1. Given, Basis $B = \{x^2, x, 1\}$

Linear Transformation T :-

$$T(x^2) = x + m$$
$$T(x) = (m-1)x$$
$$T(1) = x^2 + m$$

(a) Matrix Representation of T relative to B :-

(i) $T(x^2) = x + m = 0.x^2 + 1.x + m = \begin{bmatrix} 0 \\ 1 \\ m \end{bmatrix}$

(ii) $T(x) = (m-1)x = 0x^2 + (m-1)x + 0 = \begin{bmatrix} 0 \\ m-1 \\ 0 \end{bmatrix}$

(iii) $T(1) = x^2 + m = 1.x^2 + 0.x + m = \begin{bmatrix} 1 \\ 0 \\ m \end{bmatrix}$

$\therefore$ From (i), (ii) & (iii),

$$\boxed{\text{Transformation Matrix :- } \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix}}$$ (Ans)

(b) Find Kernel (T) for all values of m :-

Let $\{\alpha, \beta, \gamma\} \in$ Kernel (T) ; where $\alpha, \beta, \gamma \in \mathbb{R}$

$$\therefore \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, we get :-

$\gamma = 0$ --- (i)

$\alpha + (m-1)\beta = 0$ --- (ii)

$m\alpha + m\gamma = 0$ --- (iii)

From (i) & (iii),

$\alpha = -\gamma = 0$ ---- (iv)

From (ii) & (iv),

$(m-1)\beta = 0$ ---- (v)

$\therefore$ $\Rightarrow$ For $m = 1$:- Kernel (T) $= \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix}$

$\Rightarrow$ For $m \neq 1$:- Kernel (T) $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (Ans)

(c) Find the image of T for all values of m :-

(i) $m = 0$:- Transformation Matrix $\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\downarrow a_1 \quad \downarrow a_2 \quad \downarrow a_3$

Here, $a_1 + a_2 = 0$

$\therefore$ a_1, a_2, a_3 are NOT independent.

Now, By Rank - Nullity Theorem,

Rank + Nullity = 3

or, Rank = 2 ($\because$ two independent vector)

$\therefore$ $\boxed{\text{Image (T)} = \text{Span}\{(0,1,0), (1,0,0)\}}$ (Ans)

(ii) $m = 1$:- Transformation Matrix $\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\downarrow a_1 \quad \downarrow a_2 \quad \downarrow a_3$

(P.T.O)

$\therefore$ By Rank-Nullity Theorem,

~~Rank = 2~~ (as $a_2 = (0,0,0)$ and two

Nullity = 1 independent vectors)

$\therefore$ Rank = 3 - 1 = 2

$$\therefore \boxed{\text{Image}(T) = \text{Span}\{(0,1,1), (1,0,1)\}}$$ (Ans)

<u>(iii) $m \notin \{0,1\}$</u> :-

In this case, all the column vectors are independent :-

$$\begin{array}{ccc} \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} \\ \downarrow \quad \downarrow \quad \downarrow \\ \underline{a_1} \quad \underline{a_2} \quad \underline{a_3} \end{array}$$

$$\therefore \boxed{\text{Image}(T) = \text{Span}\{(0,1,m), (0,m-1,0), (1,0,m)\}}$$

(Ans)

———X———

(2). Given,

$$T(x,y,z) = (x+2y-z,\ 2x+3y+z,\ 4x+7y-z)$$

Let us take the standard basis B:

$$\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$$

$$\therefore T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\\4\end{bmatrix};\ T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\3\\7\end{bmatrix};\ T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$$

$\therefore$ Transformation Matrix, $T = \begin{bmatrix}1 & 2 & -1\\2 & 3 & 1\\4 & 7 & -1\end{bmatrix}$

(i) Kernel (T) :-

Let $(\alpha, \beta, \gamma) \in$ Kernel(T); where $\alpha, \beta, \gamma \in \mathbb{R}$.

$$\therefore \begin{bmatrix}1 & 2 & -1\\2 & 3 & 1\\4 & 7 & -1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

or,

$$\alpha + 2\beta - \gamma = 0 \quad \text{.... (i)}$$
$$2\alpha + 3\beta + \gamma = 0 \quad \text{.... (ii)}$$
$$4\alpha + 7\beta - \gamma = 0 \quad \text{.... (iii)}$$

On equating the above equations, we get:-

$$\alpha = -5\gamma;\ \beta = 3\gamma$$

$$\therefore \text{Kernel (T)} = \begin{bmatrix}-5\\3\\1\end{bmatrix}\gamma$$

where, Basis of Kernel(T) = $\{(-5, 3, 1)\}$.

(Next Page) →

$\therefore$ Geometry :- $\underline{-5x + 3y + z}$ i.e. a straight line passing through the origin. (Ans)

(ii) Range (T) :-

$$T = \begin{bmatrix}1 & 2 & -1\\2 & 3 & 1\\4 & 7 & -1\end{bmatrix}$$
$$\quad\ \ \downarrow\quad \downarrow\quad \downarrow$$
$$\quad\ \ a_1\quad a_2\quad a_3$$

We observe that :- $5a_1 - 3a_2 - a_3 = 0$

$\therefore$ They are NOT independent.

But, we cannot write a_1 in terms of a_2 or vice-versa. Therefore,

a_1 and a_2 are Independent.

$$\therefore \text{Range (T)} = \text{Span}\{(1,2,4),(2,3,7)\}$$

Hence,

Geometry :- The two vectors with equations $\underline{x + 2y + 4z}$ and $\underline{2x + 3y + 7z}$ will span a ~~plane in~~ 3-dimensional plane. (Ans)

—X—

3 (a) To prove :- A linear transformation $T: V \to W$ between two vector spaces is one-to-one iff its kernel is a singleton set of zero vector, i.e. the Nullity is zero.

Proof :-
We will prove this in two ways :-

(i) If T is One-to-One, the Nullity is Zero :-

Suppose T is One-to-One i.e. Injective.
Now, since T is a linear transformation, it sends the Zero Vector 0_V of V to the Zero Vector 0_W of W.

$$\therefore T(0_V) = T(0_V - 0_V)$$
$$= T(0_V + (-1)0_V)$$
$$= T(0_V) + (-1)T(0_V) \quad \{\text{by linearity of } T\}$$
$$= T(0_V) - T(0_V)$$
$$= 0_W$$

So, $0_V \in N(T)$ ...... (I)

Also, if $v \in N(T)$, then,
$$T(v) = 0_W = T(0_V) \quad \text{---(II)}$$

Since T is One-to-One, $\therefore v = 0_V$

$\therefore$ We got that $N(T) = \{0_V\}$ i.e.
$\boxed{\text{The Nullity of T is Zero}}$

(ii) If Nullity is Zero, then T is One-to-One :-

Here it is given, $N(T) = \{0_V\}$

Now, suppose $T(v_1) = T(v_2)$ for some $v_1, v_2 \in V$.

(Next Page) →

Then, $0_W = T(v_1) - T(v_2)$
$$= T(v_1) + (-1)T(v_2)$$
$$= T(v_1 + (-1)v_2) \quad \{\text{by linearity of } T\}$$
$$= T(v_1 - v_2)$$

$\therefore$ It shows that the vector $v_1 - v_2$ is in the Null space $N(T) = \{0_V\}$.

$\therefore v_1 - v_2 = 0_V$, or $\boxed{v_1 = v_2}$

$\therefore \boxed{\text{T is Injective i.e. One-to-One}}$

From the above two proofs (i) & (ii), we can conclude that :-

$\boxed{\text{T is One-to-One iff its Nullity is Zero i.e. its kernel is a singleton set of Zero Vector}}$

(Proved)

(b) To prove :- The above transformations i.e. One-to-One linear transformations will preserve linear independence.

Proof (by Contradiction) :-

Suppose $T(v_1), \ldots, T(v_n)$ are not linearly independent, where $(v_1, \ldots, v_n)$ is a basis for V.
Also, there are scalars $a_1, \ldots, a_n$ not all of them zeroes, such that :-

$$T(a_1 v_1 + \ldots + a_n v_n) = a_1 T(v_1) + \ldots + a_n T(v_n)$$
$$= 0$$

Now, since T is One-to-One, it follows that :-
$$a_1 v_1 + \ldots + a_n v_n = 0$$

But this is a contradiction since $(v_1, \ldots, v_n)$ was assumed to be a basis of V.

$\therefore \boxed{\text{Linear Independence is Preserved.}}$ (Proved)

4. Given, $\Lambda(V, W)$ is the set of all linear maps from V to W.

Pointwise Addition :-

$$(T_1+T_2)(v) = T_1(v) + T_2(v)$$

Scalar Multiplication :-

$$(\alpha T_1)(v) = \alpha T_1(v)$$

where $T_1, T_2 \in \Lambda$; $v \in V$; $\alpha \in \mathbb{R}$.

We will prove all the axioms of a Vector Space to show if Λ is a Vector Space or not.

(i) $(T_1+T_2)(v) = T_1(v) + T_2(v)$ $\{\forall\, T_1, T_2 \in \Lambda ; v \in V\}$
$= T_2(v) + T_1(v)$
$= (T_2+T_1)(v)$

$\therefore \boxed{T_1 + T_2 = T_2 + T_1}$ (Commutative)

(ii) $((T_1+T_2)+T_3)(v)$ $\{\forall\, T_1, T_2, T_3 \in \Lambda ; v \in V\}$
$= (T_1+T_2)(v) + T_3(v)$
$= (T_1(v) + T_2(v)) + T_3(v)$
$= T_1(v) + (T_2(v) + T_3(v))$
$= T_1(v) + (T_2+T_3)(v)$
$= (T_1 + (T_2+T_3))(v)$

$\therefore \boxed{(T_1+T_2)+T_3 = T_1+(T_2+T_3)}$ (Associative)

(iii) a) $(T_1+T_2)(u) = T_1(u) + T_2(u)$ $\{\forall\, u \in V\}$

$\boxed{T_1, T_2 \in \Lambda \Rightarrow T_1 + T_2 \in \Lambda}$ (Closure)

b) $(\alpha T_1)(u) = \alpha T_1(u)$ $\{u \in V ; \alpha \in \mathbb{R}\}$

$\boxed{T_1, \alpha T_1 \in \Lambda}$

(iv) Let $O : V \to W$.
So, $O(v) = O_W$; where $v \in V$; $O_W \in W$ is the additive identity of W.

$\therefore (T_1 + O)(v)$ $\{T_1, O \in \Lambda ; v \in V\}$
$= T_1(v) + O(v)$
$= T_1(v) + O_W$
$= T_1(v)$
$= O_W + T_1(v)$
$= O(v) + T_1(v)$
$= (O + T_1)(v)$

$\therefore \boxed{T_1 + O = T_1 = O + T_1}$

O is the Additive Identity of set Λ.

(v) $(-T_1)(v) = -(T_1(v))$ $\{\forall\, T_1, -T_1 \in \Lambda ; v \in V\}$
$\therefore (T_1 + (-T_1))(v)$
$= T_1(v) - T_1(v)$
$= O$ $\{O \in \Lambda$ is the Additive Identity$\}$

$\therefore \boxed{T_1 + (-T_1) = O}$

$-T_1$ is the Additive Inverse of T_1.

(vi) $\alpha(T_1+T_2)(v)$ $\{\forall\, T_1, T_2 \in \Lambda ; v \in V ; \alpha \in \mathbb{R}\}$
$= \alpha(T_1(v) + T_2(v))$
$= \alpha T_1(v) + \alpha T_2(v)$
$= (\alpha T_1 + \alpha T_2)(v)$

$\therefore \boxed{\alpha(T_1+T_2) = \alpha T_1 + \alpha T_2}$ (First Distributive Law)

(P.T.O.)

(vii) $((\alpha+\beta)T_1)(v)$ $\{\forall T_1 \in \Lambda; v \in V; \alpha, \beta \in \mathbb{R}\}$

$= (\alpha+\beta)T_1(v)$

$= \alpha T_1(v) + \beta T_1(v)$

$= (\alpha T_1 + \beta T_1)(v)$

$\therefore \boxed{(\alpha+\beta)T_1 = \alpha T_1 + \beta T_1}$ (Second Distributive Law)

(viii) $((\alpha\beta)T_1)(v)$ $\{T_1 \in \Lambda; v \in V; \alpha, \beta \in \mathbb{R}\}$

$= (\alpha\beta)T_1(v)$

$= \alpha(\beta T_1(v))$

$= (\alpha(\beta T_1))(v)$

$\therefore \boxed{(\alpha\beta)T_1 = \alpha(\beta T_1)}$ (Scalar Associative Law)

(ix) $(1.T_1)(v) = 1.T_1(v) = T_1(v)$ $\{T_1 \in \Lambda; v \in V\}$

$\therefore \boxed{1.T_1 = T_1}$ (Monoid Law)

Since all the properties are satisfied,

$\therefore$ $\Lambda(V, W)$ forms a Vector Space under pointwise addition & scalar multiplication.

(Proved)

X

(5) Given V is a Vector Space and $V = M \oplus N$.

(A). Show that $\forall v \in V; \exists m \in M, n \in N$ s.t. $v = m + n$:-

Since $V = M \oplus N$, so $V = M + N$ and $M \cap N = \{0\}$

Let, $v = m_1 + n_1$ and also $v = m_2 + n_2$,

where $v \in V$; $m_1, m_2 \in M$; $n_1, n_2 \in N$.

$\therefore \quad m_1 + n_1 = m_2 + n_2$

or, $\quad m_1 - m_2 = n_2 - n_1$

We can observe that $m_1 - m_2 \in M$ and $n_2 - n_1 \in N$. --- (i)

Since they are equal (i.e. $m_1 - m_2 = n_2 - n_1$), let's take $x = m_1 - m_2 = n_2 - n_1$.

Now, $x \in M \cap N$ as $x \in M$ and $x \in N$ {from (i)}

But we know that, $M \cap N = \{0\}$

$\therefore \quad x = 0$

or, $\quad m_1 - m_2 = n_2 - n_1 = 0$

or, $\boxed{m_1 = m_2}$ and $\boxed{n_1 = n_2}$

Hence,

$\boxed{\forall v \in V, \exists m \in M \text{ and } \exists n \in N \text{ s.t. } v = m + n \text{ where } m \text{ and } n \text{ are unique.}}$ (Proved)

(B). Given, $P : V \to V$ is a projection of V along M onto N as $P(v) = n$, where $v \in V$ and $n \in N$.

(a) P is Linear:-

We can represent n as :-

$n = y_1 n_1 + y_2 n_2 + \cdots + y_k n_k$, where $\{n_1, n_2, \ldots, n_k\}$ is a basis of N.

Also, $P(v) \in N$ as $P(v) = n$ & $n \in N$. --- (i)

~~Now, let $X = [n_1, n_2, \ldots, n_k]$~~

Now, let $X = [n_1 \ n_2 \ \cdots \ n_k]$

$$\therefore \quad n = [n_1 \ n_2 \ \cdots \ n_k] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$$

or, $\boxed{n = Xy}$ --- (ii)

$\therefore$ From (i) & (ii),

$P(v) = Xy$ ---- (iii)

Now, $V - P(v) \in N^{\perp}$

or, $V - P(v) \in C(X)^{\perp}$ {orthogonal to column space}

Since column space is orthogonal to the left nullspace of X,

$\therefore \quad V - P(v) \in N(X^T)$

Hence, we can say that :-

$X^T(V - P(v)) = 0$

or, $X^T V - X^T P(v) = 0$

or, $X^T V - X^T (Xy) = 0$ {From (iii)}

or, $X^T V = X^T X y$ ---- (iv)

Now, since X is the Basis-Vectors matrix which are linearly independent by definition, so :-

$X^T X$ is obviously Invertible.

$\therefore \quad (X^T X)^{-1} X^T V = (X^T X)^{-1} X^T X y$

or, $y = (X^T X)^{-1} X^T V$ ---- (v)

(P.T.O.)

$\therefore$ From (iii) and (V),

$$P(v) = X((X^TX)^{-1}X^TV)$$

$$\text{or, } P(v) = (X(X^TX)^{-1}X^T)\,V$$

Now, let $A = X(X^TX)^{-1}X^T$. So, $\boxed{P(v) = AV}$

So, clearly A is a matrix, and we know that any transformation T of the form $T(v) = AV$ (where A is a matrix) is a Linear Transformation

$\therefore$ $\boxed{\text{P is Linear}}$ // (Proved)

(b) P is Idempotent :-

P will be Idempotent if $P^2(v) = P(v)$.

We know that, $P(v) = (X(X^TX)^{-1}X^T)\,V$

$$\therefore\; P^2(v) = P(P(v))$$
$$= P((X(X^TX)^{-1}X^T)V)$$
$$= (X(X^TX)^{-1}X^T)(X(X^TX)^{-1}X^T)\,V$$
$$= (X(X^TX)^{-1}\underline{X^TX(X^TX)^{-1}}X^T)V$$
$$= (X(X^TX)^{-1}X^T)V \quad \{X^TX(X^TX)^{-1} = I\}$$
$$= P(v)$$

$$\therefore\; P^2(v) = P(v)$$

Hence, $\boxed{\text{P is Idempotent.}}$ // (Proved)

(c) Range (P) = N :-

$$\text{Range}(P) = \{P(v) : v \in V\}$$
$$= \{n : n \in N\} \quad (\text{given } P(v) = n)$$
$$= N$$

$\therefore$ $\boxed{\text{Range}(P) = N}$ // (Proved)

(d) Kernel (P) = M :-

$$\text{Kernel}(P) = \{v \in V : P(v) = 0\}$$
$$= \{v \in V : n = 0\} \quad \left(\begin{matrix} n \in N; \\ \text{Given } P(v) = n \end{matrix}\right)$$
$$= \{m + n \in V : n = 0, n \in N, m \in M\}$$
$$(\text{Given } v = m + n)$$
$$= \{m + 0 \in V : m \in M\}$$
$$= M$$

$\therefore$ $\boxed{\text{Kernel}(P) = M}$ // (Proved)

(e). To prove :- I-P is the projection of V along N onto M

Proof :-

Given $v = m + n$; $v \in V$, $m \in M$, $n \in N$.

$$\therefore\; (I-P)(v)$$
$$= (I-P)(m+n)$$
$$= I(m+n) - P(m+n) \qquad \overset{\text{Given,}}{\{P(v) = P(m+n) = n\}}$$
$$= m + n - P(v)$$
$$= m + n - n$$
$$= m \quad \text{------------ (i)}$$

$$\therefore\; \text{Range}(I-P) = \{(I-P)(v) : v \in V\}$$
$$= \{m : m \in M\}$$
$$= M \quad \text{------ (ii)}$$

$$\text{Also, Kernel}(I-P) = \{v \in V : (I-P)(v) = 0\}$$
$$= \{v \in V : m = 0\} \quad (\text{From (i)})$$
$$= \{m + n \in V : m = 0, n \in N\}$$
$$= \{n : n \in N\}$$
$$= N \quad \text{----- (iii)}$$

(P.T.O)

Also, $(I-P)^2(v) = (I-P)(I-P)v$

$= (I-P)(m)$ (From (i))

$= m - P(m)$ $\{P(m)=0;\ \text{Kernel}(P)=M\}$

$= m$

$= (I-P)(v)$ ------ $\underline{(iv)}$

Let $\alpha \in \mathbb{R};\ v_1, v_2 \in V$.

$\therefore (I-P)(\alpha v_1 + v_2)$

$= I(\alpha v_1 + v_2) - P(\alpha v_1 + v_2)$

$= \alpha(m_1+n_1) + (m_2+n_2) - P(\alpha(m_1+n_1) + (m_2+n_2))$

$\{v_i = m_i + n_i;\ m_i \in M,\ n_i \in N\}$

$= \alpha m_1 + \alpha n_1 + m_2 + n_2 - \alpha P(m_1) - \alpha P(n_1) - P(m_2) - P(n_2)$

$= \alpha m_1 + \cancel{\alpha n_1} + m_2 + \cancel{n_2} - \cancel{\alpha n_1} - \cancel{n_2}$

$$\left\{\begin{aligned} &\text{where, } P(m_i)=0;\ \text{Kernel}(P)=M \\ &\text{and, } P(n_i) = P(n_i) + 0 \\ &\qquad = P(n_i) + P(m_i) \\ &\qquad = P(n_i + m_i) \\ &\qquad = P(v_i) \\ &\qquad = n_i \end{aligned}\right\}$$

$= \alpha m_1 + m_2$

$= \alpha(I-P)(v_1) + (I-P)(v_2)$ ------ $\underline{(V)}$

$\therefore$ From $\underline{(V)}$, $I-P$ is Linear.

From $\underline{(iv)}$, $I-P$ is Idempotent.

From $\underline{(iii)}$, Kernel $(I-P) = N$.

From $\underline{(ii)}$, Range $(I-P) = M$.

Hence, $\boxed{\text{I-P is the projection of V along N onto M.}}$ (Proved)

⑥. Given, an empty $n \times n$ matrix A.
Alice wins if the determinant of the final matrix is non-zero, while Bob wins if it is Zero.

Let's assume that Alice starts first, and both of them fill the matrix entries alternatively one at a time.

(i) 'n' is even:-
Let Alice places a number "x" in one of the entries in the k-th row. Then Bob can place a number "-x" somewhere else in the same k-th row. This can go on until the whole matrix is filled up.

Now, since 'n' is even, the last entry will always be filled by Bob. So, by using this above strategy, Bob will ensure that all the row-sums of the final matrix will be Zero always. Therefore, the Determinant will always be Zero.

On the other hand, Alice can't have any strategy when 'n' is even.

∴ When 'n' is Even, Bob will have the winning strategy described above.

(ii) 'n' is Odd:-
We will try to solve this using Leibniz Formula:-

$$\det(A) = \sum_{k \in S_n} \text{sign}(k) \prod_{i=1}^{n} a_{i,k(i)}$$

(Next Page) →

Here,
No. of Matrix entries $= n \times n = n^2$
Each entry is denoted by $a_{i,j}$ where $1 \leq i,j \leq n$.
S_n = Set of all permutations of the matrix indices.
No. of permutations in $S_n = n!$

For example,
Let $n=3$.
$\therefore S_n = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$

Sign(k) is the sign function which returns +1 if the permutation 'k' is even, and -1 if it is odd.

Now, a permutation is Odd when we swap two elements once.
For example,
Let $n=3$ and we initially have (1,2,3).
Now swapping $2 \leftrightarrow 3$, we get (1,3,2).
So, (1,3,2) is an Odd permutation. Sign() function will give -1 for this.
Similarly, a permutation is Even when we swap any two elements in the initial permutation twice.
For example,
Let $n=3$ and we initially have (1,2,3).
(i) Now swapping $2 \leftrightarrow 3$, we get (1,3,2).
(ii) Again swapping $1 \leftrightarrow 3$, we finally get (3,1,2).
So, (3,1,2) is an Even permutation. Sign() function will give +1 for this.

∴ Using this formula, for $n=3$, we get:-

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

(P.T.O.)

Now coming back to the question,

(i) For Bob to win, he has to make each of these $n!$ terms Zero to get $\det(A) = 0$.

(ii) For Alice to win, she has to make atleast one term out of these $n!$ terms as Non-Zero.

As we can see using Leibniz formula, each entry a_{ij} appears in exactly $(n-1)!$ terms of the summation.

For example, when $n=3$:-

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Let's take $i=j=1$. So, a_{11} appears in exactly two terms of $\det(A)$. $\{(3-1)! = 2! = 2\}$

So, if Bob makes any $a_{ij} = 0$, he is actually making $(n-1)!$ terms of the summation as Zero.

Now, since Alice starts first, she will get $\left(\frac{n^2+1}{2}\right)$ chances to fill the matrix and Bob will get $\left(\frac{n^2-1}{2}\right)$ chances.

Let's again take the same example of $n=3$:-

Alice's chances $= \frac{3^2+1}{2} = 5$

Bob's chances $= \frac{3^2-1}{2} = 4$

No. of terms in summation $= 3! = 6$.

(Next Page) →

* For Alice to win, she has to make sure that she accesses all the elements of any particular term and make them Non-Zero.

For example:- If she is able to fill a_{11}, a_{22} & a_{33}, then she can set the term $(a_{11}a_{22}a_{33})$ as Non-Zero ~~making~~ which makes $\det(A) \neq 0$.

So, when 'n' is Odd, Alice should follow the above mentioned strategy to win.

* For Bob to win, he has to make sure that he directly or indirectly accesses atleast one element of each of the summation terms and make them Zero. For $n=3$, Bob has 4 chances. By accessing a single element & setting it to zero, he can actually set $(3-1)! = 2! = 2$ terms as Zero.

Since, there are 6 terms in the summation and Bob has 4 chances, he can make each and every term Zero in just 3 chances.

So, when 'n' is Odd, Bob should follow the above mentioned strategy to win.

—X—

7. (a) To prove :- A linear transformation between vector spaces has a left inverse iff it is injective.

Proof :-
Let the linear transformation be $T: U \to V$.
Now, T is ~~line~~ left-invertible if there exists a linear transformation $L: V \to U$ such that :
$L(T(u)) = u \;\; \forall \; u \in U$.
Here, L is called the left-inverse of T.

(i) Let L exists for T. We will prove if T is injective :-
Let $a, b \in U$ such that $T(a) = T(b)$.
So, $a = L(T(a))$
$= L(T(b))$
$= b$

$\therefore$ For $\boxed{T(a) = T(b)}$, we get $\boxed{a = b}$.
$\boxed{\text{T is Injective}}$ ---- (i)

(ii) Let T is Injective. We will prove if L exists for T or not :-
Let $\{u_i : i \in \mathbb{N}\}$ be a basis for U. Then, $\alpha = \{T(u_i) : i \in \mathbb{N}\}$ is a linearly-Independent subset of V.
So, there exists a basis $\mathcal{B}$ of V such that :-
$$\alpha \subset \mathcal{B}$$

Now, Let $L: V \to U$ be the linear transformation defined on $\mathcal{B}$ by :-
$$L(v) = \begin{cases} w \,; & \text{if } v \in \alpha \text{ with } v = T(w) \\ 0 \,; & \text{if } v \notin \alpha \end{cases} \quad \text{---} \; (1)$$

Then for $u = \sum \lambda_i u_i \in U$, we have :-
$$\begin{aligned} L(T(u)) &= L(T(\textstyle\sum \lambda_i u_i)) \\ &= \textstyle\sum \lambda_i L(T(u_i)) \\ &= \textstyle\sum \lambda_i u_i \qquad \{\text{From } (1)\} \\ &= u \end{aligned}$$
$\therefore$ $\boxed{\text{L is the left-inverse of T}}$ --- (ii)

So, from the above two proofs (i) & (ii), we can say that :-

$\boxed{\text{A linear transformation T has a left-inverse L iff T is Injective.}}$

(Proved)

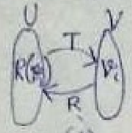
(b) To prove :- A linear transformation between vector spaces has a right inverse iff it is surjective.

Proof :-
Let the linear transformation be $T: U \to V$.
Now, T is right-invertible if there exists a Linear Transformation $R: V \to U$ such that :
$T(R(v)) = v \;\; \forall \; v \in V$.
Here, R is called the right-inverse of T.

(i) Let R exists for T. We will prove if T is Surjective :-
~~Since R exists, we get :~~
~~$R(v_i) \in U \;\; \forall \, v_i \in V$.~~

(P.T.O.)

Since R exists for T, we get $R(v_i) \in U \ \forall v_i \in V$ such that :- $\boxed{T(R(v_i)) = v_i}$



$\therefore \boxed{T \text{ is Surjective}}$ ----- (I)

(ii) Let T is Surjective. We will prove if R exists for T or not :-

Let $\alpha = \{v_i : i \in \mathbb{N}\}$ be a basis for V.

Now, T is Surjective. So, there exists $\{x_i : i \in \mathbb{N}\}$ $\forall x_i \in U$ such that :- $T(x_i) = v_i$ (Surjective).

Now, let $a_1, a_2, ---, a_n \in \mathbb{R}$ i.e scalars such that :-

$$a_1 x_1 + a_2 x_2 + --- + a_n x_n = 0$$

or, $T(a_1 x_1 + a_2 x_2 + --- + a_n x_n) = 0$

or, $a_1 T(x_1) + a_2 T(x_2) + --- + a_n T(x_n) = 0$

or, $a_1 v_1 + a_2 v_2 + --- + a_n v_n = 0$

Since, $\{v_i : i \in \mathbb{N}\} = \alpha$ = Basis for V. So,

$$a_1 = a_2 = --- = a_n = 0$$

$\therefore$ $\{x_i : i \in \mathbb{N}\}$ ~~are~~ ~~Indep~~ is a Linearly-Independent ~~subset~~ subset of U.

Now, let $R : V \to U$ be a linear transformation defined as follows :-

$$R(v) = \begin{cases} w \, ; \text{ if } v \in \alpha \text{ with } v = T(w) \\ 0 \, ; \text{ if } v \notin \alpha \end{cases} \quad ---- (1)$$

Then for $v = \sum \lambda_i v_i$, we have :-

$$\begin{aligned} T(R(v)) &= T(R(\textstyle\sum \lambda_i v_i)) \qquad \forall v_i \in V \\ &= \textstyle\sum \lambda_i T(R(v_i)) \\ &= \textstyle\sum \lambda_i v_i \qquad \{\text{From } (1)\} \\ &= v \end{aligned}$$

$\therefore \boxed{R \text{ is the right-inverse of } T}$ --- (ii)

So, from the above two proofs (i) & (ii), we can say that :-

$\boxed{\text{A linear transformation } T \text{ has a right-inverse } R \text{ iff } T \text{ is Surjective.}}$

(Proved)

—X—