1. Given, S is a linearly independent subset of a vector space V.
Also, β is a vector in V such that $\beta \notin \text{span}(S)$

Adjoining β to S :-
new set, $S_{new} = S \cup \{\beta\}$
Let $S = \{u_1, u_2, \text{------}, u_n\}$

Case-I :- Co-efficient of β is ZERO :-

Given S is linearly independent.
$\therefore \alpha_1 u_1 + \alpha_2 u_2 + \text{---} + \alpha_n u_n = 0$ if and only if $\alpha_1 = \alpha_2 = \text{---} = \alpha_n = 0$ $\{\alpha_i \Rightarrow \text{scalars}\}$ i.e. $\alpha_i \in F$ where $1 \leq i \leq n$

Now calculating for S_{new},
$\alpha_1 u_1 + \alpha_2 u_2 + \text{---} + \alpha_n u_n + \alpha_{n+1} \beta = 0$ $\{\alpha_{n+1} \in F\}$
We know that $\alpha_1 = \alpha_2 = \text{---} = \alpha_n = 0$
So, we can say that S_{new} is also linearly independent iff $\boxed{\alpha_{n+1} = 0}$ (Trivial Case)

Case-II :- Co-efficient of β is NOT zero :-

We will prove this by contradiction.
Given, $\beta \notin \text{span}(S)$
Let S_{new} be linearly dependent & $\alpha_{n+1} \neq 0$.
$\therefore \alpha_1 u_1 + \alpha_2 u_2 + \text{---} + \alpha_n u_n + \alpha_{n+1} \beta = 0$

or, $\beta = \left(\frac{-\alpha_1}{\alpha_{n+1}}\right) u_1 + \left(\frac{-\alpha_2}{\alpha_{n+1}}\right) u_2 + \text{---} + \left(\frac{-\alpha_n}{\alpha_{n+1}}\right) u_n$

So, β can be written as a linear combination of vectors present in S.

$\therefore$ This is a contradiction to the fact that $\beta \notin \text{span}(S)$.
So, by contradiction we can ~~say~~ now say that S_{new} is actually Linearly Independent

Hence, by case-I and case-II, we have proved that :-

$\boxed{S_{new} \text{ is also Linearly Independent.}}$

—X—

(2). Given :- S is a subspace of a Vector Space V.

To prove :- $span(S) = S$

Proof :-

Span(s) is the set of all finite linear combinations of vectors present in S.

So, for any vector $v \in S$, it will ~~ast~~ also satisfy $v \in span(S)$.

$$\therefore \quad S \subseteq Span(S) \quad \text{----- (i)}$$

Now, let vector $u \in span(s)$.

So,

$$u = \alpha_1 s_1 + \alpha_2 s_2 + \text{-----} + \alpha_m s_m$$

where, scalar $\alpha_i \in F$,

$s_i \in S$,

and ~~for~~ $m \in \mathbb{N}$

Since S is a subspace, so it satisfies the closure property under vector addition and scalar multiplication.

So, $u \in S$.

$\therefore$ We can say that, $Span(S) \subseteq S$ --- (ii)

From (i) and (ii),

$$\boxed{Span(S) = S}$$

———X———

3 (a) $\begin{bmatrix} 12 & 4 & 4 \\ 5 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 12 & 4 & 4 \\ 5 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\therefore$ Nullspace i.e. $N(A) = X$ such that $AX = 0$

Now, reducing the matrix A (echelon form):-

$$\begin{bmatrix} 12 & 4 & 4 \\ 5 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2 - \left(\frac{5}{12}\right)R_1} \begin{bmatrix} 12 & 4 & 4 \\ 0 & 7/3 & 10/3 \\ 2 & 3 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - \left(\frac{1}{6}\right)R_1} \begin{bmatrix} 12 & 4 & 4 \\ 0 & 7/3 & 10/3 \\ 0 & 7/3 & 10/3 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 12 & 4 & 4 \\ 0 & 7/3 & 10/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, $AX = 0$

i.e. $\begin{bmatrix} 12 & 4 & 4 \\ 0 & 7/3 & 10/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

So, $12x_1 + 4x_2 + 4x_3 = 0$

or, $3x_1 + x_2 + x_3 = 0$ ---- (i)

$7/3 x_2 + 10/3 x_3 = 0$

or, $7x_2 + 10x_3 = 0$ ---- (ii)

Let $x_3 = \alpha$. ---- (iii)

$\therefore$ From (ii) & (iii),

$x_2 = -\frac{10}{7}\alpha$ ---- (iv)

From (i), (iii) & (iv),

$$x_1 = -\frac{x_2}{3} - \frac{x_3}{3}$$

$$= \frac{10}{21}\alpha - \frac{\alpha}{3}$$

$$= \frac{\alpha}{7} \quad \text{---- (v)}$$

$\therefore$ Nullspace of A i.e. N(A):-

$X = \begin{bmatrix} \alpha/7 \\ -10\alpha/7 \\ \alpha \end{bmatrix}$ Ans

(where $\alpha = x_3$)

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\therefore$ $N(A) = X$ such that $AX = 0$

Now reducing the matrix A (echelon form):-

$$\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2 - 6R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 0 & -2 & -5 \end{bmatrix} \xrightarrow{R_3 = R_3 - \left(\frac{2}{7}\right)R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 0 & 0 & -1 \end{bmatrix}$$

(P.T.O.)

Now $AX = 0$

i.e. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore \quad x_1 + 2x_2 + 3x_3 = 0 \quad \text{---} \quad (i)$

$-7x_2 - 14x_3 = 0$

or, $x_2 + 2x_3 = 0 \quad \text{----} \quad (ii)$

$-x_3 = 0$

or, $x_3 = 0 \quad \text{----} \quad (iii)$

From (i), (ii) & (iii),

$x_1 = x_2 = x_3 = 0$

$\therefore$ Nullspace of A i.e. N(A):-

$X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Ans

(c) $\begin{bmatrix} 12 & 3 & 9 \\ 20 & 5 & 15 \\ 16 & 4 & 12 \end{bmatrix}$

Let $A = \begin{bmatrix} 12 & 3 & 9 \\ 20 & 5 & 15 \\ 16 & 4 & 12 \end{bmatrix}$ and $X' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\therefore$ $N(A) = X$ such that $AX = 0$

Now reducing the matrix A (echelon form) :-

$\begin{bmatrix} 12 & 3 & 9 \\ 20 & 5 & 15 \\ 16 & 4 & 12 \end{bmatrix} \xrightarrow{R_2 = R_2 - \left(\frac{5}{3}\right)R_1} \begin{bmatrix} 12 & 3 & 9 \\ 0 & 0 & 0 \\ 16 & 4 & 12 \end{bmatrix}$

$\xrightarrow{R_3 = R_3 - \left(\frac{4}{3}\right)R_1} \begin{bmatrix} 12 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = \left(\frac{1}{3}\right)R_1} \begin{bmatrix} 4 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \quad 4x_1 + x_2 + 3x_3 = 0 \quad \text{---} \quad (i)$

Let $x_2 = \alpha$ and $x_3 = \beta \quad \text{---} \quad (ii)$

So, $x_1 = -\frac{\alpha}{4} - \frac{3\beta}{4}$

$\therefore$ Nullspace of A i.e. N(A):-

$X = \begin{bmatrix} -\frac{\alpha}{4} - \frac{3\beta}{4} \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} -\frac{3}{4} \\ 0 \\ 1 \end{bmatrix} \beta$ Ans

(where $\alpha = x_2$ & $\beta = x_3$)

X

(4) Given, W is a subspace of V with a basis $\{\alpha_i : i \in [m]\}$ i.e. :-

$$B_W = \{\alpha_1, \alpha_2, ---, \alpha_m\}$$

and $\dim(W) = m$

Since B_W is a Linearly Independent set (by definition of Basis), so :-

$a_1\alpha_1 + a_2\alpha_2 + --- + a_m\alpha_m = 0$ iff $a_1 = a_2 = --- = a_m = 0$

where scalars $a_i \in F$,

and $1 \leq i \leq m$ ----- (i)

Now, given $\beta \in V \setminus W$.

Let $S = \{\alpha_i + \beta : i \in [m]\}$

$= \{\alpha_1 + \beta, \alpha_2 + \beta, ----, \alpha_m + \beta\}$

$\therefore$ We have to prove that :-

a) $Span(S)$ is a subspace of V.

b) $\dim(Span(S)) = m$

Proof :-

a) Since any $(\alpha_i + \beta) \in V$ as it is closed under vector addition, therefore, S is a subset of V i.e. $S \subseteq V$.

So, by definition, we know that :-

$Span(S)$ will form a subspace of V. -- (ii)

b) Checking for Linear Independence of S :-

$a_1(\alpha_1 + \beta) + a_2(\alpha_2 + \beta) + --- + a_m(\alpha_m + \beta)$

$= (a_1\alpha_1 + a_2\alpha_2 + --- + a_m\alpha_m) + (a_1 + a_2 + -- + a_m)\beta$

$= 0$ iff $(a_1 = a_2 = \cdots = a_m = 0)$ {from eq. (i)}

$\therefore$ S is a Linearly Independent Set.

So, $Span(S)$ is a subspace for which S is the basis, i.e. $Span(S)$ is the set of all finite linear combination of elements in S.

Now, S has 'm' no. of elements, and S is a basis of $Span(S)$.

$\therefore$ $\dim(Span(S)) = m$ ... (iii)

Hence, we have proved that (from (ii) & (iii)) :-

$\boxed{\text{The span of set } \{\alpha_i + \beta : i \in [m]\} \text{ i.e. } Span(S) \text{ is a } m\text{-dimensional subspace of } V.}$

—X—

(5) a) The no. of linear transformations $T: V \to V$:-

Given, F is a finite field with p^n elements where p is a prime.
V is a k-dimensional vector space over F.

Let $\{b_1, b_2, \dots, b_k\}$ be a basis of V.

$\therefore$ We can define $T: V \to V$ as :-

$$T(b_1) = \alpha_{11} b_1 + \alpha_{12} b_2 + \dots + \alpha_{1k} b_k$$
$$T(b_2) = \alpha_{21} b_1 + \alpha_{22} b_2 + \dots + \alpha_{2k} b_k$$
$$\vdots$$
$$T(b_k) = \alpha_{k1} b_1 + \alpha_{k2} b_2 + \dots + \alpha_{kk} b_k$$

So, the no. of linear transformations is the no. of different combinations of α_{ij} where $1 \le i, j \le k$.

We have p^n choices for each α_{ij} and we have total $k \times k$ no. of α_{ij}'s.

$\therefore$ Total no. of linear transformations :-

$$\Rightarrow (p^n)^{(k \times k)}$$
$$\boxed{= (p)^{nk^2}}$$ Ans

b) The no. of invertible linear transformations $T: V \to V$:-

Given F is a finite field with p^n elements where p is a prime.

V is a k-dimensional vector space over F.

We can represent the no. of invertible linear transformations in the form of $k \times k$ invertible matrices (where each of its elements $\in F$).

$$k\left\{\underbrace{\begin{bmatrix} & \\ & \end{bmatrix}}_{k}\right.$$

1st row :- It can be anything excluding the 0-vector.

$\therefore$ No. of possibilities $= (p^n)^k - 1 \quad \{1 \text{ for 0-vector}\}$
$= p^{nk} - 1$

2nd row :- It can be anything excluding the multiples of 1st row.

$\therefore$ No. of possibilities $= (p^n)^k - p^n = p^{nk} - p^n$

3rd row :- It can be anything excluding the multiples of 1st and 2nd rows.

$\therefore$ No. of possibilities $= (p^n)^k - (p^n \times p^n) = p^{nk} - p^{2n}$

$\vdots$

k^{th} row :- It can be anything excluding the multiples of the first $(k-1)$ rows.

$\therefore$ No. of possibilities $= p^{nk} - p^{(k-1)n}$

$\therefore$ Total no. of linear transformations :-

$$\Rightarrow (p^{nk} - 1)(p^{nk} - p^n)(p^{nk} - p^{2n}) \dots (p^{nk} - p^{(k-1)n})$$
$$\boxed{= (p^{nk} - 1) \times \prod_{i=1}^{k-1} (p^{nk} - p^{in})}$$ Ans

—X—