabla V$

- **For Cartesian Co-ordinate System**

$$\boxed{\nabla^2 V = \nabla \cdot \nabla V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

- **For Cylindrical Co-ordinate System**

$$\boxed{\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}}$$

- **For Spherical Co-ordinate System**

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}}$$

The Laplacian of a Vector :-

The ***laplacian*** of a vector $\vec{A}$ is defined as the gradient of divergence of $\vec{A}$ minus curl of curl of $\vec{A}$; i.e.,

$$\boxed{\nabla^2\vec{A} = \nabla\left(\nabla\cdot\vec{A}\right) - \nabla\times\left(\nabla\times\vec{A}\right)}$$

Thank you