Brief Description of Vector Calculus: Part-1

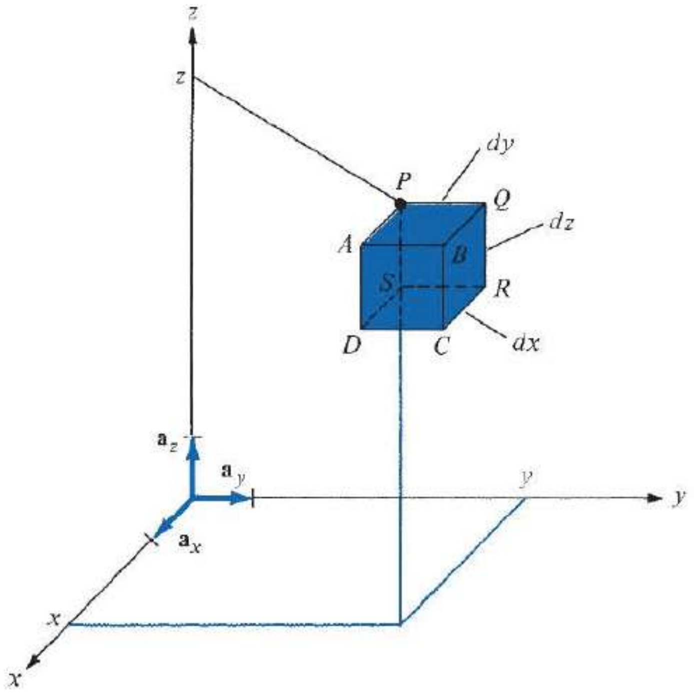
Presented by

Dr. Asim Halder
Dept. of Applied Electronics and Instrumentation Engineering
Haldia Institute of Technology

Brief Description of Vector Calculus

Here, we are going to understand mainly differentiation and integration of a vector.

Different elements in length, area, and volume are useful in vector calculus. They are defined in the Cartesian, Cylindrical, and Spherical coordinate systems.



Let's take the Cartesian coordinate system.

Suppose, the coordinate of the point "S" is (x, y, z) and of the point "B" is

$$\left(x + \Delta x, y + \Delta y, z + \Delta z\right)$$

If, the distance between "S" and "B" is $d\vec{l}$

Then the differential displacement is given by-

$$\boxed{d\vec{I} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z}$$

The differential or elementary normal surface area is given by-

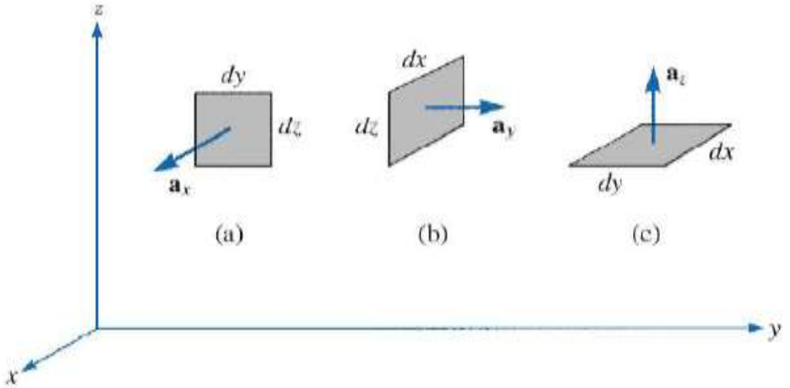
$$\boxed{\begin{aligned} d\vec{S} &= dydz\hat{a}_x \\ d\vec{S} &= dxdz\hat{a}_y \\ d\vec{S} &= dxdy\hat{a}_z \end{aligned}}$$

In general, the differential surface element is defined as- $\boxed{d\vec{S} = dS\hat{a}_n}$

Here, dS is the area of the surface element and $\hat{a}_n$ is an unit vector normal to the surface dS and directed away from the volume if dS is a part of the surface that describing the volume.

The differential or elementary volume is given by -

$$\boxed{dv = dxdydz}$$

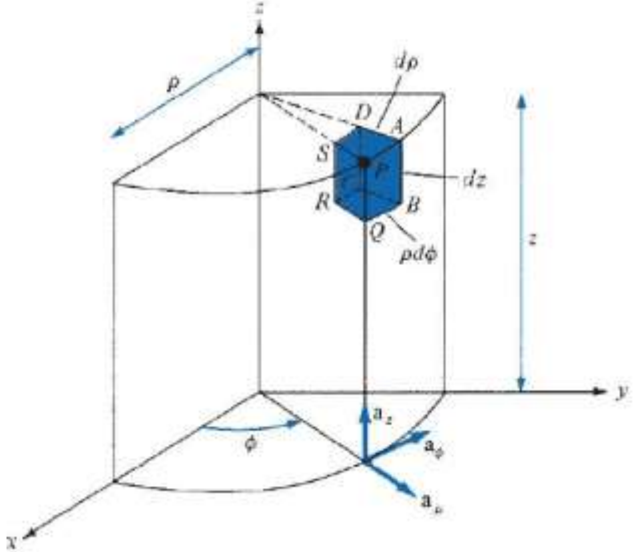


One point should be remembered here that the differential length and surface are vector, but differential volume is a scalar.

In Cylindrical Coordinate :-

The differential displacement is given by -

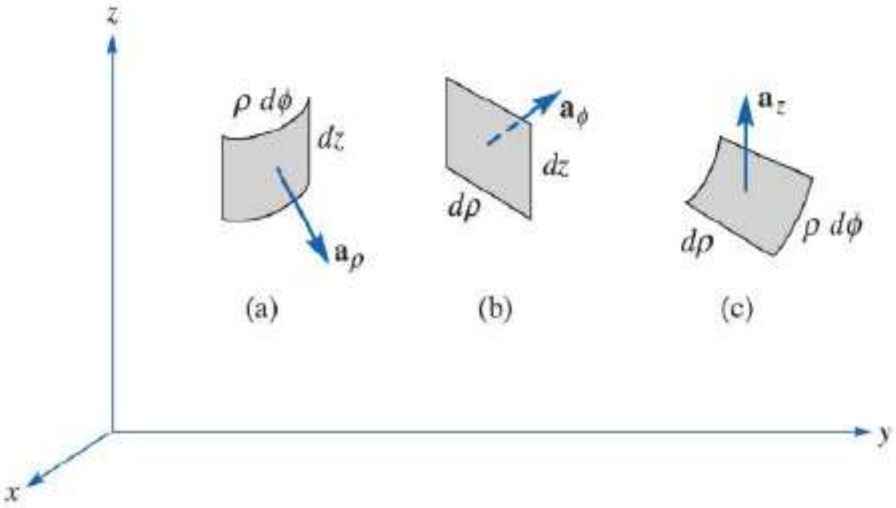
$$\boxed{d\vec{I} = dr\hat{a}_r + rd\varphi\hat{a}_\varphi + dz\hat{a}_z}$$



The differential normal surface area is given by -

The pictorial view is given in the next page

$$\boxed{\begin{aligned} d\vec{S} &= r\,d\varphi\,dz\,\hat{a}_r \\ d\vec{S} &= dr\,dz\,\hat{a}_\varphi \\ d\vec{S} &= r\,dr\,d\varphi\,\hat{a}_z \end{aligned}}$$



The differential or elementary volume is given by -

$$\boxed{dv = r\,dr\,d\varphi\,dz}$$

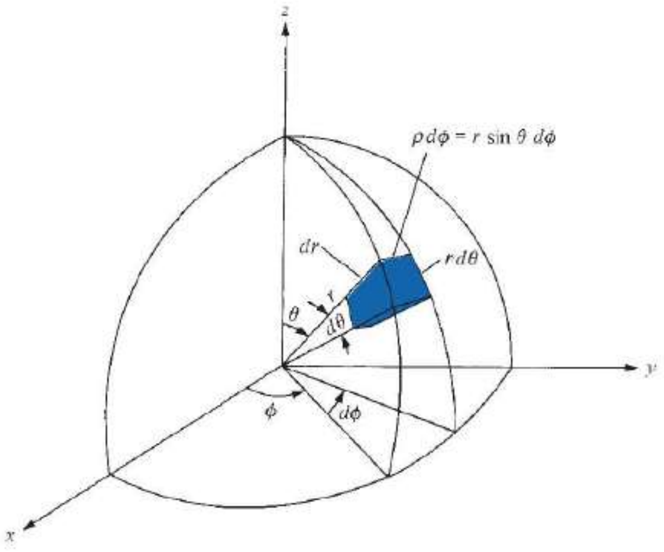
In Spherical Coordinate :-

The differential displacement is given by -

$$d\vec{l} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta d\varphi\hat{a}_\varphi$$

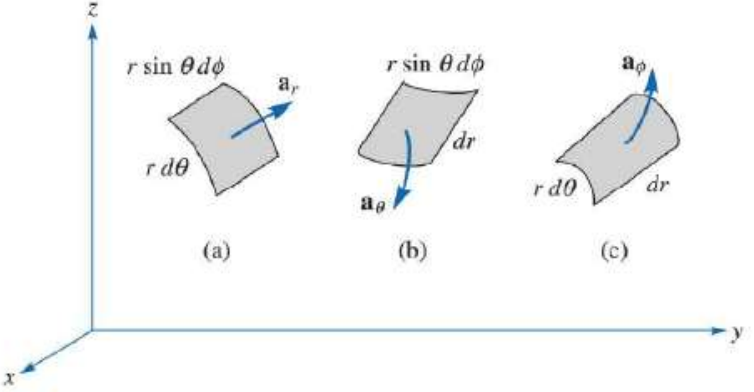
The differential normal surface area is given by -

$$\boxed{\begin{aligned} d\vec{S} &= rdrd\theta\hat{a}_\varphi \\ d\vec{S} &= r^2\sin\theta d\theta d\varphi\hat{a}_r \\ d\vec{S} &= rdr\sin\theta d\varphi\hat{a}_\theta \end{aligned}}$$



The differential or elementary volume is given by -

$$\boxed{dv = r^2\sin\theta d\theta d\varphi dr}$$



The Line Integral:-

✓ An important application of the scalar product or dot product is the line integral.

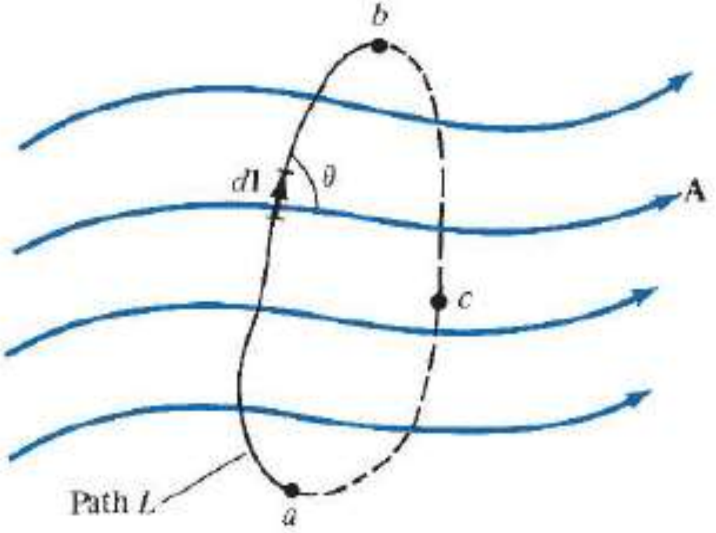
Line integral consists of evaluating the integral of the dot product of the tangential component of a vector and the differential displacement vector.

Let's consider a vector field "$\vec{A}$"

Again consider a path "a-b" in this vector field.

The magnitude of "$\vec{A}$" varies from point to point.

To find the total work done by the vector (if the vector is consider force) from "a" to "b" , we divide the path "a-b" into number of segments of lengths, i.e., $dl_1, dl_2, \ldots$



Let us consider, these small lengths as straight lines and the directions by elemental length vectors within each segment.

Again consider, the vectors $A_1, A_2, \ldots$ are acting on differential lengths $dl_1, dl_2, \ldots$

Respectively.

For a length vector *dl*, the component of the vector $\vec{A}$ acting along *dl* is $A\cos\theta$ and the work done by this vector along *dl* is $(A\cos\theta)dl$ or $\vec{A} \bullet d\vec{l}$, θ being the angle between $\vec{A}$ and *dl*.

By this process we can calculate the work done by the vector for all the elemental segments and the total work done will be -

$$W = \vec{A}_1 \bullet d\vec{l}_1 + \vec{A}_2 \bullet d\vec{l}_2 + \vec{A}_3 \bullet d\vec{l}_3 + \cdots$$

or, $$W = \sum_{i=1}^{n} \vec{A}_i \bullet d\vec{l}_i \qquad (1)$$

when the length of the segment tends to zero, then we can write (1) into integral form, i.e.,

$$W = \int_a^b \vec{A} \bullet d\vec{l} \qquad (2)$$

The integration is performed along the path "a-b" and is called "**line integral**".

For a closed path line integral is written as-

$$W = \oint \vec{A} \bullet d\vec{l}$$

For a field, if the line integral along the closed path is zero (we will find this in electric field), i.e.

$$\oint \vec{A} \bullet d\vec{l} = 0$$

Then the field is called "**Conservative field or Lamellar field**".

The Surface Integral:-

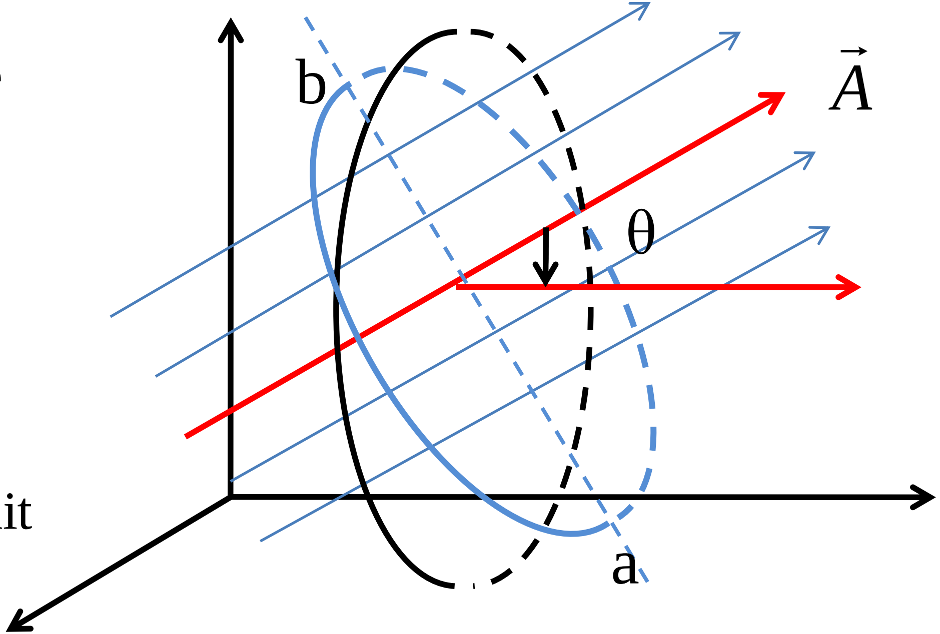
Suppose, a vector field "$\vec{A}$" is given which is continuous in a region that contains a smooth surface "S"

We define the surface integral or the **flux** of "$\vec{A}$" through "S" as -

(psi) ⇒ $$\psi = \int_S |\vec{A}| \cos\theta dS = \int_S \vec{A}.\hat{a}_n dS$$

or simply, $$\boxed{\psi = \int_S \vec{A} \bullet dS} \qquad (3)$$

where, at any point on "S" $\hat{a}_n$ is the unit normal vector to "S".



For a closed surface the above equation becomes -

$$\boxed{\psi = \oint \vec{A} \bullet ds} \qquad (4)$$

which is referred to as the net outward flux of "$\vec{A}$" from "S".

Notice that a closed path defines an open surface whereas a closed surface defines a volume.

The volume Integral:-

We define the volume integral as-

$$\int_v \rho_v \, dv$$

as the volume integral of scalar ρ_v over the volume "v".

Thank you