Ch - I Fourier Series

Q1. What is Fourier Series?

Ans The Fourier Series is representation of periodic signal in complex exponentials (Cosine + Sine wave form) or Sinusoid (Sine wave form)

Q2. What are Fourier Coefficients?

Ans. The terms which consist of the fourier series along with their sine or cosine values.

Q3. Who is the father of Fourier Series (Discover)

Ans. French mathematician Jean Baptiste Joseph Fourier Discover Fourier Series. And also researched by ~~Bods~~ Leonhard Euler.

Q4. Periodic Function.

Ans. Any Function $f(x)$ is said to be periodic function if it repeat its value after a regular interval of time i.e. $f(x) = f(x+T) = f(x+2T) = \ldots$ Then $f(x)$ is periodic function of period 'T'. 'T' is also called time interval of periodic function. For eg. Sinx, Cosx are periodic function of period '2π'

Also $\tan x$, $\cot x$ are periodic function 'π'.

5. Even and Odd functions.

Even function:- Any function $f(x)$ is said to be Even function if $f(-x) = f(x)$. For eg

$$f(x) = x^2, \cos x, |x| \text{ etc}$$

$$f(-x) = (-x)^2, \cos(-x), |-x|$$

$$= x^2, \cos x, |x| \text{ etc}$$

$$= f(x)$$

$\therefore$ $f(x)$ is even function.

Odd function:- Any function $f(x)$ is said to be Odd function if $f(-x) = -f(x)$

for eg $f(x) = \sin x, x^3$ etc

$$f(-x) = \sin(-x), (-x)^3$$

$$= -\sin x, -x^3 = -f(x)$$

$\therefore$ $f(x)$ is odd function.

Note: $\boxed{\text{Check Even or odd Function only when limit is } -\pi \le x \le \pi}$

Q6. Fourier Series representation for any periodic function. $f(x)$ in Full range* in interval $(c, c+2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where a_0, a_n, b_n are called Euler Constants.

Where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$

4

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x).\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x) \sin nx\, dx$$

These are known as Euler formula for Fourier series.

Note(I) Full range :→ If period of the function $f(x)$ is 2π = upper limit - lower li

There are two type of limits $0 \le x \le 2\pi$ & $-\pi \le x \le \pi$.

(ii) Half range :→ If period of the function $f(x)$ is π = upper limit - lower limi

Limit in this case like $0 \le x \le \pi$

Note(ii) For Even Function $b_n = 0$

(iii) For Odd Function $a_0, a_n = 0$

Q. Dirichelt's Conditions.

Ans: To express any function $f(x)$ in Fourier series. $f(x)$ has to satisfies fallowing Conditions which are known as Dirichelt Conditions

(i) $f(x)$ should be finite, single + periodic.

(ii) $f(x)$ should have finite points of maxima + minima.

(iii) $f(x)$ should have finite number of Discontinuiti

8.

Obtain Fourier series for function $f(x) = \frac{1}{4}(\pi - x)^2$ in interval $(0, 2\pi)$. Also deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

A:- The given function is $f(x) = \frac{1}{4}(\pi - x)^2$

$0 \le x \le 2\pi$.

Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Now

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4}(\pi - x)^2\, dx$$

$$= \frac{1}{4\pi} \left[\frac{(\pi - x)^3}{3(-1)} \right]_0^{2\pi} = \frac{1}{4\pi} \left[\left\{ \frac{-\pi^3}{-3} \right\} - \left\{ \frac{\pi^3}{-3} \right\} \right]$$

$$\therefore a_0 = \frac{1}{4\pi} \left[\frac{\pi^3}{3} + \frac{\pi^3}{3} \right] = \frac{1}{\cancel{4}_2 \cancel{\pi}} \left[\frac{\cancel{2}\pi^{\cancel{3}\,2}}{3} \right] = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) . \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4} (\pi - x)^2 . \cos nx\, dx$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\pi - x)^2 . \cos nx\, dx$$

(2)

$$a_n = \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi-x)(-1)\cdot\left(\frac{-\cos nx}{n^2}\right) + 2(-1)(-1)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\cdot\frac{\sin nx}{n} - 2(\pi-x)\frac{\cos nx}{n^2} + \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left\{0 - 2(-\pi)\cdot\frac{1}{n^2} + 0\right\} - \left\{0 - \frac{2\pi\cdot 1}{n^2} + 0\right\}\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{1}{\cancel{4\pi}}\left[\frac{4\pi}{n^2}\right]$$

$$a_n = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\cdot\sin nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$=$ (Solve it)

$=$

$b_n = 0$

$0, \pi, -\pi, 2\pi$

Angles for Full range Fourier Series

$\sin 0 = 0$

$\sin(\text{any }\pi) = 0$

ie $\sin\pi = \sin 2\pi = \cdots$

$\sin n\pi = 0$

$\sin\frac{n\pi}{2}$ hold it. (it may be 0, 1, or -1

$\cos 0 = 1$

$\cos(\text{any even }\pi) = 1$

$\cos 2\pi = \cos 4\pi = \ldots \cos 2n\pi = 1$

$\cos(\text{odd }\pi) = -1$

$\cos\pi, \cos 3\pi, \ldots = -1$

$\cos\frac{n\pi}{2}$ hold it

$\cos n\pi = (-1)^n$, $\cos(n+1)\pi = (-1)^{n+1}$, $\cos(n-1)\pi = (-1)^{n-1}$

Now Fourier Series for $f(x)$ is (7)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{1}{2}\left(\frac{\pi^2}{6}\right) + \sum_{n=1}^{\infty} \frac{1}{n^2}\cdot \cos nx + 0 \qquad (\because b_n = 0)$$

$$\therefore f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \quad \text{--- (2)}$$

$$\Rightarrow f(x) = \frac{\pi^2}{12} + \left[\frac{\cos 1x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} + \cdots\right] \quad \text{--- (3)}$$

<u>Deducting</u>

For $x = 0$

$$f(0) = \frac{1}{4}(\pi - 0)^2 = \frac{\pi^2}{12} + \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{\pi^2}{12} + \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{3\pi^2 - \pi^2}{12}$$

$$= \frac{2\pi^2}{12} = \frac{\pi^2}{6} \quad \text{--- (4)}$$

For $x = \pi$

$$f(\pi) = \frac{1}{4}(\pi - \pi)^2 = \frac{\pi^2}{12} + \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\Rightarrow 0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \quad \text{--- (5)}$$

Adding (4) + (5), we get

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{2\pi^2 + \pi^2}{12} = \frac{3\pi^2}{12} = \frac{\pi^2}{4}$$

2: Obtain Fourier series for function $f(x) = |x|, \; -\pi < x < \pi$

1: The given function is

$$f(x) = |x| \qquad -\pi < x < \pi$$

[Check for even or odd function]

Now $f(-x) = |-x| = |x| = f(x)$

$\therefore$ $f(x)$ is even function.

$\therefore \quad b_n = 0$

$\therefore$ Fourier series is

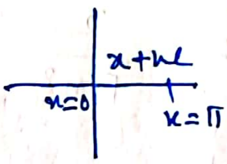
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad — (1)$$

Now ~~f(x) =~~ $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_{0}^{\pi} |x|\,dx$$

[$\because$ of property of even or odd functions $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$]

$$= \frac{2}{\pi}\int_{0}^{\pi} x\,dx$$



$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi^2 - 0\right]$$

$$a_0 = \pi$$

If $|x| = l$ then $x = \pm l$

If $x > 0$ then $x = l$

& if $x < 0$ then $x = -l$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\,dx \quad — (2)$$

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$$a_n = \frac{2}{\pi}\int_0^{\pi} |x| \cdot \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cdot \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x\left\{\frac{\sin nx}{n}\right\} - 1\left\{-\frac{\cos nx}{n^2}\right\}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 + \frac{(-1)^n}{n^2}\right\} - \left\{0 + \frac{1}{n^2}\right\}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n + 1\right]$$

Case I when n is even

then $(-1)^n = 1$

$$\therefore a_n = \frac{2}{\pi n^2}[1+1] = \frac{4}{\pi n^2}$$

[not nec to Sal

Case II when n is odd

then $(-1)^n = -1$

$$a_n = \frac{2}{\pi n^2}[-1+1] = 0$$

$\therefore$ From (1), Fourier Series is

$$\therefore f(x) = \frac{a_0}{2} + [a_1 \cos 1x + a_2 \cos 2x + a_3 \cos 3x + \dots]$$

$$= 1(\pi) + [0 + \frac{4}{} \cos 2x + 0 + 4 \cos 4x$$

$$\therefore f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \cdots\right]$$ (10)

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3. Obtain Fourier series $f(x) = e^{-ax}$, $-\pi < x < \pi$

A: The Fourier series for $f(x) = e^{-ax}$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— (1)}$$

Now $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{-a} - \left(\frac{e^{a\pi}}{-a}\right)\right]$$

$$a_0 = \frac{1}{a\pi}\left[-e^{-a\pi} + e^{a\pi}\right] = \frac{1}{a\pi}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$\therefore a_0 = \frac{1}{a\pi} 2\sinh a\pi \qquad \left[\because \sinh x = \frac{e^{x} - e^{-x}}{2}\right.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}.\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left\{-a\cos nx + n\sin nx\right\}\right]_{-\pi}^{\pi}$$

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$$\therefore a_n = \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}\{-a(-1)^n+0\} - e^{a\pi}\{-a(-1)^n-0\}\right]$$

Reduction for

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx+b\sin bx\right]$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx-b\cos bx\right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[-a(-1)^n e^{-a\pi}+a(-1)^n e^{a\pi}\right]$$

$$= \frac{a(-1)^n}{\pi(a^2+n^2)}\left[e^{a\pi}-e^{-a\pi}\right]$$

$$a_n = \frac{2a(-1)^n}{\pi(a^2+n^2)}\sinh a\pi$$

$b_n =$ Try yourself

Questions for practice

(I) obtain Fourier series for $f(x) = \frac{1}{2}(\pi - x), 0<x$

(II) obtain Fourier series for $f(x) = x^2 - x, \; -\pi < x < \pi$

(III) obtain Fourier series for $f(x) = x^2 + x, \; -\pi < x <$

(IV) obtain Fourier series for $f(x) = e^{-x}, \; 0 < x <$

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12

Obtain Fourier series for $x\sin x$, $-\pi \le x \le \pi$

Sol: Given $f(x) = x\sin x$, $-\pi \le x \le \pi$

Now $f(-x) = -x\sin(-x)$

$= -x\{-\sin x\}$

$= x\sin x$

$= f(x)$

limit is $-\pi \le x \le \pi$ so you can check function for even or odd function. It may be or not may be

$\therefore f(x)$ is even function

$\therefore b_n = 0$

$\therefore$ Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \underset{I}{x}\,\underset{II}{\sin x}\,dx$ ($\because$ of property of even & odd function)

$$= \frac{2}{\pi}\left[x\{-\cos x\} - 1\{-\sin x\}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\{-\pi(-1)+0\} - \{0+0\}\right]$$

$$= \frac{2}{\cancel{\pi}}\left[\cancel{\pi}\right] = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x.\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\{2\cos nx.\sin x\}\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \underset{\mathrm{I}}{x}\underbrace{\{\sin(n+1)x - \sin(n-1)x\}}_{\mathrm{II}}\,dx \quad (*)$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}, \ n \neq 1$$

$$= \frac{1}{\pi}\left[\left\{\pi\left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right) - (0+0)\right\} - \left\{0 - \cancel{1}(0+0)\right\}\right] \quad n \neq 1.$$

$$= \frac{1}{\pi}\left[\pi\left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right)\right], \ n \neq 1$$

$$a_n = -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}, \ n \neq 1$$

Case I when n is even then $n+1$ and $n-1$ are odd and $(-1)^{odd} = -1$

$$\therefore a_n = -\frac{(-1)}{n+1} + \frac{-1}{n-1} = \frac{1}{n+1} - \frac{1}{n-1}$$

$$= \frac{\cancel{n}-1-\cancel{n}-1}{(n+1)(n-1)}, \quad n \neq 1$$

$$a_n = \frac{-2}{n^2-1}, \ n \neq 1$$

Case II when n is odd then $n+1$ & $n-1$ are ~~odd~~ even & $(-1)^{even} = 1$

$$\therefore a_n = -\frac{1}{n+1} + \frac{1}{n-1} = \frac{-\cancel{n}+1+\cancel{n}+1}{(n+1)(n-1)}$$

$$= \frac{2}{n^2-1}, \quad n \neq 1$$

When $n=1$

From (*), put $n=1$

$$\therefore a_1 = \frac{1}{\pi}\int_0^{\pi} x\{\sin 2x - 0\}\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$a_1 = \frac{1}{\pi}\left[\left\{-\pi\cdot\frac{1}{2} + 0\right\} - \{0+0\}\right] = \frac{1}{\cancel{\pi}}\left[-\frac{\cancel{\pi}}{2}\right]$$

$$a_1 = -\frac{1}{2}$$

$\therefore$ From (1), we get

$$f(x) = \frac{a_0}{2} + a_1\cos 1x + \left[a_2\cos 2x + a_3\cos 3x + a_4\cos 4x + \dots \infty\right]$$

$$= \frac{1}{2}(2) - \frac{1}{2}\cos 1x + \left[\frac{-2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x - \frac{2}{4^2-1}\cos 4x + \dots\right]$$

$\therefore$ Fourier series for $f(x)$ is

$$f(x) = 1 - \frac{1}{2}\cos 1x \cancel{+} 2\left[\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \dots\infty\right]$$

Eg Obtain Fourier series for $|\sin x|$, $-\pi \le x \le \pi$

Sol Given $f(x) = |\sin x|$, $-\pi \le x \le \pi$

$$f(-x) = |\sin(-x)| = |-\sin x| = |\sin x|$$
$$= f(x)$$

$\therefore$ $f(x)$ is even function

$\therefore$ $b_n = 0$

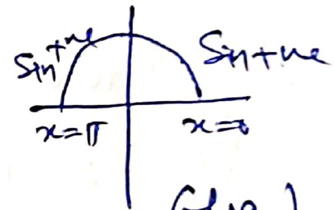
& Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |\sin x|\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$



$$= \frac{2}{\pi}\Big[-\cos x\Big]_0^{\pi} = \frac{2}{\pi}\left[-(-1)+1\right]$$

(Fig)

$$= \frac{2}{\pi}\left[1+1\right] = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x|.\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x.\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\cos nx.\sin x\,dx$$

$$\therefore\ a_n = \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx \quad \text{—} \ (*)$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}, \quad n \neq 1$$

$$= \frac{1}{\pi}\left[\left\{-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right\} - \left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\}\right], \quad n \neq 1$$

$$\therefore\ a_n = \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right], \quad n \neq 1 \quad \text{—} \ (2)$$

Two cases arises

<u>Case I</u> when n is even, then $n+1$ & $n-1$ are odd then $(-1)^{odd} = -1$

$$\therefore\ a_n = \frac{1}{\pi}\left[-\frac{(-1)}{n+1} + \frac{-1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right], \quad n \neq 1$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n+1} - \frac{2}{n-1}\right], \quad n \neq 1$$

$$a_n = \frac{2}{\pi}\left[\frac{\cancel{n}-1-\cancel{n}-1}{(n+1)(n-1)}\right] = \frac{-4}{\pi(n^2-1)}, \quad n \neq 1$$

Case II when n is odd then $n+1$ & $n-1$ are even & $(-1)^{even} = 1$

$$\therefore\ a_n = \frac{1}{\pi}\left[\cancel{-\frac{1}{n+1}} + \cancel{\frac{1}{n-1}} + \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n-1}}\right], \quad n \neq 1$$

$$a_n = 0$$

when $n=1$, from ⊛

$$a_1 = \frac{1}{\pi}\int_0^{\pi} (\sin 2x - 0)\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{1}{2}+\frac{1}{2}\right] = 0$$

$\therefore$ From (1), Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos x + \left[a_2\cos 2x + a_3\cos 3x + a_4\cos 4x + \cdots\right]$$

$$= \frac{1}{2}\left(\frac{4}{\pi}\right) + 0.\cos x + \left[\frac{-4}{\pi(2^2-1)}\cos 2x + 0 + \frac{-4}{\pi(4^2-1)}\cos 4x + \cdots\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right]$$

Q obtain Fourier series for $f(x) = |\cos x|, -\pi \le x \le \pi$

Sol: Given $f(x) = |\cos x|, \ -\pi \le x \le \pi$

$$f(-x) = |\cos(-x)| = |\cos x| = f(x)$$

$\therefore$ $f(x)$ is even function

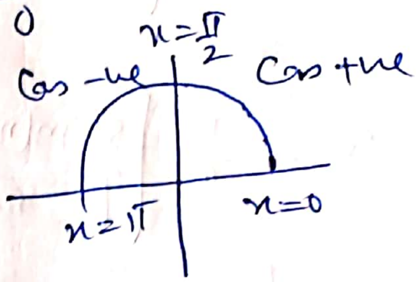
$\therefore b_n = 0$

$\therefore$ Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx$

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2}|\cos x|\,dx + \int_{\pi/2}^{\pi}|\cos x|\,dx\right]$$



$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\sin x\right)_0^{\frac{\pi}{2}} - \left(\sin x\right)_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[(1-0)-(0-1)\right] = \frac{2}{\pi}\left[1+1\right] = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\,dx$$

$$= \frac{1}{\pi}\cdot 2\int_0^{\pi}|\cos x|\cdot\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi}(-\cos x)\cdot\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} 2\cos nx\cos x\,dx - \int_{\frac{\pi}{2}}^{\pi} 2\cos nx\cos x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}}\{\cos(n+1)x+\cos(n-1)x\}\,dx - \int_{\frac{\pi}{2}}^{\pi}\{\cos(n+1)x+\cos(n-1)x\}\,dx\right]$$

*

$$a_n = \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_0^{\frac{\pi}{2}} - \left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{\frac{\pi}{2}}^{\pi}\right]$$

$$a_n = \frac{1}{\pi}\left[\left\{\left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right) - (0+0)\right\} - \left\{(0+0) - \left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)}{n-1}\right.\right.\right.$$, $n \neq 1$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} + \frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right], \quad n \neq 1$$

$$a_n = \frac{1}{\pi}\left[\frac{2\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{2\sin(n-1)\frac{\pi}{2}}{n-1}\right], \quad n \neq 1$$

$$a_n = \frac{2}{\pi}\left[\frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{2} + \cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{n+1} + \frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{2} - \cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{n-1}\right], \quad n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{0 + \cos\frac{n\pi}{2}\cdot 1}{n+1} + \frac{0 - \cos\frac{n\pi}{2}\cdot 1}{n-1}\right], \quad n \neq 1$$

$$\therefore a_n = \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]\cos\frac{n\pi}{2}$$

$$= \frac{2}{\pi}\left[\frac{\cancel{n}-1 - \cancel{n}-1}{(n+1)(n-1)}\right]\cos\frac{n\pi}{2}$$

$$a_n = \frac{-4}{\pi(n^2-1)}\cdot\cos\frac{n\pi}{2}, \quad n \neq 1$$

When $n=1$, From $*$ we get

$$a_1 = \frac{1}{\pi}\left[\int_0^{\pi/2}(\cos 2x + 1)\,dx - \int_{\pi/2}^{\pi}(\cos 2x + 1)\,dx\right]$$

$$\therefore a_1 = \frac{1}{\pi}\left[\left(\frac{\sin 2x}{2}+x\right)_0^{\frac{\pi}{2}} - \left(\frac{\sin 2x}{2}+x\right)_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\left(0+\frac{\pi}{2}\right)-(0+0)\right\} - \left\{(0+\pi)-\left(0+\frac{\pi}{2}\right)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \frac{\pi}{2}\right] = \frac{1}{\pi}\left[\frac{\pi - 2\pi + \pi}{2}\right] = 0$$

$\therefore$ From (1), Fourier series is

$$\therefore f(x) = \frac{a_0}{2} + a_1 \cos 1x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{1}{2}\left(\frac{4}{\pi}\right) + 0 + \sum_{n=2}^{\infty} \frac{-4}{\pi(n^2-1)} \cos\frac{n\pi}{2} \cdot \cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

Eg:- obtain Fourier series for $f(x) = \sqrt{1-\cos x}$, $-\pi \le x \le \pi$.

Sol: Given $f(x) = \sqrt{1-\cos x}$, $-\pi \le x \le \pi$

$$f(-x) = \sqrt{1-\cos(-x)}, \quad -\pi \le x \le \pi$$

$$= \sqrt{1-\cos x}$$

$$= f(x)$$

$\therefore f(x)$ is even function

$\therefore b_n = 0$

∴ Fourier Series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{— (1)}$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \sqrt{1-\cos x}\, dx = \frac{2}{\pi} \int_0^{\pi} \sqrt{2\sin^2 x/2}\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sqrt{2} \sin\frac{x}{2}\, dx$$

$$a_0 = \frac{2\sqrt{2}}{\pi} \left[\frac{-\cos\frac{x}{2}}{1/2} \right]_0^{\pi} = \frac{4\sqrt{2}}{\pi} [0 + 1]$$

$$= \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sqrt{1-\cos x} \cdot \cos nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sqrt{2\sin^2\frac{x}{2}} \cos nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sqrt{2} \sin x/2 \cdot \cos nx\, dx$$

$$= \frac{2\sqrt{2}}{\pi} \int_0^{\pi} \sin\frac{x}{2} \cos nx\, dx$$

$$= \frac{\sqrt{2}}{\pi} \int_0^{\pi} 2\cos nx \cdot \sin\frac{x}{2}\, dx$$

$$a_n = \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(n+\tfrac{1}{2}\right)x - \sin\left(n-\tfrac{1}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)x}{n-\frac{1}{2}}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\left\{0+0\right\} - \left\{\frac{-1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right\}\right]$$

$$\boxed{\begin{array}{l}\cos\left(n+\frac{1}{2}\right)\pi = 0 \\ \cos\left(n-\frac{1}{2}\right)\pi = 0\end{array}}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{2}{2n+1} - \frac{2}{2n-1}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{\cancel{2n}-1-\cancel{2n}-1}{(2n+1)(2n-1)}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{-2}{4n^2-1}\right] = \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

$\therefore$ From ①, fourier series is

$$f(x) = \frac{1}{2}\left(\frac{4\sqrt{2}}{\pi}\right) + \sum_{n=1}^{\infty}\frac{-4\sqrt{2}}{\pi(4n^2-1)}\cdot\cos nx$$

$$= \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx$$