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Normal Subgroup → A subgroup H of a group G is called normal subgroup of G if $\forall\ g \in G,\ h \in H, \Rightarrow ghg^{-1} \in H$

or

A subgroup H of a group G is called a normal subgroup of G iff for $g \in G$ we have $gHg^{-1} = H,\ \forall\ g \in G$.

Imp (10 Marks)

Q) Let G be the group of two by two invertible matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $ad - bc \neq 0$. Let $H = \left[\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \neq 0 \right]$. Then H is a normal subgroup of G.

Sol. We first show that H is a subgroup of G.

Let $h_1, h_2 \in H$ s.t.

$$h_1 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}; \quad h_2 = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix};$$

$a \neq 0,\ a_1 \neq 0$.

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Now $h_1 h_2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix}$

$= \begin{pmatrix} aa_1 & 0 \\ 0 & aa_1 \end{pmatrix} \in H$

i.e. H is closed under matrix multiplication

further $A \in H$, we have

$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$\Rightarrow |A| = \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = a^2$

also $A_{11} = a$; $A_{12} = 0$ $A_{21} = 0$

$A_{22} = a$

$\therefore \text{adj } A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}^t = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

Hence $\overset{-1}{A} = \dfrac{\text{adj } A}{|A|}$

$= \dfrac{1}{a^2} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \in H$; $a \neq 0$

So Each Element belonging to H has multiplicative inverse.

$\therefore$ H is subgroup of G.

further for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$,

$h = \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} \in H$

consider $ghg^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$= \begin{pmatrix} a^2 & ba \\ ca & da \end{pmatrix} \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a^2d-bac}{ad-bc} & \frac{-a^2b+a^2b}{ad-bc} \\ \frac{cad-dac}{ad-bc} & \frac{-cab+da^2}{ad-bc} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a(ad-bc)}{ad-bc} & 0 \\ 0 & \frac{a(ad-bc)}{ad-bc} \end{pmatrix}$$

$= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in H$

Hence H is normal subgroup of G under matrix multiplication.

Th^m The intersection of any number of normal subgroups of G is a normal subgroup of G.

P.Sol: let H_1, H_2, H_3 ---- be collection of normal subgroups of group G.

Let $H = \bigcap_{i=1}^{\infty} H_i$

We show that H is normal subgroup of G.

Since intersection of any collection of subgroups is a subgroup.

$\therefore$ H is a subgroup of G.

We shall show that H is normal subgroup of G.

Let $g \in G$, $h \in H$

$\Rightarrow h \in \bigcap_{i=1}^{\infty} H_i$

$\Rightarrow h \in H_i \ \forall \ i$

But H_i is normal in G.

$\therefore \ g h g^{-1} \in H_i \ \forall \ i$

$\Rightarrow g h g^{-1} \in \bigcap_{i=1}^{\infty} H_i = H$

$\Rightarrow g h g^{-1} \in H$

$\Rightarrow$ H is normal subgroup of G.

Quotient Group :- let G be a group and H is a normal subgroup of G. Let G/H denotes the right (left) cosets of H in G. Then G/H is a group called the quotient group or factor group under coset multiplication defined by

$$\boxed{(aH)(bH) = abH}$$

A group whose all elements are integral power of one or more elements called cyclic.

or

cyclic group → A group G is called cyclic if for some $a \in G$, every element $x \in G$ is of the form a^n for some $n \in \mathbb{Z}$. ~~Then~~ the element a is called generator of G.

If G is cyclic we write $G = \langle a \rangle$

for e.g. If $G = \{1, -1, i, -i\}$ then G is cyclic group generated by i

Since $i^1 = i$, $i^2 = -1$, $i^3 = -i$ $i^4 = 1$.

i.e. Every element of G is of the form i^n for some $n \in \mathbb{Z}$

$\therefore$

Hence i is a generator for the cyclic group.

Thm: Every cyclic group is abelian.

Pf: Let G be a cyclic group with a as its generator.

i.e. let $G = \langle a \rangle$ and $g_1 \in G$

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then $g_1 = a^r$ for some $r \in Z$

Let $g_2 \in G$ then $g_2 = a^s$ for some $s \in Z$.

Consider $g_1 g_2 = a^r \cdot a^s$

$= a^{r+s}$

$= a^{s+r}$ $\because$ [$r+s = s+r$ as Z is abelian]

$= a^s \cdot a^r$

$= g_2 g_1$

$\therefore \boxed{g_1 g_2 = g_2 g_1}$

$\therefore$ G is abelian.

Q) Consider the group $G = \{1, 2, 3, 4, 5, 6\}$ under multiplication modulo 7.

a) find multiplication table of G.

b) find 2^{-1}, 3^{-1}, 6^{-1}

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(c) find the order and subgroup generated by 2 & 3.

d) Is G Cyclic?

Sol. By definition $a \times_7 b$ = Then remainder when ab is divided by 7.

for eg. $5 \times_7 6 = 30 = 2$ [when 30 is divided by 7 remainder is 2]

(i) Multiplication table of G

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(b) The identity element of G is 1.
(as 1st row in table is identical with top most row)

$\therefore 2^{-1} = 4$ (as intersection of 2 & 4 is 1 in table)

$3^{-1} = 5$

$6^{-1} = 6$

(c) Since $2^1 = 2$

$2 \times_7 2 = 4$

$2 \times_7 2 \times_7 2 = 8 = 1$

$o(2) = 3$

Hence $\langle 2 \rangle$ = the subgroup generated by $2 = \{1, 2, 4\}$

$7\overline{)81}$ 11
-7
11
-7

Also $3^1 = 3$

$3 \times_7 3 = 9 = 2$

$3 \times_7 3 \times_7 3 = 27 = 6$

$3 \times_7 3 \times_7 3 \times_7 3 = 81 = 4$

$3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 =$

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Also $3^1 = 3$

$3 \times_7 3 = 9 = 2$

$3 \times_7 3 \times_7 3 = 27 = 6$

$3 \times_7 3 \times_7 3 \times_7 3 = 81 = 4$

$3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 = 243 = 5$

$3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 = 729 = 1$

~~$3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 =$~~

$O(3) = 6$

$\therefore$ the subgroup generated by 3 is

$$\langle 3 \rangle = \{1, 2, 3, 4, 5, 6\} = G$$

(d) Since a group G is cyclic if there exists an element $a \in G$ s.t. $O(a) = O(G)$

Now since $O(3) = O(G) = 6$

$\Rightarrow$ G is cyclic.

Q Let $G = \{1, 5, 7, 11\}$ under multiplication modulo 12.

(i) Find multiplication table of G

(ii) Find order of each element

(iii) Is G cyclic

Group Homomorphism

A mapping from a group $(G, \cdot)$ into a group $(\bar{G}, *)$ is said to be group homomorphism if

$$\phi(a \cdot b) = \phi(a) * \phi(b) \quad \forall\, a, b \in G$$

Group Isomorphism → A homomorphism ϕ which is one-one & onto is called isomorphism & the group G & G' are called isomorphic denoted as $G \cong G'$

A homomorphism which is onto is called epimorphism.

A homomorphism which is one-one called monomorphism.

Kernel f

If f is homomorphism of G to $\bar{G}$ then Kernel f is a set defined as

$$\operatorname{Ker} f = \{x \in G : f(x) = \bar{e},\ \bar{e} \in \bar{G}\}$$

Th^m: Let $f : G \to G'$ is a group homomorphism then

(i) $f(e) = e'$, $e \in G$, $e' \in G'$

(ii) $f(a^{-1}) = (f(a))^{-1}$, $\forall a \in G$

Pf: Given $f : G \to G'$ is a homomorphism from G to G'.

for $x \in G$, consider

$$\begin{aligned} f(x)e' &= f(x) && \because [e' \text{ is identity of } G'] \\ &= f(xe) && (f \text{ is homomorphism}) \\ &= f(x) f(e) \end{aligned}$$

$\Rightarrow \quad e' = f(e) \quad$ (by left cancellation law)

$\therefore \boxed{f(e) = e'}$

from part (a)

$$f(e) = e'$$

$\Rightarrow$ $e' = f(e)$

$= f(a\bar{a}^{1})$ $\quad \because f$ is homomorphism

$= f(a) f(\bar{a}^{1})$

$\Rightarrow$ $f(a) f(\bar{a}^{1}) = e'$

$\Rightarrow$ $(f(a))^{-1} f(a) f(\bar{a}^{1}) = (f(a))^{-1} e'$

$\cancel{f(a)}\cancel{(f(a))^{-1}}$ Cancelled

$\Rightarrow$ $\boxed{f(\bar{a}^{1}) = (f(a))^{-1}}$

Hence proved