PH3101 Classical Mechanics @2025

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Lecture 05

Conservation of total energy

If $\vec{F} = -\vec{\nabla} V$, let's see how can we use it in real life, through NLOM

$$\vec{F} = -\frac{\kappa x}{(x^2+y^2)^{\frac{3}{2}}}\hat{x} - \frac{\kappa y}{(x^2+y^2)^{\frac{3}{2}}}\hat{y}$$

Here $V = \phi(x, y) = -\dfrac{\kappa}{r}$ with $r = \sqrt{x^2 + y^2}$

As we already know polar coordinates are most suitale for this problem

V remains same in polar coordinate $V = \phi(x, y) = -\dfrac{\kappa}{r}$ with $F_r = -\dfrac{k}{r^2}$ and $F_\theta = 0$

From few days back, we get

$$\boxed{m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{k}{r^2} \quad \text{and} \quad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0}$$

Conservation of total energy

$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0 \quad \Longrightarrow \quad mr^2\dot{\theta} = L_z \quad L_z \text{ is constant}$$

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{k}{r^2} \quad \Longrightarrow \quad m\left(\ddot{r} - \frac{L_z^2}{mr^3}\right) = -\frac{k}{r^2}$$

Multiply on $\dot{r}$ on both side, it turns out that it an exact derivative

$$m\dot{r}\left(\ddot{r} - \frac{L_z^2}{mr^3}\right) = -\frac{k}{r^2}\dot{r}$$

$$\frac{d}{dt}\left[\frac{1}{2}m\,\dot{r}^2 + \frac{1}{2}\frac{L_z^2}{mr^2} - \frac{k}{r}\right] = 0 \quad \text{or} \quad \boxed{\begin{array}{l} \frac{1}{2}m\,\dot{r}^2 + \frac{1}{2}\frac{L_z^2}{mr^2} - \frac{k}{r} = E \\ mr^2\dot{\theta} = L_z \end{array}}$$

Are new EOM

System of Particles

An object may be modeled as a system of mutually interacting particles. Our goal is to understand how this system evolves over time.

The forces through which the particles of a system interact with each other are called internal forces.

External force is one vector field on given inertial frame.

For simplicity, let us consider a system of N particles that do not interact with each other, i.e., the internal forces are zero. In this case, each particle in the system, with mass m_i and position $\vec{r}_i$, evolves according to:

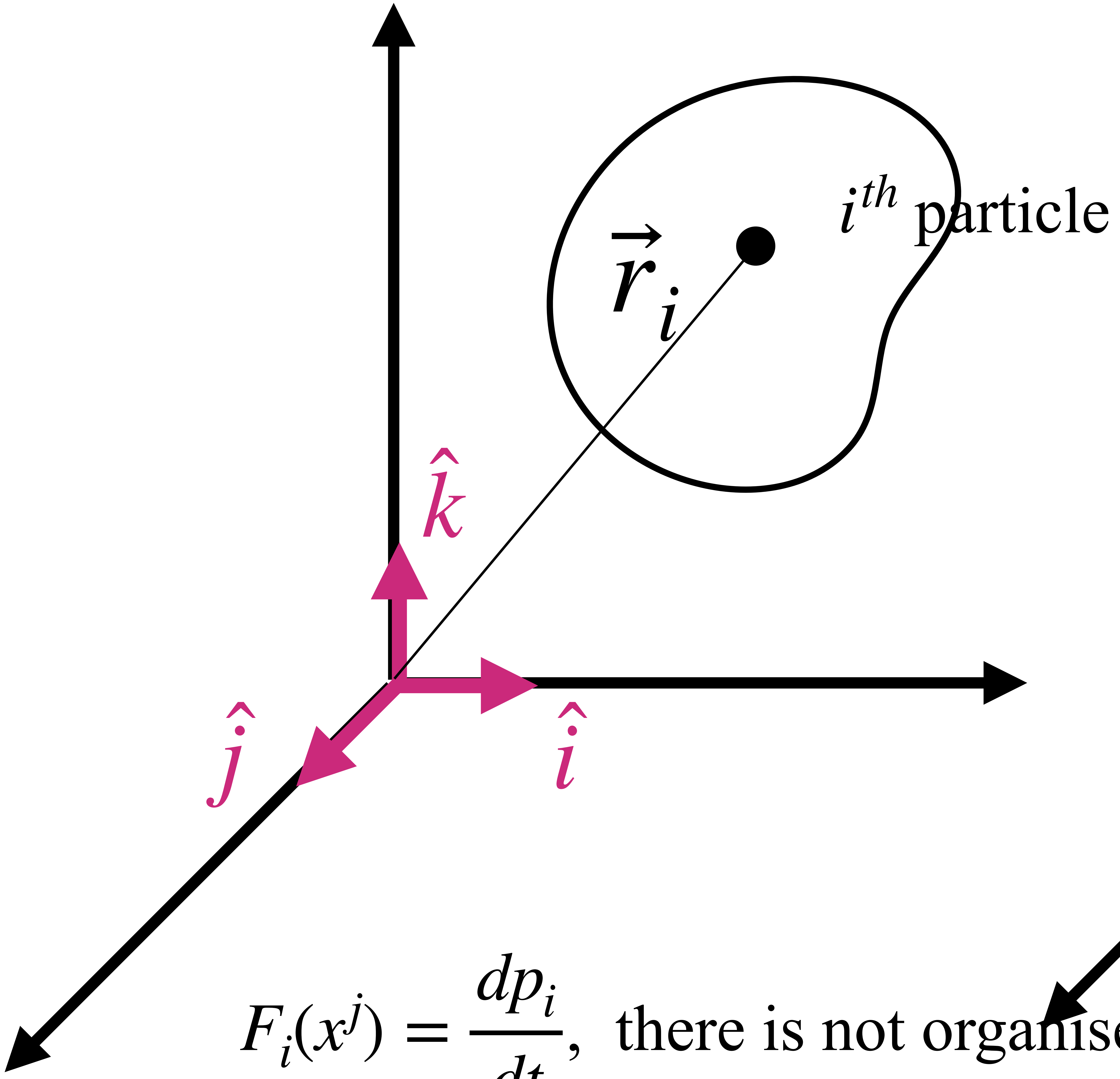
$$\overrightarrow{F^e}_i(\vec{r}_i) = \frac{d\vec{p}_i}{dt} \text{ for } i = 1, 2, \cdots N$$

In a cartesian coordinate system where we can write the equation of motion as

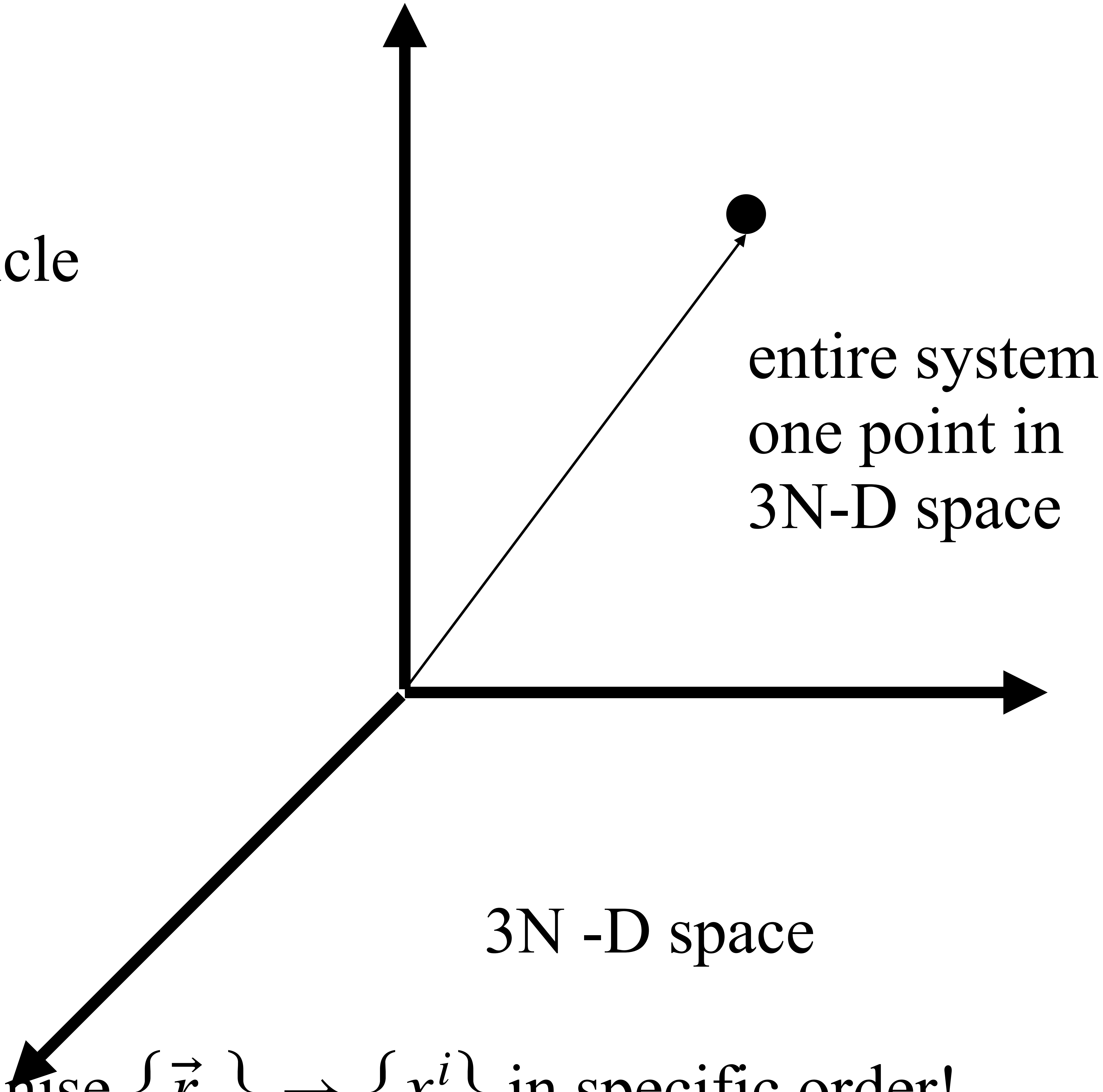
$$F_i(x^j) = \frac{dp_i}{dt} \qquad i, j = 1, 2, \cdots 3N \text{ with } \left\{x_{k+1}, x_{k+2}, x_{k+3}\right\} = \left\{x_k, y_k, z_k\right\}$$

$$\text{here } k = 1, 2, \cdots N$$

Newtonian space i.e. 3-D Eulcidian with N particle



Configuration Space, single entity with 3N dimensional space



$F_i(x^j) = \dfrac{dp_i}{dt}$, there is not organise $\{\vec{r}_k\} \to \{x^i\}$ in specific order!

Constraints

What are Constraints?

Constraints are any restrictions imposed on the motion of a particle or system. Without constraints, a particle in 3D space can move freely in all directions—described by 3 coordinates (x, y, z). Constraints limit this freedom.

$$\text{Constraints} \quad \Longrightarrow \quad \text{Some restrictions} \quad \Longrightarrow \quad \text{Some force, stopping free motion}$$

In the case of system of paerticles, where there some mutual interactions or internal forces are involved, the system as whole might not be able to move freely resulting in contraints

A paticle restricted to move in horizontal plane in the earth's gravity!

A rigid body where, the distance between aby two particle are not changing.

A bob of a pendulum

Constraints continued . . .

The presence of constraints implies a failure of Newton's first law of motion.

In the physical space, the constrained object might not move in a straight line at constant velocity, even if no other external forces are present, because the constraint force changes its motion

Presence of constraitnts means, there is force involved, which one might or might not know in advance, but know only the resulting geometric strictions.

If the forces due to constraints are unknown and not included in the analysis, the motion may appear to violate Newton's first law. In reality, the law still holds, but the missing constraint forces are responsible for the observed acceleration.

In practice, Newton's first law appears to be "violated," because the object's velocity changes even though we have not identified any external force in our equations.

In simple terms, the presence of constraints poses additional challenges in applying Newton's laws.

Types of Constrains

Holonomic Constraints

Definition: it can be expressed as equations involving only the coordinates (and possibly time):

$$f\left(x^i, t\right) = 0$$

Effect: Reduce the number of degrees of freedom. Using the constraint equation one may reduce the dimenion of the space or degree of freedom

Example: A bead on a fixed circular wire:

$$x^2 + y^2 - R^2 = 0$$

We will be mostly dealing with Holonomic Constraints! Because be can get rid of some of the mutual dependent coordinates and reduce the degree of freedom!

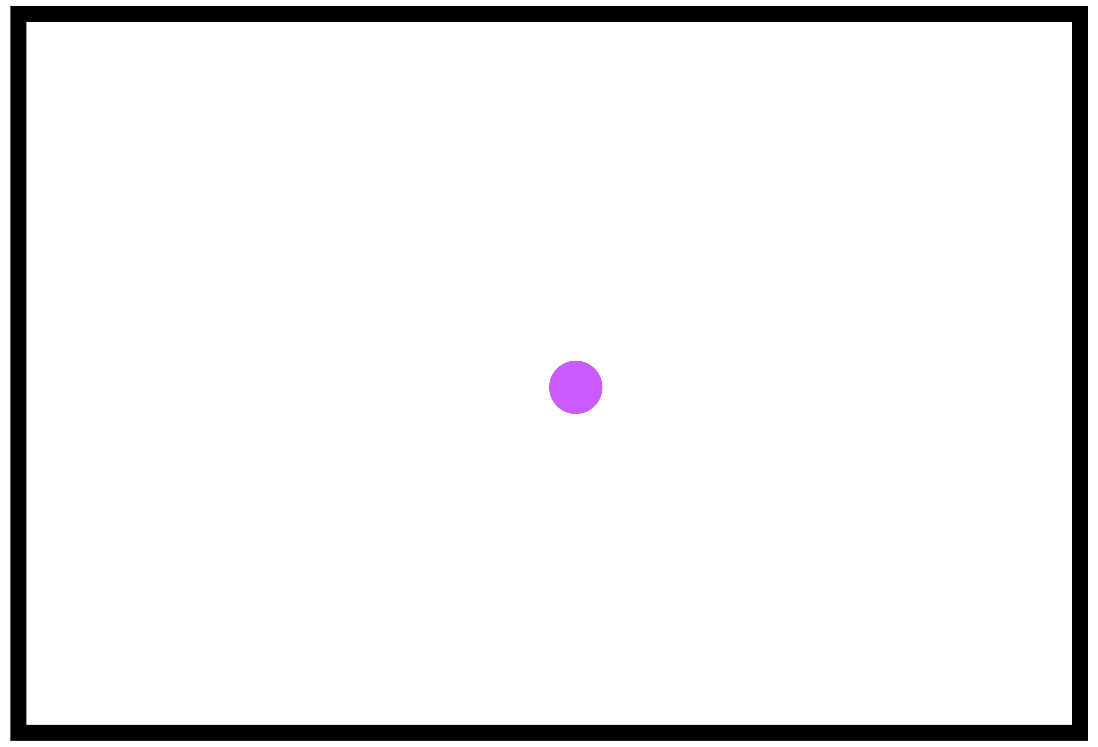
Non-Holonomic Constraints

Definition: Cannot be expressed solely in terms of coordinates; usually involve velocities or inequalities:

$$g(x^i, \dot{x}^i, t) = 0$$

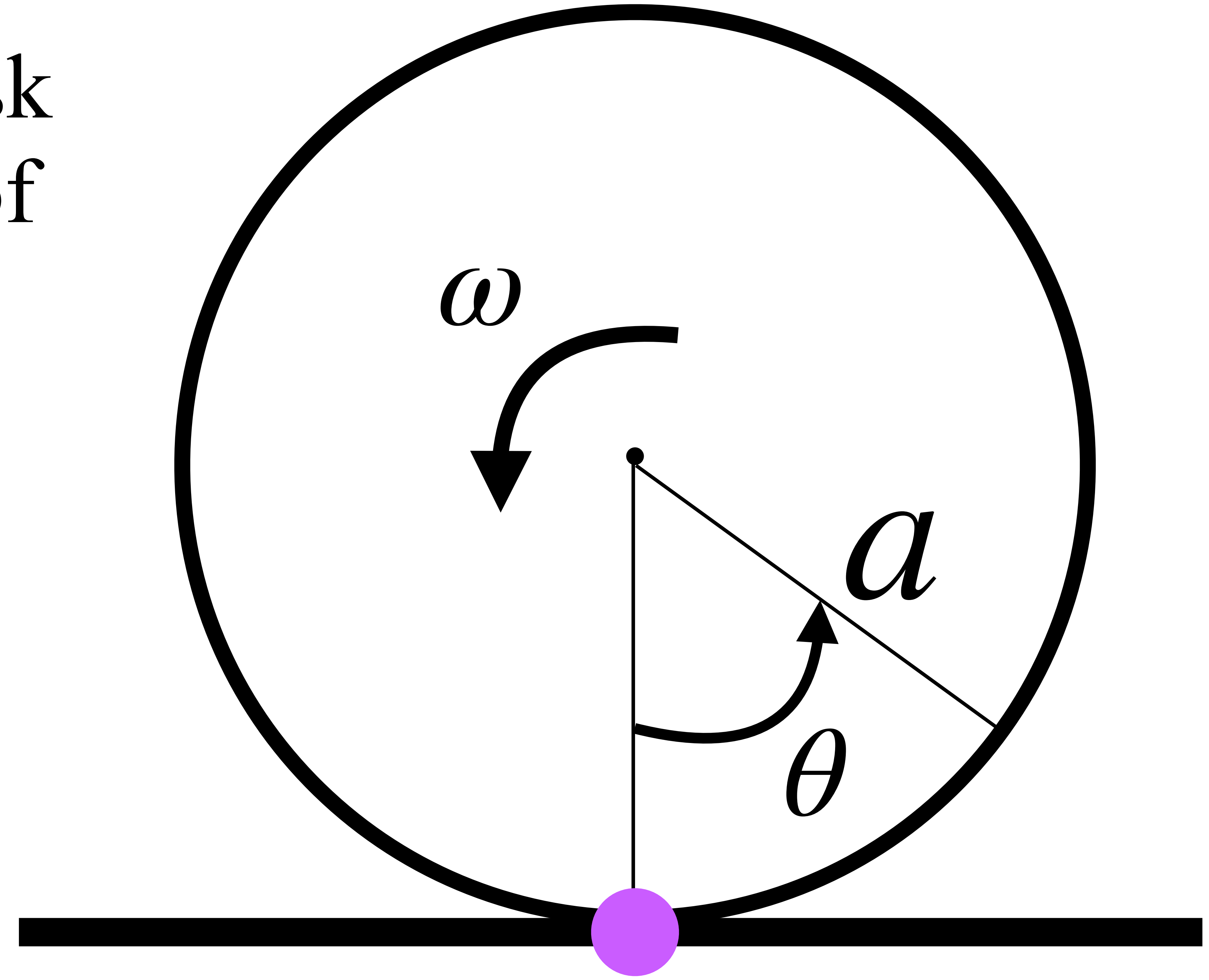
Difficulty: Cannot be reduced to a purely positional equation.

Particle in a box,



Example: Rolling without slipping: Here, we are looking

a point of contactbitween the disk and ground where the Velocity of center of disk is $v = a\dot{\theta}$ a is the radius of ring/disk



The x is the position of ring then $\dot{x} = v \sin\theta$, is the form of contraint

or we can write a $dx - v \sin\theta \, dt = 0$