PH3101 Classical Mechanics @2025

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Lecture 03

Newtonian Mechanics summary from last class

Inertial frame where Newton's law can be applied

The law of motion is given by $\frac{d\vec{P}}{dt} = \vec{F}$, where $\vec{F}$ is force and $\vec{P}$ is momentum

Force $\vec{F}$ is external influence. $\vec{P}$ physical state of the system.

Cooerdinate systems are essential for worrking out details

EOM for a particle can be reduced to:

$$\boxed{\ddot{x} = \frac{1}{m} f(x, y) \ \text{ and } \ddot{y} = \frac{1}{m} g(x, y)}$$

Two coupled differential equations.

Coupled differential equations are often difficult to solve analytically, and numerical solutions typically do not provide much insight into the underlying problem.

Newton's Second Law of Motion continued …

The role of a coordinate system

Let's look some special cases! $\ddot{x} = \dfrac{1}{m} f(x, y)$ and $\ddot{y} = \dfrac{1}{m} g(x, y)$

There are two possibilities, if $f(x, y) = 0$ and/or $g(x, y) = 0$. These are called problem with **CONSTANTS OF MOTIONS.**

Another interesting case

If $f(x, y) = f_x(x)$ and $g(x, y) = g_x(y)$ i.e. forces are separable in the coordinate variables! Then we have:

$\ddot{x} = \dfrac{1}{m} f_x(x)$ and $\ddot{y} = \dfrac{1}{m} g_y(y)$ This type of problems may be integrated!

The role of a coordinate system continued ...

Inboth the spacial cases coordinate play an important role.

One may change the coordinate system such that, in the new coordinates, the problem appears separable or reveals constants of motion.

A problem that is hard or unsolvable in one coordinate system can often be solved easily winl 😉 by switching to a more suitable coordinate system.

Coordinate system play an important role while solving problems

Role of Coordinate system and Reference Frame

In an inertial frame, we can introduce multiple coordinate systems, and the laws of physics remain unchanged.

This is called principle of Invariance.

- ✤ The equations of motion (EOM) look simpler in the Cartesian coordinate but need not be seprable or integrable.
- ✤ In an complex curvilinear coordinate the EOM themselves can be complex but may be integrable!
- ✤ If one understands all games of rules, one may even change the "frame" and make suitable changes to EOM and solve it.

This is called principle of General Covariance.

These are the freedoms available for solving/understanding problems/laws of physics.

Conservation of linear momentum

It is straightforward to show, using Newton's second law of motion, that if a particular component of force is zero, then the corresponding component of linear momentum is conserved.

$\vec{F} = F_y \hat{j} + F_z \hat{k}$ then $\dfrac{dP_x}{dt} = 0$ or $\quad$ Here the momentum $\vec{P} = P_x \hat{i} + P_y \hat{j} + P_z \hat{k}$

$P_x = constant.$

One can generalise this for a constant vector $\vec{\xi}$, i.e $\dfrac{d\vec{\xi}}{dt} = 0,$

if $\left(\vec{F} \cdot \vec{\xi}\right) = 0$ then $\dfrac{d}{dt}\left(\vec{P} \cdot \vec{\xi}\right) = 0$ or $\left(\vec{P} \cdot \vec{\xi}\right)$ is a constant of

$$\boxed{\text{If the projection of the force vanishes along a constant vector field, then the projection of linear momentum along that vector field is conserved.}}$$

⟶ see next page for derivation!

Extra Slides added later

Let $\vec{p}$ be the momentum of a particle under the influence of a force field $\overrightarrow{F}$. we have a constant vector $\vec{\xi}$ (*i.e.* $\frac{d\vec{\xi}}{dt} = 0$) then $\frac{d}{dt}\left(\vec{p} \cdot \vec{\xi}\right) = 0$, if $\overrightarrow{F} \cdot \vec{\xi} = 0$.

Which means that the projection of momentum along the vector $\vec{\xi}$ is a constant motion

Proof We start with the condition that $\overrightarrow{F} \cdot \vec{\xi} = 0$

Next, we use Newtons laws of motion *i.e.* $\overrightarrow{F} = \frac{d\vec{p}}{dt}$

Taking the dotproduct with the vector $\vec{\xi}$ both sides we get

$$0 = \overrightarrow{F} \cdot \vec{\xi} = \frac{d\vec{p}}{dt} \cdot \vec{\xi}$$

Since, $\frac{d\vec{\xi}}{dt} = 0$, we can rewrite the above equation has $\frac{d}{dt}\left(\vec{p} \cdot \vec{\xi}\right) = 0$

Hence, the proof

The role of a coordinate system continued ...

Ex: A 2-D Newtonian EOM is is given by
$$\ddot{x} = 0$$
$$\ddot{y} = y - ax$$

Then we have $\dot{x} = const = C \implies x = Ct + D$. Here C and D are constants

Substituting x in second equation we get $\ddot{y} - y = -aCt - D'$ which can be easily solved

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Ex: Force acting on a particle is given by $\ddot{x} = f(x, y) = \dfrac{GMx}{(x^2 + y^2)^{\frac{3}{2}}}$ and $\ddot{y} = g(x, y) = \dfrac{GMy}{(x^2 + y^2)^{\frac{3}{2}}}$. Solve for equation of motion. This is called central force problem in 2-D.

The Newton equation of motion in the Cartesian coordinate system are not separable. We need a more suitable coordinate system in which problem can be solved

Coordinate transformation! How to ?

When comes to Newtons's law of motion, we always start with inertial frame and Cartesian coordinate system, *i.e.* $\{x, y, z\}$, we have EOM as

$$m\ddot{x} = F_x(x, y, z)$$
$$m\ddot{y} = F_y(x, y, z)$$
$$m\ddot{z} = F_z(x, y, z)$$

Here force $\overrightarrow{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

First we need transformation rule from

$$\{x, y, z\} \leftrightarrow \{x', y', z'\}$$

We also need relationsship or transformation from basis vector $\{\hat{i}, \hat{j}, \hat{k}\}$ to new basis vector $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$, $\{\hat{i}, \hat{j}, \hat{k}\} \leftrightarrow \{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$

We express $\dot{x}, \dot{y}, \dot{z} \leftrightarrow \dot{x}', \dot{y}', \dot{z}'$ and x, y, z,

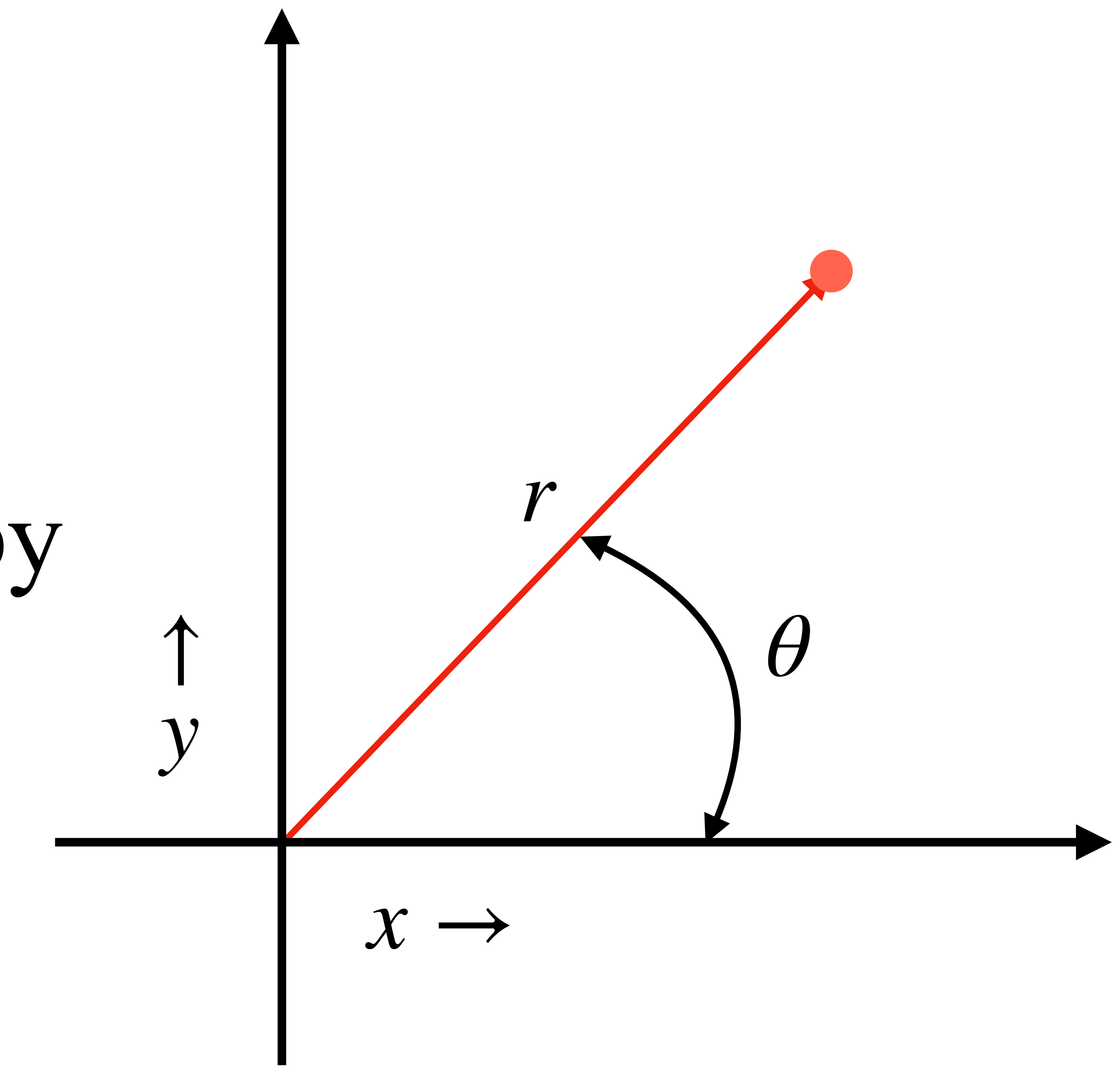
We express $\ddot{x}, \ddot{y}, \ddot{z} \leftrightarrow \ddot{x}', \ddot{y}', \ddot{z}', \dot{x}', \dot{y}', \dot{z}'$ and x, y, z,

then we can write EOM in new coordinate system

We need to express force $\overrightarrow{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ in new coordinate system

$$\overrightarrow{F} = F'_{x'}\hat{e}_x + F'_{y'}\hat{e}_y + F'_{z'}\hat{e}_x$$

Plane Polar Coordinate



The transformation is given by

$x = r\cos\theta$ and $y = r\sin\theta$

The inverse transformation is given by

$r = (x^2 + y^2)^{\frac{1}{2}}$ and $\theta = \tan^{-1}\dfrac{y}{x}$

We also need the relation between the basis vectors

$$\begin{Bmatrix} \hat{x} \\ \hat{y} \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \hat{r} \\ \hat{\theta} \end{Bmatrix} \qquad \begin{aligned} \hat{x} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \\ \hat{y} &= \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta} \end{aligned} \qquad \begin{aligned} \hat{r} &= \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \hat{\theta} &= -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \end{aligned}$$

Now we have to start with Newton's law in Cartesian coordinate, in full vector form!

$i.e.\ \vec{F} = \dfrac{d}{dt}\vec{P}$

$\vec{F}$ in Cartesian coordinate system is given by,

In polar coordinate system

$$\vec{F} = F_x\hat{x} + F_y\hat{y} \equiv F_r\hat{r} + F_\theta\hat{\theta}$$

Plane Polar Coordinate

In the polar coordinate

We start with

$$\vec{F} = F_x \hat{x} + F_y \hat{y} = F_r \hat{r} + F_\theta \hat{\theta}$$

$$\begin{aligned} \hat{x} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \\ \hat{y} &= \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta} \end{aligned}$$

$$\vec{F} = F_x \left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right] + F_y \left[\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right]$$

$$\vec{F} = \left[F_x \cos\theta + F_y \sin\theta\right] \hat{r} + \left[-F_x \sin\theta + F_y \cos\theta\right] \hat{\theta}$$

This means

$$\boxed{F_r = \left[F_x \cos\theta + F_y \sin\theta\right]}$$ and $$\boxed{F_\theta = \left[-F_x \sin\theta + F_y \cos\theta\right]}$$

are compoment of force in polar coordinate

This True for any vector transformed from cartersian coordinate to Polar

The compoments of vector $\overrightarrow{A}$, $\{A_x, A_y\}$ in cartersian system, when transformed to polar coordinate, we get $\overrightarrow{A}$, $\{A_r, A_\theta\}$, the relation is given by

$$\boxed{A_r = \left[A_x \cos\theta + A_y \sin\theta\right]}$$

and

$$\boxed{A_\theta = \left[-A_x \sin\theta + A_y \cos\theta\right]}$$

are compoment of force in polar coordinate

Now we apply to the velocity, the polar components of velocity in terms of thier cartesian counter part is give by,

$$V_r = \left[V_x \cos\theta + V_y \sin\theta\right] \qquad V_\theta = \left[-V_x \sin\theta + V_y \cos\theta\right]$$

In more complex term it is called tensor transformation rule!

Now we need rate of change of momentum

Should we do it again?

$$\frac{d}{dt}\vec{P} = m\frac{d}{dt}\left[v_x\hat{x}\right] + m\frac{d}{dt}\left[v_y\hat{y}\right]$$

$$= m\frac{d}{dt}\left[v_x\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)\right] + m\frac{d}{dt}\left[v_y\left(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right)\right]$$

$$= m\frac{d}{dt}\left[\left(v_x\cos\theta + v_y\sin\theta\right)\hat{r}\right] + m\frac{d}{dt}\left[\left(-v_x\sin\theta + v_y\cos\theta\right)\hat{\theta}\right]$$

Which we can write as, $\boxed{\frac{d}{dt}\vec{P} = m\frac{d}{dt}\left(v_r\hat{r} + v_\theta\hat{\theta}\right)}$

$$v_r = \left[v_x\cos\theta + v_y\sin\theta\right] \quad \text{and} \quad v_\theta = \left[-v_x\sin\theta + v_y\cos\theta\right]$$

here v_r and v_θ are compoment velocity components in polar coordinate.

Important property of basis vector in polar coordinate

We need to find rate of change of momentum $\frac{d}{dt}\vec{P} = m\frac{d}{dt}\left(v_r\hat{r} + v_\theta\hat{\theta}\right)$

We need rate of change of basis vector

In polar coordinate $\frac{d\hat{r}}{dt} \neq 0$ and $\frac{d\hat{\theta}}{dt} \neq 0$,

lets finndout the values

Here we start with $\frac{d\hat{r}}{dt} = \frac{d}{dt}\left[\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right] = -\sin\theta\,\dot{\theta}\,\hat{x} + \cos\theta\,\dot{\theta}\,\hat{y} \implies \boxed{\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}}$

Similarly, $\frac{d\hat{\theta}}{dt} = \frac{d}{dt}\left[-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\right] = -\cos\theta\,\dot{\theta}\,\hat{x} - \sin\theta\,\dot{\theta}\,\hat{y} \implies \boxed{\frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r}}$

Now let's find $\frac{d}{dt}\overrightarrow{P} = m\frac{d}{dt}\left(v_r\hat{r} + v_\theta\hat{\theta}\right)$

$$= m\left[\frac{dv_r}{dt}\hat{r} + v_r\frac{d\hat{r}}{dt} + \frac{dv_\theta}{dt}\hat{\theta} + v_\theta\frac{d\hat{\theta}}{dt}\right]$$

We use the relation

$$\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta} \qquad \frac{d\hat{\theta}}{dt} = -\,\dot{\theta}\,\hat{r}$$

$$= m\left[\frac{dv_r}{dt}\hat{r} + \frac{dv_\theta}{dt}\hat{\theta}\right] + m\left[v_r\dot{\theta}\,\hat{\theta} - v_\theta\dot{\theta}\,\hat{r}\right]$$

Contribution due to change in components of vecolcity

Contribution due to change of basis vectors.

$$\frac{d}{dt}\overrightarrow{P} = m\left[\frac{dv_r}{dt}\hat{r} + \frac{dv_\theta}{dt}\hat{\theta}\right] + m\left[v_r\dot{\theta}\,\hat{\theta} - v_\theta\dot{\theta}\,\hat{r}\right]$$

This is expression for derivative of a vector field when basis vectors are not constants

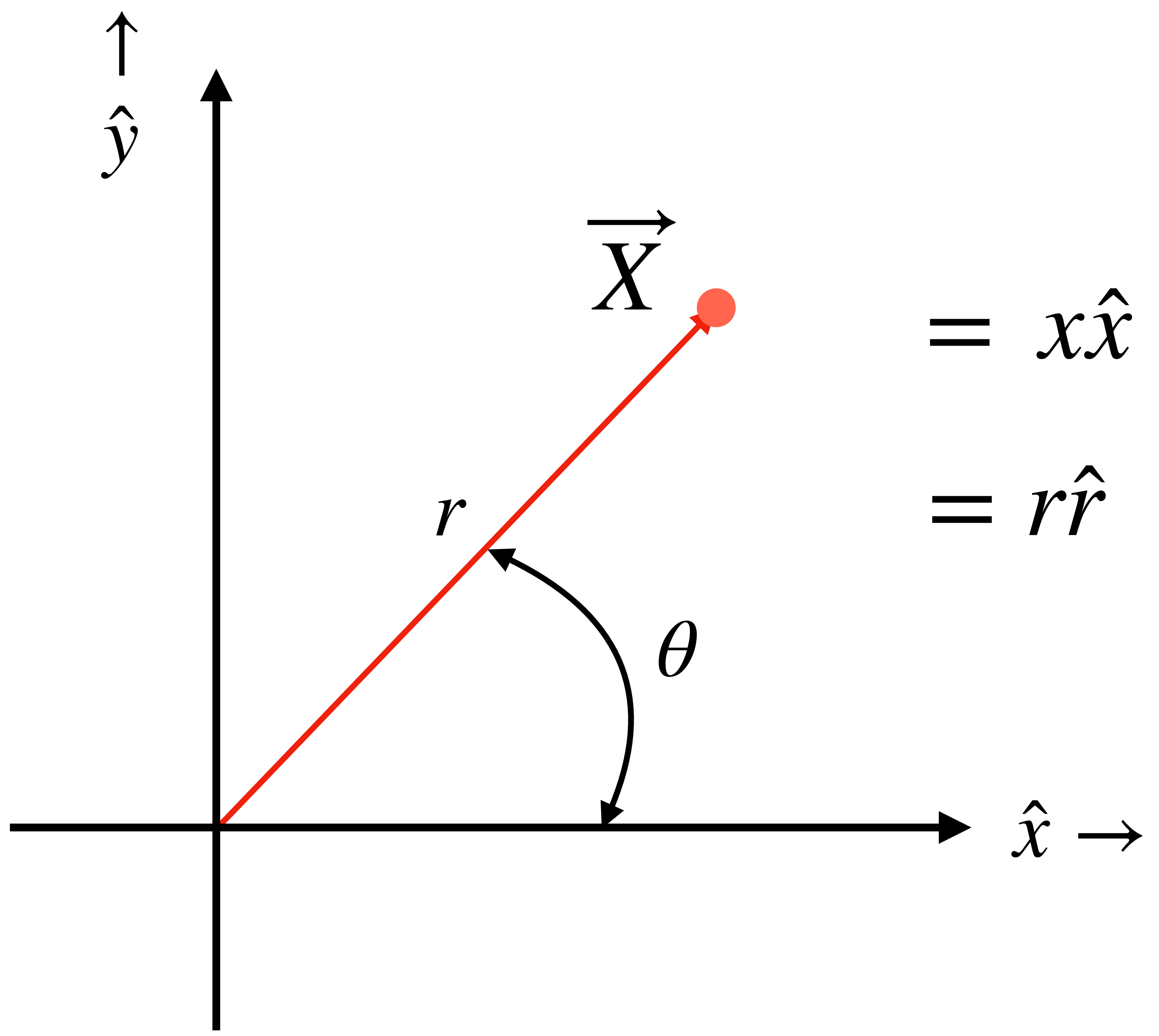
This is NLOM in polar coordinates, but not yet in a useful form

Position and Velocity in Polar coordinate?

The position is represnted by , $\{x, y\}$ or $\{r, \theta\}$

The position vector $\vec{X} = x\hat{x} + y\hat{y}$ what about in polar coordinates?

$\vec{X} = r\hat{r} + \cancel{\ldots}$!!



$= x\hat{x} + y\hat{y}$

$= r\hat{r}$

θ and $\hat{\theta}$ is not needed ? Yes, That is true atleast for position vector.

Let's check this out before proceeding

In Cartesian coordinate $\vec{v} = \dot{x}\,\hat{x} + \dot{y}\,\hat{y}$

In polar coordinate $\vec{v} = \dot{r}\,\hat{r} + r\,\dfrac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\theta\,\hat{\theta} \implies v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$

$$\frac{d}{dt}\vec{P} = m\left[\frac{dv_r}{dt}\hat{r} + \frac{dv_\theta}{dt}\hat{\theta}\right] + m\left[v_r\dot{\theta}\,\hat{\theta} - v_\theta\dot{\theta}\,\hat{r}\right]$$

We have $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$

$$= m\left[\ddot{r}\hat{r} + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta} + \dot{r}\,\dot{\theta}\,\hat{\theta} - r\,\dot{\theta}^2\,\hat{r}\right]$$

$$\frac{d}{dt}\vec{P} = m\left[\left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}\right]$$

EOM is given by $\frac{d}{dt}\vec{P} = m\left[\left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}\right] = F_r\hat{r} + F_\theta\hat{\theta}$

Component vs

$$\boxed{m\left(\ddot{r} - r\,\dot{\theta}^2\right) = F_r(r,\theta) \qquad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = F_\theta(r,\theta)}$$

This EOM in polar coordinates.

Example, 2-D Central Force

For a point particle is moving in a central force, the force field in cartesian coordinate system is given by

$$\overrightarrow{F} = -\frac{\kappa x}{(x^2+y^2)^{\frac{3}{2}}}\hat{x} - \frac{\kappa y}{(x^2+y^2)^{\frac{3}{2}}}\hat{y}$$

The EOM can be written as

$$\ddot{x} = f(x,y) = \frac{\kappa x}{(x^2+y^2)^{\frac{3}{2}}} \qquad \text{and} \qquad \ddot{y} = g(x,y) = \frac{\kappa y}{(x^2+y^2)^{\frac{3}{2}}}.$$

Solve for equation of motion. This is called central force problem in 2-D.

The Newtonian equations of motion in the Cartesian coordinate system are not separable. We need a more suitable coordinate system in which the problem can be solved.