5CS037 CONCEPTS AND TECHNOLOGIES OF AI,
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Problem Set-1 - Linear Algebra Review.

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Instructions:

- Solve all the problems.

1 Preliminary Concepts-Basic set Theory and Function

1. Suppose that:

$$A = \{x : x \in \mathbb{N} \text{ and } x \text{ is even}\},$$
$$B = \{x : x \in \mathbb{N} \text{ and } x \text{ is prime}\}, \text{ and}$$
$$C = \{x : x \in \mathbb{N} \text{ and } x \text{ is a multiple of } 5\}.$$

   Describe Following Sets:
   (a) $A \cap B$ (The intersection of sets A and B):

   (b) $B \cap C$ (The intersection of sets B and C):

   (c) $A \cup B$ (The union of sets A and B):

   (d) $A \cap (B \cup C)$ (The intersection of set A and the union of sets B and C):

2. Prove:

$$i) A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$ii) A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

3. Define a function:

$$f : \mathbb{N} \rightarrow \mathbb{N}$$

   a) that is one-to-one but not onto.
   b) that is onto but not one-to-one.

2 Matrix Operations.

1. The diagonal of a matrix A are the entries a_{ij}; where $i = j$.
   a) Write down the three-by-three matrix with ones on diagonal and zeros else where.
   b) Write down the three-by-four matrix with ones on diagonal and zeros elsewhere.
   c) Write down the four-by-three matrix with ones on the diagonal and zeros elsewhere.

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2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Compute AD and DA.
   {observe and Explain the Results}.

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3. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$. Verify that AB = AC.

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4. Prove that: $(AB)^T = B^T A^T$.
   Review some Definitions:
   - Let A be an m-by-n matrix with elements a_{ij} and b be an n-by-p matrix with elements b_{ij}, Then $C = AB$ is an m-by-p matrix and its "ij" elements are:
   $$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$
   Notice: second index of a and the first index of b are summed over.
   - $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ then $A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$.
   In other words: $a^T_{ij} = a_{ji}$

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5. Let A be a rectangular matrix given by $A = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$. Compute $A^T A$ and show that it is a symmetric square matrix.
   {Observe that the sum of the diagonal elements of $A^T A$ is the sum of squares of the elements of A.}

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6. Find the inverse of the matrices $A = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}; B = \begin{bmatrix} 6 & 4 \\ 3 & 3 \end{bmatrix}$.

   Formula Review:

   - We know:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(A)} . \begin{bmatrix} C_{11} & C_{21} \\ c_{12} & C_{21} \end{bmatrix} = \frac{1}{ad - bc} . \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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7. Given matrices A and B below, verify that they are inverses.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}; B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

   Hint: The matrices are inverses if the product:

$$AB = I_{2X2} = BA$$

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8. Verify that given matrices are inverses of each other:

   - $$\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}; \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$$

   - $$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & -4 \end{bmatrix}; \begin{bmatrix} 3 & -4 & 1 \\ 2 & -4 & 1 \\ 3 & -5 & 1 \end{bmatrix}$$

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3 Solve: System of Linear Equations

3.1 Using Inverse Methods

1. Express the system as AX = B ; then solve using matrix inverse.

   a)

$$\begin{aligned} x + 2y &= 4 \\ 3x - 5y &= 1 \end{aligned}$$

   Solutions: $x = 2; y = 1$

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   b)

$$\begin{aligned} 5x + y &= 13 \\ 3x + 2y &= 5 \end{aligned}$$

   Solutions: $x = 3; y = -2$

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   c)

$$\begin{aligned} 3x + 2y &= -2 \\ x + 4y &= 6 \end{aligned}$$

   Solutions: $x = -2; y = 2$

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2. Notes:

   - The Method for Finding the inverse of a Matrix with Elementary Row Operations:
     a) Write the augmented matrix $[A|I_n]$
     b) Write the augmented matrix in step a in reduced row echelon form.
     c) If the reduced row echelon form in b is in $[I_n|B]$,then B is the inverse of A.
     d) If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

- The Method for Solving a System of Equations when a Unique Solution Exists:
  a) Express the system in the matrix equation $AX = B$.
  b) To solve the equation $AX = B$, we multiply both sides by A^{-1}:

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \end{aligned}$$

$\{$*Where I is the identity matrix*$\}$.

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3. Solve the following system {Inverse Method}:

$$\begin{aligned} x - y + z &= 6 \\ 2x + 3y &= 1 \\ -2y + z &= 5 \end{aligned}$$

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4. Solve the following system{Gauss Elimination or Gauss-jordan Method}:

$$\begin{aligned} x + y - z &= 2 \\ x + z &= 7 \\ 2x + y + z &= 13 \end{aligned} \qquad \begin{aligned} x + y + z &= 2 \\ 3x + y &= 7 \\ x + y + 2z &= 3 \end{aligned}$$

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5. For Practise:

- Solve the following:

$$\begin{aligned} 2x + 3y &= 6 \\ x - y &= \frac{1}{2} \end{aligned} \qquad \begin{aligned} 4x + 3y &= 11 \\ x - 3y &= -1 \end{aligned}$$

- Solve the following:

$$\begin{aligned} -x - 2y + z &= 1 \\ 2x + 3y &= 2 \\ y - 2z &= 0 \end{aligned} \qquad \begin{aligned} x + 4y - z &= 4 \\ 2x + 5y + 8z &= 15 \\ x + 3y - 3z &= 1 \end{aligned} \qquad \begin{aligned} 5x + 3y + 9z &= -1 \\ -2x + 3y - z &= -2 \\ -x - 4y + 5z &= 1 \end{aligned}$$
