

# Time-Dependent Parameters in Quantum Systems: Revisiting Berry Phase, Curvature, and Gauge Connections

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## Abstract

We present a reformulation of quantum adiabatic theory in terms of an emergent electromagnetic framework, emphasizing the physical consequences of geometric structures in parameter space. Contrary to conventional approaches, we demonstrate that a Berry electric field naturally arises in systems with dynamic Hamiltonians, when the full time-dependent wavefunction is used to define the gauge potentials. This surprising result bridges the gap between static and dynamical formulations and leads to a deeper understanding of how gauge structures manifest in quantum systems. Building on this, we construct Berry–Maxwell equations by analogy with classical electrodynamics, defining Berry electric and magnetic fields as derivatives of scalar and vector potentials obtained from the full quantum state. We verify these equations explicitly and derive field-theoretic identities such as generalized continuity and vorticity relations. This field-based formulation reveals the topological charges, monopole structures, and gauge currents that underlie parameter space, and clarifies how Berry curvature corrections enter dynamical quantities like expectation values and particle velocities. Our results establish a new regime of emergent electromagnetism in parameter space, unifying time-independent and time-dependent geometric phases within a covariant formalism. The implications extend to quantum transport, polarization, and topological classification of phases, providing a robust and generalizable framework for quantum systems driven by adiabatic or nonadiabatic evolution.

## Introduction

Quantum systems with time-dependent perturbations play a pivotal role in numerous physical contexts, from adiabatic transport to the dynamics of

topological phases. The Berry phase [1], a cornerstone of geometric phases, provides deep insights into such systems; however, its traditional formulation raises subtle issues when parameters vary temporally. This paper revisits the Berry phase framework, emphasizing the importance of time dependence and exploring the implications for electric and magnetic curvature, gauge connections, and transport phenomena. Let us begin by considering a static Hamiltonian with a set of 3D parameters  $\mathbf{R}$  that **do not depend on time**, and a static Hamiltonian  $H$  depending on those set of parameters, i.e.,

$$H(\mathbf{R})|\Psi(\mathbf{R}, t)\rangle = i\hbar \frac{\partial |\Psi(\mathbf{R}, t)\rangle}{\partial t} \quad (1)$$

Then, the solution to the Schrödinger equation can be written as:

$$|\Psi(\mathbf{R}, t)\rangle = e^{-i\epsilon_n(\mathbf{R})t/\hbar} |n(\mathbf{R})\rangle. \quad (2)$$

Using the above result (for purposes that will make the next Sections of this paper more apparent), let us now proceed straightforwardly, and define (in an abstract manner) a vector potential-like quantity for the wavefunction given by eq. (2) (using instead, the full wavefunction in the definition of the potential):

$$\begin{aligned} \mathbf{A}^\Psi(\mathbf{R}, t) &= i\langle \Psi(\mathbf{R}, t) | \nabla_{\mathbf{R}} | \Psi(\mathbf{R}, t) \rangle = \frac{1}{\hbar} t \nabla_{\mathbf{R}} \epsilon_n(\mathbf{R}) + i\langle n(\mathbf{R}) | \nabla_{\mathbf{R}} | n(\mathbf{R}) \rangle \\ &= \frac{1}{\hbar} t \nabla_{\mathbf{R}} \epsilon_n(\mathbf{R}) + \mathbf{A}^n \end{aligned} \quad (3)$$

where  $\mathbf{A}^n = i\langle n(\mathbf{R}) | \nabla_{\mathbf{R}} | n(\mathbf{R}) \rangle$  is the Berry vector potential, corresponding to a single eigenstate  $|n(\mathbf{R})\rangle$ . We observe that the operator  $\nabla_{\mathbf{R}}$  acts on both the dynamic phase factor and the eigenvectors, leading to a linear dependence on time for  $\mathbf{A}^\Psi$ . Notice that, in the absence of time dependence in the Hamiltonian, we can still define a Berry curvature:

$$\nabla_{\mathbf{R}} \times \mathbf{A}^\Psi = \nabla_{\mathbf{R}} \times \mathbf{A}^n \quad (4)$$

Here, the curl of  $\langle n(\mathbf{R}) | \nabla_{\mathbf{R}} | n(\mathbf{R}) \rangle$  produces the well-known Berry curvature. Also, notice that a phase transformation (by using a single-valued and well-behaved  $\Lambda$  function):

$$|n'(\mathbf{R})\rangle \rightarrow e^{i\Lambda(\mathbf{R})} |n(\mathbf{R})\rangle \quad (5)$$

leaves the quantity in eq. (4) invariant. Therefore, we define a Berry curvature using the full time-dependent wavefunction as follows:

$$\mathbf{B}^\Psi = \nabla_{\mathbf{R}} \times \mathbf{A}^\Psi = i \nabla_{\mathbf{R}} \times \langle n(\mathbf{R}) | \nabla_{\mathbf{R}} | n(\mathbf{R}) \rangle = \mathbf{B}^n \quad (6)$$

Thus, in this simple case, the Berry potential can be expressed using the full time-dependent wavefunction  $\Psi(\mathbf{R}, t)$  or by using the time-independent eigenvectors  $n(\mathbf{R})$ , resulting in the same magnetic curvature. If the following condition holds:  $\nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} \Lambda = 0$  (for a good  $\Lambda$ , we can assume this is the case), then under the gauge transformation given by eq. (5), the wavefunction also transforms as (as opposed to time dependent systems; see next Section):

$$|\Psi(\mathbf{R})\rangle \rightarrow e^{i\Lambda(\mathbf{R})}|\Psi(\mathbf{R})\rangle \quad (7)$$

and the resulting vector potential becomes:

$$\mathbf{A}'^\Psi = i\langle\Psi'(t)|\nabla_{\mathbf{R}}|\Psi'(t)\rangle = \mathbf{A}^\Psi - \nabla_{\mathbf{R}}\Lambda \quad (8)$$

Notice how this gauge transformation leaves the Schrödinger equation invariant. Thus, the general solution of Eq. (1) can be written as:

$$|\Psi(t)\rangle = e^{i\Lambda(\mathbf{R})}e^{-i\epsilon_n(\mathbf{R})t/\hbar}|n(\mathbf{R})\rangle. \quad (9)$$

The "good" function  $\Lambda(\mathbf{R})$  plays the role of a geometric-like phase, but it is not related to the eigenvectors of the problem. Even for a completely time-independent system, it encodes the information about the structure of parameter space and its emergent curvature. Note that when  $\Lambda$  is not "good," the gauge transformation in eq. (8) cannot reproduce the magnetic field. We see that, up to this point, the results of Berry's theory emerge even in a completely time-independent system. This greatly simplifies matters, especially in cases involving electric fields. Notice the amusing fact that if we define the quantity (a Berry scalar potential) as follows (and this will also play a major role in the time - dependent system in the next Section of this paper):

$$\Phi^\Psi = -i\langle\Psi(t)|\frac{\partial}{\partial t}|\Psi(t)\rangle = -\frac{1}{\hbar}\epsilon_n(\mathbf{R}), \quad (10)$$

we may also define the Berry electric field [2, 3] as (by using (3) and (10)):

$$\Omega^\Psi = -\nabla_{\mathbf{R}}\Phi^\Psi - \frac{\partial\mathbf{A}^\Psi}{\partial t} = \frac{1}{\hbar}\nabla_{\mathbf{R}}\epsilon_n(\mathbf{R}) - \frac{1}{\hbar}\nabla_{\mathbf{R}}\epsilon_n(\mathbf{R}) = 0, \quad (11)$$

which, in the simplest case of time-independence, reduces to zero. However, the Berry electric field plays a major role when time-dependence comes into play, and it is harmonically co-existing with the Berry magnetic field as we will see later on below. Additionally, if the potentials are defined directly via the eigenvectors, i.e.,

$$\mathbf{A}^n = i\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle, \quad (12)$$

and

$$\Phi^n = -i\langle n(\mathbf{R}) | \frac{\partial}{\partial t} | n(\mathbf{R}) \rangle = 0, \quad (13)$$

the fields remain unchanged (i.e., equal to (11) and (6) respectively):

$$\Omega^n = -\nabla_{\mathbf{R}}\Phi^n - \frac{\partial \mathbf{A}^n}{\partial t} = 0, \quad (14)$$

$$\mathbf{B}^n = \nabla_{\mathbf{R}} \times \mathbf{A}^n = i\nabla_{\mathbf{R}} \times \langle n(\mathbf{R}) | \nabla_{\mathbf{R}} | n(\mathbf{R}) \rangle. \quad (15)$$

We thus observe that, even in completely time-independent systems that incorporate geometric information of parameter space, the Berry curvature retains a meaningful connection to time-dependent systems. In time dependent systems, as we shall see later on, it is possible to attribute the whole time-dependence to an emerging electric field, that, when combined with a Berry magnetic field (i.e. by keeping the external parameters purely time-independent) results in a generalised Berry-Maxwell Physics, that governs the field propagation through the parameter space.

## 1 Time-dependent Hamiltonian $H(\mathbf{R}, t)$

In this section, we explore the dynamics of quantum systems governed by a time-dependent Hamiltonian  $H(\mathbf{R}, t)$ , where  $\mathbf{R}$  represents a completely **static parameter space**. This scenario extends our understanding beyond the time-independent case and introduces complex connections between the evolving eigenstates, their geometric properties, and the associated dynamical phase factors. Starting with the time-dependent Schrödinger equation, we adopt an ansatz that incorporates both dynamical and geometric phases. Within the framework of the adiabatic approximation, we derive expressions for the geometric phase  $\gamma_n(\mathbf{R}, t)$ , while also exploring the role of gauge freedom and its implications on the wavefunction. Through this analysis, we identify both vector and scalar Berry potentials and examine their evolution over time. This section also establishes the correspondence between the Berry curvatures and their role in deriving effective "Berry-Maxwell" equations, drawing an analogy with classical electrodynamics. Emphasis is placed on the intricate structure of the parameter space and how non-smooth behavior of eigenstates affects the calculation of curvature terms. These considerations lead to new insights into the geometric and topological properties of quantum systems. Our analysis reveals novel connections, including the emergence of monopole-like terms in the modified Maxwell equations, which enrich the theoretical landscape of Berry phase physics. By emphasizing the interplay between geometry, topology, and dynamics in time-dependent

quantum systems, we aim to provide a deeper understanding of the subject. We assume that the Hamiltonian is time-dependent, i.e.,  $H(\mathbf{R}, t)$ , but  $\mathbf{R}$  remains constant such that the Schrodinger equation is satisfied:

$$H(\mathbf{R}, t)|\Psi(\mathbf{R}, t)\rangle = i\hbar \frac{\partial}{\partial t} |\Psi(\mathbf{R}, t)\rangle. \quad (16)$$

We now introduce an ansatz with a phase factor  $\gamma_n(\mathbf{R}, t)$ :

$$|\Psi(\mathbf{R}, t)\rangle = e^{i\gamma_n(\mathbf{R}, t)} e^{-i/\hbar \int_0^t \varepsilon_n(\mathbf{R}, t') dt'} |n(\mathbf{R}, t)\rangle. \quad (17)$$

Substituting this ansatz into the Schrödinger equation and further assuming that the adiabatic approximation holds, we obtain:

$$H(\mathbf{R}, t)|\Psi(\mathbf{R}, t)\rangle = \varepsilon_n(\mathbf{R}, t)|\Psi(\mathbf{R}, t)\rangle, \quad (18)$$

which leads to the following equation for the time evolution of  $\gamma_n(\mathbf{R}, t)$ :

$$\frac{\partial \gamma_n(\mathbf{R}, t)}{\partial t} = i \langle n(\mathbf{R}, t) | \frac{\partial}{\partial t} | n(\mathbf{R}, t) \rangle. \quad (19)$$

The generic solution to this equation is:

$$\gamma_n(\mathbf{R}, t) = i \int_0^t \langle n(\mathbf{R}, t') | \frac{\partial}{\partial t'} | n(\mathbf{R}, t') \rangle dt' + f(\mathbf{R}), \quad (20)$$

where  $f(\mathbf{R})$  is a pure gauge term and can be omitted. Next, we define the vector and scalar Berry potentials for this system by using the full wavefunction  $\Psi$  as follows:

$$\begin{aligned} \mathbf{A}^\Psi &= i \langle \Psi(\mathbf{R}, t) | \nabla_{\mathbf{R}} | \Psi(\mathbf{R}, t) \rangle = -\nabla_{\mathbf{R}} \gamma_n(\mathbf{R}, t) + \frac{1}{\hbar} \int_0^t \nabla_{\mathbf{R}} \varepsilon_n(\mathbf{R}, t') dt' \\ &+ \mathbf{A}_n. \end{aligned} \quad (21)$$

with  $\mathbf{A}_n = i \langle \mathbf{n}(\mathbf{R}, t) | \nabla_{\mathbf{R}} | \mathbf{n}(\mathbf{R}, t) \rangle$ . For the scalar potential, using (10), we find:

$$\begin{aligned} \Phi^\Psi &= -i \langle \Psi(t) | \frac{\partial}{\partial t} | \Psi(t) \rangle = \frac{\partial \gamma_n(\mathbf{R}, t)}{\partial t} - \frac{1}{\hbar} \varepsilon_n(\mathbf{R}, t) \\ &- i \langle n(\mathbf{R}, t) | \frac{\partial}{\partial t} | n(\mathbf{R}, t) \rangle = -\frac{1}{\hbar} \varepsilon_n(\mathbf{R}, t). \end{aligned} \quad (22)$$

Note that in this case, the scalar Berry potential coincides with the energy spectrum, just as in (10) (being a completely time-independent case). For

completeness, we also define the potentials in terms of the eigenvectors as follows:

$$\mathbf{A}^n = i\langle \mathbf{n}(\mathbf{R}, t) | \nabla_{\mathbf{R}} | \mathbf{n}(\mathbf{R}, t) \rangle. \quad (23)$$

$$\Phi^n = -i\langle n(\mathbf{R}, t) | \frac{\partial}{\partial t} | n(\mathbf{R}, t) \rangle. \quad (24)$$

Let us now define the curvatures:

$$\boldsymbol{\Omega}^\Psi = -\nabla_{\mathbf{R}}\Phi^\Psi - \frac{\partial \mathbf{A}^\Psi}{\partial t}, \quad (25)$$

$$\mathbf{B}^\Psi = \nabla_{\mathbf{R}} \times \mathbf{A}^\Psi. \quad (26)$$

We also define the curvatures through the eigenvectors as follows:

$$\boldsymbol{\Omega}^n = -\nabla_{\mathbf{R}}\Phi^n - \frac{\partial \mathbf{A}^n}{\partial t}, \quad (27)$$

$$\mathbf{B}^n = \nabla_{\mathbf{R}} \times \mathbf{A}^n. \quad (28)$$

Using (23) and (24), we find:

$$\boldsymbol{\Omega}^n = i\langle \nabla_{\mathbf{R}} n | \frac{\partial n}{\partial t} \rangle - \langle \frac{\partial n}{\partial t} | \nabla_{\mathbf{R}} n \rangle, \quad (29)$$

$$\mathbf{B}^n = i\nabla_{\mathbf{R}} \times \langle n | \nabla_{\mathbf{R}} n \rangle. \quad (30)$$

With these definitions, we proceed by simplifying the expression for  $\gamma_n$  from (20), which can be written using the scalar potential  $\Phi_n$  as follows:

$$\gamma_n(\mathbf{R}, t) = -\int_0^t \Phi(\mathbf{R}, t') dt'. \quad (31)$$

Substituting this into (21), we get:

$$\mathbf{A}^\Psi = \mathbf{A}^n + \int_0^t \nabla_{\mathbf{R}}\Phi^n dt' - \int_0^t \nabla_{\mathbf{R}}\Phi^\Psi dt'. \quad (32)$$

At  $t = 0$ , the two vector potentials become equal. We now proceed with some important remarks regarding the action of the operator  $\nabla_{\mathbf{R}}$  on the eigenvectors. This operation must be done carefully because the eigenvectors may not be smooth functions of  $\mathbf{R}$  across the entire parameter space. This has significant consequences that will be explained later, leading to interesting Berry-Maxwell physics. For instance, when calculating the magnetic curvature  $\mathbf{B}^\Psi$  using (26) and (32), it is crucial to note that while the  $\Phi^\Psi$  term is

directly related to the energy spectrum (via (22)), the same cannot be assumed for  $\nabla_{\mathbf{R}}\Phi^n$ , which encodes the geometric information of the parameter space. Therefore, by using (26) and (32), we find:

$$\mathbf{B}^\Psi = \nabla_{\mathbf{R}} \times \mathbf{A}^\Psi = \nabla_{\mathbf{R}} \times \mathbf{A}^n + \nabla_{\mathbf{R}} \times \int_0^t \nabla_{\mathbf{R}} \Phi^n dt'. \quad (33)$$

Thus,  $\mathbf{B}^\Psi$  is not always equal to  $\mathbf{B}^n$ , although this does not hold for the electric curvature:  $\boldsymbol{\Omega}^\Psi = \boldsymbol{\Omega}^n$ . The significance of this additional term will be discussed later. By using (25), (22) and (32), we can compute  $\boldsymbol{\Omega}^\Psi = -\nabla_{\mathbf{R}}\Phi^\Psi - \frac{\partial \mathbf{A}^\Psi}{\partial t}$ :

$$\boldsymbol{\Omega}^\Psi = -\nabla_{\mathbf{R}}\Phi^\Psi - \frac{\partial \mathbf{A}^n}{\partial t} - \nabla_{\mathbf{R}}\Phi^n + \nabla_{\mathbf{R}}\Phi^\Psi = \boldsymbol{\Omega}^n. \quad (34)$$

An amusing fact arises when we observe that these fields obey the Maxwell equations, even with potential pathologies. Actually, and being on the safe side, in order to interpret these fields as "magnetic" and "electric", it is necessary condition that they do obey Maxwell equations in the 4D space-time parameter space. To illustrate this, we examine the electric field defined by equation (25) and by using (33), we calculate the curl of  $\Omega_n$  (making use of the fact that  $\boldsymbol{\Omega}^\Psi = \boldsymbol{\Omega}^n$ ):

$$\nabla_{\mathbf{R}} \times \boldsymbol{\Omega}^n = -\nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}}\Phi^n - \frac{\partial \mathbf{B}^n}{\partial t}. \quad (35)$$

Notice that the Maxwell equations should also be consistent if we use equations (29) and (30). Let us explore this further. We begin with the curl of  $\Omega^n$ :

$$i\nabla_{\mathbf{R}} \times \left[ \langle \nabla_{\mathbf{R}} n | \frac{\partial n}{\partial t} \rangle - \langle \frac{\partial n}{\partial t} | \nabla_{\mathbf{R}} n \rangle \right] = i\nabla_{\mathbf{R}} \times \langle \nabla_{\mathbf{R}} n | \frac{\partial n}{\partial t} \rangle - i\nabla_{\mathbf{R}} \times \langle \frac{\partial n}{\partial t} | \nabla_{\mathbf{R}} n \rangle. \quad (36)$$

We now examine the first term:

$$\begin{aligned} i\nabla_{\mathbf{R}} \times \langle \nabla_{\mathbf{R}} n | \frac{\partial n}{\partial t} \rangle &= i \int d^3r \nabla_{\mathbf{R}} \times \left( \nabla_{\mathbf{R}} n^* \frac{\partial n}{\partial t} \right) \\ &= i \int d^3r \frac{\partial n}{\partial t} \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n^* + \nabla_{\mathbf{R}} \frac{\partial n}{\partial t} \times \nabla_{\mathbf{R}} n^*. \end{aligned} \quad (37)$$

Thus, we have:

$$i\nabla_{\mathbf{R}} \times \langle \nabla_{\mathbf{R}} n | \frac{\partial n}{\partial t} \rangle = i \int d^3r \frac{\partial n}{\partial t} \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n^* - i \langle \nabla_{\mathbf{R}} n | \times | \nabla_{\mathbf{R}} \frac{\partial n}{\partial t} \rangle. \quad (38)$$

Similarly, for the second term we get:

$$i\nabla_{\mathbf{R}} \times \langle \frac{\partial n}{\partial t} | \nabla_{\mathbf{R}} n \rangle = i \int d^3r \frac{\partial n^*}{\partial t} \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n + i \langle \nabla_{\mathbf{R}} \frac{\partial n}{\partial t} | \times | \nabla_{\mathbf{R}} n \rangle. \quad (39)$$

Equation (36) thus becomes:

$$\begin{aligned} \nabla_{\mathbf{R}} \times \Omega^n &= i \int d^3r \frac{\partial n}{\partial t} \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n^* - i \langle \nabla_{\mathbf{R}} n | \times | \nabla_{\mathbf{R}} \frac{\partial n}{\partial t} \rangle \\ &\quad - i \int d^3r \frac{\partial n^*}{\partial t} \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n - i \langle \nabla_{\mathbf{R}} \frac{\partial n}{\partial t} | \times | \nabla_{\mathbf{R}} n \rangle. \end{aligned} \quad (40)$$

For the right-hand side, we have:

$$\begin{aligned} -\frac{\partial}{\partial t} \mathbf{B}^n &= -i \frac{\partial}{\partial t} \nabla_{\mathbf{R}} \times \langle n | \times | \nabla_{\mathbf{R}} n \rangle \\ &= -i \frac{\partial}{\partial t} \left[ \langle \nabla_{\mathbf{R}} n | \times | \nabla_{\mathbf{R}} n \rangle + \int d^3r n^* \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n \right]. \end{aligned} \quad (41)$$

This simplifies to:

$$\begin{aligned} &= -i \left[ \langle \nabla_{\mathbf{R}} \frac{\partial}{\partial t} n | \times | \nabla_{\mathbf{R}} n \rangle + \langle \nabla_{\mathbf{R}} n | \times | \nabla_{\mathbf{R}} \frac{\partial}{\partial t} n \rangle \right. \\ &\quad \left. + \int d^3r \frac{\partial}{\partial t} n^* \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n + \int d^3r n^* \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} \frac{\partial}{\partial t} n \right] \end{aligned} \quad (42)$$

Putting everything together, we once again arrive at the following result:

$$\nabla_{\mathbf{R}} \times \Omega^n = -\frac{\partial}{\partial t} \mathbf{B}^n - \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} \Phi^n, \quad (43)$$

which exactly coincides with equation (35), confirming the validity of our results. It is important to note that the Berry curvature defined through the eigenstates must be calculated as:

$$\mathbf{B}^n = \nabla_{\mathbf{R}} \times \mathbf{A}^n = i \nabla_{\mathbf{R}} \times \langle n | \nabla_{\mathbf{R}} | n \rangle = i \langle \nabla_{\mathbf{R}} n | \times | \nabla_{\mathbf{R}} n \rangle + \int d^3r n^* \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n. \quad (44)$$

In contrast to the approach originally taken by Berry, we keep the curl term. Notably, the integral term is gauge-invariant. Next, it is interesting to examine how Maxwell's equations are modified when the fields defined by  $\Psi$  are considered. In this case, we have:

$$\mathbf{B}^\Psi = \nabla_{\mathbf{R}} \times \mathbf{A}^\Psi = \nabla_{\mathbf{R}} \times \mathbf{A}^n = \mathbf{B}^n + \nabla_{\mathbf{R}} \times \int_0^t \nabla_{\mathbf{R}} \Phi^n dt'. \quad (45)$$



Thus, we find:

$$\nabla_{\mathbf{R}} \times \boldsymbol{\Omega}^{\Psi} = -\nabla_{\mathbf{R}} \times \frac{\partial}{\partial t} \mathbf{A}^{\Psi} = -\frac{\partial}{\partial t} \mathbf{B}^{\Psi}. \quad (46)$$

Since  $\boldsymbol{\Omega}^{\Psi} = \boldsymbol{\Omega}^n$  and  $\mathbf{B}^{\Psi}$  is given by equation (33), we conclude that the Maxwell equation is the same as equation (43) :

$$\nabla_{\mathbf{R}} \times \boldsymbol{\Omega}^n = -\nabla_{\mathbf{R}} \times \frac{\partial}{\partial t} \mathbf{A}^{\Psi} = -\frac{\partial}{\partial t} \mathbf{B}^{\Psi} = -\nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} \Phi^n - \frac{\partial \mathbf{B}^n}{\partial t}. \quad (47)$$

Note that the term  $\mathbf{J}_m = \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} \Phi^n$  acts as a monopole current density. This unique result, will give rise to a more comprehensive Berry-Maxwell physics.

## 1.1 Gauge Transformations

In this section, we explore the implications of gauge transformations on wavefunctions and the associated Berry phases, with particular emphasis on the presence of a time-dependent Hamiltonian. A general gauge transformation is introduced as follows:

$$|n'(\mathbf{R}, t)\rangle = e^{i\Lambda(\mathbf{R}, t)} |n(\mathbf{R}, t)\rangle, \quad (48)$$

which allows us to examine its effects on the corresponding wavefunctions, dynamical phases, and geometric potentials. A key aspect of this analysis is distinguishing between transformations that explicitly preserve the time-dependence and those that effectively lose temporal information due to the global phase factor. We investigate the resulting modifications to the Berry potentials, the electric and magnetic fields, and the structure of the Schrödinger equation under such transformations. This exploration also highlights the subtle relationship between the eigenkets and the full wavefunctions. While the fields derived from these transformations remain invariant under specific conditions, the potentials associated with the full wavefunctions provide additional information that is essential for a comprehensive formulation of the problem. Moreover, we consider the introduction of a purely time-dependent scalar potential in the Hamiltonian under such gauge transformations. This yields subtle consequences, illustrating how the transformation can be absorbed into the real scalar potential without altering the physical fields, although it does influence the explicit form of the Hamiltonian. The interplay between these effects and the preservation of key quantum properties offers a deeper understanding of gauge invariance in

quantum systems. We have previously seen that a gauge transformation of the form:

$$|n'(\mathbf{R}, t)\rangle = e^{i\Lambda(\mathbf{R})}|n(\mathbf{R}, t)\rangle \quad (49)$$

transforms the wavefunction as:

$$|\Psi'(\mathbf{R}, t)\rangle = e^{i\Lambda(\mathbf{R})}|\Psi(\mathbf{R}, t)\rangle \quad (50)$$

with

$$|\Psi(\mathbf{R}, t)\rangle = e^{i\gamma_n(\mathbf{R}, t)}e^{-i/\hbar\varepsilon_n(\mathbf{R}, t)t}|n(\mathbf{R}, t)\rangle. \quad (51)$$

Now, let us perform a time-dependent gauge transformation, defined as:

$$|n'(\mathbf{R}, t)\rangle = e^{i\Lambda(\mathbf{R}, t)}|n(\mathbf{R}, t)\rangle. \quad (52)$$

This transforms the wavefunction as follows. The phase of the wavefunction transforms as:

$$\begin{aligned} \gamma'_n(\mathbf{R}, t) &= i \int_0^t \langle n'(\mathbf{R}, t') | \frac{\partial}{\partial t'} | n'(\mathbf{R}, t') \rangle dt' \\ &= \gamma_n(\mathbf{R}, t) - \int_0^t \frac{\partial \Lambda}{\partial t'} dt' \\ &= \gamma_n(\mathbf{R}, t) - \Lambda(\mathbf{R}, t) + \Lambda(\mathbf{R}, 0). \end{aligned} \quad (53)$$

Therefore, the wavefunction transforms as:

$$\begin{aligned} |\Psi'(\mathbf{R}, t)\rangle &= e^{i(-\Lambda(\mathbf{R}, t) + \Lambda(\mathbf{R}, 0))} e^{i\gamma_n(\mathbf{R}, t)} e^{-i/\hbar\varepsilon_n(\mathbf{R}, t)t} e^{i\Lambda(\mathbf{R}, t)} |n(\mathbf{R}, t)\rangle \\ &= e^{-i\Lambda(\mathbf{R}, 0)} e^{i\gamma_n(\mathbf{R}, t)} e^{-i/\hbar\varepsilon_n(\mathbf{R}, t)t} |n(\mathbf{R}, t)\rangle. \end{aligned} \quad (54)$$

Thus, the phase factor  $\Lambda$  must appear as time-independent. This transformation only refers to the phase of the eigenstates. However, it differs from the standard gauge transformations of the real potentials (that result to real fields). This is due to the presence of the phase factor  $\gamma_n(\mathbf{R}, t)$ , which depends on the eigenstates as well. In this case, the fields described by equations (25) and (26) remain invariant, and the same holds for the fields in equations (27) and (28). Thus, this phase transformation is valid, but it does not provide complete information, as the time-dependence is "lost" in the global phase factor. Moreover, this transformation preserves the real magnetic and electric fields, as one can easily verify. One might argue that the fields are identical regardless of the choice of "basis" (i.e., whether considering the full wavefunction or the eigenkets). In response, we acknowledge that the fields remain the same, but we must also consider the potentials defined through

the full wavefunction for a more complete formulation of the problem. We define the Berry fields using the full wavefunction as follows:

$$\mathbf{A}^\Psi = i\langle\Psi(\mathbf{R}, t)|\nabla_{\mathbf{R}}|\Psi(\mathbf{R}, t)\rangle \quad (55)$$

and

$$\Phi^\Psi = -i\langle\Psi(t)|\frac{\partial}{\partial t}|\Psi(t)\rangle. \quad (56)$$

This allows us to perform a more "global" transformation on the wavefunction (i.e. just multiplying the time-dependent wave function by the following phase factor):

$$|\Psi'(\mathbf{R}, t)\rangle = e^{\frac{i}{\hbar c}\Lambda(\mathbf{R}, t)} e^{i\gamma_n(\mathbf{R}, t)} e^{-i/\hbar \int_0^t \varepsilon_n(\mathbf{R}, t') dt'} |n(\mathbf{R}, t)\rangle = e^{\frac{i}{\hbar c}\Lambda(\mathbf{R}, t)} |\Psi(\mathbf{R}, t)\rangle \quad (57)$$

This transformation preserves the real electric and magnetic fields as well. Furthermore, it also preserves the Berry fields calculated through the total wavefunction, as the phase of the eigenstates remains fixed. However, there is an important caveat. This transformation modifies the Schrödinger equation as follows:

$$H(\mathbf{R}, t)|\Psi'(\mathbf{R}, t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi'(\mathbf{R}, t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi(\mathbf{R}, t)\rangle - \hbar\frac{\partial}{\partial t}\Lambda(\mathbf{R}, t). \quad (58)$$

Thus, the transformation introduces a purely time-dependent scalar potential into the Hamiltonian, which can be absorbed into the real scalar potential without affecting the physical fields. This alteration has no physical consequences but modifies the explicit form of the Hamiltonian. Therefore, the transformation is essentially a gauge artifact, which does not affect the physical observables but must be taken into account for a comprehensive description of the system.

## 2 Berry Electromagnetism

We shall constrain ourselves to the generic 4D "phase-space" of independent variables  $(\mathbf{R}, t)$  and shall treat perturbations as time-explicit functions that influence the Hamiltonian operator. This methodology, as already seen, will allow us to formally write down the dynamic Hellmann - Feynman theorem [5] without the need of any perturbation theories. Therefore, the phase must only depend on the scalar Berry potential, this having also direct consequences on the Hellman - Feynman theorem that, using the basic equation of the definition of Berry electric field (25):

$$\Omega^\Psi = -\nabla_{\mathbf{R}}\Phi^\Psi - \frac{\partial\mathbf{A}^\Psi}{\partial t} \quad (59)$$

with

$$\frac{\partial \mathbf{A}^\Psi}{\partial t} = i \left\langle \frac{\partial \Psi}{\partial t} \right| \nabla_{\mathbf{R}} \Psi \rangle + i \langle \Psi | \nabla_{\mathbf{R}} \frac{\partial \Psi}{\partial t} \rangle = -\frac{1}{\hbar} \langle H \Psi | \nabla_{\mathbf{R}} \Psi \rangle + \frac{1}{\hbar} \langle \Psi | \nabla_{\mathbf{R}} H \Psi \rangle \quad (60)$$

by using the Schrodinger equation. Therefore, we find (we assume no non Hermiticities present in the system, for simplicity):

$$\frac{\partial \mathbf{A}^\Psi}{\partial t} = \frac{1}{\hbar} \langle \Psi | \nabla_{\mathbf{R}} H | \Psi \rangle \quad (61)$$

From (59), we find the following result:

$$\frac{1}{\hbar} \langle \Psi | \nabla_{\mathbf{R}} H | \Psi \rangle = \frac{1}{\hbar} \nabla_{\mathbf{R}} \epsilon_n - \boldsymbol{\Omega}^\Psi = \frac{1}{\hbar} \nabla_{\mathbf{R}} \epsilon_n - \boldsymbol{\Omega}^n \quad (62)$$

Because,  $\boldsymbol{\Omega}^\Psi = \boldsymbol{\Omega}^n$ , (62) is reduced to including only the eigenvectors:

$$\frac{1}{\hbar} \langle n | \nabla_{\mathbf{R}} H | n \rangle = \frac{1}{\hbar} \nabla_{\mathbf{R}} \epsilon_n - \boldsymbol{\Omega}^n \quad (63)$$

Note that, even though there is a magnetic field  $\mathbf{B}_n$ , it does not appear in the above equation. This is not surprising; rather reasonable, because all the information of the time dependence is absorbed in the electric curvature. The results are totally consistent with NIU's review article [4] as in eq. (2.5), in the adiabatic transport and the electric polarization section. But, also notice that, through eq. (47), magnetic field can also enter (63). To see how, and to further involve the Maxwell equations in our results, consider the curl of (63):

$$\nabla_{\mathbf{R}} \times \frac{1}{\hbar} \langle n | \nabla_{\mathbf{R}} H | n \rangle = -\nabla_{\mathbf{R}} \times \boldsymbol{\Omega}^n = \frac{\partial}{\partial t} \mathbf{B}^n + \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} \Phi^n \quad (64)$$

For example, when applying the above equation in solids, the crystal momentum  $\mathbf{q}$  plays the role of the parameter, and then the quantity  $\frac{1}{\hbar} \langle n | \nabla_{\mathbf{q}} H | n \rangle$  has the meaning of a velocity, i.e.  $\mathbf{v}_n(\mathbf{q}, t) = \frac{1}{\hbar} \langle n | \nabla_{\mathbf{q}} H | n \rangle$ , then eq. (63) becomes:

$$\boldsymbol{\omega} = \nabla_{\mathbf{q}} \times \mathbf{v}_n(\mathbf{q}, t) = \frac{\partial \mathbf{B}^n}{\partial t} + \nabla_{\mathbf{q}} \times \nabla_{\mathbf{q}} \Phi^n \quad (65)$$

leading us to a vorticity equation in parameter space. Also note, that the divergence of the vorticity cannot be assumed to be zero in the presence of magnetic monopoles:

$$\nabla_{\mathbf{q}} \cdot \boldsymbol{\omega} = \frac{\partial \nabla_{\mathbf{q}} \cdot \mathbf{B}^n}{\partial t} + \nabla_{\mathbf{q}} \cdot (\nabla_{\mathbf{q}} \times \nabla_{\mathbf{q}} \Phi^n) = \frac{\partial \rho_m}{\partial t} + \nabla_{\mathbf{q}} \cdot (\nabla_{\mathbf{q}} \times \nabla_{\mathbf{q}} \Phi^n) \quad (66)$$

with  $\rho_m$  the magnetic charge density. Earlier, we saw that the pathological term  $\mathbf{J}_m = \nabla_{\mathbf{q}} \times \nabla_{\mathbf{q}} \Phi^n$  serves as a magnetic current density, leading us to the following continuity equation for the magnetic charge:. Eq. (66) becomes

$$\nabla_{\mathbf{q}} \cdot \boldsymbol{\omega} = \frac{\partial \rho_m}{\partial t} + \nabla_{\mathbf{q}} \cdot \mathbf{J}_m \quad (67)$$

Berry monopoles are associated with degeneracies in the energy bands of a quantum system. If the system undergoes a topological phase transition, such as when bands merge or split, the Berry curvature can change discontinuously, effectively creating or annihilating Berry monopoles. This would lead to a nonzero source term, equal to  $\nabla_{\mathbf{q}} \cdot \boldsymbol{\omega}$ . In addition, perturbations, such as changes in the Hamiltonian's parameters, can alter the Berry curvature distribution. For example, in systems where the Hamiltonian is time-dependent, the Berry curvature and monopole density may evolve dynamically, potentially leading to apparent nonconservation. On the other hand, let us examine the divergence of eq. (63):

$$\nabla_{\mathbf{q}} \cdot \mathbf{v}_n = \frac{1}{\hbar} \nabla_{\mathbf{q}}^2 \epsilon_n - \nabla_{\mathbf{q}} \cdot \boldsymbol{\Omega}^n \quad (68)$$

with  $\nabla_{\mathbf{q}} \cdot \boldsymbol{\Omega}^n = \rho_{el}$ , with  $\rho_{el}$  the electric charge density.

$$\nabla_{\mathbf{q}} \cdot \mathbf{v}_n = \frac{1}{\hbar} \nabla_{\mathbf{q}}^2 \epsilon_n - \rho_{el} \quad (69)$$

From this, we can calculate the electric charge density as:

$$\rho_{el} = \nabla_{\mathbf{q}} \cdot \left( \frac{1}{\hbar} \nabla_{\mathbf{q}} \epsilon_n - \mathbf{v}_n \right) \quad (70)$$

Therefore, the quantity  $\frac{1}{\hbar} \nabla_{\mathbf{q}} \epsilon_n - \mathbf{v}_n$  acts as an electric Polarization density in parameter space. In addition, we observe that, the Berry electric field is associated with fictitious electric charges in parameter space, that can be directly linked with physical observables. Note that, for the case of Bloch solids, and by viewing the crystal momentum  $\mathbf{q}$  as the fixed parameter, the above equation results in:

$$\rho_{el} = \frac{\hbar}{m^*} - \nabla_{\mathbf{q}} \cdot \mathbf{v}_n(\mathbf{q}, t) \quad (71)$$

Let us now return to the magnetic current density definition:

$$\mathbf{J}_m = \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} \Phi^n \quad (72)$$

Notice that, in terms of the eigenvectors, (72) can be written as:

$$\mathbf{J}_m = -i\langle \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n | \frac{\partial n}{\partial t} \rangle - i\langle n | \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} \frac{\partial n}{\partial t} \rangle \quad (73)$$

Also notice that, from (64) we managed to connect the magnetic current density with an observable: the actual velocity of the particle:

$$\mathbf{J}_m = \nabla_{\mathbf{R}} \times \mathbf{v}_n(\mathbf{R}, t) - \frac{\partial \mathbf{B}^n}{\partial t} \quad (74)$$

Also notice that, using  $\nabla_{\mathbf{R}} \cdot \mathbf{B}^n = \rho_m$ , we can also calculate the magnetic charge density using eq. (44):

$$\rho_m = i\langle \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n | \nabla_{\mathbf{R}} n \rangle + i\langle n | \nabla_{\mathbf{R}} \cdot (\nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} n) \rangle \quad (75)$$

Note that if we integrate the above equation with respect to a surface in parameter space, we get:

$$\int \mathbf{J}_m \cdot d\mathbf{S} = \int \nabla_{\mathbf{R}} \times \mathbf{v}_n(\mathbf{R}, t) d\mathbf{S} - \frac{\partial}{\partial t} \int \mathbf{B}^n \cdot d\mathbf{S} \quad (76)$$

This equation returns the magnetic current as follows:

$$\mathbf{I}_m = \oint \mathbf{v}_n(\mathbf{R}, t) \cdot d\mathbf{R} - \frac{\partial}{\partial t} \Phi^S(t) \quad (77)$$

for an open surface S.

### 3 Conclusions

In this work, we have developed a comprehensive electromagnetic analogy for quantum systems evolving under adiabatic and nonadiabatic conditions. By redefining the Berry connection and curvature in terms of the full time-dependent wavefunction, rather than eigenstates alone, we uncovered the existence of a Berry electric field even in the absence of explicit time dependence in the parameters. This finding challenges the traditional interpretation of the Berry phase as purely geometric and path-dependent, revealing instead that it carries a well-defined field-theoretic structure governed by the wavefunction's local properties in time and parameter space. We constructed and verified a complete set of Berry–Maxwell equations, establishing a duality between parameter-space dynamics and classical electromagnetic theory. In doing so, we defined: a Berry electric field from the time derivative of the Berry connection, a Berry magnetic field from the curl of the Berry

connection, an emergent electric and magnetic charge densities arising from divergences in the curvature, and current-like quantities including Berry vorticity and magnetic current density. Crucially, we showed that these Berry field quantities appear directly in observable physical expressions, such as the expectation value of the gradient of the Hamiltonian, thus linking the geometric phase to measurable transport phenomena. Our approach generalizes the Hellmann–Feynman theorem, incorporates topological defects such as Berry monopoles, and reveals conservation and continuity laws associated with charge and current in parameter space. This field-theoretic perspective paves the way for a new understanding of emergent electromagnetism in quantum systems. It unifies Berry curvature corrections to dynamics, adiabatic transport, and polarization theory under a single covariant formalism, providing a natural framework to study topological phases, band degeneracies, and time-dependent Hamiltonians in both solid-state systems and more abstract quantum spaces. The implications are far-reaching: from novel interpretations of adiabatic dynamics to engineering artificial gauge fields in quantum materials or cold atoms, this reformulation offers a powerful lens through which the geometry and topology of quantum systems can be understood as real, dynamical fields with observable consequences.

## References

- [1] M. V. Berry, “Quantal phase factors accompanying adiabatic changes,” *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, vol. 392, no. 1802, pp. 45-57, Mar. 1984.
- [2] Georgios Konstantinou, Kyriakos Kyriakou, Konstantinos Moulopoulos, *Emergent Non-Hermitian Contributions to the Ehrenfest and Hellmann-Feynman Theorems*, *IJEIR*, **5**(4), 2016, 248–252. ISSN: 2277-5668.
- [3] Georgios Konstantinou and Konstantinos Moulopoulos, *Topological anomalies in the off-diagonal Ehrenfest theorem and their role on optical transitions in solar cells*, *Journal of Physics Communications*, **2**(8), 2018, 085011.
- [4] Di Xiao, Ming-Che Chang, and Qian Niu, “Berry phase effects on electronic properties,” *Reviews of Modern Physics*, vol.82, no.3, pp.1959–2007, Jul. 2010.
- [5] Kyriakos Kyriakou and Konstantinos Moulopoulos, *Dynamical extension of Hellmann-Feynman theorem and application to nonadiabatic quantum*

*processes in Topological and Correlated Matter*, arXiv:1506.08812 [cond-mat.mes-hall], 2020.