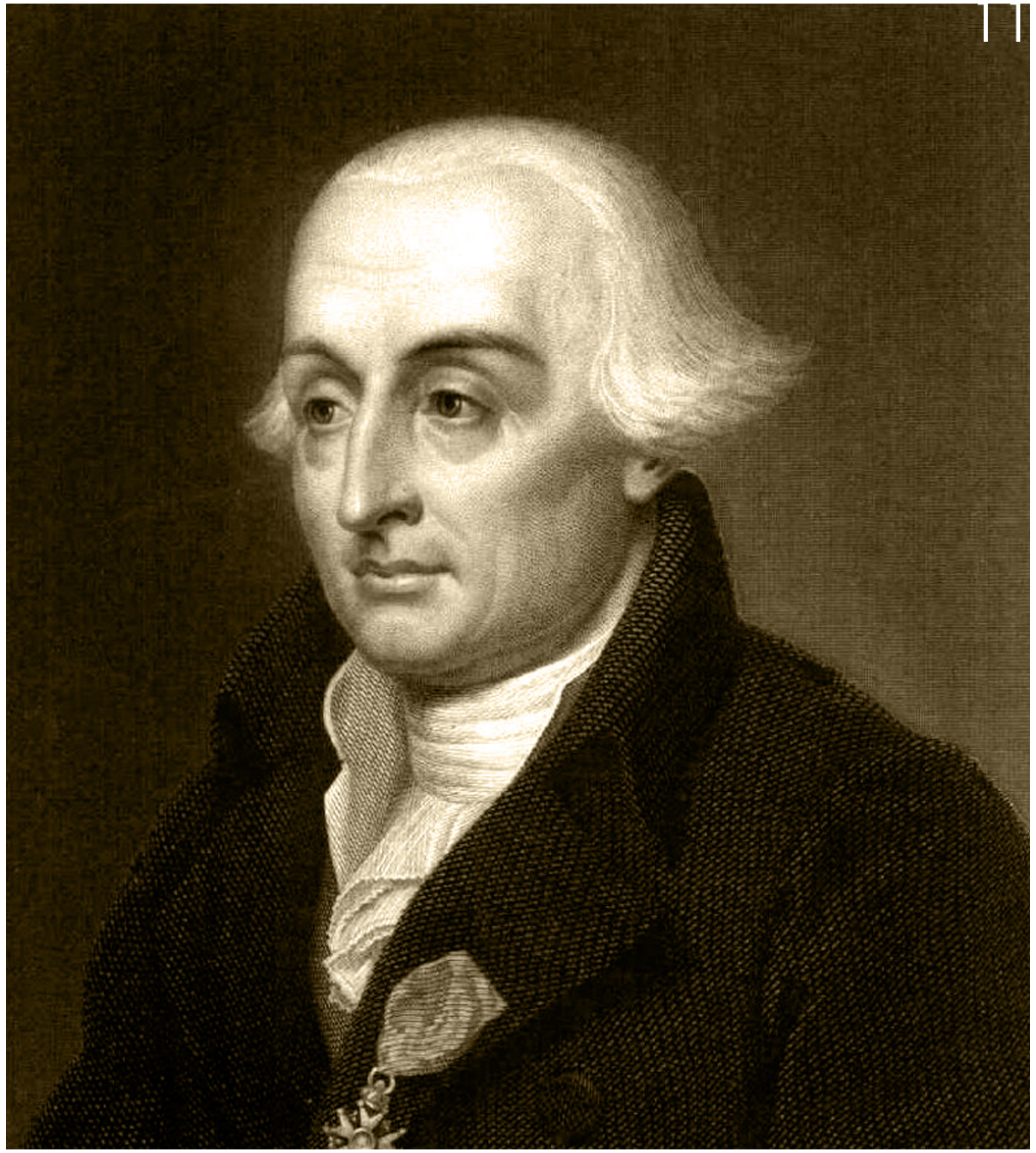

PH3101 Classical Mechanics @2025

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Lecture 07

Lagrange Equation of Motion

Newton's Equation of motion

In the last class we see that, the generalised coordinate systems are best suited for applying laws of motion, because it is free from the constraints.

These are the detailed formal steps

First we need the relation beween the coordinate system

$\vec{r}^i = \vec{r}^i(q_j)$ and $q_k = g_k(\vec{r}^i)$. $i = 1, 2, \cdots, N$ and $k = 1, 2, \cdots, n$

In addition we have M holonomic constraint $f_\alpha\left(\vec{r}^i\right) = 0$ for $\alpha = 1, 2, \cdots, M$

We also need relation between the basis vectors $\left\{\hat{e}_i\right\} \leftrightarrow \left\{\hat{q}_j\right\}$

Using this, we can relate the components of froces and momentumin the generalised coordinate

$$\vec{F} = \sum_{i=1}^{N} F_i \vec{e}_i = \sum_{kj=1}^{n} Q_k \hat{q}_k$$

F_i are components of forces

Q_k are components of forces in generalised coordinate

Newton's Equation of motion

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This step is not difficult, Next we need to transform momentum, and differentiate it substitute in NLOM.

$\vec{p} = \sum_{kj=1}^{n} p_k \hat{q}_k$ for momentum, we need $\dfrac{d\vec{p}}{dt}$

if thee basis vectors $\{\hat{q}_j\}$ are not constant, *i.e.* $\dfrac{d\hat{q}_j}{dt} \neq 0$, we have seen this results in the many terms and often difficult to deal with.

Work done

We have seen earlier, the work done $W = \overrightarrow{F} \cdot \overrightarrow{dr}$ and energy can carry information about the NLOM

Work done is a scalar, so it can be easily transformed from one coordinate system to another while retaining information about the EOM, which can then be reconstructed in the new (generalized) coordinate system. We follow this approach.

When a forcce $\overrightarrow{F}$ acting on a system with momentu

$$\overrightarrow{F} = \frac{d\vec{p}}{dt}$$

External force

Reaction offered by the physical system against force.

As per Newton, these should balance—that is, the third law of motion. The total external force and the reaction/restoring force offered by the system are in equilibrium.

Principle of Virtual Work

Let's start on an inertial frame $\vec{F} = \frac{d\vec{p}}{dt}$ let $\vec{dr}$ be a small displacement.

taking dot product on both side we, get, $\vec{F}_i \cdot \vec{dr}_i = \frac{d\vec{p}_i}{dt} \cdot \vec{dr}_i$ for N particles

System is under dynamical equilibirum, between the applied force and reaction force offered by the system. Combined with the work done, we get the condition

$$\sum_{i=1}^{N} \left(\vec{F}_i - \frac{d\vec{p}_i}{dt} \right) \cdot \vec{dr}_i = 0$$

D'Alembert's Principle: D'Alembert realised that, it is not need to have real displacement , it sufficient to use virtual displacement and virtual work results in the priciple of virtual work

D'Alembert's Principle

For a dynamical system subject to constraints, the difference between the applied forces and the inertial forces on each particle, projected onto any virtual displacement consistent with the constraints, is zero:

$$\sum_{i=1}^{N} \left(\overrightarrow{F}_i - \frac{\vec{p}_i}{dt} \right) \cdot \delta \vec{r}_i = 0$$

where $\overrightarrow{F}_i$ is the total applied force on the i-th particle, $\vec{p}_i$ its momentum , and $\delta \vec{r}_i$ is any virtual displacement allowed by the constraints.

We use this priciple for transforming EOM from inertial frame to generalised coordinate

$$\left(\overrightarrow{F}_i - \frac{d\vec{p}_i}{dt} \right) \cdot \overrightarrow{\delta r_i} = 0 \text{ and}$$

$$\left(\overrightarrow{Q} - \frac{d\vec{p}}{dt} \right) \cdot \overrightarrow{\delta q} = 0$$

Here, Coefficient of δq represent the EOM

Why Virtual Work ?

Cartesian Coordinate

The realtion with generalised coordinates

$$\vec{r}^i = \vec{r}^i(q_1, q_2, \cdots, q_n, t)$$

holonomic constraint $f_\alpha\left(\vec{r}^i, t\right) = 0$

$$\alpha = 1, 2, \cdots, M$$

work done on total system while displacement $\overrightarrow{dr_i}$

$$W = \sum_{i=1}^{N} \overrightarrow{F}_i \cdot \overrightarrow{dr_i} = \sum_{i=1}^{3N} F_i \, dx_i$$

$\overrightarrow{dr_i}$ in terms of dq_k as follows

$$\overrightarrow{dr_i} = \sum_{k=1}^{n} \frac{\partial r^i}{\partial q_k} dq_k + \frac{\partial r^i}{\partial t} dt$$

Generalised Coordinate

The realtion with Cartesian coordinates

$$q_k = g_k(\vec{r}^1, \vec{r}^2, \cdots, \vec{r}^N, t)$$

No constraints!

work done on total system in the generalised coordinate system can be expressed in displacement $\overrightarrow{dq}_k$ as

$$W = \sum_{k=1}^{n} Q_k \, dq_k$$

$\overrightarrow{dq}_k$ in terms of dq_i as follows

$$dq_k = \sum_{i=1}^{N} \frac{\partial q^k}{\partial r_i} \overrightarrow{dr_i} + \frac{\partial q^k}{\partial t} dt$$

Why Virtual Work ?

work done on total system while displacement $\overrightarrow{dr_i}$

$$W = \sum_{i=1}^{N} \overrightarrow{F}_i \cdot \overrightarrow{dr_i} = \sum_{i=1}^{3N} F_i \, dx_i$$

$\overrightarrow{dr_i}$ in terms of dq_k as follows

$$\overrightarrow{dr_i} = \sum_{k=1}^{n} \frac{\partial r^i}{\partial q_k} dq_k + \frac{\partial r^i}{\partial t} dt$$

And the virtual displacement $\overrightarrow{\delta r_i}$ is given by

$$\overrightarrow{\delta r_i} = \sum_{k=1}^{n} \frac{\partial r^i}{\partial q_k} \delta q_k$$

work done on total system while displacement $\overrightarrow{dq}_k$

$$W = \sum_{k=1}^{n} Q_k \, dq_k$$

$\overrightarrow{dq}_k$ in terms of dq_i as follows

$$dq_k = \sum_{i=1}^{N} \frac{\partial q^k}{\partial r_i} \overrightarrow{dr_i} + \frac{\partial q^k}{\partial t} dt$$

And the virtual displacement δq_i is given by

$$\delta q_k = \sum_{i=1}^{N} \frac{\partial q^k}{\partial r_i} \overrightarrow{\delta r_i}$$

Time is frozen hence no differentiation with t

Why Virtual Work ?

$$W = \sum_{i=1}^{N} \vec{F}_i \cdot \vec{dr}_i = \sum_{i=1}^{3N} F_i \, dx_i$$

$\vec{dr}_i$ in terms of dq_k as follows

$$\vec{dr}_i = \sum_{k=1}^{n} \frac{\partial r^i}{\partial q_k} dq_k + \frac{\partial r^i}{\partial t} dt$$

From this we get

$$\vec{F} \cdot \vec{dr}_i = \vec{F} \cdot \left\{ \sum_{k=1}^{n} \frac{\partial r^i}{\partial q_k} dq_k + \underline{\underline{\frac{\partial r^i}{\partial t} dt}} \right\}$$

$$W = \sum_{k=1}^{n} Q_k \, dq_k$$

$\vec{dq}_k$ in terms of dq_i as follows

$$dq_k = \sum_{i=1}^{N} \frac{\partial q^k}{\partial r_i} \vec{dr}_i + \frac{\partial q^k}{\partial t} dt$$

$$Q_k dq_k = Q_k \left\{ \sum_{i=1}^{N} \frac{\partial q^k}{\partial r_i} \vec{dr}_i + \underline{\underline{\frac{\partial q^k}{\partial t} dt}} \right\}$$

there is now. real way to compare the last term and hence may be left all together

Virtual work give's more freedom to work with only one of the virtual displacement is real

Lagrange Equation of Motion

Ref: Goldstein, Herbert; Poole, Charles; Safko, John; published by Pearson Education, Inc.,

We start with
$$\sum_{i=1}^{N}\left(\vec{F}_i - \frac{d\vec{p}_i}{dt}\right)\cdot\delta\vec{r}_i = 0$$

We have already seen,
$$\vec{\delta r_i} = \sum_{k=1}^{n}\frac{\partial r^i}{\partial q_k}\delta q_k$$

$$\sum_i \vec{F}_i\cdot\vec{\delta r_i} = \sum_{i,j}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = \sum_j\left(\sum_i\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\right)\delta q_j$$

$$\boxed{\sum_i \vec{F}_i\cdot\vec{\delta r_i} = \sum_j Q_j\,\delta q_j}$$

Where $Q_j = \sum_i \vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}$ is called compoment of generalised force.

It is just transformation of components of vector under the coordinate transformation

Lagrange Equation of Motion

The remaining term is $\sum_{i=1} \frac{d\vec{p}_i}{dt} \cdot \delta\vec{r}_i = \sum_{i=1} m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i$

For a particle $\vec{p}_i = m_i \dot{\vec{r}}_i$

Substituting for $\overrightarrow{\delta r_i} = \sum_{k=1}^{n} \frac{\partial r^i}{\partial q_k} \delta q_k$ we get,

Let's check this term

$$\sum_{i=1} \frac{d\vec{p}_i}{dt} \cdot \delta\vec{r}_i = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_j \left[\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right] \delta q_j$$

The rest is a series of algebraic steps to bring the equation into a simpler form.

Lagrange Equation of Motion

term

$$\sum_{i=1} \frac{d\vec{p}_i}{dt} \cdot \delta \vec{r}_i = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_j \left[\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right] \delta q_j$$

The rest is a series of algebraic steps to bring the equation into a simpler form.

Let's start with the coefficients of δq_j

$$\left[\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right] = \left[\sum_i m_i \frac{d\dot{\vec{r}}_i}{dt} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right]$$

expand $\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right)$ and adjust the terms we should get

$$= \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right]$$

$$\left[\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right] = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right]$$

with $v_i = \dfrac{d\vec{r}_i}{dt} = \dot{\vec{r}}_i$, we get,

$$= \sum_i \left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i v_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right]$$

2

Expanding second term, 2 , we get

$$= \sum_i \left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i v_i \left(\sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}\right)\right]$$

Note this is $\dot{r}_i = v_i$

$$\left[\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right] = \sum_i \left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i v_i \frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial t}\right)\right]$$

We need more simplification, let's look at $v_i = \frac{d\vec{r}_i}{dt} = \dot{\vec{r}}_i$ under transformation to new coodinate

$$v_i = \frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}$$

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left[\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right] = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\delta_{jk} \implies \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$

Do not deprnd on $\dot{q}_j$

Do not deprnd on $\dot{q}_j$

Going back to original equation,

$$\left[\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right] = \sum_i \left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i v_i \frac{\partial v_i}{\partial q_j}\right]$$

$$\left[\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right] = \sum_i \left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i v_i \frac{\partial v_i}{\partial q_j}\right]$$

$$= \sum_i \left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right) - m_i v_i \frac{\partial v_i}{\partial q_j}\right]$$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right] - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right)$$

$$\left[\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right] = \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j}\right] - \frac{\partial T}{\partial q_j}$$

$$\sum_{i=1}^{N} \frac{\vec{p}_i}{dt} \cdot \delta \vec{r}_i \quad \boxed{= \sum_{j=1} \left(\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j}\right] - \frac{\partial T}{\partial q_j}\right) \delta q_j}$$

Finally we have

Lagrange Equation of Motion

Cartesian Coordinate

$$\sum_{i=1}^{N}\left(F_i - \frac{dp_i}{dt}\right)\cdot \delta r_i = 0$$

Generalised Coordinate

$$\sum_{i=j}\left(Q_j - \frac{dp_j}{dt}\right)\cdot \delta q_j = 0$$

$$\Downarrow$$

$$\sum_j \left\{ Q_j - \left(\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j}\right] - \frac{\partial T}{\partial q_j}\right)\right\}\delta q_j$$

Where $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$ is called compoment of generalised force.

The equation of motion in the new coodinate system is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

This is general form of equation of motion for a given force Q_j

Lagrange Equation of Motion

When the forces are derivable from a scalar potential function V; $F_i = -\nabla_i V$

Then the generalized forces can be written as

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i -\nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

We can also directly transform $V(r^i)$ to $V(q_j)$ under coordinate transformation then we have

$$Q_j = -\frac{\partial V}{\partial q_j}$$

We define $\mathscr{L} = T - V$, which is called the Lagrangian. Then the EOM of motion takes the form

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathscr{L}}{\partial q_j} = 0}$$