Time Domain Analysis

2.1 Introduction

- ✓ The time domain analysis is the analysis of response of systems in time domain when subjected to various types of inputs.
- ✓ In a practical system, the type of input signal is not known ahead of time. However, for the purpose of analysis, systems are tested for different types of known standard input signals.
- ✓ These signals are known as test signals.
- ✓ Some of them are impulse input, step input, ramp input and parabolic input signals.
- ✓ All these input signals are also known as ***singularity functions***.
- ✓ ***Singularity functions*** can be obtained from one another by successive differentiation or integration.

2.2 Time Response

Definition: ***The time response of a system is the output of the system as a function of time, when subjected to a given input***.

What is the dynamics of the systems? ...Let's understand this

- ✓ If we inject the same type and equal amplitude input signal to different systems, some systems respond quickly, some respond slowly; some become over excited and bounce around; while others sluggishly respond.
- ✓ This behaviour of the systems is referred to as the ***dynamics of the systems***.
- ✓ Most of the times, to know the dynamics of the system, a step change is used as input signal.
- ✓ The fig. 2.1 (below) shows the variety of system responses for the same change in input level.

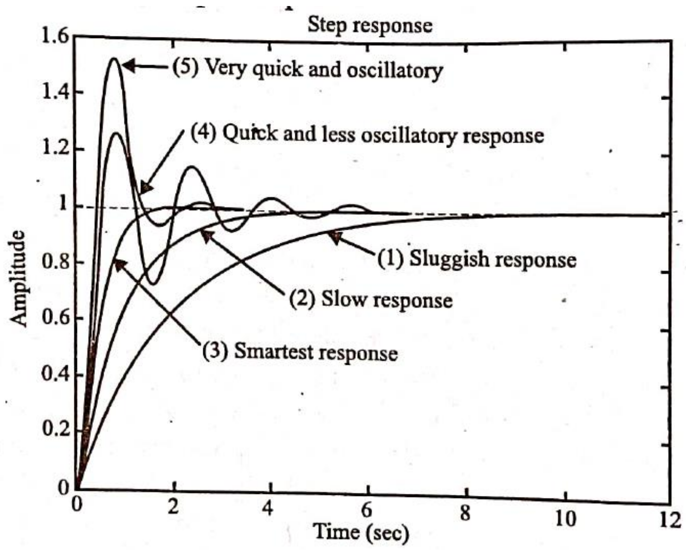


Fig. 2.1 various types of responses

- ✓ The responses are;

  (1) Very sluggish, and takes a long time to settle.

  (2) A slow response and takes less time than response (1).

  (5) Rises very quickly and oscillates around the level where it finally settles.

  (4) quicker than (1), (2) and (3) and in addition does not have much oscillation like (5).
- ✓ The response (3) is smartest among all responses.

2.2.1 Transient response and steady state response

The time response of a system consists of two parts

(i) Transient response

(ii) Steady state response

and can be represented by

Total time response = Transient response+ Steady state response

$$C(t) = C_{tr}(t) + C_{ss}(t)$$

- ✓ Transient response of the system is the portion of total time response during which the output changes from one value to another value.
- ✓ In other words, it is the response before the output reaches the steady-state value.
- ✓ Steady state response of the system is the response of the system for a given input after a very long time.
- ✓ In steady state, the output response settles to its final steady-state value or steady oscillations.
- ✓ Fig. 2.2 shows the regions of transient and steady state responses for a sluggish process (2.2(a)) and for a smart process (2.2(b)) that exhibits oscillatory behaviour.

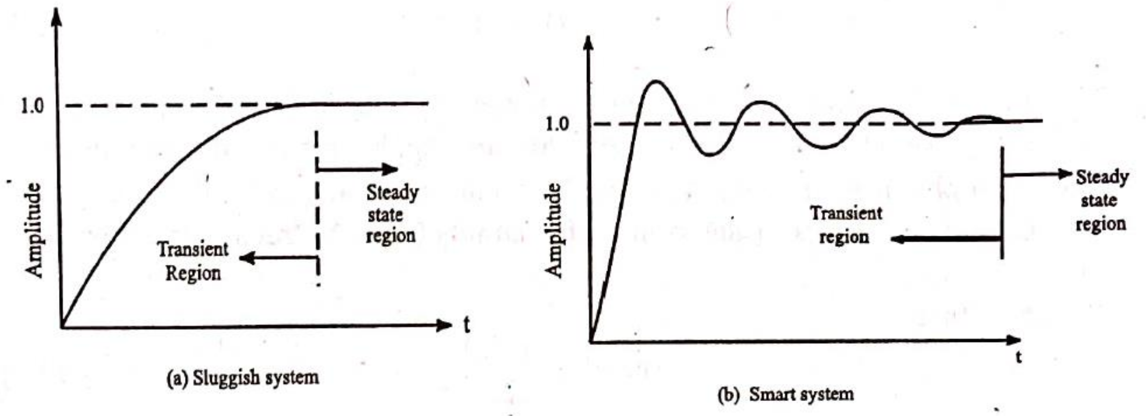


Fig. 2.2 Step response of systems

2.3 Standard Test Signals

2.3.1 Impulse signal

- ✓ The impulse signal has zero amplitude everywhere, except at the origin (shown in Fig. 2.3)
- ✓ Mathematically the impulse signal can be represented by

$$A\delta(t) = 0 \qquad \text{for } t \neq 0 \text{ and}$$

$$\int_{-\varepsilon}^{\varepsilon} A\delta(t)dt = A \qquad (2.1)$$

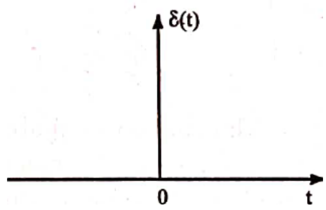


Fig. 2.3 Impulse Signal

where, 'ε' tends to zero.

- ✓ Here the value 'A' represents the area of the signal or energy content of the signal and the Laplace transform of the impulse is given by

$$L[A\delta(t)] = A$$

Unit Impulse signal

- ✓ If A = 1, then the signal is called unit impulse signal; that is, for a unit impulse signal

$$\delta(t) = 0 \qquad \text{for } t \neq 0 \text{ and}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)dt = 1 \qquad (2.2)$$

- ✓ The Laplace transform of the unit impulse signal is

$$L[\delta(t)] = 1 \qquad (2.3)$$

- ✓ The transfer function of a linear time-invariant system is the Laplace transform of the impulse response of the system.
- ✓ So, an impulse input can't be applied to a system to obtain its transfer function.
- ✓ If a unit impulse signal is applied to the system, then the Laplace transform of the output $C(t)$ is the transfer function $G(s)$ of the system.
- ✓ We know that

$$G(s) = \frac{C(s)}{R(s)} \qquad (2.4)$$

where, $G(s)$ is transfer function of the system, $C(s)$ is Laplace transfer function ofoutput of the system and $R(s)$ Laplace transform of the input signal.

- ✓ If the input $r(t) = \delta(t)$, a unit impulse signal, then

$$R(s) = L[r(t)] = L[\delta(t)] = 1 \qquad (2.5)$$

- ✓ Therefore

$$G(s) = C(s) \qquad \text{and}$$

$$c(t) = L^{-1}[G(s)] = g(t) \qquad (2.6)$$

- ✓ Thus the unit impulse response of a system is the inverse of Laplace transform of the transfer function.
- ✓ In other words, Laplace transform of unit impulse response of a system is transfer function of the system itself.

2.3.2 Step signal

- ✓ Step signal of size 'A' is a signal that changes from zero level to another level 'A' in zero time and stays there forever as shown in Fig. 2.4.
- ✓ The step signal is applied to the system to study the behaviour of the system for a sudden change in input.
- ✓ Mathematically

$$u(t) = A \qquad \text{for } t \geq 0$$
$$= 0 \qquad \text{for } t < 0 \qquad (2.8)$$

- ✓ The Laplace transform of the step signal is

$$U(s) = L[r(t)] = \frac{A}{s} \qquad (2.9)$$

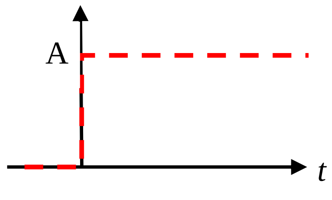


Fig. 2.3 Step Signal

Unit step signal

- ✓ If the magnitude 'A' of the step signal is unity, then the step signal is known as unit step signal as shownin Fig. 2.5 and is denoted by $u(t)$. That is

$$u(t) = 1 \qquad \text{for } t \geq 0$$
$$= 0 \qquad \text{for } t < 0$$

- ✓ The Laplace transform of the unit step signal $u(t)$ is

$$L[u(t)] = \frac{1}{s} \qquad (2.10)$$

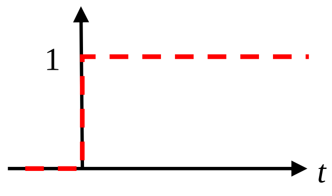


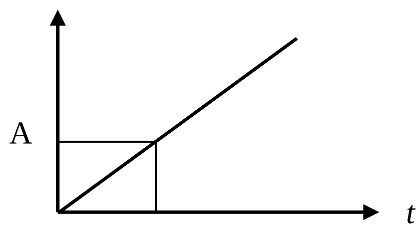
Fig. 2.4 Unit step signal

2.3.3 Ramp signal

- ✓ The ramp signal increases linearly with time from an initial value of zero at t = 0 as shown in Fig. 2.6.
- ✓ Mathematical representation of ramp signal in givenby

$$r(t) = At \qquad \text{for} \quad t \geq 0$$
$$= 0 \qquad \text{for} \quad t = 0 \qquad (2.11)$$

where 'A' represents the slope of the line.



- ✓ The Laplace transform of the ramp signal is

Fig. 2.5 Ramp signal

$$R(s) = L[r(t)] = AL[t] = \frac{A}{s^2} \qquad (2.12)$$

Unit ramp signal

- ✓ If the slope 'A' is unity, then the ramp signal is known as ramp signal

2.3.4 Parabolic signal

- ✓ The instantaneous value of a parabolic signal varies as square of the time from an initial value of zero at $t = 0$ as shown in fig. 2.6.
- ✓ Mathematical representation of the parabolic signal is

$$r(t) = At^2 \quad \text{for} \quad t \geq 0$$
$$= 0 \quad \text{for} \quad t < 0 \tag{2.13}$$

- ✓ The Laplace transform of the parabolic signal is given by

$$R(s) = L\left[At^2\right] = \frac{2A}{s^3} \tag{2.14}$$

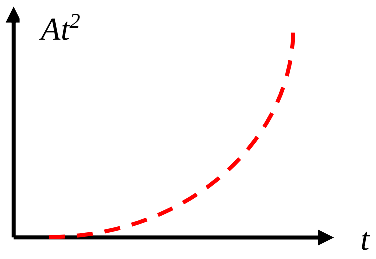


Fig. 2.6 Parabolic signal

- ✓ To have a convenient from for Laplace transform of parabolic signal, i.e. A/s^3, the parabolic signal in the time domain is often defined as $At^2/2$.

2.3 About Transfer Function

- ✓ Systems can be modelled in different ways.
- ✓ The transfer function approach is one of those types.
- ✓ In this method, first the system is represented normally by differential equations using.
- ✓ The Laplace transform is applied to the equation assuming zero initial conditions.
- ✓ Then the ratio of Laplace transform of output to Laplace transform of input is expressed as a ratio of polynomial in '*s*' and it is called the ***transfer function*** of the system.
- ✓ We know, if the denominator of a polynomial has degree '*n*', the order of the system is '*n*'.
- ✓ If *n* = 1 the system is known as first order system; for *n* = 2, the system is known as second order system and so on.
- ✓ For all practical systems '*m*' is always less than or equal to '*n*' (i.e. $m < n$).
- ✓ The transfer function can be of two types

Time Constant Form

- ✓ In this form the transfer function can be represented as follows

$$G(s) = \frac{C(s)}{R(s)} = \frac{K(T_{z1}s+1)(T_{z2}s+1)\cdots(T_{zm}s+1)}{(T_{p1}s+1)(T_{p2}s+1)\cdots(T_{pn}s+1)} \tag{2.19}$$

where $K = \dfrac{b_0}{a_0}$

- ✓ This form of representing the system is called time constant form. The gain '*K*' here is called steady-sate gain.

Pole-Zero Form

- ✓ The transfer function $G(s)$ can also be expressed in the pole-zero form as

$$G(s) = \frac{K'(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{2.20}$$

here $z_1, z_2, \cdots z_m$ are roots of the numerator of the polynomial and known as zeros of the systems.

whereas, $p_1, p_2, \cdots p_n$ are roots of the denominator polynomial known as poles of the system.

Note: the K and K' are not equal.

- ✓ The denominator polynomial of the transfer functions of eqn. (2.19) and (2.20) are known as Characteristic equation.
- ✓ Note that the roots of the characteristic equation are poles of the transfer function $G(s)$.
- ✓ Poles and zeros may be real or complex. Complex conjugate poles occur in pairs.
- ✓ The complex plane is used to represents poles and zeros. Locations of poles exhibit the stability of the system.

2.4 *S*-plane

- ✓ The variables in G(s) are complex variables and a complex variable is defined by

$$s = \sigma + j\omega \tag{2.21}$$

where 'σ' is the real part of 's' and 'ω' is the imaginary part of 's'.

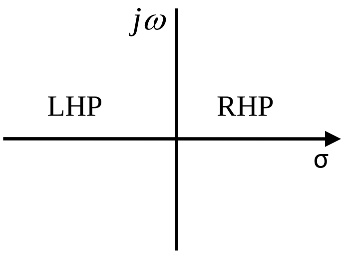
- ✓ A set of all complex coordinate (σ, ω) form a plane called s-plane. The horizontal axis of this plane is known as real axis since it is represented by the line

$$s = \sigma + j0 \tag{2.22}$$

- ✓ The vertical axis is known as the imaginary axis and is represented by the line

$$s = 0 + j\omega \tag{2.23}$$

- ✓ The origin is represented by the complex zero $s = 0 + j0$. The $j\omega$-axis (imaginary axis) divides the s-plane into two half planes.
- ✓ The region to the left of imaginary $j\omega$-axis (for $-\infty < \sigma < 0$) is termed the left half plane (LHP) and the region to the right of imaginary axis (for $0 < \sigma < \infty$) is termed the right half plane (RHP) as shown if fig. 2.7.



- ✓ The poles and zeros of the transfer function G(s) can be represented schematically in s-plane.
- ✓ The symbol $\times$ is used for a pole and '*0*' is used for a zero.

2.5 First Order System

- ✓ The most general form of a differential equation representing a first order system may be expressed as

$$\tau \frac{dy(t)}{dt} + y(t) = Kx(t) \tag{2.24}$$

where $y(t)$ and $x(t)$ are input-output variables and the coefficients 'τ' and 'K' are called the time constant and steady-state gain or simply gain of the system respectively.

- ✓ Let $x(t)$ and $y(t)$ are input and output variables of the system respectively.
- ✓ The transfer function of the system (2.24) can be written as

$$\frac{Y(s)}{X(s)} = \frac{K}{\tau s + 1} \tag{2.25}$$

- ✓ We can notice that any first order system can be represented by transfer function $G(s)$ if we know the time constant 'τ' and steady-state gain 'K' of the system.
- ✓ That is

$$G(s) = \frac{K}{\tau s + 1} \tag{2.26}$$

- ✓ The model parameters 'τ' and 'K' depend on natural behaviour of the system (i.e. in the given electrical system $K = 1$ and $\tau = RC$ respectively).

2.6 Time Response of First Order System

- ✓ From the transfer function of the first order system, we can write the Laplace transform of the output

$$C(s) = \frac{K}{\tau s + 1} R(s) \tag{2.27}$$

- ✓ If we known the type of input signal $r(t)$ and its Laplace transform $R(s)$, then we can easily obtain the output response $c(t)$ by taking the inverse Laplace transform of $C(s)$, i.e.

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{K}{\tau s + 1} R(s)\right] \tag{2.28}$$

where, $c(t)$ is the time response of the system for the input $r(t)$ provided the initial conditions are zero.

2.6.1 *Impulse response of a first order system*

- ✓ Let the transfer function of a first order system be

$$\frac{C(s)}{R(s)} = \frac{K}{\tau s + 1} \tag{2.29}$$

Now, $r(t) = \delta(t)$ and $R(s) = 1$ (2.30)

- ✓ For an impulse input

$$C(s) = \frac{K}{\tau s + 1} = \frac{K}{\tau\left(s + \frac{1}{\tau}\right)} \tag{2.31}$$

$$\text{or, } c(t) = L^{-1}\left[\frac{K}{\tau\left(s + \frac{1}{\tau}\right)}\right] = \frac{K}{\tau} e^{-t/\tau}; \quad t \geq 0 \tag{2.32}$$

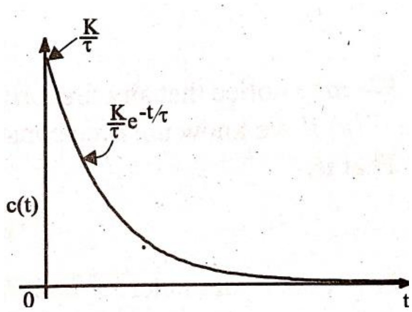


Fig. 2.7 Impulse response of first order system

At $t = 0$, $c(0) = \frac{K}{\tau}$ and as $t \to \infty$; $c(\infty) = 0$.

- ✓ Thus, the output starts at K/τ; at $t = 0$ and decreases exponentially and reaches to zero as $t \to \infty$. The output c(t) is shown in fig. 2.7.

2.6.2 *Step response of a first order system*

✓ For a step input

$$r(t) = u(t) \text{ and } R(s) = \frac{1}{s} \tag{2.33}$$

✓ Substituting (2.33) into (2.27) we get

$$C(s) = \frac{K}{s(\tau s + 1)} \tag{2.34}$$

✓ Now, using partial fraction to (2.34) we get

$$C(s) = K\left[\frac{1}{s} - \frac{\tau}{\tau s + 1}\right] = \frac{K}{s} - \frac{K}{s + \frac{1}{\tau}} \tag{2.35}$$

✓ Taking inverse Laplace transform of (2.35) we get

$$\boxed{c(t) = K\left[1 - e^{-t/\tau}\right]} \quad \text{for } t \geq 0 \tag{2.36}$$

✓ Substituting t = 0 in (2.36) gives $c(0)$ = 0.

✓ Similarly, as $t \to \infty$; the output $c(\infty) = K$, i.e. the output $c(t)$ starts at zero for t = 0, increases exponentially to a steady state value K as 't' tends to infinity.

✓ When $t = \tau$, the output becomes

$$c(t) = K\left[1 - e^{-1}\right] = 0.632K \tag{2.37}$$

✓ It means when $t = \tau$, the output reaches a value that is equal to 63.2% of the final value.

✓ The value τ is called the ***time constant*** of the system.

✓ The time constant indicates how fast the system reaches the final value and is a measure of speed of response.

Definition: *The time taken for a step response of a system to reach 63.2% of the final value is known as the time constant of the system.*

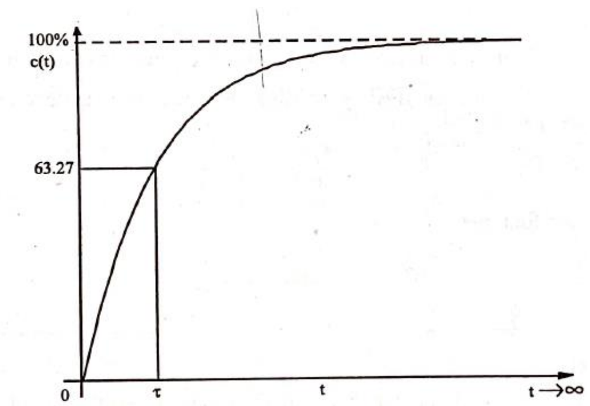


Fig 2.8 unit step response of 1st order system

- ## The Steady State Value

✓ The steady state value C_{ss} of the response is the value of the output as $t \to \infty$.

- ✓ In this case, $$C_{ss} = \lim_{t\to\infty}[c(t)] = \lim_{t\to\infty}\left[K(1-e^{-t/\tau})\right] = K \tag{2.38}$$
- ✓ This can also be found easily by using the ***Final Value Theorem*** as follows

$$C_{ss} = \lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s)$$

▪ Various form of first order transfer function

- ✓ The transfer function of the 1st order system in the form

$$G(s) = \frac{K}{\tau s + 1} \tag{2.29}$$

is known as gain-time constant form, where ‘K’ is the steady-state gain and τ is the time constant of the system.

- ✓ Another form of the representation of transfer function is

$$G(s) = \frac{K'}{s + a} \tag{2.30}$$

- ✓ It is known as the pole-zero form. *One point to be noted here that K' is not the gain of the system.*
- ✓ The eqn. (2.30) can be obtained by dividing both the numerator and denominator of (2.29) by ‘τ’, i.e.

$$G(s) = \frac{K/\tau}{s + \frac{1}{\tau}} = \frac{K'}{s + a} \tag{2.31}$$

where $K' = \frac{K}{\tau}$ and $a = \frac{1}{T}$.

2.7 Speed of Response

- ✓ We can found from (2.29) that the open loop gain ‘K’ has an effect on the steady-state value of the output and it has no effect on the speed of the open loop response.
- ✓ Now, let us take three systems having same open loop gain but different time constants

System 1 $$G_1(s) = \frac{5}{0.5s + 1} \tag{2.32}$$

System 2 $$G_2(s) = \frac{5}{2s + 1} \tag{2.33}$$

System 3 $$G_3(s) = \frac{5}{5s + 1} \tag{2.34}$$

- ✓ The time constants of these systems are $\tau_1 = 0.5$; $\tau_2 = 2$ and $\tau_3 = 5$ respectively.
- ✓ The unit step response of these systems are

$$c_1(t) = 5\left[1 - e^{-t/0.5}\right] = 5\left[1 - e^{-2t}\right] \tag{2.35}$$

$$c_2(t) = 5\left[1 - e^{-t/2}\right] = 5\left[1 - e^{-0.5t}\right] \tag{2.36}$$

$$c_3(t) = 5\left[1 - e^{-t/5}\right] = 5\left[1 - e^{-0.2t}\right] \tag{2.37}$$

✓ Now, we can plot the unit step responses of these systems as follows

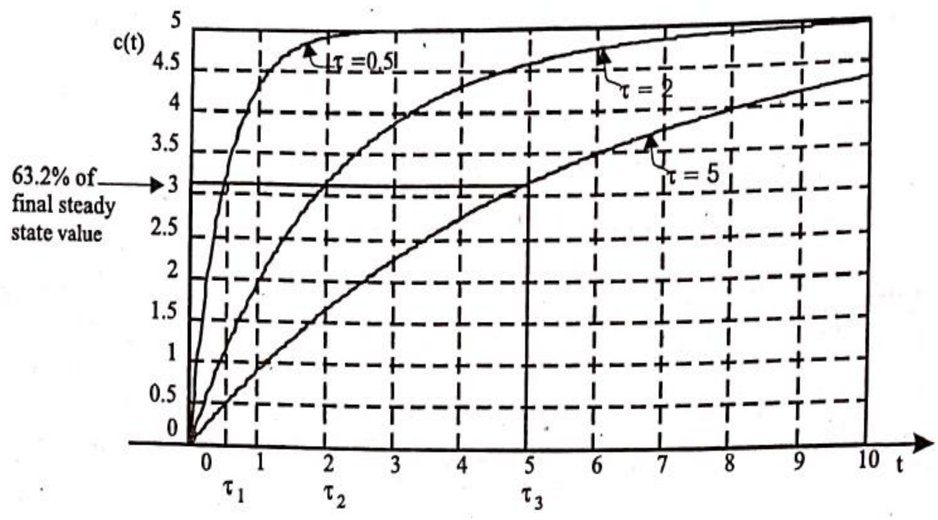


Fig. 2.9 1st order system's response for different time constant

✓ We can observe from these responses that the steady-state values of all the systems are same, i.e. $C_{ss} = 5$, but the time taken to reach the steady-state value differs.

✓ Moreover, we can observe that the system with smallest time constant ($\tau_1 = 0.5$) reaches very quickly, whereas the system with larger time constant ($\tau_2 = 5$) reaches steady-state slowly.

✓ Hence, we can conclude that as 'τ' increases the response becomes sluggish.

2.8 Second Order System

✓ The most general form of a differential equation for a 2nd order system may be expressed as

$$\frac{1}{\omega_n^2}\frac{d^2y(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy(t)}{dt} + y(t) = Kx(t) \tag{2.8.1}$$

where ω_n and 'K' depends on the characteristics of the system under consideration; $x(t)$ and $y(t)$ are the input and output respectively; ω_n, K and ζ are called natural frequency, steady-state gain and damping ratio of the system respectively.

✓ The transfer function of the system described in (2.8.1) can be derived by taking the Laplace transform on both sides of the above equation considering initial conditions are zero, i.e.

$$\dot{y}(0) = 0 \text{ and } y(0) = 0 \tag{2.8.2}$$

✓ Applying Laplace transform to (2.8.1) we get

$$L\left[\frac{1}{\omega_n^2}\frac{d^2y(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy(t)}{dt} + y(t)\right] = L[Kx(t)] \tag{2.8.3}$$

$$\left[\frac{1}{\omega_n^2}s^2Y(s)+\frac{2\zeta}{\omega_n}sY(s)+Y(s)\right]=KX(s)$$

$$\frac{Y(s)}{X(s)}=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad (2.8.4)$$

- ✓ The expression (2.8.4) is the general form of the transfer function of a second order system.
- ✓ The derivation of an electrical and mechanical second order systems are shown in Fig. 2.10 and Fig. 2.11 respectively.

<table>
<tr><th>Electrical System</th><th>Mechanical System</th></tr>
<tr>
<td>

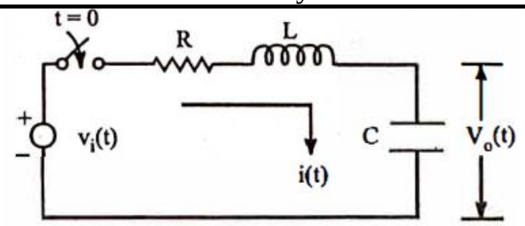


Fig. 2.10

Applying Kirchhoff's voltage law we have

$$v_i(t)=Ri(t)+L\frac{di(t)}{dt}+\frac{1}{C}\int_0^t i(\tau)d\tau$$

$$V_i(s)=RI(s)+LsI(s)+\frac{1}{Cs}I(s)$$

$$I(s)=\frac{V_i(s)}{R+Ls+\frac{1}{Cs}}$$

$$v_0(t)=\frac{1}{C}\int_0^t i(t)dt$$

$$V_0(s)=\frac{I(s)}{Cs}$$

$$V_0(s)=\frac{V_i(s)}{Cs\left(R+Ls+\frac{1}{Cs}\right)}$$

$$\frac{V_0(s)}{V_i(s)}=\frac{1}{LCs^2+RCs+1}$$

$$\frac{V_0(s)}{V_i(s)}=\frac{1/LC}{s^2+RCs+1} \qquad (2.8.5)$$

Comparing (2.8.4) and (2.8.5) we can find that $\omega_n^2=\frac{1}{LC}$; i.e. $\omega_n=\frac{1}{\sqrt{LC}}$ and $2\zeta\omega_n=\frac{R}{L}$ and from this we have $\zeta=R/2\omega_n L=R/2\sqrt{C/L}$

</td>
<td>

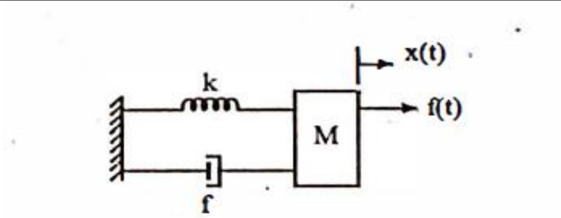


Fig. 2.11

Applying Newton's law, we have

$$f(t)=M\frac{d^2x(t)}{dt^2}+f\frac{dx(t)}{dt}+kx(t)$$

$$F(s)=Ms^2X(s)+fsX(s)+kX(s)$$

$$\Rightarrow\frac{X(s)}{F(s)}=\frac{1/M}{s^2+\frac{f}{M}s+\frac{k}{M}}$$

$$=K'\frac{k/M}{s^2+\frac{f}{M}s+\frac{k}{M}} \qquad (2.8.6)$$

where $K'=1/k$

Comparing (2.8.4) and (2.8.6), we can find that $\omega_n^2=\frac{k}{M}$ or $\omega_n=\sqrt{\frac{k}{M}}$ and $2\zeta\omega_n=\frac{f}{M}$

which gives $\zeta=\frac{f}{2\omega_n M}=\frac{f}{2\sqrt{kM}}$

</td>
</tr>
</table>

✓ From the above two examples we may find that any second order system can be represented in the general transfer function model as

$$G(s)=\frac{K'\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{2.8.7}$$

2.8.1 Impulse response of a 2nd order system

✓ Let's consider the generalized transfer function of the second order system provided K = 1 as

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{2.8.8}$$

✓ If we apply the Laplace transform unit impulse input $r(t)=\delta(t)$, i.e., R(s) = 1. Then the response of the system for unit impulse input is

$$C(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{2.8.9}$$

and the roots of the characteristic equation $s^2+2\zeta\omega_n s+\omega_n^2=0$ are given as

$$s_1,s_2=\frac{-2\zeta\omega_n\pm\sqrt{4\zeta^2\omega_n^2-4\omega_n^2}}{2} \tag{2.8.10}$$

$$=-\zeta\omega_n\pm\omega_n\left(\sqrt{\zeta^2-1}\right) \tag{2.8.11}$$

✓ The roots s_1 and s_2 will be real' or complex depending on the parameter ζ .

- If ζ > 1; the roots are real and distinct.
- If ζ = 1, s_1 and s_2 are real and equal.
- If ζ < 1, the roots s_1 and s_2 are complex conjugate.
- $0<\zeta<1$, the system is called under-damped.
- If ζ = 1, the system is called critically damped system.
- If ζ > 1 the system is over damped.

✓ Now let us find the response $c(t)$ for different values of ζ .

CASE I: Impulse response for ζ > 1

✓ When ζ > 1 the values of s_1 and s_2 are given by

$$s_1=-\zeta\omega_n+\omega_n\left(\sqrt{\zeta^2-1}\right)$$

$$s_2=-\zeta\omega_n-\omega_n\left(\sqrt{\zeta^2-1}\right)$$

$$C(s)=\frac{A}{s-\left(-\zeta\omega_n+\omega_n\left(\sqrt{\zeta^2-1}\right)\right)}+\frac{B}{s-\left(-\zeta\omega_n-\omega_n\left(\sqrt{\zeta^2-1}\right)\right)}$$

$$A = s - \left(-\zeta\omega_n + \omega_n\left(\sqrt{\zeta^2 - 1}\right)\right)\frac{\omega_n^2}{\left[s - \left(-\zeta\omega_n + \omega_n\left(\sqrt{\zeta^2 - 1}\right)\right)\right]\left[s - \left(-\zeta\omega_n - \omega_n\left(\sqrt{\zeta^2 - 1}\right)\right)\right]}\Bigg|_{s = -\zeta\omega_n + \omega_n\left(\sqrt{\zeta^2 - 1}\right)}$$

$$= \frac{\omega_n^2}{2\omega_n\sqrt{\zeta^2 - 1}} = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}$$

✓ Similarly

$$B = s - \left(-\zeta\omega_n - \omega_n\left(\sqrt{\zeta^2 - 1}\right)\right)\frac{\omega_n^2}{\left[s - \left(-\zeta\omega_n + \omega_n\left(\sqrt{\zeta^2 - 1}\right)\right)\right]\left[s - \left(-\zeta\omega_n - \omega_n\left(\sqrt{\zeta^2 - 1}\right)\right)\right]}\Bigg|_{s = -\zeta\omega_n - \omega_n\left(\sqrt{\zeta^2 - 1}\right)}$$

$$= \frac{-\omega_n^2}{2\omega_n\sqrt{\zeta^2 - 1}} = \frac{-\omega_n}{2\sqrt{\zeta^2 - 1}}$$

$$C(s) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}\left[s - \left(-\zeta\omega_n + \omega_n\left(\sqrt{\zeta^2 - 1}\right)\right)\right]} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}\left[s - \left(-\zeta\omega_n - \omega_n\left(\sqrt{\zeta^2 - 1}\right)\right)\right]}$$

✓ Taking inverse Laplace transform on both sides, we have

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left[e^{\left(-\zeta\omega_n + \omega_n\left(\sqrt{\zeta^2 - 1}\right)\right)t} + e^{\left(-\zeta\omega_n - \omega_n\left(\sqrt{\zeta^2 - 1}\right)\right)t}\right]$$

$$= \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_n t} \sinh\left(\omega_n\sqrt{\zeta^2 - 1}\right)t \qquad (2.8.10)$$

CASE II: Impulse response for $\zeta < 1;\ t \geq 0$

✓ For $\zeta < 1$, the roots s1 and s2 are given by

$$s_1 = -\zeta\omega_n + j\omega_n\left(\sqrt{1 - \zeta^2}\right)$$

$$s_2 = -\zeta\omega_n - j\omega_n\left(\sqrt{1 - \zeta^2}\right)$$

$$C(s) = \frac{A}{s - \left(-\zeta\omega_n + j\omega_n\left(\sqrt{1 - \zeta^2}\right)\right)} + \frac{B}{s - \left(-\zeta\omega_n - j\omega_n\left(\sqrt{1 - \zeta^2}\right)\right)}$$

$$= \frac{\omega_n}{2j\sqrt{1 - \zeta^2}}\left[\frac{1}{\left[s - \left(-\zeta\omega_n + j\omega_n\left(\sqrt{1 - \zeta^2}\right)\right)\right]} - \frac{1}{\left[s - \left(-\zeta\omega_n - j\omega_n\left(\sqrt{1 - \zeta^2}\right)\right)\right]}\right]$$

- Taking inverse Laplace transform on both sides, we have

$$c(t)=\frac{\omega_n}{2j\sqrt{1-\zeta^2}}\left[e^{\left(-\zeta\omega_n+j\omega_n\left(\sqrt{1-\zeta^2}\right)\right)t}-e^{-\left(\zeta\omega_n+j\omega_n\left(\sqrt{1-\zeta^2}\right)\right)t}\right]$$

$$=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\frac{e^{j\omega_n\left(\sqrt{1-\zeta^2}\right)t}-e^{-j\omega_n\left(\sqrt{1-\zeta^2}\right)t}}{2j}\right]$$

$$=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t;\ t\geq 0 \qquad (2.8.11)$$

CASE III: Impulse response for ζ = 1

$$C(s)=\frac{\omega_n^2}{(s+\omega_n)^2}$$

Taking inverse Laplace transform we get

$$c(t)=\omega_n^2 t e^{-\omega_n t} \qquad (2.8.12)$$

CASE IV: Impulse response for ζ = 0

$$C(s)=\frac{\omega_n^2}{s^2+\omega_n^2}$$

Taking inverse Laplace transform we get

$$c(t)=\omega_n\sin(\omega_n t);\ t\geq 0 \qquad (2.8.13)$$

- The impulse response for different values of ζ is shown in Fig. 2.12.
- Note that for critically damped and over-damped cases, the unit impulse response is always positive or zero.
- For the under-damped case, the unit impulse response oscillates about zero and takes both positive and negative values.
- Response has sustained oscillation when ζ = 0.

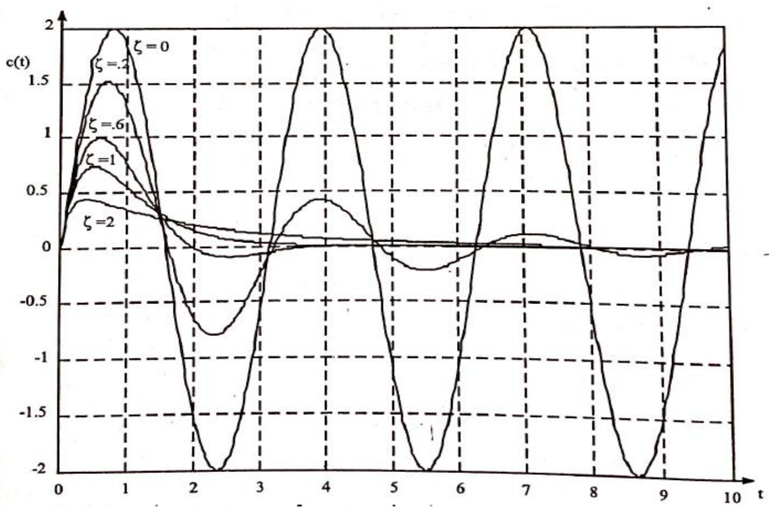


Fig. 2.12 Impulse response of 2nd order system for different ζ

2.8.2 Unit Step response of a 2nd order system

- In this section let us see the analysis of the second order system's response to the unit step signal.

$$r(t) = u(t)$$

- Laplace transform of $r(t)$ is

$$R(s) = \frac{1}{s} \tag{2.8.14}$$

- Substituting Eq. (2.8.14) in Eq. (2.8.8) gives in

$$C(s) = \frac{\omega_n}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} \tag{2.8.15}$$

- In this case also we will obtain three types of responses depending on the values of ζ .

CASE I: Step response for $\zeta < 1$

- For $\zeta < 1$, the roots of the characteristic eqn. are

$$s_1 = -\zeta\omega_n + j\omega_d$$
$$s_2 = -\zeta\omega_n + j\omega_d$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$; ω_d is known as ***damped frequency*** of oscillation; under this condition

- Now,

$$C(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)}$$

- So, we can write

$$C(s) = \frac{\omega_n^2}{s[s-(-\zeta\omega_n + j\omega_d)][s-(-\zeta\omega_n - j\omega_d)]}$$
$$= \frac{A}{s} + \frac{B}{[s-(-\zeta\omega_n + j\omega_d)]} + \frac{B'}{[s-(-\zeta\omega_n - j\omega_d)]} \tag{2.8.16}$$

$$A = s\frac{\omega_n^2}{s[s-(-\zeta\omega_n + j\omega_d)][s-(-\zeta\omega_n - j\omega_d)]}\Bigg|_{s=0} = \frac{\omega_n^2}{(\zeta\omega_n - j\omega_d)(\zeta\omega_n + j\omega_d)} = \frac{\omega_n^2}{\zeta^2\omega_n^2 + \omega_d^2}$$

$$= \frac{\omega_n^2}{\zeta^2\omega_n^2 + \omega_n^2\left(1-\zeta^2\right)} = 1$$

$$B = [s-(-\zeta\omega_n + j\omega_d)]\frac{\omega_n^2}{s[s-(-\zeta\omega_n + j\omega_d)][s-(-\zeta\omega_n - j\omega_d)]}\Bigg|_{s=-\zeta\omega_n + j\omega_d}$$

$$= \frac{\omega_n^2}{(-\zeta\omega_n + j\omega_d)(2j\omega_d)} = \frac{\omega_n^2}{-2\omega_n^2\left(1-\zeta^2\right) - j2\zeta\omega_n^2\sqrt{1-\zeta^2}} = \frac{-1}{2\left[\left(1-\zeta^2\right) + j\zeta\sqrt{1-\zeta^2}\right]}$$

$$= \frac{-\left[\left(1-\zeta^2\right) - j\zeta\sqrt{1-\zeta^2}\right]}{2\left[\left(1-\zeta^2\right)^2 + \zeta^2\left(1-\zeta^2\right)\right]} = \frac{-\left[\left(1-\zeta^2\right) - j\zeta\sqrt{1-\zeta^2}\right]}{2\left[1+\zeta^4 - 2\zeta^2 + \zeta^2 - \zeta^4\right]} = -0.5 + j0.5\frac{\zeta}{\sqrt{1-\zeta^2}}$$

✓ Similarly, we can find

$$B' = -0.5 - j0.5\frac{\zeta}{\sqrt{1-\zeta^2}}$$

✓ Substituting the values of A, B and B' into (2.8.16) we get

$$C(s) = \frac{1}{s} + \frac{-0.5 + j0.5\frac{\zeta}{\sqrt{1-\zeta^2}}}{s - (-\zeta\omega_n + j\omega_d)} + \frac{-0.5 - j0.5\frac{\zeta}{\sqrt{1-\zeta^2}}}{s - (-\zeta\omega_n - j\omega_d)}$$

✓ Now, taking inverse Laplace transform

$$c(t) = 1 + \left(-0.5 + j0.5\frac{\zeta}{\sqrt{1-\zeta^2}}\right)e^{(-\zeta\omega_n + j\omega_d)t} + \left(-0.5 - j0.5\frac{\zeta}{\sqrt{1-\zeta^2}}\right)e^{(-\zeta\omega_n - j\omega_d)t}$$

$$= 1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[-0.5e^{j(\omega_d)t}\left(\sqrt{1-\zeta^2}\right) + 0.5j\zeta e^{j(\omega_d)t} - 0.5e^{-j(\omega_d)t}\left(\sqrt{1-\zeta^2}\right) - 0.5j\zeta e^{-j(\omega_d)t}\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos(\omega_d)t + \zeta\sin(\omega_d)t\right] = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\phi\cos(\omega_d)t + \cos\phi\sin(\omega_d)t\right]$$

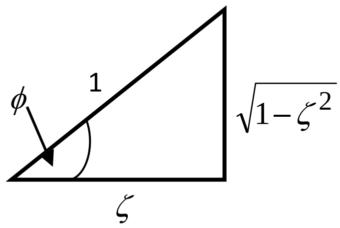
$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\omega_d t + \phi)\right]$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$; $t \ge 0$

Point to be noted

$e^{j\theta} + e^{-j\theta} = 2\cos\theta$

$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$



$\cos\phi = \zeta$

$\sin\phi = \sqrt{1-\zeta^2}$

$\tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta}$

CASE II: Critically damped (ζ = 1).

✓ If ζ = 1 in Eq. (2.8.15), then the transfer function becomes

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2} \qquad (2.8.17)$$

$$A = s.\frac{\omega_n^2}{s(s+\omega_n)^2}\bigg|_{s=0} = 1; \; B = \frac{1}{1!}\frac{d}{ds}\left[(s+\omega_n)^2 \frac{\omega_n^2}{s(s+\omega_n)^2}\right]\Bigg|_{s=-\omega_n} = \frac{1}{1!}\left(\frac{-\omega_n^2}{\omega_n^2}\right) = -1$$

$$C = \left[(s+\omega_n)^2 \frac{\omega_n^2}{s(s+\omega_n)^2}\right]\Bigg|_{s=-\omega_n} = -\omega_n$$

✓ Substituting A, Band C values in Eq. (2.8.17) yields

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

✓ Taking inverse Laplace transform

$$c(t) = 1 - e^{-\zeta\omega_n t} - \omega_n t e^{-\zeta\omega_n t}$$

$$= 1 - (1+\omega_n t)e^{-\zeta\omega_n t} \quad \text{for } t \geq 0 \qquad (2.8.18)$$

CASE III: Overdamped (ζ > 1)

✓ In the case of the overdamped system, the roots of the characteristic equation are real and non-repeated. That is

$$s_1 = -\zeta\omega_n + \omega_n\left(\sqrt{\zeta^2 - 1}\right)$$

$$s_2 = -\zeta\omega_n - \omega_n\left(\sqrt{\zeta^2 - 1}\right)$$

$$C(s) = \frac{\omega_n^2}{s\left(s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)\left(\left(s+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)\right)}$$

$$= \frac{A}{s} + \frac{B}{s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}} + \frac{C}{s+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}}$$

$$A = s\frac{\omega_n^2}{s\left(s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)\left(\left(s+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)\right)}\Bigg|_{s=0} = 1$$

$$B = \left(s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)\frac{\omega_n^2}{s\left(s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)\left(s+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)}\Bigg|_{s=-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}}$$

$$= \frac{-1}{2\sqrt{\zeta^2-1}\left(\zeta - \sqrt{\zeta^2-1}\right)}$$

$$C = \left(s+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)\frac{\omega_n^2}{s\left(s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)\left(s+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)}\Bigg|_{s=-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}}$$

$$= \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta+\sqrt{\zeta^2-1}\right)}$$

✓ Substituting the values of A, B and C and taking inverse Laplace transform on both sides, we have

$$C(t) = 1 - \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta-\sqrt{\zeta^2-1}\right)} e^{\left(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t} + \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta+\sqrt{\zeta^2-1}\right)} e^{\left(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2-1}} \left(\frac{e^{\left(\omega_n\sqrt{\zeta^2-1}\right)t}}{\zeta-\sqrt{\zeta^2-1}} + \frac{e^{\left(-\omega_n\sqrt{\zeta^2-1}\right)t}}{\zeta+\sqrt{\zeta^2-1}} \right) \qquad (2.8.10)$$

✓ The underdamped second order system is a common model for many second order systems.

✓ Therefore the transient response of a second order underdamped system for a unit step input is to be studied in detail.

✓ In this section our objective is to define transient response specification of a second order underdamped system.

✓ We know already that the response of an underdamped second order system for a unit step input is given by

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \phi\right)$$

where $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$.

The plot of the response of the 2nd order system for step input is shown in fig. Below for various values of ζ.

2.9 Time Domain Specifications

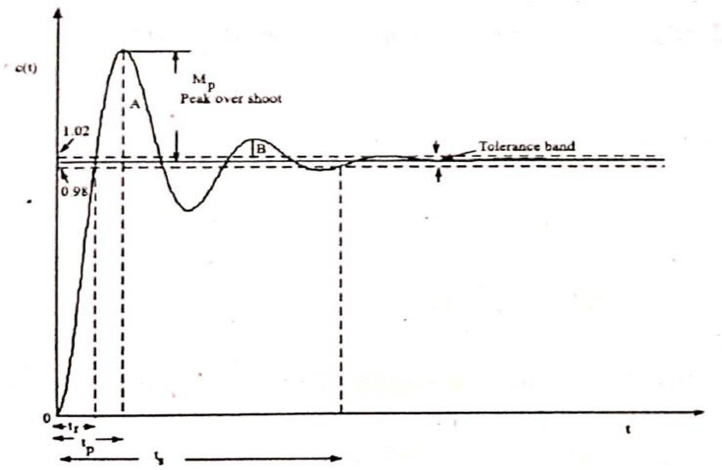


Fig. 2.13 The underdamped response of a 2nd order system

- ✓ One typical response is shown in Fig. 2.13.
- ✓ From Fig. 2.16, we can observe that the step response has a number of overshoots and undershoots before reaching steady state and these overshoots and undershoots decay exponentially.
- ✓ The following are the time domain specifications.

1. **Rise time** (t_r): The rise time is the time required for the system to reach 90% of its final value first time from the initial value of 10%.
2. **Peak time** (t_p)**:** The peak time is the time required for the response to reach the peak of the first overshoot.
3. **Maximum overshoot:** Maximum Overshoot is a measure of how much the response exceeds the final value following a step change. The percentage of maximum overshoot can be calculated as

$$M_p = \frac{\textit{First peak value} - \textit{Final Value}}{\textit{Final Value}} \times 100\%$$

$$= \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\% \qquad (2.9.1)$$

4. **Settling time** (t_s)**:** The settling time is the time required for the response to reach and stay within a specified tolerance band of its final value. The tolerance band is usually 2% to 5%.
5. **Decay ratio:** The decay ratio is defined as the ratio of the amplitude of first two successive peaks about steady state value.

Note that the response shown in Fig 2.13 has the first overshoot of amplitude A and second overshoot of amplitude B. Therefore the decay ratio is B/A.

2.9.1 Evaluation of time domain specifications

Rise time (t_r):

- ✓ Consider a unit step response of a second order under damped system.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi) \qquad (2.9.2)$$

where $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$

- ✓ Let us assume that t_r is the time required for the system to reach 100% for the first time.

$$c(t_r) = 1 \text{ (Since final value = 1)}$$

$$\text{but } \; c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \phi)$$

$$\therefore \quad \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \phi) = 0$$

$$\sin(\omega_d t_r + \phi) = 0; \qquad \text{Since} \quad \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \neq 0$$

this implies that $\omega_d t_r + \phi = n\pi$

For first peak n = l, therefore, $\boxed{t_r = \dfrac{\pi - \phi}{\omega_d} = \dfrac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}}$ (2.9. 3)

Peak time (t_p):

The peak time t_p is found by differentiating $c(t)$ in Eq. (2.9.2) and letting this derivative to zero.

$$\left.\frac{dc}{dt}\right|_{t=t_p} = -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\left[\cos(\omega_d t_p + \phi)\right]\omega_d - \left[\sin(\omega_d t_p + \phi)\right]\frac{-\zeta\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} = 0$$

$$\Rightarrow -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\left[\cos(\omega_d t_p + \phi)\omega_n\sqrt{1-\zeta^2} + \sin(\omega_d t_p + \phi)\zeta\omega_n\right] = 0$$

$$\Rightarrow -\frac{\omega_n e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\left[\zeta\sin(\omega_d t_p + \phi) - \sqrt{1-\zeta^2}\cos(\omega_d t_p + \phi)\right] = 0$$

Exponential term is not equal to zero at $t = t_p$

$\therefore \qquad \zeta\sin(\omega_d t_p + \phi) - \sqrt{1-\zeta^2}\cos(\omega_d t_p + \phi) = 0$; but $\zeta = \cos\phi$ and $\sqrt{1-\zeta^2} = \sin\phi$

$\therefore$ The above expression becomes

$$\sin(\omega_d t_p + \phi)\cos\phi - \cos(\omega_d t_p + \phi)\sin\phi = 0$$
$$\Rightarrow \sin(\omega_d t_p + \phi - \phi) = 0$$
$$\Rightarrow \sin(\omega_d t_p) = 0$$

Or, $\qquad \omega_d t_p = 0, \pi, 2\pi, 3\pi$

Since the peak time corresponds to first peak overshoot, we have

$$\omega_d t_p = \pi$$

Therefore, $\qquad t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ or $\boxed{t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$ (2.9.4)

Maximum· peak over shoot (M_p):

The maximum peak over shoot occurs at $t = t_p$. The final value of the output is unity.

$$\therefore \qquad M_p = c(t_p) - 1$$

i.e. $$M_p = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p + \phi) - 1$$

$$= -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \phi\right) = -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} (\sin \pi \cos\phi + \cos\pi \sin\phi)$$

$$= -\frac{e^{-\zeta\omega_n \cdot \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} (-\sin\phi)$$

$$M_p = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \qquad \left(\because \sin\phi = \sqrt{1-\zeta^2}\right)$$

The maximum percentage in overshoot is

$$\boxed{\%M_p = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100\%} \tag{2.9.5}$$

Settling time (t_s):

- ✓ To evaluate the settling time, we must find the time for which $c(t)$ in Eq. (2.9.2) reaches and stay within ±2% (for 2 % criterion) or ±5% (for 5 % criterion) of the steady state value.
- ✓ Using the definition, the settling time is the time it takes for the amplitude of the decaying sinusoidal to decay to 0.02 (for 2% criterion), i.e.

$$\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$$

$$e^{-\zeta\omega_n t_s} = 0.02\sqrt{1-\zeta^2}$$

$$-\zeta\omega_n t_s = \ln\left(0.02\sqrt{1-\zeta^2}\right)$$

$$t_s = -\frac{\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

The numerator varies from 3.91 to 4.74 as ζ varies from 0 to 0.9.

Therefore an approximation for the settling time for all values of ζ can be

$$\boxed{t_s = \frac{4}{\zeta\omega_n}} \quad \text{(Approximately)} \tag{2.9.6}$$

Similarly settling time for 5% criterion is

$$\boxed{t_s = \frac{3}{\zeta\omega_n}} \quad \text{(Approximately)} \tag{2.9.7}$$

2.10 Steady State Error and Error Constants

- ✓ In section 2.9 we have studied the different types of time domain specifications of a second order system.

- ✓ In this section let us study one more time domain specification of the system known as steady state-error.
- ✓ Steady state error is an extremely important aspect of system behaviour.
- ✓ The steady state error e_{ss} is the difference between the input (or desired value) and the output of a closed loop system for a known input as $t \to \infty$.
- ✓ Mathematically

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} \left[r(t) - c(t)\right]$$

- ✓ By using the final value theorem,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \qquad (2.10.1)$$

- ✓ This is valid provided that $sE(s)$ has no poles on the $j\omega$-axis, no multiple poles on the $j\omega$ axis and on the right half plane.
- ✓ The steady state error is a measure of system accuracy.
- ✓ Generally steady state behaviour of the system is judged by its steady state error for step, ramp and parabolic inputs.
- ✓ Consider a system shown in Fig. 2.14. The input is $R(s)$ and the output is $C(s)$.
- ✓ $E(s)$, $B(s)$, $G(s)$ and $H(s)$ are error signal, feedback signal, open-loop transfer function and feedback transfer function respectively.

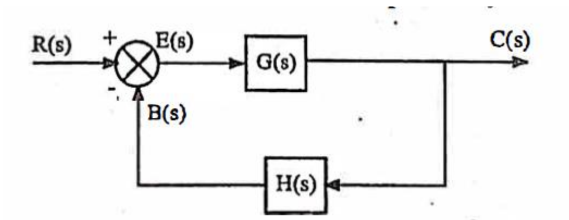


Fig. 2.14 A closed loop system

- ✓ From Fig. 2.14, we have

$$E(s) = R(s) - H(s)C(s) \qquad (2.10.2)$$

$$\text{and} \quad C(s) = G(s)E(s) \qquad (2.10.3)$$

- ✓ Substituting (2.10.3) into (2.10.2), we have

$$E(s) = R(s) - H(s)G(s)E(s)$$
$$E(s)\left[1 + H(s)G(s)\right] = R(s)$$
$$E(s) = \frac{R(s)}{\left[1 + H(s)G(s)\right]} \qquad (2.10.4)$$

- ✓ The steady state error is

$$e_{ss} = \lim_{s\to 0} sE(s)$$
$$= \lim_{s\to 0} \frac{sR(s)}{1 + G(s)H(s)} \qquad (2.10.5)$$

- ✓ The above said relation shows that the steady state error is the function of input signal $R(s)$, the forward transfer function $G(s)$ and the feedback transfer function $H(s)$.

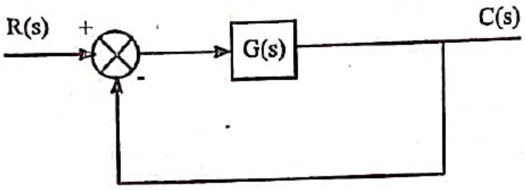


Fig. 2.15 Unity feedback closed loop system

- ✓ For a unity feedback system shown in Fig. 2.15, the feedback transfer function is H (s) = 1. Hence
- ✓ The steady state error for various types of standard test signals and their error constants are derived below.

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

2.10.1 Unit step input and position errors

- ✓ The steady state error of the system for a unit step input is called position error.
- ✓ For input $r(t) = u(t)$ and $R(s) = 1/s$.
- ✓ From Eq. (2.10.5)

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s \to 0} \frac{s \cdot \frac{1}{s}}{1+G(s)H(s)} = \lim_{s \to 0} \frac{1}{1+G(s)H(s)} = \frac{1}{1+\lim_{s \to 0} G(s)H(s)} = \frac{1}{1+k_p} \quad (2.10.6)$$

where $k_p = \lim_{s \to 0} G(s)H(s)$ is defined as the ***Position Error Constant***.

- ✓ For a unity feedback system $k_p = \lim_{s \to 0} G(s)$; since $H(s) = 1$.

2.10.2 Unit ramp input and velocity errors

- ✓ The steady state error of the system for unit ramp input is called ***velocity error***.
- ✓ For input $r(t) = tu(t)$ and $R(s) = 1/s^2$.
- ✓ From Eq. (2.10.5)

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s \to 0} \frac{s \cdot \frac{1}{s^2}}{1+G(s)H(s)} = \lim_{s \to 0} \frac{1}{s+sG(s)H(s)} = \frac{1}{\lim_{s \to 0} sG(s)H(s)} = \frac{1}{k_v} \quad (2.10.7)$$

where $k_v = \lim_{s \to 0} sG(s)H(s)$ is defined as the ***Velocity Error Constant***.

- ✓ For a unity feedback system $k_v = \lim_{s \to 0} sG(s)$; since $H(s) = 1$.

2.10.3 Unit parabolic input and acceleration error

- ✓ The steady state error of the system for unit parabolic input is called acceleration error.

- ✓ For input $r(t)=t^2/2$ and $R(s)=1/s^3$.
- ✓ From Eq. (2.10.5)

$$e_{ss}=\lim_{s\to 0}\frac{sR(s)}{1+G(s)H(s)}=\lim_{s\to 0}\frac{s\cdot\frac{1}{s^3}}{1+G(s)H(s)}=\lim_{s\to 0}\frac{1}{s^2+s^2G(s)H(s)}=\frac{1}{\lim_{s\to 0}s^2G(s)H(s)}=\frac{1}{k_a} \quad (2.10.7)$$

where $k_a=\lim_{s\to 0}s^2G(s)H(s)$ is defined as the ***Acceleration Error Constant***.

- ✓ For a unity feedback system $k_a=\lim_{s\to 0}s^2G(s)$; since $H(s)$ = 1.

2.11 Types of Feedback Control Systems

- ✓ The open loop transfer function G(s) of a unity feedback system can be written in two standard forms namely the time-constant form and the pole zero form.
- ✓ The general representation of these two forms are:

$$G(s)=\frac{K(T_{z_1}s+1)(T_{z_2}s+1)(T_{z_3}s+1)\cdots(T_{z_m}s+1)}{s^n(T_{p_1}s+1)(T_{p_2}s+1)(T_{p_3}s+1)\cdots(T_{p_m}s+1)}\cdots\text{(Time-constant form)} \quad (2.11.1)$$

$$=\frac{K'(s+z_1)(s+z_2)(s+z_3)\cdots(s+z_m)}{s^n(s+p_1)(s+p_2)(s+p_3)\cdots(s+p_m)}\cdots\text{(Pole-zero form)} \quad (2.11.2)$$

where the relation between K and K' is

$$K=K'\frac{Product\ of\ all\ zeros}{Product\ of\ all\ zeros} \quad (2.11.3)$$

- ✓ In both representations, the terms $\boldsymbol{s^n}$ in the denominator represent the n-poles at the origin, which represents the type number of the system.
- ✓ If n = 0 then the system is known as type-0 system and if n = 1 then the system is known as type-1 system and so on.

Definition: **The type number of a system is defined as the number of poles of G(s) at the origin or number of integrators present in the system.**

- ✓ In Eq. (2.11.2) the number of poles at the origin is 'n'. So, the type number of the system is 'n'.
- ✓ For example

$$G(s)=\frac{(s+1)}{(s+2)(s+3)} \qquad \text{is a type 0 system}$$

$$G(s)=\frac{(s+1)}{s(s^2+2s+2)} \qquad \text{is a type-1 system}$$

$$G(s)=\frac{1}{s^2(s+2)} \qquad \text{is a type-2 system}$$

2.11.1 Steady state errors of type -0 unity feedback system

- ✓ Let $G(s)$ be an open loop transfer function of a system with no poles at the origin ($n = 0$).
- ✓ The steady state errors for the standard inputs are as follows;
- ✓ Position error

$$e_{ssp}(t) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{s\cdot\frac{1}{s}}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+k_p} = \frac{1}{1+K} \tag{2.11.4}$$

- ✓ Velocity error

$$e_{ssv}(t) = \lim_{s\to 0} \frac{s\cdot\frac{1}{s^2}}{1+G(s)} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{0} = \infty \tag{2.11.5}$$

- ✓ Acceleration error

$$e_{ssa}(t) = \lim_{s\to 0} \frac{s\cdot\frac{1}{s^3}}{1+G(s)} = \frac{1}{\lim_{s\to 0} s^2G(s)} = \frac{1}{0} = \infty \tag{2.11.6}$$

2.11.2 Steady state errors of type -1 unity feedback system

- ✓ Let $G(s)$ be an open loop transfer function of a system in pole-zero form with one pole at the origin ($n = 1$).
- ✓ The steady state errors for various inputs are as follows;
- ✓ Position error

$$e_{ss}(p) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{s\cdot\frac{1}{s}}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+\infty} = 0 \tag{2.11.7}$$

- ✓ Velocity error

$$e_{ss}(v) = \lim_{s\to 0} \frac{s\cdot\frac{1}{s^2}}{1+G(s)} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{K} \tag{2.11.8}$$

- ✓ Acceleration error

$$e_{ss}(a) = \lim_{s\to 0} \frac{s\cdot\frac{1}{s^3}}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2+s^2G(s)} = \frac{1}{\lim_{s\to 0} s^2G(s)} = \frac{1}{0} = \infty \tag{2.11.9}$$

2.11.3 Steady state errors of type -3 unity feedback system

- ✓ Let $G(s)$ be an open loop transfer function of a system in pole-zero form with two poles at the origin ($n = 2$).
- ✓ The steady state errors for various inputs are as follows;
- ✓ Position error

$$e_{ss}(p) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{s\cdot\frac{1}{s}}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+\infty} = 0 \qquad (2.11.10)$$

✓ Velocity error

$$e_{ss}(v) = \lim_{s\to 0} \frac{s\cdot\frac{1}{s^2}}{1+G(s)} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{\infty} \qquad (2.11.11)$$

✓ Acceleration error

$$e_{ss}(a) = \lim_{s\to 0} \frac{s\cdot\frac{1}{s^3}}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2+s^2G(s)} = \frac{1}{\lim_{s\to 0} s^2G(s)} = \frac{1}{K} \qquad (2.11.12)$$

Table 2.11 Steady state error for various inputs and systems

Input	**Error**	**Steady-State Error**		
		Type-0 system	Type-1 system	Type-2 system
Unit step $u(t)$	Position $e_{ss}(p) = \frac{1}{1+k_p}$	$\frac{1}{1+k_p}$	0	0
Unit ramp (t)	Velocity $e_{ss}(v) = \frac{1}{k_v}$	∞	$\frac{1}{k_v}$	0
Unit parabolic $t^2/2$	Velocity $e_{ss}(a) = \frac{1}{k_a}$	∞	∞	$\frac{1}{k_a}$

where $k_p = \lim_{s\to 0} G(s)$; $\quad k_v = \lim_{s\to 0} sG(s)$; $\quad k_a = \lim_{s\to 0} s^2G(s)$

✓ From the table 2.11 it can be noticed that as the type number increases the steady state error decreases.

✓ The steady state errors so far we have seen do not give details of dynamics of error.

✓ So, these errors are called as ***static errors*** and the constants k_p, k_v and k_a are called as ***static error coefficients***.

2.12 Disadvantage of static error constants