Electrostatics: Part-4

Presented by

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Gauss's law

Statement :-It states that the total electric flux ψ through any closed surface is equal to the total charge enclosed by that surface.

Thus,

$$\boxed{\psi = Q_{enc}} \qquad (34)$$

Q_{enc} = Q charge enclosed by a surface.

$$\psi = \oint_s d\psi = \oint_s \vec{D} \bullet dS = Q_{enc} = \int_v \rho_v dv$$

or,

$$\boxed{\oint_s \vec{D} \bullet dS = \int_v \rho_v dv} \qquad (35)$$

By applying divergence theorem we get -

$$\oint_s \vec{D} \bullet dS = \int_v \nabla \cdot \vec{D} dv \qquad (36)$$

Comparing (35) and (36) we get

$$\int_v \nabla \cdot \vec{D} dv = \int_v \rho_v dv$$

or,

$$\boxed{\nabla \cdot \vec{D} = \rho_v} \qquad (37)$$

✓ The equations (37) is the **first** of the four **Maxwell's equations** which we are going to derive.

✓This equation (37) states that the volume charge density is the same as the divergence of the electric flux density.

Note :-

1. Equation (35) and (37) are basically stating Gauss's law in different forms; where (35) is the integral form and (37) is the differential or point form of Gauss's law.
2. Gauss's law is an alternative statement of the Coulomb's law; proper application of the divergence theorem to Coulomb's law results in Gauss's law.
3. Gauss's law provides as easy means of finding electric field intensity $\vec{E}$ or the electric flux density $\vec{D}$ for symmetrical charge distributions such as a point charge, an infinite line of charge, an infinite surface charge and a spherical distribution of charge.

Applications of Gauss's Law

- ✓Before we apply the Gauss's law to calculate electric field, we should know whether the symmetry exist or not.
- ✓Once it has been found that the symmetric charge distribution exists, we will construct a mathematical closed surface (known as **Gaussian surface**).
- ✓The surface should be chosen in such a way that the electric flux density $\vec{D}$ will be normal or tangential to the Gaussian surface.
- ✓ when $\vec{D}$ is normal to the surface, then $\vec{D} \cdot dS = DdS$ because $\vec{D}$ is constant on the surface.
- ✓ when $\vec{D}$ is tangential to the surface; then $\vec{D} \cdot dS = 0$. Thus we must choose a surface that has some of the symmetry exhibited by the charge distribution.

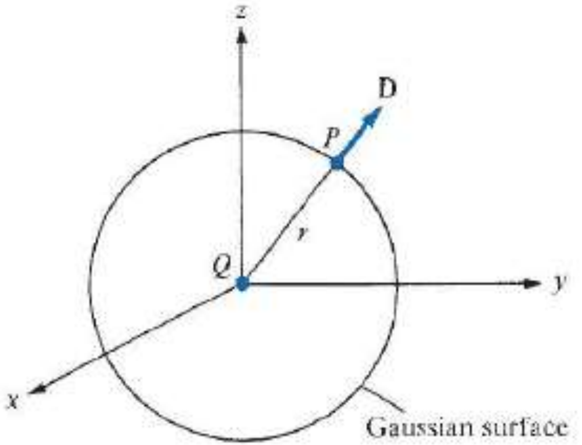
Let's now apply these ideas to the following cases :-

(A) A point Charge

Suppose, a point charge +Q is located at the origin of the Cartesian coordinate system.

Now, if we want to determine $\vec{D}$ at point '*P*'; we should choose a spherical surface containing '*P*' on the surface of this sphere.

This sphere can satisfy the symmetrical condition.



Thus a spherical surface centered at the origin of the Cartesian coordinate system is the Gaussian surface.

Since $\vec{D}$ is normal everywhere to the Gaussian surface, i.e. $\vec{D} = D_r \hat{a}_r$ then applying Gauss's law $(\psi = Q_{enc})$ gives -

$$Q = \oint_S \vec{D} \cdot dS = \oint_S D_r \hat{a}_r dS = D_r \oint_S dS = D_r \left(4\pi r^2\right)$$

Total Surface area of the Sphere

or, $D_r = \dfrac{Q}{4\pi r^2}$ where, $\oint dS = \int\limits_{\varphi=0}^{2\pi}\int\limits_{\theta=0}^{\pi} r^2 \sin\theta d\theta d\varphi = 4\pi r^2$ is the surface area of the Gaussian surface

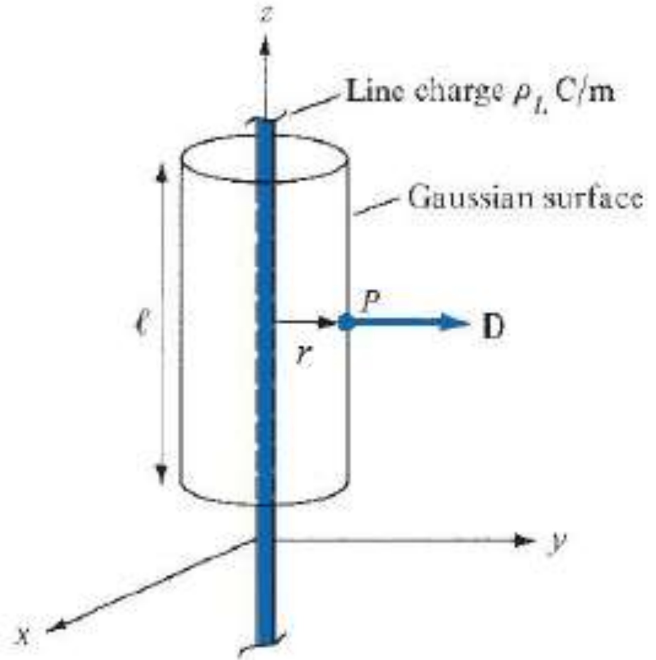
Hence,

$$\boxed{\begin{aligned}\vec{D} &= D_r \hat{a}_r \\ \vec{D} &= \frac{Q}{4\pi r^2}\hat{a}_r\end{aligned}} \qquad (38)$$

(B) Infinite Line Charge

Suppose, the infinite line of uniform charge ρ_L C/m lies along the z-axis.

To determine $\vec{D}$ at the point 'P', we choose a cylindrical surface containing 'P' to satisfy the symmetrical condition.



The electric flux density $\vec{D}$ is constant on and normal to the cylindrical Gaussian surface, i.e.

$$\vec{D} = D_r \hat{a}_r \qquad (39)$$

Now, let us apply Gauss's law for an arbitrary length 'l' of the line.

Hence,

$$\rho_L l = Q = \oint_s \vec{D} \cdot dS = \oint_s D_r \hat{a}_r dS = D_r \oint_s dS = D_r 2\pi r l$$

or, $$D_r = \frac{Q}{2\pi r l} \qquad (40)$$

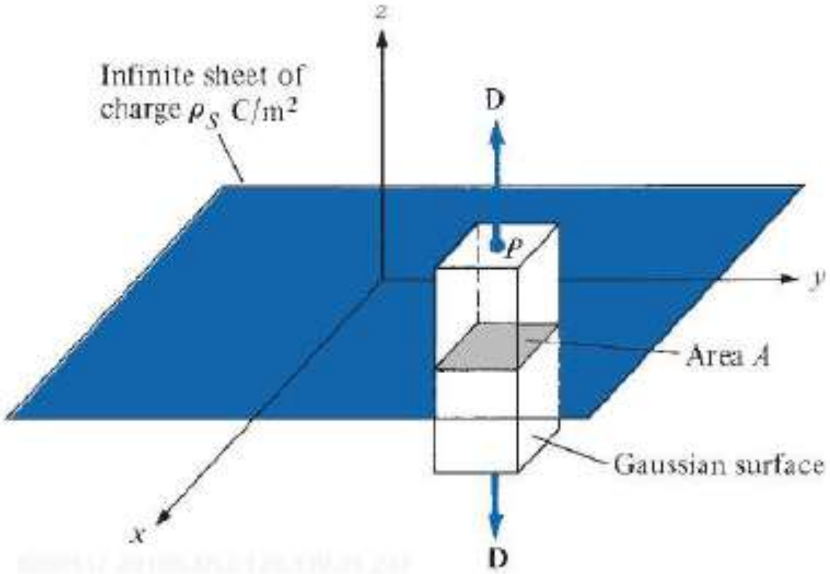
Where $\oint_s dS = 2\pi r l$ is the surface area of the Gaussian surface

Note that $\oint_s \vec{D} \cdot dS$ evaluated on the top and bottom surface's of the cylinder is zero. Since D has no z-component that means $\vec{D}$ is tangential to those surfaces.

Thus, $\vec{D} = \frac{\rho_L l}{2\pi r l}\hat{a}_r$ or $\boxed{\vec{D} = \frac{\rho_L}{2\pi r}\hat{a}_r}$ (41)

(C) Infinite Sheet of Charge

Let us consider an infinite sheet of uniform charge ρ_S C/m^2 lying on $z = 0$ plane. To determine $\vec{D}$ at the point 'P', we choose a rectangular box that cuts the sheet symmetrically and has two of its faces parallel to the sheet.



As $\vec{D}$ is normal to the sheet then-

$$\vec{D} = D_z\hat{a}_z \qquad (42)$$

Now, applying Gauss's law we have -

$$Q = \int_s \rho_s dS = \oint_s \vec{D}\cdot dS = D_z\left[\int_{top} dS + \int_{bottom} dS\right] \qquad (43)$$

Not that $\oint_s \vec{D}\cdot dS$ evaluated on the sides of the box is zero because $\vec{D}$ has no components along *x*- and *y*-axis.

If the each of the top and bottom area of the box is taken as '*A*'. Then (42) becomes -

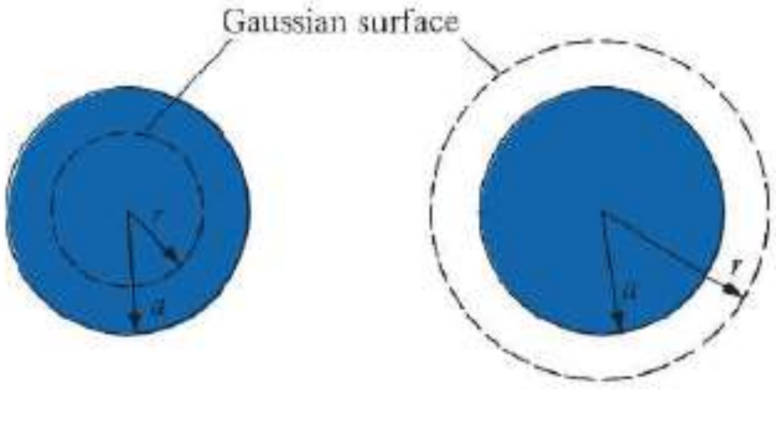
$$\rho_S A = D_z(A+A) \quad \text{or,} \quad D_z = \frac{\rho_s}{2}$$

Hence, $$\boxed{\vec{D} = \frac{\rho_s}{2}\hat{a}_z} \qquad (44)$$

(D) Uniformly Charged Sphere

Let us consider a sphere of radius '*a*' with uniformly distribution of charge is ρ_0 C/ .

To determine $\vec{D}$ anywhere, let us construct Gaussian surface for -



(a) (b)

Case-I $\longrightarrow$ $r \le a$

Case-II $\longrightarrow$ $r \ge a$

Since the charge has spherical symmetry; it is obvious that a spherical surface is an appropriate Gaussian surface.

For, $r \le a$, the total charge enclosed by the spherical surface of radius 'r' is -

$$Q_{enc} = \int_v \rho_v dv = \rho_0 \int_v dv = \rho_0 \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta dr d\theta d\varphi = \rho_0 \frac{4}{3}\pi r^3 \qquad (45)$$

and

$$\psi = \oint_s \vec{D} \cdot dS = D_r \oint_s dS = D_r \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta d\theta d\varphi = D_r 4\pi r^2 \qquad (46)$$

Hence $(\psi = Q_{enc})$ gives -

$$\rho_0 \frac{4}{3}\pi r^3 = D_r 4\pi r^2$$

$$D_r = \frac{1}{3} r\rho_0$$

or, $$\boxed{\vec{D} = D_r \hat{a}_r = \frac{r}{3}\rho_0 \hat{a}_r} \qquad (47)$$

for $0 \le r \le a$

For, $r \ge a$, the total charge enclosed by the surface is the entire charge in this case, i.e.

$$Q_{enc} = \int_v \rho_v dv = \rho_0 \int_v dv$$

$$= \rho_0 \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta dr d\theta d\varphi = \rho_0 \frac{4}{3}\pi a^3 \quad (48)$$

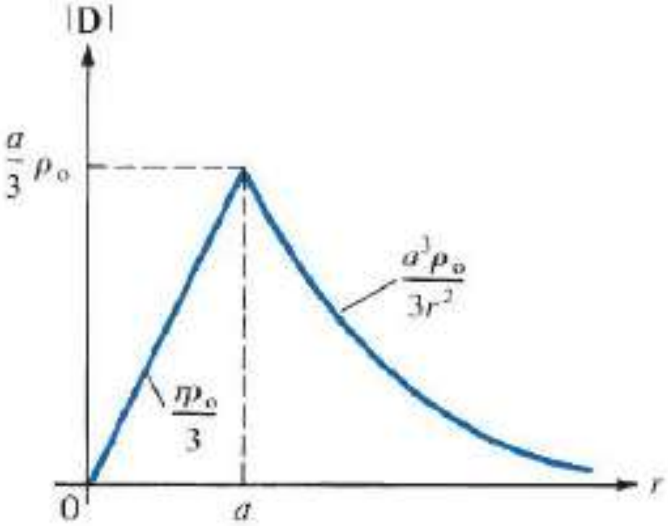
while $\psi = \oint_s \vec{D} \cdot dS = D_r 4\pi r^2$

So, $\vec{D} = \dfrac{a^3 \rho_0}{3r^2} \hat{a}_r \quad$ for $\quad r \geq a$

Thus, $\vec{D}$ everywhere is given by -

$$\vec{D} = \begin{cases} \dfrac{r}{3} \rho_0 \hat{a}_r & 0 \leq r \leq a \\[2ex] \dfrac{a^3 \rho_0}{3r^2} \hat{a}_r & r \geq a \end{cases} \qquad (49)$$

and $|\vec{D}|$ is given by -



Thus, from (38)-(49) we can see that the ability to take $\vec{D}$ out of the integral sign is the key to finding $\vec{D}$ using Gauss's law.
In other words, $\vec{D}$ must be constant on the Gaussian Surface.

Instead of Coulomb's law and Gauss's law, there is another way of finding electric flux intensity or flux density, i.e. Electric Scalar Potential (V).

Thank you