PH3101 Classical Mechanics @2025

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Lecture 06

Generalise coordinates

Example

Let's consider a system with a particle confined to move in a circle in two-dimensional space.

In a two dimensional Cartesian coordinate system, the constraint equation can be written as $x^2 + y^2 = a$, where a is the radius of the circle. Let $\vec{F}$ be the force acting on the particle. The system can then be expressed using Newton's law, combined with the constraint equation, as follows.

$$\ddot{x} = \frac{1}{m} f(x, y) \; , \; \ddot{y} = \frac{1}{m} g(x, y) \text{ and } x^2 + y^2 = a. \text{ Where } \vec{F} = f(x, y)\,\hat{i} + g(x, y)\,\hat{j}.$$

Here, even if the two differential equation are separable, the constraints are coupled and equation of motion cannot be solved.

Constrains and Coordinate system

Since the equation of constraint takes a simpler form in the plane polar coordinate system, it might be a suitable choice for solving this problem. Let us reconsider the problem in the plane polar coordinate system.

In the plain polar coordinate system, we have

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = F_r(r,\theta)\,, \qquad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = F_\theta(r,\theta)$$

The constraint is the simpler form $r = a$ simple implication is $\Longrightarrow \dot{r} = \ddot{r} = 0$

Using this we can simplify the EOM to

$$-m\,a\,\dot{\theta}^2 = F_r(r = a, \theta)\,, \qquad \text{and} \qquad ma\,\ddot{\theta} = F_\theta(r = 0, \theta)$$

Differentiate first equation *w.r.t.* time, we get $-2ma\dot{\theta}\,\ddot{\theta} = \dfrac{\partial F_r}{\partial \theta}\,\dot{\theta},$

Simplifies to $-2ma\,\ddot{\theta} = \dfrac{\partial F_r}{\partial \theta}$ and $ma\,\ddot{\theta} = F_\theta$

Constraints and Coordinate system

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Finally we get one single equation $-ma\,\ddot{\theta} = \dfrac{\partial F_r}{\partial \theta} + F_\theta$

Here θ is only a coordinate, it can solved as system with one-degree of freedom.

There is a effective EOM, which need to be determined!

✓ *Suitable coordinate system can simplify the effect of holonomic constraint.*

✓ *With the new coordinate system it is possible to reduce the degree of freedom or number of equation of motion*

Generalised Coordinate System

One easy way to handle holonomic constraints is to perform a coordinate transformation and reduce the degrees of freedom of the system. We then need to write the transformed equations of motion(EOM).

It is important to understand that coordinate transformations are, in principle, performed in configuration space and not in real 3D space. This is because the constraints might affect certain components of the equations of motion or the degrees of freedom.

Example:

A bead of mass m slides without friction on a circular wire of radius a, lying in a vertical plane.

The bead's position is described by $\left(x, y, z\right)$ The holonomic constraint is

$x^2 + y^2 + z^2 = a^2$, and $z = 0$

These equations live in physical space but don't directly reduce the degrees of freedom — they're just conditions on the Cartesian coordinates.

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In configuration space

We note that the bead's motion can be described by a single generalized coordinate: θ

The coordinate transformation is: $x = a\cos\theta, y = a\sin\theta$ and $z = 0$.

- ✤ The constraint is automatically satisfied in this new coordinate, and the degrees of freedom reduce from 3 to 1.
- ✤ Equations of motion can now be written directly in terms of $\theta(t)$ without explicitly involving constraint forces.

Holonomic Constraints and Generalised Coordinates

Physical system: We have a system of N particles in 3D space.

The Position vectors are given by $\vec{r}_i\left(x_i, y_i, z_i,\right)$ for $i = 1, 2, \cdots, N$

Total number of Cartesian coordinates or the configuration space is $3N$

Holonomic Constraints We have m - holonomic constraints given by the equations,

$$f_j(x^1, x^2, \ldots, x^{3N}, t) = 0, \quad j = 1,2,\ldots, m$$

Where x^k where $k = 1,\ldots,3N$ are Cartesian coordinates of all particles.

Note:

- *Holonomic* means constraints can be expressed as equations relating coordinates and possibly time.
- Each independent constraint reduces the number of independent coordinates by **1**.

Holonomic Constraints and Generalised Coordinates

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Degrees of Freedom

The degrees of freedom are: $n = 3N - m$

n coordinates completely describe the system's configuration space.

Transformation to Generalized Coordinates

We introduce generalized coordinates: $\left(q_1, q_2, \cdots, q_n\right)$,

and the transformation between x_i and q_i are given by

$x^k = x^k(q_1, q_2, \ldots, q_n, t), \quad k = 1,\ldots,3N$

Here:

- q_i are independent variables.

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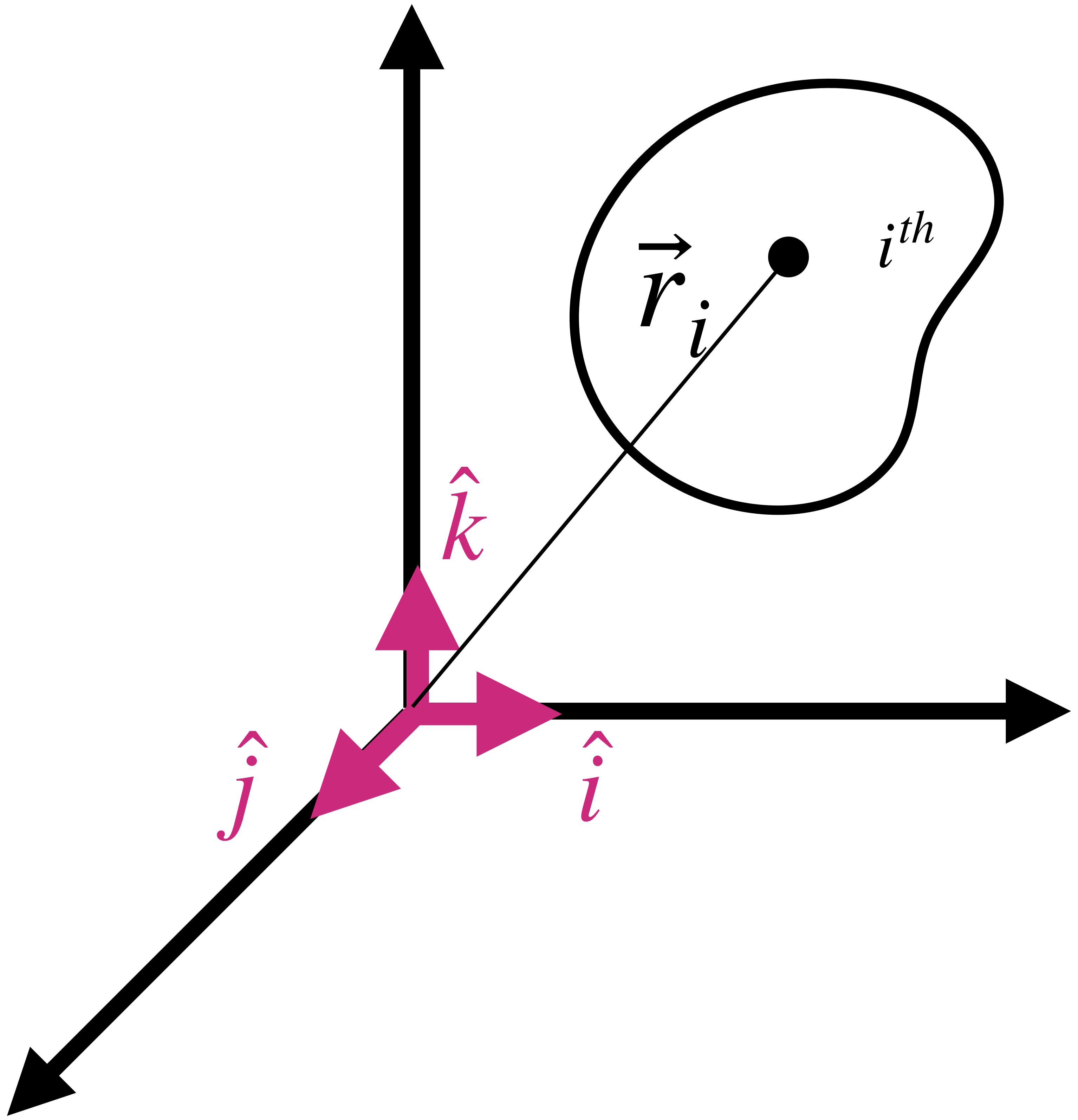
Here:

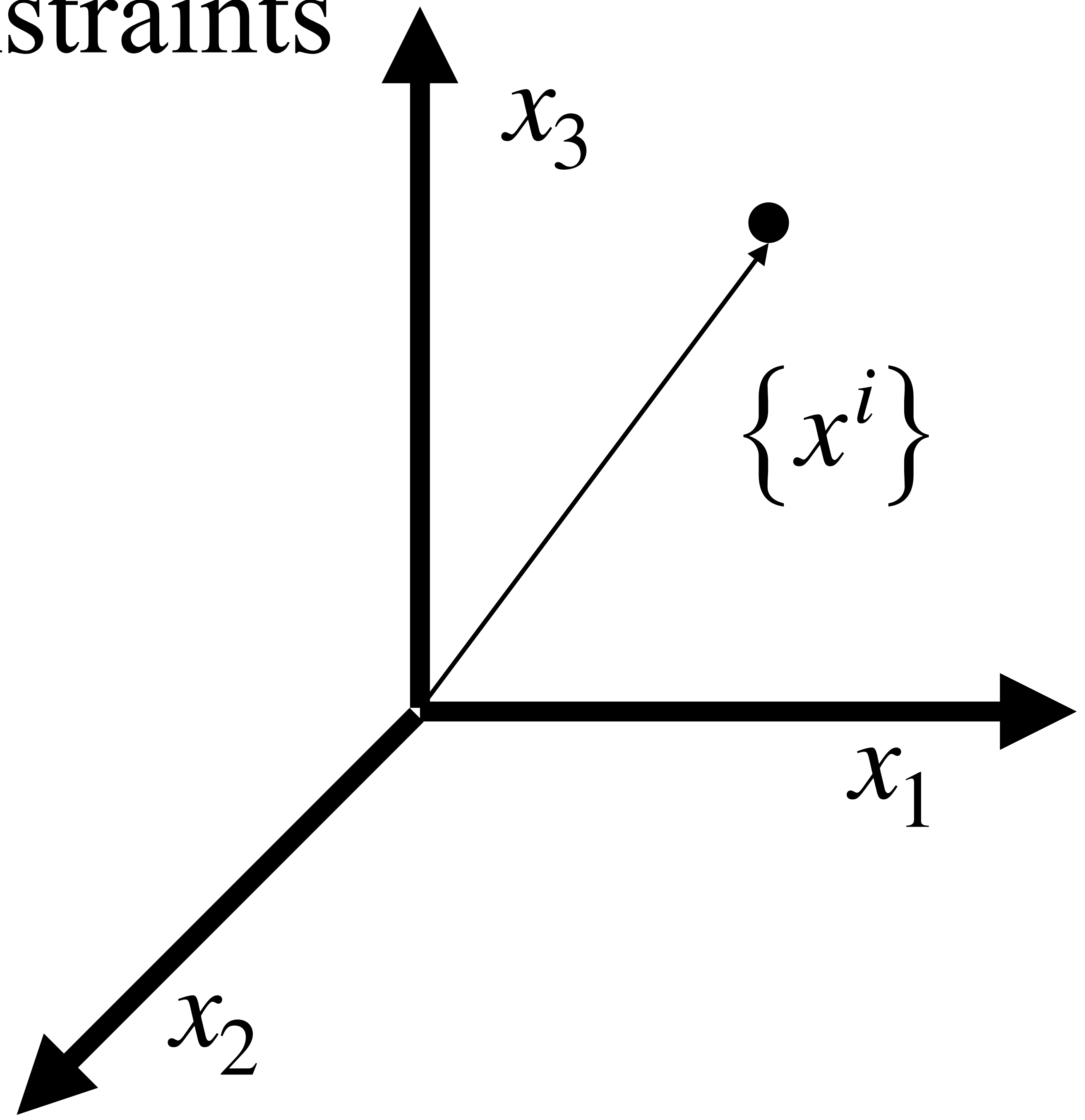
- q_i are independent variables.
- The functional form is chosen so that all constraints are automatically satisfied.

Generalised Coordinate System

Absolute 3-D space

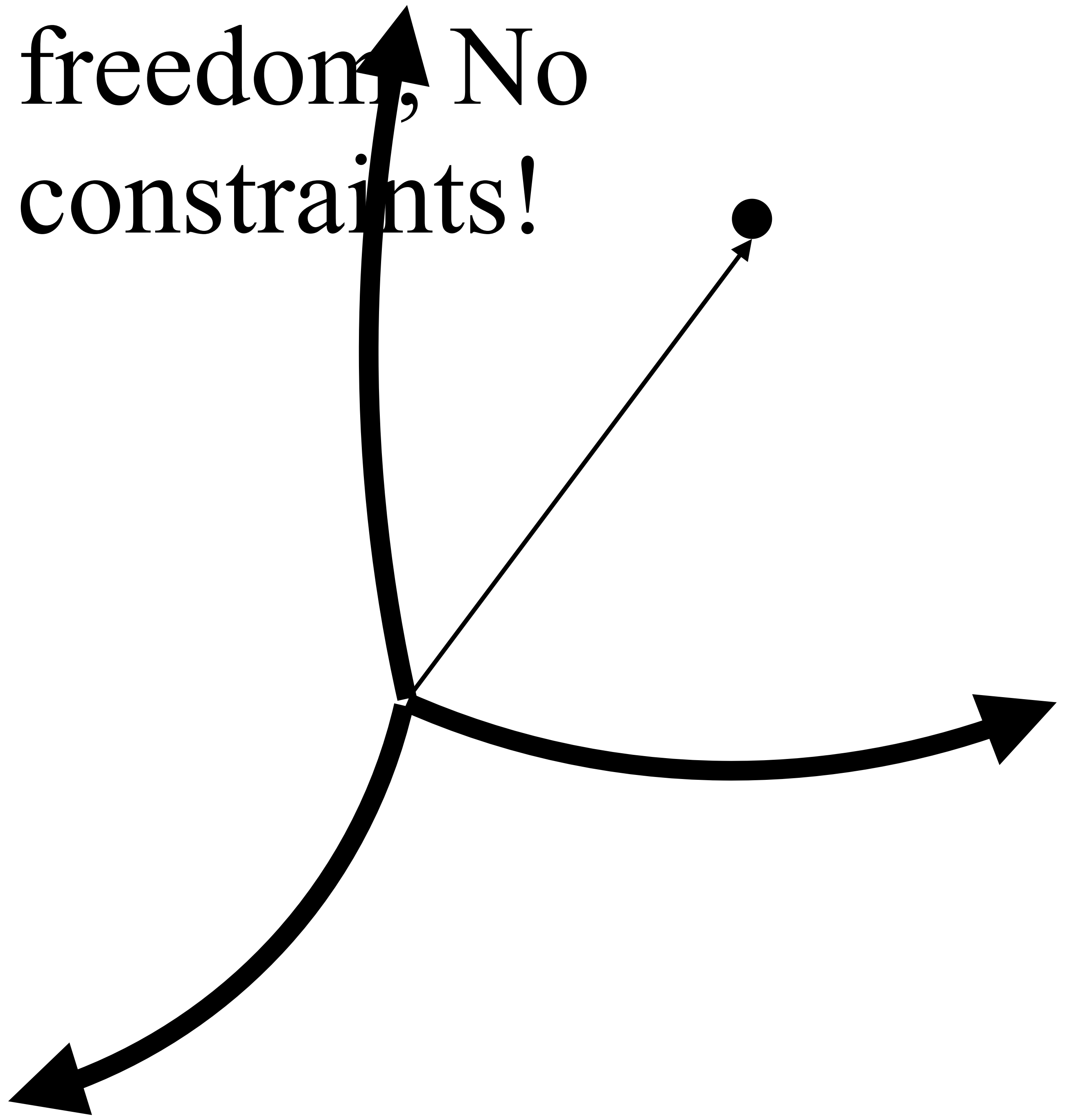


Configuration space 3N-D with m constraints



Generalise coordinate

n=3N-m degree of freedom, No constraints!



$$\underbrace{(x_1, y_1, z_1, \ldots, x_N, y_N, z_N)}_{\text{3N coordinates}} \xrightarrow{\text{Apply m constraints}} \underbrace{(q_1, q_2, \ldots, q_n)}_{\text{n = 3N - m degrees of freedom}}$$