Chapter 1

Introduction of Signal and Systems

A system is any process that produces an output signal in response to an input signal.

Continuous systems input and output continuous signals, such as in analog electronics.

Discrete systems input and output discrete signals, such as computer programs that manipulate the values stored in arrays.

Properties of Signals: A signal can be classified as periodic or aperiodic; discrete or continuous time; discrete of continuous valued; or as a power or energy signal. The following defines each of these terms. In addition, the signal-to-noise ratio of a signal corrupted by noise is defined.

Periodic / Aperiodic: A periodic signal repeats itself at regular intervals. In general, any signal $x(t)$ for which

x(t) = x(t-T)

for all t is said to be *periodic*.

The fundamental period of the signal is the minimum positive, non-zero value of T for which above equation is satisfied. If a signal is not periodic, then it is *aperiodic*.

Symmetric / Asymmetric: There are two types of signal symmetry: odd and even. A signal$x(t)$ has *odd symmetry* if and only if $x(-t) = -x(t)$ for all t. It has *even symmetry* if and only if$x(-t) = x(t)$.

Discrète / Continuous Time : A continuous time signal is defined for all values of t. A discrete time signal is only defined for discrete values of $t = \ldots, t_{-1}, t_0, t_1, \ldots, t_n, t_{n+1}, t_{n+2}, \ldots$ It is uncommon for the spacing between t_n and $t_{n+1}$to be change with n. The spacing is most often some constant value referred to as the sampling rate,

$T_S = t_{n+1} - t_n$.

It is often convenient to express discrete time signals as $x(nT_s) = x[n]$.

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That is, if $x(t)$ is a continuous-time signal, then $x[n]$ can be considered as the n^{th} sample of$x(t)$.

Sampling of a continuous-time signal $x(t)$ to yield the discrete-time signal $x[n]$ is an important step in the process of digitising a signal.

Discrete/Continuous Valued: A continuous valued signal can take on any real (or complex) value. Examples include the water level in a water tank or the speed of a vehicle. A discrete valued signal can only take on only a predefined subset of real (or complex) values. An example is the output of an analogue to digital converter (*i.e.* a continuous-time, continuous valued signal that has been digitized.) Such a signal consists of integers only. Note that although a signal might be discrete-valued, it is not necessarily a discrete-time time signal.

Signal Energy and Power: When the strength of a signal is measured, it is usually the signal power or signal energy that is of interest. The signal power of $x(t)$ is defined as

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$$

and the signal energy as

$$E_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$$

A signal for which P_x is finite and non-zero is known as a *power signal*.

A signal for which E_x is finite and non-zero is known as an *energy signal*.

Px is also known as the *mean-square* value of the signal.

Signal power is often expressed in the units of decibels (dB).

The decibel is defined as

$$P_{xdB} = 10\log\left(\frac{P_x}{P_0}\right)$$

where P_0 is a reference power level, usually equal to one squared SI unit of the signal.

For example if the signal is a voltage then the P_0 is equal to one square Volt.

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As an example, the sinusoidal test signal of amplitude A,

$$x(t)=A\sin(\omega t)$$

has energy E_x that tends to infinity and power

$$P_x = \frac{1}{2}A^2,$$

or in decibels (dB):

$$20\log(A)-3$$

The signal is thus a power signal.

Signal to Noise Ratio: Any measurement of a signal necessarily contains some random noise in addition to the signal. In the case of additive noise, the measurement is

$$x(t)=s(t)+n(t)$$

where $s(t)$ is the signal component and $n(t)$ is the noise component.

The signal to noise ratio is defined as

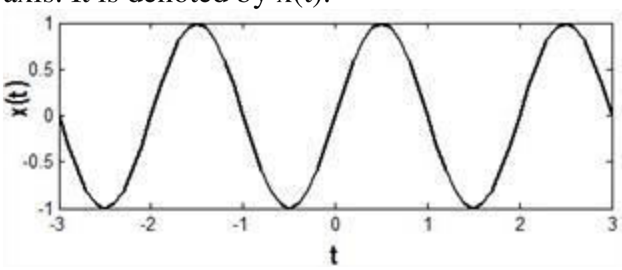
$$SNR_x = \frac{P_s}{P_n}$$

or in decibels,

$$SNR_x = 10\log\left(\frac{P_s}{P_n}\right)$$

The signal to noise ratio is an indication of how much noise is contained in a measurement.

Continuous Time Signals: A continuous time signal is one which is defined for all values of time. A continuous time signal does not need to be continuous (in the mathematical sense) at all points in time. A continuous-time signal contains values for all real numbers along the X-axis. It is denoted by x(t).



Example: A rectangular wave is discontinuous at several points but it is continuous time signal.

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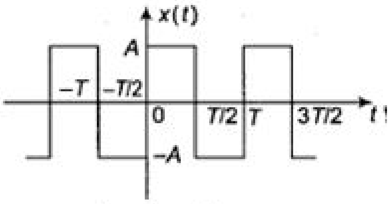


A rectangular wave

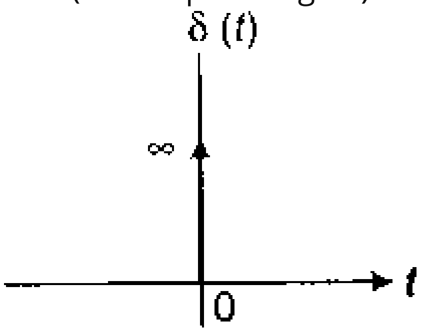
Standard Continuous Time Signals

- **Impulse Signal**

$$\delta(t) = \begin{cases} \infty \; ; t = 0 \\ 0 \; ; t = 0 \end{cases}$$

and $$\int_{-\infty}^{\infty} \delta(t) \, dt = A$$

When A = 1 (unit impulse signal)

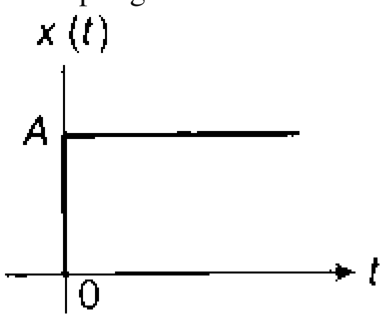


Impulse function

- **Step Signal**

$$x(t) = \begin{cases} A \; ; t \geq 0 \\ 0 \; ; t = 0 \end{cases}$$

Unit Step Signal $$x(t) = u(t) = \begin{cases} 1 \; ; t > 0 \\ 0 \; ; t < 0 \end{cases}$$



Step function

- **Ramp Signal**

$$x(t) = \begin{cases} At \; ; t \geq 0 \\ 0 \; ; t = 0 \end{cases}$$

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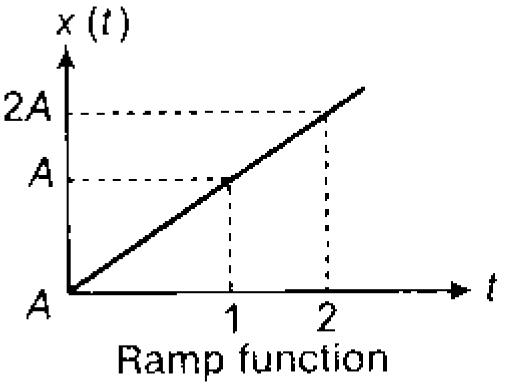
Unit Ramp Signal (A=1)

$$x(t) = r(t) = \begin{cases} t \; ; \; t \geq 0 \\ 0 \; ; \; t < 0 \end{cases}$$

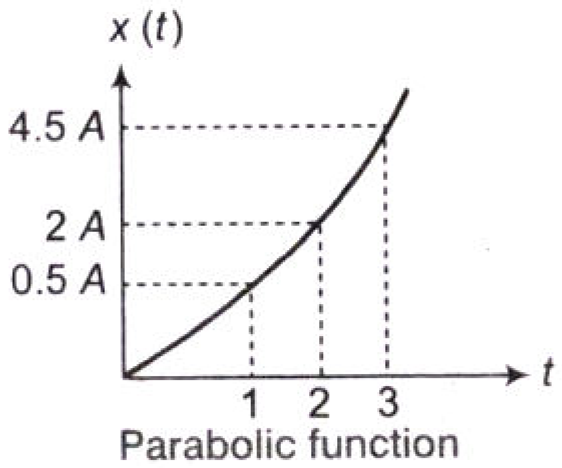


- **Parabolic Signal**

$$x(t) = \begin{cases} \dfrac{At^2}{2} \; ; \; t \geq 0 \\ 0 \; ; \; t < 0 \end{cases}$$

Unit Parabolic Signal:

$$x(t) = \begin{cases} \dfrac{t^2}{2} \; ; \; t \geq 0 \\ 0 \; ; \; t < 0 \end{cases}$$



- **Unit Pulse Signal**

$$x(t) = \pi(t)$$
$$= u(t + 1/2) - u(t - 1/2)$$

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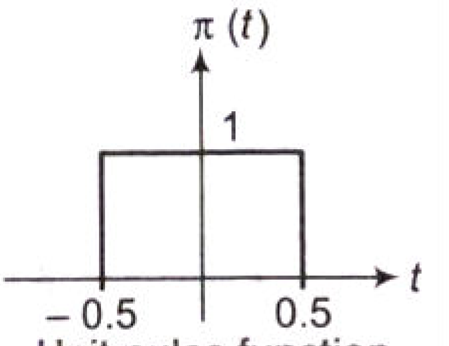


Unit pulse function

Sinusoidal Signal

- **Co-sinusoidal Signal:**

$x.(t) = A\cos(\omega_0 t + \phi)$

Where, $\omega_0 = 2\pi t_0 = \frac{2\pi}{T} =$ angular frequency in rad/sec

$f0$ = frequency in cycle/sec or Hz

T = time period in second

When $\phi = 0,\ x(t) = A\cos\omega_0 t$

When ϕ = positive, $x(t) = A\cos\left(\omega_0 t + \phi\right)$

When ϕ = negative, $x(t) = A\cos\left(\omega_0 t - \phi\right)$

- **Sinusoidal Signal:**

$x(t) = A\sin\left(\omega_0 t + \phi\right)$

Where, $\omega_0 = 2\pi t_0 = \frac{2\pi}{T} =$ Angular frequency in red/sec

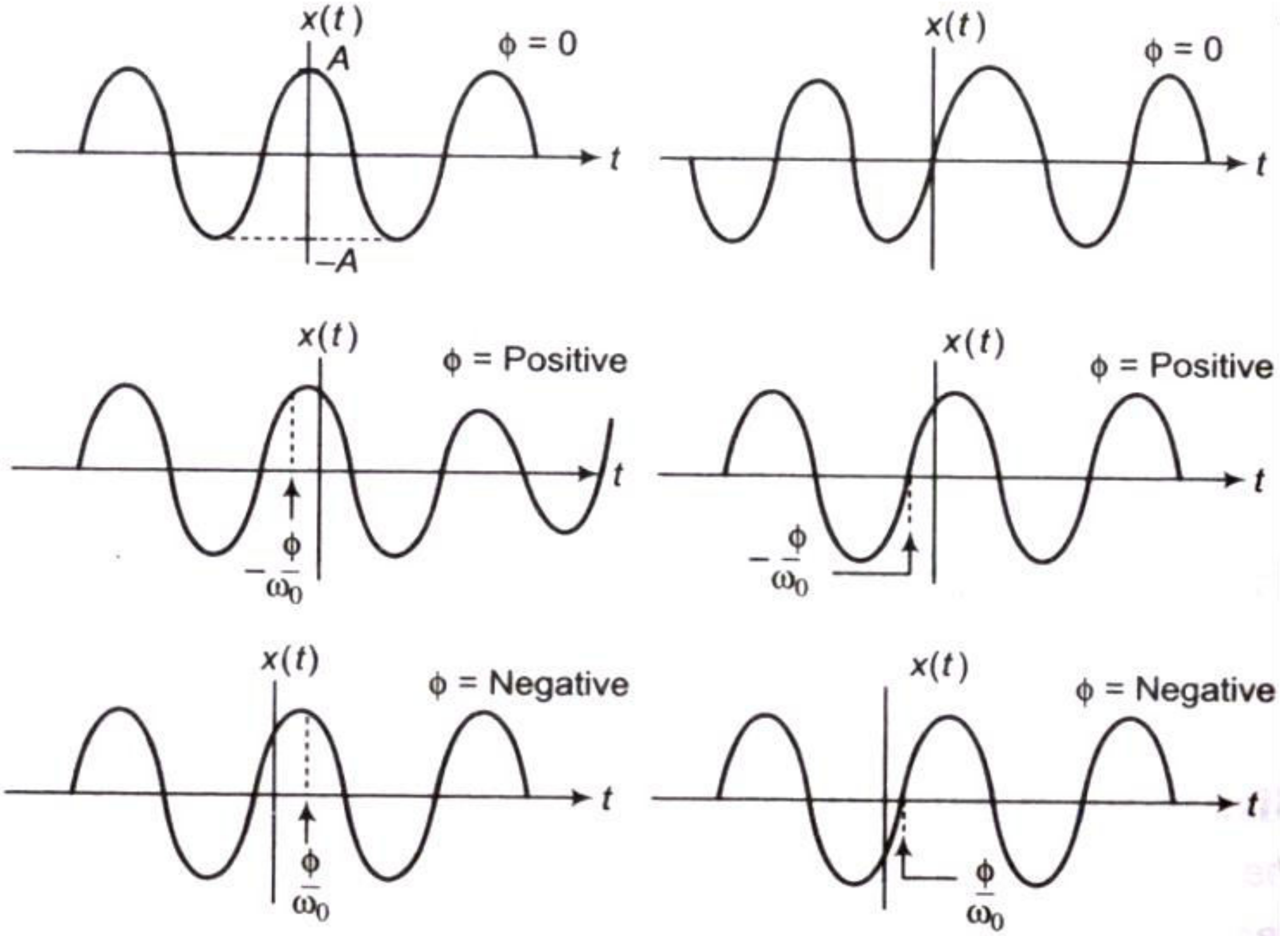
$f0$ = frequency in cycle/sec or Hz

T = time period in second

When $\phi = 0,\ x(t) = A\sin(\omega_0 t)$

When ϕ = positive, $x(t) = A\sin(\omega_0 t + \phi)$

When ϕ = negative, $x(t) = A\sin(\omega_0 t - \phi)$



Co-sinusoidal signal

Sinusoidal signal

Exponential Signal:

- **Real Exponential Signal**

$x(t) = A\, e^{bt}$; where, A and b are real.

Exponential signal

Exponential signal when $b < 0$

- **Complex Exponential signal**

$x(t) = A\, e^{j\omega_0 t}$

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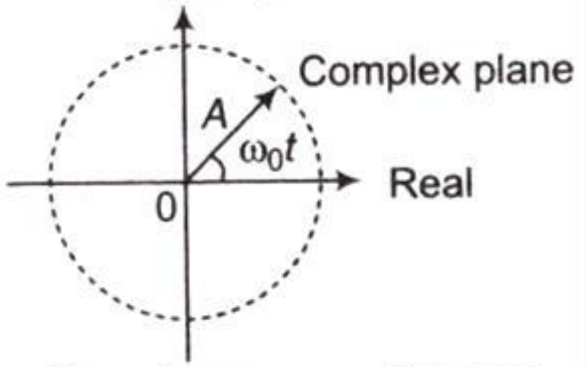
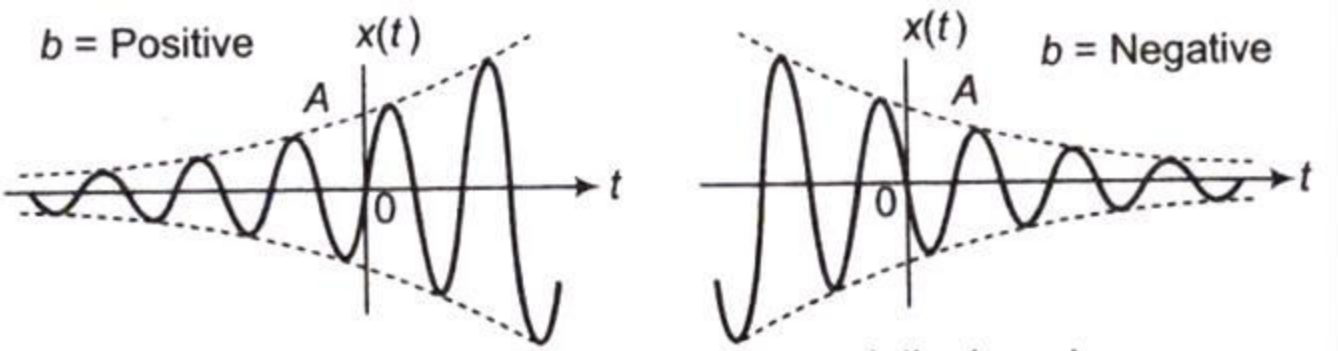
The complex exponential signal can be represented in a complex plane by a rotating vector, which rotates with a constant angular velocity of ω_0 red/sec.



Complex exponential as a complex plane

Exponentially Rising/Decaying Sinusoidal Signal

$$x(t) = A\, e^{bt} \sin \omega_0 t$$

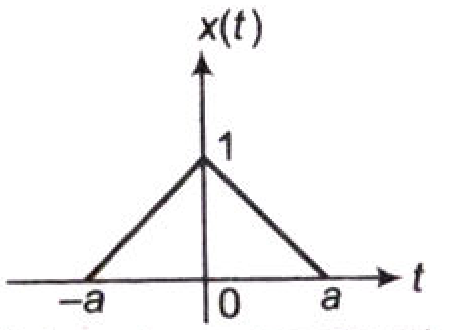


Exponentially rising

Exponentially decaying

Triangular Pulse Signal

$$x(t) = \triangle a(t) = \begin{cases} 1 - \frac{|t|}{a};\ |t| \le a \\ 0\ ;\ |t| > a \end{cases}$$



Triangular pulse signal

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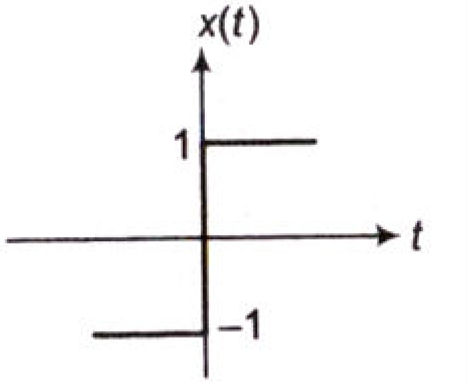
Signum Signal

$$x(t) = Sgn(t) = \begin{cases} 1 \ ; \ t > 0 \\ -1 ; \ t < 0 \end{cases}$$

$$Sgn(t) = 2u\,(t) - 1$$

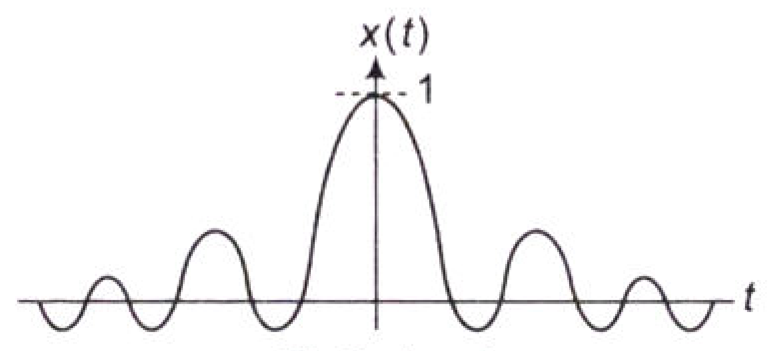
$$Sgn(t) = u\,(t) - u(-t)$$



Signum signal

SinC Signal

$$x(t) = \sin C\,(t) = \frac{\sin t}{t}; \infty < t < \infty$$



SinC signal

Gaussian Signal

$$x(t) = g_a(t) = e^{-a^2 t^2}; \infty < t < \infty$$

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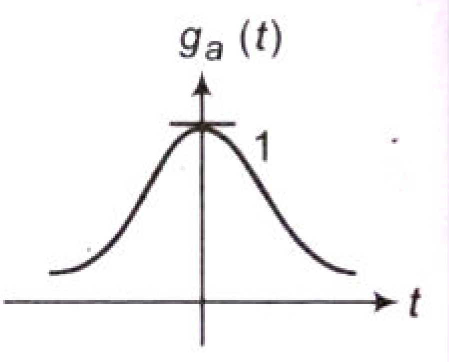

Gaussian signal

Important points:

- The sinusoidal and complex exponential signals are always periodic.
- The sum of two periodic signals is also periodic if the ratio of their fundamental periods is a rational number.
- Ideally, an impulse signal is a signal with infinite magnitude and zero duration.
- Practically, an impulse signal is a signal with large magnitude and short duration.

Classification of Continuous Time Signal: *The continuous time signal can be classified as*

1. **Deterministic and Non-deterministic Signals:**
   - The signal that can be completely specified by a mathematical equation is called a deterministic signal. The step, ramp, exponential and sinusoidal signals are examples of deterministic signals.
   - The signal whose characteristics are random in nature is called a non-deterministic signal. The noise signal from various sources like electronic amplifiers, oscillator etc., are examples of non-deterministic signals.
   - Periodic and Non-periodic Signals
   - A periodic signal will have a definite pattern that repeats again and again over a certain period of time.

$$x(t+T) = x(t)$$

2. **Symmetric (even) and Anti-symmetric (odd) Signals**

When a signal exhibits symmetry with respect to $t = 0$, then it is called an **even signal.**

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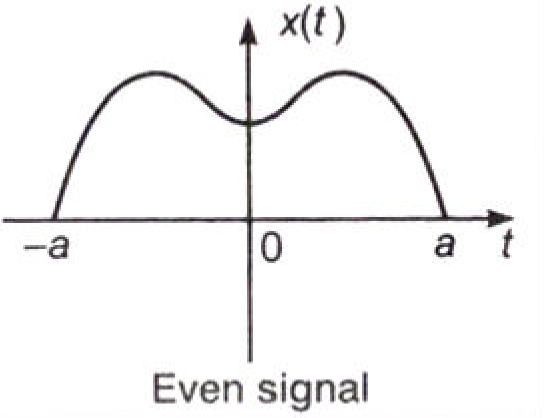
$$x(-t) = x(t)$$



When a signal exhibits anti-symmetry with respect to $t = 0$, then it is called an **odd signal.**

$$x(-t) = -x(t)$$

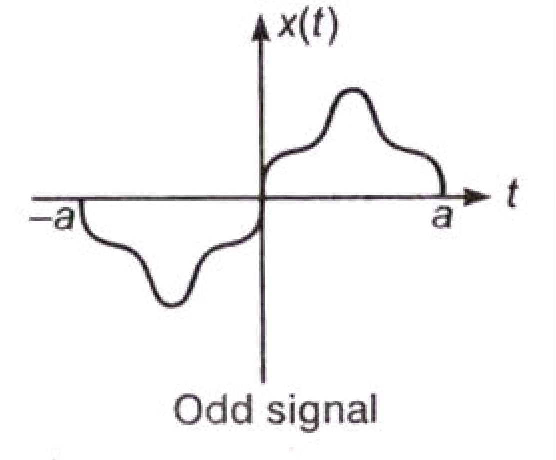
Let $X(t) = X_e(t) + X_0(t)$

Where, $X_e(t) =$ even part of $X(t)$

$X_0(t) =$ odd part of $X(t)$

$$X_e(t) = \frac{1}{2}[X(t) + X(-t)]$$

$$X_0(t) = \frac{1}{2}[X(t) - X(-t)]$$



Discrete Time Signals and System: The discrete signal is a function of a discrete independent variable. In a discrete time signal, the value of discrete time signal and the independent variable time are discrete. The digital signal is same as discrete signal except that the magnitude of the signal is quantized. Basically discrete time signals can be obtained by sampling a continuous-time signal. It is denoted as x(n).

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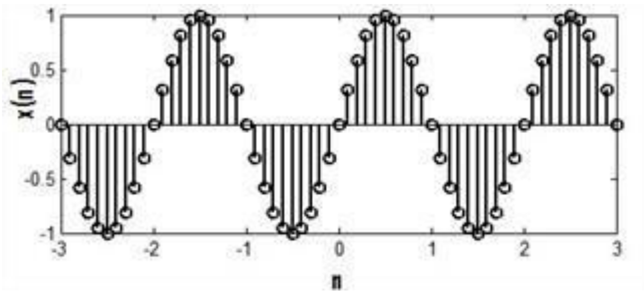
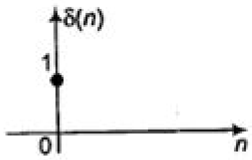


Standard Discrete Time Signals

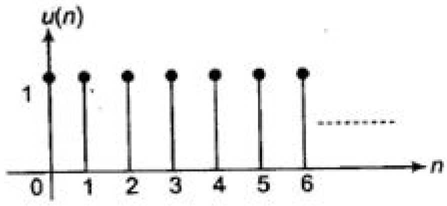
- **Digital Impulse Signal or Unit Sample Sequence**

  Impulse signal, $$\delta(n)=\begin{cases}1; & n=0\\ 0; & n\neq 0\end{cases}$$



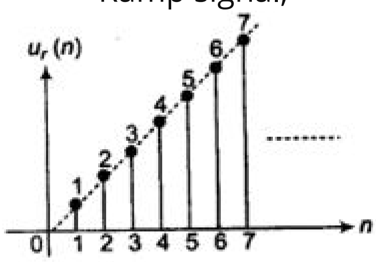
- **Unit Step Signal**

  $$u(n)=\begin{cases}1; & n\geq 0\\ 0; & n<0\end{cases}$$



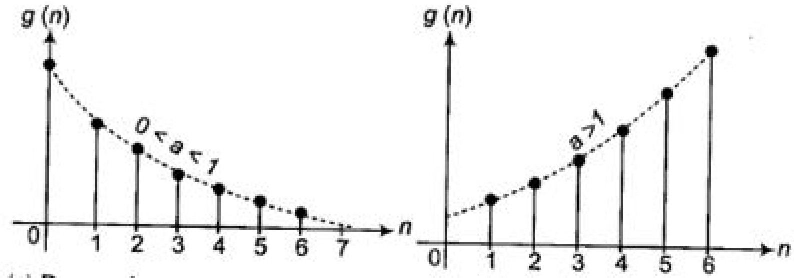
- **Ramp Signal**

  Ramp signal, $$u_r(n)=\begin{cases}n; & n\geq 0\\ 0; & n<0\end{cases}$$



- **Exponential Signal**

  Exponential signal, $$g(n)=\begin{cases}a^n; & n\geq 0\\ 0; & n<0\end{cases}$$

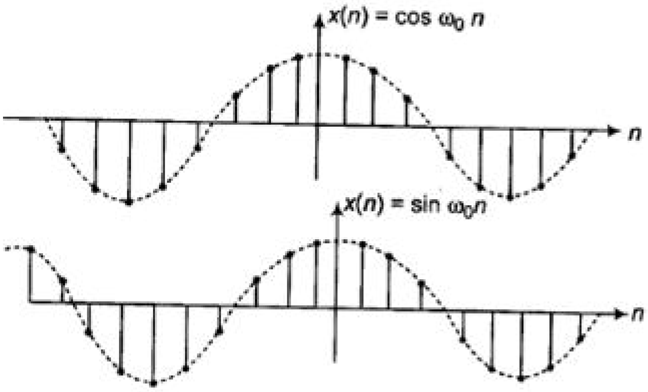


(a) Decreasing exponential signal (b) Increasing exponential signal

Discrete Time Sinusoidal Signal

$x[n] = A\cos(\omega_0 n + \theta)$; For n in the range $-\infty < n < \infty$

$x[n] = A\sin(\omega_0 n + \theta)$; For n in the range $-\infty < n < \infty$



- A discrete time sinusoid is periodic only if its frequency is a rational number.
- Discrete time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.

Shifting and Scaling Operations

There are some important properties of signal such as amplitude-scaling, time-scaling and time-shifting.

Amplitude Scaling:

Consider a signal *x(t)* which is multiplying by a constant *'A'* and this can be indicated by a notation *x(t)* → *Ax(t)*.

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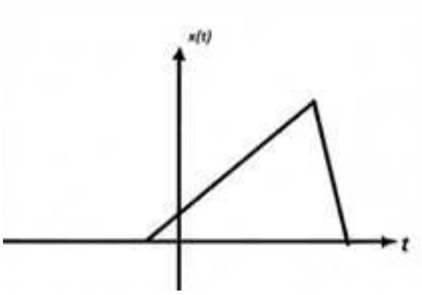
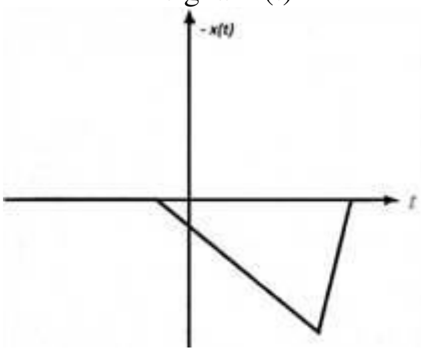
For any arbitrary *'t'* this multiplies the signal value $x(t)$ by a constant *'A'*.

Thus, $x(t) \rightarrow Ax(t)$multiplies $x(t)$ at every value of *'t'* by a constant *'A'*. This is called amplitude-scaling.

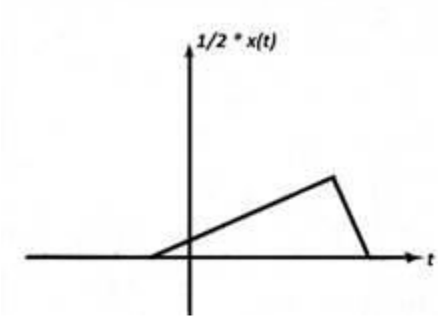
- If the amplitude-scaling factor is negative then it flips the signal with the t-axis as the rotation axis of the flip.
- If the scaling factor is -1 then only the signal will be flip.



A signal x(t)



A signal x(t) scaled by *-1*



A signal x(t) scaled by *1/2*

Time-Scaling of Signal:

Time scaling compresses or dilates a signal by multiplying the time variable by some quantity.

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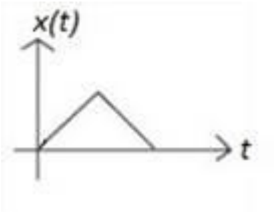
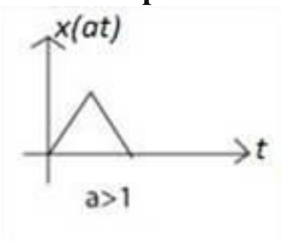


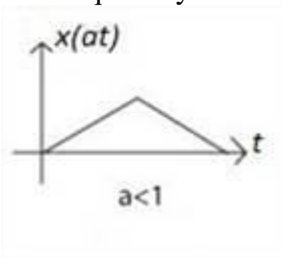
Signal x(t)

If that quantity is greater than one, the signal becomes narrower and the operation is called **compression**.



Compression of signal

If that quantity is less than one, the signal becomes wider and the operation is called**dilation**.

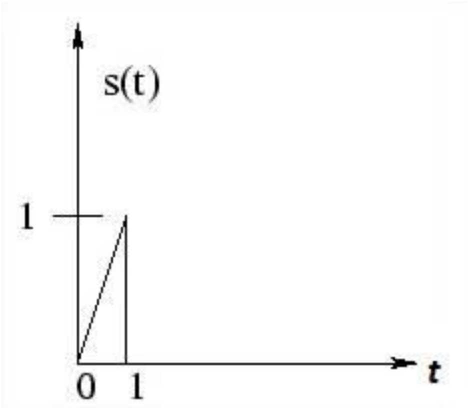


Dilation of signal

Time-Shifting of Signal:

If a continuous time signal is defined as $x(t) = s(t - t1)$. Then we can say that $x(t)$ is the time shifted version of $s(t)$.

Consider a simple signal $s(t)$ for $0 < t < 1$



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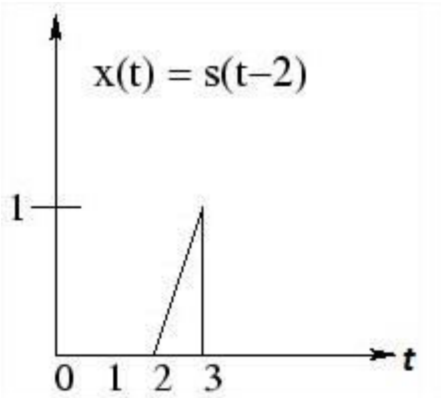
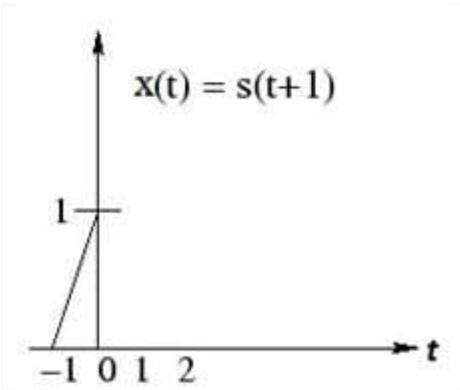
Signal within $0 < t < 1$



Signal shifted by *2* sec.



Signal shifted by -1 sec.

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Now shifting the function by time $t_1 = 2$ sec.

$x(t) = s(t-2) = t-2$; for $0 < (t - 2) < 1$

$= t-2$; for $2 < (t - 2) < 3$

This is simply signal s(t) with its origin delayed by 2 sec.

Now if we shift the signal by $t_1 = -1$ sec,

then $x(t) = s\,(t+1) = t+1$ *for* $0 < (t+1)$

$= t+1$ *for* $-1 < t < 0$.

This is simply s(t) with its origin shifted to the left or advance in time by 1 seconds.

Linear, Time-Invariant and Causal Systems

Linear Time Invariant Systems: A system can be regarded as the transformation of a signal according to some specified rule. Linear time invariant systems are a special class of system that can be described by constant-coefficient difference or differential equations, or equivalently by a convolution sum or integral.

Classification of Systems: Systems can be classified by a number of properties. These include memory; invertibility; causality; stability; time invariance; and linearity.

Memory / Memoryless: A system without memory does not depend on past or future inputs. The output is determined entirely by the present input. An example is a resistor at low frequencies, where the voltage across its terminals is directly proportional to the current entering / leaving its terminals.

$v(t)=Ri(t)$

An example of a system with memory is a delay element:
$y(t)=x(t-T)$

Another example is a capacitor, which has a voltage across its terminals proportional to the time-average of the current entering / leaving its terminals:

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

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Invertibility: A system is invertible if for every input there is a unique output. In that way the input signal can be inferred for any output signal. An example is a resistor with finite, non-zero resistance. The inverse system for the resistor is

$$i(t) = \frac{1}{R} v(t)$$

An example of a non-invertible system is:

$y(t)$=tan[$x(t)$]

Causality: A causal system is one that does not depend on future inputs. Thus a causal system cannot respond before the input is applied.

Examples of non-causal systems include image processing, where the independent variable is not time, but space. In that case the interpretation of future inputs simply refers to a point in space ahead of the present position. Another example is the post-processing of an audio signal. In that case the signal is recorded onto some media before it is processed.

Stability: A system can be considered stable if for any bounded input, the output is also bounded. An example of an unstable system is exponential population growth. Systems consisting of passive elements only such as resistors and capacitors are all stable systems. Care must be taken when designing various classes of control systems and filters to ensure that they behave in a stable manner.

Time Invariance: A system is time invariant if its characteristics do not change with time. Stated mathematically, if $y(t)$ is the response to $x(t)$, then the system is time invariant if and only if for any given T, $y(t–T)$ is the response to $x(t–T)$ for all t.

Let

$x(t) \Leftrightarrow y(t)$

denote that $y(t)$ is the response of the system to $x(t)$.

Suppose that

$x_1(t) \Leftrightarrow y_1(t)$

and

$x_2(t) \Leftrightarrow y_2(t)$

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Then the system is linear if and only if

$$ax_1(t)+bx_2(t) \Leftrightarrow ay_1(t)+by_2(t)$$

As an example, the system

$y(t)=x(t)+1$

is not linear.

Consider the signals $x_1(t) = 1$ and $x_2(t) = 2$. In that case, $y_1(t) = 2$ and $y_2(t) = 3$. The response to $x(t) = x_1(t)+x_2(t) = 3$ is $y(t) = 4$, which is clearly not equal to $y_1(t)+y_2(t) = 5$. Hence the system is non-linear.

There are very few truly linear systems in the physical world. However, many systems can be considered linear over a limited range of inputs and to a specified degree of accuracy, or can be linearized through mathematical manipulation. An example is a thermocouple.

Linear time invariant systems are conveniently described by the operation of convolution, which is discussed for discrete and continuous time systems below.

The convolution sum: Consider a discrete time signal $x[n]$. It is sometimes useful to express $x[n]$ as the sum of a series of weighted, time-shifted pulses.

For example, suppose a signal has values 1, -2 and 5 at samples $n = 0$, 1 and 2 and is zero for all other n. Then signal can be described by

$$x[n]=p[n]-2p[n-1]+5p[n-2]$$

where

$$p[k] = \begin{cases} 1, & k = 0 \\ 0, & otherwise \end{cases}$$

In general, for a signal of infinite duration, $x[n]$ is given by the infinite sum

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]p[n-k]$$

where $p[n–k]$ is only ever non-zero when $k = n$.

In that way, the correct sample value is picked out for any given n.

An interesting situation occurs when the elementary pulse, $p[n]$, is replaced with a signal in its own right, $h[n]$.

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The resulting signal is

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

It is possible that $h[n–k]$ is non-zero for a number of different k. Thus for any given n, $y[n]$ is actually a weighted average of $x[k]$. The weights are $h[n–k]$.

The operation described by the above equation is called the *convolution sum*, often denoted in abbreviated form by

$$y[n]=x[n]*h[n]$$

As an example, consider the signal

$$h[n]=\begin{cases} 2 & n=0 \\ -1 & n=1 \\ 0 & otherwise \end{cases}$$

which is more conveniently expressed in terms of $p[n]$ by

$$h[n]=2p[n]-p[n-1]$$

The convolution sum reduces to

$$y[n]=2x[k]-x[k-1]$$

If the convolution sum is regarded as a system with $p[n]$ as the input signal, then the output of the system is $h[n]$. For this reason, $h[n]$ is known as the *impulse response* of the system.

The convolution integral: The convolution sum can be extended to encompass continuous time signals and systems to yield the convolution integral.

If the discrete-time signal $x[n]$ is considered to be samples of the continuous-time signal$x(t)$ then

$$x(t)\approx\sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

where $\delta_{\Delta}(t)$ is unit impulse. In the limit as Δ approaches zero, it becomes a continuous integral and the approximation becomes an equality:

$$x(t)=\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

As for the discrete-time case, if the impulse is replace by an arbitrary signal $h(t)$, the*convolution integral* results:

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$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

which is often denoted as

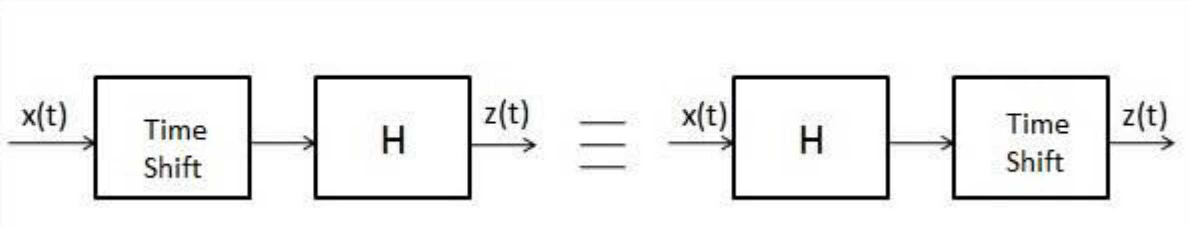
$$y(t)=x(t)*h(t)$$

If the convolution integral is regarded as a system, then $h(t)$ is the impulse response of the system.

Time-variant and Time invariant property:

- A system is said to be time invariant if the response of the system to an input is not a function of time.
- A system is time variant if the response to an input alters with time i.e. the system has varying response to the same input at different instants of time.
- One consequence of time variance or time invariance property is that a shifted input will produce a same amount of shift in the output response in a linear time invariant system for any given shift.
- Time varying system will produce a response which does not match with the shifted version of the original unshifted response shifted by same amount.

The following block diagram shows the time-invariance property of a system.

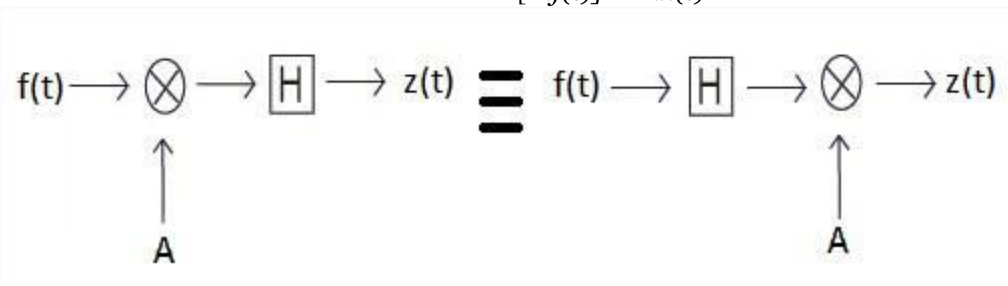


Time-invariance property of a system

- If the system is time invariant, the two waveforms will match when the input and output shifts match.
- If time variant, the waveforms will not match.

Linearity and Non-linearity property: A linear system is any system that obeys the superposition principle means if it satisfies the **additivity** and **homogeneity**.
To show that a system 'H' obeys the scaling or homogeneity property is to show that

$$H[A.f(t)] = A.x(t)$$



Homogeneity property of a system

If $f(t)$ and $g(t)$ are two distinct signals and $x(t) = H[f(t)]$ and $y(t) = H[g(t)]$

To demonstrate that a system H obeys the additivity property is to show that

$$H[f(t) + g(t)] = x(t) + y(t)$$

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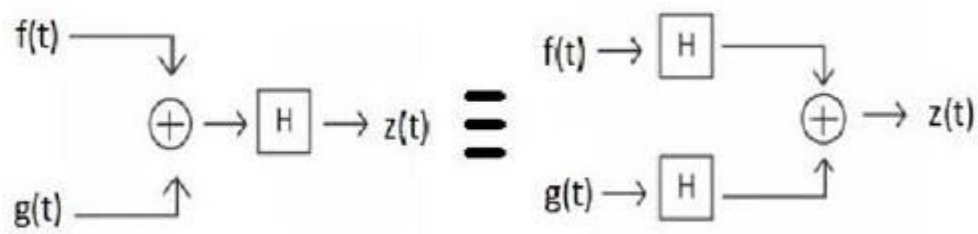


Additivity property of a system

If f(t) and g(t) are two distinct signals and x(t) = H[f(t)] and y(t) = H[g(t)], 'A', 'B' are constants.

It is possible to check a system for linearity in a single (though larger) step. To do this, simply combine the two steps to get

$$H[A.f(t) + B.g(t)] = A.x(t) + B.y(t)$$

Now, if we have a setup where the linear combination of two inputs f(t) and g(t) with gains 'A' and 'B' respectively can be provided to a system and if we can now linearly combine the responses x(t) of f(t) and y(t) of g(t) with gain factors 'C' and 'D' respectively and compare the two waveforms, we can test the linearity or non linearity of a system by matching the gains A = C and B = D. If the two waveforms match when A = C and B = D, the system is linear, otherwise it is not.

A nonlinear system is any system that does not have at least one of these properties.

Now if x(t) = H[f(t)] is the response of the system to an input f(t), 'A' is a constant

Properties of Linear Time Invariant Systems:

- Any linear time invariant system is characterized by its impulse response and its behaviour is described by the convolution operation, the properties of such systems are equivalent to the properties of convolution.

  The convolution is associative, commutative and distributive.

  A system can also be causal or non-causal.

Associative property: The associative property of convolution can be stated as $\bar{x}(t)$

In terms of linear systems, any cascade of systems with impulse responses $h_1(t)$ and $h_2(t)$ is equivalent to a single system with impulse response $h_1(t)*h_2(t)$.

Commutative property: The commutative property can be stated as

$h1(t)*h_2(t)=h_2(t)*h_1(t)$

It states that the order in which any two linear systems are applied to a signal is irrelevant. An alternative interpretation is allowed if we consider

$x(t)*h(t)=h(t)*x(t)$,

which implies that the impulse response of a system can be interchanged with its input signal without affecting the output.

Distributive property: The distributive property is

$$\tilde{x}(t) = \sum_{-\infty}^{\infty} \frac{1}{T_0} X(kf_0) e^{j2\pi kf_0 t}$$

$$= \sum_{k=-\infty}^{\infty} X(kf_0) e^{j2\pi kf_0 t} f_0$$

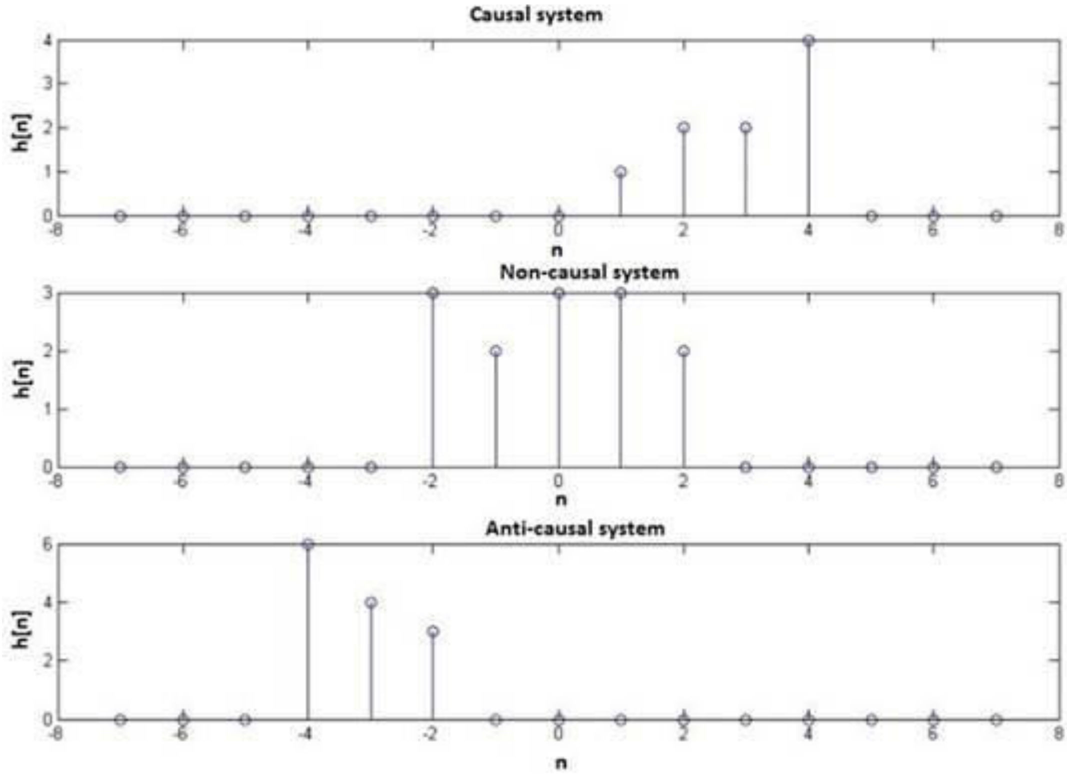
It implies that if signal is applied to two separate systems and the outputs are summed together, the result is equivalent to applying the same signal to a single system with an impulse response equal to the sum of the impulse responses of the individual systems.

Causality: Not all linear time-invariant systems are necessarily causal. The causality of a system is reflected by the impulse response. For any causal system, the impulse response is zero for all $t < 0$.

Causality and Non-causality property:

- A causal system is a system in which the output depends only on current or past inputs, but not on future inputs.
- An anti-causal system is a system in which the output depends only on current or future inputs, but not past inputs. Finally, a non-causal system is a system in which the output depends on both past and future inputs.

The following figure shows impulse responses of discrete-time linear time invariant causal, non-causal and anti-causal systems.



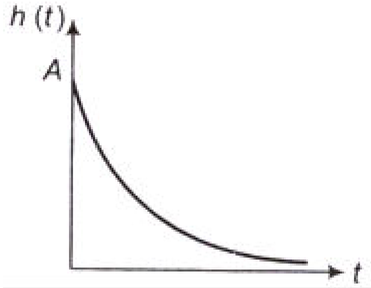
Impulse response of causal, non-causal and anti-causal system

- Real time systems in which time is the independent variable must always be causal because no system can depend on a future input value.
- Non-causality can however exist in domains such as image processing where independent variable is the pixel position.
- A non-causal or anti-causal system can be converted to a causal system by introducing appropriate delay in the system.

Impulse Response and Location of Poles of Transfer Function in s-plane

- **Impulse Response h(t):**

h(t) Ae^{-at} u(t); a>0



Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{s+a}$$

Pole at s = -a
ROC is σ > -a
Where, σ is real part of s.

Location of Poles in s-plane and ROC:

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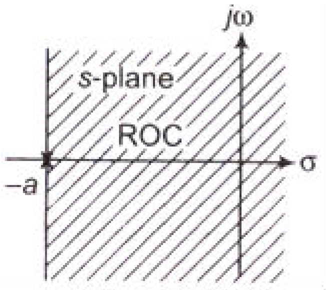
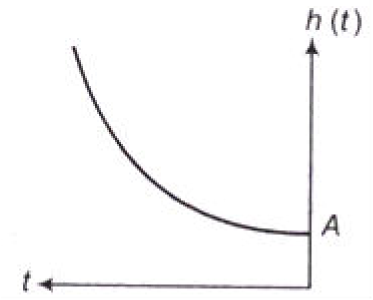


The pole at lies on left half of s-plane. ROC includes imaginary axis, causal system. Since poles lies on LHP and the imaginary axis is included in ROC, the system is stable.

- **Impulse Response h(t):** h(t) Ae-at u(-t); a>0

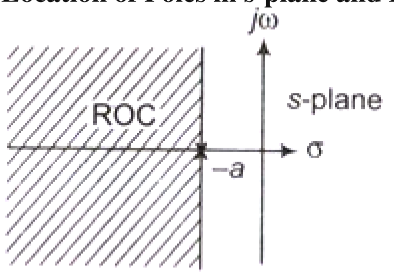


Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{s+a}$$

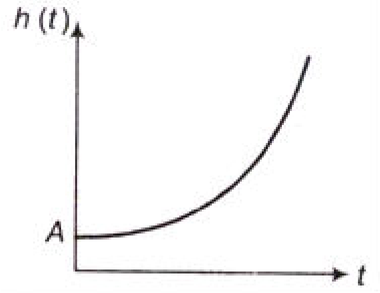
Pole at s = -a
ROC is σ < -a
Where, σ is real part of s.

Location of Poles in s-plane and ROC:



The pole at s = -a lies on left half of s-plane. The ROC does not include imaginary axis, non-causal system. Since, imaginary axis, is not included in ROC, the system is unstable.

- **Impulse Response h(t):** h(t) Ae^{-at} u(t); a>0



Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{s-a}$$

Pole at s = -a

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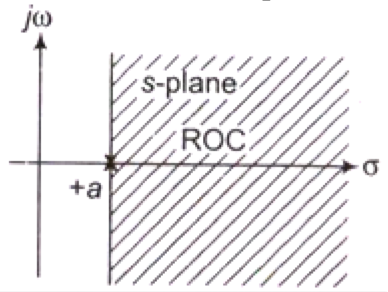
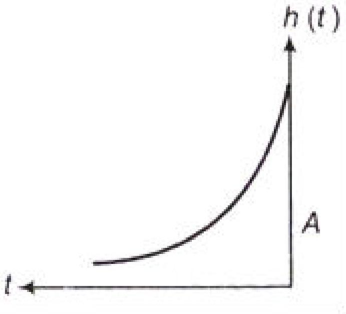
ROC is σ < -a
Where, σ is real part of s.

Location of Poles in s-plane and ROC:



The pole at lies on right half of s-plane. ROC does not include imaginary axis, causal system. Since, pole lies on RHP and imaginary axis, is not included in ROC, the system is unstable.

- **Impulse Response h(t):** h(t) Ae^{-at} u(-t); a>0

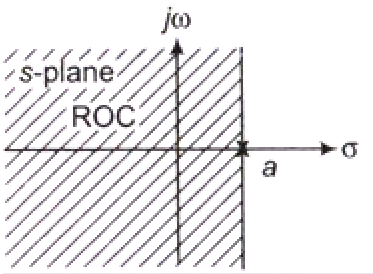


Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{s-a}$$ Pole at s = +a

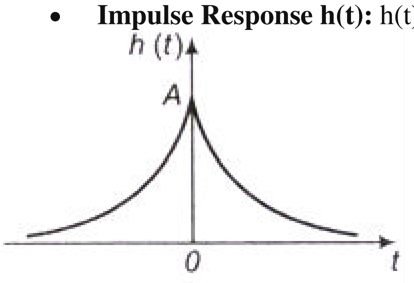
ROC is σ < +a
Where, σ is real part of s.

Location of Poles in s-plane and ROC:



The pole at lies on right half of s-plane. The ROC include imaginary axis, non-causal system. Since, imaginary axis, is included in ROC, the system is stable.

- **Impulse Response h(t):** h(t) Ae$^{-a|t|}$; a>0

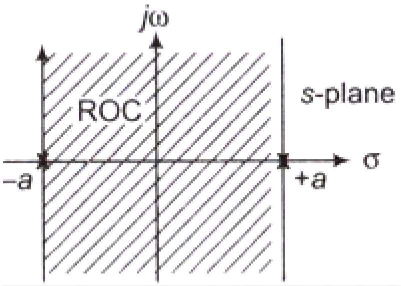


Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{s+a} - \frac{A}{s-a}$$

Pole at s = -a, +a
ROC is -a < σ < + a
Where, σ is real part of s.

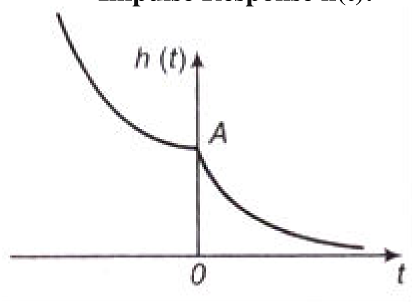
Location of Poles in s-plane and ROC:



The pole at lies on RHP and pole at s = – a, lies on LHP. ROC includes imaginary axis, non-causal system. Since, imaginary axis, is included in ROC, the system is stable.

$$h(t) Ae^{-at}u(t) + A^{-bt}u(-t)$$

- **Impulse Response h(t):** $where\ a > 0,\ b > 0,\ a > b$

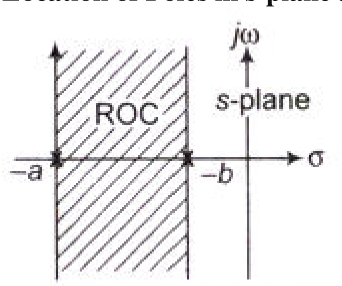


Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{s+a} - \frac{A}{s+a}$$

Pole at s = -a, -b
ROC is -a < σ < +a
Where, σ is real part of s.

Location of Poles in s-plane and ROC:



The pole at lies on LHP. The ROC does includes imaginary axis, non-causal system. Since, imaginary axis, is not included in ROC, the system is unstable.

- **Impulse Response h(t):** $h(t)\ A\ u(t)$

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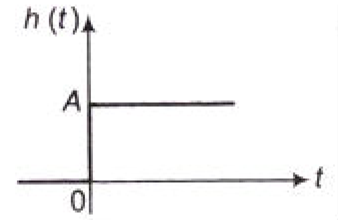
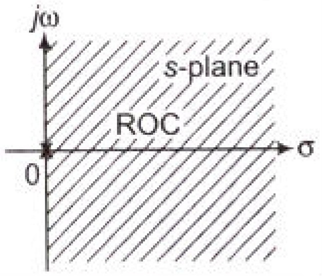


Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{s}$$

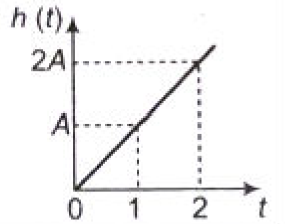
Pole at s = 0
ROC is σ > 0
Where, σ is real part of s.

Location of Poles in s-plane and ROC:



The pole at lies on imaginary axis. The ROC does not include the imaginary axis, causal system. Since, imaginary axis, is not included in ROC, the system is unstable.

- **Impulse Response h(t):** *h(t) = At(u)t*

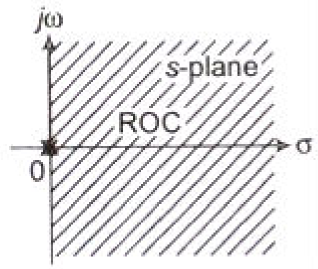


Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{s^2}$$

Double Pole at s = 0
ROC is σ > 0
Where, σ is real part of s.

Location of Poles in s-plane and ROC:



s = -a – jb, +jb, The pole at lic on imaginary axis. The ROC does not include the imaginary axis, causal system. Since, the imaginary axis, is not included in ROC, the system is unstable.

- **Impulse Response h(t):** *h(t) = 2A cos bt u(t)*

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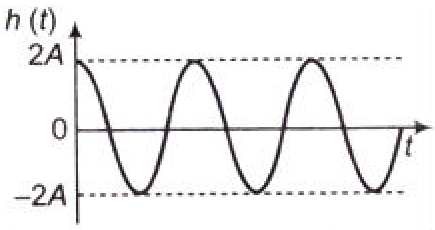
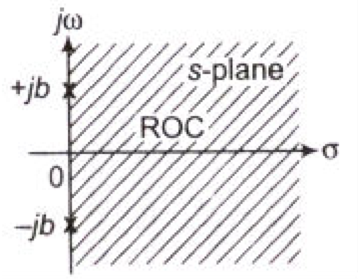


Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{s + jb} + \frac{A}{s - jb}$$

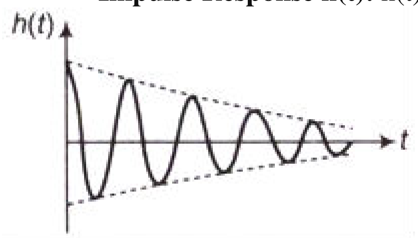
Pole at s = -jb, +jb
ROC is σ > 0
Where, σ is real part of s.

Location of Poles in s-plane and ROC:



The pole at lies on imaginary axis. The ROC does not include the imaginary axis, causal system. Since, imaginary axis, is not included in ROC, the system is unstable.

- **Impulse Response h(t):** h(t) = $2Ae^{bt}$ cos bt u(t) *where a > 0*

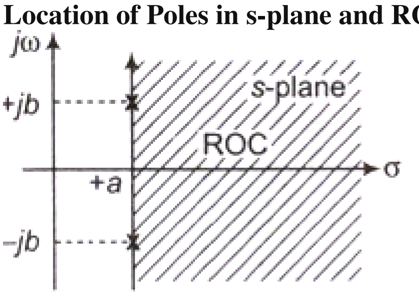


Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{s - a + jb} + \frac{A}{s - a - jb}$$

Pole at s = a-jb, -a+jb
ROC is σ > -a
Where, σ is real part of s.

Location of Poles in s-plane and ROC:



The pole at lies on right half of s-plane. The ROC does not includes imaginary axis, causal system. Since, poles lie on. RHP and the imaginary axis, is included in ROC, the system is stable.

- **Impulse Response h(t):** *h(t) = 2At cos bt u(t)*

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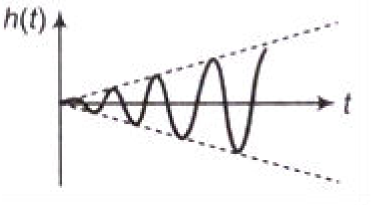
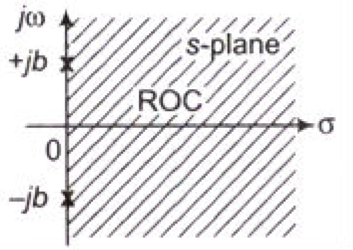


Transfer Function H(s) = L{h(t)}:

$$H(s) = \frac{A}{(s+jb)^2} + \frac{A}{(s-jb)^2}$$

Double Pole a s = -jb, +jb
ROC is σ > a
Where, σ is real part of s.

Location of Poles in s-plane and ROC:



The pole at s = -jb, +jb, lies on imaginary axis. The ROC does not includes imaginary axis, causal system. Since, the imaginary axis, is included in ROC, the system is unstable.

Basic Elements of Block Diagram in Time Domain and s-Domain

Elements of Block Diagram	Time Domain Representation
Differentiator	$x(t) \rightarrow \boxed{d/dt} \rightarrow \frac{d}{dt}x(t)$
Integrator (with zero initial condition)	$x(t) \rightarrow \boxed{\int} \rightarrow \int x(t)\,dt$
Constant multiplier	$x(t) \rightarrow \boxed{a} \rightarrow a\,x(t)$
Signal adder	

Key Points

- Since, the signal X(s) attains infinite values at poles, the RCO of X(s) does not include poles.
- In a realizable system, the number of zeros will be less than or equal to number of poles.
- For a stable LTI continuous time system, the ROC should include the imaginary axis of s-plane.
- For a stable LTI continuous time causal system, the poles should lie on the left half of s-plane and the imaginary axis should be included in the ROC.

- For a stable LTI continuous time non-causal system, the imaginary axis should be included in the ROC.

Fourier Series Representation of Continuous Periodic Signals

Fourier Series: The Fourier series is the frequency domain representation of periodic signals.

Trigonometric Form of Fourier Series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Note: Here $a_0/2$ is the value of constant component of the signal $x(t)$.

The Fourier coefficients a_n and b_n are maximum amplitude of nth harmonic component.

Determination of Fourier Coefficient

The Fourier coefficient can be evaluated using the following formulae.

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \text{ or } a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\,\omega_0\,t dt$$

$$\text{or } a_n = \frac{2}{T}\int_0^T x(t)\cos n\,\omega_0\,t dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\,\omega_0\,t dt$$

$$\text{or } b_n = \frac{2}{T}\int_0^T x(t)\sin n\,\omega_0\,t dt$$

Note: The Fourier series exists only if Dirichlet conditions are satisfied.

Exponential Form of Fourier Series: The signals with negative frequency are required for mathematical representation of real signals in terms of complex signals. In exponential form of Fourier series, $|C_n|$ represents the magnitude of nth harmonic component.

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$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t).e^{-jn\omega_0 t}\,dt \text{ or } c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}\,dt \;;\; c_n = |c_n|\angle c_n$$

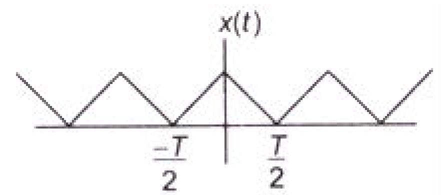
$|C_n|$ Magnitude of cn; $\angle c_n$ = Phase of c_n

- The plot of harmonic magnitude/phase versus harmonic number n (or harmonic frequency) is called frequency spectrum.
- The frequency spectrum obtained from Fourier series is also called line spectrum.
- The plot of magnitude versus n (or ω_0) is called magnitude (line) spectrum.
- The plot of phase versus n (or ω_0) is called phase (line) spectrum.

Fourier Coefficient of Signal with Symmetry

Even Symmetry

$$\boldsymbol{x(t) = x(-t)}$$



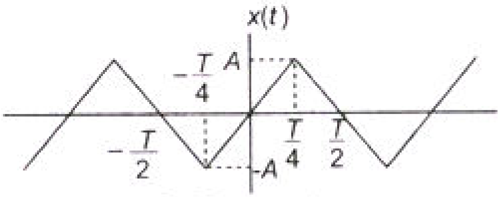
Even symmetry

For signals with even symmetry, the Fourier coefficient b_n are zero.

Odd Symmetry

$$\boldsymbol{x(t) = x(-t)}$$

For signals with odd symmetry, the Fourier coefficient a_0 and a_n are zero.

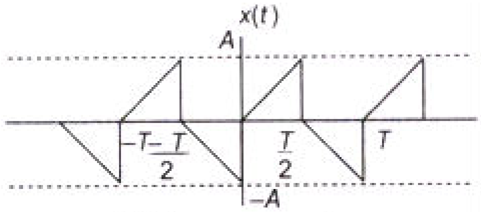


Odd symmetry

Half-wave Symmetry (or Alternation Symmetry)

$$x\left(t \pm \frac{T}{2}\right) = -x(t)$$

For signals with half-wave symmetry, the Fourier series will consist of odd harmonic terms alone.



Half-wave symmetry

Relation between Fourier Coefficients of Trigonometric and Exponential Form

$$c_0 = a_0 / 2$$

$$c_n = \frac{1}{2}(a_n - j b_n) \; for \; n = 1, 2, 3, 4, ...$$

$$c_n = \frac{1}{2}(a_n - j b_n) \; for - n = -1, -2, -3, ...$$

$$\therefore |c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$ for all values of n, except when n = 0.

Note: c_n and d_n are exponential form of Fourier series coefficients of $x(t)$ and $y(t)$respectively.

Properties of Exponential Form of Fourier Series Coefficients

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Property	Continuous Time Periodic Signal	Fourier Series Coefficients
Linearity	$A\,x(t)+B\,y(t)$	Ac_n+Bd_n
Time Shifting	$x\,(t-t_0)$	$c_n e^{-jm\omega_0 t_0}$
Frequency shifting	$e^{-jm\omega_0 t}\,x(t)$	c_{n-m}
Conjugation	$x^*(t)$	c^*_{-n}
Time reversal	$x(-t)$	c_{-n}
Time scaling	$x(\alpha t)\,;\,\alpha>0\,[x(t)$ Is period with period $T/\alpha]$	c_n (No change in Fourier coefficient)

Property	Continuous Time Periodic Signal	Fourier Series Coefficients
Multiplication	$x(t)\,y(t)$	$\sum_{m=-\infty}^{+\infty} c_m\,d_{n-m}$
Differentiation	$\frac{d}{dt}x(t)$	$jn\omega_0 c_n$
Integration	$\int_{-\infty}^{t} x(t)\,dt$ (Finite valued and periodic only if $a_0=0$)	$\frac{1}{jn\omega_0}c_n$
Periodic convolution	$\int_T x(\tau)\,y(t-\tau)\,d\tau$	$Tc_n d_n$

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Property	Continuous Time Periodic Signal	Fourier Series Coefficients
Symmetry of real signals.	$x(t)$ is real	$c_n = c_{-n}$ $\lvert c_n \rvert = \lvert c_{-n} \rvert ; \angle c_n = -\angle c_{-n}$ $\mathrm{Re}\{c_n\} = \mathrm{Re}\{c_{-n}\}$ $\mathrm{Im}\{c_n\} = -\mathrm{Im}\{c_{-n}\}$
Real and even	$x(t)$ is real and even	c_n Are real and even
Real and odd	$x(t)$ is real and odd	c_n Are imaginary and odd
Parseval's relation	Average power, P of $x(t)$ is defined as	The average power, P in terms of Fourier series coefficients is $P = \sum_{N=-\infty}^{+\infty} \lvert c_n \rvert^2$
	$\frac{1}{T}\int_T \lvert x(t) \rvert^2 \, dt = \sum_{n=-\infty}^{+\infty} \lvert c_n \rvert^2$	

Key Points

- The term $|c_n|^2$ represent the power in nth harmonic component of $x(t)$. The total average power in a periodic signal is equal to the sum of power in all of its harmonics.
- The term $|c_n|^2$ for n=0,1,2,… is the distribution of power as a function of frequency and so it is called power density spectrum or power spectral density of the periodic signal.
- The Fourier transform is frequency domain representation of non-periodic signals.

Fourier Transform: The Fourier Transform converts a set of time domain data vectors into a set of frequency (or per time) domain vectors. The Fourier Transform is a generalization of the Fourier series. Strictly speaking it only applies to continuous and aperiodic functions, but the use of the impulse function allows the use of discrete signals.

Fourier Transform can be given as

$$X(\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}\, dt$$

$$or\ X(f) = \int_{-\infty}^{\infty} x(t).e^{-j2\pi f t}\, dt$$

X(ω), X(f)…. represents Fourier transform of *x(t)* signal.

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Inverse Fourier Transform

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty} X(f).e^{j2\pi ft}\,df$$

$$X(t) \leftrightarrow X(\omega) \text{ or } X(t) \leftrightarrow X(f)$$

Existence of Fourier Transform-Dirichlet Conditions

- Single valued property
- Finite discontinuities
- Finite peaks
- Absolute integrability

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty,$$

These conditions are sufficient, but not necessary for the signal to be Fourier transformable.

$$Let\ F\{x(t)\} = X(j\omega);\ F\{x_1(t)\} = X_1(j\omega); F\{x_2(t)\} = X_2(j\omega)$$

Sampling Theorem

Sampling Theorem: A band limited continuous time signal with highest frequency (bandwidth) f_m hertz can be uniquely recovered from its samples provided that the sampling rate f_s is greater than or equal to $2f_m$ samples per second.

$$fs \leq 2fm$$

The sampling is the process of conversion of continuous time signal into discrete time signal. The time interval between successive samples is called sampling time or sampling period. The inverse of sampling period is called sampling frequency.

The sampling frequency of a particular situation, which may exceed by quite a bit the maximum frequency in the signal, is the Nyquist frequency. Twice the maximum frequency of the signal is called the Nyquist rate, and is the minimum sampling rate that can resolve the signal. i.e. when sampling frequency f_s is equal to $2f_m$. the sampling rate is called Nyquist rate.

- Nyquist rate $f_N = 2f_m$ hz
- Nyquist interval = $1/f_N = 1/(2fm)$ seconds

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The phenomenon of high frequency component getting the identity of low frequency component during sampling is called aliasing. For frequencies just above the half the sampling rate, up to the sampling rate, the aliased frequency $f_{alias} = f_N - |f_{actual} - f_N|$

Applications of Fourier Transform

Summary of Properties of Fourier Transform:

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Property	**Time Domain Signal**	**Frequency Domain Signal**
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(j\omega)+a_2X_2(j\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-j\omega)$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$Xj(\omega-\omega_0)$
Time differentiation	$\frac{d}{dt}x(t)$	$j\omega\, X(j\omega)$

Property	**Time Domain Signal**	**Frequency Domain Signal**
Time integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(j\omega)}{j\omega}=\pi X(0)\delta(\omega)$
Frequency differentiation	$t\ x(t)$	$j\frac{d}{d\omega}X(j\omega)$
Time convolution	$x_1(t)*x_2(t)$ $\int_{-\infty}^{+\infty} x_1(\tau)x_2(t-\tau)\,d\tau$	$X_1(j\omega)X_2(j\omega)$
Frequency convolution (or multiplication)	$x_1(t)x_2(t)$	$\frac{1}{2\pi}\int_{\lambda=+\infty}^{\lambda=+\infty} X_1(j\lambda)X_2[j(\omega-)]\delta\lambda$

Property	**Time Domain Signal**	**Frequency Domain Signal**
Symmetry of real signals	$x(t)$ is real	$X(j\omega)=X^*(j\omega)$ $\lvert X(j\omega)\rvert=\lvert X(-j\omega\rvert;$ $\angle X(j\omega)=-\angle X(-j\omega)$ $\mathrm{Re}\{X(j\omega)\}=\mathrm{Re}\{X(-j\omega)\}$ $\mathrm{Im}\{X(j\omega)=-\mathrm{Im}\{\mathrm{Im}\{X(-j\omega)\}$
Real and even	$x(t)$ is real and even	$X(j\omega)$ is real and even
Real and odd	$x(t)$ is real and odd	$X(j\omega)$ is imaginary and odd
Duality	If $x_2(t)\cong X_1(j\omega)$ [i.e., $x_2(t)$ and $X_1(j\omega)$ are similar functions] then and are similar functions]	

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Applications of Fourier Transform:

- Designing and using antennas
- Image Processing and filters
- Transformation, representation, and encoding
- Smoothing and sharpening
- Restoration, blur removal, and Wiener filter
- Data Processing and Analysis
- Seismic arrays and streamers
- Multibeam echo sounder and side scan sonar
- Interferometers
- Synthetic Aperture Radar (SAR) and Interferometric SAR (InSAR)
- High-pass, low-pass, and band-pass filters
- Cross correlation; transfer functions; Coherence
- Signal and noise estimation; encoding time series.

Laplace Transform and Z-Transform

Laplace Transform

The Laplace transform is used to transform a time domain signal to complex frequency domain.

$L\{X(t)\} = X(s) = \int_{-\infty}^{\infty} X(t)\,.\,e^{-st}\,dt$ is also called bilateral or two-sided Laplace transform, If $x(t)$ is defined for $t \geq 0$, [i.e., if $x(t)$ is causal],

then $L\{x(t)\} = X(s) = \int_{0}^{\infty} x(t)\,.\,e^{-st}\,dt$ is also called unilateral or one-sided Laplace transform.

The signal Ke^{st} can be thought of as an universal signal which represents all types of signals.

- A causal signal $x(t)$ is said to be of exponential order if a real, positive constant σ (where σ is real part of s) exists such that the function, $e^{-\sigma t}|X(t)|$ approaches zero as t approaches infinity.
- For a causal signal, if $\lim e^{-\sigma t}|x(t)| = 0$, for $\sigma > \sigma_c$ and if $\lim_{t \to \infty} e^{-\sigma t}|x(t)| = \infty$, for $\sigma > \sigma_c$ then σ_c is called abscissa of convergence, (where σ_c is a point on real axis in s-plane).

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- The value of s for which the integral $\int_{-\infty}^{\infty} x(t)\,.\,e^{-st}\,dt$ converges is called Region of Convergence (ROC).
- For a causal signal, the ROC includes all points on the s-plane to the right of abscissa of convergence.
- For an anti-causal signal, the ROC includes all points on the s-plane to the left of abscissa of convergence.
- For a two-sided signal, the ROC includes all points on the s-plane in the region in between two abscissa of convergence.

Properties of the ROC

The region of convergence has the following properties

- ROC consists of strips parallel to the $j\omega$-axis in the s-plane.
- ROC does not contain any poles.
- If $x(t)$ is a finite duration signal, $x(t) \neq 0,\ t_1 < t < t_2$ and is absolutely integrable, the ROC is the entire s-plane.
- If $x(t)$ is a right sided signal, $x(t) = 0,\ t_1 < t_0$, the ROC is of the form $R\{s\} > \max\{R\{p_k\}\}$
- If $x(t)$ is a left sided signal $x(t) = 0,\ t_1 > t_0$, the ROC is of the form $R\{s\} > \min\{R\{p_k\}\}$
- If $x(t)$ is a double sided signal, the ROC is of the form $p_1 < R\{s\} < p_2$
- If the ROC includes the $j\omega$-axis. Fourier transform exists and the system is stable.

Partial Fraction Expansion in Laplace Transform

All poles have multiplicity of 1.

$$X(s) = \frac{c_1}{s - p_1} + \ldots + \frac{c_n}{s - p_n}$$

where, $c_k = (s - p_k)\,X(s)\,|_{s = p_k}$

When one or more poles have multiplicity r.

In this case, $X(s)$ has the term (s-p)'.

$$\therefore X(s) = \frac{\lambda_r}{s - p} + \frac{\lambda_2}{(s - p)^2} + \ldots + \frac{\lambda_r}{(s - p)^r}$$

The coefficient λ_k can be found as

$$\lambda_k = \left[\frac{1}{(r-k)!}\frac{d^{r-k}}{ds^{r-k}}\left((s-p_k)^r X(s)\right)\right]_{s=p}$$

Inverse Laplace Transform

It is the process of finding $x(t)$ given $X(s)$

$$X(t) = L^{-1}\{X(s)\}$$

There are two methods to obtain the inverse Laplace transform.

Inversion using Complex Line Integral

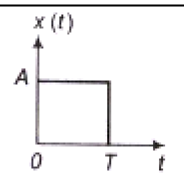
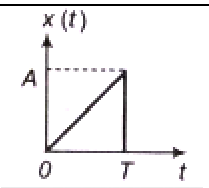
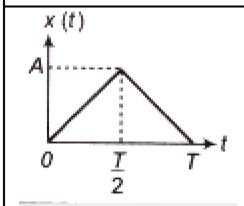
$$x(t) = \frac{1}{2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st} ds$$

Inversion using Laplace Table

Laplace transform can be written as $X(s)$ $\dfrac{\text{Nemerator (s)}}{\text{Denominator(s)}}$

$$= K\,\frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)}$$

Laplace Transform of Some Standard Signals

Waveform	$x(t)$	$x(s) = L\{x(t)\}$
	$x(t) = A;\ 0 < t < T$ $= 0;\ t > T$	$X(t) = \frac{A}{s}(1 - e^{-st})$
	$x(t) = \frac{At}{T}$ $0 < t < T$ $= 0;\ t > T$	$X(t) = \frac{A}{Ts^2}$ $[1 - e^{-st}(1 + sT)]$
	$x(t) = \frac{At}{T};0<t<\frac{T}{2}$ $= 2A - \frac{2At}{T}$ $= 0;\ t > T$	$X(t) = \frac{A}{Ts^2}\left(1 - e^{\frac{-sT}{2}}\right)^2$

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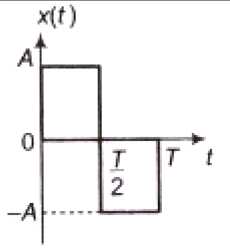
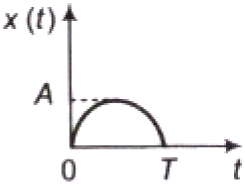
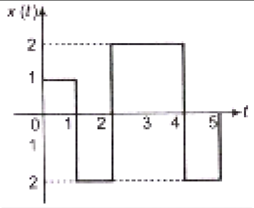
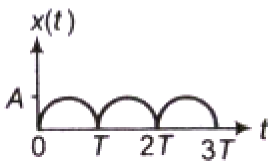
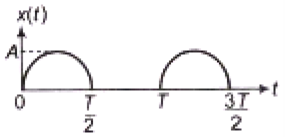
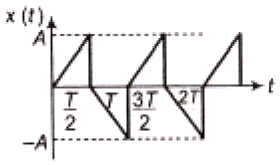
Waveform	$x(t)$	$x(s) = L\{x(t)\}$
	$x(t)= A; 0< t< \frac{T}{2}$ $= -A; \frac{T}{2}< t< T$	$X(t) = \frac{A}{s}\left(1-e^{\frac{-sT}{2}}\right)^2$
	$x(t)= A\sin t;$ $0< t< T$ $= 0; T> T$	$X(t) = \frac{A}{s^2+1}\left(e^{-sT}+1\right)$
	$x(t)= 1; 0< t< 1$ $= -2; 1< t< 2$ $= -2; 4< t< 5$ $= 0; t> 5$	$X(s)= \frac{1}{s}(1-3e^{-s}+$ $4e^{-2s}-4e^{-4s}+2e^{-2s})$

Waveform	$x(t)$	$x(s) = L\{x(t)\}$
	$x(t)= A\sin t;$ $0< t< T$ *and* $x(t+nT)= x(t)$	$X(t) = \frac{A}{\left(s^2+1\right)}\left(\frac{1-e^{-sT}}{1-e^{-sT}}\right)$
	$x(t)\cong A\sin t;$ $0< t< \frac{T}{2}$ $= 0; \frac{T}{2}< t< T$ *and* $(t+nT)= x(t)$	$X(t) = \frac{A}{\left(s^2+1\right)\left(1-e^{\frac{-sT}{2}}\right)}$
	$x(t)= \frac{2At}{T}; 0< t< \frac{T}{2}$ $= A-\frac{2At}{T}; \frac{T}{2}< t< T$ *and* $x(t+nT)= x(t)$	$X(s) = \frac{2A\left[1-\left(1+\frac{Ts}{2}\right)e^{-\frac{sT}{2}}\right]}{Ts^2\left(1-e^{\frac{sT}{2}}\right)}$

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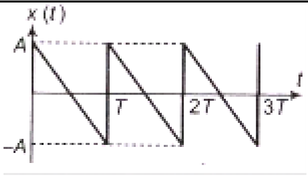
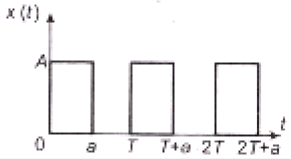
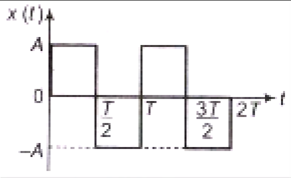
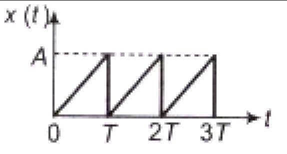
Waveform	$x(t)$	$x(s) = L\{x(t)\}$
	$x(t)= A\frac{2At}{T};$ $0< t< T$ and $x(t+nT)= x(t)$	$X(s) = \frac{2A}{Ts}\left(\frac{T 1+e^{-sT}}{2 1-e^{-sT}} - \frac{1}{s}\right)$
	$x(t)= A; 0< t< a$ $= 0; a< t< T$ and $(t+nT)= x(t)$	$X(s) = \frac{A}{s}\left(\frac{1-e^{-as}}{1-e^{-sT}}\right)$
	$x(t)= A; 0< t< \frac{T}{2}$ $= A; \frac{T}{2}< t< T$ and $(t+nT)= x(t)$	$X(t) = \frac{A}{s}\left(\frac{1-e^{-\frac{sT}{2}}}{1+e^{-\frac{sT}{2}}}\right)$
	$x(t)= \frac{At}{T}; 0< t< T$ and $(t+nT)= x(t)$	$X(s) = \frac{A}{Ts^2}$ $\left[\frac{1-e^{-sT}(1+sT)}{1+e^{sT}}\right]$

Some Standard Laplace Transform Pairs

$x(t)$	$x(s)$	**ROC**
$\delta(t)$	1	Entire s-plane
$u(t)$	$1/s$	$\sigma > 0$
$tu(t)$	$1/s^2$	$\sigma > 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$, where $n = 1, 2, 3$	$1/s^n$	$\sigma > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\sigma > -a$

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$x(t)$	$x(s)$	**ROC**
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	σ < -a
$t^n u(t)$ where, $n=1, 2, 3$	$\frac{n!}{s^{n+1}}$	σ > 0
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	σ > -a
$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t)$ where, $n= 1, 2, 3, \ldots$	$\frac{1}{(s+a)^n}$	σ > -a

$x(t)$	$x(s)$	**ROC**
$t^n e^{-at}u(t)$ where, $n= 1, 2, 3, \ldots$	$\frac{n!}{(s+a)^{n+1}}$	σ > -a
$\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(s^2+\omega_0^2)}$	σ > 0
$\cos\omega_0 t\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	σ > 0
$\sin h\ \omega_0 t\ u(t)$	$\frac{\omega_0}{s^2-\omega_0^2}$	σ > ω_0

$x(t)$	$x(s)$	**ROC**
$\cos h\ \omega_0 t\, u(t)$	$\frac{s}{s^2-\omega_0^2}$	σ > ω_0
$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	σ > -a
$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	σ > -a

Note: $L\{x(t)\}=X(s); L\{x_1(t)\}=X_1(s); L_2\{X_2(t)\}=X_2(s)$

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Properties of Laplace Transform

Property	Time Domain Signal	s-domain Signal
Amplitude scaling	$A\,x(t)$	$A\,X(s)$
Linearity	$a_1x_1(t) \pm a_2x_2(t)$	$a_1X_1(s) \pm a_2X_2(s)$
Time differentiation	$\frac{d}{dt}x(t)$	$sX(s) - x(0)$
	$\frac{d^n}{dt^n}x(t)$, where $n = 1, 2, 3, \ldots..$	$s^n X(s) - \sum_{k=1}^{n} s^{n-k} \left.\frac{d^{(k-1)}x(t)}{dt^{k-1}}\right\|_{t=0}$

Property	Time Domain Signal	s-domain Signal
Time integration	$\int x(t)\,dt$	$\frac{X(s)}{s} + \frac{[\int x(t)\,dt)]_{t=0}}{s}$
	$\int \ldots \pm \int x(t)\,(dt)^n$, where $n = 1,2,3,\ldots.$	$\frac{X(s)}{s} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}}$ $[\int \ldots \int x(t)dt)^k]]_{t=0}$
Frequency shifting	$e^{\pm at}\,x(t)$	$X(s \pm a)$
Time shifting	$x(t \pm \alpha)$	$e^{\pm as}\,X(s)$
Frequency differentiation	$t\,x(t)$	$-\frac{dX(s)}{ds}$
	$t^n\,x(t)$, where $n = 1,2,3,\ldots.$	$(-1)^n \frac{d^n}{ds^n} X(s)$

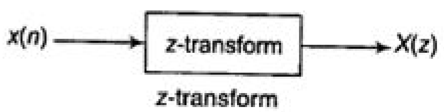
Property	Time Domain Signal	s-domain Signal
Frequency integration	$\frac{1}{t}x(t)$	$\int_s^{\infty} X(s)\,ds$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{s}{a}\right)$
Periodicity	$x(t+nT)$	$\frac{1}{1-e^{-sT}} \int_0^T x_1(t)e^{-st}dt$ where, $x_1(t)$ is one period of $x(t)$
Initial value Theorem	$\lim_{t\to 0} x(t) = x(0)$	$\lim_{s\to 0} s\ X(s)$
Final value theorem	$\lim_{t\to 0} x(t) = x(\infty)$	$\lim_{s\to 0} s\,X(s)$
Convolution theorem	$x_1(t)*x_2(t)$ $=\int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)d\lambda$	$X_1(s)X_2(s)$

Key Points

- The convolution theorem of Laplace transform says that, Laplace transform of convolution of two time domain signals is given by the product of the Laplace transform of the individual signals.
- The zeros and poles are two critical complex frequencies at which a rational function of a takes two extreme value zero and infinity respectively.

z–Transform: The z-transform is discrete counterpart of Laplace transform. The DTFT can be applied only to stable systems. But z-transform can be calculated for unstable systems as well.

The solution of linear difference equation becomes easy with the help of z-transform. The linear difference equation is converted to algebraic equation with the help of z-transform.



$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n}$$ z-transform pair $x(n) \xleftrightarrow{z} X(z)$

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The z-transform defined above has both sided summation. It is called bilateral or both sided z-transform.

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$ is called unilateral or one sided z-transform.

Region of Convergence (ROC): ROC is the region where z-transform converges. It is clear that z-transform is an infinite power series. The series is not convergent for all values of z.

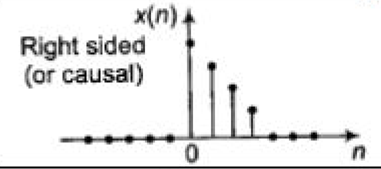
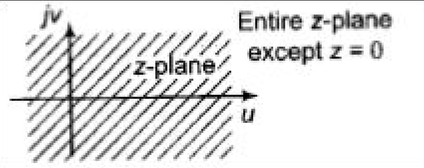
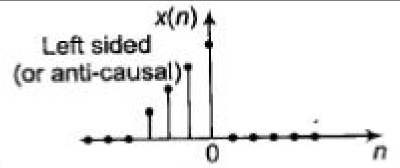
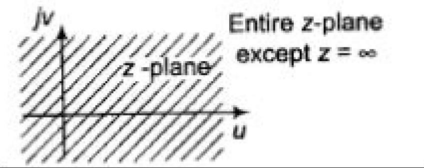
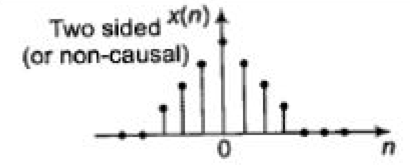
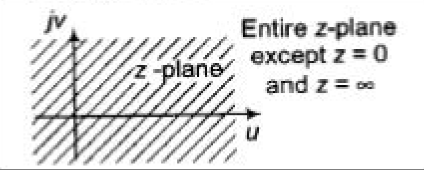
Significance of ROC

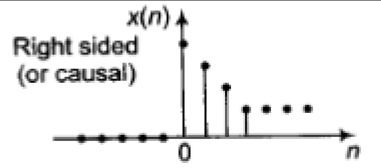
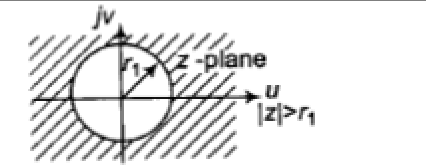
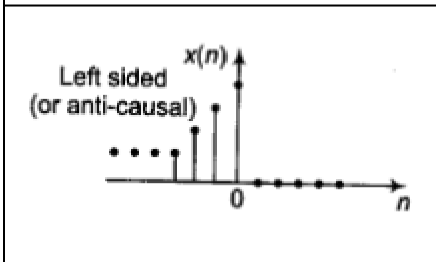
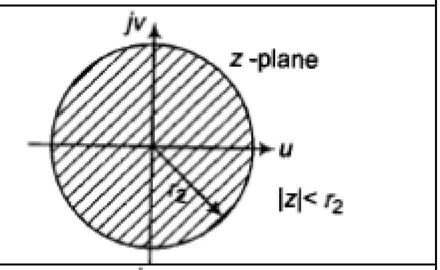
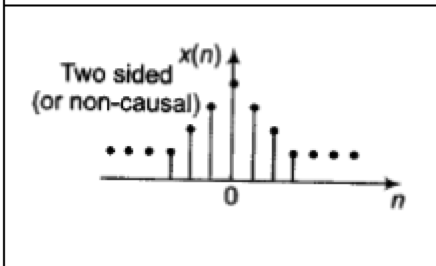
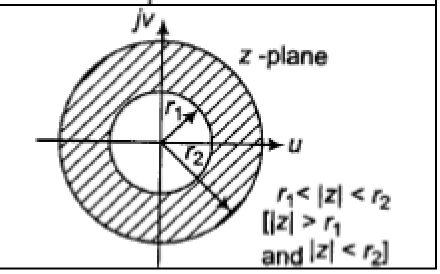
- ROC gives an idea about values of z for which z-transform can be calculated.
- ROC can be used to determine causality of the system.
- ROC can be used to determine stability of the system.

Summary of ROC of Discrete Time Signals for the sequences

Sequence	**ROC**
Finite, right sided (causal)	Entire z-plane except $z = 0$
Finite, left sided (anti-causal)	Entire z- plane except $z = \infty$
Finite, two sided (non-causal)	Entire z-plane except $z = 0$ and $z = \infty$
Infinite, right sided (causal)	Exterior of circle of radius r_1, where $\lvert z\rvert > r_1$
Infinite, left sided (anti-causal)	Interior of circle of radius r_2, where $\lvert z\rvert < r_2$
Infinite, two sided (non-causal)	The area between two circles of radius r_2 and r where, $r_2 > r_1$ and $r_1 < \lvert z\rvert\ r_2$, (i. e., $\lvert z\rvert > r_1$ and $\lvert z\rvert < r_2$)

Characteristic Families of Signals and Corresponding ROC

Signal	ROC in z-plane
Finite Duration Signals	
	
	
	

Signal	ROC in z-plane
Infinite Duration Signals	
Signal	**ROC in z-plane**
	
	
	

Note: $X(z) = z\{x(n)\}$; $X_1(z) = Z\{x_1(n)\}$; $X_2(z) = z\{x_2(n)\}$; $Y(z) = z(y(n))$

Summary of Properties of z-Transform:

Property		Discrete Time Signal	z-Transform
Linearity		$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$
Shifting $(m \geq 0)$	x(n) for $n \geq 0$	$x(n-m)$ $x(n+m)$	$z^{-m}X(z) + \sum_{i=1}^{m} x(-i)z^{-(m-i)}$ $z^{m}X(z) = \sum_{i=0}^{m-1} x(i)z^{m-i}$
	x(n) for all n	$x(n-m)$ $x(n+m)$	$z^{-m}X(z)$ $z^{m}X(z)$

Property	Discrete Time Signal	z-Transform
Multiplication by n^m (or differentiation in z-domain)	$n^m x(n)$	$\left(-z\frac{d}{dz}\right)^m X(z)$
Scaling in z-domain (or multiplication by a^n)	$a^n x(n)$	$X(a^{-1}z)$
Time reversal	$x(-n)$	$X(z^{-1})$
Conjugation	$x^*(n)$	$X^*(z^*)$
Convolution	$x_1(n)^* x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)x_2(n-m)$	$X_1(z)X_2(z)$

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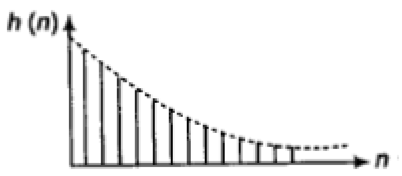
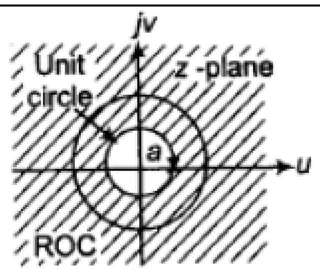
Property	**Discrete Time Signal**	**z-Transform**
Correlation	$r_{xy}(m) = \sum_{m=-\infty}^{+\infty} x(n)y(n-m)$	$X(z)Y(z^{-1})$
Initial value	$x(0) = \lim_{z\to\infty} X(z)$	
Final value	$x(\infty) = \lim_{z\to 1}(1-z^{-1})X(z)$ $= \lim_{z\to 1}\frac{(z-1)}{z}X(z)$ If $X(z)$ is analytic for $\lvert z\rvert > 1$	
Complex convolution theorem	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_c X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$
Parseval's relation	$\sum x_1(n)x_2^*(n) = \frac{1}{2\pi j}\int_c X_1(z)X_2^*\left(\frac{1}{z^*}\right)z^{-1}dz$	

Impulse Response and Location of Poles

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = a^n u(n); 0 < a < 1$  $\sum_{n=0}^{+\infty} \lvert h(n)\rvert < \infty$; stable system	$H(z) = \frac{z}{z-a}$ ROC is $\lvert z\rvert > a$ Pole at $z = a$	 Since $0 < a < 1$, the pole $z = a$, lies inside the unit circle. The ROC contains the unit circle.

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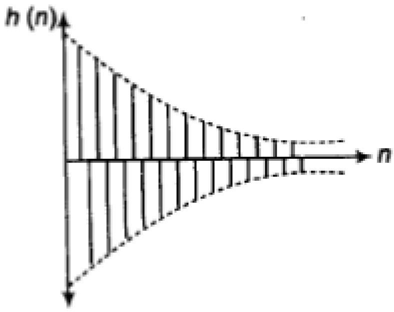
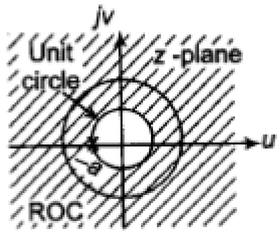
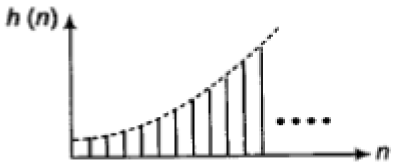
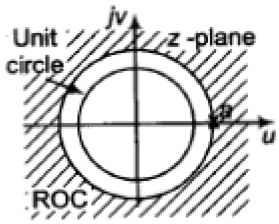
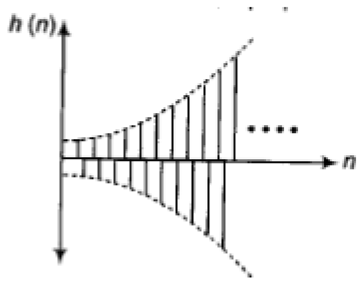
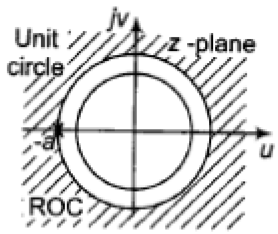
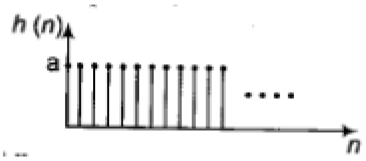
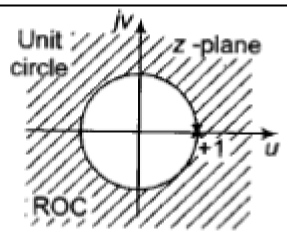
Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n)=(-a)^{n}u(n);0<\|-a\|<1$  $\sum_{n=0}^{+\infty}\|h(n)\|<\infty$; stable system	$H(z)=\frac{z}{z+a}$ ROC is $\|z\|>\|-a\|$ Pole at z = -a	 Since $0<\|-a\|<1$, the pole z = -a, lies inside the unit circle. The ROC contains the unit circle.

Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n)=a^{n}u(n);a>1$  $\sum_{n=0}^{+\infty}\|h(n)\|=\infty$; unstable system	$H(z)=\frac{z}{z-a}$ ROC is $\|z\|>a$ Pole at z = a	 Since a > 1, the pole z = +a, lies outside the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = -a^{n}u(n); \lvert -a \rvert > 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{z}{z-a}$ ROC is |z| > |-a| Pole at z = -a	 Since 0 <|-a| > 1, the pole z = -a, lies outside the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = au(n); a > 0$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{az}{z-1}$ ROC is |z| > 1 Pole at z = 1	 The pole z = 1, lies on the unit circle. The ROC does not contain the unit circle.

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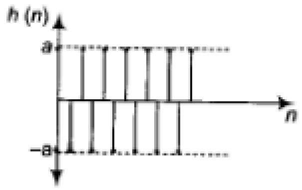
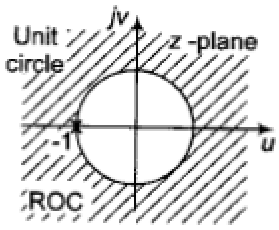
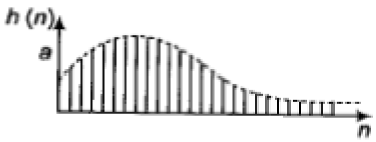
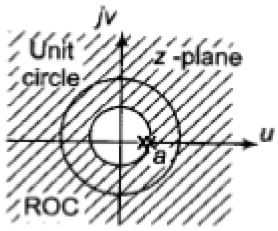
Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = a(-1)^n u(n); a > 0$ (i.e., a is positive)  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{az}{z+1}$ ROC is $\lvert z \rvert > 1$ Pole at z = -1	 The pole z = -1, lies on the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = na^n u(n); 0 < a < 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert < \infty$; stable system	$H(z) = \frac{az}{(z-a)^2}$ ROC is $\lvert z \rvert > 1$ Two Pole at z = a	 Since 0 < a < 1, the two poles at z = a, lie inside the unit circle. The ROC contains the unit circle.

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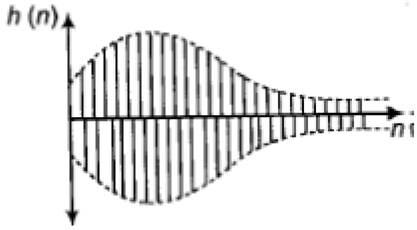
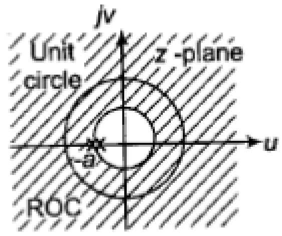
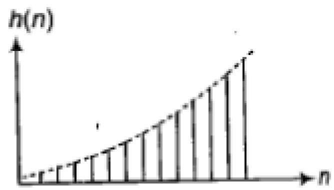
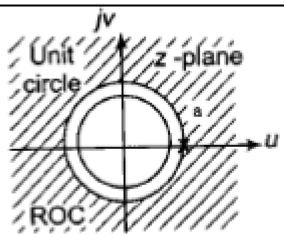
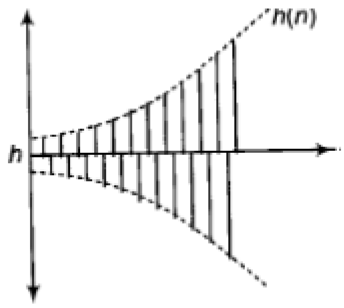
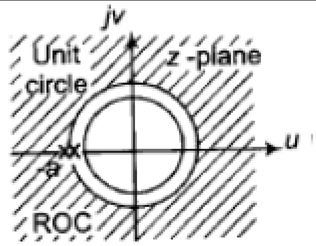
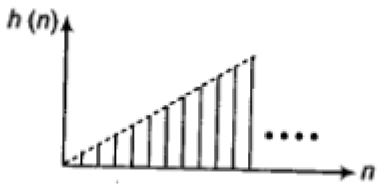
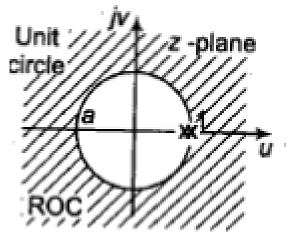
Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = n(-a)^n u(n); 0 < \|-a\| < 1$  $\sum_{n=0}^{+\infty} \|h(n)\| < \infty$; stable system	$H(z) = \frac{az}{(z+a)^2}$ ROC is $\|z\| > 1$ Two Pole at z = -a	 Since 0 <|-a| < 1, the two poles at z = -a, lie inside the unit circle. The ROC contains the unit circle.

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = na^n u(n); a > 1$  $\sum_{n=0}^{+\infty} \|h(n)\| = \infty$; unstable system	$H(z) = \frac{az}{(z-a)^2}$ ROC is $\|z\| > a$ Two Pole at z = a	 Since a > 1, the two poles at z = a, lie outside the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = n(-a)^n u(n); \lvert -a \rvert > 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{az}{(z+a)^2}$ ROC is $\lvert z \rvert > \lvert -a \rvert$ Two Pole at z = -a	 Since $\lvert -a \rvert > 1$, the two poles at z = -a, lie outside the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = nu(n)$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{z}{(z-1)^2}$ ROC is $\lvert z \rvert > 1$ Two Pole at z = 1	 The two pole z = 1, lie on the unit circle. The ROC does not contain the unit circle.

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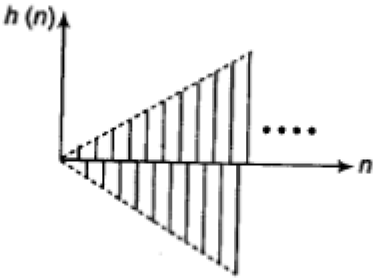
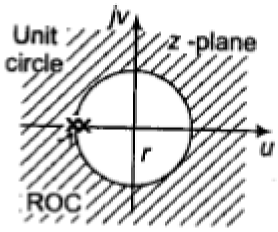
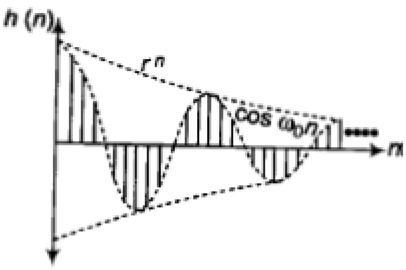
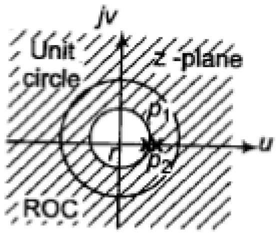
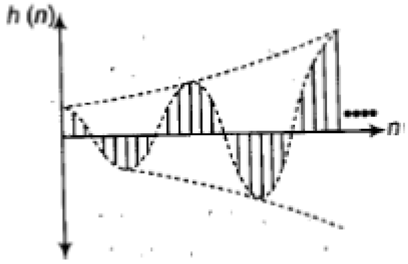
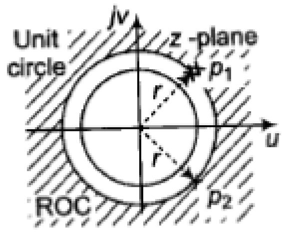
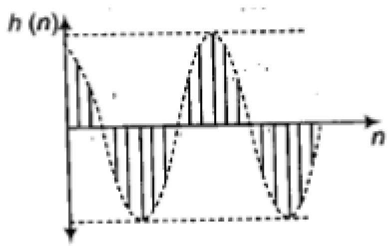
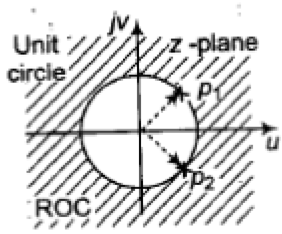
Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = n(-1)^n u(n); \|-a\| > 1$  $\sum_{n=0}^{+\infty} \|h(n)\| = \infty;$ unstable system	$H(z) = \frac{z}{(z+1)^2}$ ROC is $\|z\| > 1$ Two Pole at $z = -1$	 The two pole z = -1, lie on the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = r^n \cos \omega_0 n u(n); 0 < r < 1$  $\sum_{n=0}^{+\infty} \|h(n)\| < \infty;$ stable system	$H(z)$ $= \frac{z(z - r\cos\omega_0)}{(z - r\cos\omega_0)}$ $jr \sin \omega_0)$ $(z - r\cos\omega_0$ $+ jr \sin \omega_0)$ ROC is $\|z\| > r$ A pair of conjugate poles at $z = P_1 = r\cos\omega_0 + jr\sin\omega_0$ $z = P_2 = r\cos\omega_0 - jr\sin\omega_0$	 Since $0 < r < 1$, the conjugate pole pairs lie inside the unit circle. The ROC contains the unit circle.

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = r^n \cos \omega_0 n u(n); r > 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty;$ unstable system	$H(z)$ $= \frac{z(z - r\cos\omega_0)}{(z - r\cos\omega_0)}$ $jr \sin \omega_0)$ $(z - r\cos\omega_0$ $+ jr \sin \omega_0)$ ROC is $\lvert z \rvert \geq r$ A pair of conjugate poles at $z = P_1 = r\cos\omega_0 + jr\sin\omega_0$ $z = P_2 = r\cos\omega_0 - jr\sin\omega_0$	 Since $r > 1$, the conjugate pole pairs lie outside the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = \cos \omega_0 n u(n)$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty;$ unstable system	$H(z)$ $= \frac{z(z - \cos\omega_0)}{(z - \cos\omega_0)}$ $j\sin\omega_0)$ $(z - \cos\omega_0$ $+ j\sin\omega_0)$ ROC is $\lvert z \rvert \geq r$ A pair of conjugate poles on unit circle at, $z = P_1 = \cos\omega_0 + j\sin\omega_0$ $z = P_2 = \cos\omega_0 - j\sin\omega_0$	 The conjugate pole pairs lie on the unit circle. The ROC does not contain the unit circle.