1. Let λ be an eigenvalue of Hermitian matrix H, and let y be the corresponding eigen vector.

$$\therefore \quad Hy = \lambda y$$

Multiplying by $\bar{y}^T$, we obtain :-

$$\bar{y}^T(Hy) = \bar{y}^T(\lambda y)$$
$$= \lambda \bar{y}^T y$$
$$= \lambda |y| \quad \text{---- (i)}$$

Also, $\bar{y}^T(Hy) = (Hy)^T \bar{y} = y^T H^T \bar{y}$ ---- (ii)

From (i) & (ii),

$$y^T H^T \bar{y} = \lambda |y|$$

Taking the complex conjugate of this equation, we get :-

$$\bar{y}^T \bar{H}^T y = \bar{\lambda} |y| \quad \text{---- (iii)}$$

Now we also know that, $\bar{\bar{y}} = y$;

and $\overline{|y|} = |y|$ (Since $|y|$ is a real number)

$$\therefore \quad \bar{\lambda}|y| = \bar{y}^T H y \quad (\bar{H}^T = H \text{ as } H \text{ is a Hermitian matrix})$$
$$= \bar{y}^T \lambda y$$
$$= \lambda |y| \quad \text{------- (iv)}$$

Since we know that y is an eigenvector, so it is not a zero vector. Hence, the length $|y| \neq 0$.

$\therefore$ We divide eq. (iv) by $|y|$ on both sides,

(Next Page) →

We get :- $\dfrac{\bar{\lambda}|y|}{|y|} = \dfrac{\lambda|y|}{|y|}$

or, $\boxed{\bar{\lambda} = \lambda}$

$\therefore$ λ is a real number.

Now since λ is an arbitrary eigenvalue of H, we can conclude that :-

$\boxed{\text{All the eigen values of } H \text{ are real numbers.}}$

Proved

X

2. (a) Given $A, B \in C^{n\times n}$ such that $AB = BA$. λ is an eigenvalue of A and V_λ is the subspace of all eigen vectors corresponding to this eigenvalue.

Let a basis of $V_\lambda = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$

Let's take $\vec{v}_1 \in V_\lambda$, we get:-

$$A\vec{v}_1 = \lambda\vec{v}_1 \quad \text{----- (I)}$$

Since $AB = BA$, we get:-

$$AB\vec{v}_1 = BA\vec{v}_1 = B\lambda\vec{v}_1$$

$$\text{or, } AB\vec{v}_1 = \lambda B\vec{v}_1 \quad \text{---- (II)}$$

$\therefore$ From (II), we can say that $B\vec{v}_1$ is an eigenvector of A i.e.:-

$$\boxed{B\vec{v}_1 \in V_\lambda} \quad \text{---- (III)}$$

Now we can also write it as:-

$$B\vec{v}_1 = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_k\vec{v}_k \quad \text{(spanned using basis)}$$

where, $\alpha_1, \alpha_2, \dots, \alpha_k$ are scalars.

As all the vectors in the basis are linearly independent, we can take $\alpha_2 = \alpha_3 = \dots \alpha_k = 0$ to get:-

$$B\vec{v}_1 = \alpha_1\vec{v}_1$$

i.e. $\vec{v}_1$ is an eigen vector of B.

Similarly, we can show that:-

$$\boxed{\exists \vec{v} \in V_\lambda \text{ such that it is an eigen vector of } B}$$ (Proved)

But α_1 is not necessarily same as λ.

$\therefore$ The eigenvalues are NOT necessarily same. (Ans)

(Next Page)

(b) Suppose $\vec{v}$ is an eigen vector of A with eigenvalue λ.

$$\therefore A\vec{v} = \lambda\vec{v}$$

Since $AB = BA$,

$$AB\vec{v} = BA\vec{v}$$

$$\text{or, } AB\vec{v} = \lambda B\vec{v} \quad \text{---- (I)}$$

$\therefore B\vec{v}$ is also an eigen vector of A with eigen value λ.

Since A has n no. of distinct eigen values, all of them have the multiplicity 1. So, all the eigen spaces of A are One-Dimensional.

$\therefore B\vec{v}$ and $\vec{v}$ are dependent on each other.

Now let $B\vec{v} = \alpha\vec{v}$ for some scalar α.

$\therefore \vec{v}$ is an eigen vector of B with eigenvalue α.

Since A has n no. of distinct eigen values, it implies: n independent eigen vectors i.e. it is diagonalizable.

Since we have already proved that eigen vectors of A are also eigen vectors of B,

$\therefore$ There exists a basis such that A and B are simultaneously diagonal in that basis. (Proved)

—X—

(3) Given matrix A where all its eigenvalues $\lambda_i < 1$.

<u>To prove</u> :- $\sum_{k=0}^{\infty} A^k = (I-A)^{-1}$

<u>Sol^n</u> :- Let $S = \sum_{k=0}^{\infty} A^k$ ------ (a)

and, S_n = Sum of first n terms $= \sum_{k=0}^{n} A^k$

$\therefore S_n = A^0 + A^1 + \cdots + A^n = S_n I$ {I = Identity matrix}

or, $S_n I = I + A^1 + \cdots + A^n$ ---- (i)

Multiplying both sides by A to the left,

$S_n A = A + A^2 + \cdots + A^{n+1}$ ---- (ii)

From (i) & (ii),

$S_n(I-A) = I - A^{n+1}$ ------ (iii)

Now, we know that:-

$|A^{n+1}| \leq |A|^n |A| = |A|^{n+1}$

$\therefore \lim_{n\to\infty} A^{n+1} = 0$ when $|A| < 1$

(since all $\lambda_i < 1$, so $|A| < 1$ because $|A| = \lambda_1 . \lambda_2 . \cdots$)

So, $S(I-A) = \lim_{n\to\infty} S_n(I-A) = \lim_{n\to\infty}(I - A^{n+1})$ {From (iii)}

$= I - \underbrace{\lim_{n\to\infty} A^{n+1}}_{0} = I$ ---- (iv)

From (iv), we got $S(I-A) = I$ i.e. S is a left inverse of $(I-A)$. ---- (v)

Similarly, we can prove that S is a right inverse of $(I-A)$ using the same technique as we used above. ---- (vi)

(Next Page) →

~~From (v) & (vi)~~,

From (iv),

$S(I-A) = I$

Multiplying both sides by $(I-A)^{-1}$,

$S(I-A)(I-A)^{-1} = I(I-A)^{-1}$

or, $S = (I-A)^{-1}$ ---- (vii)

From (a) and (vii), we get:-

$$\boxed{\sum_{k=0}^{\infty} A^k = (I-A)^{-1}}$$ (Proved)

X

(4) Given, A is an $n \times n$ complex matrix.

(a) To prove :- Trace(A) is the sum of eigenvalues.

Soln:- Consider the minor cofactor expansion of $\det(A-\lambda I)$ which gives a sum of terms.

Each term is a product of n factors comprising one entry from each row and each column.

Now consider the minor cofactor term containing members of the diagonal $(a_{11}-\lambda)(a_{22}-\lambda)\dots(a_{nn}-\lambda)$.

The coefficient for the λ^{n-1} term will be :-

$$(-1)^n \left(\sum_{i=1}^{n}(-\lambda_i)\right) = (-1)^{n+1}\sum_{i=1}^{n}\lambda_i$$

We can observe here that this minor cofactor term is the only one that will contribute to the λ^{n-1} order terms.

$\therefore$ Trace(A) = Coefficient of the λ^{n-1} term.

We have the characteristics polynomial,

$$f(\lambda) = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_n)$$

$\therefore$ Trace(A) = The λ^{n-1} coefficient

or, $\text{Trace}(A) = (-1)^n \sum_{i=1}^{n}(-\lambda_i)$

$$= (-1)^{2n}\sum_{i=1}^{n}\lambda_i$$

or, $\boxed{\text{Trace}(A) = \sum_{i=1}^{n}\lambda_i}$ (Proved)

(b) To prove :- Det(A) is the product of eigenvalues.

Soln:- Since $(-1)^n$ is the highest order term coefficient and $\lambda_1, \lambda_2, \dots, \lambda_n$ are solutions to the characteristics polynomial f.

(Next Page) →

$\therefore f(\lambda) = (-1)^n(\lambda-\lambda_1)\dots(\lambda-\lambda_n)$

If we evaluate f at Zero, we get :-

$$f(0) = \text{Det}(A - 0I) = \text{Det}(A)$$

$\therefore \text{Det}(A) = f(0) = (-1)^n(0-\lambda_1)(0-\lambda_2)\dots(0-\lambda_n)$

$$= (-1)^{2n}\prod_{i=1}^{n}\lambda_i$$

or, $\boxed{\text{Det}(A) = \prod_{i=1}^{n}\lambda_i}$ (Proved)

— X —

(5) Given, A is a diagonalizable $n \times n$ matrix.

Let us first assume that A is a diagonal matrix, say $D = \text{diagonal}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

So, $\exp(D)$ is also diagonal matrix which is :-

$(\exp(\lambda_1), \exp(\lambda_2), \ldots, \exp(\lambda_n))$.

$\therefore \ \text{Det}(\exp(D)) = (\exp(\lambda_1))(\exp(\lambda_2)) \ldots (\exp(\lambda_n))$ {Proved in Q4}

or, $\text{Det}(\exp(D)) = \exp(\lambda_1 + \lambda_2 + \ldots + \lambda_n)$

or, $\text{Det}(\exp(D)) = \exp(\text{tr}(D))$ {Trace property proved in Q4} ---- (i)

Now we will consider A. So, let P be an invertible matrix such that $A = PDP^{-1}$

Now the exponential function for a square matrix is similar to the exponential function of a real number i.e. :-

$$\exp(A) = I + A + \left(\frac{1}{2!}\right)A^2 + \left(\frac{1}{3!}\right)A^3 + \ldots$$

where, I = Identity matrix

Now we substitute PDP^{-1} into A^2 :-

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) \\ &= PD(P^{-1}P)DP^{-1} \\ &= PD^2P^{-1} \end{aligned}$$

Similarly we reduce all the other powers of A. Therefore,

$$\exp(A) = P\left[I + D + \left(\frac{1}{2!}\right)D^2 + \left(\frac{1}{3!}\right)D^3 + \ldots\right]P^{-1}$$

(Next Page) →

$\therefore \ \exp(A) = P(\exp(D))P^{-1}$

Also, the determinant of a product of square matrices is the product of their determinants. So,

$$\begin{aligned} \text{Det}(\exp(A)) &= \text{Det}(P(\exp(D))P^{-1}) \\ &= \text{Det}(P).\text{Det}(\exp(D)).\text{Det}(P^{-1}) \\ &= \text{Det}(P).\text{Det}(P^{-1}).\text{Det}(\exp(D)) \\ &= \text{Det}(PP^{-1}).\text{Det}(\exp(D)) \\ &= \text{Det}(I).\text{Det}(\exp(D)) \\ &= \exp(\text{tr}(D)) \quad \{\text{From (i)}\} \end{aligned}$$

---- (ii)

Now, we also know that the trace of a matrix is the sum of its eigenvalues.

So, $\text{tr}(D) = \text{tr}(A)$ ... (iii)

$\therefore$ From (ii) & (iii),

$\boxed{\text{Det}(\exp(A)) = \exp(\text{tr}(A))}$ (Proved)

—X—

(6). Given points on a 2D plane :-

$$\left\{ \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

Mean :- $\mu_x = \dfrac{4-2-1}{3} = \dfrac{1}{3}$

$\mu_y = \dfrac{4-3-1}{3} = 0$

Covariance :- $\sigma_{xy} = \sum_{i=1}^{n} \dfrac{(x_i-\mu_x)(y_i-\mu_y)}{n}$

$= \dfrac{(4-\frac{1}{3})(4-0)+(-2-\frac{1}{3})(-3-0)+(-1-\frac{1}{3})(-1-0)}{3}$

$= \dfrac{23}{3}$

Variance :- $Var(x) = \sigma_x^2 = \sum_{i=1}^{n} \dfrac{(x_i-\mu_x)^2}{n}$

$= \dfrac{(4-\frac{1}{3})^2+(-2-\frac{1}{3})^2+(-1-\frac{1}{3})^2}{3}$

$= \dfrac{62}{9}$

$Var(y) = \sigma_y^2 = \sum_{i=1}^{n} \dfrac{(y_i-\mu_y)^2}{n}$

$= \dfrac{(4-0)^2+(-3-0)^2+(-1-0)^2}{3}$

$= \dfrac{26}{3}$

$\therefore$ Co-variance matrix :-

$$M = \begin{bmatrix} \frac{62}{9} & \frac{23}{3} \\ \frac{23}{3} & \frac{26}{3} \end{bmatrix} = \begin{bmatrix} 6.889 & 7.667 \\ 7.667 & 8.667 \end{bmatrix}$$

(Next Page) →

Now, $|M-\lambda I| = 0$

$$\begin{vmatrix} 6.889-\lambda & 7.667 \\ 7.667 & 8.667-\lambda \end{vmatrix} = 0$$

or, $\lambda^2 - 15.5\lambda + 0.905 = 0$

$\therefore$ $\boxed{\lambda_1 = 0.0585, \lambda_2 = 15.5}$

Calculating the eigenvectors :-

(i) $(A-\lambda_1 I)\vec{v_1} = 0$

$$\begin{bmatrix} 6.82 & 7.66 \\ 7.66 & 8.60 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

We get :- $\left. \begin{array}{l} 6.82x_1 + 7.66x_2 = 0 \\ 7.66x_1 + 8.60x_2 = 0 \end{array} \right\} \Rightarrow \begin{array}{l} x_1 = -1.12x_2 \\ x_2 = x_2 \end{array}$

$\therefore$ $\vec{v_1} = \begin{bmatrix} -1.12 \\ 1 \end{bmatrix}$

$\therefore$ The equation for the line corresponding to $\vec{v_1}$:- $\boxed{x = -1.12y}$ $(\lambda_1 = 0.0585)$

(ii) $(A-\lambda_2 I)\vec{v_2} = 0$

$$\begin{bmatrix} -8.6 & 7.66 \\ 7.66 & -6.82 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

We get :- $\left. \begin{array}{l} -8.6x_1 + 7.66x_2 = 0 \\ 7.66x_1 - 6.82x_2 = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} x_1 = 0.891x_2 \\ x_2 = x_2 \end{array} \right\}$

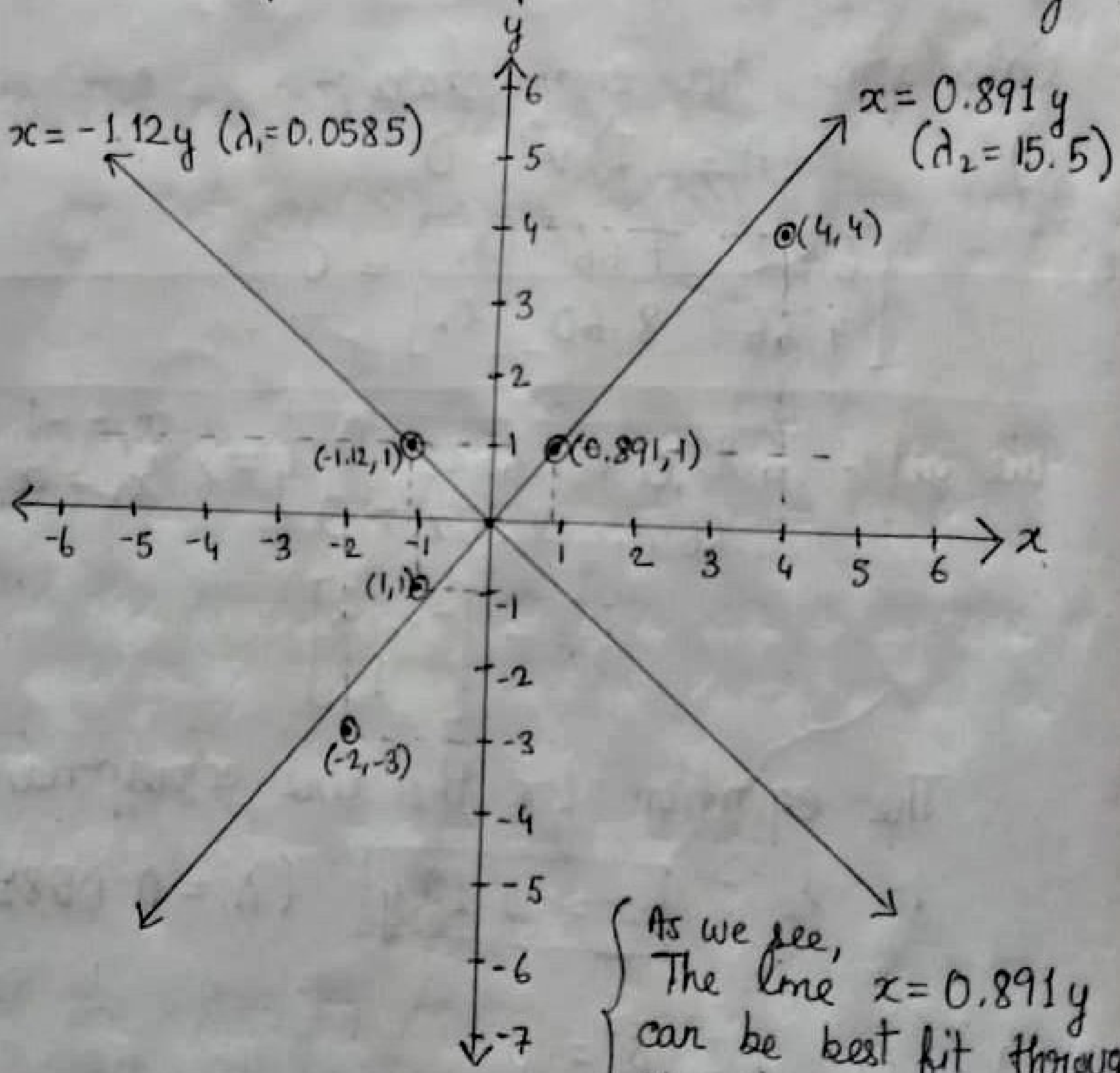
$\therefore$ The equation of the line corresponding to $\vec{v_2} = \begin{bmatrix} 0.891 \\ 1 \end{bmatrix}$ is :- $\boxed{x = 0.891y}$ $(\lambda_2 = 15.5)$

(P.T.O.)

So, the eigenvectors are :-

$$\boxed{\vec{v}_1 = \begin{bmatrix} -1.12 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0.891 \\ 1 \end{bmatrix}} \quad \text{(Ans)}$$

Now we will plot these points on the same graph :-



As we see,
The line $x = 0.891y$
can be best fit through
the given points.

——— X ———

7. (a) As we can observe, A is symmetric.

We will now verify all the properties of a scalar product.

Since dot product is commutative, we have:-

$$\begin{aligned}(x,y) &= x^T Ay = x\cdot(Ay)\\ &= (Ay)\cdot x\\ &= (Ay)^T x\\ &= y^T A^T x\\ &= y^T A x \quad \{A \text{ is symmetric}\}\\ &= (y,x)\end{aligned}$$

Thus the function (x,y) is symmetric. ---(i)

Now, for any vectors x, y, z and any real number r, we get:-

$$(rx, y) = (rx)^T Ay = rx^T Ay = r(x,y)$$

and,

$$\begin{aligned}(x+y, z) &= (x+y)^T Az\\ &= (x^T+y^T)Az\\ &= x^T Az + y^T Az\\ &= (x,y) + (y,z) \quad \text{----(ii)}\end{aligned}$$

Thus, linearity is satisfied.

If x is a non-zero vector in $\mathbb{R}^3$, then we have:- $(x,x) = x^T Ax > 0$ since A is positive definite.

We also have:- $(0,0) = 0^T A 0 = 0$

It follows that $(x,x) \geq 0$ for any vector $x \in \mathbb{R}^3$.

(Next Page) →

Since A is positive definite, it happens iff $x=0$.

Hence $(x,x) = 0$ iff $x=0$.

$\therefore$ Positive-definiteness of the function (x,y) is satisfied. ----- (iii)

From (i), (ii) and (iii),

all the properties are verified.

$$\boxed{\text{Hence, the bilinear map } R^3 \to R : (x,y) \to x^T Ay \text{ gives a scalar product.}}$$

(Proved)

(b) Linear function, $\alpha(x_1, x_2, x_3) \rightarrow x_1 + x_2$.

Representing α in matrix form,

$$\alpha : [1 \;\; 1 \;\; 0]$$

Hence, for $(x_1, x_2, x_3) \in \mathbb{R}^3$;

$$[1 \;\; 1 \;\; 0]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1 + x_2$$

Now, Kernel of α,

Since $\alpha : [1 \;\; 1 \;\; 0]_{3\times 3}$

Hence Kernel will be a 2-dimensional subspace.

i.e. $\text{Kernel}(\alpha) = k_1\begin{bmatrix}1\\-1\\0\end{bmatrix}, k_2\begin{bmatrix}0\\0\\1\end{bmatrix}$ --- (b)

Also, given $v_1 = (1,1,1)$, $v_2 = (2,2,0)$, $v_3 = (1,0,0)$

We have to find e_1, e_2, e_3 such that all are orthogonal basis of $\mathbb{R}^3$.

(P.T.O.)

$e_1 \in \text{Span}(v_1)$

$\therefore e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now, $e_2 \in \text{Kernel}(\alpha)$

$\therefore$ From (i), $e_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

To find e_3, $e_1^T e_3 = 0$ and $e_2^T e_3 = 0$.

Suppose $e_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Now, $e_1^T e_3 = 0$

$\therefore \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

or, $x + y + z = 0$ ---- (ii)

Now, $e_2^T e_3 = 0$

$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

or, $x - y = 0$ ---- (iii)

Solving (ii) & (iii),

$x = -z/2$; $y = -z/2$; $z = z$

Hence, $e_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

Therefore,

$$\boxed{e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; e_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}}$$ (Ans)

—— X ——

(8). Given, $A = \begin{bmatrix} 15 & 0 & 6 \\ 0 & 15 & 3 \\ 6 & 3 & 27 \end{bmatrix}$

(a) Calculating the spectral decomposition and writing it in the form: $A = UDU'$:-

We know, $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 15-\lambda & 0 & 6 \\ 0 & 15-\lambda & 3 \\ 6 & 3 & 27-\lambda \end{vmatrix} = 0$$

or, $(15-\lambda)((15-\lambda)(27-\lambda) - (3)(3)) + 6(-6(15-\lambda)) = 0$

or, $-\lambda^3 + 57\lambda^2 - 990\lambda + 5400 = 0$

or, $-(\lambda-12).(\lambda^2 - 45\lambda + 450) = 0$

or, $-(\lambda-12).(\lambda-15).(\lambda-30) = 0$

$\therefore \boxed{\lambda_1 = 12, \lambda_2 = 15, \lambda_3 = 30}$

(i) Eigenvector for $\lambda_1 = 12$:-

$(A - \lambda_1 I)\vec{v}_1 = 0$

or, $\begin{bmatrix} 3 & 0 & 6 \\ 0 & 3 & 3 \\ 6 & 3 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

We get:- $\left.\begin{aligned} 3x_1 + 6x_3 &= 0 \\ 3x_2 + 3x_3 &= 0 \\ 6x_1 + 3x_2 + 15x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= -2x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned}$

$\therefore \vec{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

(ii) Eigenvector for $\lambda_2 = 15$:-

$(A - \lambda_2 I)\vec{v}_2 = 0$

or, $\begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 3 \\ 6 & 3 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

We get:- $\left.\begin{aligned} 6x_3 &= 0 \\ 3x_3 &= 0 \\ 6x_1 + 3x_2 + 12x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= -\frac{1}{2}x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \end{aligned}$

$\therefore \vec{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$

(iii) Eigenvector for $\lambda_3 = 30$:-

$(A - \lambda_3 I)\vec{v}_3 = 0$

or, $\begin{bmatrix} -15 & 0 & 6 \\ 0 & -15 & 3 \\ 6 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

We get:- $\left.\begin{aligned} -15x_1 + 6x_3 &= 0 \\ -15x_2 + 3x_3 &= 0 \\ 6x_1 + 3x_2 - 3x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= \frac{2}{5}x_3 \\ x_2 &= \frac{1}{5}x_3 \\ x_3 &= x_3 \end{aligned}$

$\therefore \vec{v}_3 = \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$

Now, the diagonal matrix, $D = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 30 \end{bmatrix}$

(P.T.O.)

Let U be a matrix with eigenvectors as its columns,

$$U = \begin{bmatrix} -2 & -1/2 & 2/5 \\ -1 & 1 & 1/5 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\therefore U' = \begin{bmatrix} -1/3 & -1/6 & 1/6 \\ -2/5 & 4/5 & 0 \\ 1/3 & 1/6 & 5/6 \end{bmatrix}$$

So, in the form $A = UDU'$:-

$$\begin{bmatrix} 15 & 0 & 6 \\ 0 & 15 & 3 \\ 6 & 3 & 27 \end{bmatrix} = \begin{bmatrix} -2 & -1/2 & 2/5 \\ -1 & 1 & 1/5 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} -1/3 & -1/6 & 1/6 \\ -2/5 & 4/5 & 0 \\ 1/3 & 1/6 & 5/6 \end{bmatrix}$$

(Ans)

(b) Writing in the form $A = \sum_{i=0}^{rank} \lambda_i u_i u_i^T$:-

We will find the normalized eigenvectors.

(i) $\lambda_1 = 12$; $\vec{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

(ii) $\lambda_2 = 15$, $\vec{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$

(iii) $\lambda_3 = 30$, $\vec{v}_3 = \begin{bmatrix} 2/5 \\ 1/5 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2/\sqrt{30} \\ 1/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix}$

So, in the form $A = \sum_{i=0}^{rank} \lambda_i u_i u_i^T$:-

$$A = 12 \begin{bmatrix} -2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} + 15 \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix}$$

$$+ 30 \begin{bmatrix} 2/\sqrt{30} \\ 1/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix} \begin{bmatrix} 2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \end{bmatrix}$$

(Ans)

X