Assignment 5

LA Monsoon 2020, IIIT Hyderabad

November 16, 2020

Due date: November 23, 2020

General Instructions: All symbols have the usual meanings (example: F is an arbitrary field, $\mathbb{R}$ is the set of reals, $\mathbb{N}$ the set of natural numbers, and so on.) Remember to prove all your intermediate claims, starting from basic definitions and theorems used in class to show whatever is being asked. You may use any other non-trivial theorems not used in class, as long as they are well known and a part of basic linear algebra texts. It is always best to try to prove everything from definitions. Arguments should be mathematically well formed and concise.

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1. [1 point] Say H is a hermitian $n \times n$ matrix. Show that all its eigenvalues are real.
2. [2 points] Let $A, B \in \mathbb{C}^{n \times n}$ such that $AB = BA$. Let λ is an eigenvalue of A, let V_λ be the subspace of all eigenvectors corresponding to this eigenvalue. Answer the following. Prove all claims.
   (a) Show that $\exists v \in V_\lambda$ such that it is an eigenvector of B. Are the eigenvalues necessarily same?
   (b) If the spectrum of A is non-degenerate, prove that there exists a basis such that A and B are simultaneously diagonal in that basis.
3. [2 points] Let A be a matrix such that all of its eigenvalues $\lambda_i < 1$. Prove that
$$\sum_{k=0}^{\infty} A^k = (\mathbb{I} - A)^{-1}$$
4. [2 points] Suppose A is an $n \times n$ complex matrix. Show the following.
   (a) trace(A) is the sum of its eigenvalues.
   (b) determinant(A) is the product of eigenvalues.
5. [2 points] Show that if A is a diagonalizable $n \times n$ matrix, then determinant$(\exp(A)) = \exp(\text{trace}(A))$.
6. [3 points] Consider the following points on a 2D plane: $\left\{ \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$

   Find the eigenvectors of the Covariance Matrix (wiki) of the points, and plot them, label them with the corresponding eigenvalues. Also plot the points on the same graph. *Aside: Notice how the eigenvector corresponding to the larger eigenvalue is aligned in a direction such that a line can be best fit through the points. You can now try to reason why PCA and other techniques work as they do.*

7. [4 points] Let $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (a) Show that the bilinear map $\mathbb{R}^2 \to \mathbb{R} : (x, y) \to x^T A y$ gives a scalar product.

   (b) Let $\alpha : \mathbb{R}^3 \to \mathbb{R}$ be a linear functional defined as $\alpha : (x_1, x_2, x_3) \to x_1 + x_2$ and let $v_1 = (1, 1, 1)$, $v_2 = (2, 2, 0)$, $v_3 = (1, 0, 0)$ be a basis of $\mathbb{R}^3$. Using the scalar product from the previous part, find an orthonormal basis e_1, e_2, e_3 of $\mathbb{R}^3$ with $e_1 \in \text{span}(v_1), e_2 \in \ker(\alpha)$.

8. [2 points] For the given symmetric matrix $A = \begin{bmatrix} 15 & 0 & 6 \\ 0 & 15 & 3 \\ 6 & 3 & 27 \end{bmatrix}$, calculate its spectral decomposition and write it in following forms:

   (a) $A = UDU'$

   (b) $A = \sum_{i=0}^{rank} \lambda_i u_i u_i^T$, where u_i is i^{th} eigenvector (normalized) of the matrix.