1. Given set, $F = \{0, 1\}$
Given operations :- $+ \rightarrow \oplus$ (XOR)
$\cdot \rightarrow \cdot$ (AND)

a) Closure Property :-

$$0 \oplus 1 = 1 \in F$$
$$0 \cdot 1 = 0 \in F$$

$\therefore$ Set F satisfies the Closure property with the operations XOR and AND.

b) Commutative Property :-

(i) $0 \oplus 1 = 1 \oplus 0 = 1$
(ii) $0 \cdot 1 = 1 \cdot 0 = 0$

$\therefore$ Set F satisfies this property with XOR and AND operations.

c) Associative Property :-

(i) $0 \oplus (1 \oplus 0) = (0 \oplus 1) \oplus 0 = 1$
(ii) $0 \cdot (1 \cdot 0) = (0 \cdot 1) \cdot 0 = 0$

{NOTE :- We can try this for other combinations too.}

$\therefore$ Set F also follows this property with XOR and AND operations.

d) Identity Element :-

(i) For AND operation :-

$$0 \cdot \underline{1} = 0 = \underline{1} \cdot 0$$
$$1 \cdot \underline{1} = 1 = \underline{1} \cdot 1$$

So, here 1 is the Identity element.

(ii) For XOR operation :-

$$0 \oplus 1 = 1 = 1 \oplus \underline{0}$$
$$\underline{0} \oplus \underline{0} = 0 = 0 \oplus \underline{0}$$

So, here 0 is the Identity element.

e) Inverse Element :-

(i) For AND operation :-
Identity Element = 1
So,
$$0 \cdot 1 = 1 = 1 \cdot 0$$
$$1 \cdot 0 = 1 = 0 \cdot 1$$

$\therefore$ Inverse of $\underline{0}$ is 1 and Inverse of $\underline{1}$ is 0

(ii) For XOR operation :-
Identity element = 0
So,
$$0 \oplus 0 = 0 = 0 \oplus 0$$
$$1 \oplus 1 = 0 = 1 \oplus 1$$

$\therefore$ Inverse of $\underline{0}$ is 0 and Inverse of $\underline{1}$ is 1

f) Distributive Property :-

(i) $0 \cdot (1 \oplus 0) = (0 \cdot 1) \oplus (0 \cdot 0) = 0$
(ii) $(0 \oplus 1) \cdot 0 = (0 \cdot 0) \oplus (1 \cdot 0) = 0$

{NOTE :- We can do this for other combinations too.}

So, this property is also satisfied.

$\therefore$ Since all the above properties are satisfied, so we can say that :

$\boxed{\text{Set F is a Field with XOR and AND operations}}$

———X———

2\. (a) $V = \mathbb{R}$ and $F = \mathbb{N}$:-

Firstly checking if $\mathbb{N}$ is a field or not.

(i) $a + b \in \mathbb{N} \quad \forall\, a, b \in \mathbb{N}$
$a.b \in \mathbb{N} \quad \forall\, a, b \in \mathbb{N}$
So, Closure Property is satisfied.

(ii) $a + b = b + a \quad \forall\, a, b \in \mathbb{N}$
$a.b = b.a \quad \forall\, a, b \in \mathbb{N}$
So, it is Commutative also.

(iii) $a + (b + c) = (a + b) + c \quad \forall\, a, b, c \in \mathbb{N}$
$a.(b.c) = (a.b).c \quad \forall\, a, b, c \in \mathbb{N}$
So, it is Associative too.

(iv) $a + 0 = a = 0 + a$ but $0 \notin \mathbb{N}$
So, the set of natural numbers $\mathbb{N}$ does NOT have an Additive Identity.

$\therefore$ $\mathbb{N}$ is NOT a field.

$\boxed{\text{So, this will NOT form a valid Vector Space}}$

(b) $V = \mathbb{Q}$ and $F = \mathbb{R}$:-

Checking if set $\mathbb{R}$ is a field or not.

(i) $\left.\begin{array}{l} a + b \in \mathbb{R} \\ a.b \in \mathbb{R} \end{array}\right\} \forall\, a, b \in \mathbb{R}$

(ii) $\left.\begin{array}{l} a + b = b + a \\ a.b = b.a \end{array}\right\} \forall\, a, b \in \mathbb{R}$

(iii) $\left.\begin{array}{l} a + (b + c) = (a + b) + c \\ a.(b.c) = (a.b).c \end{array}\right\} \forall\, a, b, c \in \mathbb{R}$

(iv) $a + 0 = a = 0 + a$ where, $0, a \in \mathbb{R}$
$\therefore$ 0 is the Additive Identity.
Also, $a.1 = a = 1.a$ where $1, a \in \mathbb{R}$
$\therefore$ 1 is the Multiplicative Identity.

(v) $a + (-a) = 0 = (-a) + a \quad \forall\, -a, a \in \mathbb{R}$
$\therefore$ $-a$ is the Additive Inverse of a.
Also, $a.\left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right).a \quad \forall\, a, \frac{1}{a} \in \mathbb{R}$
$\therefore$ $\frac{1}{a}$ is the Multiplicative Inverse of a.

(vi) $a.(b + c) = a.b + a.c \quad \forall\, a, b, c \in \mathbb{R}$

Since all the properties are satisfied, so we can say that:-
Set $\mathbb{R}$ is a field.

Now, checking if $\mathbb{Q}$ forms a vector space over $\mathbb{R}$ or not.

Let us look at the following property:-
We know that $\sqrt{2} = \mathbb{R}$
$\therefore$ For $\bar{\alpha} \in \mathbb{Q}$,
$\sqrt{2}.\bar{\alpha} \notin \mathbb{Q}$ as it is Irrational.

So, scalar multiplication property is NOT satisfied.

$\therefore$ $\boxed{\text{This will NOT form a valid Vector Space}}$

(c) $V = \mathbb{R}$ and $F = \mathbb{Q}$:-

Checking if $\mathbb{Q}$ is a field or not.

(P.T.O.)

(i) $\left.\begin{array}{l} a+b \in Q \\ a.b \in Q \end{array}\right\} \forall\, a,b \in Q$

(ii) $\left.\begin{array}{l} a+b = b+a \\ a.b = b.a \end{array}\right\} \forall\, a,b \in Q$

(iii) $\left.\begin{array}{l} a+(b+c) = (a+b)+c \\ a.(b.c) = (a.b).c \end{array}\right\} \forall\, a,b,c \in Q$

(iv) $a+0 = a = 0+a$ where, $0, a \in Q$

$\therefore$ 0 is the Additive Identity.

Also, $a.1 = a = 1.a$ where $1, a \in Q$

$\therefore$ 1 is the Multiplicative Identity.

(v) $a+(-a) = 0 = (-a)+a$ $\forall\, -a, a \in Q$

$-a$ is the Additive Inverse of a.

Also, $a.\left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right).a$ $\forall\, \frac{1}{a}, a \in Q$

$\frac{1}{a}$ is the Multiplicative Inverse of a.

(vi) $(a+b).c = a.c + b.c$ $\forall\, a,b,c \in Q$

Since all the properties are satisfied, hence

Q is a field.

Now checking if $\mathbb{R}$ forms a Vector Space over Q or not.

(i) $\bar{\alpha} + \bar{\beta} \in \mathbb{R}$ $\forall\, \bar{\alpha}, \bar{\beta} \in \mathbb{R}$

(ii) $\bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\alpha}$ $\forall\, \bar{\alpha}, \bar{\beta} \in \mathbb{R}$

(iii) $\bar{\alpha} + (\bar{\beta} + \bar{\gamma}) = (\bar{\alpha} + \bar{\beta}) + \bar{\gamma}$ $\forall\, \bar{\alpha}, \bar{\beta}, \bar{\gamma} \in \mathbb{R}$

(iv) $\bar{\alpha} + 0 = \bar{\alpha} = 0 + \bar{\alpha}$ where $0, \bar{\alpha} \in \mathbb{R}$

0 is the Additive Identity.

(v) $\bar{\alpha} + (-\bar{\alpha}) = 0 = (-\bar{\alpha}) + \bar{\alpha}$ $\forall\, \bar{\alpha}, -\bar{\alpha} \in \mathbb{R}$

$-\bar{\alpha}$ is the Additive Inverse of $\bar{\alpha}$.

(vi) $a(\bar{\alpha} + \bar{\beta}) = a.\bar{\alpha} + a.\bar{\beta}$ $\forall\, a \in Q$ and $\bar{\alpha}, \bar{\beta} \in \mathbb{R}$

(vii) $a.\bar{\alpha} \in \mathbb{R}$ $\forall\, a \in Q$ and $\bar{\alpha} \in \mathbb{R}$

(viii) $(a+b)\bar{\alpha} = a.\bar{\alpha} + b.\bar{\alpha}$ $\forall\, a,b \in Q$ and $\bar{\alpha} \in \mathbb{R}$

(ix) $(ab).\bar{\alpha} = a.(b.\bar{\alpha})$ $\forall\, a,b \in Q$ and $\bar{\alpha} \in \mathbb{R}$

(x) $1.\bar{\alpha} = \bar{\alpha}$ where unit scalar $1 \in Q$ and $\bar{\alpha} \in \mathbb{R}$

Since all the properties are satisfied,

$\therefore$ $\boxed{\text{V=}\mathbb{R}\text{ and F=Q FORMS a Vector Space}}$

(d) V=$\mathbb{R}$ and F=C :-

Let us just assume that the set of Complex Numbers C is a field for a moment.

Now, let's look at the following property :-

Let's $(a+ib) \in C$ and $\bar{\alpha} \in \mathbb{R}$

So, $(a+ib).\bar{\alpha} = a\bar{\alpha} + i(b.\bar{\alpha}) \notin \mathbb{R}$ since it is also a Complex Number.

Hence, the scalar multiplication property is NOT satisfied.

$\therefore$ $\boxed{\text{V=}\mathbb{R}\text{ and F=C will NOT form a Vector Space.}}$

—— X ——

③ a) To prove :- Field F is a vector space over itself.

Since F is a field (given), so all the properties of a field are valid over F i.e. :-

(i) $a+b \in F \ \forall a,b \in F$

$a.b \in F \ \forall a,b \in F$

(ii) $\left.\begin{array}{l} a+b = b+a \\ a.b = b.a \end{array}\right\} \forall a,b \in F$

(iii) $\left.\begin{array}{l} a+(b+c) = (a+b)+c \\ a(b.c) = (a.b).c \end{array}\right\} \forall a,b,c \in F$

(iv) $a+0 = a = 0+a$ where $0, a \in F$

0 is the Additive Identity.

$a.1 = a = 1.a$ where $1, a \in F$

1 is the Multiplicative Identity

(v) $a+(-a) = 0 = (-a)+a$ where $a, -a \in F$

$-a$ is the Additive Inverse of a

$a.\left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right).a$ where $a, \frac{1}{a} \in F$

$\frac{1}{a}$ is the Multiplicative Inverse of a.

(vi) $a.(b+c) = a.b + a.c \ \forall a,b,c \in F$.

Now, let's have a look at the properties of a Vector Space which are remaining :-

(i) $\alpha . a \in F \ \forall \alpha, a \in F$

(ii) $a.(\alpha+\beta) = a.\alpha + a.\beta \ \forall a,\alpha,\beta \in F$

(iii) $(a+b).\alpha = a\alpha + b.\alpha \ \forall a,b,\alpha \in F$

(iv) $(a.b).\alpha = a.(b.\alpha) \ \forall a,b,\alpha \in F$

(v) $1.\alpha = \alpha$ where unit scalar $1 \in F$ and $\alpha \in F$.

Since all the properties are satisfied, we can say that :-

$\boxed{\text{F is a vector space over itself}}$ //

b) To prove :- The direct sums of a field F will form a vector space V over F.

The direct sums of a field F (as we have proved F is a vector space over itself) :-

$F \oplus F \oplus F \oplus \cdots = F^n$ as per the definition of direct sum of modules.

~~Let $V = F^n$~~

Properties :-

(i) Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in F^n$,

$\therefore (a_1, \ldots, a_n) + (b_1, \ldots, b_n)$
$= (a_1+b_1, \ldots, a_n+b_n) \in F^n$

(ii) Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n), (c_1, \ldots, c_n) \in F^n$,

$\therefore (a_1, \ldots, a_n) + ((b_1, \ldots, b_n) + (c_1, \ldots, c_n))$
$= (a_1, \ldots, a_n) + (b_1+c_1, \ldots, b_n+c_n)$
$= (a_1+b_1+c_1, \ldots, a_n+b_n+c_n)$
$= ((a_1+b_1)+c_1, \ldots, (a_n+b_n)+c_n)$
$= (a_1+b_1, \ldots, a_n+b_n) + (c_1, \ldots, c_n)$
$= ((a_1, \ldots, a_n) + (b_1, \ldots, b_n)) + (c_1, \ldots, c_n)$

(iii) Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in F^n$

$\therefore (a_1, \ldots, a_n) + (b_1, \ldots, b_n)$
$= (a_1+b_1, \ldots, a_n+b_n)$
$= (b_1+a_1, \ldots, b_n+a_n)$
$= (b_1, \ldots, b_n) + (a_1, \ldots, a_n)$

(P.T.O.)

(iv) Let n-times 0 i.e. $(0,0,\ldots,0) \in F^n$ and let $(a_1, a_2, \ldots, a_n) \in F^n$.

$\therefore (0,0,\ldots,0) + (a_1, a_2, \ldots, a_n)$
$= (0+a_1, 0+a_2, \ldots, 0+a_n)$
$= (a_1, a_2, \ldots, a_n)$

$\therefore \underline{(0,0,\ldots,0)}$ is the Identity element.

(v) Let $(a_1, a_2, \ldots, a_n), (-a_1, -a_2, \ldots, -a_n) \in F^n$.

$\therefore (a_1, a_2, \ldots, a_n) + (-a_1, -a_2, \ldots, -a_n)$
$= (a_1 - a_1, a_2 - a_2, \ldots, a_n - a_n)$
$= (0, 0, \ldots, 0)$ {n-times 0}

So, $(-a_1, -a_2, \ldots, -a_n)$ is the Inverse of $(a_1, a_2, \ldots, a_n)$.

(vi) Let $c \in F$ and $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in F^n$

$\therefore c((a_1, \ldots, a_n) + (b_1, \ldots, b_n))$
$= c(a_1 + b_1, \ldots, a_n + b_n)$
$= (ca_1 + cb_1, \ldots, ca_n + cb_n)$
$= (ca_1, \ldots, ca_n) + (cb_1, \ldots, cb_n)$
$= c(a_1, \ldots, a_n) + c(b_1, \ldots, b_n)$

(vii) Let $c \in F$ and $(a_1, \ldots, a_n) \in F^n$.

$\therefore c(a_1, \ldots, a_n)$
$= (ca_1, \ldots, ca_n) \in F^n$

(viii) Let $c, d \in F$ and $(a_1, \ldots, a_n)$, ~~$(b_1, \ldots, b_n)$~~ $\in F^n$

$\therefore (c+d).(a_1, \ldots, a_n)$
$= (ca_1 + da_1, \ldots, ca_n + da_n)$
$= (ca_1, \ldots, ca_n) + (da_1, \ldots, da_n)$
$= c(a_1, \ldots, a_n) + d(a_1, \ldots, a_n)$

(ix) Let $c, d \in F$ and $(a_1, \ldots, a_n) \in F^n$.

$\therefore c.(d.(a_1, \ldots, a_n))$
$= c.(da_1, \ldots, da_n)$
$= ((cd)a_1, \ldots, (cd)a_n)$
$= (cd)(a_1, \ldots, a_n)$

(x) Let $(a_1, a_2, \ldots, a_n) \in F^n$ and 1 is the Unit scalar.

$\therefore 1.(a_1, a_2, \ldots, a_n)$
$= (1a_1, 1a_2, \ldots, 1a_n)$
$= (a_1, a_2, \ldots, a_n)$

Since all the properties are satisfied, so we can say that :-

$\boxed{\text{The direct sume of a field } F \text{ i.e. } F^n \text{ is a vector space over } F.}$

—X—

④ Given, $F = \mathbb{R}$
$V = \{(x,y);\ x,y \in \mathbb{R}\}$

Given operations:-
$(x,y) + (x_1,y_1) = (x+x_1, 0)$
$c(x,y) = (cx, 0)$

Properties:-
Let $\vec{a} = (x_1,y_1)$, $\vec{b} = (x_2,y_2)$, $\vec{c} = (x_3,y_3)$
where $(x_i,y_i) \in V$ for $i \geq 1$

(i) $\vec{a}+\vec{b} = (x_1,y_1) + (x_2,y_2)$
$= (x_1+x_2, 0) \in V$
This property holds TRUE.

(ii) Commutative Property:-
$\vec{a}+\vec{b} = (x_1,y_1) + (x_2,y_2)$
$= (x_1+x_2, 0)$
$= (x_2+x_1, 0)$
$= \vec{b}+\vec{a}$
This also holds TRUE.

(iii) Associative Property:-
$\vec{a}+(\vec{b}+\vec{c}) = (x_1,y_1) + ((x_2,y_2)+(x_3,y_3))$
$= (x_1,y_1) + (x_2+x_3, 0)$
$= (x_1+(x_2+x_3), 0)$
$= ((x_1+x_2)+x_3, 0)$
$= (x_1+x_2, 0) + (x_3,y_3)$
$= ((x_1,y_1)+(x_2,y_2)) + (x_3,y_3)$
$= (\vec{a}+\vec{b})+\vec{c}$

So, this also holds TRUE.

(iv) Additive Identity:-
Let $\vec{e} = (0,0) \in V$ for $0 \in \mathbb{R}$
$\therefore\ \vec{a}+\vec{e} = (x_1,y_1) + (0,0)$
$= (x_1+0, 0)$
$= (x_1, 0)$
$\neq \vec{a}$

So, the Identity Element does not exist for V

$\therefore$ We can say that,

$\boxed{\text{V is NOT a Vector Space}}$

X

5. Let the set of vectors $\alpha = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ be X.

(a) all α such that $a_1 \geq 0$:-

Taking $\alpha = (x, 0, \ldots, 0) \in X$ where $x \in \mathbb{R}$

So the Inverse of α :-

$$-\alpha = (-x, 0, \ldots, 0) \notin X$$

$\therefore$ $\boxed{\text{This set is NOT a subspace of } \mathbb{R}^n}$

(b) all α such that $a_1 + 3a_2 = a_3$:-

Let $\alpha = (a_1, a_2, \ldots, a_n)$ and

$\beta = (b_1, b_2, \ldots, b_n)$

Now, $\cancel{a_3 + b_3 =}$

$(a_1 + b_1) + 3(a_2 + b_2)$

$= (a_1 + 3a_2) + (b_1 + 3b_2)$

$= a_3 + b_3$

$\therefore \alpha + \beta \in X$ ----- (I)

Also, $ca_1 + 3ca_2$

$= c(a_1 + 3a_2)$

$= ca_3$ where $c \in \mathbb{R}$

$\therefore c\alpha \in X$ ------- (II)

Since, it follows both the subspace properties of addition and scalar multiplication,

$\therefore$ $\boxed{\text{This set IS a subspace of } \mathbb{R}^n}$

(c) all α such that $a_2 = a_1^2$:-

~~Given $a_2 = a_1^2$~~ Let $a_1 = 1$.

Let $\alpha = (1, 1, a_3, \ldots, a_n) \in X$ $\{as\ a_1 = 1\}$

Now, $2\alpha = (2, 2, 2a_3, \ldots, 2a_n) \notin X$

So, the scalar property of multiplication is NOT satisfied here.

$\therefore$ $\boxed{\text{This set is NOT a subspace of } \mathbb{R}^n}$

(d) all α such that $a_1 a_2 = 0$:-

Let $\alpha = (0, a_2, a_3, \ldots, a_n)$ where $a_1 = 0$

and, $\beta = (b_1, 0, b_3, \ldots, b_n)$ where $b_2 = 0$

Here: $a_2, a_3, \ldots, a_n, b_1, b_3, \ldots, b_n \in \mathbb{R}$

and; $\alpha, \beta \in X$.

Now, $\alpha + \beta$

$= (b_1, a_2, a_3 + b_3, \ldots, a_n + b_n) \notin X$

So, it does NOT satisfy the addition property.

$\therefore$ $\boxed{\text{This set is NOT a subspace of } \mathbb{R}^n}$

(e) all α such that a_2 is rational :-

Let $\alpha = (a_1, 1, a_3, \ldots, a_n) \in X$

where; $a_2 = 1$ and $a_1, a_3, \ldots, a_n \in \mathbb{R}$

Now, $\sqrt{2}\alpha = (\sqrt{2}a_1, \sqrt{2}, \sqrt{2}a_3, \ldots, \sqrt{2}a_n) \notin X$

So, it does NOT satisfy the scalar multiplication property.

$\therefore$ $\boxed{\text{This set is NOT a subspace of } \mathbb{R}^n}$

X