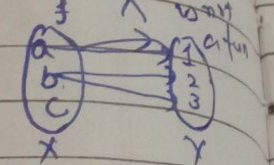
functions

function: A function f from a set A into a set B is a relation from A to B such that each element of A is related to exactly one element of set B. It is denoted as $f: A \to B$ & read as "f is a function from A to B"



Domain of function.
Let f be a function from A to B. The set A is called domain of the function f.

Codomain of a function.
Let f be a function from A to B. The set B is called codomain of function f.

Image of an Element

If element x of A corresponds to y under function f, then y is image of x under f & it is written as $f(x) = y$.

If $f(x) = y$ then we can say x is preimage of y.

If $f: X \to Y$ then each element of X has unique image in Y whereas every element in Y need not be image of some x in X.

Preimage or inverse image

Let $f: A \to B$ be a function from A to B. Set T is subset of B. Then inverse image or preimage of T under f is denoted by $f^{-1}(T)$ defined as

$$f^{-1}(T) = \{ a \in A ; f(a) \in T \}.$$

Or $f^{-1}(T)$ consist of Elements of A whose image belong to T.

Range of function:

The range of function is the set of images of its domain.

Also we can say range is subset of its Codomain

If $f: A \to B$ then $f(A) = \{ f(x) : x \in A \}$

$\cancel{= \{ y : y \in B \}}$

$= \{ y : y \in B \mid \exists \, x \in A \text{ s.t } f(x) = y \}$

e.g. let $P = \{x, y, z, u\}$ & $Q = \{a, b, c, d\}$ & $f: P \to Q$

s.t. $f = \{ (x,a) \ (y,b) \ (z,c) \ (u,c) \}$

find Domain, Codomain & range of function

Sol

Domain $= \{x, y, z, u\}$

Codomain $= \{a, b, c, d\}$

Range $= \{a, b, c\}$

Types of functions

Injective or One to one functions:

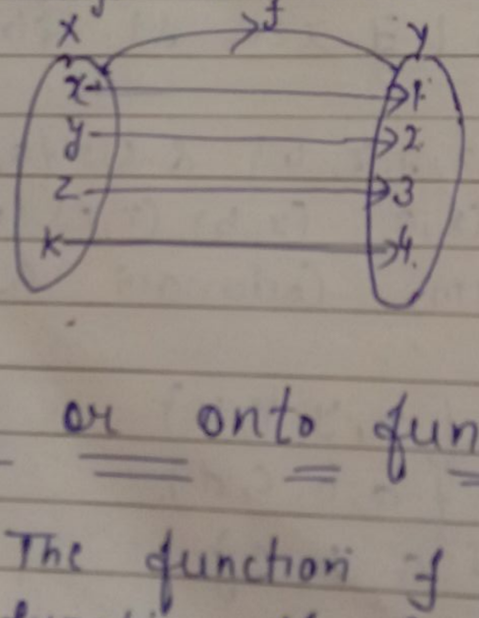
Let $f : X \to Y$. The function f is called one one or injective if different elements in X have different images in Y. i.e. if $f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2 \quad \forall \; x_1, x_2 \in X$

for eg. Let $X = \{x, z, y, h\}$ & $Y = \{1, 2, 3, 4\}$ and f is a function from X to Y such that

$f = \{(x,1), (y,2), (z,3), (h,4)\}$

The function f is injective as Every element of domain X has unique image in codomain Y.

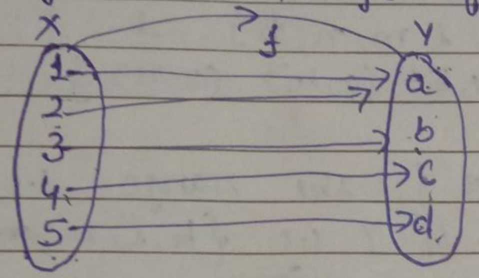


Surjective or onto function

Let $f : X \to Y$. The function f is Called Surjective function if Each Element in Y, is the image of atleast one Element in X.

In other words in surjective function the range of f is equal to co-domain Y.
i.e. $\forall\, y \in Y$; $y = f(x)$ for some $x \in X$.

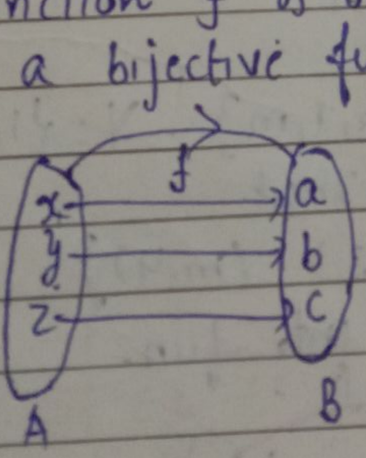
e.g. Consider $X = \{1, 2, 3, 4, 5\}$, $Y = \{a, b, c, d\}$
& $f = \{(1,a), (2,a), (3,b), (4,c), (5,d)\}$
It is a surjective function as every element of Y is image of some element of X.



Bijective (one-to one onto) functions

A function which is both injective & surjective is called bijective function.

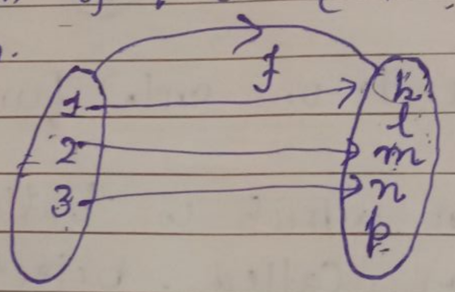
for e.g.
let $A = \{x, y, z\}$ $B = \{a, b, c\}$
and $f : A \to B$ such that
$f = \{(x,a), (y,b), (z,c)\}$
The function f is one-one and onto.
So it is a bijective function.



Into functions: let $f : X \to Y$. The function f is called an into function if the range of f is not equal to the codomain Y. $\therefore$ there must be an element of codomain Y which is not the image of any element of domain X.

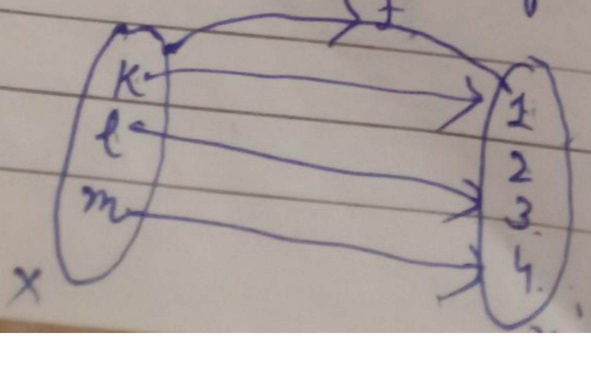
for E.g. consider $X = \{1, 2, 3\}$
$Y = \{k, l, m, n, p\}$ & $f : X \to Y$ s.t.
$f = \{(1, k)\ (2, m)\ (3, n)\}$

In function f, the range i.e. $\{k, m, n\} \neq$ codomain of Y i.e. $\{k, l, m, n, p\}$.
$\therefore$ It is into fun.



One-one Into function: let $f : X \to Y$. The function f is called one one into function if different elements of X have different unique images of Y.

for eg. Consider $X = \{k, l, m\}$ $Y = \{1, 2, 3, 4\}$ &
$f : X \to Y$ such that
$f = \{(k, 1)\ (l, 3)\ (m, 4)\}$
$\therefore$ fun. is one one into fun.



Many one functions: let $f: X \to Y$. The function f is said to be many one function if there exists two or more than two elements in X having the same image in Y.

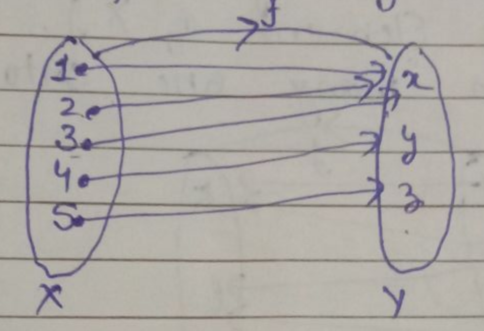
for e.g.

consider $X = \{1, 2, 3, 4, 5\}$ $Y = \{x, y, z\}$

& $f: X \to Y$ s.t

$f = \{(1, x), (2, x), (3, x), (4, y), (5, z)\}$

The function f is many one function.



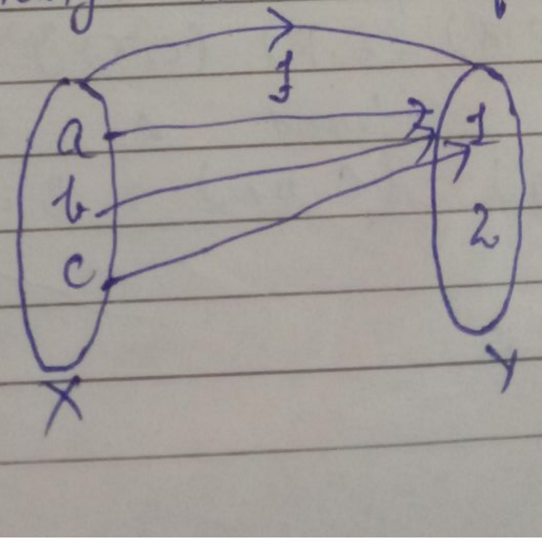
Many one into functions: let $f: X \to Y$

The function f is called many one into function if & only if it is both many one & into function.

e.g. consider $X = \{a, b, c\}$ $Y = \{1, 2\}$ & $f: X \to Y$

s.t. $f = \{(a, 1)\ (b, 1)\ (c, 1)\}$

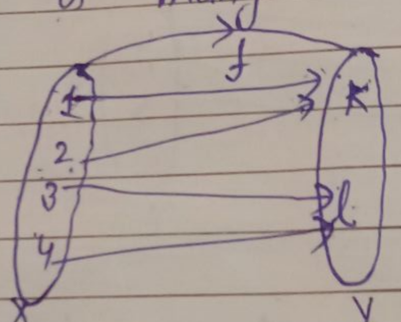
As fun f is many one & into so it is many one into function.



Many one onto functions :→ Let $f : X \to Y$.
The function f is called many one onto function if & only if it is both many one & onto.

e.g. $X = \{1, 2, 3, 4\}$ & $Y = \{k, l\}$ & $f : X \to Y$ s.t. $f = \{(1,k)\ (2,k)\ (3,l)\ (4,l)\}$

The function f is many one (as two elements have same image in Y} & it is onto (Every element of Y is image of some element of X}.
So it is many one onto function.



$f : X \to Y$

Everywhere defined function.

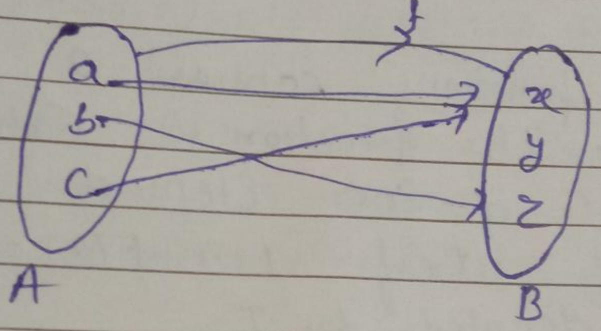
Consider a function f from A to B then function f is everywhere defined if $dom(f) = A$.

e.g. Let $B = \{a, b, c\}$, & $C = \{\alpha, \beta, \gamma)$ and f is a function from B to C s.t.
$f = \{(a,\alpha)\ (b,\beta)\ (c,\gamma)\}$ then f is everywhere defined as
$dom(f) = (a, b, c) = B$.

Representation of a function.

The two set A and B are represented by two circles. The function $f: A \to B$ is represented by collection of arrows joining the pts. which represent the elements of A & corresponding elements of B.

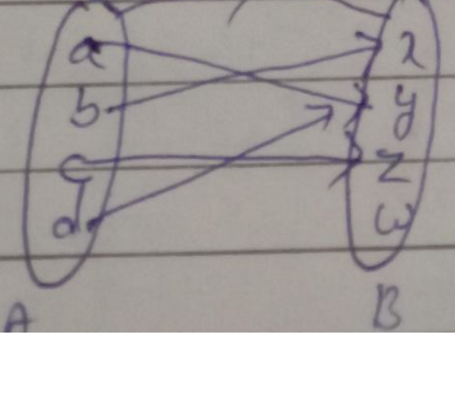
e.g. $A = \{a, b, c\}$ $\quad B = \{x, y, z\}$
& $f: A \to B$ s.t. $f = \{(a,x)\ (b,z)\ (c,x)\}$



<u>Graph</u> <u>of</u> a <u>function</u>: let $f: A \to B$ is a function then graph of f, denoted by graph f is a subset of $A \times B$ given by

$$\text{graph } f = \{(a, f(a)) : a \in A\}$$

for e.g. $A = \{a, b, c, d\}$ $\quad B = \{x, y, z, w\}$
defined as



* Equal functions : consider two functions f & g from set X to set Y. The functions f & g are called Equal functions if & only if $f(a) = g(a)$ for every $a \in$ Domain f, Domain g

eg. $X = \{1, 2, 3\}$ & $Y = \{a, b, c\}$ consider $f: X \to Y$ & $g: X \to Y$ and $h: X \to Y$ s.t.

$f = \{(1,a)\ (2,a)\ (3,c)\}$

$g = \{(1,b)\ (2,a)\ (3,c)\}$

$h = \{(1,a)\ (2,a)\ (3,c)\}$

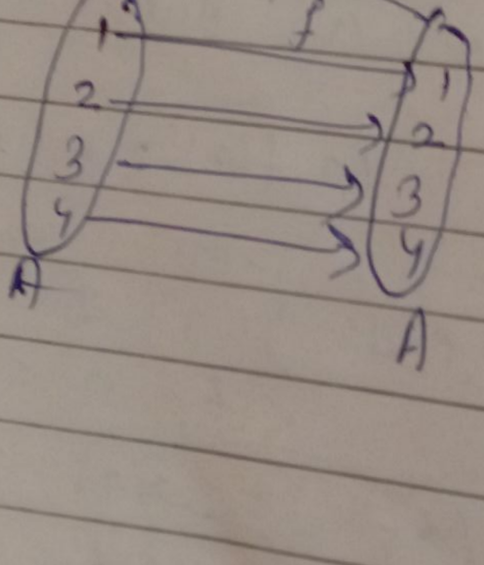
then f & h are Equal functions.

* Identity functions : consider a set A. Let function $f: A \to A$. The function is called identity function if each element of set A has image of itself i.e. $f(a) = a \ \forall a \in A$.
It is denoted by I.

for eg. consider $A = \{1, 2, 3, 4\}$ & $f: A \to A$ s.t.

$f = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

The function f is identity fun as each element of A is mapped onto itself.



Composition of functions:

Def: let $f: A \to B$ & $g: B \to C$ be two functions then the function $gof: A \to C$ defined by

$$(gof)(x) = g(f(x)) \text{ for all } x \in A,$$

is called composition of f and g.

further we say that gof is defined only if Range R_f is a subset of domain D_g.

Also gof is a function from A to C.

for e.g. let $f: \{1,2,3\} \to \{a,b\}$ is a function defined by $f(1)=a$, $f(2)=a$ $f(3)=b$ and $g: \{a,b\} \to \{5,6,7\}$ is defined by $g(a)=5$, $g(b)=7$. we find gof

Ans) Here Range $R_f = \{a,b\}$, Domain $D_g = \{a,b\}$

Since Range $R_f \subseteq$ Domain D_g

$\therefore$ gof is defined & $gof: \{1,2,3\} \to \{5,6,7\}$

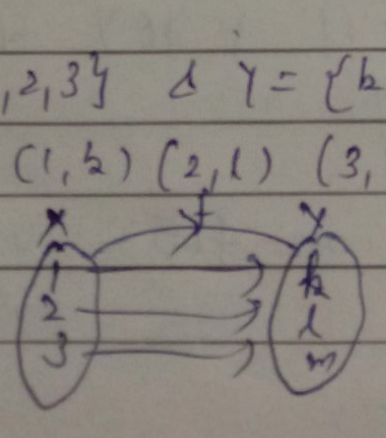
$$(gof)(1) = g(f(1)) = g(a) = 5$$
$$(gof)(2) = g(f(2)) = g(a) = 5$$
$$(gof)(3) = g(f(3)) = g(b) = 7$$

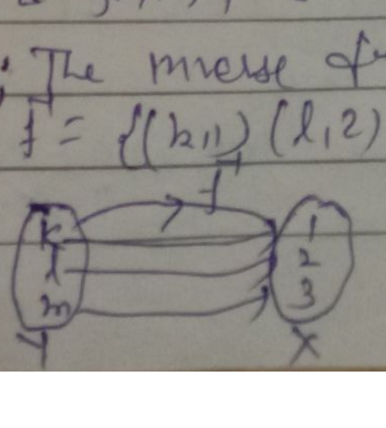
* Invertible or Inverse functions

A function $f: X \to Y$ is invertible if & only if it is bijective function. The Inverse function for f if exists is f^{-1} is a function from Y to X

e.g. $X = \{1,2,3\}$ & $Y = \{k, l, m\}$ & $f: X \to Y$ s.t.

$f = \{(1,k)\ (2,l)\ (3,m)\}$; The inverse fn. is $f^{-1} = \{(k,1)\ (l,2)\ (m,3)\}$





Q. Consider the function $f: N \to N$ where N is set of natural no. defined by $f(x) = n^2+n+1$ show that f is one-one but not onto

Sol. Given a function $f: N \to N$ where N is set of natural no is defined by
$f(n) = n^2 + n + 1$.
We know that a function $f: X \to Y$ is one one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in X$

One-one : let $n_1, n_2 \in N$ s.t. $f(n_1) = f(n_2)$

$\Rightarrow n_1^2 + n_1 + 1 = n_2^2 + n_2 + 1$

$\Rightarrow n_1^2 - n_2^2 + n_1 - n_2 = 0$

$\Rightarrow (n_1 - n_2)(n_1 + n_2) + (n_1 - n_2) = 0$

$\Rightarrow (n_1 - n_2)(n_1 + n_2 + 1) = 0$

$\Rightarrow$ Either $n_1 - n_2 = 0$ or $n_1 + n_2 + 1 = 0$

If $n_1 - n_2 = 0 \Rightarrow \boxed{n_1 = n_2}$

If $n_1 + n_2 + 1 = 0 \Rightarrow n_1 = 1 - n_2 \notin N$ for $n_2 \in N$

$\therefore \boxed{n_1 = n_2}$

Hence $f: N \to N$ is one-one.

Also $f: X \to Y$ is onto if $\forall \ y \in Y \ \exists$ atleast one $x \in X$ st. $f(x) = y$.

Onto: let $m \in N$ s.t. $m = f(n) \ \forall \ n \in N$

or $m = n^2 + n + 1$

$\Rightarrow n^2 + n + (1-m) = 0$

$\Rightarrow n = \frac{-1 \pm \sqrt{1 - 4(1-m)}}{2}$

$n = \frac{-1 \pm \sqrt{1 + 4(m-1)}}{2}$

$n = \frac{-1 \pm \sqrt{4m-3}}{2} \notin N$

Now for $m \in N$ there exist $n \notin N$.
Hence f is not onto.

(Q) Let $A \subseteq Z$ & $f : A \to N$ be a one-one function where Z is set of integers & N is set of natural no.'s. Let R be a relation on A defined as $(x,y) \in R$ iff $f(y) = k f(x)$ for $k \in N$. Prove that R is partial order relation.

Sol. Given $f : A \to N$ is a one-one function where $A \subseteq Z$, the set of integers.
The relation R is defined as
$(x,y) \in R \Leftrightarrow f(y) = k f(x) \ \forall\ x, y \in Z$
We shall prove that R is partial order relation.

i.e. (i) R is reflexive
ii) R is Antisymmetric
iii) R is transitive

(1) Reflexive: Consider $f(x) = 1 \cdot f(x)$ where $1 = k \in$

$\therefore (x, x) \in R$

$\therefore R$ is Reflexive

ii) Antisymmetric: let $(x,y) \in R$ & $(y,x) \in R$.

$\therefore f(y) = k_1 f(x) \rightarrow (1)$ & $f(x) = k_2 f(y) \rightarrow (2)$

where $k_1, k_2 \in N$

Now $f(y) = k_1 (k_2 f(y))$ [Sub. value of $f(x)$ in (1) eq^n]

$\Rightarrow f(y) = k_1 k_2 f(y)$

$\Rightarrow f(y)[1 - k_1 k_2] = 0$

$\Rightarrow 1 - k_1 k_2 = 0$ $\because$ [if $y \in Z$ $f(y) \in N$ since $f: Z \rightarrow N$ so $f(y) \neq 0$] $f: A \rightarrow N$

$\Rightarrow k_1 k_2 = 1 \rightarrow (3)$

$\Rightarrow k_2 = \frac{1}{k_1}$

But $k_1, k_2 \in N$ so Eq^n (3) implies

$\boxed{k_1 = k_2 = 1}$

$\therefore$ from eq^n (2)

$f(x) = f(y)$ $\because$ (f is one-one)

$\Rightarrow \boxed{x = y}$

Hence R is Antisymmetric

iii) Transitive: let $(x,y) \in R$ & $(y,z) \in R$

$\therefore f(y) = k_1 f(x)$ & $f(z) = k_2 f(y)$

for $x, y \in Z$ & $k_1, k_2 \in N$

Now $f(z) = k_2 [k_1 f(x)]$ $\because (f(y) = k_1 f(x))$

$\Rightarrow f(z) = k_1 k_2 f(x)$ (where $k = k_1 k_2 \in N$)

$\Rightarrow f(z) = k f(x)$

$\Rightarrow (x,z) \in R$

$\therefore$ R is transitive.

Hence relation R is <u>partial</u> ~~order~~ relation

Q) Let $A = B = \{1, 2, 3, 4, 5\}$ Define functions $f: A \to B$ (if possible) s.t that

(a) f is one-one & onto

(b) f is neither one-one nor onto

(c) f is one-one but not onto

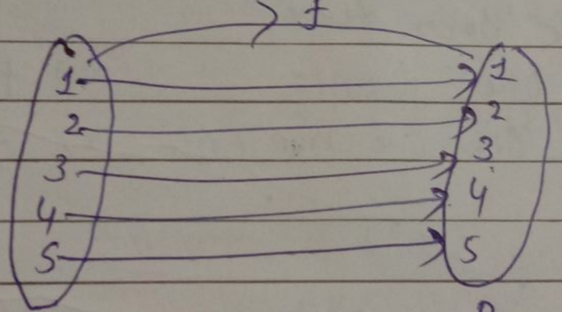
(d) f is onto but not one one

Sol. (a) Define $f: A \to B$ s.t

$f = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (5,5)\}$

i.e. different Elements of A have different images in B. $f(1) = 1$

$\therefore$ $f: A \to B$ is one-one



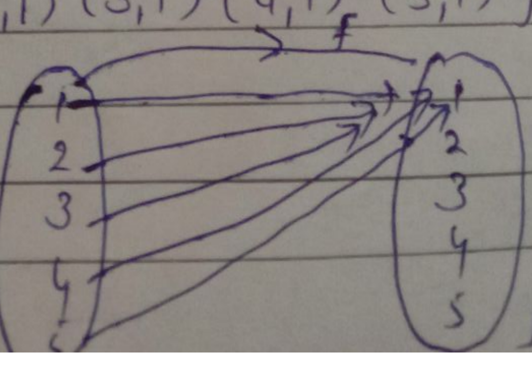
A B

Also $\forall\ y \in B, \exists\ x \in A$ s.t $f(x) = y$.

Hence f is onto also.

(b) Define $f: A \to B$ s.t.

$f = \{(1,1)\ (2,1)\ (3,1)\ (4,1)\ (5,1)\}$



B

Different Element of Set A has Same images in B.

$\therefore f: A \to B$ is not one-one

Also 2, 3, 4, 5 have no image in A.

$\therefore f: A \to B$ is not onto.

(c) Since $A = B$.

$\therefore$ There is not function which is one-one but not onto.

(d) Since $A = B$

$\therefore$ There is no function which is onto but not one-one.

Q) Let $A = B = \{1, 2, 3, 4\}$; Define functions $f: A \to B$ if possible such that

(i) f is one-one & onto

(ii) f is neither one-one nor onto

(iii) f is onto but not one-one.

(iv) f is one-one but not onto.