Introduction to Biomedical Engineering

Section 2: Control theory

Lecture 2.1: Introduction to the control theory

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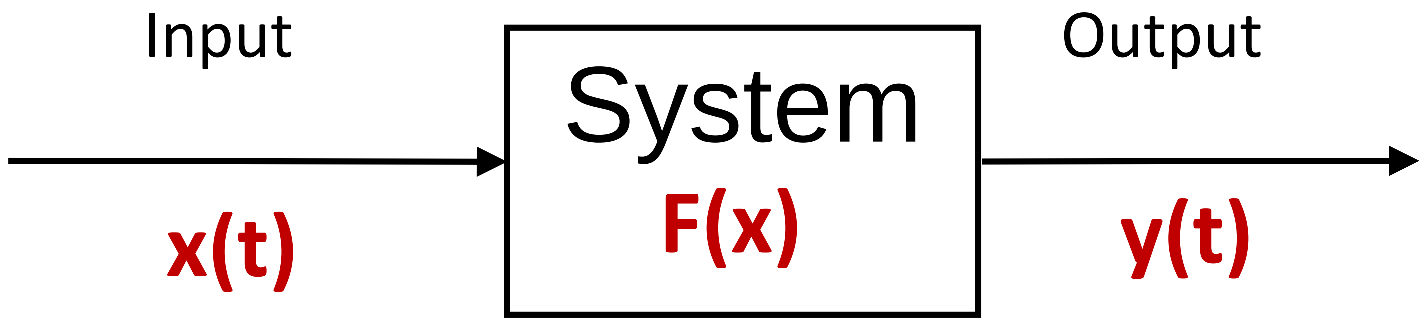
Modelling Electrical & Mechanical systems

We are considering systems in the general sense, with a generic function block approach.



We will investigate how we model physical systems, and obtain the function block F(x) for electrical and mechanical components.

Linear Time Invariant (LTI) Systems

The focus of this course, and indeed much of control theory itself focuses on modelling physical systems as *linear* and *time invariant*.

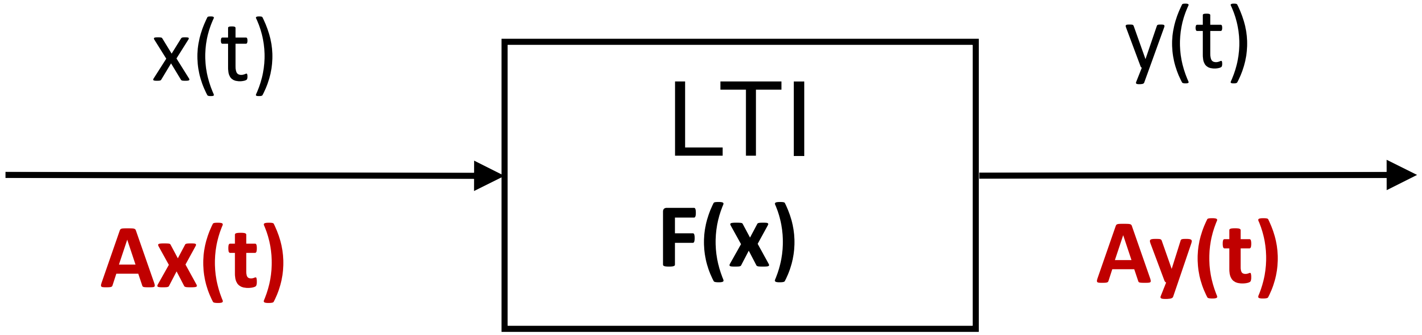
LTI systems have three key properties:

- Obey principle of superposition
- Homogeneity
- Time Invariance

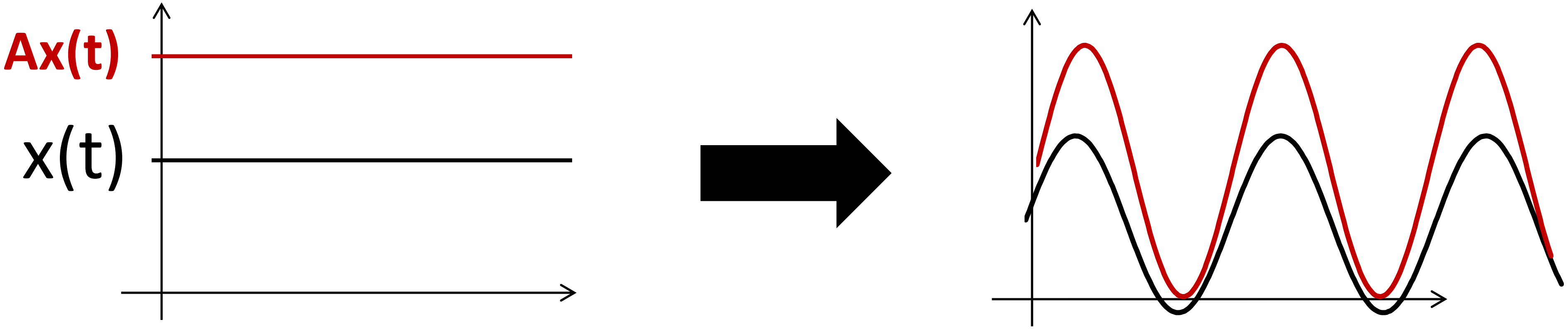
Linearity

Homogeneity

If the input to the system x(t) is scaled by a magnitude scale factor *A*, then the output y(t) is also scaled by the same factor



For example, consider a system which generates a sine wave at a given amplitude, with a set frequency:



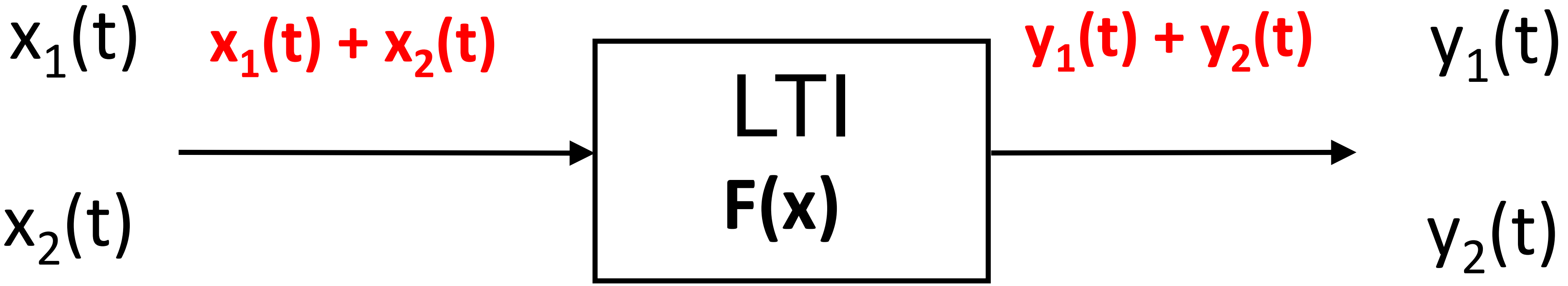
Electrical versions of this circuit are used to generate the carrier frequency for radio transmitters.

They could also represent the volume knob on a synthesizer

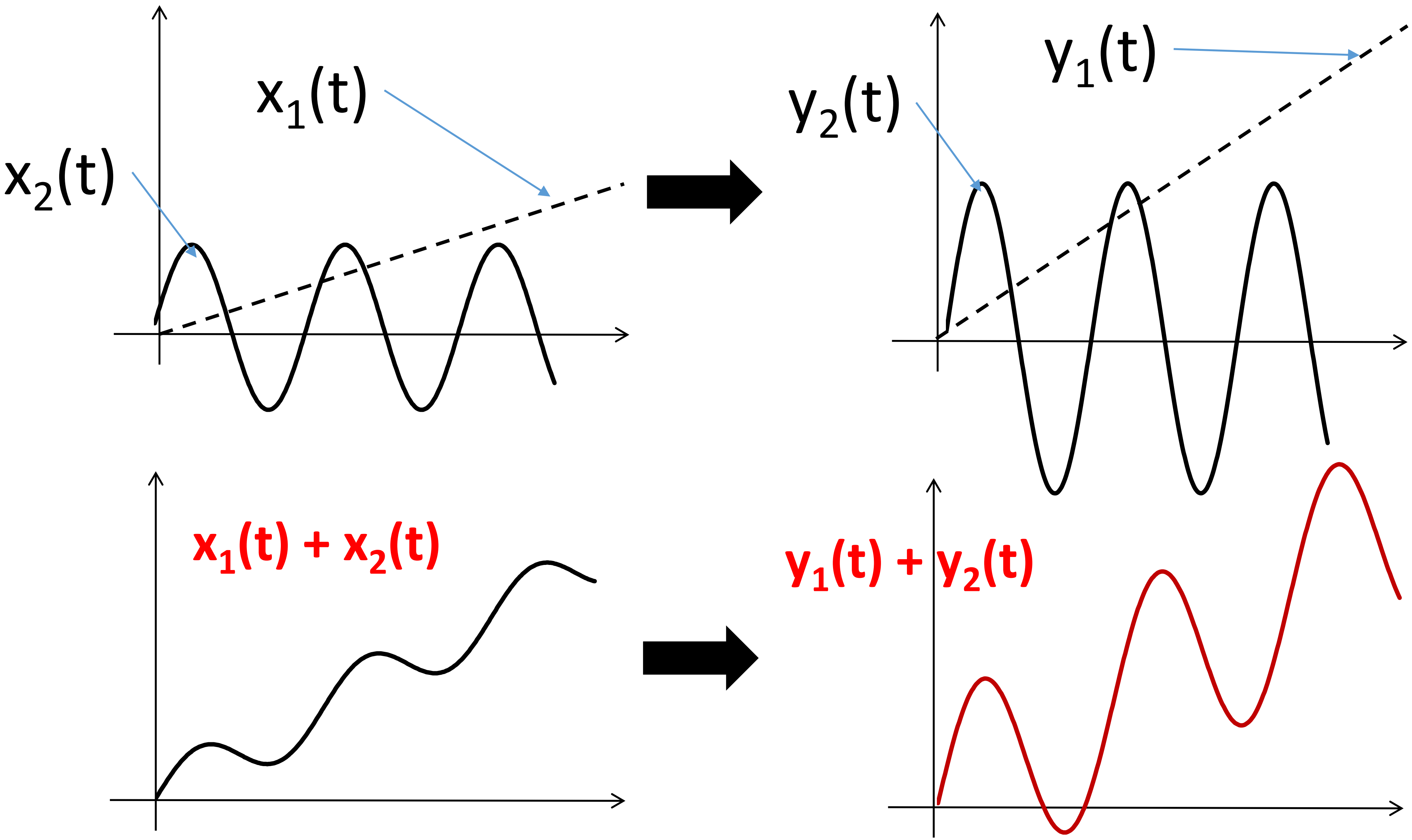
Or as the inputs to a mechanical shaker table, used for robustness testing

Superposition

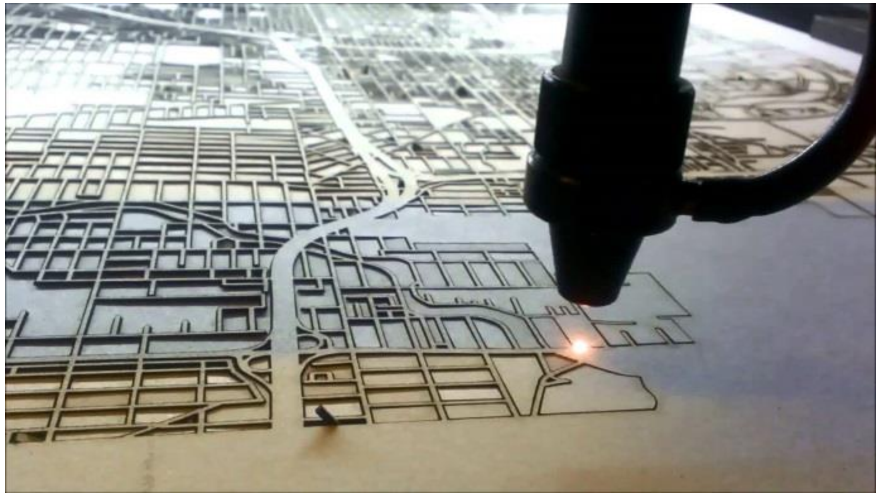
If input $x_1(t)$ produces output $y_1(t)$, and input $x_2(t)$ produces $y_2(t)$, then input $x_1(t) + x_2(t)$ produces output $y_1(t) + y_2(t)$



Say for a system which doubles the input F(x) =2x :



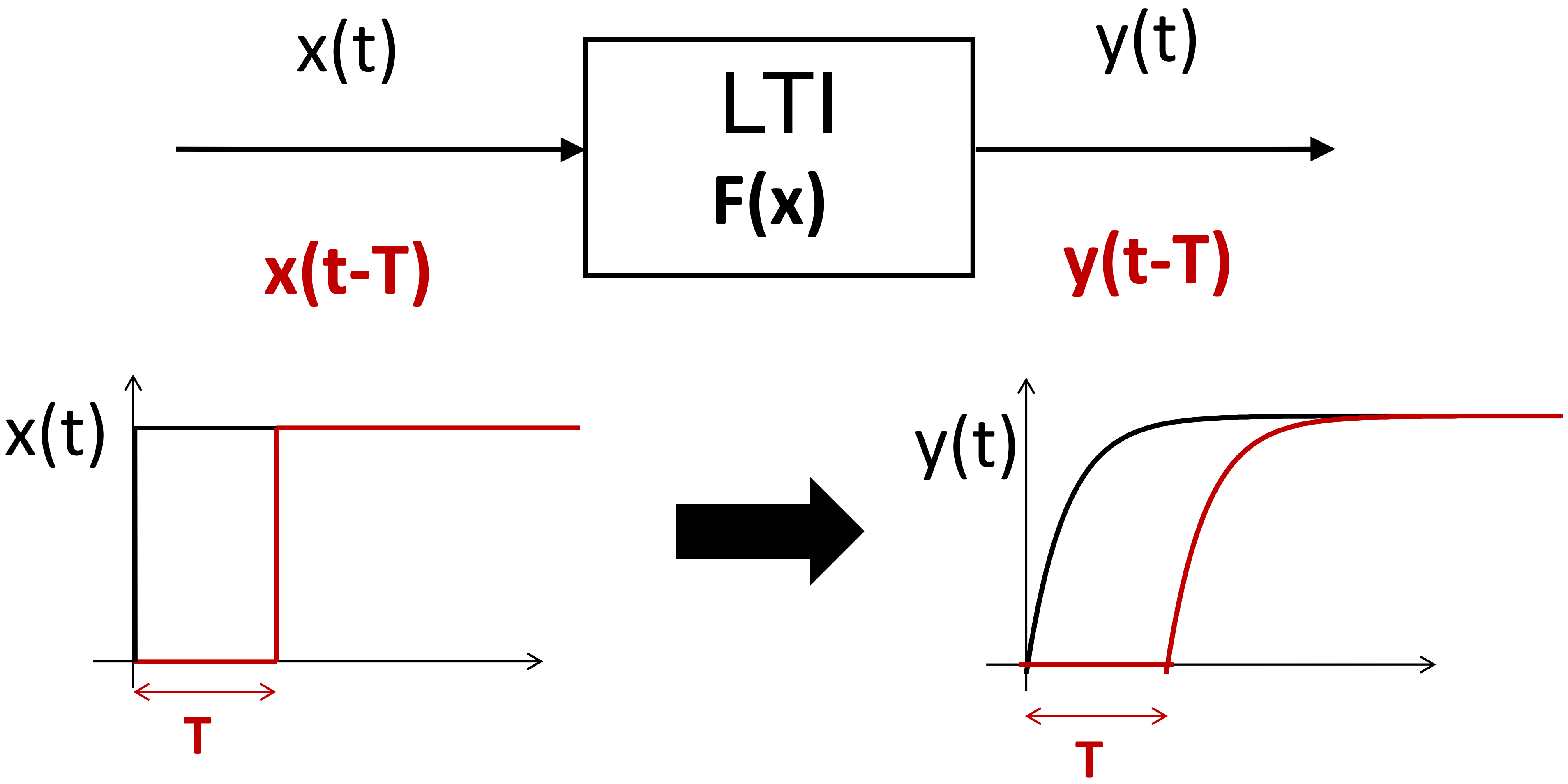
Superposition like this is important in many systems. A loud speaker should be able to play two notes at once in order to be useful!



Or a laser cutter tool which is able to cut a complex shape whist moving in a constant speed in one direction

Time invariance

If input is applied at time t=0 or *T* seconds from now, the output is identical with the exception of a delay of *T* seconds

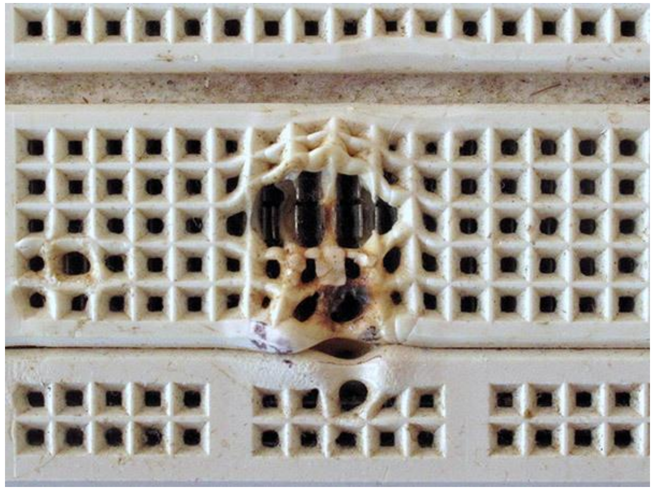


Are these models suitable for physical systems?

These three requirements, whilst simple, are so stringent that *almost no physical LTI system truly exists!*

Consider a car engine – the performance deteriorates over time, to stretch it further, would you expect a system to give the same output after a time delay *T* of 10 years?

Even simple systems such a resistor in an electrical circuit have non-linearities – a scaling factor *A* could be chosen for x(t) which would mean too much current flows and the resistor melts.



So why do we use them?

Most practical systems are not linear, but often we can assume they behave linearly *under certain conditions*

Linear systems are *much* easier to solve! There are *analytic* solutions with standard tools used solve the equations. Whereas for non-linear problems it is often necessary to solve them numerically

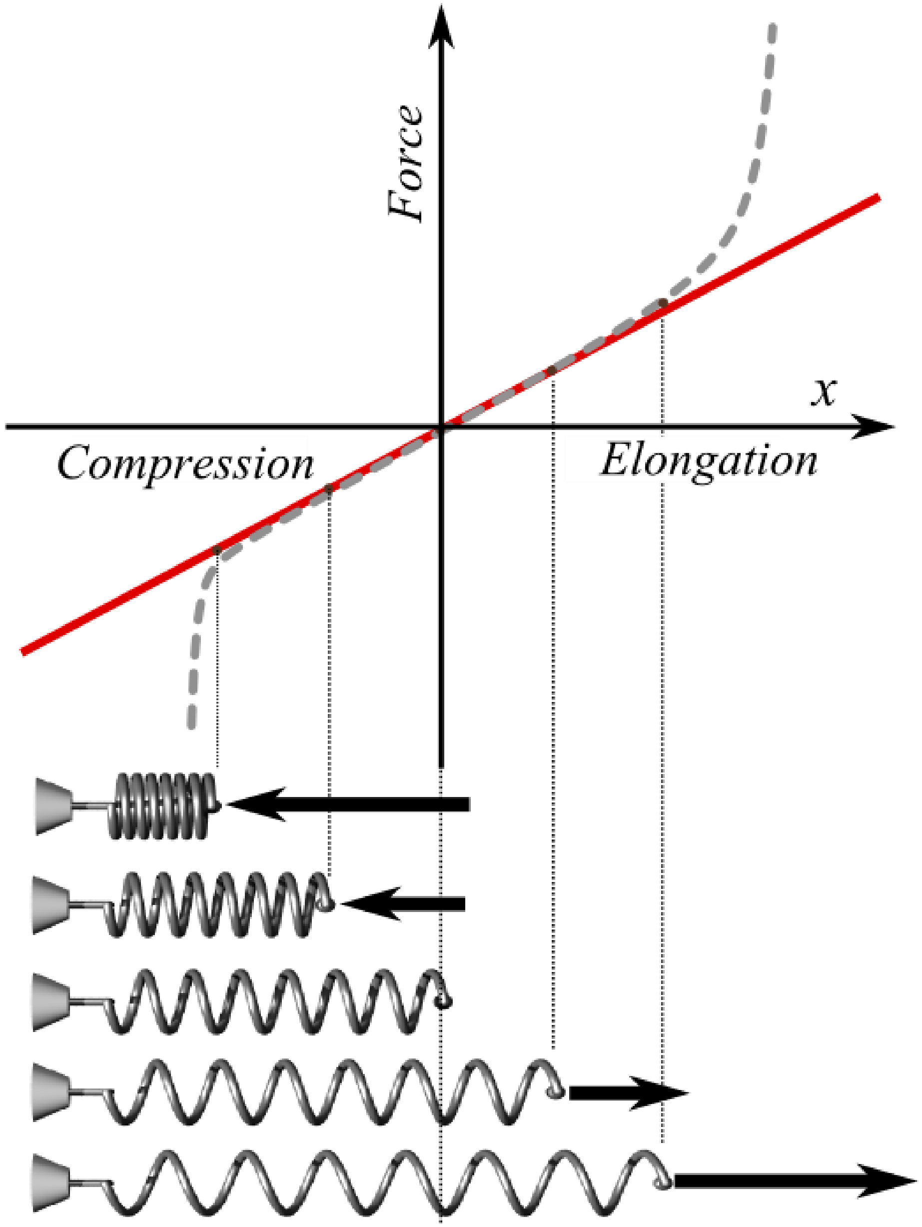
Linearisation examples

For a simple system such as a spring, across all possible compressions or extensions the response is non linear:

$$F = kX$$

Hookes law is only a linear approximation of the true response

However, if we choose the operating range of the spring correctly, the response is within the linear region

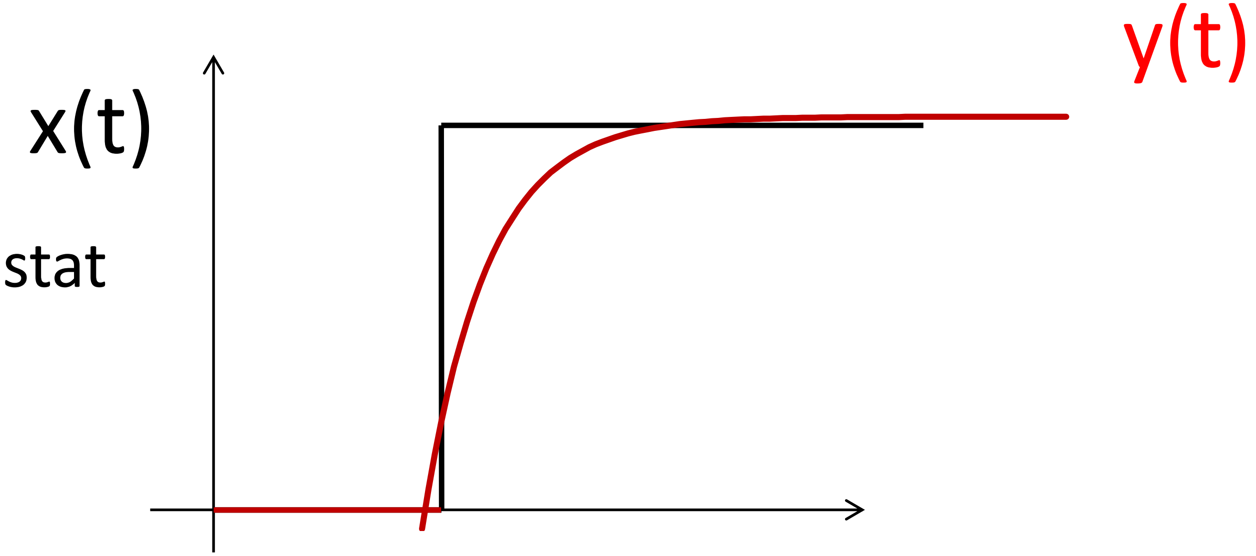


And the approximation is valid

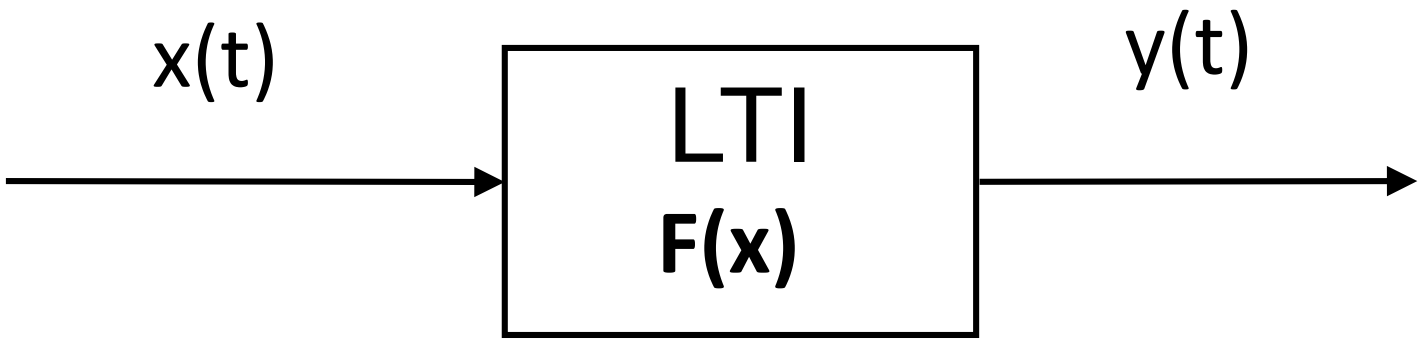
Dynamic Systems as ODEs

Ideal systems would respond *instantaneously* to inputs, however real world systems require some time to adjust to changes an are thus known as *dynamic systems* as the output changes over time

e.g. Thermostat setting



As we are interested in describing something that *changes* with time, it is useful to express the function block of the system F(t) as an ordinary differential equation (ODE)



$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_2\frac{d^2 y}{dt^2} + a_1\frac{dy}{dt} + a_o = bx$$

X is input function or forcing function

Y is output

N is *order* of the ODE

A0… are coefficients. These *completely characterise the system*

Laplace Transforms

Because of our **linear assumptions** we can use Laplace transforms to simplify solving the ODEs (no really, it does!)

The Laplace transform of *x(t)* is defined by

$$L\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

s is a complex variable (it has real + imaginary parts), so we will often use this form: *s* = *s* + *jw*

The Laplace variable, s, can be considered to represent the differential operator (VERY useful for control engineering):

$$\boxed{s \equiv \frac{d}{dt}}$$

$$\boxed{\frac{1}{s} \equiv \int_{0^-}^{\infty} dt}$$

How is this easier exactly!?

As we shall see: **You can look up the transforms in a table!**

- Differential operators are replaced with algebraic variables
- Algebraic equations are much easier to manipulate & solve
- Standard forms exist for many physical systems

Laplace transform of time derivative dx/dt:

$$L\left\{\frac{dx}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st} dt$$

Integrating by parts:

$$L\left\{\frac{dx}{dt}\right\} = s\int_{0^-}^{\infty} x(t) e^{-st} dt + \left[x(t)e^{-st}\right]_{0^-}^{\infty}$$

The initial condition $x(0^-)$ is often zero in practice

$$\boxed{L\left\{\frac{dx}{dt}\right\} = sX(s) - x(0^-)}$$

Laplace Transforms

We can substitute this result to solve higher order derivatives:

$$L\left\{\frac{d^2x}{dt^2}\right\} = sL\left\{\frac{dx}{dt}\right\} - \frac{dx}{dt}(0^-)$$

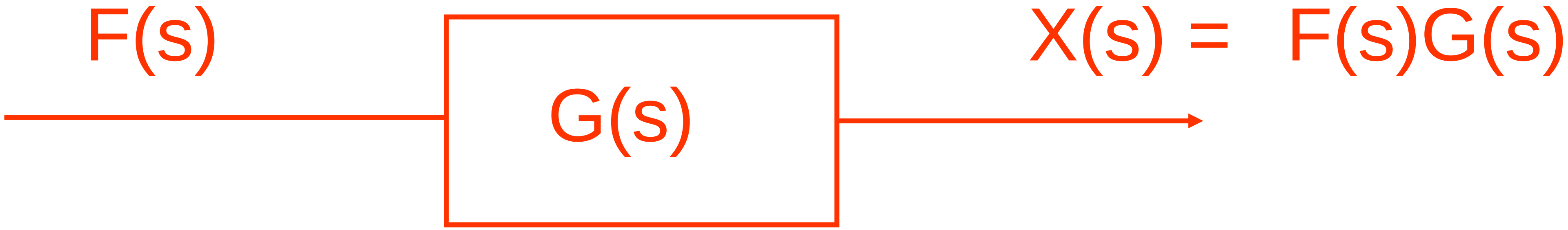
$$\boxed{L\left\{\frac{d^2x}{dt^2}\right\} = s^2X(s) - sx(0^-) - \frac{dx}{dt}(0^-)}$$

So more generally, with all initial conditions set to zero:

$$L\left\{\frac{d^nx}{dt^n}\right\} = s^nX(s)$$

Transfer Functions

After we have taken the Laplace transform of the differential equation of a system, it's useful to rearrange to give the system **output** as the product of the system **input** and the system **transfer function**.

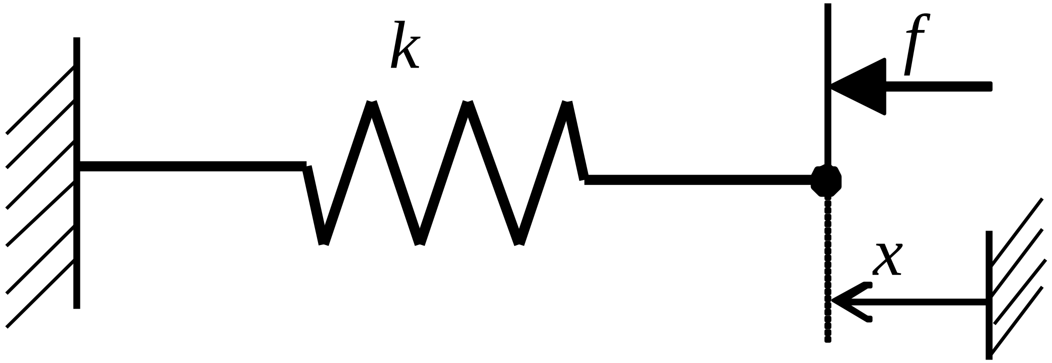


The transfer function of a linear system is defined as the **ratio** of the Laplace transform of the output variable to the Laplace transform of the input variable, *with all initial conditions assumed to be zero*.

Transfer Functions of Mechanical Components

Convention is input force, output displacement. Balance **forces**

Spring



Time domain equation:
Hooke's law
k is stiffness in N/m

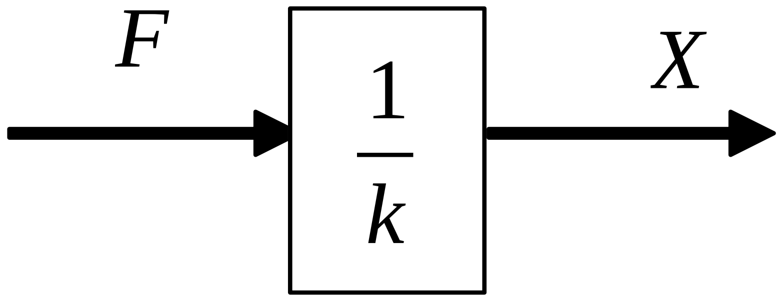
$$f(x) = kx$$

Laplace domain equation:

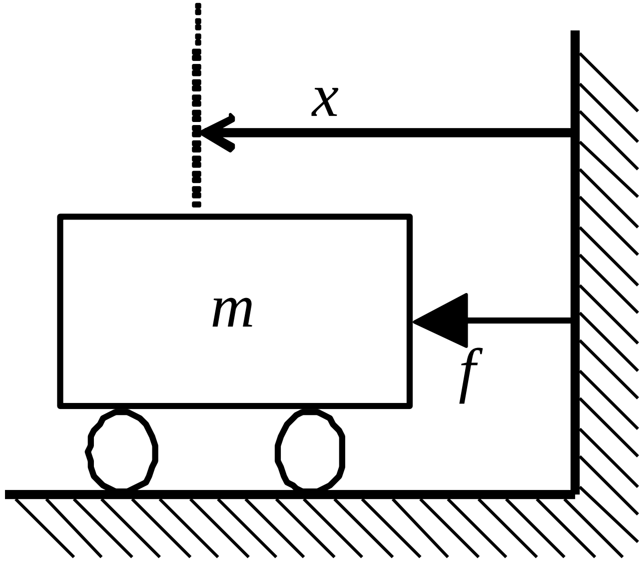
$$F(s) = kX(s)$$

Transfer function

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{k}$$



Inertial load - mass



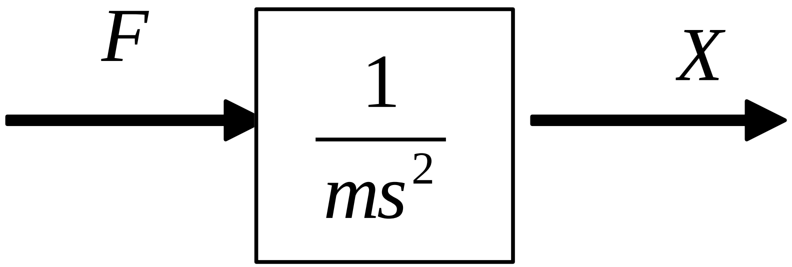
Time domain equation:

From Newton's 2nd law of motion: $$f(t) = m\frac{d^2x(t)}{dt^2}$$

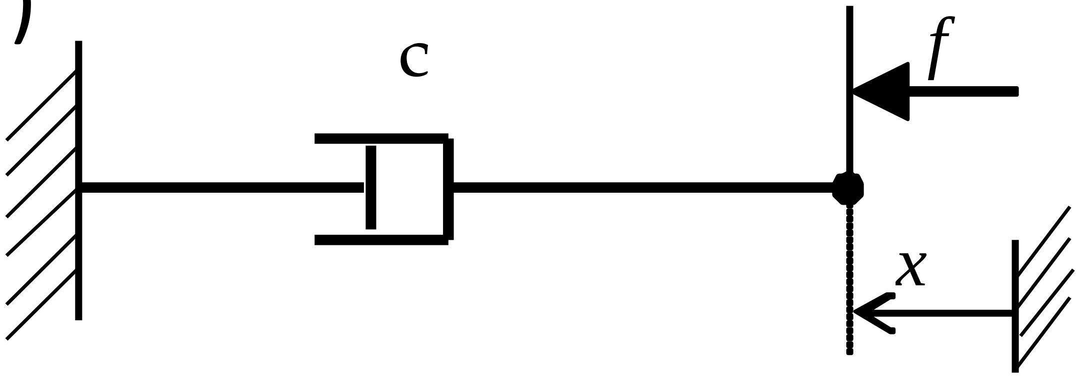
Laplace domain equation: $$F(s) = ms^2X(s)$$

Transfer function

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2}$$



Dashpot (viscous damper)



Dashpot resists motion through friction.
Damping coefficient *c* in N/ms or kg/s

Time Domain equation:

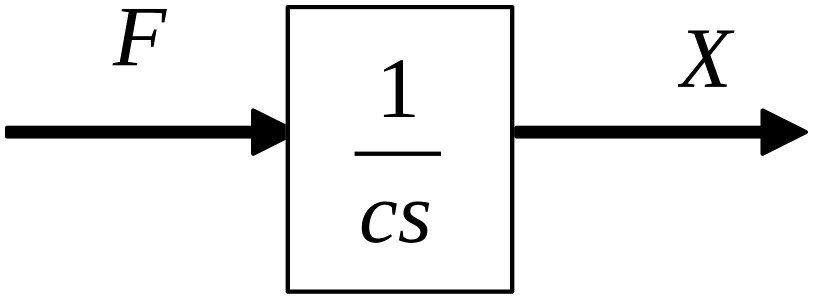
$$f(x) = c\frac{dx}{dt} = csx$$

Laplace Domain equation:

$$F(s) = csX(s)$$

Transfer function:

$$\frac{X(s)}{F(s)} = \frac{1}{cs}$$

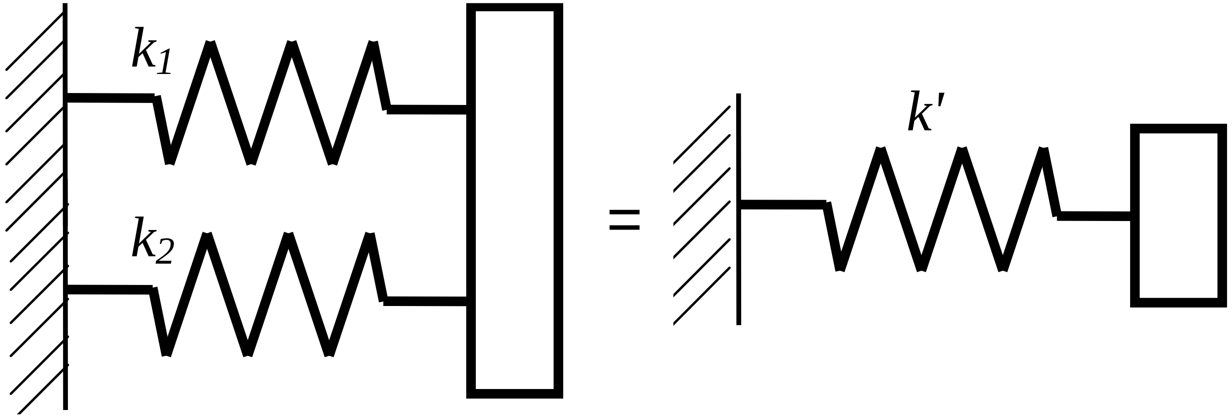


Combining components

Springs in parallel

$$k' = k_1 + k_2$$

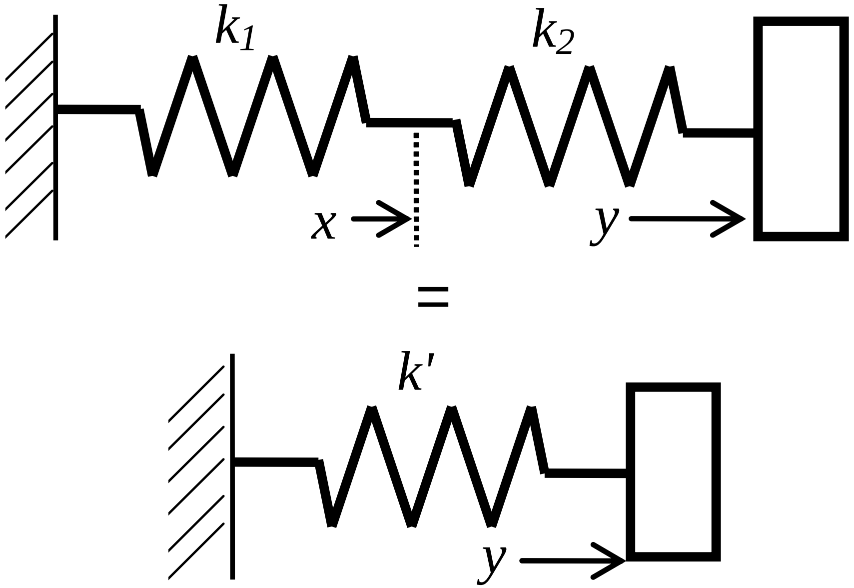
$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{k'}$$



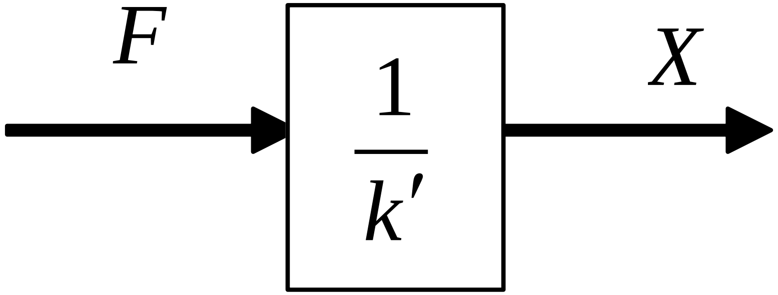
Springs in series

$$\frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{k'}$$



Combining components



Dashpots behave is a similar way to springs except the rule is inverted:

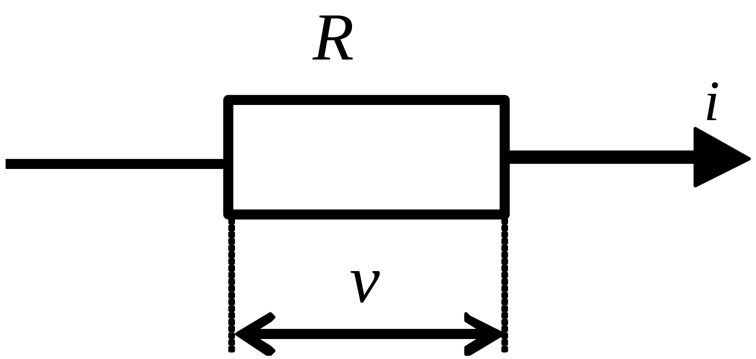
parallel $$c' = c_1 + c_2$$

and in series $$\frac{1}{c'} = \frac{1}{c_1} + \frac{1}{c_2}$$

Transfer Functions of Electrical Components

Convention is input voltage, output current. Balance **Voltages**

Resistor



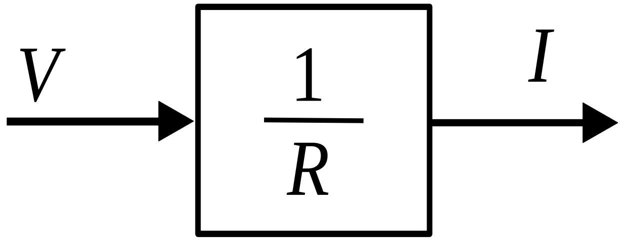
Ohm's law V=IR:

Time domain $$v(t) = i(t)R$$

Laplace domain $$V(s) = I(s)R$$

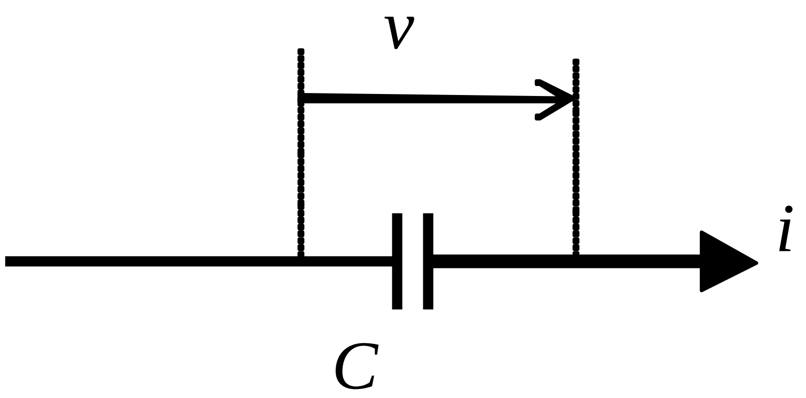
Transfer function:

$$G(s) = \frac{I(s)}{V(s)} = \frac{1}{R}$$



Capacitor

Either definition of current/voltage relationship gives same result

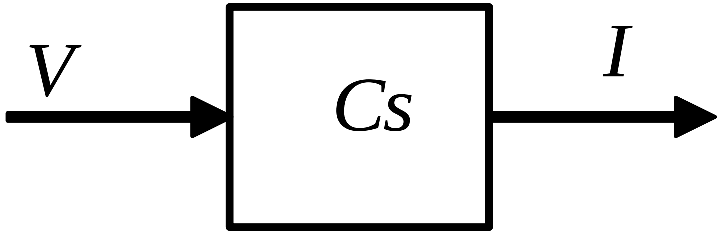


Time domain $i(t) = C\frac{dv}{dt}$ $v(t) = \frac{1}{C}\int i(t)dt$

Laplace domain $I(s) = CsV(s)$ $V(s) = \frac{1}{C}\frac{1}{s}I(s)$

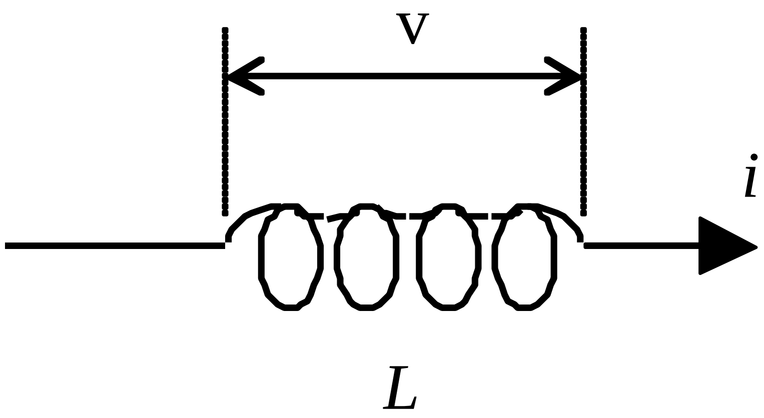
Transfer function:

$$G(s) = \frac{I(s)}{V(s)} = Cs$$



Inductor

An inductor resists changes of current by generating a voltage in opposition via magnetic induction



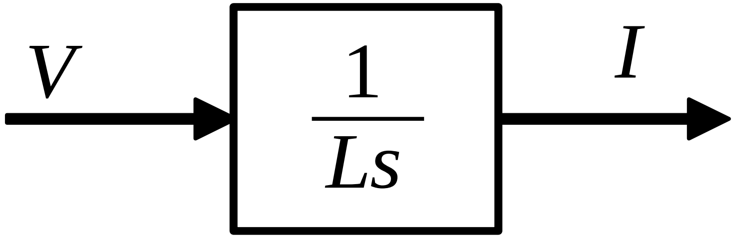
From Faraday's Law: $v(t) = L\dfrac{di}{dt}$

Time domain

Laplace domain $V(s) = LsI(s)$

Transfer function

$$G(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls}$$

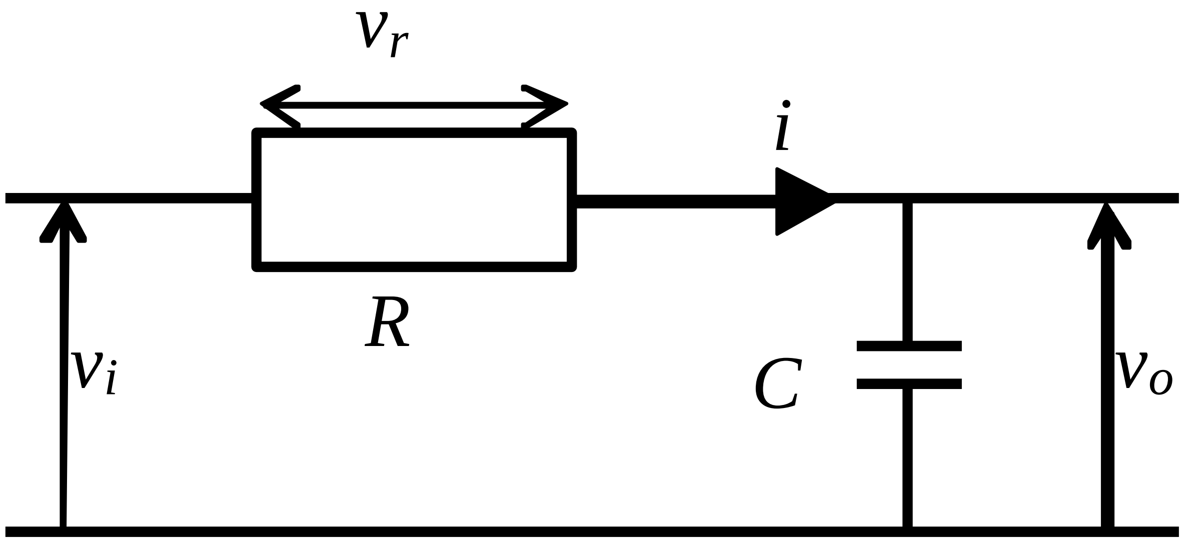


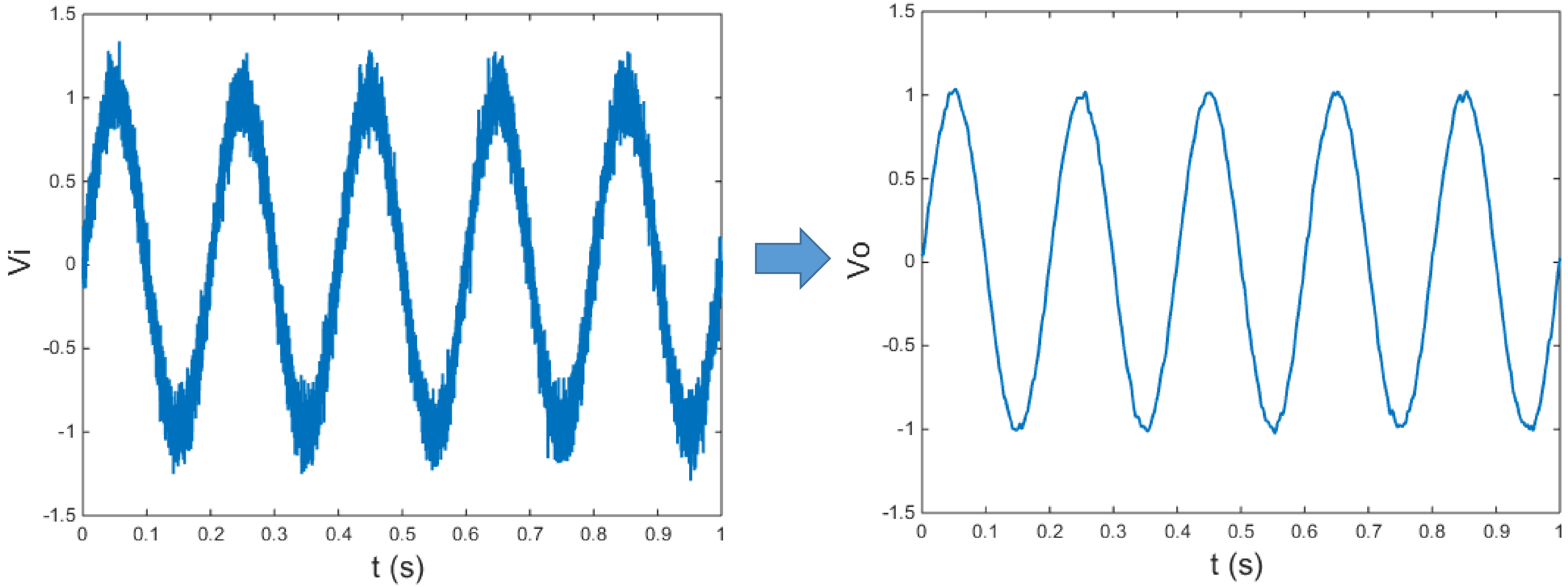
As with Mechanical Components, these can be combined:

	Resistor	Inductor	Capacitance
Series	$R' = R_1 + R_2$	$L' = L_1 + L_2$	$\frac{1}{C'} = \frac{1}{C_1} + \frac{1}{C_2}$
Parallel	$\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2}$	$\frac{1}{L'} = \frac{1}{L_1} + \frac{1}{L_2}$	$C' = C_1 + C_2$

Examples – RC Filter

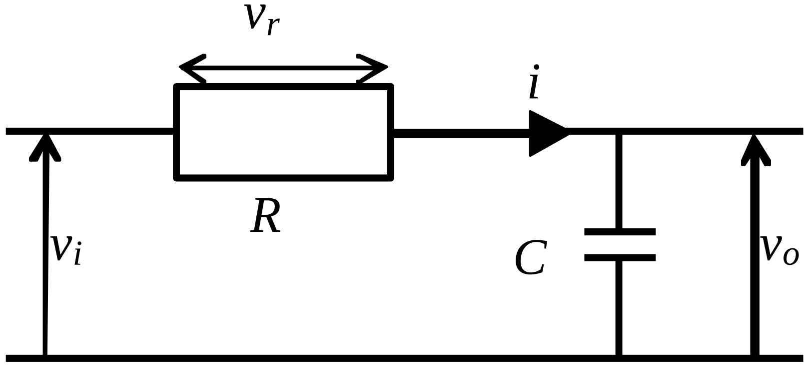
This simple circuit filters out high frequencies from v_i but allows low frequency voltages to pass.





The goal is the transfer function, with v_o the output of interest:

$$G(s) = \frac{V_0(s)}{V_i(s)}$$



Input voltage is sum of voltage drops in circuit

$$V_i = V_r + V_o$$

Kirchhoff's Voltage Law

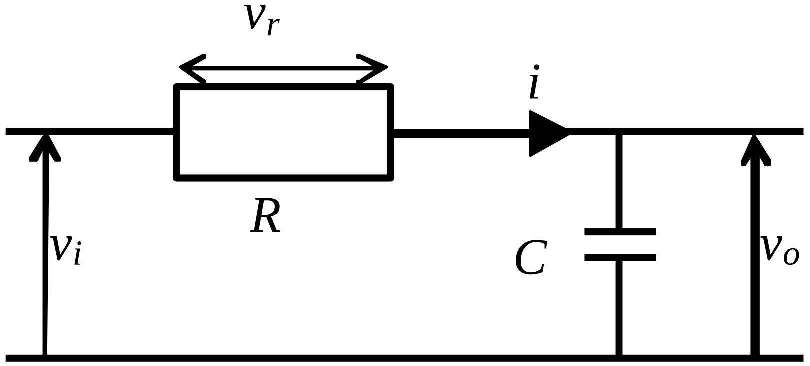
Starting with Time domain equations:

$$v_i(t) = v_r(t) + v_o(t) \quad v_r(t) = i(t)R \quad i(t) = C\frac{dv_o}{dt}$$

We can write v_i in terms of v_o

$$v_i(t) = RC\frac{dv_o}{dt} + v_o(t)$$

$$v_i(t) = RC\frac{dv_o}{dt} + v_o(t)$$



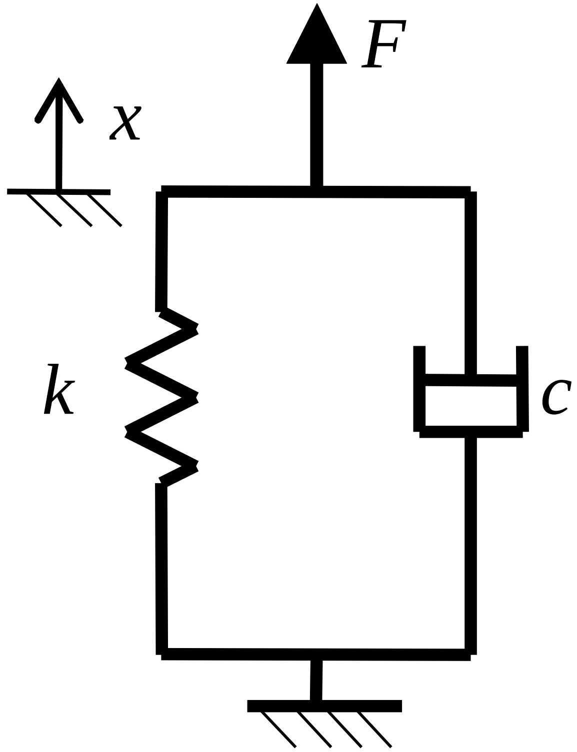
Solving the above in time domain requires integration/ differentiation.
The problem is simplified using the Laplace transform:

$$V_i(s) = RCsV_o(s) + V_o(s) = V_o(s)(RCs+1) \qquad s \equiv \frac{d}{dt}$$

Transfer function w.r.t. voltage:

$$G(s) = \frac{V_0(s)}{V_i(s)} = \frac{1}{RCs+1} = \boxed{\frac{\alpha}{Ts+1}} \qquad \begin{array}{l} \alpha = 1 \\ T = RC \end{array}$$

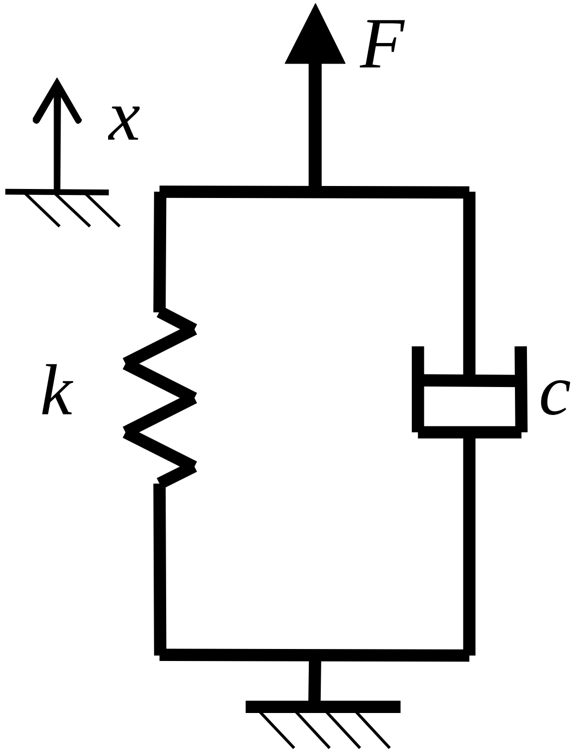
Examples – Parallel Spring & Damper



In a shock absorber in a car, damping added in parallel to spring suspension in vehicles to damp oscillations and absorb impulses

Transfer function desired:

$$G(s) = \frac{X(s)}{F(s)}$$



Balancing forces as function of time:

$$f(t) = f_{spring}(t) + f_{damper}(t) = kx(t) + c\frac{dx}{dt}$$

Rewriting as function of s

$$F(s) = kX(s) + csX(s) = X(s)(k + cs)$$

Transfer function is thus:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{(k+cs)} = \boxed{\frac{\alpha}{(1+Ts)}}$$

$$\alpha = {}^{1}\!/_{k}$$

$$T = {}^{c}\!/_{k}$$

First Order Systems

All first order systems i.e. those with only $\dfrac{dx}{dt}$
Take the following "standard" forms

$$\frac{X(s)}{Y(s)} = \frac{\alpha}{(1+Ts)} = \frac{\gamma}{1+\tau s}$$

α, γ Gain

T, τ Time Constant

This function is commonly known as an exponential time delay, or lag.
This is an incredibly common function, they turn up everywhere!

How do I go back to the time domain?

To get the response of the system in the time domain we need to convert the transfer function from the Laplace domain.

Luckily, people have already made tables of common inverse Laplace functions – "*all" we need to do arrange our transfer function into one of these standard forms (e.g. from Dorf and Bishop)*

How do I go back to the time domain?

f(t)	F(s)
Step function, u(t)	$\frac{1}{s}$
$\boxed{e^{-at}}$	$\boxed{\frac{1}{s+a}}$
sin ωt	$\frac{\omega}{s^2+\omega^2}$

So in this case: $a = 1/\tau$

$$\frac{X(s)}{Y(s)} = \frac{\gamma/\tau}{1/\tau + s} = \frac{\gamma}{\tau}\frac{1}{1/\tau + s}$$

$$Gain\left(\frac{1}{s+a}\right)$$

$$x(t) = L^{-1}\left\{\frac{\gamma}{\tau}\frac{1}{1/\tau + s}\right\} = \frac{\gamma}{\tau}L^{-1}\left\{\frac{1}{1/\tau + s}\right\}$$

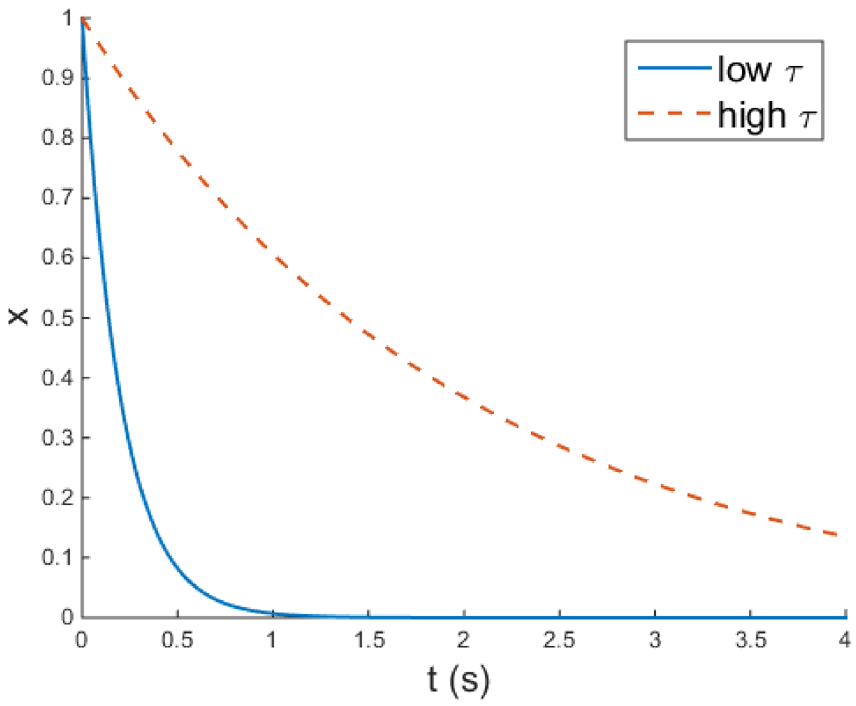
$$= \frac{\gamma}{\tau}e^{-t/\tau}$$

Gain included no s terms,
So can be taken outside transform

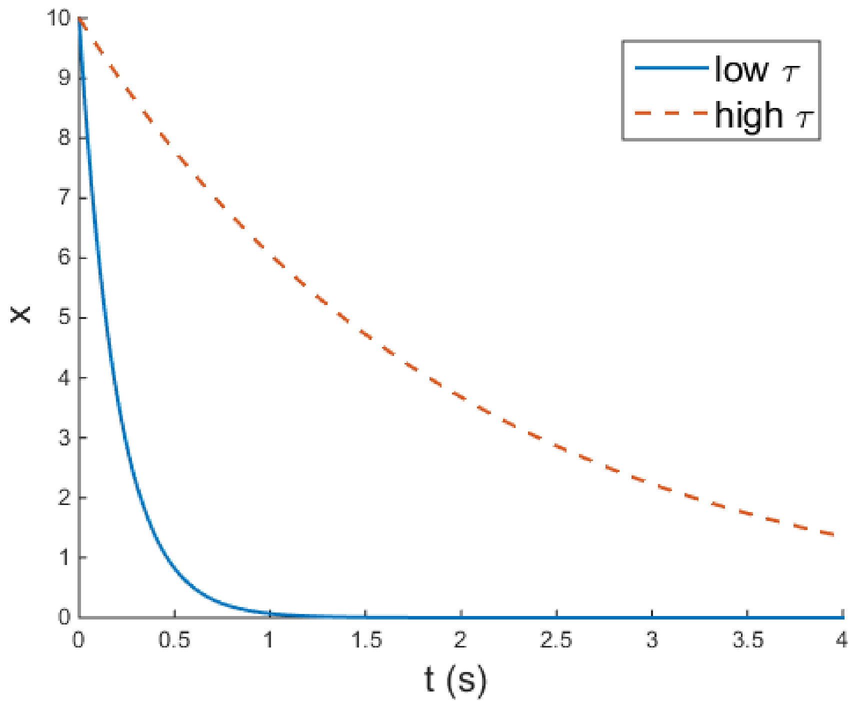
What does this equation look like?

$$x(t) = \frac{\gamma}{\tau} e^{-t/\tau}$$

This is an exponential decay – the only variable needed to define the system is the time constant. The gain merely scales the response



Gain =1

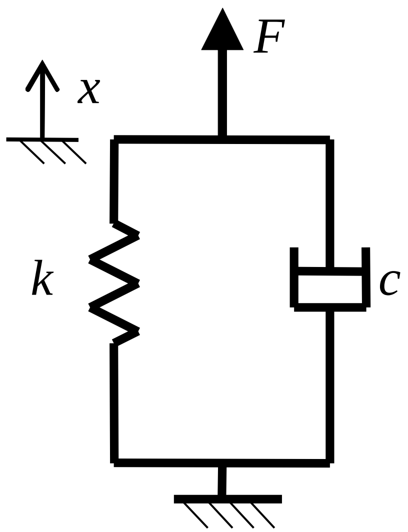


Gain =10

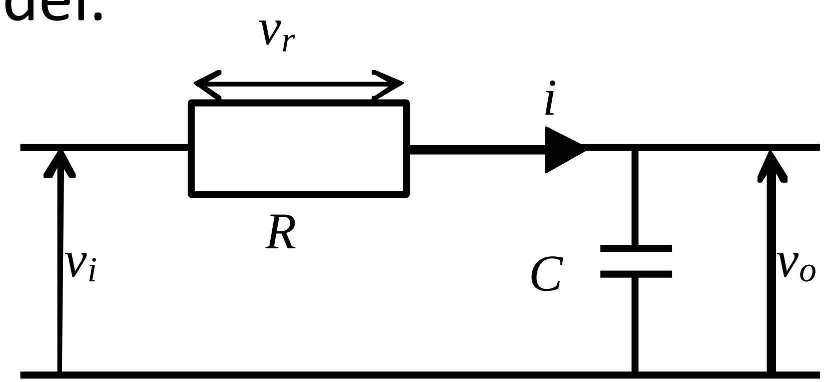
Having a negative time constant makes no physical sense, so the exception of T=0, the response will always be an exponential decay

Remember, we have modelled *mechanical* and *electrical* systems, and arrived at the same equations. This could be biological, financial, meteorological…

This is good, because you only need to understand *one* system to understand hundreds. However, this means the models can appear abstract (or possibly arbitrary!), so it is important to keep in mind the physical systems we are trying to model.



$$\tau = c/k$$



$$\tau = RC$$

Rate of decay determined by ratio of spring return force to viscous friction

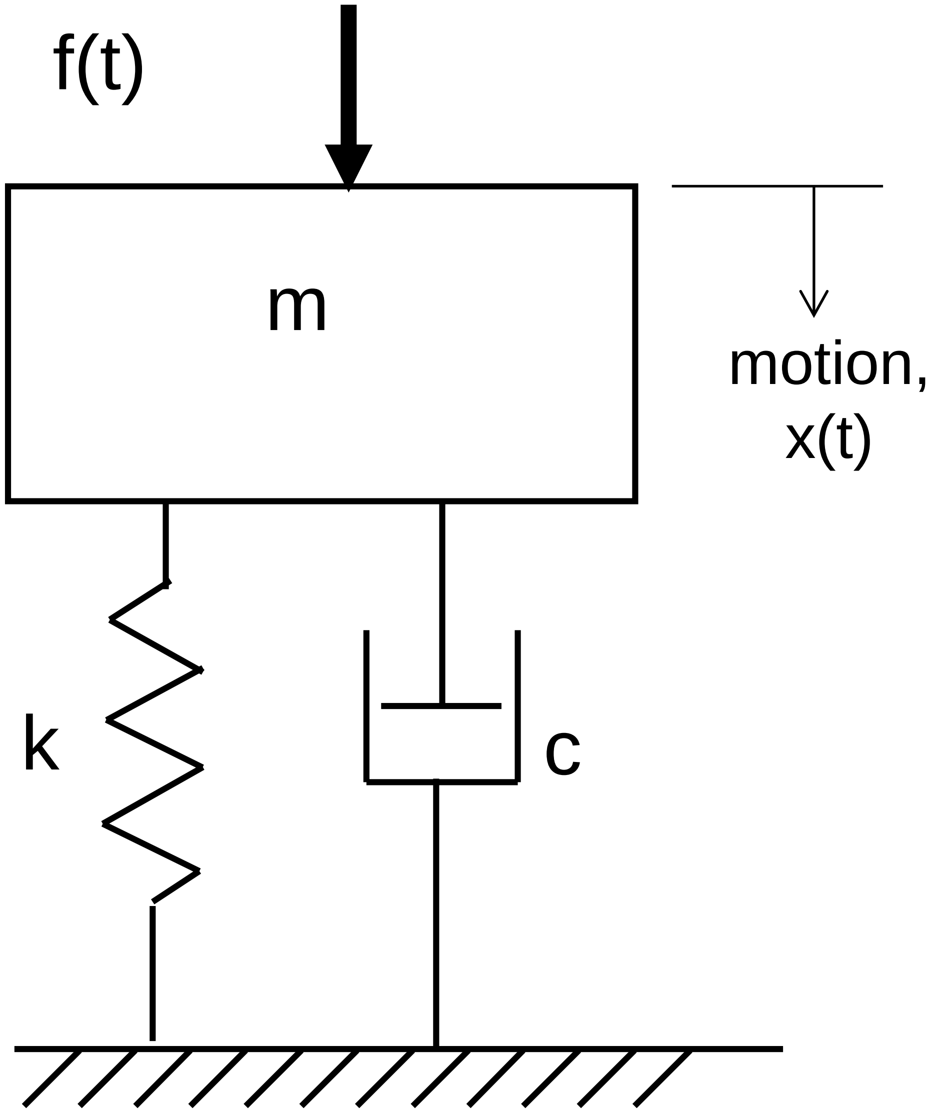
Rate of decay determined by total impedance of R and C

Examples – Mass Spring Damper

Consider a simple mechanical system Mass/Spring/Damper (MSD)

As before, we wish to relate the input force to the output displacement i.e. Transfer function desired:

$$G(s) = \frac{X(s)}{F(s)}$$



Using Newton's Second Law Balancing forces as function of time:

$$f(t) = f_S(t) + f_D(t) + f_I(t)$$

Examples – Mass Spring Damper

$$f_S(t) = kx(t) \qquad f_D(t) = c\frac{dx(t)}{dt} \qquad f_I(t) = m\frac{d^2x(t)}{dt^2}$$

Combining yields the following time domain equations

$$f(t) = kx(t) + c\frac{dx(t)}{dt} + m\frac{d^2x(t)}{dt^2}$$

Initial conditions zero

$$x(0) = \frac{dx(0)}{dt} = 0$$

Rewriting as function of s

$$F(s) = kX(s) + csX(s) + ms^2X(s)$$

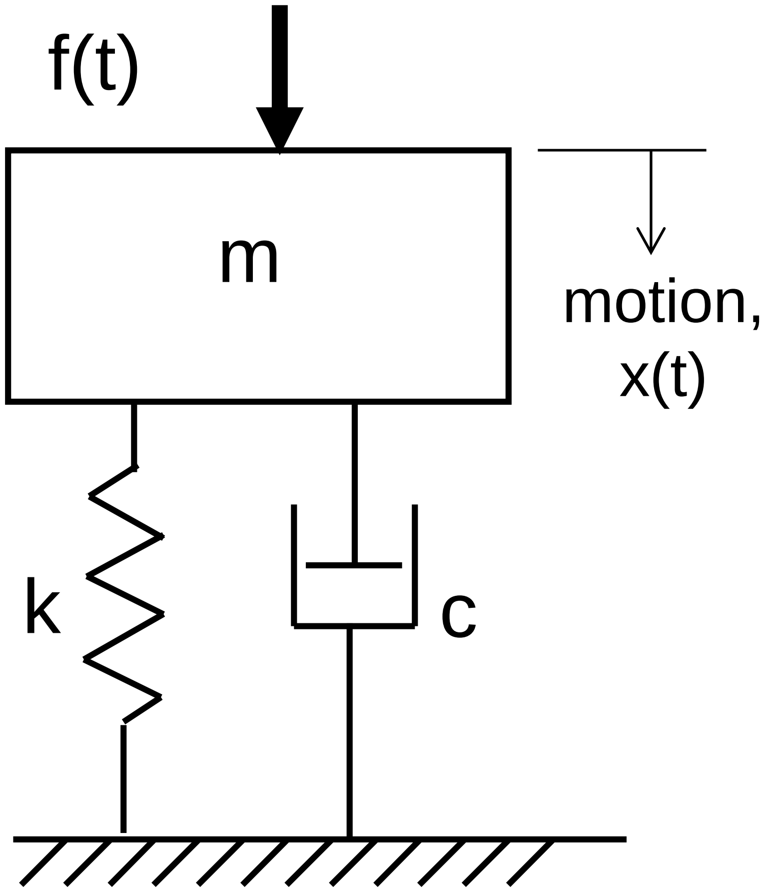
$$F(s) = X(s)\left(k + cs + ms^2\right)$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} = G(s)$$

Difficult looking 2nd order ODE, converted to quadratic equation

Worked Examples – Mass Spring Damper

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$



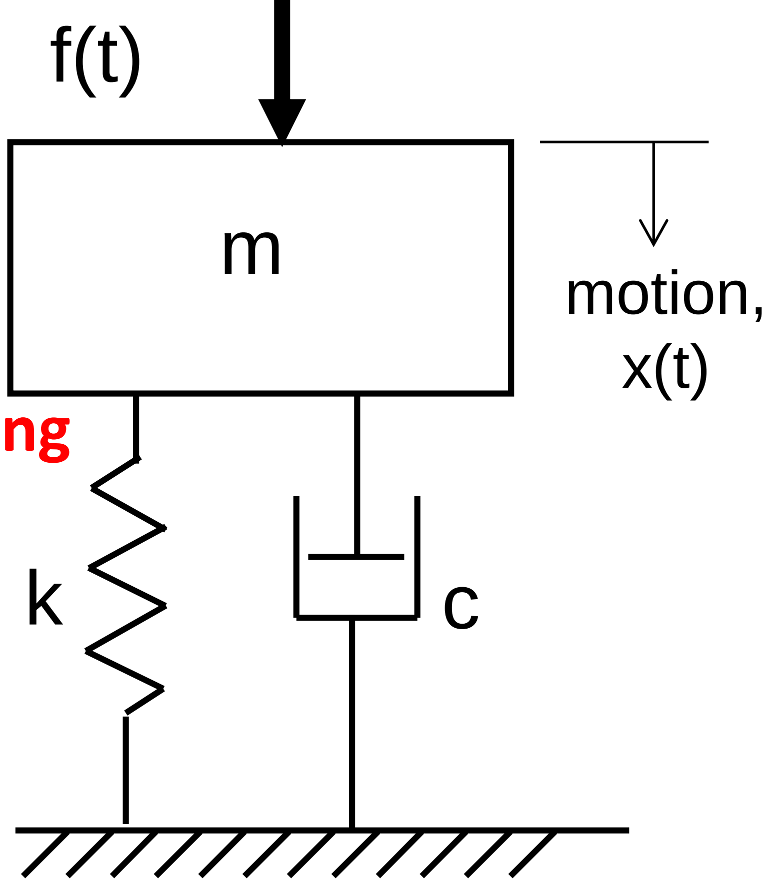
However, we want to write it in a "standard form", not least because that's what it will look like in the Laplace transfer tables! This means the coefficient of the highest order of s on the denominator is 1

$$\boxed{\frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}}$$

$$G(s) = \frac{k/m}{s^2 + \frac{c}{m}s + \frac{k}{m}} \boxed{\frac{1}{k}}$$

Just a scaling factor
Or gain

$$= \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}} \frac{1}{k}$$



Where,

$$\omega_n = \sqrt{\frac{k}{m}}$$ Natural frequency

$$\zeta = \frac{c}{2\sqrt{km}}$$ Damping ratio

These have physical meanings

Second Order systems

The standard form for second order systems is shown below:

$$G(s) = \gamma \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

γ **Gain**

ω_n **Natural Frequency**

ζ **Damping Ratio**

This function is known as a damped oscillator, in that it produces harmonic sinusoidal oscillations which decay over time. This type of system appears *everywhere* in physics, as well as in engineering. Even to the extent that some higher order systems are simplified to become second order, just because it is so well understood

You might be sick of seeing it by the end of your degree!

How do I go back to the time domain?

As before, we use the inverse Laplace transform to get the time domain response. To reiterate, the benefit of standard forms is that the transforms are given in the tables: *(e.g. from Dorf and Bishop)*

$$\boxed{\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t, \zeta < 1}$$

$$\boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

γ not shown as it is unaffected by Laplace transform

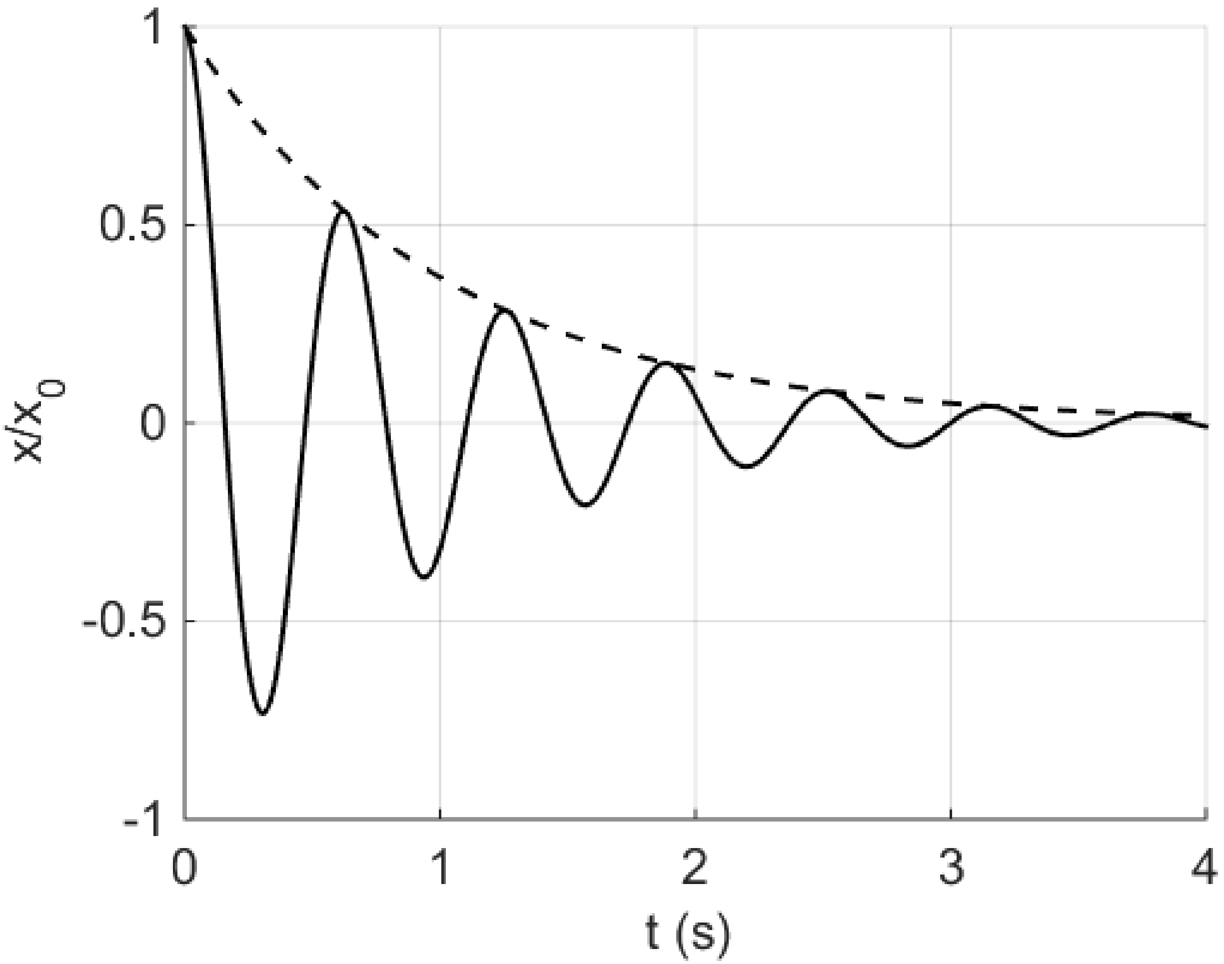
$$\frac{x(t)}{x(0)} = \gamma \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2 t}\right)$$

What does this equation look like?

$$\frac{x(t)}{x(0)} = \underline{\gamma \frac{\omega_n}{\sqrt{1-\zeta^2}}}\, \underline{e^{-\zeta\omega_n t}}\, \underline{\sin\left(\omega_n \sqrt{1-\zeta^2} t\right)}$$

Gain **+ Exponential** **+Sine Wave**

This is assuming $\boxed{\zeta < 1}$ – we will look at when it isn't next week



This is an exponentially decaying sinusoidal oscillation, with

Frequency: $\omega_n\sqrt{1-\zeta^2 t}$

Decay: $e^{-\zeta\omega_n t}$

Gain: $\gamma\dfrac{\omega_n}{\sqrt{1-\zeta^2}}$

This is assuming $\boxed{\zeta < 1}$

MUCH MORE DETAIL LATER!

Summary

Mechanical and Electrical components can be modelled as Linear Time Invariant systems, through approximation by linearisation

The change in the output of a LTI systems over time is described by *linear* ODEs in the form

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_2\frac{d^2 y}{dt^2} + a_1\frac{dy}{dt} + a_o = bx$$

The *transfer function* of these systems is the relation between the input and output, which can be described in the Laplace domain:

$$G(s) = \frac{X(s)}{Y(s)}$$

Summary

A *second order* system has an ODE with a highest order n=2, the transfer functions of which take the standard form

$$G(s) = \gamma \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

γ Gain

ζ Damping Ratio

ω_n Natural Frequency

Which describes an exponentially decaying sinusoidal oscillation

$$\frac{x(t)}{x(0)} = \gamma \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2 t}\right)$$

Both Electrical and Mechanical components take the same form

Thank you for your attention!

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