Chapter 10
Orthogonalization

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Orthogonality of the Four Subspaces

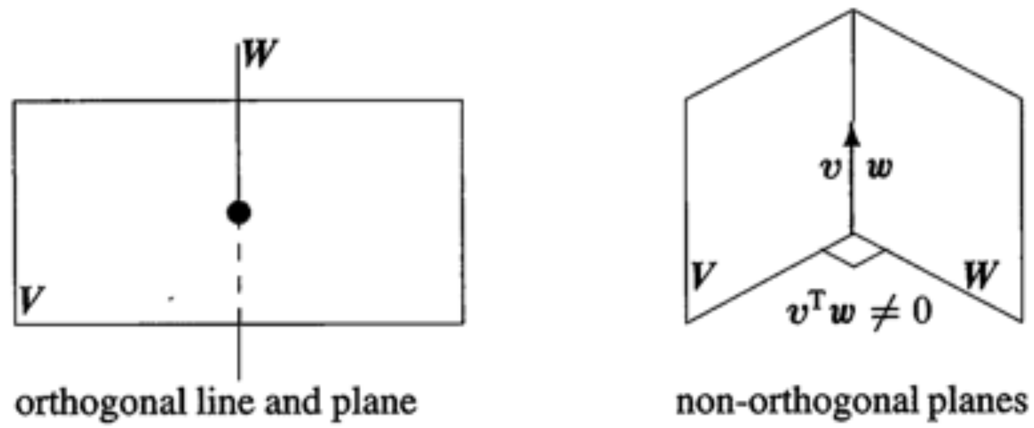

Figure 4.1: Orthogonality is impossible when dim V +dim W > dimension of whole space.

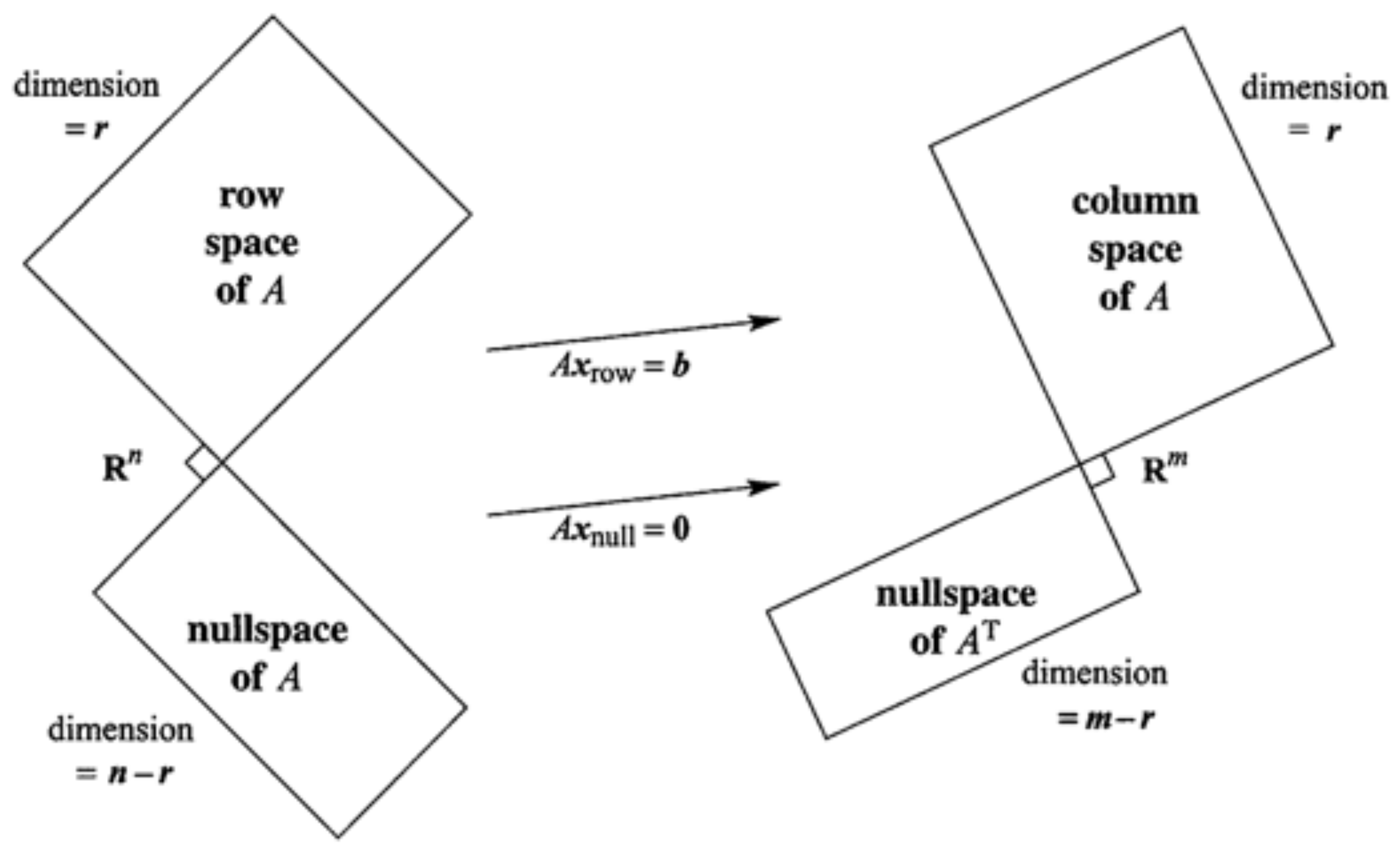

Figure 4.2: Two pairs of orthogonal subspaces. The dimensions add to n and add to m. **This is an important picture**—one pair of subspaces is in $\mathbf{R}^n$ and one pair is in $\mathbf{R}^m$.

Orthogonal Complements

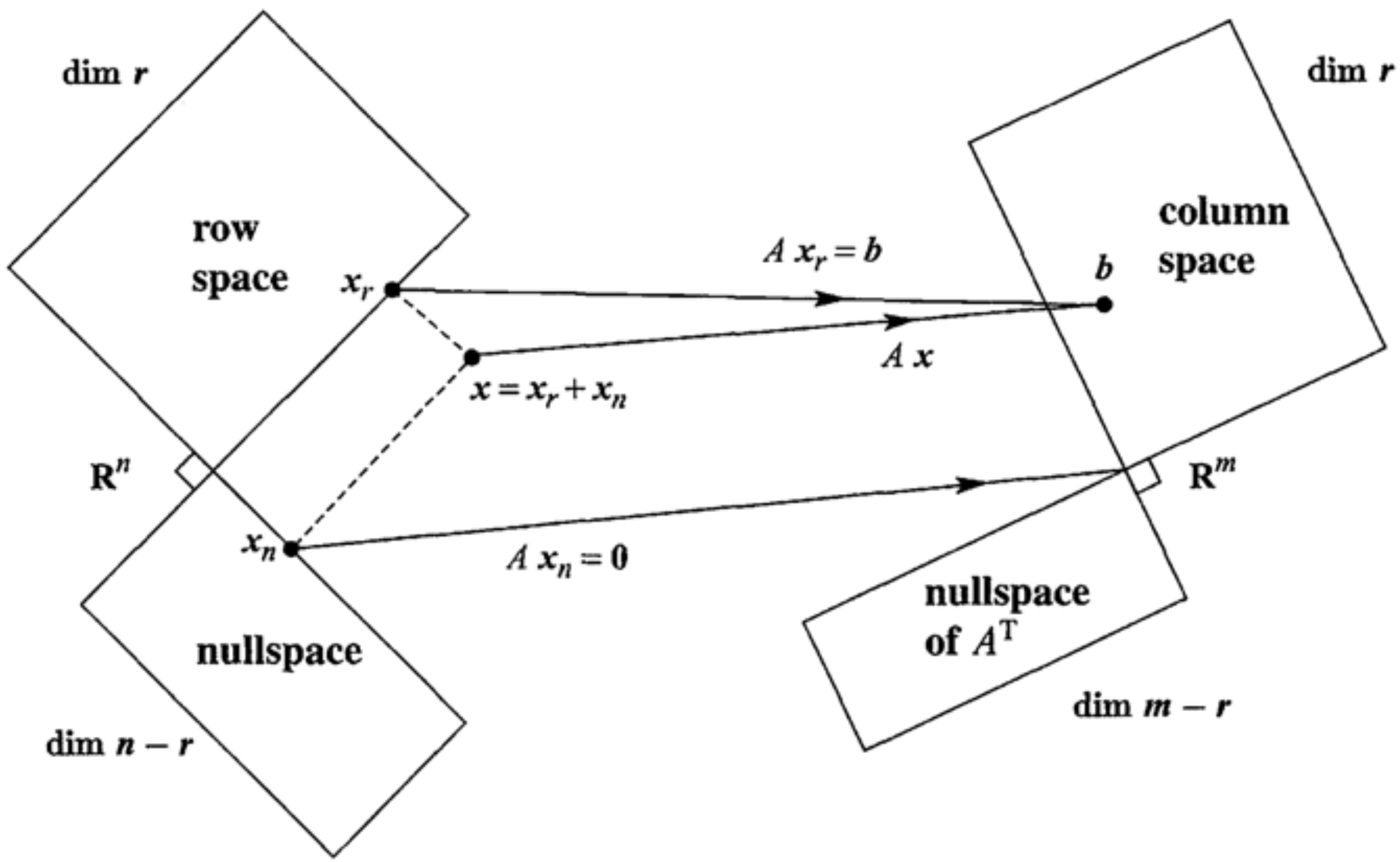

Figure 4.3: This update of Figure 4.2 shows the true action of A on $x = x_r + x_n$. Row space vector x_r to column space, nullspace vector x_n to zero.

Projections

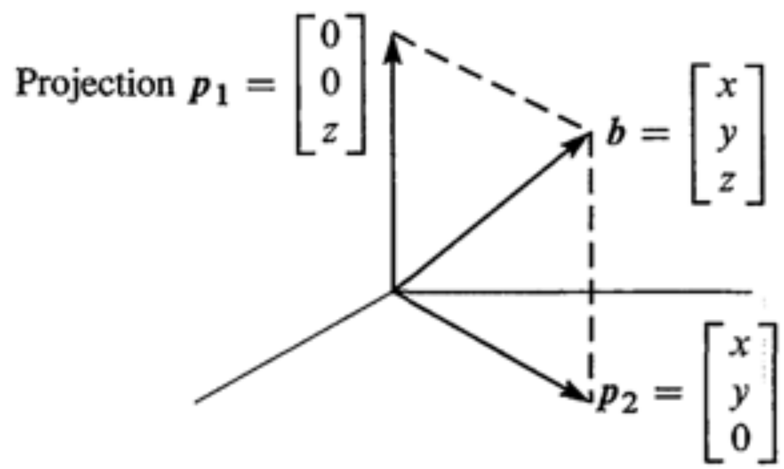

Figure 4.4: The projections $\boldsymbol{p}_1 = P_1\boldsymbol{b}$ and $\boldsymbol{p}_2 = P_2\boldsymbol{b}$ onto the z axis and the xy plane.

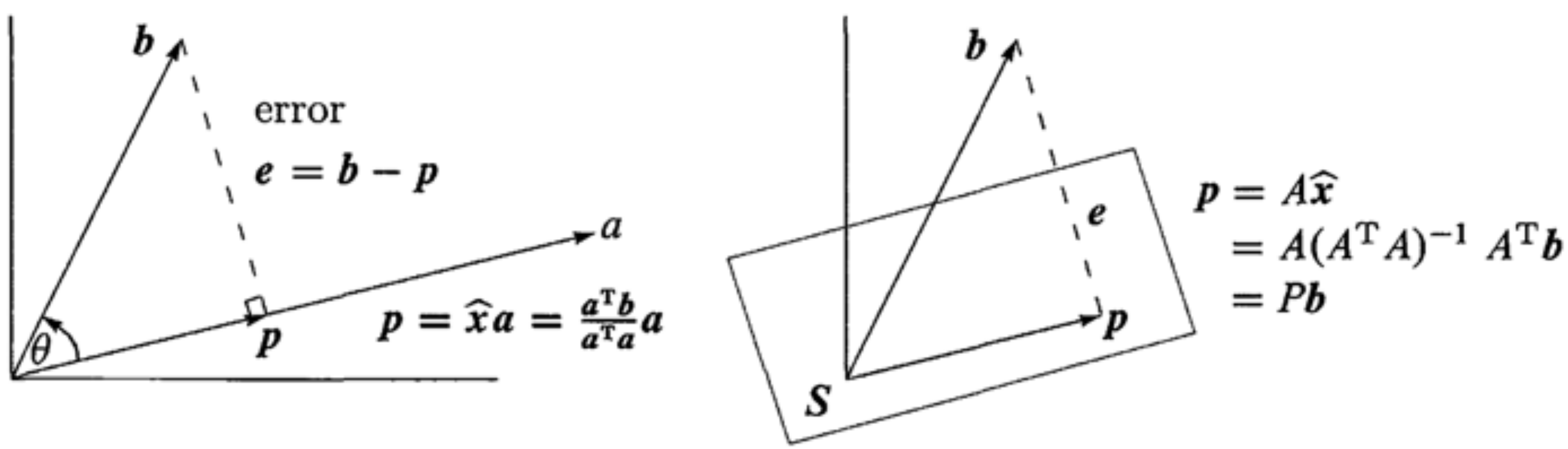

Figure 4.5: The projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto a line and onto $\boldsymbol{S}$ = column space of A.

Least Squares Approximation

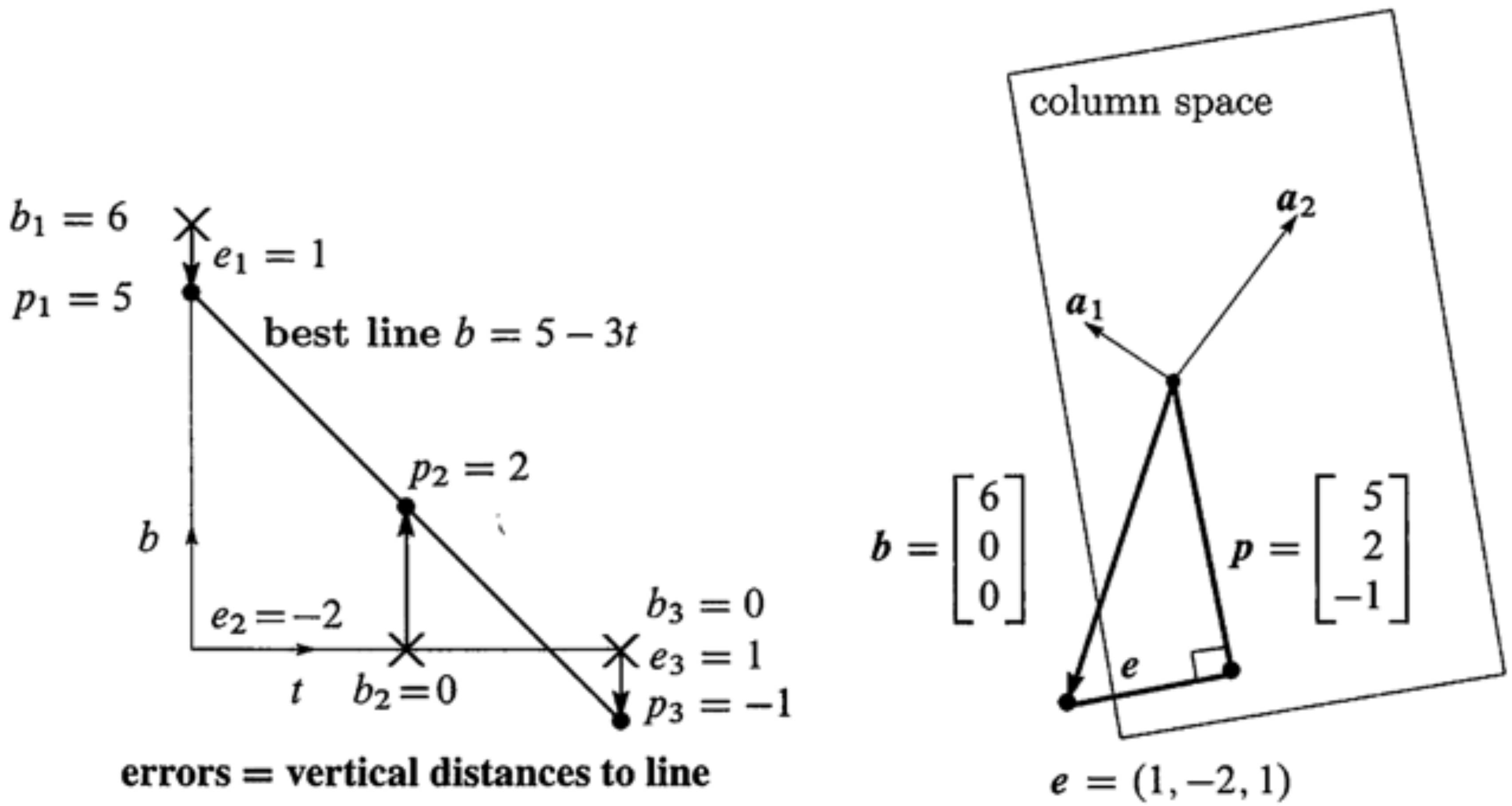

Figure 4.6: **Best line and projection: Two pictures, same problem.** The line has heights $\boldsymbol{p} = (5, 2, -1)$ with errors $\boldsymbol{e} = (1, -2, 1)$. The equations $A^{\mathrm{T}} A \widehat{\boldsymbol{x}} = A^{\mathrm{T}} \boldsymbol{b}$ give $\widehat{\boldsymbol{x}} = (5, -3)$. The best line is $b = 5 - 3t$ and the projection is $\boldsymbol{p} = 5\boldsymbol{a}_1 - 3\boldsymbol{a}_2$.

Least Squares Approximation

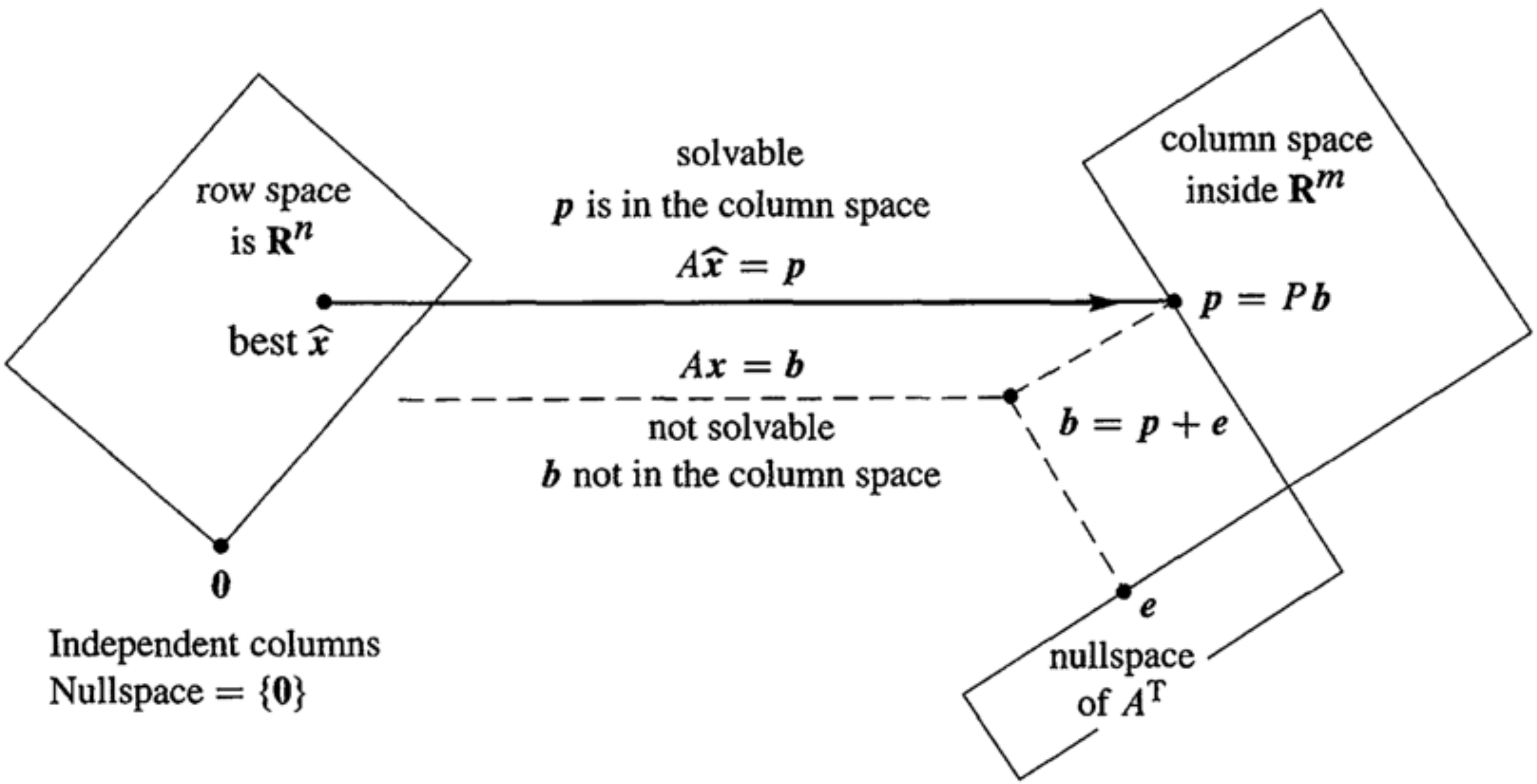

Figure 4.7: The projection $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$ is closest to $\boldsymbol{b}$, so $\widehat{\boldsymbol{x}}$ minimizes $E = \|\boldsymbol{b} - A\boldsymbol{x}\|^2$.

Least Squares Approximation

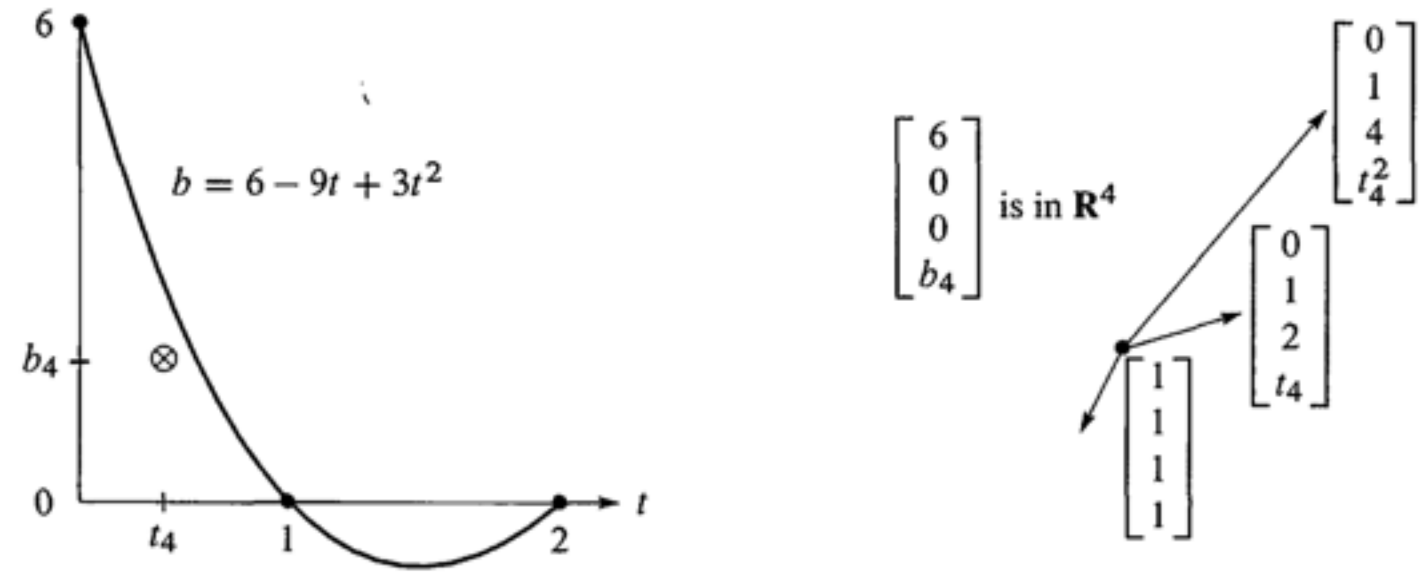

Figure 4.8: From Example 3: An exact fit of the parabola at $t = 0, 1, 2$ means that $\boldsymbol{p} = \boldsymbol{b}$ and $\boldsymbol{e} = \mathbf{0}$. The point b_4 off the parabola makes $m > n$ and we need least squares.

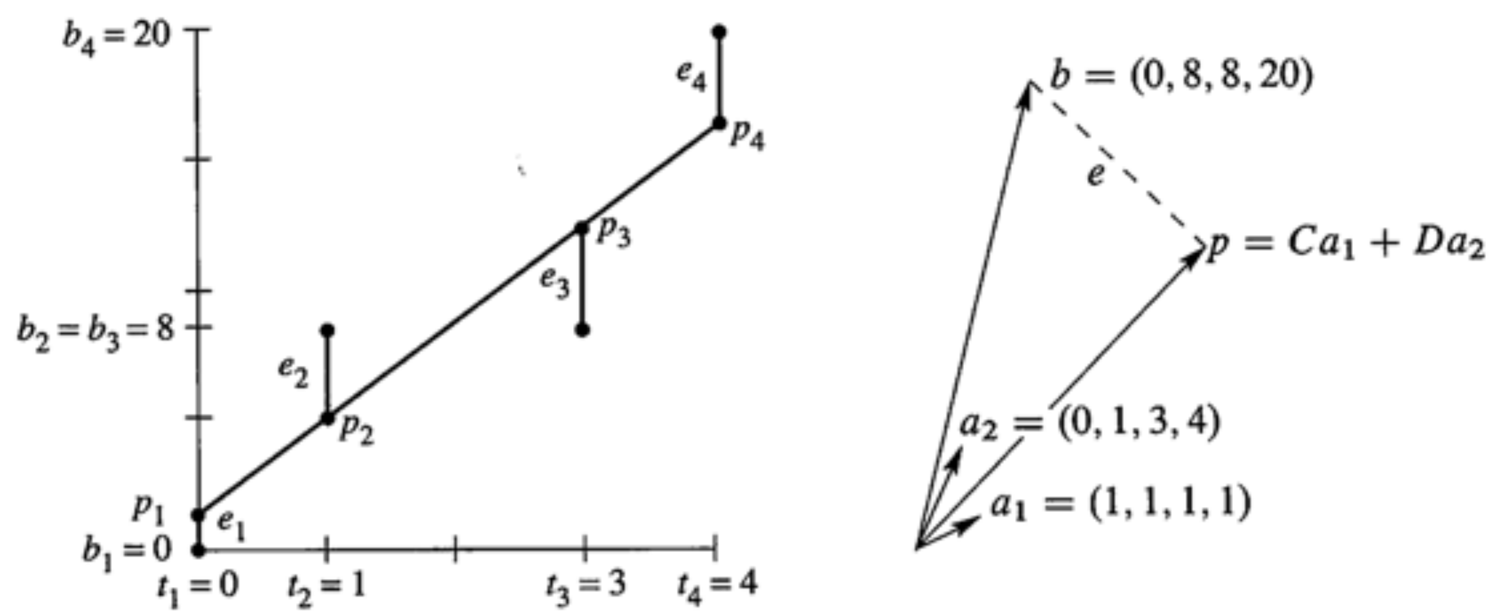

Figure 4.9: **Problems 1–11**: The closest line $C + Dt$ matches $C\boldsymbol{a}_1 + D\boldsymbol{a}_2$ in $\mathbf{R}^4$.

Orthogonal Bases and Gram-Schmidt

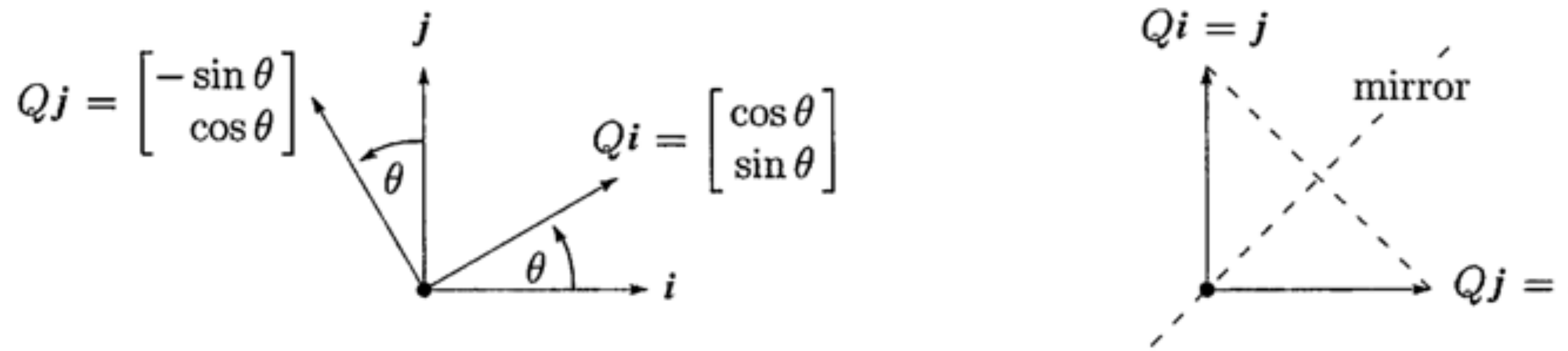

Figure 4.10: Rotation by $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ and reflection across 45° by $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

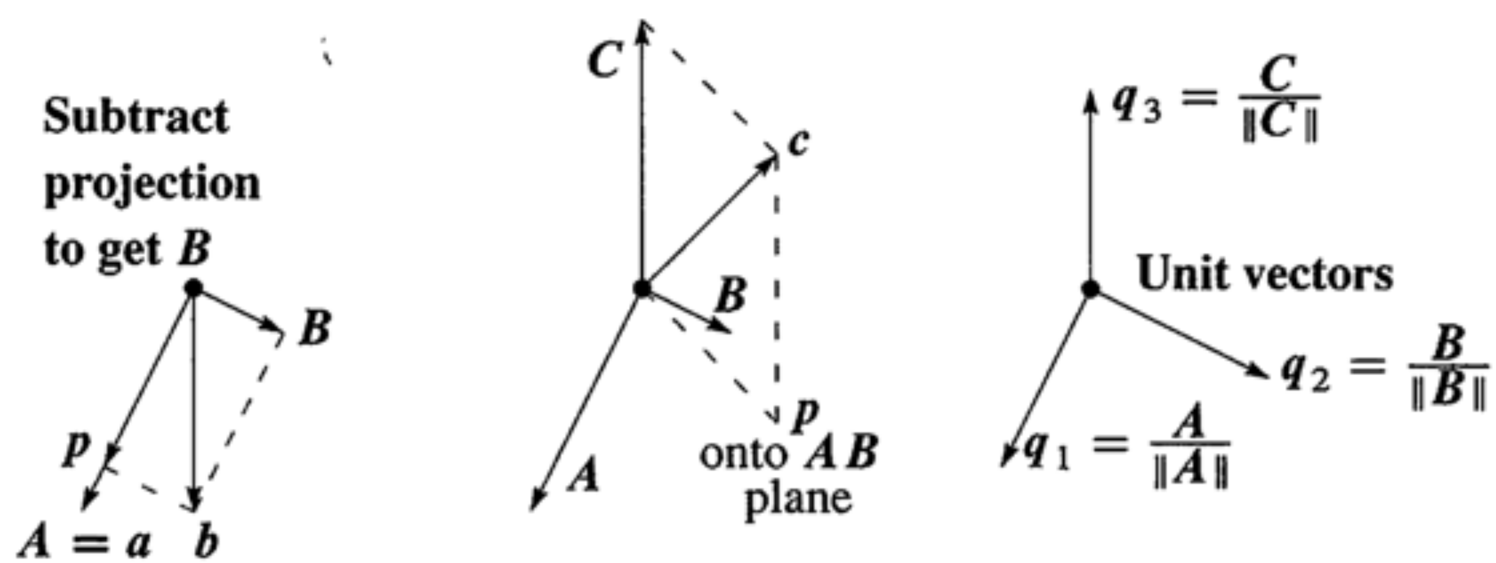

Figure 4.11: First project b onto the line through a and find the orthogonal B as $b - p$. Then project c onto the AB plane and find C as $c - p$. Divide by $\|A\|$, $\|B\|$, $\|C\|$.

QR Factorization

When $Ax = b$ has no solution, multiply by A^T and solve $A^T A\hat{x} = A^T b$.

A matrix Q with orthonormal columns satisfies $\boldsymbol{Q^T Q = I}$ **:**

$$Q^T Q = \begin{bmatrix} - \boldsymbol{q}_1^T - \\ - \boldsymbol{q}_2^T - \\ \\ - \boldsymbol{q}_n^T - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & & \boldsymbol{q}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I. \tag{1}$$

$$\begin{bmatrix} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \\ \\ \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^T \boldsymbol{a} & \boldsymbol{q}_1^T \boldsymbol{b} & \boldsymbol{q}_1^T \boldsymbol{c} \\ & \boldsymbol{q}_2^T \boldsymbol{b} & \boldsymbol{q}_2^T \boldsymbol{c} \\ & & \boldsymbol{q}_3^T \boldsymbol{c} \end{bmatrix} \quad \text{or} \quad A = QR. \tag{9}$$

Least squares $\quad R^T R\hat{\boldsymbol{x}} = R^T Q^T \boldsymbol{b}$ or $R\hat{\boldsymbol{x}} = Q^T \boldsymbol{b}$ or $\hat{\boldsymbol{x}} = R^{-1} Q^T \boldsymbol{b}$ (10)