Relations

Ordered pair: let A and B be any two sets then by an ordered pair of elements, we mean a pair (x,y) where $x \in A$, $y \in B$.

for e.g. $(1,2)$, $(3,6)$ $(1,9)$

Cartesian product of sets

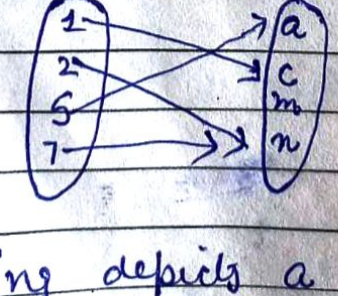
Let A and B be any two non empty sets then the cartesian product of the sets A & B is the set of all ordered pairs (x,y) such that $x \in A$, $y \in B$ & it is denoted by $A \times B$.

$\therefore$ $A \times B = \{(x,y) : x \in A \text{ and } y \in B\}$

e.g. If $A = \{1,2\}$ & $B = \{3,4,5\}$ then $A \times B$ is $\{(1,3)\ (1,4)\ (1,5)\ (2,3)\ (2,4)\ (2,5)\}$

Relation:

A set of ordered pairs is defined as a relation



This mapping depicts a relation from a

set A into set B.
A relation from A to B is a subset of $A \times B$.
The ordered pairs are $(1,c)$ $(2,n)$ $(5,a)$ $(7,n)$ where set $\{1,2,5,7\}$ represents the domain
set $\{c,n,a\}$ represents the ~~domain~~ range.

Domain and Range of a relation

Let R be a relation from set A to set B. Then domain of R is the set of all first coordinates of the ordered pairs belonging to R.
Illy Range is set of all second coordinates of ordered pairs belonging to R.

for e.g. consider $A = \{1,3,5,7\}$ & $B = \{2,4,6,8,10\}$
& $R = \{(1,8), (3,6)\ (5,2), (1,4)\}$ be a relation from A to B. then
Domain of $R = \{1,3,5\}$
Range of $R = \{2,4,6,8\}$

Inverse Relation: Let R be a relation from set A to set B. then the inverse relation is a relation from B to A.
It is denoted by R^{-1}.
Thus $R = \{(a,b) : a \in A,\ b \in B\}$ then

$$R^{-1} = \{(b,a) : b \in B,\ a \in A\}$$

Also Domain of R^{-1} = Range of R
& Range of R^{-1} = Domain of R.

Types of Relations

1) **Reflexive relation**: A relation R on a set A is said to be reflexive relation if every element of A is related to itself.

Thus R is reflexive iff $(x, x) \in R \ \forall x \in A$.

A relation R on a set A is not reflexive if there is an element $x \in A$ such that $(x, x) \notin R$.

for eg. consider $A = \{1, 2, 3\}$ then the relation R_1 defined by $R_1 = \{(1,1), (2,2), (3,3), (1,3), (2,1)\}$ is a reflexive relation on A.

Also $R_2 = \{(1,1)\ (3,3)\ (2,1)\ (3,2)\}$ is not reflexive relation as $(2,2) \notin R_2$.

2) **Symmetric relation**: let A be any set then the relation R on the set A is called symmetric iff $(x, y) \in R \Rightarrow (y, x) \in R$ $\forall x, y \in A$.

for Eg. If $A = \{1, 2, 3, 4\}$ and define relation R_1 & R_2 as

$$R_1 = \{(1,3)\ (1,4)\ (3,1)\ (2,2), (4,1)\}$$

$$R_2 = \{(1,1)\ (2,2)\ (3,3)\ (1,3)\}$$

In R_1; $(1,3) \in R_1 \Rightarrow (3,1) \in R_1$

$(1,4) \in R_1 \Rightarrow (4,1) \in R_1$

& $(2,2) \in R_1 \Rightarrow (2,2) \in R_1$

$\therefore$ R_1 is a symmetric relation on set A.

In R_2 $(1,3) \in R_2$ but $(3,1) \notin R_2$

Thus R_2 is not a sym. relation.

3) Transitive relation : let A be any set. Then a relation R is said to be transitive iff

$(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R \ \forall x,y,z \in A$.

for E.g. If $A = \{1,2,3,4\}$ and define a relation $R = \{(2,1)\ (3,1)\ (3,2)\ (4,1)\ (4,2)\ (4,3)\}$

R is transitive as

$(3,2)$ and $(2,1) \in R \Rightarrow (3,1) \in R$

$(4,3)$ and $(3,1) \in R \Rightarrow (4,1) \in R$

$(4,2)$ and $(2,1) \in R \Rightarrow (4,1) \in R$

$(4,3)$ and $(3,2) \in R \Rightarrow (4,2) \in R$.

Consider set A of straight lines in a plane. define a relation "is parallel to" on A.

take $l_1, l_2, l_3 \in A$

Then l_1 is $\parallel$ to l_2 & l_2 to $\parallel$ to l_3

$\Rightarrow$ l_1 is $\parallel$ to l_3.

$\therefore$ relation parallel to is transitive.

If $A = \{1,2,3,4\}$
and $R_1 = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,4), (4,1)\ (4,4)\}$

R_1 is not transitive as $(3,4) \in R$ and $(4,1) \in R$
but $(3,1) \notin R$.

Examples:

Q1) Let R be a relation on $A = \{2,3,4,5,6\}$ defined by "x is relative prime to y". Write R as set of ordered pairs.

Sol. We know two integers (x,y) are said to be prime iff $(x,y) = 1$ i.e gcd of x & y is 1.

$\therefore R = \{(2,3)\ (2,5)\ (3,4)\ (3,5)\ (4,5)\ (5,6)\}$

Q) Let R be the Relation on set $X = \{0,1,2,3 \dots\}$ of non negative integers defined by the ~~relation~~ Eqn $x^2+y^2 = 25$. Write R as set of ordered pairs.

Sol Given $x^2+y^2 = 25 \Rightarrow y^2 = 25 - x^2$

$x = 0 \Rightarrow y = 5$
$x = 3 \Rightarrow y = 4$
$x = 4 \Rightarrow y = 3$
$x = 5 \Rightarrow y = 0$
$\therefore R = \{(0,5)\ (3,4)\ (4,3)\ (5,0)\}$

Date / /
Page No.

ii) Let S be a relation on a set N of +ve integers defined by Equation $3x+4y=17$. Write set S as set of ordered pairs.

Sol Since $3x+4y=17$

$\Rightarrow 3x = 17-4y$

If $y=1 \Rightarrow x = \frac{11}{3} \notin Z$

If $y=2 \Rightarrow x = 3$

If $y=3 \Rightarrow x = \frac{5}{3} \notin Z$

If $y=4 \Rightarrow x = \frac{1}{3} \notin Z$

If $y=5 \Rightarrow x = -1$ not a +ve int.

$\therefore S = \{(2,3)\}$

iii) Let R be a relation on a set N of +ve integers defined by $x^2+2y=100$ find domain & Range of R.

Sol Since $x^2+2y=100$

$2y = 100-x^2$

If $x=1 \Rightarrow y = \frac{99}{2} \notin Z$

$x=2 \Rightarrow y = \frac{96}{2} = 48$

$x=3 \Rightarrow y = \frac{91}{2} \notin Z$

$x=4 \Rightarrow y = \frac{84}{2} = 42$

$x=5 \Rightarrow y = \frac{75}{2}$

$x=6 \Rightarrow y = \frac{64}{2} = 34$

$x=7 \Rightarrow y = \frac{51}{2}$

$x=8 \Rightarrow y = \frac{36}{2} = 18$

$x=9 \Rightarrow y = \frac{19}{2}$

$x=10 \Rightarrow y = 0$

$\therefore R = \{(2,48)\ (4,42),\ (6,32)\ (8,18)\}$

$\therefore$ Domain $= \{2,4,6,8\}$

Range $= \{32,18,42,48\}$

6

Antisymmetric relation

Let A be any set then a relation R on a set A is said to be antisymmetric iff $(x,y) \in R$ & $(y,x) \in R \rightarrow \boxed{x = y}$, $\forall x, y \in A$

for eg. If $A = \{1, 2, 3, 4\}$

and $R_1 = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\}$

$R_2 = \{(1,1)\ (2,2)\ (3,3)\}$ $\cancel{(4,4)\}}$

R_1 is not antisymmetric as $(1,2) \in R_1$ & $(2,1) \in R_1$ but $1 \neq 2$.

R_2 is an antisymmetric relation i.e. Identity relation is antisymmetric.

Equivalence relation:

Let A be any set then a Relation R on a set A is said to be an Equivalence relation on A iff

(i) It is reflexive

(ii) It is symmetric

(iii) It is transitive

e.g. If $A = \{a, b, c\}$ & $R = \{(a,a)(b,b)\ (c,c)\ (a,b)\ (b,a)\}$

Then R is an Equivalence relation on A.

Th: If R and S are two Equivalence relations on a set A, then (a) $R \cap S$ is also equivalence relation on A.

(b) $R \cup S$ may or may not be an equivalence relation on A.

Sol. Given R is a relation on A. It means R is a subset of $A \times A$.

i.e. $R \subseteq A \times A \rightarrow ①$

Also $S \subseteq A \times A \rightarrow ②$

from Eqⁿ ① & ②

$R \cap S \subseteq A \times A$

i.e. $R \cap S$ is a subset of $A \times A$

$\therefore$ $R \cap S$ is a relation on A.

Now we shall show that $R \cap S$ is an equivalence relation on A.

(i) Reflexivity: let x be an arbitrary Element of A.

As R and S are equivalence relations on A.

$\therefore$ R and S are reflexive relations on A.

If $x \in A \Rightarrow (x,x) \in R$ and $(x,x) \in S$

$\Rightarrow (x,x) \in R \cap S \;\; \forall x \in A$.

Hence $R \cap S$ is reflexive on A.

Symmetry: let $x, y \in A$ such that $(x,y) \in R \cap S$.

$\Rightarrow (x,y) \in R$ and $(x,y) \in S$

as R and S are Symmetric relations on A.

$\therefore (y,x) \in R$ & $(y,x) \in S$

$\Rightarrow (y,x) \in R \cap S$.

Date / /
Page No.

$\therefore (x,y) \in R\cap S \Rightarrow (y,x) \in R\cap S \quad \forall\ x,y \in A$

$\therefore R\cap S$ is symmetric on A.

iii) Transitivity: let $x,y,z \in A$ s.t.
$(x,y) \in R\cap S$, $(y,z) \in R\cap S$
We shall show that $(x,z) \in R\cap S$.

Now $(x,y) \in R\cap S$ & $(y,z) \in R\cap S$
$\Rightarrow (x,y) \in R$ and $(x,y) \in S$
Also $(y,z) \in R$ and $(y,z) \in S$
Also R and S are transitive on A
$\therefore (x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$
Also $(x,y) \in S$ & $(y,z) \in S \Rightarrow (x,z) \in S$

$\Rightarrow (x,z) \in R\cap S$
$\therefore R\cap S$ is transitive on A
Hence $R\cap S$ is an Equivalence relation on A.

(b) Now we shall show that $R\cup S$ may or may not be equivalence relation on A.

consider $A = \{a,b,c\}$
Define relation R and S by

$R = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}$
$S = \{(a,a)\ (b,b)\ (c,c)\ (b,c)\ (c,b)\}$
clearly R and S are Equivalence relation on A.
but $R\cup S = \{(a,a)\ (b,b)\ (c,c)\ (a,b)\ (b,a)$
$(b,c)\ (c,b)\}$

Here $(a,b) \in R \cup S$, $(b,c) \in R \cup S$
but $(a,c) \notin R \cup S$
$\therefore$ $R \cup S$ is not transitive.
Hence $R \cup S$ can't be equivalence relation on A.

~~Tute~~ Partial order relation : A relation R on a set A is called partial relation if it is
(i) reflexive
(ii) Anti-symmetric
(iii) Transitive

for E.g. Define a relation $\subseteq$ on A set.
We shall show that $\subseteq$ is a partial relation on A.
(i) ~~Let~~ $A \subseteq A$ for any set A. i.e. $\subseteq$ is reflexive
ii) If $A \subseteq B$ & $B \subseteq A$ then $A = B$ i.e. $\subseteq$ is anti symmetric
iii) If $A \subseteq B$ & $B \subseteq C$ then $A \subseteq C$ i.e. $\subseteq$ is transitive

$\therefore$ The relation $\subseteq$ is partial order relation.

Identity Relation : let A be any set. Then the relation $R = \{(x,x) : x \in A\}$ on A is called identity relation on A. Thus in identity relation, every element is related to itself only.

for eg: $A = \{a, b, c\}$ & define two relations R_1 & R_2
$R_1 = \{(a,a)\ (b,b)\ (c,c)\}$
& $R_2 = \{(a,a)\ (b,b)\ (c,c)\ (a,b)\}$
$\therefore$ R_1 is identity relation on A but R_2 is not an identity relation as a is related to a & b.

Date / /
Page No.

Empty or void relation → Let A be any set. Thus $\phi \subset A \times A$ & hence it is a relation on A. This relation is called void or Empty relation.

Universal relation: Let A be any set. Then $A \times A \subseteq A \times A$ & hence it is a relation on A. This relation is called universal relation on A.

Q (a) Give an Example to show that a reflexive relation on a set A is not necessarily symmetric.

(b) Prove that a relation R on a set A is symmetric iff $R = R^{-1}$.

Sol. Consider $A = \{1, 2, 3, 4\}$

and define a relation R by

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,2)\}$ ~~(3,4)~~

Here $(x,x) \in R \ \forall\ x \in A$

$\therefore$ The relation R is reflexive.

But $(1,2) \in R \Rightarrow (2,1) \notin R$

$\therefore$ Relation R is not symmetric.

(b) Let R is symmetric on A

We shall show that $R = R^{-1} \rightarrow \underline{R \subseteq R^{-1}}$ & $R^{-1} \subseteq R$

Let $(x,y) \in R \ \forall\ x, y \in A$

$\Rightarrow (y,x) \in R$ $\because$ (R is symmetric)

$\Rightarrow (x,y) \in R^{-1}$ [By definition of inverse relation]

$\Rightarrow$

$\therefore (x,y) \in R \Rightarrow (x,y) \in R^{-1}$

$\therefore$

$\Rightarrow \boxed{R \subseteq R^{-1}} \rightarrow (1)$

...ritsar-Jalandhar, G.T. Road, Amritsar-143001 (Pb.) INDIA
Tel : 0183-5069530, 5069532, 5069534, 5069527, 5010423

$(x,y) \in R \Rightarrow (y,x) \in R$

Date / /

Page No.

Again let $(x,y) \in R^{-1}$ $\forall$ $x,y \in A$

$\Rightarrow$ $(y,x) \in R$ (by def of inverse relation)

$\Rightarrow$ $(x,y) \in R$. $\because$ (R is symmetric)

$\therefore$ $(x,y) \in R^{-1} \Rightarrow (x,y) \in R$

$\Rightarrow$ $\boxed{R^{-1} \subseteq R}$ $\rightarrow$ (2)

from (1) & (2) $\boxed{R = R^{-1}}$

converse : let $R = R^{-1}$.

We shall show that R is symmetric

let $(x,y) \in R$

$\Rightarrow$ $(x,y) \in R^{-1}$ $\because$ $[R = R^{-1}]$

$\Rightarrow$ $(y,x) \in R$ $\forall$ $x,y \in A$ (By def of inverse relation)

$\therefore$ R is symmetric.

Q) Each of following defines a relation on the set N of +ve integers

$R: x > y$

$S: x+y = 10$

$T: x + 4y = 10$ for all $x,y \in N$.

Determine which of ~~following~~ relations are

(i) reflexive ii) Symmetric iii) Transitive

iv) Anti Symmetric

Sol: Reflexive : R, S and T are not reflexive.

for eg. $(1,1) \notin R, S$ and T.

ii) Symmetric:

Let $x=3$ and $y=6$
Since $y>x$ but $x \not> y$
$\therefore$ R is not symmetric

If $(x,y) \in S$ then $x+y=10$
$\Rightarrow y+x=10$
$\Rightarrow (y,x) \in S$
Hence S is Symmetric.

Also if $x=6$ & $y=1$ then $x+4y=10$ holds
i.e. $(6,1) \in T$ but $(1,6) \notin T$ $\because (1+24 \neq 10.)$
$\therefore$ T is not symmetric.

(c) Transitive:
Let $(x,y), (y,z) \in R$
$\therefore x>y$ and $y>z$
$\Rightarrow x>z$
$\therefore (x,z) \in R \quad \forall x,y,z \in N$
$\therefore$ R is transitive.

$x=5$ $y=4$ $z=2$
$5>4$
$4>2$

Let $(3,7) \in S$ since $3+7=10$ and $(7,3) \in S$ as $7+3=10$
but $(3,3) \notin S$ as $3+3=6 \neq 10$
$\therefore$ S is not transitive.

further if $(x,y) \in T$ & $(y,z) \in T$
it means $x+4y=10$ & $y+4z=10$

Consider $x+4z = x+10-y$
$$= x+10-\left(\frac{10-x}{4}\right)$$
$$= \frac{4x+40-10+x}{4}$$

Amritsar-Jalandhar, G.T. Road, Amritsar

$= \frac{5x+30}{4} \neq 10$

$\therefore (x,2) \notin T$

$\therefore$ T is not transitive.

(d) <u>Antisymmetric</u>: if $x>y$ & $y>x$ then $x=y$ $\forall x,y \in N$

$\therefore$ R is antisymmetric.

let $(2,8) \in S$ & $(8,2) \in S$ but $2 \neq 8$

$\therefore$ S is not antisym.

If $(x,y) \in T$ then $x+4y=10$

$(y,x) \in T$ then $y+4x=10$

$\therefore \quad x+4y = y+4x$

$\Rightarrow \quad 3y = 3x \Rightarrow x=y \quad \forall x,y \in N$

$\therefore$ T is antisym.

Q) Let A be set of integers & $\sim$ be relation on $A \times A$ defined by $(a,b) \sim (c,d)$ if $a+d = b+c$. Prove $\sim$ is an Equivalence relation.

Sol. <u>Reflexive</u>: $(a,b) \sim (a,b)$ if $a+b = b+a$ which is true

Hence $\sim$ is reflexive

<u>Sym</u>: let $(a,b) \sim (c,d) \Rightarrow a+d = b+c$

or $c+b = d+a$

$\Rightarrow (c,d) \sim (a,b)$

$\therefore$ $\sim$ is symmetric

<u>Transitive</u>: let $(a,b) \sim (c,d)$ & $(c,d) \sim (e,f)$

$\Rightarrow a+d = b+c$ & $c+f = d+e$

Adding; $a+\cancel{d}+\cancel{c}+f = b+\cancel{c}+\cancel{d}+e \Rightarrow a+f = b+e$

$\Rightarrow (a,b) \sim (e,f) \Rightarrow$ $\sim$ is transitive

hence $\sim$ is Equivalence relation.

Assym

Asymmetric relation → A relation R is called asymmetric relation if for every $(x,y) \in R$ ⇒ $(y,x) \notin R$ $\forall x,y \in A$
for E.g. $A = \{1,2\}$
$R = \{(1,1)\ (2,2)\ (1,2)\}$
Relation R is asymmetric as $(1,2) \in R$ but $(2,1) \notin R$

Irreflexive: A relation R is called irreflexive if for every $x \in A$, $(x,x) \notin R$
for Eg. $A = \{1,2\}$ & $R = \{(1,2)\ (2,1)\}$
R is irreflexive since $(x,x) \notin R$ $\forall x \in A$.

I. Imp

Q. What are Properties of relations. Explain with Examples.

There are many properties of relation. These properties tell us about the nature & type of relations. The following are main properties.

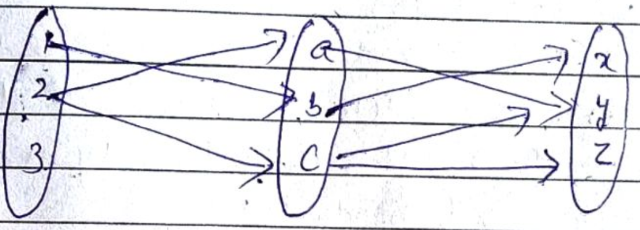
- Reflexive relation
- Irreflexive relation
- Symmetric Relation
- Asymmetric relation
- Antisymmetric relation
- Transitive relation

Explain Each def. with Example

Composition of relations: Let R and S are two relations from sets A to B & B to C resp. Then a relation SoR is called Composite relation from A to C where $(a,c) \in SoR$ iff we can find $b \in B$ s.t $(a,b) \in R$ & $(b,c) \in S$.
The relation SoR is read as Composition of R & S.

e.g Let $A=\{1,2,3\}$ & $B=\{a,b,c\}$, $C=\{x,y,z\}$
Consider Relation R from A to B & S from B to C
s.t $R=\{(1,b)\ (2,a)\ (2,c)\}$; $S=\{(a,y)\ (b,x)\ (c,y)\ (c,z)\}$

To find RoS draw arrow diagram from R to S.

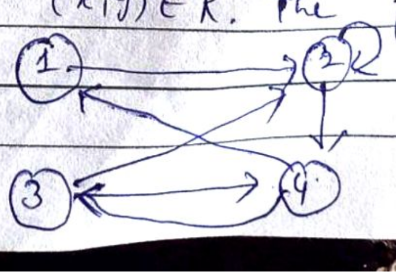


Since there is an arrow from 1 to b & b to x
$\therefore (1,x) \in RoS$.
|||ly $(2,y)\ (2,z) \in RoS$

$\therefore RoS = \{(1,x)\ (2,y)\ (2,z)\}$

Directed graph or Digraph of a relation

Consider relation R on $A=\{1,2,3,4\}$ defined as
$R=\{(1,2)\ (2,2)\ (2,4)\ (3,2)\ (3,4)\ (4,1)\ (4,3)\}$
Draw an ~~Elemen~~ arrow from an Element x to Element y when $(x,y) \in R$. The following diagram show directed graph.



Date / /
Page No.

a) Consider the set Z of integers and an integer $m > 1$. We say that x is Congruent to y modulo m written as $x \equiv y \pmod m$ if $x - y$ is divisible by m or $x - y = km$ for some integer k. Show that $\equiv$ is an Equivalence relation on Z.

Sol (i) Reflexive : Since $x - x = 0$ is divisible by m

$\therefore x \equiv x \pmod m \ \forall x \in Z$

$\therefore \equiv$ is relflexive.

ii) Symmetric : Let $x \equiv y \pmod m$

$\Rightarrow x - y$ is divisible by m

$\Rightarrow x - y = km$ for some int k.

or ~~$-(x-y)$~~ $-(y - x) = km$

$\Rightarrow y - x = -km$

$\Rightarrow y - x = rm$ where $r = -k \in Z$

$\therefore y - x$ is divisible by m

$\Rightarrow y \equiv x \pmod m$

$\therefore \equiv$ is symmetric.

iii) Transitive. let $x \equiv y \pmod m$ & $y \equiv z \pmod m$

$\therefore x - y$ and $y - z$ are divisible by m.

$\Rightarrow x - y + y - z$ is also divisible by m

$\Rightarrow x - z$ is divisible by m

$\Rightarrow x \equiv z \pmod m$

$\therefore \equiv$ is transitive.

Hence $\equiv$ is an Equivalence relation.