(1) a) Using Gaussian Elimination :-

$$x + 3y + 5z = 14$$
$$2x - y - 3z = 3$$
$$4x + 5y - z = 7$$

Augmented Matrix :- $\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right]$

Converting it into Reduced Row Echelon Form :-

(i) $R_2 = R_2 - 2R_1$:- $\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 4 & 5 & -1 & 7 \end{array}\right]$

(ii) $R_3 = R_3 - 4R_1$:- $\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right]$

(iii) $R_3 = R_3 - R_2$:- $\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & 0 & -8 & -24 \end{array}\right]$

∴ The equations are :-

(i) $-8z = -24 \Rightarrow \boxed{z = 3}$

(ii) $-7y - 13z = -25$

or, ~~-21~~ $-7y - 39 = -25$

or, $\boxed{y = -2}$

(iii) $x + 3y + 5z = 14$

or, $x + 3(-2) + 5(3) = 14$

or, $\boxed{x = 5}$

∴ $\boxed{x = 5,\ y = -2,\ z = 3}$ Ans

b) Using Gauss-Jordan Elimination :-

$$y + z = 4$$
$$3x + 6y - 3z = 3$$
$$-2x - 3y + 7z = 10$$

Augmented Matrix :- $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 7 & 10 \end{array}\right]$

Converting it into Reduced Row Echelon Form :-

(i) $R_2 \leftrightarrow R_1$:- $\left[\begin{array}{ccc|c} 3 & 6 & -3 & 3 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right]$

(ii) $R_1 = \frac{R_1}{3}$:- (Normalization) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right]$

(iii) $R_3 = R_3 - (-2)R_1$:- (Reduction) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right]$

(iv) $R_2 = R_2$:- (Normalization) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right]$

(v) $R_1 = R_1 - 2R_2$, $R_3 = R_3 - R_2$:- (Reduction) $\left[\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right]$

(P.T.O.)

(vi) $R_3 = \frac{R_3}{4}$:- (Normalization)

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

(vii) $R_1 = R_1 - (-3)R_3$:-
$R_2 = R_2 - R_3$
(Reduction)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\therefore$ The equations are :-

$$\boxed{x = -1,\ y = 2,\ z = 2}$$ Ans

X

(2). 1) Absolutely summable sequences ((x_i) such that $\sum_{i=1}^{\infty} |x_i| < \infty$, i.e the sum is ~~infi~~ finite):-

(i) Taking the Zero Vector $\{0, 0, \ldots\}$,
$$\sum_{i=1}^{\infty} 0 = 0$$
$\therefore$ $\boxed{\text{Zero Vector is Absolutely Summable}}$

(ii) Let S = Set of Absolutely Summable Sequences.
Let $X, Y, Z \in S$ such that :-
$$\sum_{i=1}^{\infty} |x_i| < s_x \quad \forall x_i \in X ;$$
$$\sum_{i=1}^{\infty} |y_i| < s_y \quad \forall y_i \in Y$$
Now, let $Z = X + Y$
$$\therefore Z = (x_1, x_2, \ldots) + (y_1, y_2, \ldots)$$
$$= (x_1 + y_1, x_2 + y_2, \ldots)$$
$$\therefore \sum_{i=1}^{\infty} |z_i| = |x_1 + y_1| + |x_2 + y_2| + \ldots \quad (\text{where all } z_i \in Z)$$
$$\leq |x_1| + |y_1| + |x_2| + |y_2| + \ldots \quad (|p+q| \leq |p| + |q| \ \forall p, q \in \mathbb{R})$$
$$\leq |x_1| + |x_2| + \ldots + |y_1| + |y_2| + \ldots$$
$$\leq s_x + s_y$$
$\therefore$ Z is absolutely summable.
$\therefore$ $\boxed{\text{S is closed under Vector Addition}}$

(iii) Let S = Set of Absolutely Summable Sequences.
Let $X \in S$ such that :-
$$\sum_{i=1}^{\infty} |x_i| < s_x \quad \forall x_i \in X$$

(Next Page →)

$\therefore kX = (kx_1, kx_2, kx_3, \ldots) \quad \forall k \in \mathbb{R}$ (i.e. k is a scalar)
$$\text{So, } \sum_{i=1}^{\infty} |kx_i| = |kx_1| + |kx_2| + \ldots$$
$$\leq |k||x_1| + |k||x_2| + \ldots \quad (|p.q| \leq |p|.|q| \ \forall p, q \in \mathbb{R})$$
$$\leq |k| (|x_1| + |x_2| + \ldots)$$
$$\leq |k| s_x$$
$\therefore$ kX is absolutely summable
$\therefore$ $\boxed{\text{S is closed under Scalar Multiplication}}$

$\therefore$ $\boxed{\text{S forms a subspace of V}}$

---

2) Bounded Sequences ((x_i) such that $\exists M > 0$ such that $|x_i| \leq M \forall i$, i.e the sum is bounded):-

(i) The Zero Vector $Z = (0, 0, \ldots)$;
$$z_i \leq 1 \ \forall z_i \in Z \text{ and } M = 1$$
$\therefore$ $\boxed{\text{Zero Vector is Bounded}}$

(ii) Let B = Set of Bounded Sequences.
Let $X, Y \in B$ such that :-
$$|x_i| \leq M_x \ \forall x_i \in X ; \ (i \in \{1, 2, 3, \ldots\})$$
$$|y_i| \leq M_y \ \forall y_i \in Y$$
Now, let $C \in B$ such that :-
$$C = X + Y$$
$$= (x_1 + y_1, x_2 + y_2, \ldots)$$
$$\therefore c_i = x_i + y_i \quad (\forall c_i \in C, \forall x_i \in X, \forall y_i \in Y)$$
or, $|c_i| = |x_i + y_i|$
or, $|c_i| \leq |x_i| + |y_i| \quad (|p+q| \leq |p| + |q| \ \forall p, q \in \mathbb{R})$

(P.T.O.)

or, $|c_i| \leq M_x + M_y$

$\therefore$ C is a Bounded Sequence.

$\therefore$ $\boxed{\text{B is closed under Vector Addition.}}$

(iii) Let B = Set of Bounded Sequences.

Let $X \in B$ such that $|x_i| \leq M_x \; \forall x_i \in X$.

So, $kX = (kx_1, kx_2, ---) \; \forall k \in \mathbb{R}$ (i.e. k is a scalar)

$\therefore \quad |kx_i| \leq |k||x_i| \quad \forall kx_i \in kX$
$\leq M'$

$\therefore$ X is a Bounded Sequence.

Hence, $\boxed{\text{B is closed under Scalar Multiplication}}$

$\therefore$ $\boxed{\text{B forms a subspace of V.}}$

3) Arithmetic Sequences $((x_i)$ such that $x_i = a + di$ for some fixed a and d$)$:-

(i) If $a = 0$ and $d = 0$; then all $x_i = 0$.

$\therefore$ $\boxed{\text{Zero Vector is an Arithmetic Sequence.}}$

(ii) Let A = Set of all Arithmetic Sequences.

Let $X, Y \in A$ such that:- $(i \in \{1, 2, 3, ---\})$

$x_i = a + di \quad \forall x_i \in X$
$y_i = b + ei \quad \forall y_i \in Y$ {a, d, b, e are fixed}

So, $X + Y = (a+d, a+2d, ---) + (b+e, b+2e, ---)$
$= (a+b+d+e, a+2d+b+2e, ----)$
$= ((a+b)+(d+e), (a+b)+2(d+e), ---)$

$\therefore$ X+Y is also an Arithmetic Sequence.

$\therefore$ $\boxed{\text{A is closed under Vector Addition}}$ (Common Difference $= (d+e)$)

(iii) Let A = Set of all Arithmetic Sequences.

Let $X \in A$ such that:-

$x_i = a + di \; \forall x_i \in X$ {a, d are fixed}

So, $kX = (ka + kd, ka + k(2)d, ka + k(3)d, ---)$
$= (ka + kd, ka + 2kd, ka + 3kd, ----)$

$\therefore$ kX is also an Arithmetic Sequence (Common Difference = kd)

$\therefore$ $\boxed{\text{A is closed under Scalar Multiplication}}$

$\therefore$ $\boxed{\text{A forms a subspace of V}}$

4) Geometric Sequences $((x_i)$ such that $x_i = ar^i$ for some fixed a and r$)$:-

Directly checking for the Vector Addition property,

Let G = Set of all Geometric Subsequences.

Let $A, B \in G$ (Taking $i \in \{0, 1, 2, ---\}$)

$A = (1, 1, 1, ----)$ {where $a = 1, r = 1$}

$B = (2, 4, 8, ----)$ {where $a = 2, r = 2$}

$\therefore \; A + B = (3, 5, 9, ----)$

As we can see that the common ratio is not equal i.e. $\frac{9}{5} \neq \frac{5}{3}$.

$\therefore$ $\boxed{\text{G is NOT a subspace of V}}$

— X —

(3) Let V = Set of all continuous real valued functions on the domain $[0,1] \subset \mathbb{R}$

$= \mathbb{R}[0,1]$

$= \{f : [0,1] \to \mathbb{R} ; f \text{ is continuous}\}$

We will prove all the properties of a Vector space to show if V is a Vector Space or not.

(i) $(f+g)(x) = f(x) + g(x)$ $\{\forall f,g \in V ; x \in [0,1]\}$

$= g(x) + f(x)$

$= (g+f)x$

$\therefore \boxed{f+g = g+f}$ (Commutative)

(ii) $((f+g)+h)(x) = (f+g)(x) + h(x)$ $\{\forall f,g,h \in V ; x \in [0,1]\}$

$= (f(x) + g(x)) + h(x)$

$= f(x) + (g(x) + h(x))$

$= f(x) + (g+h)(x)$

$= (f + (g+h))(x)$

$\therefore \boxed{(f+g)+h = f+(g+h)}$ (Associative)

(iii) Let 0 denotes the constant function with value 0;

$\therefore (f+0)x = f(x) + 0 = f(x)$ $\{\forall f \in V ; x \in [0,1]\}$

$\therefore \boxed{f+0 = f}$ (Additive Identity)

(iv) $(-f)(x) = -(f(x))$ $\{\forall f, -f \in V ; x \in [0,1]\}$

(Next Page →)

$\therefore (f+(-f))(x) = f(x) - f(x) = 0$

$\therefore \boxed{f+(-f) = 0}$ (Additive Inverse)

(V) $\alpha(f+g)(x)$ $\{f,g \in V ; x \in [0,1] ; \alpha \in \mathbb{R}\}$

$= \alpha(f(x) + g(x))$

$= \alpha f(x) + \alpha g(x)$

$= (\alpha f + \alpha g)(x)$ (First Distributive Law)

(vi) $((\alpha+\beta)g)(x)$ $\{g \in V ; x \in [0,1] ; \alpha,\beta \in \mathbb{R}\}$

$= (\alpha+\beta)g(x)$

$= \alpha g(x) + \beta g(x)$

$= (\alpha g + \beta g)(x)$ (Second Distributive Law)

(vii) $((\alpha\beta)f)(x)$ $\{\forall f \in V ; x \in [0,1] ; \alpha,\beta \in \mathbb{R}\}$

$= (\alpha\beta)f(x)$

$= \alpha(\beta f(x))$

$= (\alpha(\beta f))(x)$ (Scalar Associative Law)

$\therefore \boxed{(\alpha\beta)f = \alpha(\beta f)}$

(viii) $(1f)(x) = 1f(x) = f(x)$ $\{f \in V ; x \in [0,1]\}$

$\therefore \boxed{1f = f}$ (Monoid law)

Since, all the properties are satisfied,

$\therefore$ **V forms a Vector Space over $\mathbb{R}$.**

—×—

4. Given, three vectors f, g, h as $f(x) = x$, $g(x) = e^x$, $h(x) = e^{-x}$; $x \in [0, 1]$ from $\mathbb{R}[0, 1]$.

Let us assume that f, g and h are linearly dependent.

$\therefore \alpha_1 x + \alpha_2 e^x + \alpha_3 e^{-x} = 0$ for some $\alpha_i \; (i \in \{1,2,3\})$ where not all α_i are Zeroes.

Since $x \in [0, 1]$,

(i) For $x = 0$, $\alpha_2 + \alpha_3 = 0$

or, $\alpha_2 = -\alpha_3$ ------ (a)

(ii) For $x = 1$, $\alpha_1 + \alpha_2 e + \alpha_3 e^{-1} = 0$ ---- (b)

(iii) For $x = 0.5$, $0.5\alpha_1 + \alpha_2 \sqrt{e} + \frac{\alpha_3}{\sqrt{e}} = 0$ --- (c)

From (a) and (b),

$$\alpha_1 + \alpha_2 e + \frac{(-\alpha_2)}{e} = 0$$

or, $\alpha_1 = -\alpha_2 \left(e - \frac{1}{e}\right)$ ---- (d)

From (a) and (c),

$$\frac{\alpha_1}{2} + \alpha_2 \sqrt{e} - \frac{\alpha_2}{\sqrt{e}} = 0$$

or, $\frac{\alpha_1}{2} + \alpha_2 \left(\sqrt{e} - \frac{1}{\sqrt{e}}\right) = 0$ ---- (e)

From (d) and (e),

$$-\frac{\alpha_2}{2}\left(e - \frac{1}{e}\right) + \alpha_2 \left(\sqrt{e} - \frac{1}{\sqrt{e}}\right) = 0$$

or, $\alpha_2 \left(\sqrt{e} - \frac{1}{\sqrt{e}} - \frac{e}{2} + \frac{1}{2e}\right) = 0$

or, $\boxed{\alpha_2 = 0}$

Substituting this to (d) & ~~(e)~~ (a), we get :-

$$\boxed{\alpha_1 = \alpha_2 = \alpha_3 = 0}$$

$\therefore$ $\boxed{\text{$f, g$ and h are Linearly Independent.}}$

—X—