Tutorial 01

In a generalised coordinate system $\{\sigma, \tau\}$ the Lagrangian is given by

$$\mathcal{L} = \frac{m}{2}\left(\frac{1}{\sigma^2+\tau^2}\right)\left[\dot{\sigma}^2+\dot{\tau}^2\right]$$

Find the Lagrange equation of motion.

$$\mathcal{L} = \frac{m}{2}\,\frac{1}{\sigma^2+\tau^2}\left[\dot{\sigma}^2+\dot{\tau}^2\right]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\sigma}} = \frac{m\dot{\sigma}}{\sigma^2+\tau^2} \qquad \frac{\partial \mathcal{L}}{\partial \sigma} = -\frac{m\sigma}{(\sigma^2+\tau^2)^2}\left[\dot{\sigma}^2+\dot{\tau}^2\right]$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\sigma}}\right) = \frac{m}{\sigma^2+\tau^2}\ddot{\sigma} - \frac{m\dot{\sigma}}{(\sigma^2+\tau^2)^2}\left[2\sigma\dot{\sigma}+2\tau\dot{\tau}\right]$$

The equation of motion

$$\frac{m\ddot{\sigma}}{(\sigma^2+\tau^2)} - \frac{2m\sigma\dot{\sigma}^2 - 2m\tau\dot{\tau}\dot{\sigma}}{(\sigma^2+\tau^2)^2} + \frac{m\sigma}{(\sigma^2+\tau^2)^2}\left[\dot{\sigma}^2+\dot{\tau}^2\right] = 0$$

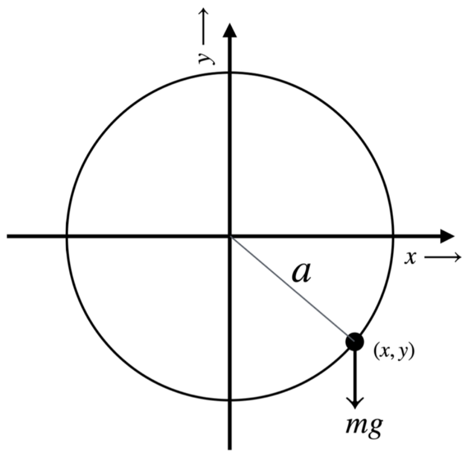
$$m\ddot{\sigma} - \frac{m\sigma\dot{\sigma}^2 + m\sigma\dot{\tau}^2 - 2m\tau\dot{\tau}\dot{\sigma}}{\sigma^2+\tau^2} = 0$$

$$m(\sigma^2+\tau^2)\ddot{\sigma} + m\sigma(\dot{\tau}^2-\dot{\sigma}^2) - 2m\tau\dot{\tau}\dot{\sigma} = 0$$

Similarly

$$m(\sigma^2+\tau^2)\ddot{\tau} + m\tau(\dot{\sigma}^2-\dot{\tau}^2) - 2m\sigma\dot{\tau}\dot{\sigma} = 0$$

Q - 2(20 Marks)



A particle of mass m is confined to move along a vertically oriented circle under the influence of gravity. Find a generalised coordinate for this problem and write the Lagrangian for it.

On an inertial frame y axis along vertical direction we have

$$K.E = T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

constraints $\longrightarrow$ $x^2+y^2=a^2$

$$P.E = V = mg(a+y)$$

Now use a transformation

$$x = a\cos\theta \qquad y = a\sin\theta$$

Satisfies constraint $x^2+y^2=a^2$

in the new coordinate system only coordinate is θ and a is constant

$$T = \frac{1}{2} m \left(a^2 \dot{\theta}^2\right)$$

$$V = mg\left(a + a\sin\theta\right)$$

$$= mga\left(1 + \sin\theta\right)$$

$$L = \frac{1}{2} m a^2 \dot{\theta}^2 - mga\left(1 + \sin\theta\right) \quad \text{or}$$

or

$$L = \frac{1}{2} m a^2 \dot{\theta}^2 - mga\sin\theta$$

Consider a free particle in two-dimensional inertial frame,

(1) Write the equation of motion in a plane polar coordinates.

(2*) Show that the particle moves along a straight line.

we can start with Lagrangian on plane polar coordinate

$$\mathcal{L} = \frac{1}{2} m \left[\dot{r}^2 + r^3 \dot{\theta}^2 \right]$$

for free particle

$$\frac{\partial \mathcal{L}}{\partial r} = m r \dot{\theta}^2 \qquad \frac{\partial \mathcal{L}}{\partial \dot{r}} = m \dot{r}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = m\ddot{r} - m r \dot{\theta}^2 = 0$$

$$\boxed{\ddot{r} - r\dot{\theta}^2 = 0} \text{ — } ①$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \qquad m\, 2 r \dot{r} \dot{\theta} + m r^2 \ddot{\theta} = 0$$

$$\boxed{\ddot{\theta} r + 2 \dot{r} \dot{\theta} = 0} \text{ — } ②$$

① & ② are EOM for free particle in planepolar coordinate ?

② The st. line is given by the eqn

$$y = m'x + c$$

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$ $\quad$ m' is slope

$$r\sin\theta = m' r\cos\theta + c$$

diff w.r.t once

$$\dot{r}\sin\theta + r\cos\theta\,\dot{\theta} = m'\dot{r}\cos\theta - m' r\sin\theta\,\dot{\theta}$$

diff. once again w.r.t and we get

$$\ddot{r}\sin\theta + \dot{r}\cos\theta\,\dot{\theta} + \dot{r}\cos\theta\,\dot{\theta} - r\sin\theta\,\dot{\theta}^2$$

$$+ r\cos\theta\,\ddot{\theta} = m'\ddot{r}\cos\theta - m'\dot{r}\sin\theta\,\dot{\theta}$$

$$- m\dot{r}\sin\theta\,\dot{\theta} - m r\cos\theta\,\dot{\theta}^2 - m r\sin\theta\,\ddot{\theta}$$

Collect the coefficient of sin and cos we get

$$\ddot{r} - r\dot{\theta}^2 + 2m'\dot{r}\dot{\theta} + m'r\ddot{\theta} = 0$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} - m'\ddot{r} + m'r\dot{\theta}^2 = 0$$

This can be shown using eq ① and eq ②.

Hence $y = mx + c$ is consistent with EOM given in ① & ②