Recurrence Relation

Recurrence relation is an Equation that defines a Sequence or Express each Element of a Sequence as a function of the preceding ones.

Order of a Recurrence relation

order of reccurrence relation is the difference between the highest and lowest Subscripts of S(K).

For e.g. consider The recurrence relation

$$F_K = F_{K-2} + F_{K-1} \; ; \quad K \geq 2$$

$$\text{Order} = K-(K-2) = 2$$

Degree of recurrence relation

It is highest power of S(K) occuring in relation.

for E.g. consider $\cancel{S^4(K) + 3S^3(K-1) + 6S^2(K-2) + 4S(K-3) = 0}$

Degree = 2

$\cancel{\text{Order} = K-(K-3) = 3.}$

$$S^3(K+3) + 2S^2(K+2) + 2S(K+1) = 0$$

$$\text{order} = K+3-K-1 = 2$$

Homogeneous Eq^n :→ If we have zero on right hand side we called it as hom. Eq^n.

Q1) Solve the recurrence relation

$$S_n - 5S_{n-1} + 6S_{n-2} = 0$$

Sol. The given recurrence relation is

$S_n - 5S_{n-1} + 6S_{n-2} = 0$

$S_n \to x^n$
$S_{n-1} \to x^{n-1}$
$S_{n-2} \to x^{n-2}$

Replace S_n by x^n

$\therefore$ Characteristic Eq^n is

$x^n - 5x^{n-1} + 6x^{n-2} = 0$

Div. b/s by x^{n-2}

Order $= n-(n-2) = 2$
$\Downarrow$
2 Roots

$\Rightarrow \frac{x^n}{x^{n-2}} - \frac{5x^{n-1}}{x^{n-2}} + \frac{6x^{n-2}}{x^{n-2}} = 0$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x^2 - 3x - 2x + 6 = 0$

$\Rightarrow x(x-3) - 2(x-3) = 0$

$\Rightarrow (x-2)(x-3) = 0$

$\Rightarrow \boxed{x = 2, 3}$

Since characteristic Eq^n has different roots so the homogeneous sol^n of this Eq^n is

$S_n = C_1(2)^n + C_2(3)^n$.

(2) Solve recurrence relation $a_r - 7a_{r-1} + 10a_{r-2} = 0$ given that $a_0 = 0$ & $a_1 = 3$.

Sol The given recurrence relation is $a_r - 7a_{r-1} + 10a_{r-2} = 0$

Replace a_r by x^r so its characteristics Eq^n is

$x^r - 7x^{r-1} + 10x^{r-2} = 0$

Divide b/s by x^{r-2}

$\frac{x^r}{x^{r-2}} - \frac{7x^{r-1}}{x^{r-2}} + \frac{10x^{r-2}}{x^{r-2}} = 0$

$\Rightarrow x^2 - 7x + 10 = 0$

$\Rightarrow x^2 - 5x - 2x + 10 = 0$

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$x(x-5) - 2(x-5) = 0$

$\Rightarrow (x-2)(x-5) = 0$

$\Rightarrow x = 2, 5$

Since roots are different so Homogeneous Solⁿ of this Eqⁿ is

$a_n = c_1(2)^n + c_2(5)^n \rightarrow (1)$

$a_0 = c_1(2)^0 + c_2(5)^0$

put $a_0 = 0$ in Eqⁿ (1)

$\Rightarrow 0 = c_1(2)^0 + c_2(5)^0$

$\Rightarrow \boxed{c_1 + c_2 = 0} \rightarrow (2)$

Again put $a_1 = 3$ in (1) Eqⁿ $\qquad a_1 = c_1(2)^1 + c_2(5)^1$

$\therefore 3 = c_1(2) + c_2(5)$

$\Rightarrow \boxed{2c_1 + 5c_2 = 3} \rightarrow (3)$

from Eqⁿ (2) & (3)

$2c_1 + 5(-c_1) = 3 \qquad \because \left[\begin{array}{l} c_1 + c_2 = 0 \\ c_2 = -c_1 \end{array}\right]$

$\Rightarrow -3c_1 = 3$

$\boxed{c_1 = -1}$

put value of c_1 in Eqⁿ no. (2)

$\therefore \boxed{c_2 = 1}$

$\therefore$ from Eqⁿ (1), the required Solⁿ is

$$\boxed{a_n = -(2)^n + (5)^n}$$

(3) Solve recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = 0$

Sol The given recurrence relation is

$a_n - 4a_{n-1} + 4a_{n-2} = 0$

Replace a_n by x^n so its characteristic Eqⁿ is

$$x^r - 4x^{r-1} + 4x^{r-2} = 0$$

divide b/s by x^{r-2}, we get

$$\Rightarrow \frac{x^r}{x^{r-2}} - \frac{4x^{r-1}}{x^{r-2}} + \frac{4x^{r-2}}{x^{r-2}} = 0$$

$$\Rightarrow x^2 - 4x + 4 = 0$$

$$\Rightarrow (x-2)^2 = 0 \quad \text{i.e} \quad (x-2)(x-2) = 0$$

$$\Rightarrow x = 2 \ \& \ 2$$

Since characteristic Eqⁿ has Equal roots so homogeneous Solⁿ of this Equation is

$$\boxed{a_r = (C_1 + rC_2)(2)^r}$$

(4) Solve the recurrence relation $a_r + a_{r-1} + a_{r-2} = 0$.

Sol. The given recurrence relation is

$$a_r + a_{r-1} + a_{r-2} = 0$$

replace a_r by x^r, so its characteristic Eqⁿ is

$$x^r + x^{r-1} + x^{r-2} = 0$$

Divide b/s by x^{r-2}, we get

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x^2 + x + 1 = 0$$

roots of this characteristic Eqⁿ are imaginary i.e. $\frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2}$

$\frac{-1 \pm \sqrt{-3}}{2}$ $-1 \pm \sqrt{3i^2}$

i.e $\frac{-1 + \sqrt{3}\,i}{2}$ & $\frac{-1 - \sqrt{3}\,i}{2}$

$\therefore$ Homogeneous sol of given Eqⁿ is

$$a_r = C_1 \left(\frac{-1 + \sqrt{3}\,i}{2}\right)^r + C_2 \left(\frac{-1 - \sqrt{3}\,i}{2}\right)^r$$

Note: If characteristics Eqⁿ has repeated imaginary roots say $\alpha \pm i\beta$, $\alpha \pm i\beta$ then hom. solⁿ is

$$S(K) = (C_1 + C_2 K)(\alpha + i\beta)^K + (C_3 + C_4 K)(\alpha - i\beta)^K$$

e.g. Solve the recurrence relation

$$a_{r+4} + 2a_{r+3} + 3a_{r+2} + 2a_{r+1} + a_r = 0$$

Sol. The given recurrence relation is

$$a_{r+4} + 2a_{r+3} + 3a_{r+2} + 2a_{r+1} + a_r = 0$$

Replace a_r by x^r so characteristics Eqⁿ is

$$x^{r+4} + 2x^{r+3} + 3x^{r+2} + 2x^{r+1} + x^r = 0$$

Div. by sides by x^r

$$\therefore x^4 + 2x^3 + 3x^2 + 2x + 1 = 0$$

$$x^4 + x^2 + 2x^2 + 2x^3 + 2x + 1 = 0$$

$$\Rightarrow (x^4 + x^2 + 1) + (2x^3 + 2x + 2x^2) = 0$$

$$\Rightarrow (x^2 + x + 1)^2 = 0 \qquad \because \left[(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca\right]$$

$x^2 + x + 1 = 0$ is a quad Eqⁿ with roots $\frac{-1 \pm \sqrt{3}i}{2}$

$$\therefore x = \frac{-1 \pm \sqrt{3}i}{2}, \frac{-1 \pm \sqrt{3}i}{2}$$

Since Eqⁿ has repeated imag. roots so hom. solⁿ of this relation is

$$a_r = (C_1 + C_2 r)\left(\frac{-1 + \sqrt{3}i}{2}\right)^r + (C_3 + C_4 r)\left(\frac{-1 - \sqrt{3}i}{2}\right)^r$$

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Solve the recurrence relation
$a_n = 6a_{n-1} - 8a_{n-2}$; $a_0 = 1$ & $a_1 = 10$
The given recurrence relation is
$a_n = 6a_{n-1} - 8a_{n-2}$
i.e $a_n - 6a_{n-1} + 8a_{n-2} = 0$
Replace a_n by x^n, we get

$$x^n - 6x^{n-1} + 8x^{n-2} = 0$$

Div. b/s by x^{n-2};

$$x^2 - 6x + 8 = 0$$
$$x^2 - 4x - 2x + 8 = 0$$
i.e. $x(x-4) - 2(x-4) = 0$
$\Rightarrow (x-4)(x-2) = 0$
$\Rightarrow \boxed{x = 2, 4}$

Since roots are different
$\therefore$ Homogeneous solⁿ of given relation is
$$a_n = C_1(2)^n + C_2(4)^n \rightarrow (1)$$
put $a_0 = 1$ in Eqⁿ (1)
$\therefore (1) \Rightarrow 1 = C_1(2)^0 + C_2(4)^0$
$\Rightarrow C_1 + C_2 = 1 \rightarrow (2)$

Also $a_1 = 10$
$\therefore (1) \Rightarrow 10 = 2C_1 + 4C_2 \rightarrow (3)$; xply Eqⁿ (2) by 2 & sub from (3) Eqⁿ
$\therefore$ from (2) & (3)
$$2C_1 + 4C_2 = 10$$
$$\underline{-2C_1 \pm 2C_2 = 2}$$
$$2C_2 = 8$$
$$C_2 = \frac{8}{2} = 4$$

put value of C_2 in (2) Eqⁿ $\Rightarrow C_1 = 1 - 4 = -3$
$\therefore$ The required solⁿ is
$a_n = -3(2)^n + 4(4)^n$.
$\therefore \boxed{a_n = -3(2)^n + (4)^{n+1}}$

Q) Solve $S(k)-7S(k-2)+6S(k-3)=0$ where $S(0)=8$, $S(1)=6$, $S(2)=22$.

Sol The given recurrence relation is
$S(k)-7S(k-2)+6S(k-3)=0$
Replace $S(k)$ by a^k so the characteristic Eqn is

$$a^k-7a^{k-2}+6a^{k-3}=0$$

Divide b/s by a^{k-3}, we get the characteristic Eqn

$$a^3-7a+6=0 \rightarrow (1)$$

By inspection $a=1$ is root of Eqn (1)
The remaining roots can be calculated as

$$\begin{array}{c|cccc} 1 & 1 & 0 & -7 & 6 \\ & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

$$a^2+a-6=0$$
$$\Rightarrow a^2+3a-2a-6=0$$
$$\Rightarrow a(a+3)-2(a+3)=0$$
$$\Rightarrow (a-2)(a+3)=0$$
$$\Rightarrow a=2,-3$$

$\therefore$ Roots are $1, 2, -3$
Since all the roots are different so its homogeneous soln is

$$S(k)=C_1(1)^k+C_2(2)^k+C_3(-3)^k$$

i.e. $S(k)=C_1+C_2(2)^k+C_3(-3)^k \rightarrow (2)$

Given $S(0) = 8$

$\therefore$ (2) $\Rightarrow$ $S(0) = C_1 + C_2(2)^0 + C_3(-3)^0$

$$\boxed{8 = C_1 + C_2 + C_3} \rightarrow (4)$$

also $S(1) = 6$

$\therefore$ (2) $\Rightarrow$ $S(1) = C_1 + C_2(2) + C_3(-3)$

$$\boxed{6 = C_1 + 2C_2 + 3C_3} \rightarrow (5)$$

And $S(2) = 22$

$\therefore$ (2) $\Rightarrow$ $S(2) = C_1 + C_2(2)^2 + C_3(-3)^2$

$\Rightarrow$ $$\boxed{22 = C_1 + 4C_2 + 9C_3} \rightarrow (6)$$

Subtract Eq^n (4) & (5)

$$C_2 + 4C_3 = -2 \rightarrow (7)$$

Again Subtract Eq^n (5) & (6)

$$2C_2 + 12C_3 = 16 \rightarrow (8)$$

Xply Eq^n (7) by 2 & sub from Eq^n no. (8)

$$\cancel{2C_2 + 8C_3 = 12}$$
$$\cancel{2C_2 + 4C_3 = -4}$$
$$\cancel{2C_3 = 16}$$
$$\cancel{\boxed{C_3 = 3}}$$

$$2C_2 + 12C_3 = 16$$
$$2C_2 - 8C_3 = -4$$
$$+$$
$$20C_3 = 20$$

$$C_3 = \frac{20}{20} = 1$$

$$\boxed{C_3 = 1}$$

$\therefore$ Eq^n no (7) $\Rightarrow$ $C_2 - 4(1) = -2$

$$C_2 = -2 + 4 = 2$$

& (4) $\Rightarrow$ $C_1 + C_2 + C_3 = 8$ i.e. $C_1 = 8 - 2 - 1 = 5$

$\therefore$ Required Sol^n is $S(k) = 5 + 2(2)^k + (1)(-3)^k$.

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Algorithm for Solving Hom. Linear recurrence relation:

Consider Hom. ~~Rectur~~ recurrence relation of order n. $S(k)+C_1 S(k-1)+C_2 S(k-2)+\dots+C_n S(k-n)=0 \rightarrow (1)$

Step 1: Write the characteristics Eq^n:

Step 2: find the roots of characteristics Eq^n. obtained in Step 1.

Case 1: If characteristics Eq^n has n distinct roots, Say $m_1, m_2, m_3 \dots m_n$ the Sol^n of (Eq^n (1)) recurrence relation is.

$$S(k) = C_1 m_1^k + C_2 m_2^k + C_3 m_3^k + \dots + C_n m_n^k.$$

Case 2: If characteristics Eq^n has two Equal roots Say $m_1 = m_2$ then Sol^n of (1) Eq^n is

$$S(k) = (C_1 + C_2 k) m_1^k + C_3 m_3^k + \dots + C_n m_n^k.$$

Case 3: If characteristics Eq^n has three Equal roots. Say $m_1 = m_2 = m_3$ then Sol of (1) Eq^n is

$$S(k) = (C_1 + C_2 k + C_3 k^2) m_1^k + C_4 m_4^k + \dots + C_n m_n^k$$

Case 4: If characteristics Eq^n has Imaginary roots Say $\alpha \pm i\beta$ Then Sol^n of (1) Eq^n is

$$S(k) = C_1 (\alpha + i\beta)^k + C_2 (\alpha - i\beta)^k$$

Case 5: If characteristics Eq^n has repeated imag. roots

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Say $\alpha \pm i\beta$, $\alpha \pm i\beta$ then Sol of ① Eqn is

$$S(k) = (C_1 + C_2 k)(\alpha + i\beta)^k + (C_3 + C_4 k)(\alpha - i\beta)^k.$$

Non-homogeneous recurrence relation

~~if $S_n + C_1 S C$~~

let $S_n + a_1 S_{n-1} + a_2 S_{n-2} + \dots + a_r S_{n-r} = f(n) \rightarrow ①$

It Consist of 2 parts:

Homogeneous Soln of corresponding homogeneous Recurrence relation.

Particular Sol of given non homogeneous recurrence relation that depend upon the function.

Step 1: find the homogeneous Soln of recurrence relation taking $f(n) = 0$

i.e. $S_n + a_1 S_{n-1} + a_2 S_{n-2} + \dots + a_r S_{n-r} = 0$.

Step 2: find particular Sol of given non-homogeneous recurrence relation as Explained below.

Case 1: when $f(n)$ = Constant then assume particular Solution $S_n = p$ (constant)

$$p + a_1 p + a_2 p + \dots + a_r p = f(n)$$

$$p = \frac{f(n)}{a_1 + a_2 + \dots + a_r}$$

Now $a_1 + a_2 + \dots + a_r \neq 0$

If $a_1 + a_2 + \dots + a_r = 0$ then S_n can't be Equal to p

In this case put $f(n) = np$ in ① Eqn.

If again not possible put $S_n = n^2 p$.

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Q:1 Solve $S_n + 5S_{n-1} = 9 \quad S(0) = 6$.

Sol^n The given non homogeneous recurrence relation is

$$S_n + 5S_{n-1} = 9 \rightarrow (1)$$

The homogeneous sol^n is obtained as

$$x^n + 5x^{n-1} = 0$$

$$x^{n-1}(x+5) = 0$$

$$x + 5 = 0$$

$$\boxed{x = -5}$$

$\therefore$ Homogeneous Sol is $C_1(-5)^n$.

Since R.H.S. of Eq^n (1) is Constant so assume particular sol as $S_n = p$.

$\therefore$ Eq^n (1) becomes

$$p + 5p = 9$$

$$6p = 9$$

$$\Rightarrow p = \frac{3}{2}$$

$\therefore$ General Sol^n is

$$S_n = C_1(-5)^n + \frac{3}{2} \rightarrow (2)$$

$\because$ [Gen Sol^n = Hom Sol + Part Sol
i.e. $S_n = S_h(n) + S_p(n)$]

Since $S(0) = 6$

put $n = 0$

$\therefore$ (2) $\Rightarrow S_0 = C_1(-5)^0 + \frac{3}{2}$

$$6 = C_1 + \frac{3}{2}$$

$$\Rightarrow C_1 = 6 - \frac{3}{2} = \frac{9}{2}$$

$\therefore$ $S_n = \frac{9}{2}(-5)^n + \frac{3}{2}$ is the required sol^n

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Q.2) Solve $S_n - 2S_{n-1} + S_{n-2} = 12$

Sol. The given non-homogeneous recurrence relation is $S_n - 2S_{n-1} + S_{n-2} = 12$ — ① So its sol^n consist of two parts i.e. homogeneous sol^n & particular sol^n.

The homogeneous relation is

$$S_n - 2S_{n-1} + S_{n-2} = 0$$

The Corresponding characteristics Eq^n is

$$x^n - 2x^{n-1} + x^{n-2} = 0$$

$$x^{n-2}(x^2 - 2x + 1) = 0$$

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

$$\Rightarrow x = 1, 1$$

Since roots are equal so the homogeneous sol^n is $S_n^{(h)} = (C_1 + nC_2)(1)^n$

Since R.H.S of Eq^n ① is Constant i.e. $f(n) = 12$ so assume particular sol^n as $S_n = p$.

$$\therefore ① \Rightarrow \quad p - 2p + p = 12$$

$0 = 12$ which is not possible.

So Consider $S_n = np$

$$\therefore ① \Rightarrow \quad np - 2(n-1)p + (n-2)p = 12$$

$$\Rightarrow \cancel{np} - \cancel{2np} + \cancel{2p} + \cancel{np} - \cancel{2p} = 12$$

$0 = 12$ which is again not possible

So put $S_n = n^2 p$

$$\therefore ① \Rightarrow \quad n^2p - 2(n-1)^2p + (n-2)^2p = 12$$

$$\Rightarrow n^2p - 2(n^2 + 1 - 2n)p + (n^2 + 4 - 4n)p = 12$$

$$\Rightarrow \cancel{n^2p} - \cancel{2n^2p} - 2p + \cancel{4np} + \cancel{n^2p} + 4p - \cancel{4np} = 12$$

$$2p = 12$$

$$\boxed{p = 6}$$

$\therefore$ Particular sol^n is $S_n^{(p)} = 6n^2$

$\therefore$ General Sol $= (C_1 + nC_2)\,(1)^n + 6n^2$

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* Case 2: If $f(n) = an+b$ i.e. a linear polynomial then particular Sol^n is

$$S_n = d_0 + d_1 n$$

Similarly if $f(n) = an^2 + bn + c$ is quadratic then $S_n = p_0 + p_1 n + p_2 n^2$.

(Q:1) Solve $S_n - 7S_{n-1} + 10S_{n-2} = 6 + 8n$

Sol The given non-homogeneous recurrence relation is

$$S_n - 7S_{n-1} + 10S_{n-2} = 6 + 8n \rightarrow (1)$$

The homogeneous relation is

$$S_n - 7S_{n-1} + 10S_{n-2} = 0$$

The corresponding characteristic Eq^n is

$$\cancel{S_n} = x^n - 7x^{n-1} + 10x^{n-2} = 0$$

$$x^{n-2}(x^2 - 7x + 10) = 0$$

$$x^2 - 7x + 10 = 0$$

$$x^2 - 5x - 2x + 10 = 0 \quad \text{i.e } x(x-5) - 2(x-5) = 0$$

$$(x-5)(x-2) = 0$$

$$x = 2, 5$$

Since roots are different so the homogeneous Sol^n is $S_n^{(h)} = C_1(2)^n + C_2(5)^n$.

Since R.H.S of Eq^n (1) is linear Eq^n of form $6+8n$ so assume particular Sol^n as

$$S_n^{p} = d_0 + d_1 n$$

Substitute value in Eq^n (1)

$$(d_0 + d_1 n) - 7[d_0 + d_1(n-1)] + 10[d_0 + d_1(n-2)] = 6 + 8n$$

$$\Rightarrow d_0 + d_1 n - 7d_0 - 7d_1 n + 7d_1 + 10d_0 + 10d_1 n - 20d_1 = 6 + 8n$$

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$4d_0 + 4d_1 n - 13 d_1 = 6 + 8n$

Equate the coefficient of constant term & n

$4d_0 - 13 d_1 = 6$ & $4d_1 = 8$

$\boxed{d_1 = 2}$

$4d_0 - 26 = 6$

$\Rightarrow 4d_0 = 32$

$\boxed{d_0 = 8}$

$\therefore S_n^{(p)} = 8 + 2n$

Now the general Sol is $S_n = S_n^{(h)} + S_n^{(p)}$

i.e. $S_n = C_1(2)^n + C_2(5)^n + 8 + 2n$ is the required Solⁿ.

Case 3: When $f(n)$ is an Exponential function $f(n) = k a^n$

then assume particular Solⁿ as $S_n = p \cdot a^n$.

If homogeneous solⁿ contains a term a^n Then particular Solⁿ is $S_n^{(p)} = p n (a)^n$

Case 4: If $f(n) = \phi(n) a^n$ where $\phi(n)$ is a polynomial of degree m then particular Solution $S_n = f(n)$ where $f(n)$ is also a polynomial of same degree as of $\phi(n)$.

If base a of Exponential $f(n)$ is also a roots of $\phi(n)$ repeated k times Then

$S_n = p(n) n^k \cdot a^n$ will be particular solⁿ.

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Q: Solve $S_n - 4S_{n-1} + 3S_{n-2} = 3^n$.

Sol. The given non-homogeneous recurrence relation is

$$S_n - 4S_{n-1} + 3S_{n-2} = 3^n \rightarrow (1)$$

The homogeneous relation is

$$S_n - 4S_{n-1} + 3S_{n-2} = 0.$$

The corresponding characteristic Eq^n is

$$x^n - 4x^{n-1} + 3x^{n-2} = 0$$

$$\Rightarrow x^{n-2}(x^2 - 4x + 3) = 0$$

$$\Rightarrow x^2 - 4x + 3 = 0$$

$$\Rightarrow x^2 - 3x - x + 3 = 0$$

$$\Rightarrow x(x-3) - 1(x-3) = 0$$

$$\Rightarrow (x-1)(x-3) = 0 \quad \text{i.e. } x = 1, 3$$

Since roots are real & different so the homogeneous sol^n is

$$S_n^{(h)} = C_1(1)^n + C_2(3)^n.$$

Since R.H.S of given relation is an Exponential function i.e. $f(n) = 3^n$ so assume its particular sol^n as. $S_n = p \cdot n(3)^n$ [as homogeneous sol contains a term $(3)^n$ so we have particular sol $S_n = np(3)^n$ instead of $p(3)^n$]

$$\therefore (1) \Rightarrow np(3)^n - 4(n-1)p(3)^{n-1} + 3(n-2)p(3)^{n-2} = 3^n$$

$$\Rightarrow np(3)^n - 4np(3)^{n-1} + 4p(3)^{n-1} + 3np(3)^{n-2} - 6p(3)^{n-2} = 3^n$$

Divide b/s by 3^n we get

$$np - \frac{4np}{3} + \frac{4p}{3} + \frac{3np}{9} - \frac{6p}{9} = 1$$

$$\left(\cancel{np} - \cancel{\frac{4}{3}np} + \cancel{\frac{np}{3}}\right) + \left(\frac{4p}{3} - \frac{2p}{3}\right) = 1$$

$$\Rightarrow \frac{2p}{3} = 1 \quad \Rightarrow \boxed{p = \frac{3}{2}}$$

$\therefore$ particular Sol^n is

$S_n^{(p)} = \frac{3n}{2}(3)^n = \frac{n}{2}(3)^{n+1}$

Hence the general Sol^n is

$S_n = S_n^{(h)} + S_n^{(p)}$

i.e. $S_n = C_1(1)^n + C_2(3)^n + \frac{n}{2}(3)^{n+1}$

Q) Find particular Solution of $a_{r+2} - 3a_{r+1} + 2a_r = z^r$
where z is some constant.

Sol the given relation is $a_{r+2} - 3a_{r+1} + 2a_r = z^r \rightarrow ①$
So its solution will be of the form $a_r = p \cdot z^r$

$\therefore ① \Rightarrow p \cdot z^{r+2} - 3pz^{r+1} + 2pz^r = z^r$

$\Rightarrow p\cancel{z^r}(z^2 - 3z + 2) = \cancel{z^r}$

$\Rightarrow (z^2 - 3z + 2)p = 1$

$\Rightarrow p = \frac{1}{z^2 - 3z + 2}$

$\Rightarrow p = \frac{1}{z^2 - 2z - z + 2}$

$p = \frac{1}{z(z-2) - 1(z-2)}$

$\Rightarrow p = \frac{1}{(z-1)(z-2)}$; $(z \neq 1, z \neq 2)$

$\therefore$ particular Sol is $\frac{z^r}{(z-1)(z-2)}$

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Q) find homogeneous & particular Sol^n of recurrence relation

$$S_{n+2} - 4S_n = n^2+n-1.$$

Sol The given recurrence relation is

$$S_{n+2} - 4S_n = n^2+n-1 \rightarrow (1)$$

The homogeneous relation is

$$S_{n+2} - 4S_n = 0$$

The Corresponding characteristics Eq^n is

$$x^{n+2} - 4x^n = 0$$

$$\Rightarrow x^n(x^2-4)=0$$

$$\Rightarrow x^2-4=0$$

$$\Rightarrow x^2-(2)^2=0$$

$$\Rightarrow (x-2)(x+2)=0$$

$$\Rightarrow x=2,-2$$

$\therefore$ Homogeneous Sol^n is $S_n^{(h)} = C_1(2)^n + C_2(-2)^n$

to find the particular Sol^n; let

$$S_n = d_0 + d_1 n + d_2 n^2$$

$$\therefore (1) \Rightarrow d_0 + d_1(n+2) + d_2(n+2)^2 - 4[d_0 + d_1 n + d_2 n^2] = n^2+n-1$$

$$\Rightarrow d_0 + d_1 n + 2d_1 + d_2 n^2 + 4d_2 + 4nd_2 - 4d_0 - 4d_1 n - 4d_2 n^2 = n^2+n-1$$

$$\Rightarrow -3d_0 - 3d_1 n - 3d_2 n^2 + 2d_1 + 4d_2 + 4nd_2 = n^2+n-1$$

$$\Rightarrow -3d_2 n^2 + (4d_2 - 3d_1)n + (2d_1 + 4d_2 - 3d_0) = n^2+n-1$$

Equate Coeff of n^2, n & Constant term, we get

$-3d_2 = 1 \Rightarrow \boxed{d_2 = \frac{-1}{3}}$

$4d_2 - 3d_1 = 1$

$\Rightarrow -\frac{4}{3} - 3d_1 = 1 \quad \Rightarrow -\frac{4}{3} - 1 = 3d_1$

$\Rightarrow 3d_1 = -\frac{7}{3}$

$\Rightarrow \boxed{d_1 = \frac{-7}{9}}$

and $2d_1 + 4d_2 - 3d_0 = -1$

$\Rightarrow 2\left(\frac{-7}{9}\right) + 4\left(\frac{-1}{3}\right) - 3d_0 = -1$

$\Rightarrow -\frac{14}{9} - \frac{4}{3} + 1 = 3d_0$

$\Rightarrow \frac{-14 - 12 + 9}{9} = 3d_0$

$\Rightarrow -\frac{17}{9} \times \frac{1}{3} = d_0$

$\Rightarrow \boxed{d_0 = \frac{-17}{27}}$

$\therefore$ Particular Solⁿ is $S_n^{(p)} = \frac{-17}{27} + \left(\frac{-7}{9}\right)^n + \left(\frac{-1}{3}\right)^{n^2}$

$S_n^{(p)} = -\frac{n^2}{3} - \frac{7}{9}n - \frac{17}{27}$

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Q. Solve $S_n - S_{n-1} - 12S_{n-2} = (-3)^n + 6\cdot(4)^n$

Sol. The given relation is

$$S_n - S_{n-1} - 12S_{n-2} = (-3)^n + 6(4)^n \rightarrow (1)$$

The characteristic Eq^n is

$$x^2 - x - 12 = 0$$

$$x^2 - 4x + 3x - 12 = 0$$

$$x(x-4) + 3(x-4) = 0$$

$$(x+3)(x-4) = 0$$

$$\Rightarrow x = -3, 4$$

Since roots are different so hom. Sol^n is $S_n^{(h)} = C_1(-3)^n + C_2(4)^n$

To find particular Sol^n, we observe R.H.S of (1) Eq^n is combination of $(-3)^n$ & $(4)^n$.

But these terms already appear in hom. Sol^n.

$\therefore$ particular Sol^n of (1) Eq^n is assumed as

$$S_n = np_1(-3)^n + np_2(4)^n$$

$\therefore (1) \Rightarrow np_1(-3)^n + np_2(4)^n - [(n-1)p_1(-3)^{n-1} + (n-1)p_2(4)^{n-1}] - 12[(n-2)p_1(-3)^{n-2} + (n-2)p_2(4)^{n-2}] = (-3)^n + 6(4)^n$

$\Rightarrow np_1(-3)^n + np_2(4)^n - np_1(-3)^{n-1} + p_1(-3)^{n-1} - np_2(4)^{n-1} + p_2(4)^{n-1}$
$- 12np_1(-3)^{n-2} + 24p_1(-3)^{n-2} - 12np_2(4)^{n-2} + 24p_2(4)^{n-2}$

$$= (-3)^n + 6(4)^n$$

$\Rightarrow p_1(-3)^{n-2}\left[9n - \frac{n}{(-3)} + \frac{1}{(-3)} - 12n + 24\right] + p_2(4)^{n-2}$

$\left[\frac{n}{16} - \frac{n}{4} + \frac{1}{4} - 12n + 24\right] = (-3)^n + 6\cdot(4)^n$

$p_1(-3)^{n-2}\left[9n-(-3)n+(-3)-12n+24\right]+$

$p_2(4)^{n-2}\left[16n-4n+4-12n+24\right]=(-3)^n+6(4)^n$

$\Rightarrow p_1(-3)^{n-2}\left(\cancel{9n}+\cancel{3n}-3-\cancel{12n}+24\right)+p_2(4)^{n-2}\left[\cancel{16n}-\cancel{4n}+4-\cancel{12n}+24\right]$

$=(-3)^n+6\cdot(4)^n$

$\Rightarrow 21p_1(-3)^{n-2}+28\cdot p_2(4)^{n-2}=(-3)^n+6\cdot(4)^n$

Equate Coeff of $(-3)^n$, we get

$\frac{21p}{9}=1 \Rightarrow p=\frac{9}{21}=\frac{3}{7}$

Equate Coeff of $(4)^n$

$\frac{28p_2}{16}=6$

$\Rightarrow p_2=\frac{6\times 16}{28}=\frac{24}{7}.$

$\therefore$ particular sol is $S_n^{p}=\frac{3}{7}n(-3)^n+\frac{24}{7}n(4)^n$

$\therefore$ The general solⁿ of Eqⁿ (1) is

$S_n=S_n^{(h)}+S_n^{(p)}$

$=C_1(-3)^n+C_2(4)^n+\frac{3}{7}n(-3)^n+\frac{24}{7}n(4)^n.$

12 − 1.05
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Generating functions

Let $\langle S_n \rangle$ be a sequence ~~of~~ defined for $n \geq 0$ then infinite sum $G(S,z)$ where

$$G(S,z) = S_0 + S_1 z + S_2 z^2 + S_3 z^3 + \cdots \infty$$

$$= \sum_{n=0}^{\infty} S_n z^n$$ is called generating function or numeric funⁿ of the seqⁿ $\langle S_n \rangle$.

Generating function of Some Standard Sequences

(1) $S_n = a \; ; \; n \geq 0$

$$G(S,z) = \sum_{n=0}^{\infty} S_n z^n$$

$$= \sum_{n=0}^{\infty} a z^n$$

$$= a \sum_{n=0}^{\infty} z^n$$

$$= a\left[1 + z + z^2 + \cdots \infty\right]$$ which is infinite G.P. Series

$$\boxed{G(S,z) = a\left(\frac{1}{1-z}\right)}$$

$$\because \left(S_\infty = \frac{a}{1-r}\right)$$

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ii) $S_n = b^n, \; n \geq 0$

Since $G(S,z) = \sum_{n=0}^{\infty} S_n z^n$

$= \sum_{n=0}^{\infty} b^n z^n$

$= \sum_{n=0}^{\infty} (bz)^n$

$= \left[1 + bz + b^2 z^2 + \cdots \infty\right]$

$\because \left[S_\infty = \frac{a}{1-r}\right]$

$$\boxed{G(S,z) = \frac{1}{1-bz}}$$

iii) $S_n = c \cdot b^n$

$G(S,z) = \sum_{n=0}^{\infty} S_n z^n = \sum_{n=0}^{\infty} c \cdot b^n z^n$

$= c \sum_{n=0}^{\infty} (bz)^n$

$= c \left[1 + bz + b^2 z^2 + b^3 z^3 + \cdots\right]$

$$\boxed{G(S,z) = c\left(\frac{1}{1-bz}\right).}$$

iv) $S_n = n$

$G(S,z) = \sum_{n=0}^{\infty} S_n z^n$

$= \sum_{n=0}^{\infty} n z^n$

$= \left[z + 2z^2 + 3z^3 + 4z^4 + \cdots\right]$

$= z\left[1 + 2z + 3z^2 + 4z^3 + \cdots\right]$

$= z\left[(1-z)^{-2}\right] \rightarrow$ (formula of binomial $(1-z)^{-2}$)

$= \frac{z}{(1-z)^2}$

(v) $S_n = \frac{1}{n!}$

$$G(S,z) = \sum_{n=0}^{\infty} S_n z^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

$$= 1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \dots$$

$$\boxed{G(S,z) = e^z}$$

Method for Solution of Recurrence relation by Generating function.

Step 1: xply b/s of given recurrence relation by z^n & take summation from $n=k$ to ∞ where k is order of relation

Step 2: Express each term in form of $\sum_{n=0}^{\infty} S_n z^n$ we will get solution in term of $G(S,z)$.

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Q1) Solve Recurrence relation $S_n - 3S_{n-1} - 2 = 0$ $n \geq 1$ where $S_0 = 1$ by using generating fun.

Sol Given Eq^n. is $S_n - 3S_{n-1} = 2$
Xply b/s by z^n, we get

$$S_n z^n - 3S_{n-1} z^n = 2z^n$$

Take summation from $n=1$ to ∞ as order $= n-(n-1) = 1$

$$\therefore \sum_{n=1}^{\infty} S_n z^n - 3\sum_{n=1}^{\infty} S_{n-1} z^n = 2\sum_{n=1}^{\infty} z^n \rightarrow (1)$$

Consider $G(S,z) = \sum_{n=0}^{\infty} S_n z^n$

$$= S_0 + S_1 z + S_2 z^2 + S_3 z^3 + \cdots$$

$$G(S,z) = S_0 + \sum_{n=1}^{\infty} S_n z^n$$

$$\Rightarrow \sum_{n\geq 1}^{\infty} S_n z^n = G(S,z) - S_0$$

$$\therefore (1) \Rightarrow G(S,z) - S_0 - 3\left[\sum_{n\geq 1}^{\infty} S_{n-1} z^n \cdot \frac{z}{z}\right] = 2\left[z + z^2 + z^3 + \cdots\right]$$

$$\Rightarrow G(S,z) - 1 - 3z\sum_{n=1}^{\infty} S_{n-1} z^{n-1} = 2\left[\frac{z}{1-z}\right]$$

$\because \left(S_\infty = \frac{a}{1-r}\right)$ and $S_0 = 1$ given

$$\Rightarrow G(S,z) - 1 - 3z\,G(S,z) = \frac{2z}{1-z}$$

$$G(S,z)(1-3z) = \frac{2z}{1-z} + 1$$

$$\Rightarrow (1-3z)G(S,z) = \frac{2z+1-z}{1-z}$$

$$\Rightarrow (1-3z)\,G(S,z) = \frac{z+1}{1-z}$$

$$\Rightarrow G(S,z) = \frac{z+1}{(1-z)(1-3z)} \rightarrow (2)$$

consider
$$\frac{1+z}{(1-z)(1-3z)} = \frac{A}{1-z} + \frac{B}{1-3z} \rightarrow (3)$$

$$(1+z) = A(1-3z) + B(1-z) \rightarrow (4)$$

put $1-3z=0 \Rightarrow z = \frac{1}{3}$

$\therefore$ Eq^n (4) $\Rightarrow$
$$1+\frac{1}{3} = A\left(1-3\times\frac{1}{3}\right) + B\left(1-\frac{1}{3}\right)$$

$$\Rightarrow \frac{4}{3} = B\left(\frac{2}{3}\right)$$

$$\Rightarrow \boxed{B=2}$$

put $1-z=0 \Rightarrow z=1$

$\therefore$ (4) $\Rightarrow$
$$1+1 = A(1-3) + B(1-1)$$

$$\Rightarrow 2 = -2A$$

$$\Rightarrow \boxed{A=-1}$$

$\therefore$ (3) $\Rightarrow$
$$\frac{1+z}{(1-z)(1-3z)} = \frac{-1}{1-z} + \frac{2}{1-3z}$$

put this value in Eq^n no (2)

$$\therefore G(S,z) = \frac{-1}{1-z} + \frac{2}{1-3z}.$$

$$\therefore \left\{ G(S,z) = \frac{1}{1-az},\ S_n = a^n \right.$$

Hence required sol^n is

$$S_n = -(1)^n + 2\cdot(3)^n \ ; \ n \geq 0$$

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Q) find generating function for foll. relation

$S(n) - 6S(n-1) + 5S(n-2) = 0$ where $S(0)=1$ & $S(1)=2$

Sol. The Given relation is

$S(n) - 6S(n-1) + 5S(n-2) = 0 \rightarrow (1)$

Order $= n-(n-2) = 2$

Xply b/s of Eq^n (1) by z^n & take Summation for $n=2$ to ∞, we get

$$\sum_{n=2}^{\infty} S(n) z^n - 6\sum_{n=2}^{\infty} S(n-1) z^n + 5\sum_{n=2}^{\infty} S(n-2) z^n = 0 \rightarrow (2)$$

Consider $G(S,z) = \sum_{n=0}^{\infty} S(n) z^n$

$= S(0) + S(1)z + S_2(z)^2 + S_3(z)^3 + \ldots$

$= S(0) + zS(1) + \sum_{n=2}^{\infty} S_n (z)^n$

$\therefore \sum_{n=2}^{\infty} S(n) z^n = G(S,z) - 1 - 2z$ $\quad \because \{S(0)=1$ & $S(1)=2\}$ given

$\therefore$ Eq^n (2) becomes

$$G(S,z) - 1 - 2z - 6\left[z\sum_{n=1}^{\infty} S(n-1) z^{n-1} - S_0\right] + 5z^2\sum_{n=2}^{\infty} S(n-2) z^{n-2} = 0$$

$\Rightarrow G(S,z) - 1 - 2z - 6z\,[G(S,z) - 1] + 5z^2 G(S,z) = 0$

$\Rightarrow G(S,z)\,[1 - 6z + 5z^2] - 1 - 2z + 6z = 0$

$$\Rightarrow \boxed{G(S,z) = \frac{1-4z}{1-6z+5z^2}}$$

Ans

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(3) find gen. fun from relation
$S(n-2) = S(n-1) + S(n)$ where $S(0)=1$, $S(1)=1$; $n \geq 0$

Sol The given relation is
$$S(n) + S(n-1) - S(n-2) = 0 \rightarrow (1)$$

Order $n-(n-2) = 2$

Aply b/s of Eq^n (1) by z^n & take summation from $n=2$ to ∞

$$\sum_{n=2}^{\infty} S(n) z^n + \sum_{n=2}^{\infty} S(n-1) z^n - \sum_{n=2}^{\infty} S(n-2) z^n = 0$$

$$\left[\sum_{n=2}^{\infty} S(n) z^n + S(0) + S(1) z\right] - S(0) - S(1) z + z\left[\sum_{n=2}^{\infty} S(n-1) z^{n-1} + S(0) - S(0)\right] - z^2 \sum_{n=2}^{\infty} S(n-2) z^{n-2} = 0$$

$\because (S(0)=1 \ \& \ S(1)=1)$

$$\Rightarrow \sum_{n=0}^{\infty} S(n) z^n - 1 - z + z \sum_{n=1}^{\infty} S(n-1) z^{n-1} - z - z^2 \sum_{n=2} S(n-2) z^{n-2} = 0$$

$$\Rightarrow G(S,z) - 1 - z + z\,G(S,z) - z - z^2 G(S,z) = 0$$

$$\Rightarrow G(S,z)(1 + z - z^2) = 1 + 2z$$

$$\Rightarrow \boxed{G(S,z) = \frac{1+2z}{1+z-z^2}}$$

Ans

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Q) find closed form Exp. (Generating function) for terms of fibonacci sequence.

Sol. Consider Fibonacci sequence

$$F(n) = F(n-1) + F(n-2) \; ; \; n \geq 2 \rightarrow (1)$$

where $F(0) = 1$ & $F(1) = 1$

Its order $= n - (n-2) = 2$

Xply b/s by z^n & taking summation from $n = 2$ to ∞.

$$\therefore (1) \Rightarrow \sum_{n=2}^{\infty} F(n) z^n - \sum_{n=2}^{\infty} F(n-1) z^n + \sum_{n=2}^{\infty} F(n-2) z^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} F(n) z^n + [F(0) + F(1) z - F(0) - F(1) z] - \left\{ z \sum_{n=2}^{\infty} F(n-1) z^{n-1} \right\}$$

$$+ z^2 \sum_{n=2}^{\infty} F(n-2) z^{n-2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} F(n) z^n - 1 - z - \left[z \left\{ \sum_{n=2}^{\infty} F(n-1) z^{n-1} + F(0) - F(0) \right\} \right]$$

$$+ z^2 \sum_{n=2}^{\infty} F(n-2) z^{n-2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} F(n) z^n - 1 - z - z \sum_{n=1}^{\infty} F(n-1) z^{n-1} + z + z^2 \sum_{n=2}^{\infty} F(n-2) z^{n-2} = 0$$

$$\Rightarrow G(S,z) - 1 - \cancel{z} - z\, G(S,z) + \cancel{z} + z^2 G(S,z) = 0$$

$$G(S,z) (1 - z + z^2) = 1$$

$$\boxed{G(S,z) = \frac{1}{1 - z + z^2}} \quad \text{Ans}$$