Spherical Coordinate System

Presented by

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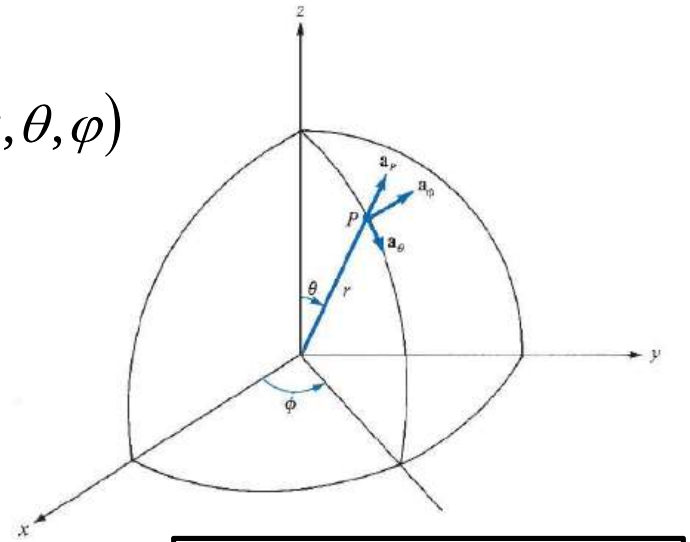
Spherical Coordinate System:-

The spherical coordinate system is most appropriate when we are dealing with problems having a degree of spherical symmetry.

The point "*P*" , in the fig. can be represented as $P(r,\theta,\varphi)$



r = distance from the origin to the point *"P"* or radius of the sphere centre at the origin and passing through *"P"*

θ= colatitudes which is the angle between z-axis and the position vector *"P"* .

φ= azimuthal angle; same as cylindrical coordinate system.

Just like previous cases, any point vector can be represented here.

Range of coordinates

$$0 \le r \le \infty$$

$$0 \le \theta \le \pi$$

$$\infty \le \varphi \le 2\pi$$

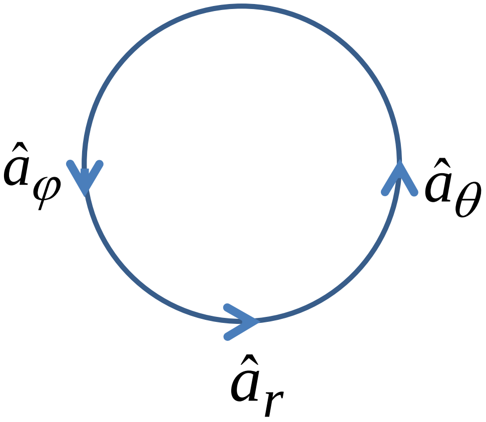
Say, a vector $\vec{A}$, can be represented in this coordinate system as

$$\vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\varphi\hat{a}_\varphi$$

where, $\hat{a}_r$, $\hat{a}_\theta$ and $\hat{a}_\varphi$ are unit vector along "r", θ and φ direction respectively

The magnitude of $\vec{A}$ is -

$$\left|\vec{A}\right| = \sqrt{A_r^2 + A_\theta^2 + A_\varphi^2}$$



Cross product :-

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\varphi \qquad\qquad \hat{a}_r \times \hat{a}_\varphi = -\hat{a}_\theta$$

$$\hat{a}_\theta \times \hat{a}_\varphi = \hat{a}_r \quad \text{or} \quad \hat{a}_\varphi \times \hat{a}_\theta = -\hat{a}_r$$

$$\hat{a}_\varphi \times \hat{a}_r = \hat{a}_\theta \qquad\qquad \hat{a}_\theta \times \hat{a}_r = -\hat{a}_\varphi$$

Dot product :-

$$\hat{a}_r \bullet \hat{a}_r = \hat{a}_\theta \bullet \hat{a}_\theta = \hat{a}_\varphi \bullet \hat{a}_\varphi = 1$$

$$\hat{a}_r \bullet \hat{a}_\theta = \hat{a}_\theta \bullet \hat{a}_\varphi = \hat{a}_\varphi \bullet \hat{a}_r = 0$$

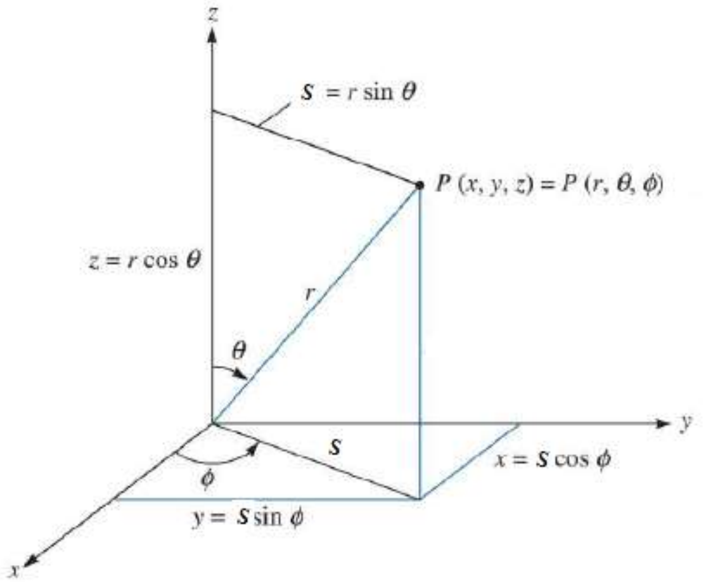
Relationship between components of Cartesian and Spherical coordinate system

Here, z-axis is perpendicular to *x*-*y* plane. So if we resolve vector into two components along *"z"* and "S" direction, then-

$$z = r\cos\theta \quad \text{and} \quad S = r\sin\theta$$

Now, if we resolve $\vec{S}$ into two components along x and y axis, we get-

$$\boxed{\begin{aligned} x &= S\cos\varphi = r\cos\varphi\sin\theta \\ y &= S\sin\varphi = r\sin\varphi\sin\theta \\ z &= r\cos\theta \end{aligned}}$$



Spherical to Cartesian coordinate transformation

We can also write-

$$r^2 = z^2 + S^2 \quad \text{and} \quad S^2 = x^2 + y^2$$

$$r = \sqrt{z^2 + S^2}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{S}{z} = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \quad \text{and} \quad \varphi = \tan^{-1}\frac{y}{x}$$

$$\boxed{\begin{gathered} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \\ \varphi = \tan^{-1}\frac{y}{x} \end{gathered}}$$

Cartesian to Spherical transformation

Relationship between unit vectors

Let's resolve $\hat{a}_r$;

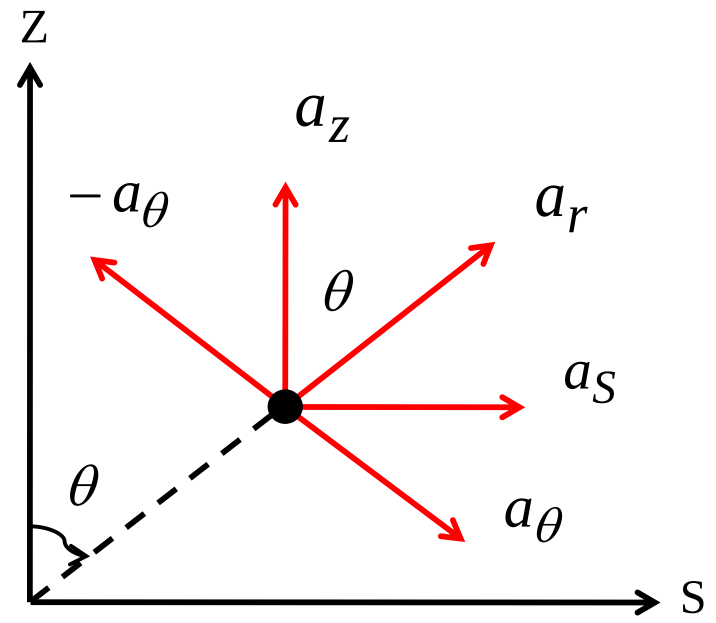
$$a_z = a_r \cos\theta$$
$$a_s = a_r \sin\theta$$

Thus,

$$a_x = a_s \cos\varphi = a_r \sin\theta \cos\varphi$$
$$a_y = a_s \sin\varphi = a_r \sin\theta \sin\varphi$$

In this way, we can get

$$a_x = \sin\theta\cos\varphi a_r + \cos\theta\cos\varphi a_\theta - \sin\varphi a_\varphi$$
$$a_y = \sin\theta\sin\varphi a_r + \cos\theta\sin\varphi a_\theta + \cos\varphi a_\varphi$$
$$a_z = \cos\theta a_r - \sin\theta a_\theta$$



or
$$a_r = \sin\theta\cos\varphi a_x + \sin\theta\sin\varphi a_y + \cos\theta a_z$$
$$a_\theta = \cos\theta\cos\varphi a_x + \cos\theta\sin\varphi a_y - \sin\theta a_z$$
$$a_\varphi = -\sin\varphi a_x + \cos\varphi a_y$$

Now, the relationship between components of Cartesian and spherical coordinate systems can be written as:-

$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$$
$$= A_x\left(\sin\theta\cos\varphi\hat{a}_r + \cos\theta\cos\varphi\hat{a}_\theta - \sin\varphi\hat{a}_\varphi\right) + A_y\left(\sin\theta\sin\varphi\hat{a}_r + \cos\theta\sin\varphi\hat{a}_\theta + \cos\varphi\hat{a}_\varphi\right)$$
$$+ A_z\left(\cos\theta\hat{a}_r - \sin\theta\hat{a}_\theta\right)$$
$$= \left(A_x\sin\theta\cos\varphi + A_y\sin\theta\sin\varphi + A_z\cos\theta\right)\hat{a}_r + \left(A_x\cos\theta\cos\varphi + A_y\cos\theta\sin\varphi - A_z\sin\theta\right)\hat{a}_\theta$$
$$+ \left(A_y\cos\varphi - A_x\sin\varphi\right)\hat{a}_\varphi$$

$$\therefore \vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\varphi\hat{a}_\varphi$$

$$A_r = \left(A_x \sin\theta\cos\varphi + A_y \sin\theta\sin\varphi + A_z\cos\theta\right)$$

$$A_\theta = \left(A_x \cos\theta\cos\varphi + A_y \cos\theta\sin\varphi - A_z\sin\theta\right)$$

$$A_\varphi = \left(A_y \cos\varphi - A_x \sin\varphi\right)$$

In Matrix form :-

$$\begin{bmatrix} A_r \\ A_\theta \\ A\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

or,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_r \\ A_\theta \\ A\varphi \end{bmatrix}$$

Thank you