POLYTECH
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Course

# « Introduction to Biomedical Engineering»

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## Section 2: Control theory
## Lecture 2.1: Introduction to the control theory

# First and second order systems

In this section we are going to talk about the control theory, which is crucial part of any engineering, especially electronics, and robotics, which both are part of biomedical engineering.

We will start the lecture from reminding ourselves about transfer functions which we touched in the first section. Remember the transfer function of an RC filter? Also, remember the second order RC filter where we cascaded 2 of them through a buffer? Here we are going to learn how to analyze those transfer functions, and how to design effective controllers in order to improve their performance.

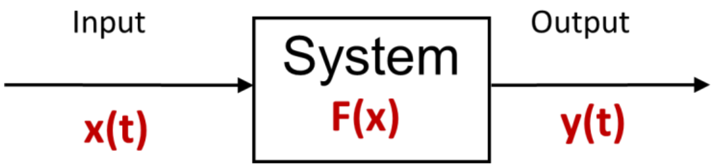


*Figure 1 - transfer function*

First, let's remind ourselves how the first order transfer functions look like. If we apply the inverse laplace transform, which we can simply find in the standard table, we can see that the solution to the ODE, or system response in time, is always structurally the same for any first order system, and equals to some exponent multiplied by a coefficient. If the tau is positive, which is always the case for an RC filter, it is basically decaying exponent, with the gain defining the starting point, and time constant defining the rate of decay.

$$\frac{X(s)}{Y(s)}=\frac{\alpha}{(1+Ts)}=\frac{\gamma}{1+\tau s}$$

$\alpha, \gamma$ Gain

$T, \tau$ Time Constant

*Figure 2 - First Order Systems*

As an example of second order transfer function, let's consider a mechanical system of a mass supported by spring and damper – essentially a shock absorber on a car. This system gives very good intuitive understanding of any second order system as all of the constants have real physical meaning. First, we as usual write the equation of motion, which includes inertial force, hooks law for a spring, and viscous damping – the one proportional to the velocity, or first derivative of a deflection. Here external force is the input, and mass deflection is the output.

The laplace transform replaces all derivatives to the s-multiplication, and some rearranging brings us to a transfer function for this system. Now, let's re-define some constants and now we have this written in standard form: Gain, Natural Frequency, and Demping ratio. You can instantly see the amount of useful intuitive information you can get about the system just by knowing those constants.

$$G(s) = \gamma \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

$\gamma$ **Gain**

$\omega_n$ **Natural Frequency**

$\zeta$ **Damping Ratio**

*Figure 3 - The standard form for second order systems*

This function is known as a damped oscillator, in that it produces harmonic sinusoidal oscillations which decay over time. This type of system appears everywhere in physics, as well as in engineering. Even to the extent that some higher order systems are simplified to become second order, just because it is so well understood.

Another useful example for us is a servo motor: as you can imagine, those are used everywhere in robotics, and for us it is important as it is a key component for active prosthetics. Here we have the angle of rotation as the output, a motor torque as input, which we can control by voltage or current directly, and it has viscous friction in the bearings, and inertia. The ODE can be derived through momentum balance equation, and by taking Laplace we can see, that the transfer function has similar structure, however there is no spring element.

Several systems can be obviously combined together, and it is easy to see that a combined transfer function of a chain of several transfer functions can be found by a simple multiplication.

The crucial component of the control theory is the feedback. As the name suggest, to organize the feedback, you need to measure the output, transform it somehow, and feed it back to the input. Remember the OpAmp, and a multitude of nice properties that we have achieved by doing so? In control theory to describe this process we add a summation element, where we put + or – near the signal path to indicate whether we add it or subtract. In most cases we tend to use negative feedback loops, which means we subtract the modified output signal from the input.

The mathematics of computing the feedback, or closed-loop, transfer function is also straightforward: By careful tracing of all the inputs and outputs, we can do basic algebra and arrive to the equation.

Similar, singe transfer function can represent any complicated system, which involves electro-mechanical components, measurement of several parameters, and decision making integrated into continuous process control. For example, part of tesla autopilot, which measures GPS speed and acceleration in order to maintain constant velocity, can be gradually unwrapped into a second order transfer function. The rule here is to start working from the inner most loop progressing outwards. Handy!

Remembering the transfer function for servo motor, we can see how a simplest feedback of just subtracting unmodified output from the input, changes radically its transfer function. Now the system can oscillate (as if it had an artificial spring!), and we can control the frequency and damping ratio by adjusting the parameters, for example gain on the motor.

The last bit to consider about the second order transfer functions is the generalized solution. If we look at the table of inverse Laplace transforms, we can construct the solution in time, and see that it has a constant, multiplied by exponent, which in turn is multiplied by a sinusoid. This lets us generalize the entire class of second order systems, be it car shock absorber, second order RC filter, or servo motor: Their dynamic response contains an oscillatory behavior bound by the exponential amplitude. If damping ratio is less than 1, the solution is decaying sinusoid, where frequency and the rate of decay is fully defined by the constants of the standard representation of the second order transfer function.

$$\frac{x(t)}{x(0)} = \gamma \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2 t}\right)$$

*Figure 4 - time domain response*

You can see now this system step response – the usual measure of system dynamic performance, which we can easily compute knowing the transfer function and supplying step function to the input. The step input in Laplace domain is equal to A over s, where A is the step size. After multiplying, you can find the appropriate equations in the Laplace table, or use digital Laplace transform to evaluate the output. The damping ratio here basically controls how fast the system is reacting the target VS how many oscillations you will get after the kick-off.

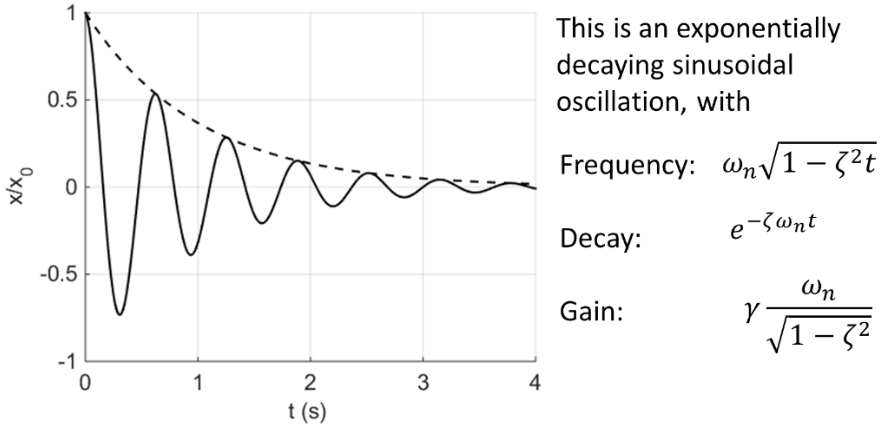


*Figure 5 - exponentially decaying sinusoidal oscillation*

More generally, the roots of the denominator actually control this, with standard classification of the system being under or over-damped (exponentially-bound oscillations), critically damped (decaying exponent) or undamped (constant oscillations) depending on damping ratios.

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