Sandwich test for Quantum Phase Estimation

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Quantum Phase Estimation (QPE) has potential for a scientific revolution through numerous practical applications like finding better medicines, batteries, materials, catalysts etc. Many QPE algorithms use the Hadamard test to estimate $\langle \psi | U^k | \psi \rangle$ for a large integer k for an efficiently preparable initial state $|\psi\rangle$ and an efficiently implementable unitary operator U. The Hadamard test is hard to implement because it requires controlled applications of U^k which increase the circuit depth T_{max} by a factor of $\mathcal{O}(n)$ where n is the system size. But the total run time T_{tot} of the Hadamard test scales only as $\mathcal{O}(k/\epsilon^2)$ where ϵ is the desired accuracy of estimation. Recently, a Sequential Hadamard test (SHT) was proposed (arXiv:2506.18765) which requires controlled application of U only, improving T_{max} by a factor of $\mathcal{O}(k)$. But the bottleneck is that T_{tot} of SHT scales as $\mathcal{O}(k^3/\epsilon^2 r_{\text{min}}^2)$ where r_{min} is the minimum value of $|\langle \psi | U^{k'} | \psi \rangle|$ among all integers $k' \leq k$. Typically r_{min} is exponentially low and SHT becomes too slow.

We present a new algorithm, the SANDWICH test to address this bottleneck. Our algorithm uses efficient preparation of the initial state $|\psi\rangle$ to efficiently implement the SPROTIS operator R_{ψ}^{ϕ} where SPROTIS stands for the Selective Phase Rotation of the Initial State. It sandwiches the SPROTIS operator between U^a and U^b for integers $\{a,b\} \leq k$ to estimate $\langle \psi | U^k | \psi \rangle$. The circuit depth of the Sandwich test is almost same as that of SHT. The total run time $T_{\rm tot}$ is $\mathcal{O}(k^2 \ln k/\epsilon^2 s_{\rm min}^6)$. Here $s_{\rm min}$ is the minimum value of $|\langle \psi | U^{\hat{k}} | \psi \rangle$ among all integers \hat{k} which are values of the nodes of a random binary sum tree whose root node value is k and leaf nodes' values are 1 or 0. It is difficult to analytically prove that $s_{\rm min} \not \ll 1$.But it can be reasonably expected in typical cases because there is extremely wide freedom in choosing the random binary sum tree. Numerical experiments are needed to confirm this.

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I. INTRODUCTION

Quantum Phase Estimation (QPE) is the holy grail of quantum simulation. Here the goal is to estimate the eigenvalues of a quantum system. In general, this is a hard problem [1–7]. But it is tractable in many cases of interest where we can efficiently prepare a quantum state with non-negligible overlap with the relevant eigenstates. It is widely believed that using QPE, a Quantum Computer (QC) can provide exponential advantage over the classical computers to find the eigenvalues but this belief is also questioned in [8]. QPE has potential to bring a scientific revolution through numerous practical applications like discovery of better medicines, batteries, materials, catalysts etc [9–17]. More details are provided in the review articles [18–20].

There are several algorithms for QPE using various methods including Quantum Fourier Transform [21], semiclassical Fourier transform [22], variational methods [23–25], Fourier-Filtering methods [26–30], matrix pencil methods [31, 32], quantum imaginary time evolution (QITE) [33], robust phase estimation [34–38], Krylov-subspace methods [39, 40], and other important methods [41–44]. Apart from quantum simulation, QPE also has important applications in quantum al-

gorithms [45–51], quantum metrology [52–55], quantum field theory [56], error mitigation [57] etc.

Many algorithms for QPE use a simple quantum algorithm, the Hadamard test, to estimate the complex number $\langle \psi | U^k | \psi \rangle$ for a large integer k, an efficiently preparable initial state $|\psi\rangle$, and an efficiently implementable unitary operator U, whose eigenvalues need to be estimated through QPE. Typically, it is the time evolution operator corresponding to a time-independent Hamiltonian of a quantum system.

The actual physical implementation of the Hadamard test is very hard because of the necessity of the c– U^k operator, the controlled application of U^k using an ancilla qubit, also known as the control qubit. Usually there is a constraint of locality on a Quantum Computer. It means that in one time step, we can implement only one local unitary operator which acts only on the neighboring qubits. The c– U^k operator is not a local operator because it acts on all qubits. The control qubit can be distributed onto a GHZ state of O(n) ancilla qubits [58–63] so that each of the system's n qubits has a neighboring ancilla qubit. But this will be very challenging as so much of ancilla qubits greatly adds to the decoherence of Quantum Computer.

To precisely explain the locality constraint, let $T_{\max}(V)$ be the circuit depth required to implement any unitary operator V on a Quantum Computer with the locality constraint. Then

$$T_{\max}(U^k) = kT_{\max}(U). \tag{1}$$

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We note that the efficient preparation of $|\psi\rangle$ means that we can efficiently implement a unitary operator W which transforms $|0^n\rangle$ to $|\psi\rangle$ where $|0^n\rangle$ is the all-zero state in which all n qubits are in $|0\rangle$ state. Typically

$$T_{\text{max}}(W) \ll T_{\text{max}}(U) \ll T_{\text{max}}(c-U).$$
 (2)

So the circuit depth of the Hadamard test is

$$T_{\text{max}}(\text{Hadamard}) \approx T_{\text{max}}(\text{c-U}^{\text{k}}) = \text{kT}_{\text{max}}(\text{c-U})$$
 (3)

To find $T_{\max}(c-U)$, we note that U can be decomposed into $L = T_{\max}(U)$ layers where each layer is a unitary operator U_l $(l \in \{1, 2, ..., L\})$ of circuit depth 1. So

$$U = \prod_{l=1}^{L} U_l, \quad T_{\text{max}}(U_l) = 1, \quad l \in \{1, 2, \dots L\}.$$
 (4)

Each U_l consists of local unitary operators which can be applied in parallel. The only way to locally implement c-U is to perform the following operations for each U_l :

- 1. We apply $\mathcal{O}(n)$ swaps to make the control qubit a neighbor of each of the n qubits one-by-one. The control qubit has different neighbors after each swap.
- 2. Between each of these swaps, we use the control qubit for a controlled application of the local unitary operator which acts on the neighboring qubits of the control qubit in U_l .
- 3. Then we apply the swap operators again to bring back the control qubit to its original position.

These steps cannot be applied in parallel. Hence the circuit depth of the controlled application of each layer is $\mathcal{O}(n)$ and the total circuit depth increases by the same factor. So Eq. (3) can be rewritten as

$$T_{\text{max}}(\text{Hadamard}) \approx kT_{\text{max}}(c-U) = \mathcal{O}(kn)T_{\text{max}}(U).$$
 (5)

Without the locality constraint, it would have been only $kT_{\max}(U)$. Hence the circuit depth increases by the factor of $\mathcal{O}(n)$ because of the locality constraint. This is a huge increase in view of the fact that it is extremely difficult to develop a Quantum Computer of large depth because of decoherence.

To address these bottlenecks, it is necessary to reduce the requirement of controlled operators. Several algorithms have been presented to do this but they have their own limitations. In [64, 65], it was shown that if we can prepare a superposition of $|\psi\rangle$ with an eigenstate of U with a known eigenvalue then such controlled operations are not needed. But preparing such superpositions is itself highly susceptible to noise [60–62, 66–68]. Another recent approach [69] is to use classical phase-retreival methods. But its performance has been demonstrated only numerically, not analytically.

A better algorithm is presented in [70] using the Quantum Imaginary time evolution (QITE) [33]. It requires

 $|\psi\rangle$ to have a short correlation length and U to be a time evolution operator of a local Hamiltonian. Very recently, an algorithm of the Sequential Hadamard test (SHT) was presented in [71] to relax these requirements also. It requires controlled application of U only, improving $T_{\rm max}$ by a factor of $\mathcal{O}(k)$. To estimate $\langle \psi|U^k|\psi\rangle$, SHT has to find and sum over the phase-differences between $\langle \psi|U^{k'-1}|\psi\rangle$ and $\langle \psi|U^{k'}|\psi\rangle$ for all integers $k' \leq k$. The bottleneck is that the total run time $T_{\rm tot}$ of SHT scales as $\mathcal{O}(k^3/\epsilon^2 r_{\rm min}^2)$ where ϵ is the desired accuracy of estimation and $r_{\rm min}$ is minimum value of $|\langle \psi|U^{k'}|\psi\rangle|$ among all integers $k' \leq k$. Typically $r_{\rm min}$ is exponentially low so SHT will become very slow.

We point out that in original paper [71], SHT was presented in a different context where a unitary operator U was written as a product of N_{gates} local gates u_l . In this context, they found the total sample complexity (total number of projective measurements) to be $\mathcal{O}(N_{\text{gates}}^2 \epsilon^{-2} r_{\min}^{-2})$. This can be easily generalized to the case of QPE where U^k can be written as a product of k operators U. The only difference is that in QPE, the total run time is not quantified in terms of the total number of measurements but the total number of required applications of U. In the k'^{th} iteration of SHT ($k' \leq k$), it will need k' applications of U. Furthermore, SHT requires a total of k iterations for all k'. This is why the total run time scales as k^3 , not k^2 as it prima facie appears from N_{gates}^2 dependance shown in the original paper [71].

In this paper, we present a new quantum algorithm which we name as the SANDWICH test. It is based on a crucial finding that so far, we have not completely harnessed our ability to efficiently prepare $|\psi\rangle$. This ability is used only at the *state* level to prepare $|\psi\rangle$ or at the measurement level to perform the projective measurement onto $|\psi\rangle\langle\psi|$. We point out that this ability can also be used to implement the SPROTIS operator R_{ψ}^{ϕ} where SPROTIS stands for the Selective Phase Rotation Of The Initial State. Mathematically,

$$R_{n}^{\phi} = 1_N + \Phi |\psi\rangle\langle\psi| = W R_{0n}^{\phi} W^{\dagger}, \tag{6}$$

where 1_N is the $N \times N$ identity operator $(N = 2^n)$, $R_{0^n}^{\phi}$ is the selective phase rotation of all-zero state $|0^n\rangle$, and Φ is a complex number given by

$$\Phi = e^{i2\phi} - 1 = 2\sin\phi e^{i(\phi + \pi/2)}.$$
 (7)

We show that the SPROTIS operator is very useful to estimate $\langle \psi | U | \psi \rangle$. Our algorithm is named the Sandwich test because it sandwiches the SPROTIS operator between two unitary operators U^a and U^b for integers $\{a,b\} \leq k$ to implement the Sandwich operator $S = U^a R^{\phi}_{sb} U^b$.

Like SHT, the Sandwich test also needs controlled application of only U, not for U^k for k>1 as required in the Hadamard test. The only additional operator in the Sandwich test is $R_{0^n}^{\phi}$ which is a multi-qubit controlled 2×2 unitary gate. This can be efficiently implemented

with a circuit depth of $\Theta(n)$ using the algorithms presented in [72, 73] if there is no locality constraint. But the locality constraint increases the depth by a factor of $\mathcal{O}(n)$ because of the similar reasons as given for Eq. (5). Typically $T_{\text{max}}(c-U) \gg \mathcal{O}(n^2)$ hence this overhead in the circuit depth is negligible compared to other unitary operators.

The total run time of the Sandwich test scales as $\mathcal{O}(k^2/\epsilon^2 s_{\min}^6)$ where we have ignored polylog(k) factors). Here s_{\min} is the minimum value of $|\langle \psi| U^{\hat{k}} |\psi \rangle$ among all integers \hat{k} which are values of the nodes of a random binary sum tree whose root node value is k and leaf nodes' values are 1 or 0. It is difficult to analytically prove that s_{\min} is not very small. But it can be reasonably expected in typical cases because there is an extremely wide freedom in choosing the random binary sum tree. Numerical experiments are needed to confirm this.

In the context of QPE with an inexact (approximate) eigenstate, the importance of the SPROTIS operator was shown by the author long back in [74]. There it was used to improve the spatial complexity of the Eigenpath Traversal Algorithm by Boixo, Knill, and Somma [75] with applications to quantum simulation and optimization. At that time, not much attention was given in the quantum computing community for QPE with inexact eigenstates. But now this topic has received wide attention. Hence the importance of the SPROTIS operator for this task needs to be understood in a better way as done in this paper.

The paper is organized as following. The Sandwich test is presented in the next Section. Then we conclude with some discussion in Section III.

II. THE SANDWICH TEST

A. Basic Idea

The basic ingredient of the Sandwich test is the Sandwich operator S. For any two integer powers, U^a and U^b , of unitary operator U, the Sandwich operator is given by

$$S = U^a R^{\phi}_{ab} U^b. \tag{8}$$

So, without any sandwich, S is simply U^{a+b} . We define

$$r_{a}e^{i\theta_{a}} = \langle \psi | U^{a} | \psi \rangle,$$

$$r_{b}e^{i\theta_{b}} = \langle \psi | U^{b} | \psi \rangle,$$

$$r_{a+b}e^{i\theta_{a+b}} = \langle \psi | U^{a+b} | \psi \rangle,$$

$$s_{ab} = |\langle \psi | S | \psi \rangle|$$
(9)

The phase of $\langle \psi | S | \psi \rangle$ is irrelevant for the Sandwich test. The quantity s_{ab} can be simplified using Eqs. (6-9) as

$$s_{ab} = \left| \langle \psi | U^a R_{\psi}^{\phi} U^b | \psi \rangle \right|$$
$$= \left| \langle \psi | U^a (1_N + \Phi | \psi \rangle \langle \psi |) U^b | \psi \rangle \right|$$

$$= \left| \langle \psi | U^{a+b} | \psi \rangle + \Phi \langle \psi | U^{a} | \psi \rangle \langle \psi | U^{b} | \psi \rangle \right|$$

$$= \left| r_{a+b} e^{i\theta_{a+b}} + \Phi r_{a} r_{b} e^{i(\theta_{a}+\theta_{b})} \right|$$

$$= \left| r_{a+b} + 2r_{a} r_{b} \sin \phi e^{i(\theta_{a}+\theta_{b}-\theta_{a+b}+\phi+\pi/2)} \right|$$

$$= \sqrt{r_{a+b}^{2} + 4r_{a}^{2} r_{b}^{2} \sin^{2} \phi - 4r_{a+b} r_{a} r_{b} \sin \phi \sin \omega_{a+b}}.$$
(10)

So we have

$$\sin \omega_{a+b} = \frac{4r_a^2 r_b^2 \sin^2 \phi + r_{a+b}^2 - s_{ab}^2}{4r_{a+b}r_a r_b \sin \phi}.$$
 (11)

The quantity ω_{a+b} in above equations satisfies the following equation

$$\theta_{a+b} = \theta_a + \theta_b - \omega_{a+b} + \phi. \tag{12}$$

Thus we can estimate θ_{a+b} using the estimates of the quantities $\{\theta_a, \theta_b, \omega_{a+b}\}$ as ϕ is already exactly known to us. The estimates of the quantities θ_a and θ_b are assumed to be available to us through further Sandwich tests as explained later. Eq. (11) implies that estimates of the quantities $\{r_a, r_b, r_{a+b}, s_{ab}\}$ can be used to get an estimate of $\sin \omega_{a+b}$. There can be two possible estimates of ω_{a+b} for a given estimate of $\sin \omega_{a+b}$. Two different values of ϕ can be used to resolve this ambiguity.

Eq. (9) implies that the quantities $\{r_a, r_b, r_{a+b}, s_{ab}\}$ can be estimated by repeatedly preparing $|\psi\rangle$, applying the corresponding unitary operator $\{U^a, U^b, U^{a+b}, S\}$ on it and then averaging over the projective measurements of the resultant state onto $|\psi\rangle\langle\psi|$. These projective measurements are done by applying W^{\dagger} on the resultant state and then measuring it in the computational basis of all n qubits. We define

$$s_{\min} = \min\{r_a, r_b, r_{a+b}\}.$$
 (13)

We can always choose ϕ to be $\Theta(1)$. Then Eq. (11) and the basic principles of statistics imply that $\Theta(ms_{\min}^{-6})$ projective measurements are required to get an estimate of ω_{a+b} with a variance of 1/m. Thus $\Theta(1/\epsilon^2 s_{\min}^2)$ measurements will give an estimate of ω_{a+b} to an accuracy of ϵ

B. Random Binary sum tree

The Sandwich test can be used to estimate $\langle \psi | U^k | \psi \rangle$ by estimating its argument θ_k defined by

$$r_k e^{i\theta_k} = \langle \psi | U^k | \psi \rangle. \tag{14}$$

The modulus r_k can always be estimated using projective measurements onto $|\psi\rangle\langle\psi|$ as explained earlier. Neither Sandwich test nor Hadamard test is needed to estimate r_k . It is θ_k which is really challenging to estimate wihout any controlled application of U^k .

To present the Sandwich test to estimate θ_k , let us consider a random binary sum tree. The value of a parent node of this sum tree is equal to the sum of the values of the two children nodes of that parent node. Also, children nodes are one step higher in position than their parent node. Let the nodes at height h be denoted by indices p_h . As there are 2^h nodes at height h, we have

$$p_h \in \{1, 2, \dots, H\}, \quad H = 2^h.$$
 (15)

The height of the tree is h_{max} , the maximum value of h, and the total number of nodes of this tree is

$$\sum_{h=0}^{h_{\text{max}}} 2^h = 2^{h_{\text{max}}+1} - 1. \tag{16}$$

The value of the root node at height 0 of this sum tree is chosen to be k. We also denote it by $\hat{k}_0^{p_0=1}$. The hat notation over k is used to avoid confusion with the notation k^2 or k^3 used for integer powers of k in the context of the total run time complexity.

To choose the values of other nodes, we randomly choose numbers $x_h^{p_h}$ and $y_h^{p_h}$ satisfying

$$x_h^{p_h} + y_h^{p_h} = 1 (17)$$

where

$$0 < x_{\min} \le x_h^{p_h} \le y_h^{p_h} < y_{\max} = 1 - x_{\min} < 1.$$
 (18)

The values of the children nodes are randomly chosen integers (including 0) using the following recursive relations

$$\hat{k}_{h+1}^{p_{h+1}=2p_h-1} = \lceil x_h^{p_h} \hat{k}_h^{p_h} \rceil, \quad \hat{k}_{h+1}^{p_{h+1}=2p_h} = \lfloor y_h^{p_h} \hat{k}_h^{p_h} \rfloor$$
 (19)

Note that the nodes $p_{h+1} = \{2p_h - 1, 2p_h\}$ located at height h+1 denote the children nodes of the parent node denoted by the index p_h located at height h.

The above recursive relations ensure that the tree is a random binary sum tree. Also, if the value of parent node is 2 then both of its children nodes have value 1. But if the value of parent node is 1 then one of its children node has value 1 while other has value 0. We choose $h_{\rm max}$ to be the minimum value of h for which values of all leaf nodes at height h are either 0 or 1 but never more than 1. We note that Eq. (18) imply that with each step, the value of node decreases by the maximum factor of $y_{\rm max}$ which can always be chosen such that

$$y_{\text{max}}^{h_{\text{max}}} k = 1 \Longrightarrow h_{\text{max}} = \Theta(\ln k).$$
 (20)

Thus h_{max} scales logarithmically with k.

C. Sandwich test using the Sum tree

In the Sandwich test, we start with the leaf nodes at height h_{max} . As the values of leaf nodes are either 0 or 1, the corresponding unitary operators are either $1_{N\times N}$

or U. We have already mentioned in Section I that like SHT, the Sandwich test also estimates $\langle \psi | U | \psi \rangle$ using the Hadamard test by controlled applications of U. After getting this estimate, we use the Sandwich tests to estimate $\langle \psi | U^{\hat{k}} | \psi \rangle$ for those values of \hat{k} corresponding to the parent nodes at height $h_{\text{max}} - 1$. Then we use these estimates to estimate $\langle \psi | U^{\hat{k}} | \psi \rangle$ for those values of \hat{k} corresponding to the parent nodes at height $h_{\text{max}} - 2$. We continue this till we get to the root node which gives us an estimate of the desired quantity $\langle \psi | U^k | \psi \rangle$. Precisely, let us define

$$\hat{\theta}_h^{p_h} = \arg \langle \psi | U^{\hat{k}_h^{p_h}} | \psi \rangle, \tag{21}$$

Then Eqs. (12), (17), and (19) imply that

$$\hat{\theta}_{h}^{p_{h}} = \theta_{h+1}^{p_{h+1}=2p_{h}-1} + \theta_{h+1}^{2p_{h}} - \hat{\omega}_{h}^{p_{h}} + \phi, \tag{22}$$

where $\hat{\omega}_h^{p_h}$ denotes the random variable obtained during the Sandwich test to estimate $\hat{\theta}_h^{p_h}$. The above recursive relation can be easily solved to get

$$\theta_k = \hat{\theta}_0^1 = k\theta_1 + \sum_{q=1}^{N_s} \omega_q + \sum_{q=1}^{N_s} \phi_q.$$
 (23)

Here N_s denotes the total number of Sandwich tests needed to go to the root node. This is equal to the number of non-trivial parent nodes in the sum tree which are those parent nodes whose both children nodes have value different from 0. This is because if value of any of the two children nodes of a parent node is 0 then the $\hat{\theta}$ value of a parent node is same as that of its non-zero children node. In this case, no Sandwich test is needed to estimate it.

To put an upper bound on N_s , note that at any particular height, the sum of values of all nodes is equal to k. Furthermore, a non-trivial parent node must have a minimum value of 2. So at any particular height, there can be a maximum of k/2 non-trivial parent nodes. Hence Eq. (20) implies that

$$N_s \le \frac{k}{2} h_{\text{max}} = \Theta(k \ln k). \tag{24}$$

This upper bound may be improved to $\mathcal{O}(k)$ but in this paper, our main aim is to present the Sandwich test.

So there are only $\Theta(k \ln k)$ independent random variables, apart from θ_1 , which are needed to estimate θ_k . The quantity $k\theta_1$ can be estimated to an accuracy of ϵ by estimating θ_1 to an accuracy of ϵ/k which requires $\Theta(k^2\epsilon^{-2})$ Hadamard tests. This requires same total run time also as each Hadamard test required only one application of U. As explained in Subsection II.A, $\Theta(ms_{\min}^{-6})$ measurements are needed to estimate any ω_q with a variance of 1/m. As the variance of sum of independent random variables is the sum of variances of these variables, we need to choose $m \gg k \ln k$ to get an estimate of ϕ_k with a small variance. Precisely if ϵ is the desired accuracy of estimation then ϵ^2 is the desired variance and m

should be chosen to be $\Theta(k \ln k/\epsilon^2)$. Furthermore, Sandwich tests needed to estimate θ_k require a maximum of $\Theta(k)$ applications of U. Summing these facts, we find that the total run time needed to estimate θ_k is

$$T_{\text{tot}} = \Theta\left(\frac{k^2 \ln k}{\epsilon^2 s_{\min}^6}\right). \tag{25}$$

Obviously, it is very difficult to put a lower bound on the quantity s_{\min} but certainly it is much larger than r_{\min} defined in the context of the Sequential Hadamard test. Note that Eq. (13) implies that s_{\min} is the minimum value of $|\langle \psi | U^{\hat{k}} | \psi \rangle|$ among all values of \hat{k} corresponding to all nodes of our sum tree. We can randomly choose any sum tree and we can reasonably expect it to be large enough in typical cases. Numerical experiments are needed to confirm this.

In QPE algorithms, we randomly choose k to estimate a Fourier convolution function to an accuracy of $\mathcal{O}(\eta)$ where η is the minimum overlap of the initial state with the ground state of the system and it is assumed that $\eta \not \ll 1$ [26–30]. Obviously if $|\langle \psi|U^k|\psi \rangle| \ll 1$ then it does not contribute much to the estimate of convolution function and hence such k's can be safely ignored. Thus, for practical purpose, s_{\min} is actually the minimum value of $|\langle \psi|U^{\hat{k}}|\psi \rangle|$ among all values of \hat{k} corresponding to all nodes of our sum tree except the root node.

To give a plausible argument, suppose the quantity $|\langle\psi|U^{k'}|\psi\rangle| \ll 1$ for $k' \in \{1,2,\ldots,100\}$ and we want to estimate $\langle\psi|U^k|\psi\rangle$ for k=300. Then if we are using the Sequential Hadamard test (SHT), a small value of $|\langle\psi|U^{k'}|\psi\rangle|$ for k'=101 is enough to slow down the algorithm. This is because the SHT has to cross through all integers $k' \leq k$. But the Sandwich test allows us to jump over the bad k''s. In this example, even if all $|\langle\psi|U^{k'}|\psi\rangle|$ for $k' \in \{101,102,\ldots,198\}$ are very small, it does not matter to us as we can use Sandwich test to estimate $\langle\psi|U^{k'}|\psi\rangle$ for k'=99 and k'=100. Furthermore, the quantities $\langle\psi|U^{k'}|\psi\rangle$ for bad set $k' \in \{101,102,\ldots,198\}$ are not relevant for QPE as they do not contribute much to the convolution function which is being estimated.

We can also use multi-layer Sandwich tests if required.

For example, we can apply two-layer Sandwich operators like $U^a R_{\psi}^{\phi_1} U^b R_s^{\phi_2} U^c$. It will help us to estimate θ_{a+b+c} in terms of θ_a , θ_b , and θ_c . The angles ϕ_1 and ϕ_2 can be chosen to have multiple values to refine our estimate. In above example, it will help us to estimate $\langle \psi | U^{k'} | \psi \rangle$ for k' = 199 if the quantities $|\langle \psi | U^{k'} | \psi \rangle| \not \ll 1$ for $k' \in \{1,2,\ldots,66\}$. We can easily generalize it to higher-layer Sandwich operators. But their analysis will be increasingly complicated. Numerical experiments can be used to study them.

III. CONCLUSION AND DISCUSSION

We have presented a new quantum algorithm, the Sandwich test, to estimate $\langle s|U|s\rangle$ for an efficiently preparable initial state $|s\rangle$ and an efficiently implementable unitary operator U. The Sandwich test provides $\mathcal{O}(n)$ factor improvement of circuit depth complexity over the Hadamard test. The Sandwich test is as simple as the Hadamard test. The main difference is that it uses the SPROTIS operator which is not used by the Hadamard test. We have shown that the SPROTIS operator is a useful quantum resource for the important task of estimating $\langle s|U|s\rangle$. We only need to sandwich the SPROTIS operator between properly chosen unitary operators.

Though we have presented the Sandwich test in the context of QPE, it can easily be generalized to the case considered in [71] while presenting the Sequential Hadamard test. This is because the Sandwich test works if the SPROTIS operator is sandwiched between any two unitary operators, not just two integer powers of a particular unitary operator.

Furthermore, we can also easily generalize the Sandwich test to continuous time setting even though we have presented it only in a discrete-time setting.

We also note that if we have a Quantum Computer of large enough circuit depth then we can use Hadamard test to estimate $\langle \psi | U^{k'} | \psi \rangle$ for $k' \geq 1$ also, not just for k' = 1 as discussed in the paper. Then we can use these values of k' for the leaf nodes of our sum tree. It is easy to show that this will reduce the total run time by a factor of $\Theta((k')^2)$ as the Hadamard test is more economic as far as the total run time is concerned.

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