PH3102 Quantum Mechanics   Assignment 2

Instructor: Dr. Siddhartha Lal   Autumn Semester, 2025

Start Date: August 28, 2025   Submission Deadline: September 4, 2025 .

Submit your answers to the Tutor at the start of the tutorial.

1 Time Evolution in Heisenberg picture

I. Particle under a constant force

Consider a particle moving under constant force F.

(a) Write down the corresponding Hamiltonian for the particle.

(b) Using the Heisenberg equation of motion, show that:

$$\hat{p}_H(t) = \hat{p}_0 + Ft \quad , \quad \hat{x}_H(t) = \hat{x}(0) + \frac{\hat{p}(0)t}{m} + \frac{Ft^2}{2m} \; . \tag{1}$$

Note how they are in complete analogy with equation from classical mechanics:

$$v = u + at \quad , \quad s = s_0 + ut + \frac{1}{2}at^2 \; . \tag{2}$$

(c) Consider a particle described by the wave function $\psi(x) \equiv \langle x|\psi\rangle = Ne^{-\frac{x^2}{2\Delta^2}}$, where N is the appropriate normalisation constant. The particle at $t = 0$ is centered around $x = 0$. Show how does this changes with time. **Hint:** Find expectation value of $\hat{x}_H(t)$ in $\psi(x)$.

II. Particle in a Harmonic Potential

Consider a particle under the potential $V(\hat{x}) = \frac{1}{2}k\hat{x}^2$.

(a) Show that using the Heisenberg equation of motion:

$$\frac{\mathrm{d}\hat{x}_H(t)}{\mathrm{d}t} = \frac{\hat{p}_H(t)}{m} \quad , \quad \frac{\mathrm{d}\hat{p}_H(t)}{\mathrm{d}t} = -k\hat{x}_H(t) \tag{3}$$

(b) Solve the above two coupled differential equation, and show that:

$$\hat{x}_H(t) = \hat{x}(0)\cos\omega t + \frac{\hat{p}(0)}{m\omega}\sin\omega t \tag{4}$$

$$\hat{p}_H(t) = \hat{p}(0)\cos\omega t - mw\hat{x}(0)\sin\omega t \tag{5}$$

where $\omega = \sqrt{\frac{k}{m}}$.

Note: Operators in Heisenberg picture and Schrödinger picture coincides at time t=0. This implies $\hat{x}_H(0) = \hat{x}(0)$ and $\hat{p}_H(0) = \hat{p}(0)$.

(c) Show that $\hat{x}_H(t)$ and $\hat{p}_H(t)$ satisfies the equation (4) and (5) respectively but this time using:

$$\hat{x}_H(t) = e^{\frac{i\hat{H}t}{\hbar}}\hat{x}e^{\frac{-i\hat{H}t}{\hbar}} \quad , \quad \hat{p}_H(t) = e^{\frac{i\hat{H}t}{\hbar}}\hat{p}e^{\frac{-i\hat{H}t}{\hbar}} \tag{6}$$

and the Hadamard Lemma

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \frac{1}{1!}[\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ... \tag{7}$$

2 Playing with an Operator.

Consider the operator

$$\hat{O} = |\phi\rangle\langle\psi| \ ,$$

where $|\phi\rangle$ and $|\psi\rangle$ are two vectors of the state space.

(a) Give the condition for the operator $\hat{O}$ to be Hermitian.

(b) Calculate $\hat{O}^2$. State the condition for which $\hat{O}$ can be a valid projection operator?

(d) Show that $\hat{O}$ can always be written in the form of $\hat{O} = \lambda P_1 P_2$, where λ is a constant to be determined and P_1 and P_2 are projection operators corresponding to the vectors $|\phi\rangle$ and $|\psi\rangle$ respectively.

3 Bound & Scattering states with Dirac Delta Potential

Consider a particle described by the Hamiltonian:

$$\mathcal{H} = \frac{\hat{p}^2}{2m} - V_0\delta(x) \tag{8}$$

(a) Let E be the energy of the particle. Write down the condition on E for the bound state and the scattering state solution of the Schrödinger equation.

(b) Solve the Schrödinger equation for the bound state and find the energy eigenvalue and eigenfunction. How many bound states exist?

(c) Solve the Schrödinger equation for the scattering state and find the reflection and the transmission coefficient.