POLYTECH
Peter the Great
St.Petersburg Polytechnic
University

Course
« Introduction to Biomedical Engineering»

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Section 1: Basic electronics
Lecture 1.3: Filters

Filters

In this lecture we are going to talk about filters and specifically electronic filters.

First, again, why do we need them? Well, it all comes down to a point we made in the second lecture, our signals are small, and whilst they are travelling through various cables they get contaminated with all sorts of noises. Add to this parasitics, 50Hz mains frequency noise, and all sorts of unwanted signals which are present during the signal acquisition, and we end up with unwanted mess on top of our small but valuable signal. So we use electronic filter in order to essentially, 'filter' out signal out of the unwanted components.

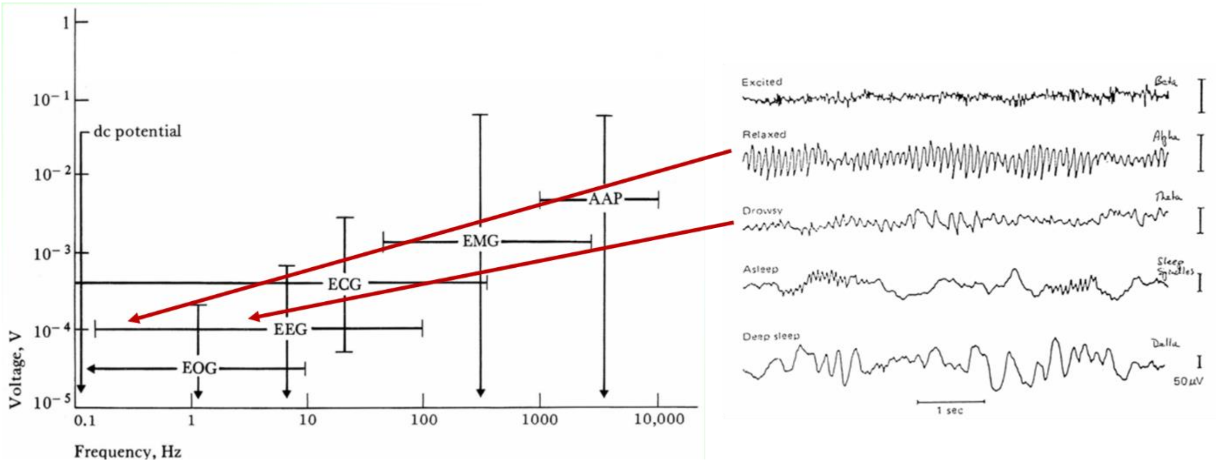


Figure 1 - Different EEG "rhythms" occur at different bands

Most commonly in engineering we are talking about filters in terms of the frequency content. So instead of common time domain (signal changes in time), we use frequency domain. You can represent a single frequency sinusoid as a point in frequency domain, with the x-axis denoting the frequency of the sinusoid, and y-axis representing the amplitude of the signal at that frequency.

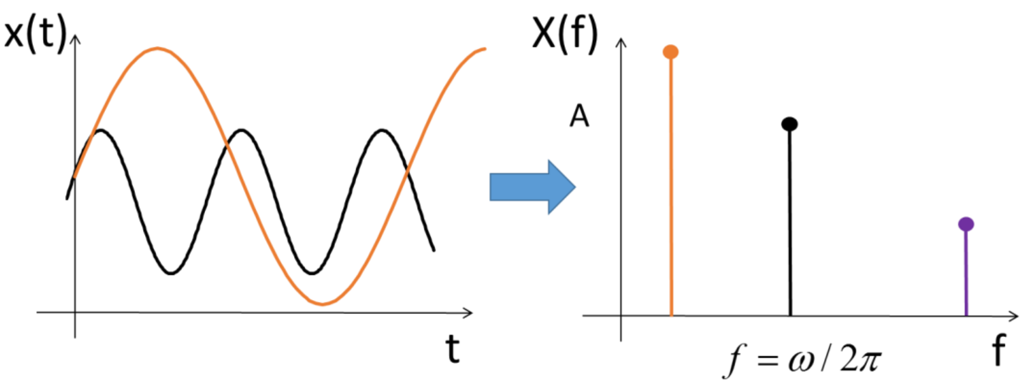


Figure 2 - Frequency vs Time domains

If you feed the sinusoidal signals with unitary amplitude to the system, you will get so-called Bode plots or Bode diagrams, which show Amplitude and Phase response of the system at any given frequency.

If you look at this example of a first order low pass filter, you can see that at small frequencies a sine wave will pass through the system almost undistorted and will have the same amplitude and phase as the original signal. If you take a very high frequency, you can see that the sine wave at that frequency will be reduced by approximately 70 dB and will be anti-phase, which means that the voltage is inverted.

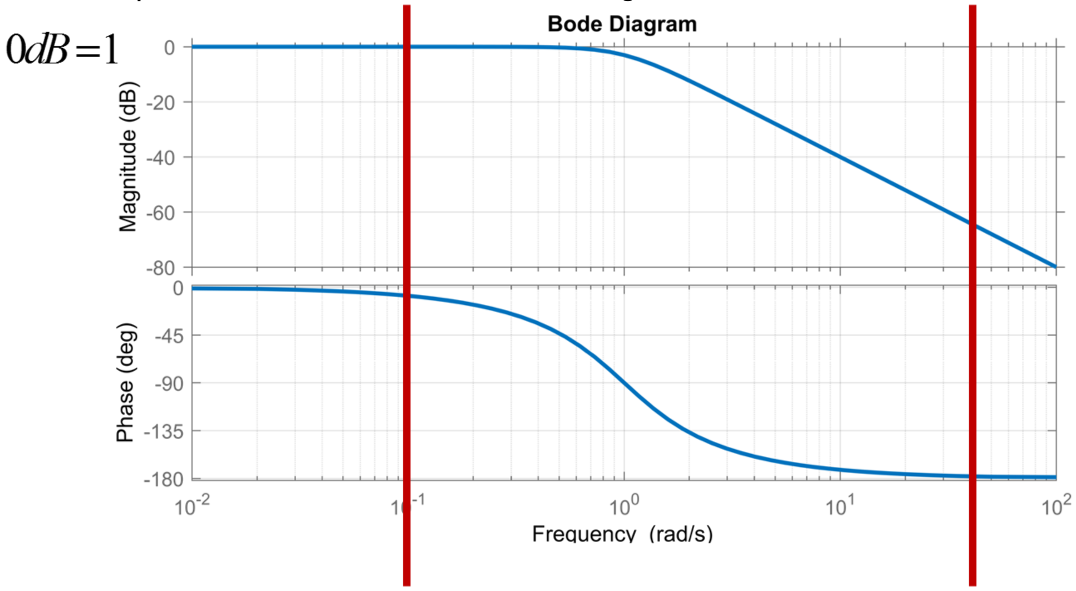


Figure 3 - Bode Plots – Frequency Response

This brings us straight to the classes of ideal filters that are there. The first one example of which we have just seen, preserves, or passes, all the frequencies up to a certain one, which is called the cut-off frequency, and does not let any other higher frequency to go through. This is called low-pass. Depending on the location or frequency content of stuff that we want VS stuff that we do not want, there can also be High-pass, Bandpass, Band stop or notch. The ideal filters are called 'brick wall filters', obviously, and do not exist.

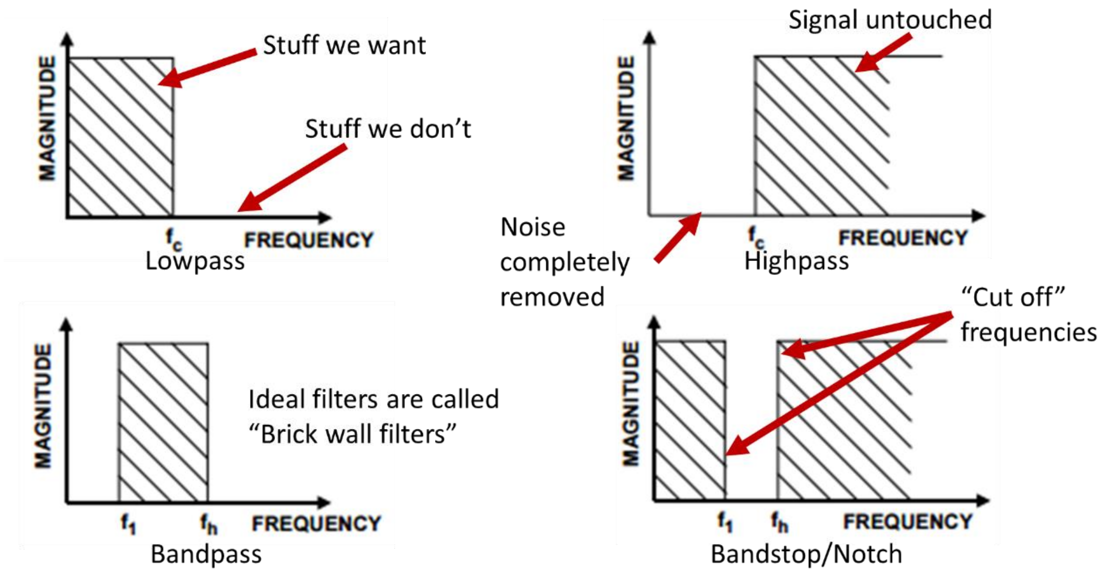


Figure 4 - Classes of ideal filter

As we know parasitics and physics limits us, so what we end up with is something on the diagram, where we have certain characteristics of a filter, which tell us how good, or bad it is. The common are Stopband attenuation, cut-off frequency (or -3dB point, similar to the amplifiers, denotes the frequency at which signal drops by 3 dB), passband ripple, and stopband frequency. Obviously they are all related to each other, and in reality we need to pick the parameters that are more important than other in order to find the right compromise.

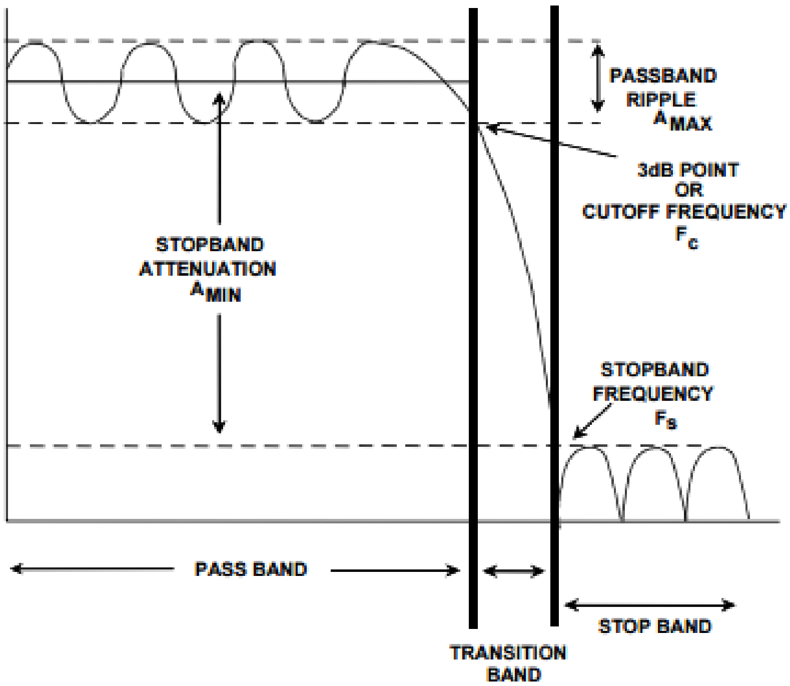


Figure 5 - Stopband attenuation

Besides the obvious, another crucial application of filtering is the reduction of artefacts caused by sampling, or aliasing. For a given sampling rate, there is a maximum frequency which can be sampled at that rate, it is equal to a half of that frequency theoretically, and is called a Nyquist frequency.

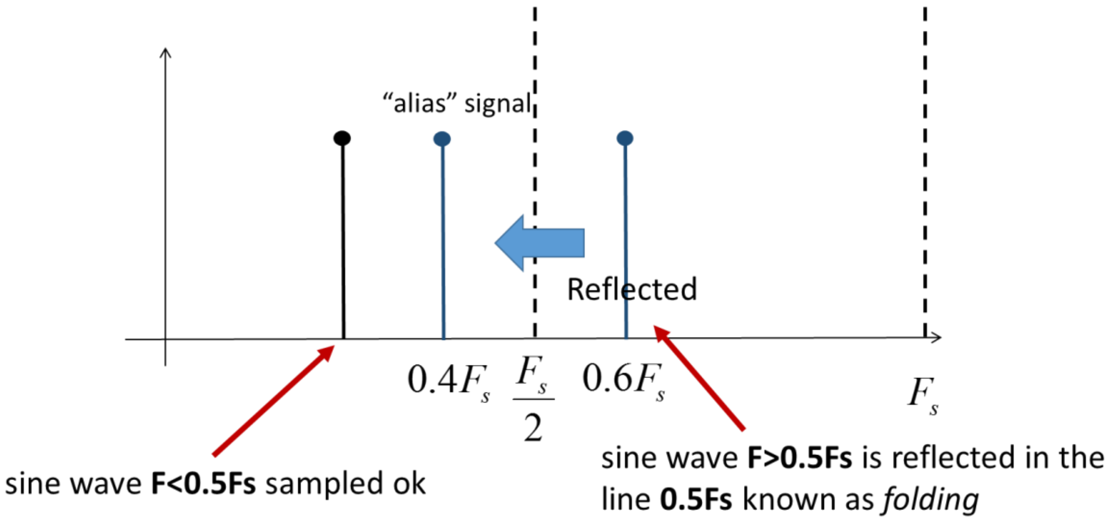


Figure 6 - Nyquist Frequency

Below this frequency everything is ok, and we can sample, above - and horrible artefacts arise! It is known as 'folding'. The mechanism of these is illustrated at the diagram, where in the bottom case you start sampling things slower than sinusoid makes a complete cycle. This can cause all sorts of 'artificial' signals contaminating your original, just because they have higher frequencies than Nyquist. To avoid aliasing, you can simply implement a low-pass filter just before sampling, with cut-off at a half of the sampling rate, thus removing troublemakers before they cause any damage.

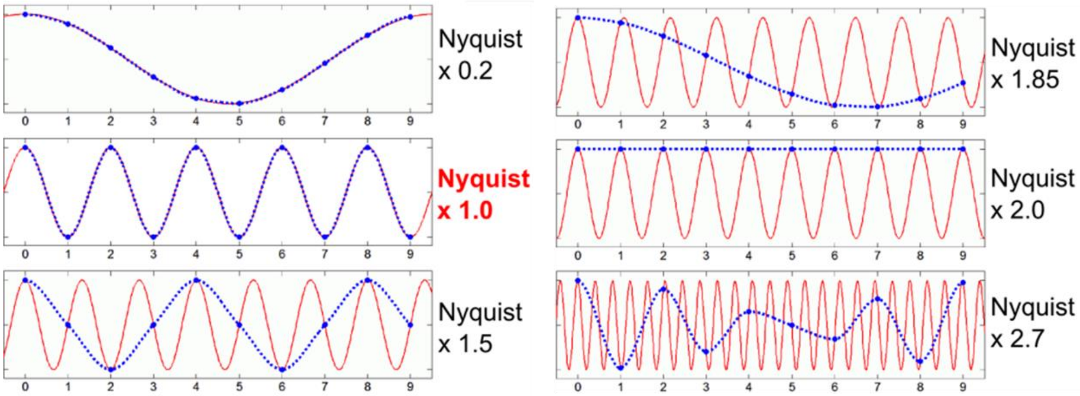
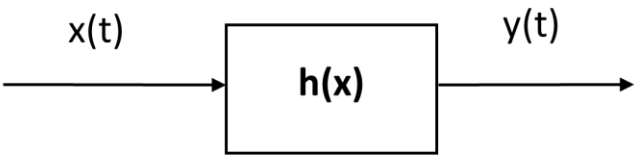


Figure 7 - Nyquist Frequency results

The industry standard however, implements harsher trouble maker removal techniques, and sets cut-off somewhere between a third and a quarter of a sampling frequency, depending on the application.

At this point we will take a slight U-turn, and talk about Transfer Functions. Do not worry, we will come back to practical stuff shortly, just bear with me for couple of minutes. The need for this particular abstraction comes from our desire to describe systems not only in frequency domain, but also changing in time. Things start when we describe the system behavior as an ordinary differential equation with respect to time (think of a second newton's law, where you have acceleration – second derivative, and velocity – first derivative of position x). The input in this case resembles right hand side of the equation (in Newton's law it is the force), and output is the stuff we need to find (in Newton's equation it is the position). So for a known input, we can compute the output, if we know how to solve those. However this is bulky and does not let simple system analysis and manipulation. Instead, we will introduce something new.

We will only consider linear time invariant systems, which most of electronic components are, and the ones which are not, can be linearized.



$$y = h * x$$

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_o = bx$$

X is input function
Y is output
N is *order* of the ODE
A0… are coefficients

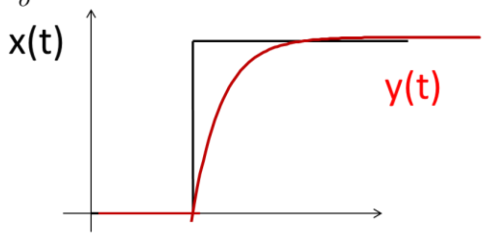


Figure 8 - Transfer functions

If you have this system, and have its ODE, then you can perform a so-called Laplace transform. This is weird transform which is equal to the integral from zero to infinity of the function multiplied by e to the power of -st, and takes us to a different space, called Laplace domain. It is the extension of the Fourier transform, and similar to a Fourier transform, gives us a powerful tool to analyze dynamic systems. The principle variable s is an extension of Fourier's frequencies (w) on the complex plane together with some exponents (it is easy to see, when exponential term is zero, i.e. we are on the imaginary axis of s-plane, and the transform simply becomes a Fourier series).

The beauty of Laplace transform is that it transforms all derivatives to a multiplications by s. Same is true for n-th derivative, it becomes multiplication by s^n. Also, it transforms all integrals to a division by s! This is very handy! Now, all your complex ODEs become algebraic equations! And on top of this, there are standard transforms for many functions, which you can use to get back to time domain, after you have solved algebraic equation (YAY). For us, however, the Laplace transform itself has much bigger value, which we will see later.

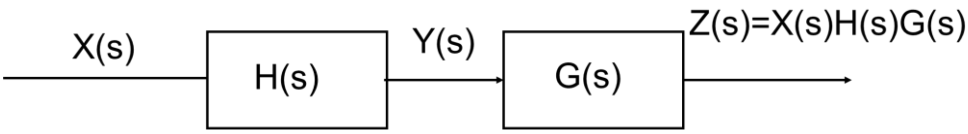


Figure 9 - Transfer functions – Laplace

However, enough theory lets practice some Laplace: Consider a spring mounted to the wall – for me it is always easier and more intuitive to analyze the mechanical system. Time domain equation is well known, it's a Hooke's law: force is proportional to deflection (position, assuming zero is where the spring is not deflected). K here is stiffness, coefficient of proportionality. Let's take Laplace transform, remembering that force is the input, and X is the output. The transform is actually quite easy in this case as there is no derivatives. Then our task is to find the ratio between the output input, and vuala, we have found a transfer function of this system, which definition is exactly this: It is the ratio between Output and Input in Laplace domain. Now, to find an output of the system to any input, we just need to multiply the Transfer Function by the input, and take inverse transform. So Transfer Function for the spring is just 1 over k.

Exponential Decays

$$s = \sigma + j\omega$$

Sine Waves

$$L\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

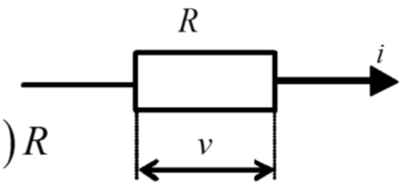
$$\boxed{s \equiv \frac{d}{dt}}$$

$$\boxed{\frac{1}{s} \equiv \int_{0^-}^{\infty} dt}$$

Let's see what happens with electronic components. Resistor, following the Ohms law, and assuming the input is Voltage, and output is the Current, has the transfer function 1 over the resistance R.

$$L\left\{\frac{d^n x}{dt^n}\right\} = s^n X(s)$$

Ohm's law V=IR:



Time domain $v(t) = i(t)R$

Laplace domain $V(s) = I(s)R$

Transfer function:

$$G(s) = \frac{I(s)}{V(s)} = \frac{1}{R}$$

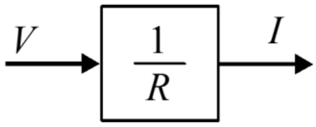
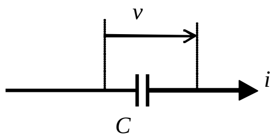


Figure 10 - Transfer functions of Resistor

Similar, capacitor, with a mild complication in replacing the time derivative for multiplication by s, has a transfer function Cs. $L\left\{\frac{d^n x}{dt^n}\right\} = s^n X(s)$

Time domain $i(t) = C\frac{dv}{dt}$ $v(t) = \frac{1}{C}\int i(t)dt$

Laplace domain $I(s) = CsV(s)$ $V(s) = \frac{1}{C}\frac{1}{s}I(s)$



Transfer function:

$$G(s) = \frac{I(s)}{V(s)} = Cs$$

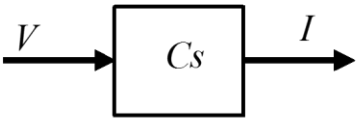


Figure 11 - Transfer functions of Capacitor

The inductor - 1 over Ls.

From Faraday's Law:

Time domain $v(t) = L\frac{di}{dt}$

Laplace domain $V(s) = LsI(s)$

Transfer function

$$G(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls}$$

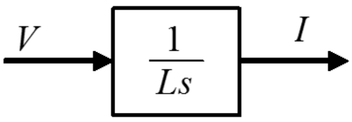


Figure 12 - Transfer functions of Inductor

Let's see some combination of components, which is a bit more complicated, since here we would like to find the ratio between input and output voltage. Right, we will start with expressing what we know, and Note where we measure all voltages. Applying Kirchhoff's law and Ohm's law, we substitute and get a time domain ODE, which can be easily converted to Laplace domain, and Finally, the transfer function can be expressed using some constant substitutions as gamma over 1 + tau s, where gamma, which is called the gain, is 1, and tau, which is commonly called the time constant, is equal to RC. OK, Sherlock, now what?

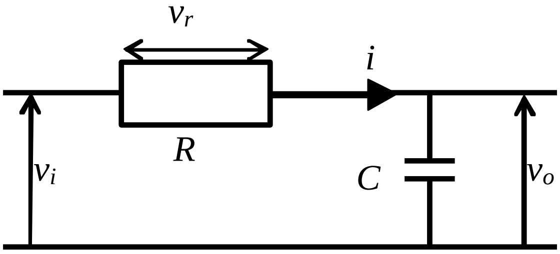


Figure 13 - Transfer function RC filter example

Well, it gets very cool when you consider the fact that you can instantly get a frequency domain response - a bode plot - by ignoring real component of s. Substitute jw into the transfer function, and here you go, you have not only found the entire frequency response, you have also instantly found the -3dB point, or cut-off frequency (I will leave it for home work to digest), which in this case is simply 1/RC. Or, in terms of actual hertz frequencies, 1 over 2 pi R C. Yes, as you have probably noted, this is a low-pass filter after all. Remember the statements about capacitors, well now we know much more about how exactly the current does or does not flow through the capacitor, and at which frequencies.

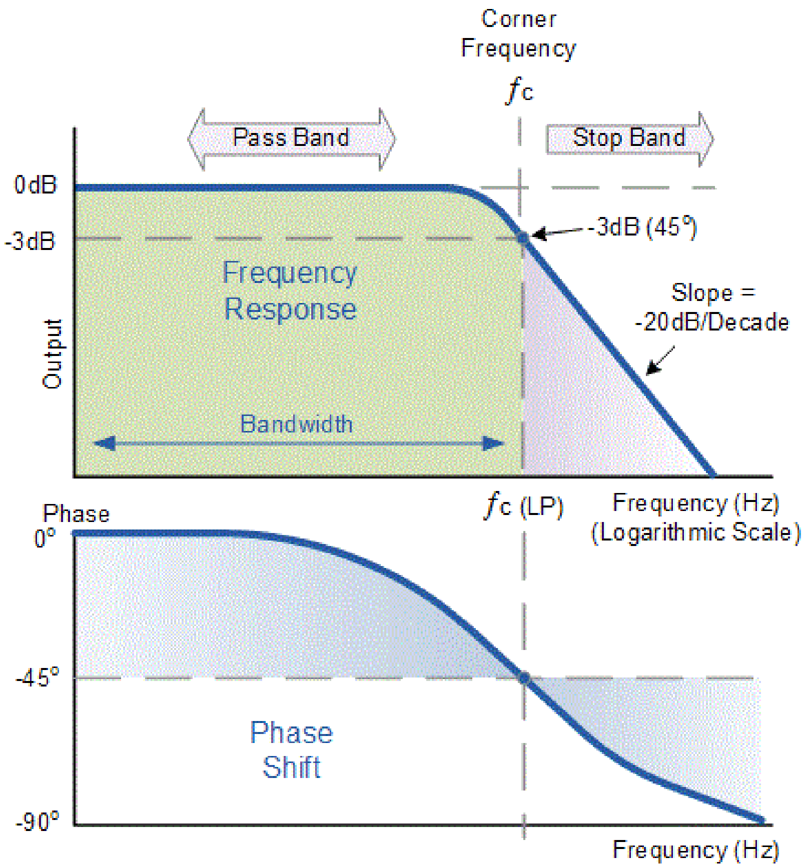


Figure 14 - Transfer function – Freq Response

Moving further, using the resistor and capacitor, it is easy to create a high-pass filter as well, Logic tells us: low frequency current will not flow through a capacitor, high frequency-will. Transfer function tells us the exact value of the cut-off frequency, which is 1 over R C. And lets us to plot Bode diagram of the filter.

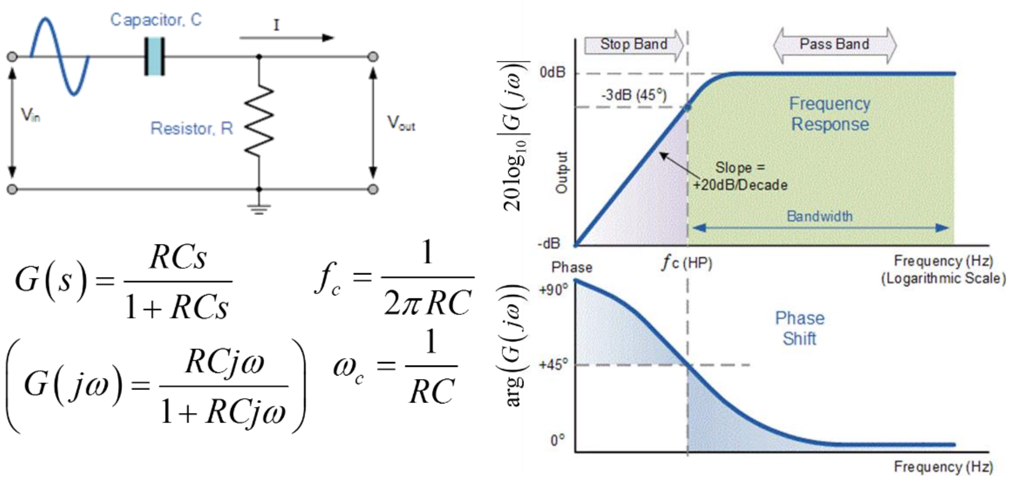


Figure 15 - Passive Filters – High Pass

But the beauty of the Laplace does not end there, If you do not drop the real part of s, but instead give the system some extreme input, like a step function, for example, the transfer function can tell you what exactly your system would do, in case of first order filter, for example, we can observe the time delay. And we can say a lot about this delay by looking at the transfer function itself. For first order transfer functions, there is a reason that tau is called a time constant! We will consider later what happens exactly when you increase the order of the system, but the general rule of thumb for filters is that if you improve the frequency response, but you also increase the initial delay, and increase the time for system to settle down from a kick.

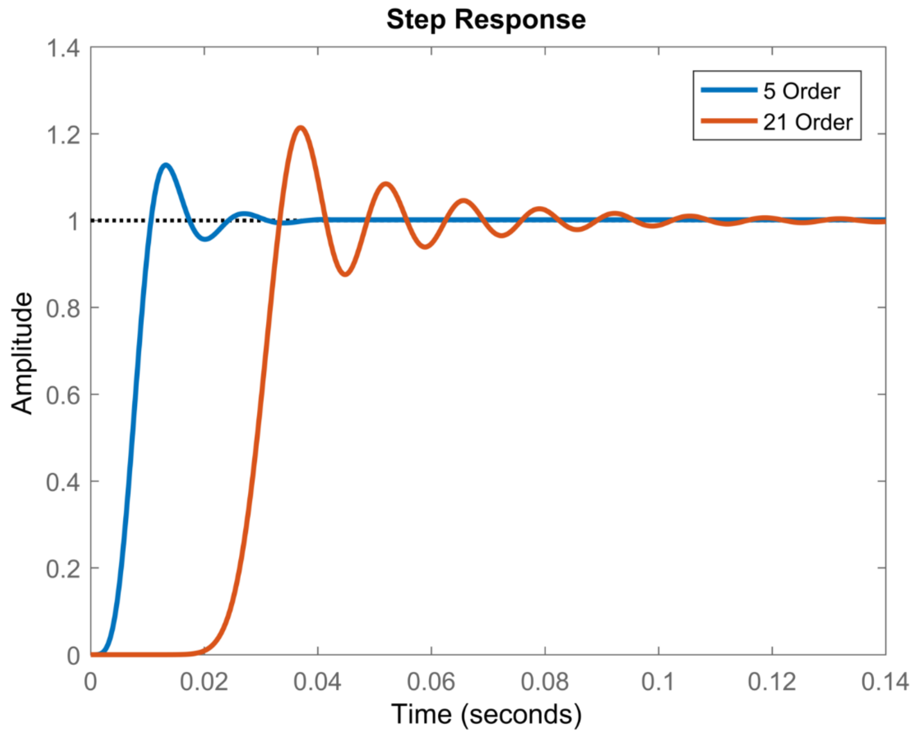


Figure 16 - Transfer functions – Step response

Well, we say, let’s see, if you have an RC filter, quickly enough you realize that it is too simple, and lets too many frequencies in, and you might want to double it. Logic tells you that it will work twice as good, right? In fact you might think infinite number of them would produce you an ideal filter, right (this is also called cascading)? First thing to consider is that gain is not actually 1, and there is a reason for it to decay towards the -

3dB point, so every time you do that, amplitude decays ever so slightly (or not slightly, depending on the situation).

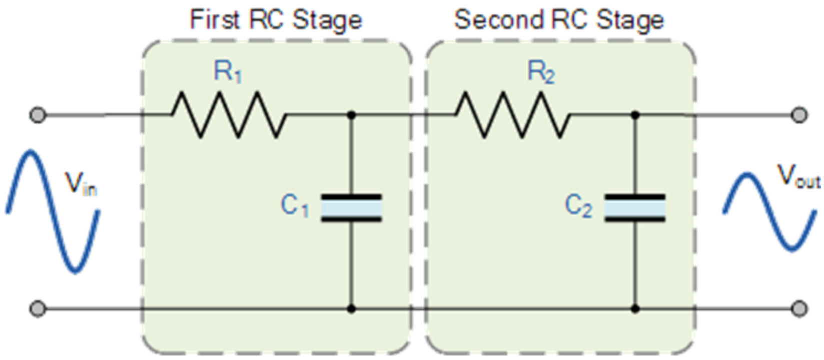
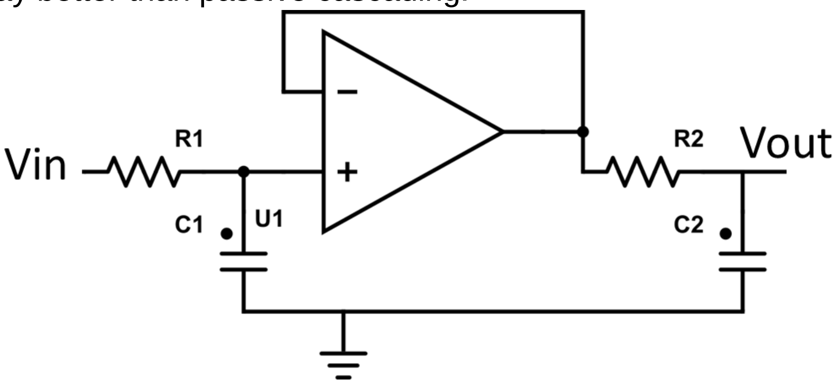


Figure 17 - Cauer Topology

Second thing is this. Let's see what we want to get, second order RC filter,, Vin, Vout, 1 transfer function, another transfer function Is it not just a multiplication? Well……

Intuitively you understand, that voltage at point A actually depends on all four components. Let's formally derive this: From Kirchoffs laws at point B we get the following. Same for point A: Substitute (1) into (2), and after rearranging into transfer function you get the following monster, which after tidying up looks nothing as we have hoped for. In fact, it is actually a bit tricky to analyze.

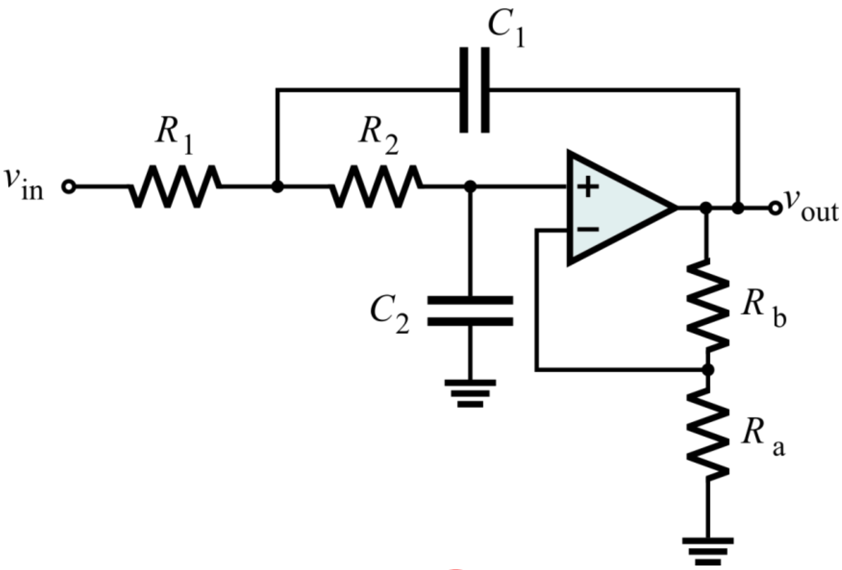
Can we make a better system? Well, yes, that's why we have introduced buffer amplifier earlier. Let's stick it between the first and second filter stages. Now, everything is independent, and we can simply multiply 2 transfer functions together! This is the answer we want, and now we can control each filter independently as well. However, still, some poor performance due to the output impedance – the one that loses us some voltage – being determined by the second RC filter, and thus poor (not the end of the world, but not perfect). Oh well, the glory of adding another buffer at the end, and making everything right again. Do not get me wrong, this is not a perfect system, and had drawbacks, but way better than passive cascading.



$$\frac{V_{out}}{V_{in}} = \frac{1}{s^2\left(C_1C_2R_1R_2\right)+s\left(C_1C_2+C_2R_2\right)+1}$$

Figure 18 - Order filter

To solve the issues of this system however, we can introduce more sophisticated filters: Sallen Key for example, has only 1 OpAmp, and also has zero output impedance. It can also be implemented in combination with the amplification in the same circuit. Gain here is defined by the ratio of 2 additional resistors, however it alters the frequency response. There are many other topologies of active filters all of which would have slightly different characteristics, and I am sure you can find your ideal filter now when you know the basics, and after we have talked a bit more about the Transfer functions in the control theory section.



$$\frac{V_{out}}{V_{in}} = \frac{K}{s^2\left(C_1C_2R_1R_2\right)+s\left(C_2R_1+C_2R_2+R_1C_1(1-K\right)+1}$$

$$K = 1+\frac{R_b}{R_a}$$

Gain also alters the roots of the denominator, so alters frequency response

Figure 19 - Sallen Key

Ok, so we have talked about filtering in the frequency domain, but what if that is not enough? What if our noise exists in the same bandwidth as our signal? Well, the solution to this is a coherent averaging: if your signal repeats, you can lock into the event you are interested in, and add together several of those. The idea being that you signal is locked to the particular event (like a heartbeat), and your noise is not, so it will be random with respect to the event, thus eliminating as 1 over root square of N, where N is the number of repeats (you can read more about this technique if and when you come across the specific problem you need to solve)

Finally, and I must stress this intentionally, please always bear in mind that

1. Filters are not magic, and nothing can recover missing data.

2. Filters can only reduce the noise by limited amount, so if you have enormous amplification, and then filter, there is a possibility that noise will still be an issue, so overall remember the following rule: **Filters are not a substitute for a good data.**

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