Example 12. *Give an example of a graph that has an Euler circuit which is also a Hamiltonian circuit.*

Sol. The graph having an Euler circuit which is also a Hamiltonian circuit is shown in Fig. 11.131.

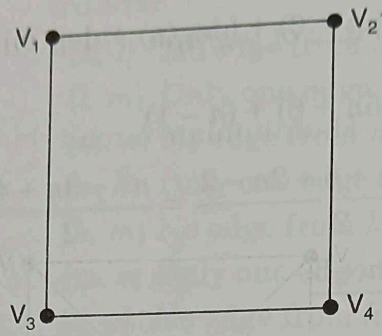

Fig. 11.131

In this graph V_1, V_2, V_3, V_4, V_1 is both an Euler circuit as well as Hamiltonian circuit. Since using this path, we can traverse both vertices and edges exactly once.

Example 13. *Give an example of a graph that has an Euler circuit and a Hamiltonian circuit, which are distinct.* (P.T.U., M.C.A. Dec. 2006)

Sol. The graph having an Euler circuit and a Hamiltonian circuit which are distinct is shown in Fig. 11.132.

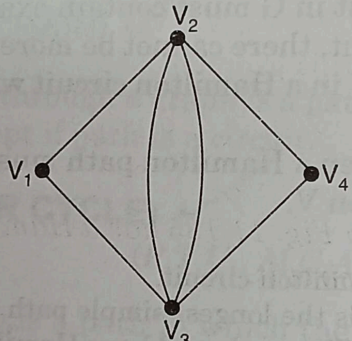

Fig. 11.132

The Euler circuit is V_1, V_3, V_2, V_3, V_4, V_2, V_1, which visits each edge exactly once.
The Hamiltonian circuit is V_1, V_2, V_4, V_3, V_1, which visits each vertex exactly once.

Example 14. *Give an example of a graph which has an Euler circuit but not a Hamiltonian circuit.*

Sol. The graph having an Euler circuit but not a Hamiltonian circuit is shown in Fig. 11.133.

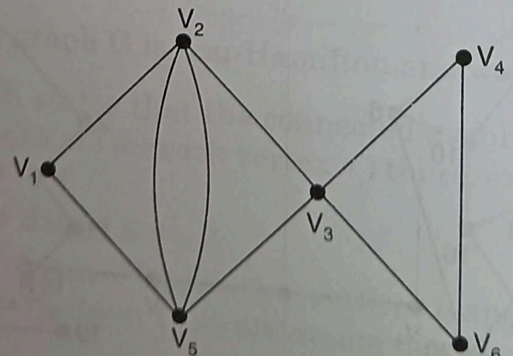

Fig. 11.133

The Euler circuit is $V_1, V_5, V_2, V_5, V_3, V_4, V_6, V_3, V_2, V_1$.

There is no Hamiltonian circuit. Since it is not possible to traverse each vertex of this graph exactly once.

Example 15. *Give an example of a graph which has a Hamiltonian circuit but not an Euler circuit.* (*P.T.U., M.C.A. May 2008*)

Sol. The graph having a Hamiltonian circuit but not an Euler circuit is shown in Fig. 11.134.

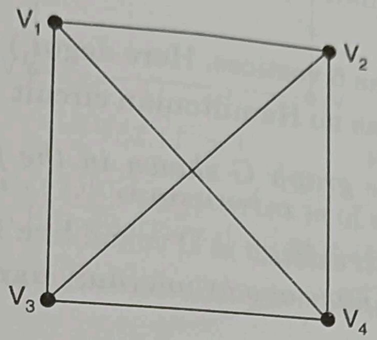

Fig. 11.134

The Hamiltonian circuit is V_1, V_2, V_4, V_3, V_1. There is no Euler circuit. Since it is not possible to traverse each edge of this graph exactly once.

Example 16. (*a*) *Give an example of a graph that has neither an Euler circuit nor a Hamiltonian circuit.*

(*b*) *Show that the graphs in Fig. 11.135 has a Hamiltonian circuit where as the graph in Fig. 11.136 has no Hamiltonian circuit.*

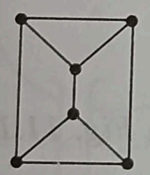

Fig. 11.135

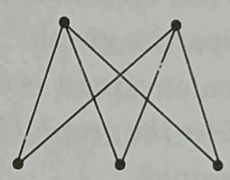

Fig. 11.136

Sol. (*a*) The graph having neither an Euler circuit nor a Hamiltonian circuit is shown in Fig. 11.137. It does not contain Euler circuit since each vertex is not of even degree.

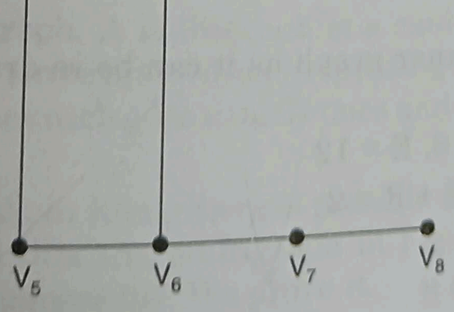

Fig. 11.137

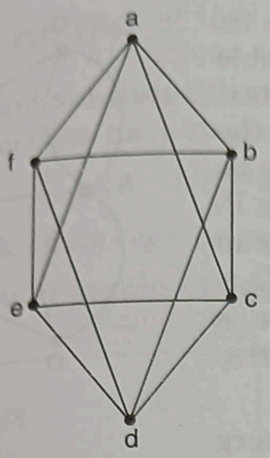

Fig. 11.141

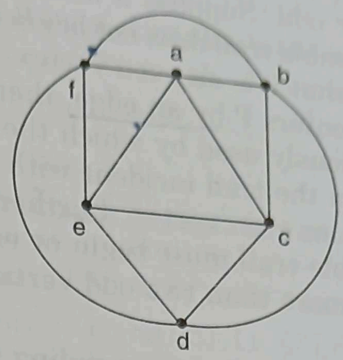

Fig. 11.142

(d) Given graph is 4-regular. $\therefore$ every vertex is of degree 4 (even)

$\therefore$ G has an Euler's circuit and hence G is an Eulerian graph.

Example 19. *State and prove Eulerian theorem on graph to show that Königsberg's graph is not proved to a solution.*

Sol. The word Königsberg is the name of a town, situated on the bank of a river, Pregel in Germany. This city has seven bridges. In 1736, L. Euler, the father of graph theory, proved that *it was not possible to cross each of the seven bridges once and only once in a walking tour.* A map of the Königsberg is shown in the following Fig. 11.143.

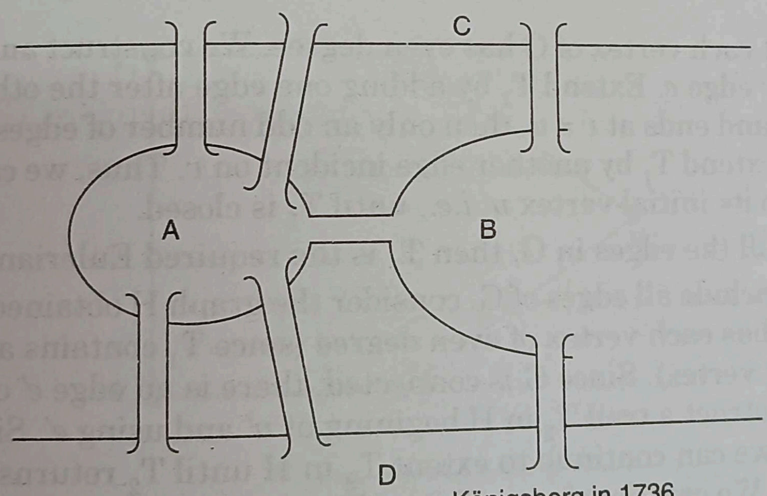

Fig. 11.143

Euler replaced the islands and the two sides of the river by points and the bridges by curves as shown in Fig. 11.144.

Figure 11.144 is a multigraph. A multigraph is a said to be **traversable** if it can be drawn without any breaks in the curve and without repeating any edges. *i.e.*, if there is a path which includes all vertices and uses each edge exactly once and such a path is called Travesable Trial.

According to Euler, the walk in Königsberg is possible iff the multigraph in Fig. 11.144 is traversable. But Euler proved that the multigraph in Fig. 11.144 is not traversable and hence the walk in Königsberg is impossible. We prove it.

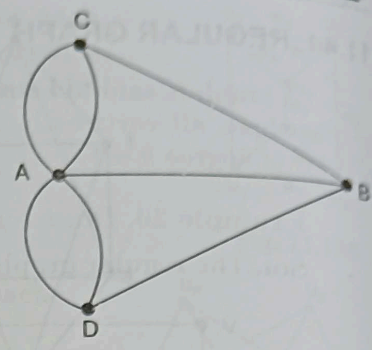

Fig. 11.144

We know that a vertex is even or odd according as its degree is even or odd. Suppose a multigraph is travesable and that a traversable trail does not begin or end at a vertex, say, P. We claim that P is an even vertex. For whenever the traversable trail enters P by an edge, there must always be an edge not previously used by which the trail can leave P. Thus, the edges in the trail incident with P must appear in pairs and so P is an even vertex. Further, if a vertex, Q is odd, the traversable trail must begin or end at Q. Hence, a multigraph with more than two odd vertices cannot be traversable.

Now the multigraph corresponding to the Königsberg bridge problem has four odd vertices. Thus, one cannot walk through Königsberg so that each bridge is crossed exactly once.

Euler actually proved the converse of the above statement, which is contained in the following theorem, called Euler theorem.

Theorem V. *A finite connected graph is Eulerian iff each vertex has even degree.*

(P.T.U., M.C.A. May 2008, B.Tech. Dec. 2012)

Proof. We know that a graph G is called an Eulerian graph if there exists a closed traversable Trial, called an Eulerian Trial. Suppose G is Eulerian and T is a closed Eulerian trial. Let v be any vertex of G. We show the vertex v is of even degree. Since the trail T enters and leaves the vertex v the same number of times without repeating any edge. $\therefore$ v is of even degree.

Conversly, Let each vertex of G has even degree. We construct an Eulerian Trial. Start with a trial T_1 at any edge e. Extend T_1 by adding one edge after the other. If T_1 is not closed *i.e.*, If T_1 begins at u and ends at $v \neq u$, then only an odd number of edges incident on v appear in T_1. Hence we can extend T_1 by another edge incident on v. Thus, we can continue to extend T_1 until T_1 returns to its initial vertex u. *i.e.*, until T_1 is closed.

If T_1 includes all the edges in G, then T_1 is the required Eulerian Trial.

If T_1 does not include all edges of G, consider the graph H obtained by deleting all edges of T_1 from G. Now H has each vertex of even degree (since T_1 contains an even number of the edges incident on any vertex). Since G is connected, there is an edge e' of H which has an end point u' in T_1. We construct a trail T_2 in H begining at u' and using e'. Since all the vertices in H have even degree, we can continue to extent T_2 in H until T_2 returns to u' as shown in the following Fig. 11.145. We can clearly put T_1 and T_2 together to form a larger closed trial in G. Proceeding the above process until all the edges of G are used, we finally obtain an Eulerian trial and hence G is Eulerian.

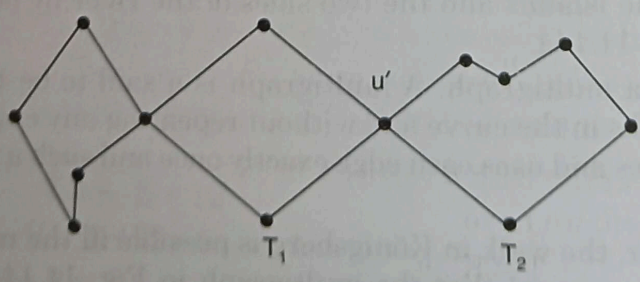

Fig. 11.145