Unit-3
Transportation and Assignment Problem

Degeneracy in Transportation Problem

→ If no. of occupied cell equals $m+n-1$, then it is called non-degenerate problem.

→ If no. of occupied cell is not equal to $m+n-1$, then it is called degenerate problem.

Example:

To / From	Bhairahawa	Birgunj	Biratnagar	Supply
Kathmandu	2	5	7	60
Banepa	2	3	4	70
Nuwakot	5	8	11	80
Demand	50	80	80	210

Solution:

Here, $\sum D = 50+80+80 = 210$

$\sum S = 60+70+80 = 210$

Since, $\sum D = \sum S$. so, given problem is balanced.

Calculation of Initial solution from VAM, we get,

no. of occupied cell(4) $\neq m+n-1$ (5)

So, the problem is degenerate.

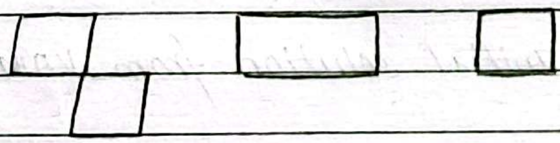
To / From	Bhairahawa	Birgunj	Biratnagar	Supply	Cost Difference I	II	III
Kathmandu	2 $\boxed{50}$	5	7 $\boxed{10}$	60→10→0	3←	2	2
Banepa	2	3	4 $\boxed{70}$	70→0	1	1	–
Nuwakot	5	8 $\boxed{80}$	11	80→0	3	3	3
Demand	50→0	80→0	80→70→0	210			
Cost Difference I	0	2	3				
Cost Difference II	–	2	3↑				
Cost Difference III	–	3	4↑				

Note: In this case allocated cell is not equal to m+n-1 so we have to assign small allocation θ(≈0). To assign θ in unallocated cell following steps should be done:

Converting the necessary no. of unallocated cell into allocated cell to satisfy the above condition.

- Starting from the least value of unallocated cell.
- Check the loop formation one by one.
- There should be no closed loop formation.
- Select that cell as the new allocated cell and assign θ.

For example:

- Open loop: 
- Closed loop: 

The table is drawn below:-

To / From	Bhairahawa	Birgunj	Biratnagar	Supply
Kathmandu	2 [50]	5	7 [10]	60
Banepa	2	3	4 [70]	70
Nuwakot	5	8 [80]	11	80
Demand	50	80	80	210

By selecting the least element (2) or $c_{21} = 2$, there is closed loop formed so we cannot assign new allocation θ. To assign the new allocation further we look next least element which is $c_{22} = 3$.

To / From	Bhairahawa	Birgunj	Biratnagar	Supply	R_i
Kathmandu	2 [50]	5	7 [10]	60	$R_1 = 0$
Banepa	2	3 [θ]	4 [70]	70	$R_2 = -3$
Nuwakot	5	8 [80]	11	80	$R_3 = 2$
Demand	50	80	80	210	
k_j	$k_1 = 2$	$k_2 = 6$	$k_3 = 7$		

Calculation:-

$R_1 = 0$ (Assumed)

For the calculation of R_i and k_j value, we have formula,

$$c_{ij} = R_i + k_j \quad [c_{ij} = \text{Allocated cell}].$$

or, $c_{11} = R_1 + k_1 \Rightarrow 2 = 0 + k_1 \Rightarrow k_1 = 2$

or, $c_{13} = R_1 + k_3 \Rightarrow 7 = 0 + k_3 \Rightarrow k_3 = 7$

or, $c_{23} = R_2 + k_3 \Rightarrow 4 = R_2 + 7 \Rightarrow R_2 = -3$

or, $c_{22} = R_2 + k_2 \Rightarrow 3 = -3 + k_2 \Rightarrow k_2 = 6$

or, $c_{32} = R_3 + k_2 \Rightarrow 8 = R_3 + 6 \Rightarrow R_3 = 2$

The improvement index for each unoccupied cell,

$$\Delta_{ij} = c_{ij} - (R_i + k_j) \quad [c_{ij} = \text{unoccupied cell}]$$

i.e. $\Delta_{12} = c_{12} - (R_1 + k_2) = 5 - (0+6) = 5 - 6 = -1$

$\Delta_{21} = c_{21} - (R_2 + k_1) = 2 - (-3+2) = 2 - (-1) = 3$

$\Delta_{31} = c_{31} - (R_3 + k_1) = 5 - (2+2) = 5 - 4 = 1$

$\Delta_{33} = c_{33} - (R_3 + k_3) = 11 - (2+7) = 11 - 9 = 2$

Optimal solution is found if all $\Delta_{ij} \geq 0$ but in our case some values are found to be negative. So, in order to obtain the optimal solution. We have to further go through loop method by selecting the most negative values from the improvement index as the starting point of loop method.

To / From	Bhairhawa	Birgunj	Biratnagar	Supply
Kathmandu	2 [50]	5 +	− 7 [10]	60
Banepa	2	3 − [θ]	+ 4 [70]	70
Nuwakot	5	8	11	80
Demand	50	80	80	210

Choose negative minimum allocation ⟹ $(\theta, 10) = \theta \approx 0$

To / From	Bhairahawa	Birgunj	Biratnagar	Supply	R_i
Kathmandu	2 [50]	5 [θ]	7 [10]	60	$R_1 = 0$
Banepa	2	3	4 [70]	70	$R_2 = -3$
Nuwakot	5	8 [80]	11	80	$R_3 = 3$
Demand	50	80	80	210	
k_j	$k_1 = 2$	$k_2 = 5$	$k_3 = 7$		

Calculation of R_i and k_j

$$C_{ij} = R_i + k_j \quad [C_{ij} = \text{Allocated cell}]$$

i.e $C_{11} = R_1 + k_1 \Rightarrow 2 = 0 + k_1 \Rightarrow k_1 = 2$

$C_{12} = R_1 + k_2 \Rightarrow 5 = 0 + k_2 \Rightarrow k_2 = 5$

$C_{13} = R_1 + k_3 \Rightarrow 7 = 0 + k_3 \Rightarrow k_3 = 7$

$C_{23} = R_2 + k_3 \Rightarrow 4 = R_2 + 7 \Rightarrow R_2 = -3$

$C_{32} = R_3 + k_2 \Rightarrow 8 = R_3 + 5 \Rightarrow R_3 = 3$

The improvement index for each unoccupied cell.

$$\Delta_{ij} = C_{ij} - (R_i + k_j)$$

i.e $\Delta_{21} = C_{21} - (R_2 + k_1) = 2 - (-3+2) = 2 - (-1) = 3$

$\Delta_{22} = C_{22} - (R_2 + k_2) = 3 - (-3+5) = 3 - 2 = 1$

$\Delta_{31} = C_{31} - (R_3 + k_1) = 5 - (3+2) = 5 - 5 = 0$

$\Delta_{33} = C_{33} - (R_3 + k_3) = 11 - (3+7) = 11 - 10 = 1$

Since all the improvement indices are zero or positive so the solution is optimal.

The minimum total transportation cost is given by

Min TC $= \sum$ cell cost X Quantity Allocated

$= C_{11} X_{11} + C_{12} X_{12} + C_{13} X_{13} + C_{23} X_{23} + C_{32} X_{32}$

$= 2 \times 50 + 5 \times 0 + 7 \times 10 + 4 \times 70 + 8 \times 80$

= Rs. 1090 Ans.

Numericals

1. A company has to transport its production from three plants to three warehouses. The unit transportation cost is given in the table given below. Determine the optimal solution of the transportation problem.

Warehouse / Plant	W_1	W_2	W_3	Supply
P_1	46	50	52	27
P_2	34	42	24	22
P_3	58	60	38	26
Demand	24	18	33	

Find the minimum cost for the transportation problem.

Solution:

Here, we have, $\Sigma D = 24+18+33 = 75$

$\Sigma S = 27+22+26 = 75$

Since, $\Sigma D = \Sigma S$. So, the given problem is balanced.

Step 1: Calculation of Initial Solution from VAM

Warehouse / Plant	W_1	W_2	W_3	Supply	Cost Difference I	II	III	IV
P_1	46 [9]	50 [18]	52	~~27~~ → ~~9~~ → 0	4	4	4	4
P_2	34 [15]	42	24 [7]	~~22~~ → ~~15~~ → 0	10	10	8	-
P_3	58	60	38 [26]	~~26~~ → 0	22←	-	-	-
Demand	~~24~~ → ~~9~~ → 0	~~18~~ → 0	~~33~~ → ~~7~~ → 0	75				
Cost Difference I	12	8	14					
II	12	8	28↑					
III	12↑	8	-					
IV	46	50↑	-					

Minimum Total Cost = ΣCell cost × Quantity Allocated

$= C_{11}X_{11} + C_{12}X_{12} + C_{21}X_{21} + C_{23}X_{23} + C_{33}X_{33}$

$= 46 \times 9 + 50 \times 18 + 34 \times 15 + 24 \times 7 + 38 \times 26$

= Rs 2980

Step 2: Testing of optimality by MODI method.

Plant \ Warehouse	W_1	W_2	W_3	Supply	R_i
P_1	46 [9]	50 [18]	52	27	$R_1 = 0$
P_2	34 [15]	42	24 [7]	22	$R_2 = -12$
P_3	58	60	38 [26]	26	$R_3 = 2$
Demand	24	18	33	75	
k_j	$k_1 = 46$	$k_2 = 50$	$k_3 = 36$		

Since, no. of occupied cells = 5 is equal to $m+n-1 = 3+3-1 = 5$, so, the solution is non-degenerate.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (k_j) can be calculated using the relation:

$C_{ij} = R_i + k_j$ [C_{ij} = Allocated cell].

i.e $C_{11} = R_1 + k_1 \Rightarrow 46 = 0 + k_1 \Rightarrow k_1 = 46$

$C_{12} = R_1 + k_2 \Rightarrow 50 = 0 + k_2 \Rightarrow k_2 = 50$

$C_{21} = R_2 + k_1 \Rightarrow 34 = R_2 + 46 \Rightarrow R_2 = -12$

$C_{23} = R_2 + k_3 \Rightarrow 24 = -12 + k_3 \Rightarrow k_3 = 36$

$C_{33} = R_3 + k_3 \Rightarrow 38 = R_3 + 36 \Rightarrow R_3 = 2$

The improvement index for each unoccupied cells can be calculated by using the relation

$\Delta_{ij} = C_{ij} - (R_i + k_j)$

$\Delta_{13} = C_{13} - (R_1 + k_3) = 52 - (0+36) = 16$

$\Delta_{22} = C_{22} - (R_2 + k_2) = 42 - (-12+50) = 4$

$\Delta_{31} = C_{31} - (R_3 + k_1) = 58 - (2+46) = 10$

$\Delta_{32} = C_{32} - (R_3 + k_2) = 60 - (2+50) = 8$

Since, all the improvement indices are positive, so the solution is optimal. The minimum total cost is given by

Min Total Cost = Σ Cell cost X Quantity Allocated

$= 46 \times 9 + 50 \times 18 + 34 \times 15 + 24 \times 7 + 38 \times 26$

$=$ Rs. 2980 Ans.

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2. A company has factories A, B, C and D which supply to warehouses X, Y and Z. The information related to supply from factories, requirement to warehouses and unit transportation cost from various factories to various warehouse are given below:

To / From	X	Y	Z	Supply
A	40	20	50	10
B	45	35	45	80
C	30	25	40	15
D	20	25	40	40
Demand	75	20	50	145

Determine the optimal distribution for this company to minimize the total transportation cost.

Solution:

Here, we have, $\Sigma D = 75 + 20 + 50 = 145$

$\Sigma S = 10 + 80 + 15 + 40 = 145$

Since, $\Sigma D = \Sigma S$. So, the given problem is balanced.

Step 1: Calculation of Initial solution from VAM

From \ To	X	Y	Z	Supply	Cost Difference I	II	III	IV	V
A	40	20 [10]	50	~~10~~→0	20	–	–	–	–
B	45 [20]	35 [10]	45 [50]	~~80~~→30→10→0	10	10	10	10	10
C	30 [15]	25	40	~~15~~→0	5	5	5	–	–
D	20 [40]	25	40	~~40~~→0	5	5	–	–	–
Demand	~~75~~→35→20→0	20→10→0	50→0	145					
Cost Difference I	10	5	0						
II	10	0	5						
III	15 ↑	10	5						
IV	45	35	45						
V	45	35	–						

Minimum Total Cost = Σ Cell Cost X Quantity Allocated

$= C_{12}X_{12} + C_{21}X_{21} + C_{22}X_{22} + C_{23}X_{23} + C_{31}X_{31} + C_{41}X_{41}$

$= 20 \times 10 + 45 \times 20 + 35 \times 10 + 45 \times 50 + 30 \times 15 + 20 \times 40$

$=$ Rs. 4950

Step 2: Testing of optimality by MODI Method.

From \ To	X	Y	Z	Supply	R_i
A	40	20 [10]	50	10	$R_1 = -15$
B	45 [20]	35 [10]	45 [50]	80	$R_2 = 0$
C	30 [15]	25	40	15	$R_3 = -15$
D	20 [40]	25	40	40	$R_4 =$
Demand	75	20	50	145	
k_j	$k_1 = 45$	$k_2 = 35$	$k_3 = 45$		

no. of occupied cells = 6

or, $m+n-1 = 6$

or, $4+3-1=6$
or, $7-1=6$
or, $6=6$

Since, no. of occupied cells $= m+n-1$ so, the solution is non-degenerate.

To calculate the row value and column value, let us assume $R_2 = 0$ for reference. Then, the remaining row value (R_i) and column value (k_j) can be calculated using the relation;

$$C_{ij} = R_i + k_j \quad [\because C_{ij} = \text{Allocated cell}].$$

i.e $C_{12} = R_1 + k_2 \Rightarrow 20 = R_1 + 35 \Rightarrow R_1 = -15$
$C_{21} = R_2 + k_1 \Rightarrow 45 = 0 + k_1 \Rightarrow k_1 = 45$
$C_{22} = R_2 + k_2 \Rightarrow 35 = 0 + k_2 \Rightarrow k_2 = 35$
$C_{23} = R_2 + k_3 \Rightarrow 45 = 0 + k_3 \Rightarrow k_3 = 45$
$C_{31} = R_3 + k_1 \Rightarrow 30 = R_3 + 45 \Rightarrow R_3 = -15$
$C_{41} = R_4 + k_1 \Rightarrow 20 = R_4 + 45 \Rightarrow R_4 = -25$

The improvement index for each unoccupied cells can be calculated by using the relation:

$$\Delta_{ij} = C_{ij} - (R_i + k_j) \quad [C_{ij} = \text{Unallocated cell}]$$

i.e $\Delta_{11} = C_{11} - (R_1 + k_1) = 40 - (-15+45) = 10$
$\Delta_{13} = C_{13} - (R_1 + k_3) = 50 - (-15+45) = 20$
$\Delta_{32} = C_{32} - (R_3 + k_2) = 25 - (-15+35) = 5$
$\Delta_{33} = C_{33} - (R_3 + k_3) = 40 - (-15+45) = 10$
$\Delta_{42} = C_{42} - (R_4 + k_2) = 25 - (-25+35) = 15$
$\Delta_{43} = C_{43} - (R_4 + k_3) = 40 - (-25+45) = 20$

Since, all the improvement indices are positive, so the solution is optimal. The minimum total cost is given by

Min Total Cost $= C_{12}X_{12} + C_{21}X_{21} + C_{22}X_{22} + C_{23}X_{23} + C_{31}X_{31} + C_{41}X_{41}$
$= 20\times10 + 45\times20 + 35\times10 + 45\times50 + 30\times15 + 20\times40$
$=$ Rs. 4950 Ans//

3. 'Nepal Transport' own several trucks used to deliver crushed stones to road projects in the region. The company has received the delivery schedule for next week as follows:

Project	Requirement per week	Plant	Available per week
A	50	W	55
B	75	X	60
C	50	Y	60

The cost matrix for above schedule is given as follows:

Plant	Cost information ('000 Rs.)		
	Project A	Project B	Project C
W	4	8	3
X	6	7	9
Y	8	2	5

Find an optimum solution to minimize the transportation cost.

Solution:

Let us merge the above table in 1 table, we get.

Plant	Project A	Project B	Project C	Supply
W	4	8	3	55
X	6	7	9	60
Y	8	2	5	60
Demand	50	75	50	175

Here, $\Sigma S = 55+60+60 = 175$

$\Sigma D = 50+75+50 = 175$

Since, $\Sigma D = \Sigma S$. So, the given problem is balanced.

Calculation of Initial solution from VAM

Plant		Project A	Project B	Project C	Supply	I	II	III
W		4 [5]	8	3 [50]	55→5→0	1	1	4←
X		6 [45]	7 [15]	9	60→45→0	1	1	1
Y		8	2 [60]	5	60→0	3	–	–
Demand		50→45→0	75→15→0	50→0				
Cost Difference	I	2	5↑	2				
	II	2	1	6↑				
	III	2	1	–				

Minimum Total Cost = ΣCell cost × Quantity Allocated

$= 4\times5 + 3\times50 + 6\times45 + 7\times15 + 2\times60$

= Rs. 665

Step 2: Testing of optimality by MODI method

Plant	Project A	Project B	Project C	Supply	R_i
W	4 [5]	8	3 [50]	55	$R_1 = 0$
X	6 [45]	7 [15]	9	60	$R_2 = 2$
Y	8	2 [60]	5	60	$R_3 = -3$
Demand	50	75	50		
k_j	$k_1 = 4$	$k_2 = 5$	$k_3 = 3$		

no. of occupied cells = 5

or, $m+n-1 = 5$

or, $3+3-1 = 5$

or, $6-1 = 5$

$\therefore 5 = 5$

Since, no. of occupied cells = $m+n-1$, so the solution is non-degenerate.

To calculate the row value and column value, let us assume

$R_1 = 0$ for reference. Then, the remaining row value (R_i) and column value (k_j) can be calculated using the relation;

$$c_{ij} = R_i + k_j \quad [c_{ij} = \text{Allocated cell}].$$

i.e. $c_{11} = R_1 + k_1 \Rightarrow 4 = 0 + k_1 \Rightarrow k_1 = 4$

$c_{13} = R_1 + k_3 \Rightarrow 3 = 0 + k_3 \Rightarrow k_3 = 3$

$c_{21} = R_2 + k_1 \Rightarrow 6 = R_2 + 4 \Rightarrow R_2 = 2$

$c_{22} = R_2 + k_2 \Rightarrow 7 = 2 + k_2 \Rightarrow k_2 = 5$

$c_{32} = R_3 + k_2 \Rightarrow 2 = R_3 + 5 \Rightarrow R_3 = -3$

The improvement index for each unoccupied cells can be calculated by using the relation:

$$\Delta_{ij} = c_{ij} - (R_i + k_j) \quad [c_{ij} = \text{Unallocated cell}]$$

i.e. $\Delta_{12} = c_{12} - (R_1 + k_2) = 8 - (0 + 5) = 8 - 5 = 3$

$\Delta_{23} = c_{23} - (R_2 + k_3) = 9 - (2 + 3) = 9 - 5 = 4$

$\Delta_{31} = c_{31} - (R_3 + k_1) = 8 - (-3 + 4) = 8 - 1 = 7$

$\Delta_{33} = c_{33} - (R_3 + k_3) = 5 - (-3 + 3) = 5 - 0 = 5$

Since, all the improvement indices are positive, so the solution is optimal.

The minimum total cost is given by:

Min. Total cost $= c_{11}x_{11} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{32}x_{32}$

$= 4 \times 5 + 3 \times 50 + 6 \times 45 + 7 \times 15 + 2 \times 60$

$=$ Rs 665 Ans.

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4. Obtain the minimum transportation cost for the following transportation problem.

Source / Determination	S_1	S_2	S_3	Units demanded
W_1	90	100	100	5
W_2	100	140	80	20
W_3	130	100	80	20
Units available	20	15	10	45

Solution:

Here, $\Sigma S = 20+15+10 = 45$

$\Sigma D = 5+20+20 = 45$

Since, $\Sigma D = \Sigma S$. So, the given problem is balanced.

Step 1: Calculation of initial solution from VAM.

Destination \ Source	S_1	S_2	S_3	Units Demanded	Cost Difference I	II	III
W_1	90 [5]	100	100	~~5~~ → 0	10	10	10
W_2	100 [10]	140	80 [10]	~~20~~ → ~~10~~ → 0	20	40 ←	–
W_3	130 [5]	100 [15]	80	~~20~~ → ~~15~~ → 0	20	30	30
Units Available	~~20~~ → ~~10~~ → ~~5~~ → 0	~~15~~ → 0	~~10~~ → 0	45			
Cost Difference I	10	0	0				
II	10	0	–				
III	40 ↑	0	–				

Minimum Total Cost $= \Sigma$ cell cost × Quantity Allocated

$= 90 \times 5 + 100 \times 10 + 80 \times 10 + 130 \times 5 + 100 \times 15$

$= Rs.\ 4400$

Step 2: Testing of optimality by MODI Method.

Destination \ Source	S_1	S_2	S_3	Units Demanded	R_i
W_1	90 [5]	100	100	5	$R_1 = 90$
W_2	+ 100 [10]	140	− 80 [10]	20	$R_2 = 100$
W_3	− 130 [5]	100 [15]	+ 80	20	$R_3 = 130$
Units Available	20	15	10	45	
k_j	$k_1 = 0$	$k_2 = -30$	$k_3 = -20$		

no. of occupied cells $= 5$

or, $m+n-1 = 5$

or, $3+3-1 = 5$

or, $6-1 = 5$

or, $5 = 5$

Since, no. of occupied cells $= m+n-1$, so the solution is non degenerate.

To calculate the row value and column value, let us assume $k_1 = 0$ for reference. Then, the remaining row value (R_i) and column value (k_j) can be calculated using the relation:

$$C_{ij} = R_i + k_j \quad [C_{ij} = \text{Allocated cell}]$$

i.e $C_{11} = R_1 + k_1 \Rightarrow 90 = R_1 + 0 \Rightarrow R_1 = 90$

$C_{21} = R_2 + k_1 \Rightarrow 100 = R_2 + 0 \Rightarrow R_2 = 100$

$C_{31} = R_3 + k_1 \Rightarrow 130 = R_3 + 0 \Rightarrow R_3 = 130$

$C_{32} = R_3 + k_2 \Rightarrow 100 = 130 + k_2 \Rightarrow k_2 = -30$

$C_{23} = R_2 + k_3 \Rightarrow 80 = 100 + k_3 \Rightarrow k_3 = -20$

The improvement index for each unoccupied cells can be calculated by using the relation:

$$\Delta_{ij} = C_{ij} - (R_i + k_j) \quad [C_{ij} = \text{unallocated cell}]$$

i.e $\Delta_{12} = C_{12} - (R_1 + k_2) = 100 - (90 - 30) = 40$

$\Delta_{13} = C_{13} - (R_1 + k_3) = 100 - (90 - 20) = 30$

$\Delta_{22} = C_{22} - (R_2 + k_2) = 140 - (100 - 30) = 30$

$\Delta_{33} = C_{33} - (R_3 + k_3) = 80 - (130 - 20) = -30$

Since, the improvement index $\Delta_{33} = -30 < 0$, the solution is not optimal. So, we should go to loop by selecting the most negative value.

Min $(10, 5) = 5$.

We should add min 5 to positive corner in loop and subtract 5 to negative corner in loop.

Source / Destination	S_1	S_2	S_3	Units Demanded	R_i
W_1	90 $\boxed{5}$	100	100	5	$R_1 = 90$
W_2	100 $\boxed{15}$	140	80 $\boxed{5}$	20	$R_2 = 100$
W_3	130	100 $\boxed{15}$	80 $\boxed{5}$	20	$R_3 = 100$
Units Available	20	15	10	45	
k_j	$k_1 = 0$	$k_2 = 0$	$k_3 = -20$		

no. of occupied cells = 5

or, $m+n-1 = 5$

or, $3+3-1 = 5$

or, $6-1 = 5$

$\therefore 5 = 5$

So, the problem isn't degenerate

So here, allocated cell is not equal to $m+n-1$ so we have to assign small allocation $\theta (\approx 0)$ at the least unoccupied cell where open loop is formed.

Source / Destination	S_1	S_2	S_3	Units Demanded	R_i
W_1	90 $\boxed{5}$	100	100	5	$R_1 = 90$
W_2	100 $\boxed{15}$	140	80 $\boxed{5}$	20	$R_2 = 100$
W_3	130	100 $\boxed{20}$	80 $\boxed{\theta}$	20	$R_3 = 100$
Units Available	20	15	10	45	
k_j	$k_1 = 0$	$k_2 = 0$	$k_3 = -20$		

no. of occupied cells = 5

or, $m+n-1 = 5$

or, $3+3-1 = 5$

or, $6-1 = 5$

$\therefore 5 = 5$

So the solution is non degenerate.

13. A company has four factories manufacturing the same commodity, which are required to be transpolted to meet the demands in four warehouses. The supplies, demands and the cost per transportation from factory to warehouse in rupees per unit of product are given in the following table.

Factory	Warehouse				Supply
	X	Y	Z	W	
A	25	55	40	60	60
B	35	30	50	40	140
C	36	45	26	66	150
D	35	30	41	50	50
Demand	90	100	120	140	450 / 400

a. Determine an optimal strategy of the transportation of goods from factories to warehouse and assess the optimal cost.

b. If a new transporter agrees to transport goods from factory C to warehouse W at a unit cost of Rs. 50, analyze the impact of this on your current optimal solution.

Solution:

Here, $\Sigma S = 60+140+150+50 = 400$

$\Sigma D = 90+100+120+140 = 450$

Since, $\Sigma S \neq \Sigma D$ so the problem is not balanced. so we have to add dummy row to balance this unbalanced problem.

Factory	Warehouse				Supply
	X	Y	Z	W	
A	25	55	40	60	60
B	35	30	50	40	140
C	36	45	26	66	150
D	35	30	41	50	50
E	0	0	0	0	50
Demand	90	100	120	140	450

Now, $\Sigma D = \Sigma S = 450$ so it is balanced problem.

Step 1: Calculation of Initial Solution using VAM.

Factory	Warehouse X	Y	Z	W	Supply	Cost Difference I	II	III	IV	V
A	25 [60]	55	40	60	~~60~~→0	15	15 ←	–	–	–
B	35	30 [50]	50	40 [90]	~~140~~→50→0	5	5	5	5	5
C	36 [30]	45	26 [120]	66	~~150~~→30→0	10	10	10	9	9 ←
D	35	30 [50]	41	50	~~50~~→0	5	5	5	5	5
E	0	0	0	0 [50]	~~50~~→0	0	–	–	–	–
Demand	~~90~~→30→0	~~100~~→50→0	~~120~~→0	~~140~~→90→0	450					
Cost Difference I	25	30	26	40 ↑						
II	10	0	14	10						
III	0	0	15 ↑	10						
IV	0	0	–	10 ↑						
V	0	0	–	–						

no. of occupied cell = 7

or, $m+n-1 = 7$

or, $5+4-1 = 7$

or, $8 \neq 7$

Min TC $= 25\times60 + 30\times50 + 40\times90 + 36\times30 + 26\times120 + 30\times50$

$=$ Rs. 12300

So, the problem is degenerate so we have to assign small allocation $\theta(\approx 0)$ in the least unoccupied cell where open loop is formed.

Factory \ Warehouse	X	Y	Z	W	Supply
A	25 [60]	55	40	60	60
B	35	30 [50]	50	40 [90]	140
C	36 [30]	45	26 [120]	66	150
D	35	30 [50]	41	50	50
E	0	0	0 [θ]	0 [50]	50
Demand	90	100	120	140	450

no. of occupied cells = 8

or, $m+n-1 = 8$

or, $5+4-1 = 8$

or, $9-1 = 8$

$\therefore$ $8 = 8$

so, the problem is non degenerate.

Step 2: Test of optimality by MODI Method.

Factory \ Warehouse	X	Y	Z	W	Supply	R_i
A	25 $\boxed{60}$	55	40	60	60	$R_1=25$
B	35	30 $\boxed{50}$	50	40 $\boxed{90}$	140	$R_2=50$
C	36 $\boxed{30}$	45	26 $\boxed{120}$	66	150	$R_3=36$
D	35	30 $\boxed{50}$	41	50	50	$R_4=50$
E	0	0	0 $\boxed{\theta}$	0 $\boxed{50}$	50	$R_5=10$
Demand	90	100	120	140	450	
k_j	$k_1=0$	$k_2=-20$	$k_3=-10$	$k_4=-10$		

Calculation of row value and column value :

$$C_{ij} = R_i + k_j$$

$C_{11} = R_1 + k_1 \Rightarrow 25 = R_1 + 0 \Rightarrow R_1 = 25$

$C_{31} = R_3 + k_1 \Rightarrow 36 = R_3 + 0 \Rightarrow R_3 = 36$

$C_{33} = R_3 + k_3 \Rightarrow 26 = 36 + k_3 \Rightarrow k_3 = -10$

$C_{53} = R_5 + k_3 \Rightarrow 0 = R_5 - 10 \Rightarrow R_5 = 10$

$C_{54} = R_5 + k_4 \Rightarrow 0 = 10 + k_4 \Rightarrow k_4 = -10$

$C_{24} = R_2 + k_4 \Rightarrow 40 = R_2 - 10 \Rightarrow R_2 = 50$

$C_{22} = R_2 + k_2 \Rightarrow 30 = 50 + k_2 \Rightarrow k_2 = -20$

$C_{42} = R_4 + k_2 \Rightarrow 30 = R_4 - 20 \Rightarrow R_4 = 50$

The Improvement indices for all unoccupied cells are calculated by the relation:

$$\Delta_{ij} = C_{ij} - (R_i + K_j)$$

i.e $\Delta_{12} = C_{12} - (R_1 + K_2) = 55 - (25 - 20) = 55 - 5 = 50$

$\Delta_{13} = C_{13} - (R_1 + K_3) = 40 - (25 - 10) = 40 - 15 = 25$

$\Delta_{14} = C_{14} - (R_1 + K_4) = 60 - (25 - 10) = 60 - 15 = 45$

$\Delta_{21} = C_{21} - (R_2 + K_1) = 35 - (50 + 0) = 35 - 50 = -15$

$\Delta_{23} = C_{23} - (R_2 + K_3) = 50 - (50 - 10) = 50 - 40 = 10$

$\Delta_{32} = C_{32} - (R_3 + K_2) = 45 - (36 - 20) = 45 - 16 = 29$

$\Delta_{34} = C_{34} - (R_3 + K_4) = 66 - (36 - 10) = 66 - 26 = 40$

$\Delta_{41} = C_{41} - (R_4 + K_1) = 35 - (50 + 0) = 35 - 50 = -15$

$\Delta_{43} = C_{43} - (R_4 + K_3) = 41 - (50 - 10) = 41 - 40 = 1$

$\Delta_{44} = C_{44} - (R_4 + K_4) = 50 - (50 - 10) = 50 - 40 = 10$

$\Delta_{51} = C_{51} - (R_5 + K_1) = 0 - (10 + 0) = -10$

$\Delta_{52} = C_{52} - (R_5 + K_2) = 0 - (10 - 20) = 10$

Here, the improvement index $\Delta_{21} = -15$, $\Delta_{41} = -15$ and $\Delta_{51} = -10$ are less than zero so we should select the most negative value from improvement index i.e -15 and make a loop.

Factory \ Warehouse	X	Y	Z	W	Supply	R_i
A	25 $\boxed{60}$	55	40	60	60	R_1:
B	35 +	30 $\boxed{50}$	50	40 − $\boxed{90}$	140	R_2:
C	− 36 $\boxed{30}$	45	26 + $\boxed{120}$	66	150	R_3:
D	35	30 $\boxed{50}$	41	50	50	R_4:
E	0	0	− 0 $\boxed{\theta}$	+ 0 $\boxed{50}$	50	R_5:
Demand	90	100	120	140	450	
K_j	K_1:	K_2:	K_3:			

Minimum negative allocation: Min $(30, 90, \theta) = \theta$

Factory \ Warehouse	X	Y	Z	W	Supply	Ri
A	25 [60]	55	40	60	60	$R_1 = 25$
B	35 [θ]	30 [50]	50	40 [90]	140	$R_2 = 35$
C	36 [30]	45	26 [120]	66	150	$R_3 = 36$
D	35	30 [50]	41	50	50	$R_4 = 35$
E	0	0	0	0 [50]	50	$R_5 = -5$
Demand	90	100	120	140	450	
kj	$k_1 = 0$	$k_2 = -5$	$k_3 = -10$	$k_4 = 5$		

no. of occupied cells = 8

or, $m+n-1 = 8$

or, $5+4-1 = 8$

$\therefore$ $8 = 8$

so, the problem is non-degenerate.

Calculation of row value and column value,

$$C_{ij} = R_i + k_j$$

i.e $C_{11} = R_1 + k_1 \Rightarrow 25 = R_1 + 0 \Rightarrow R_1 = 25$

$C_{21} = R_2 + k_1 \Rightarrow 35 = R_2 + 0 \Rightarrow R_2 = 35$

$C_{31} = R_3 + k_1 \Rightarrow 36 = R_3 + 0 \Rightarrow R_3 = 36$

$C_{22} = R_2 + k_2 \Rightarrow 30 = 35 + k_2 \Rightarrow k_2 = -5$

$C_{24} = R_2 + k_4 \Rightarrow 40 = 35 + k_4 \Rightarrow k_4 = 5$

$C_{33} = R_3 + k_3 \Rightarrow 26 = 36 + k_3 \Rightarrow k_3 = -10$

$C_{42} = R_4 + k_2 \Rightarrow 30 = R_4 - 5 \Rightarrow R_4 = 35$

$C_{54} = R_5 + k_4 \Rightarrow 0 = R_5 + 5 \Rightarrow R_5 = -5$

The improvement indices can be calculated by:

$$\Delta_{ij} = C_{ij} - (R_i + k_j)$$

i.e $\Delta_{12} = C_{12} - (R_1 + k_2) = 55 - (25-5) = 55 - 20 = 35$

$\Delta_{13} = C_{13} - (R_1 + k_3) = 40 - (25-10) = 40 - 15 = 25$

$\Delta_{14} = C_{14} - (R_1 + k_4) = 60 - (25+5) = 60 - 30 = 30$

$\Delta_{23} = C_{23} - (R_2 + k_3) = 50 - (35 - 10) = 50 - 25 = 25$

$\Delta_{32} = C_{32} - (R_3 + k_2) = 45 - (36 - 5) = 45 - 31 = 14$

$\Delta_{34} = C_{34} - (R_3 + k_4) = 66 - (36 + 5) = 66 - 41 = 25$

$\Delta_{41} = C_{41} - (R_4 + k_1) = 35 - (35 + 0) = 35 - 35 = 0$

$\Delta_{43} = C_{43} - (R_4 + k_3) = 41 - (35 - 10) = 41 - 25 = 16$

$\Delta_{44} = C_{44} - (R_4 + k_4) = 50 - (35 + 5) = 50 - 40 = 10$

$\Delta_{51} = C_{51} - (R_5 + k_1) = 0 - (-5 + 0) = 5$

$\Delta_{52} = C_{52} - (R_5 + k_2) = 0 - (-5 - 5) = 10$

$\Delta_{53} = C_{53} - (R_5 + k_3) = 0 - (-5 - 10) = 15$

Since, for all $\Delta_{ij} \geq 0$ so the optimal solution is obtained.

a). Minimum Total Cost (TC) = Σ Unit cell cost X Allocated Quantity

$= C_{11}X_{11} + C_{21}X_{21} + C_{22}X_{22} + C_{24}X_{24} + C_{31}X_{31} + C_{33}X_{33} + C_{42}X_{42} + C_{54}X_{54}$

$= 25 \times 60 + 35 \times 0 + 30 \times 50 + 40 \times 90 + 36 \times 30 + 26 \times 120 + 30 \times 50 + 0 \times 50$

$= Rs.\ 12300$

b.

Maximization of Transportation Problem.

14. From the following profit matrix, find the maximum profit by using transportation model.

Supply & Demand.

Plant	Unit Available	Project	Unit Demanded
A	1700	W	1300
B	2500	X	2000
C	1000	Y	1900

Profit in '000' rupees

From	To Project W	To Project X	To Project Y
Plant A	120	80	50
Plant B	110	150	100
Plant C	20	170	60

Solution:

Here, we have, $\Sigma D = 1300 + 2000 + 1900 = 5200$

$\Sigma S = 1700 + 2500 + 1000 = 5200$

Since, $\Sigma D = \Sigma S$, so given problem is balanced.

The given profit table is:

From \ To	W	X	Y	Supply
A	120	80	50	1700
B	110	150	100	2500
C	20	170	60	1000
Demand	1300	2000	1900	5200

Since, it is the case of profit maximization, we should construct opportunity loss table. It can be obtained by subtracting each element from the highest element 170.

We get the loss table as:

From \ To	W	X	Y	Supply
A	50	90	120	1700
B	60	20	70	2500
C	150	0	110	1000
Demand	1300	2000	1900	5200

Step 1: Calculation of Initial Solution using VAM.

From \ To	W	X	Y	Suppy	I	II	III
A	50 [1300]	90	120 [400]	~~1700~~ > ~~400~~ > 0	40	40	70
B	60	20 [1000]	70 [1500]	~~2500~~ > ~~1500~~	40	40	10
C	150	0 [1000]	110	1000 > 0	110	–	–
Demand	~~1300~~ > 0	~~2000~~ > ~~1000~~ > 0	~~1900~~ > ~~1500~~	5200			
I	10	20	40				
II	10	70 ↑	50				
III	10	–	50				

Step 2: Testing of Optimality by MODI Method.

From \ To	W	X	Y	Supply	R_i
A	50 [1300]	90	120 [400]	1700	$R_1 = 0$
B	60	20 [1000]	70 [1500]	2500	$R_2 = -50$
C	150	0 [1000]	110	1000	$R_3 = -70$
Demand	1300	2000	1900	5200	
k_j -	$k_1 = 50$	$k_2 = 70$	$k_3 = 120$		

no. of occupied cells = 5

or, $m+n-1 = 5$

or, $3+3-1 = 5$

$\therefore 5 = 5$

so, the solution is non-degenerate.

Calculation of row value and column value.

$$C_{ij} = R_i + k_j$$

i.e. $C_{11} = R_1 + k_1 \Rightarrow 50 = 0 + k_1 \Rightarrow k_1 = 50$

$C_{13} = R_1 + k_3 \Rightarrow 120 = 0 + k_3 \Rightarrow k_3 = 120$

$C_{23} = R_2 + k_3 \Rightarrow 70 = R_2 + 120 \Rightarrow R_2 = -50$

$C_{22} = R_2 + k_2 \Rightarrow 20 = -50 + k_2 \Rightarrow k_2 = 70$

$C_{32} = R_3 + k_2 \Rightarrow 0 = R_3 + 70 \Rightarrow R_3 = -70$

The improvement indices for all unoccupied cells are given by:

$$\Delta_{ij} = C_{ij} - (R_i + k_j)$$

i.e. $\Delta_{12} = C_{12} - (R_1 + k_2) = 90 - (0 + 70) = 90 - 70 = 20$

$\Delta_{21} = C_{21} - (R_2 + k_1) = 60 - (-50 + 50) = 60 - 0 = 60$

$\Delta_{31} = C_{31} - (R_3 + k_1) = 150 - (-70 + 50) = 150 - (-20) = 170$

$\Delta_{33} = C_{33} - (R_3 + k_3) = 110 - (-70 + 120) = 110 - 50 = 60$

Since, all improvement indices is greater than zero so the optimal solution is obtained.

~~Minim~~ Maximum Profit = Σ cell profit $\times$ Allocated Quantity

$= C_{11}X_{11} + C_{13}X_{13} + C_{22}X_{22} + C_{23}X_{23} + C_{32}X_{32}$

$= 120 \times 1300 + 50 \times 400 + 150 \times 1000 + 100 \times 1500 + 170 \times 1000$

$=$ Rs. 646000

The maximum profit is Rs. 646000.

15. Using transportation problem, determine the maximum profit from the information given below:

Warehouse	Factory (profit in rupees)			Capacity
	X	Y	Z	
A	130	170	180	30
B	110	140	180	30
C	150	120	150	40
D	200	130	120	50
Demand	20	60	70	

Solution:

Here, we have, $\sum D = 20+60+70 = 150$

$\sum S = 30+30+40+50 = 150$

Since, $\sum D = \sum S$ the problem is balanced.

Since, it is the case of profit maximization so we should construct opportunity loss table. It can be obtained by subtracting each element from the highest element.

Opportunity loss table

Warehouse \ Factory	X	Y	Z	Capacity
A	70	30	20	30
B	90	60	20	30
C	50	80	50	40
D	0	70	80	50
Demand	20	60	70	150

Step 1: Calculation of Initial solution using VAM.

Warehouse \ Factory	X	Y	Z	Capacity	I	II	III	IV
A	70	30 [30]	20	30→0	10	10	10	–
B	90	60	20 [30]	30→0	40	40	-	–
C	50	80	50 [40]	40→0	0	50	30	30
D	0 [20]	70 [30]	80	50→30→0	70	10	10	10
Demand	20→0	60→30→0	70→40→0	150				
I	50	30	0					
II	–	30	0					
III	–	40↑	30					
IV	–	10	30					

no. of occupied cell = 5

or, $m+n-1 = 5$

or, $4+3-1 = 5$

or, $6 \neq 5$

so, the problem is degenerate.

Here, allocated cell is not equal to $m+n-1$ so we have to assign small allocation θ (≈ 0) at the least unoccupied cell where open loop is formed.

Warehouse \ Factory	X	Y	Z	Capacity
A	70	30 [30]	20 [θ]	30
B	90	60	20 [30]	30
C	50	80	50 [40]	40
D	0 [20]	70 [30]	80	50
Demand	20	60	70	150

no. of allocated cell = 6

or, $m+n-1 = 6$

or, $4+3-1 = 6$

$\therefore$ $6 = 6$

so the problem is non-degenerate.

Max Total profit = $\sum$ unit cell cost X Allocated Quantity

$= C_{12}X_{12} + C_{13}X_{13} + C_{23}X_{23} + C_{33}X_{33} + C_{41}X_{41} + C_{42}X_{42}$

$= 170 \times 30 + 180 \times 0 + 180 \times 30 + 150 \times 40 + 200 \times 20 + 130 \times 30$

$=$ Rs 24400

Step 2: Test of Optimality by MODI method.-

Factory / Warehouse	X	Y	Z	Capacity	R_i
A	70	30 [30]	20 [θ]	30	$R_1 = 20$
B	90	60	20 [30]	30	$R_2 = 20$
C	50	80	50 [40]	40	$R_3 = 50$
D	0 [20]	70 [30]	80	150	$R_4 = 60$
Demand	20	60	70	150	
k_j	$k_1 = -60$	$k_2 = 10$	$k_3 = 0$		

Calculation of row value and column value

$C_{ij} = R_i + k_j$

or, $C_{13} = R_1 + k_3 \Rightarrow 20 = R_1 + 0 \Rightarrow R_1 = 20$

or, $C_{12} = R_1 + k_2 \Rightarrow 30 = 20 + k_2 \Rightarrow k_2 = 10$

or, $C_{23} = R_2 + k_3 \Rightarrow 20 = R_2 + 0 \Rightarrow R_2 = 20$

or, $C_{33} = R_3 + k_3 \Rightarrow 50 = R_3 + 0 \Rightarrow R_3 = 50$

or, $C_{42} = R_4 + k_2 \Rightarrow 70 = R_4 + 10 \Rightarrow R_4 = 60$

or, $C_{41} = R_4 + k_1 \Rightarrow 0 = 60 + k_1 \Rightarrow k_1 = -60$

The improvement indices for all unoccupied cells are given by relation:

$$\Delta_{ij} = C_{ij} - (R_i + k_j)$$

i.e. $\Delta_{11} = C_{11} - (R_1 + k_1) = 70 - (20 - 60) = 70 - (-40) = 110$

$\Delta_{21} = C_{21} - (R_2 + k_1) = 90 - (20 - 60) = 90 - (-40) = 130$

$\Delta_{22} = C_{22} - (R_2 + k_2) = 60 - (20 + 10) = 60 - 30 = 30$

$\Delta_{31} = C_{31} - (R_3 + k_1) = 50 - (50 - 60) = 50 - (-10) = 60$

$\Delta_{32} = C_{32} - (R_3 + k_2) = 80 - (50 + 10) = 80 - 60 = 20$

$\Delta_{43} = C_{43} - (R_4 + k_3) = 80 - (60 + 0) = 80 - 60 = 20$

Since, all values of $\Delta ij \geq 0$, our optimal solution is obtained.

Maximum Profit = Σ unit cell cost X Allocated Quantity

$= C_{12}X_{12} + C_{13}X_{13} + C_{23}X_{23} + C_{33}X_{33} + C_{41}X_{41} + C_{42}X_{42}$

$= 170 \times 30 + 180 \times 0 + 180 \times 30 + 150 \times 40 + 200 \times 20 + 130 \times 30$

= Rs. 24400 Ans.

16. The following table gives the profit of a company, find the maximum profit by using transportation model.

Destination \ Source	S_1	S_2	S_3	Unit demanded
A	490	480	480	5
B	480	440	600	20
C	450	480	500	20
Units Available	20	15	10	45

Solution:

Here, we have, $\Sigma D = 5 + 20 + 20 = 45$

$\Sigma S = 20 + 15 + 10 = 45$

Since, $\Sigma D = \Sigma S$ the problem is balanced.

Since, it is the case of profit maximization so we should construct opportunity loss table. It can be obtained by subtracting each element from highest element.

Since, for all $\Delta_{ij} \geq 0$ the optimal solution is obtained.

Max Profit = Σ cell unit profit X Allocated Quantity

$= C_{11}X_{11} + C_{12}X_{12} + C_{13}X_{13} + C_{22}X_{22} + C_{31}X_{31} + C_{41}X_{41}$

$= 12\times50 + 15\times0 + 14\times70 + 10\times80 + 20\times50 + 0\times50$

= Rs. 3380 Ans//

Assignment Problem.

- Hungarian Method

→ It involves following steps:

i) Row Reduction: Subtract the smallest element in each row from all elements in that row.

ii) Column Reduction: Subtract the smallest element in each column from all elements in that column.

iii) Zero Coverage: Draw lines (horizontal or vertical) through the rows and columns to cover all zeroes with the minimum no. of lines.

iv) Optimality Check: If the minimum no. of lines equals the order of matrix (no. of rows or columns), an optimal assignement exists & the process is complete. If not proceed to next step.

v) Matrix Revision: Find the smallest uncovered element & subtract it from uncovered elements. Add it to all elements at the intersection of lines. Leave the other elements untouched.

vi) Iterate: Repeat steps 3,4 & 5 until an optimal assignment is found. (i.e. the minimum no. of lines equals the matrix order).

Assignment Problem.

1. The three technicians have to be assigned three machines. Each technician can operate any machine but one technician can be used in one machine only. The operating cost in rupees required to different technician to different machines are given below:

Technician	Machine		
	A	B	C
P	400	600	800
Q	200	300	400
R	400	800	500

Make the assignment of technicians to the machines in order to minimize the operating cost.

Solution:

Here, no. of rows = 3, no. of columns = 3

Since, no. of rows = no. of columns. So, given Assignment problem is balanced.

The given cost table is

Technician	Machine			Row minimum
	A	B	C	
P	400	600	800	400
Q	200	300	400	200
R	400	800	500	400

Row Operation: Select the minimum element from each row and subtract these minimum element from each respective row elements.

Technician	Machine		
	A	B	C
P	0	200	400
Q	0	100	200
R	0	400	100
Column minimum	0	100	100

Column Operation: Select the minimum element from each column and subtract these minimum element from each respective column elements.

Technician	Machine		
	A	B	C
P	$\boxed{0}$	100	300
Q	0	$\boxed{0}$	100
R	0	300	$\boxed{0}$

Since, no. of lines (3) = no. of rows / column (3). Hence, optimum solution is obtained.

Hence, the total minimum cost to the machines ~~for~~ is calculated as:

Technician	Machine	Cost
P	A	400
Q	B	300
R	C	500
Total minimum cost		1200

2. The cost in rupees required for using four machines to four plants is given. There is no restriction in using the machines in different plant except one machine to one plant. Suggest optimal assignment of machines to plants.

Machines	Plant			
	1	2	3	4
P	220	125	120	105
Q	115	150	140	70
R	90	165	45	155
S	60	150	105	75

Solution:

Here, no of rows = 4, no. of columns = 4

Since, no. of rows = no. of columns so given problem is balanced.

The given cost table is:

Machines	Plant				Row minimum
	1	2	3	4	
P	220	125	120	105	105
Q	115	150	140	70	70
R	90	165	45	155	45
S	60	150	105	75	60

Row Operation

Machines \ Plants	1	2	3	4
P	115	20	15	0
Q	40	80	70	0
R	45	120	0	110
S	0	90	45	15
Col. minimum	0	20	0	0

Column operation

Machines \ Plants	1	2	3	4
P	115	$\boxed{0}$	15	0
Q	40	60	70	$\boxed{0}$
R	45	100	$\boxed{0}$	110
S	$\boxed{0}$	70	45	15

Since, no. of lines (4) = no. of rows / column so optimum solution is obtained.
So, Total cost is calculated as.

Machines	Plants	Cost
P	2	125
Q	4	70
R	3	45
S	1	60
Total cost		300

3. Find the optimal assignment for the following cost matrix.

Salesman	Areas			
	A_1	A_2	A_3	A_4
P	11	17	8	16
Q	9	7	12	10
R	13	16	15	12
S	14	15	12	11

Solution:
Here, no. of rows = 4, no. of columns = 4
Since, no. of rows = no. of columns so given problem is balanced.

The given cost table is

Salesman \ Areas	A_1	A_2	A_3	A_4	Min Row
P	11	17	8	16	8
Q	9	7	12	10	7
R	13	16	15	12	13
S	14	15	12	11	11

Row operation:

Salesman \ Areas	A_1	A_2	A_3	A_4
P	3	9	0	8
Q	2	0	5	3
R	0	3	2	1
S	3	4	1	0
Min. column	0	0	0	0

Column Operation

Salesman \ Areas	A_1	A_2	A_3	A_4
P	3	9	$\boxed{0}$	8
Q	2	$\boxed{0}$	5	3
R	$\boxed{0}$	3	2	1
S	3	4	1	$\boxed{0}$

Since, no. of lines (4) = no. of rows/column (4) so optimal solution is obtained.
so, total cost is calculated as:

Salesman	Areas	Cost
P	A_3	8
Q	A_2	7
R	A_1	13
S	A_4	11
Total cost		39

4. You are given the information relating to the Bike company and bikers for a racing club. The assignment costs are given. Find the optimal assignment.

Bikers	Bike company			
	Yamaha	Honda	Hero	Bajaj
Ram	2200	3400	1600	3200
Laxman	1800	1400	2400	2000
Bharat	2600	3200	3000	2400
Hari	2800	3000	2400	2200

Solution:

Here, no. of rows = 4, no. of columns = 4

Since, no. of rows = no. of columns so given problem is balanced.

The given cost table is:

Bikers \ Bike Company	Yamaha	Honda	Hero	Bajaj	Min Row
Ram	2200	3400	1600	3200	1600
Laxman	1800	1400	2400	2000	1400
Bharat	2600	3200	3000	2400	2400
Hari	2800	3000	2400	2200	2200

Row Operation

Bikers \ Bike Company	Yamaha	Honda	Hero	Bajaj
Ram	600	1800	0	1600
Laxman	400	0	1000	600
Bharat	200	800	600	0
Hari	600	800	200	0
Min column	200	0	0	0

Column Operation

Bikers \ Company	Yamaha	Honda	Hero	Bajaj
Ram	400	1800	$\boxed{0}$	1600
Laxman	200	$\boxed{0}$	1000	600
Bharat	$\boxed{0}$	800	600	0
Hari	400	800	200	$\boxed{0}$

Since, no. of lines (4) = no. of rows/column (4) so optimum solution is obtained.

So,

Bikers	Bikers Company	Cost
Ram	Hero	1600
Laxman	Honda	1400
Bharat	Yamaha	2600
Hari	Bajaj	2200
Total Cost		7800

5. Three customers in a certain sales territory have requested technical assistance. Three technicians are available for assignment with the distance in km from each technician to each customer being as follows:

Technician	Customer		
	A	B	C
P	10	12	15
Q	20	25	18
R	17	16	13

If it costs Rs. 2 per km for travel, find the assignment of technicians to customers that will result in a minimum travel cost.

Performers	Tasks	Cost
W	S	10
X	Q	22
Y	R	30
Z	P	9
Minimum Total Cost		71

Performers	Tasks	Cost
W	S	10
X	R	28
Y	Q	24
Z	P	9
Total Minimum Cost		71

∴ Total Minimum Cost = 71000

8. A city corporation has decided to carry out road repairs on main four arteries of the city. The government has agreed to make a special grant of 50 lakhs with a condition that the repair is done at the lowest cost and quickest time. If the condition warrant, a supplementary token grant will also be favorable. The corporation has floated tenders and five contractors have sent in their bids. In order to expedite work, one road will be awarded to only one contractor.

Cost of repair in lakh

Contractor	Road			
	R_1	R_2	R_3	R_4
A	9	14	19	15
B	7	17	20	19
C	9	18	21	18
D	10	12	18	19
E	10	15	21	16

a) Find the best way of assigning repair work to the contractors that will minimize the cost.

b) If it is necessary to seek supplementary grant, what should be the amount sought?

c) Which of the five contractors will be unsuccessful in this bid?

Solution:

In above question, no. of row ≠ no. of column so we have to add dummy column to balance this problem.

The cost table is:

Contractor \ Road	R_1	R_2	R_3	R_4	R_5	Min Row
A	9	14	19	15	0	0
B	7	17	20	19	0	0
C	9	18	21	18	0	0
D	10	12	18	19	0	0
E	10	15	21	16	0	0

Row Operation.

Contractor \ Road	R_1	R_2	R_3	R_4	R_5
A	9	14	19	15	0
B	7	17	20	19	0
C	9	18	21	18	0
D	10	12	18	19	0
E	10	15	21	16	0
Min column	7	12	18	15	0

Column Operation

Contractor \ Road	R_1	R_2	R_3	R_4	R_5
A	2	2	1	$\boxed{0}$	0
B	$\boxed{0}$	5	2	4	0
C	2	6	3	3	$\boxed{0}$
D	3	$\boxed{0}$	0	4	0
E	3	3	3	1	0

no. of lines (4) ≠ no. of rows/column (5) so we have to select the minimum uncovered element (1) and add it to the intersecting element.

Contractor \ Road	R_1	R_2	R_3	R_4	R_5
A	2	2	1	$\boxed{0}$	1
B	$\boxed{0}$	5	2	4	1
C	1	5	2	2	$\boxed{0}$
D	3	$\boxed{0}$	0	4	1
E	2	2	2	0	0

no. of lines (4) ≠ no. of rows/column (5) so we have to select the minimum uncovered element (1) and add it to the intersecting element.

Contractor \ Road	R_1	R_2	R_3	R_4	R_5
A	2	1	$\boxed{0}$	0	1
B	$\boxed{0}$	4	1	4	1
C	1	4	1	2	$\boxed{0}$
D	1	$\boxed{0}$	0	5	2
E	2	1	1	$\boxed{0}$	0

no. of lines (5) = no. of rows/column (5). Hence, optimum solution is obtained.

a)

Contractor	Road	Cost
A	R_3	19
B	R_1	7
C	R_5	0
D	R_2	12
E	R_4	16
Total minimum cost		Rs. 54

b) The minimum cost of road construction according to bid applied is Rs 54 lakhs but the available funds Rs 30 lakhs only so the supplement grant of Rs 54-50 = Rs 4 lakhs should be sought.

c) The contractor C is unsuccessful on his bid because he has been assigned to dummy Road (R_5).

9. A company solicits bids on each of four projects from five contractors. Only one project may be assigned to any contractor. The bids received in thousands of rupees are given in the table given below. Contractor D feels unable to

Faculties \ Sections	A	B	C	D
P	32	$\boxed{0}$	0	22
Q	$\boxed{0}$	14	12	26
R	0	8	$\boxed{0}$	4
S	50	0	0	$\boxed{0}$

no. of lines (4) = no. of rows/column (4). Hence optimal solution is obtained.

The maximum performance is calculated as:

Faculties	Sections	Performance
P	B	78
Q	A	85
R	C	69
S	D	84
Total Max Performance		316

Alternative solution of Assignment Problem.

11. A company has four districts to sell its products and four salesmen A, B, C and D for it. The districts sale-record of each salesman is as given in the table. Determine the areas allocation so as to make the maximum sales.

Salesman	District			
	Kathmandu	Lalitpur	Bhaktapur	Gorkha
A	4200	3500	2800	2100
B	3000	2500	2000	1500
C	3000	2500	2000	1500
D	2400	2000	1600	1200

Solution:

Here no. of rows (4) = no. of columns so the problem is balanced.

Since this is the case of maximization we have to construct opportunity cost table by subtracting highest with every element

District / Salesman	Ktm	Lalitpur	Bkt	Gorkha	Min Row
A	0	700	1400	2100	0
B	1200	1700	2200	2700	1200
C	1200	1700	2200	2700	1200
D	1800	2200	2600	3000	1800

Row Operation

District / Salesman	Kathmandu	Lalitpur	Bhaktapur	Gorkha
A	0	700	1400	2100
B	0	500	1000	1500
C	0	500	1000	1500
D	0	400	800	1200
Min Column	0	400	800	1200

Column Operation

District / Salesman	Ktm	Ltp	Bkt	Gorkha
A	$\boxed{0}$	300	600	900
B	0	100	200	300
C	0	100	200	300
D	0	$\boxed{0}$	0	0

no. of lines (2) ≠ no. of rows/column (4)

so we have to select the minimum uncovered element (100), and subtract least uncovered element (100) from each uncovered element and add it to intersecting element.

District / Salesman	Ktm	Lalitpur	Bhaktapur	Gorkha
A	$\boxed{0}$	200	500	800
B	0	$\boxed{0}$	100	200
C	0	0	100	200
D	100	0	$\boxed{0}$	0

no. of lines (3) ≠ no. of rows/column
so, we have to select the minimum uncovered element (100) and subtract it from each uncovered element and add it to the intersecting element and leave other element.

District / Salesman	Ktm	Lalitpur	Bhaktapur	Gorkha
A	$\boxed{0}$	200	400	700
B	0	0	$\boxed{0}$	100
C	0	$\boxed{0}$	0	100
D	200	100	0	$\boxed{0}$

no. of lines (4) = no. of rows/column (4). Hence, optimum solution is obtained.

Salesman	District	Sales
A	Ktm	4200
B	Bkt	2000
C	Lalitpur	2500
D	Gorkha.	1200
Max. sales		9900

Salesman	District	Sales
A	Ktm	4200
B	Lalitpur	2500
C	Bkt	2000
D	Gorkha	1200
Max Sales		9900

Machine \ Profit	A	B	C	D
X	7	1	1	$\boxed{0}$
Y	5	0	$\boxed{0}$	1
Z	5	$\boxed{0}$	0	0
M	$\boxed{0}$	0	4	2

no. of lines (4) = no. of rows/column. Hence, optimum soln is obtained.

Machine	Jobs	Performance
X	D	32
Y	B	13
Z	C	19
M	A	0
Max Profit		64

Machine	Jobs	Performance
X	D	32
Y	C	17
Z	B	15
M	A	0
Max Profit		64

Constraint Assignment Problem.

13. In the modification of a plant layout of a factory, four new machines A, B, C and D are to be installed in five vacant plants P, Q, R, S and T. Because of limited space, machine B can not be installed at plant R and machine C cannot be installed at plant P. The profit (Rs. 000 contribution by different machines at different plants are given below.

Machines	Vacant Plants				
	P	Q	R	S	T
A	89	91	95	90	91
B	92	89	-	90	89
C	-	91	94	91	87
D	94	88	92	81	88

Find the optimal assignment schedule that would maximize the profit.

Solution:

Here, no. of rows = 4

no. of columns = 5

Here, no. of rows ≠ no. of columns. so we have to add a dummy row to balance the problem.

Machines \ Plants	P	Q	R	S	T
A	89	91	95	90	91
B	92	89	–	90	89
C	–	91	94	91	87
D	94	88	92	81	88
E	0	0	0	0	0

This is case of maximization so we have to construct the opportunity loss table by subtracting every element with the highest element (95).

Machines \ Plants	P	Q	R	S	T	Min Row
A	6	4	0	5	4	0
B	3	6	–	5	6	3
C	–	4	1	4	8	1
D	1	7	3	8	7	1
E	95	95	95	95	95	95

Row Operation

Machine \ Plants	P	Q	R	S	T
A	6	4	0	5	4
B	0	3	–	2	3
C	–	3	0	3	7
D	0	6	2	7	6
E	0	0	0	0	0
Min Col	0	0	0	0	0

Column Operation:

Machine \ Plants	P	Q	R	S	T
A	6	4	$\boxed{0}$	5	4
B	$\boxed{0}$	3	1	2	3
C	1	3	0	3	7
D	0	6	2	7	6
E	0	$\boxed{0}$	0	0	0

no. of lines (3) ≠ no. of rows/column (5)
so, we have to select the minimum uncovered element (2), and subtract 2 from each uncovered element and add it to the intersecting element.

Machine \ Plants	P	Q	R	S	T
A	6	2	$\boxed{0}$	3	2
B	0	1	1	$\boxed{0}$	1
C	1	1	0	1	5
D	$\boxed{0}$	4	2	5	4
E	2	$\boxed{0}$	2	0	0

no. of lines (4) ≠ no. of rows/column (5).
so we have to select the minimum uncovered element (1), & subtract 1 from each uncovered element & add it to intersecting element.

Machine \ Plants	P	Q	R	S	T
A	6	1	$\boxed{0}$	2	1
B	1	1	1	$\boxed{0}$	1
C	1	$\boxed{0}$	0	0	4
D	$\boxed{0}$	3	2	4	3
E	3	0	3	0	$\boxed{0}$

no. of lines (5) = no. of rows/column (5). Hence optimum soln is obtained.

Machine	Vacant Plant	Profit
A	R	95
B	S	90
C	Q	91
D	P	94
E	T	0
Total Maximum Profit		370

14. A production manager wants to assign one of the five new methods to each of the four operations. The following table summarizes the weekly output in units:

Operations	Methods				
	P	Q	R	S	T
A	4	6	11	16	9
B	5	8	16	19	9
C	9	13	21	21	13
D	6	7	9	11	7

If production cost per unit is Rs.10 and selling price per unit is Rs. 35, find the maximum profit per month.

Solution:

By question:

Production cost per unit = Rs.10

Selling price = Rs 35

Profit per unit = 35 - 10 = Rs 25.

so, we should multiply each cell in the above table by 25 to get the profit per assignment.

Chapter-4
Queuing Theory

Single Channel Queuing Model

- Arrivals are served on first in first out (FIFO) discipline.
- Every arrivals waits to be served regardless of the length of line.
- Arrivals are independent of preceeding arrivals.
- Arrivals are described by a Poisson distribution and come from an infinite or very large population.
- Service time also varies from one customer to another.
- Service follows the negative exponential poision distribution.

Operating characteristics of Queuing System.

i) Queue Length

→ The average no. of customer waiting in line for getting service in the service centre is called Queue length.

$$L_q = \left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu-\lambda}\right)$$

Where, λ = average arrival rate.
μ = average service rate.

ii) System length

→ The average no. of customers waiting in the line and getting service at the counter is called system length.

$$L_s = \frac{\lambda}{\mu-\lambda}$$

iii) Waiting time in queue

→ The average time spent by customer in waiting line for getting service is called average waiting time.

$$T_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu-\lambda}\right)$$

iv) Waiting Time in the System.

→ The average time spent by the customer in waiting line and for receiving service at the counter is called average waiting time.

$$T_s = \frac{1}{\mu-\lambda}$$

v) Service Facility Utilization

→ It is the proportion of time that a server actually spends with a customer.

$$\rho = \frac{\lambda}{\mu}$$

Working Formulae.

1. The mean or expected waiting time in the system i.e the time that a customer spend the system before completing his service is given by

$$T_s = \frac{1}{\mu-\lambda}$$

2. The mean or expected waiting time in queue i.e the time that a customer spent the queue before starting his service is given by;

$$T_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu-\lambda}\right)$$

3. Expected or mean no. of customers in the system (average length of system) is given by

$$L_s = \frac{\lambda}{\mu - \lambda}$$

4. Expected or mean no. of customers in the queue (average length of queue) is given by

$$L_q = \left(\frac{\lambda}{\mu-\lambda}\right)\left(\frac{\lambda}{\mu}\right) = \left(\frac{\rho^2}{1-\rho}\right)$$

5. The utilization factor or traffic intensity which denotes the probability that the service facility is busy, is calculated as

$$\rho = \frac{\lambda}{\mu}$$

6. The probability that service facility is idle (i.e. the probability of no units in the system)

$$P(n=0) = 1 - \frac{\lambda}{\mu} = 1 - \rho.$$

7. The probability that there are "n=r" units or customers in the system (units in the queue or waiting line + no. of customers being served).

$$P(n=r) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^r = (1-\rho)(\rho)^r$$

8. The probability that there are at least 'r' customers in the system is given by

$$P(n \geq r) = \left(\frac{\lambda}{\mu}\right)^r = \rho^r$$

∴ The probability of non-empty queue $(n \geq 2)$ is given by

$$P(n \geq 2) = \left(\frac{\lambda}{\mu}\right)^2 = \rho^2$$

9. The probability that there are more than 'r' customers in the system is given by

$$p(n>r) = \left(\frac{\lambda}{\mu}\right)^{r+1} = \rho^{r+1}$$

10 The average no. of customers in the non empty queue i.e the average length of non empty queue is calculated as:

$$L_{nq} = \frac{\mu}{\mu-\lambda}$$

11. The average no. of customers in the non empty system i.e the average length of non empty system is calculated as:-

$$L_{ns} = \frac{\mu}{\mu-\lambda} + \frac{\lambda}{\mu}$$

$$= L_{nq} + \rho.$$