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# Mathematical Part

## Problem 1: Calculate derivate and double derivative of Gaussian filter

2D Gaussian filter is given by:

$$G(x,y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Ignoring the constant term $\frac{1}{\sqrt{2\pi}\sigma}$

$$G(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}}$$

let $g'x = \frac{\partial G}{\partial x}$ and $g'y = \frac{\partial G}{\partial y}$

$$g'x = e^{-\frac{x^2+y^2}{2\sigma^2}} \left[\frac{-2x}{2\sigma^2}\right] \longrightarrow$$ expanded through Chainrule

$$\boxed{g'x = \frac{-x}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}}$$

$$g'y = e^{-\frac{x^2+y^2}{2\sigma^2}} \left[-\frac{2y}{2\sigma^2}\right] \longrightarrow$$ expanded through Chain rule

$$\boxed{g'y = \frac{-y}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}}$$

Hence first derivative of gaussian filter is given by

$$g'x = \frac{-x}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$g'y = \frac{-y}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

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# Problem 1: Second derivate of Gaussian

2nd derivate of gaussian is given by taking the first derivative of $g'x$ with respect to $x$ and $g'y$ with respect to $y$

let $g''x = \frac{\partial^2 G}{\partial x^2}$ and $g''y = \frac{\partial^2 G}{\partial y^2}$

$g'x = \frac{-x}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$ and $g'y = \frac{-y}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$

then:

$$g''x = \frac{-1}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} + \left(\frac{-x}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}\left(-\frac{2x}{2\sigma^2}\right)\right) \longrightarrow$$

① Apply product rule: $f(x)G(x) = f'(x)g(x) + f(x)g'(x)$

② Apply chain rule

$$g''x = \frac{-1}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} + \frac{x^2}{\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\boxed{g''x = \frac{x^2-\sigma^2}{\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}}$$

$$g''y = \frac{-1}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} + \left(\frac{-y}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}\left(\frac{-2y}{2\sigma^2}\right)\right) \longrightarrow$$

① Apply product rule: $f(x)G(x) = f'(x)g(x) + f(x)g'(x)$

② Apply chain rule

$$g''y = \frac{-1}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} + \frac{y^2}{\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\boxed{g''y = \frac{y^2-\sigma^2}{\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}}$$

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Problem 2: Prove that Convolutions are commutative $a*b = b*a$

Convolutions are weighted summation, can be proved mathematicaly

$$a(x) * b(x) = b(x) * a(x)$$

$$a(x) * b(x) = \sum_{j=-\infty}^{\infty} a(j) . b(x-j)$$

if $x-j = z$ then $j = x-z$

then if $j = -\infty \Rightarrow z = \infty$

or $j = \infty \Rightarrow z = -\infty$

$$a(x) * b(x) = \sum_{z=\infty}^{-\infty} a(x-z) . b(z)$$

$$= \sum_{z=-\infty}^{\infty} a(x-z) . b(z)$$

$$= \sum_{j=-\infty}^{\infty} a(x-j) . b(j)$$

$$= \sum_{j=-\infty}^{\infty} b(j) . a(x-j)$$

$$\boxed{a(x) * b(x) = b(x) * a(x)}$$

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Problem 3: Calculate the rank of given matrices

A = [[1 2 3] [3 6 9]]

B = [[1 2 3] [4 5 6] [7 8 9]]

Taking determinant of A

order of A = 2*3 hence max rank of A = 2

by taking determinant of $\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix}$ sub matrix of A

determinant = (2×9) - (3×6) = 0

Rank of matrix A = 1

this is also obvious by the only one ~~linearly~~ independent row [1, 2, 3]

$\boxed{\text{Rank of } A = 1}$

Taking determinant of B $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$|B| = 1((5\times9) - (6\times8)) - 2((4\times9) - (6\times7))$
$+ 3((4\times8) - (5\times7))$

$|B| = 1(-3) - 2(-6) + 3(-3)$

$|B| = 0 \Rightarrow$ take determinant of submatrix 2×2 i.e $\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$

$|B| = (5\times9) - (6\times8) \Rightarrow |B| = -13$

Hence $\boxed{\text{Rank of } B = 2}$