6.2. EULER'S METHOD

Consider the reaction $$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \quad ...(1)$$

Let $y = g(x)$ be the solution of (1). Let $x_0, x_1\ (= x_0 + h), x_2\ (= x_0 + 2h)$,... be equidistant values of x.

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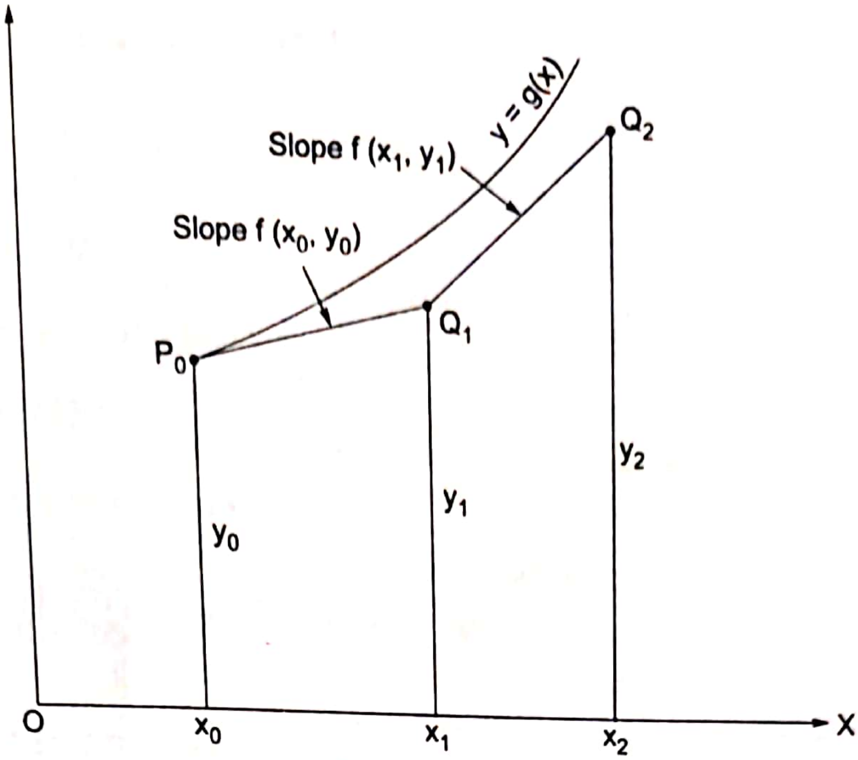

In a small interval, a curve is nearly a straight line. This is the property used in Euler's method.

The equation of the tangent of P_0 (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{P_0} \cdot (x - x_0)$$

or

$$y = y_0 + f(x_0, y_0).\,(x - x_0) \qquad \ldots(2) \qquad \left[\because \frac{dy}{dx} = f(x, y)\right]$$

This gives the y co-ordinate of any point on the tangent. Since, the curve is approximated by the tangent in the interval (x_0, x_1) the value of y on the curve corresponding to $x = x_1$ is given by the above value of y in (2) approximately.

$\therefore$ Putting $x = x_1 = x_0 + h$ in (2), we have $\mathbf{y_1 = y_0 + hf\,(x_0, y_0)}$

Thus Q_1 is (x_1, y_1). Similarly, approximating the curve in the next interval (x_1, x_2) by a line through Q_1 (x_1, y_1) whose slope is $\left(\frac{dy}{dx}\right)_{Q_1} = f(x_1, y_1)$, we get $\mathbf{y_2 = y_1 + hf\,(x_1, y_1)}$

Repeating this process, we have $y_3 = y_2 + hf(x_2, y_2)$; $y_4 = y_3 + hf(x_3, y_3)$ and so on.

In general, $\mathbf{y_{n+1} = y_n + h\,f\,(x_n, y_n)}$. This is called *Euler's algorithm*.

Note 1. In Euler's method, the curve of actual solution $y = g(x)$ is approximated by a sequence of short li...

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Example 1. *Using Euler's method, solve for y at x = 0.1 from*

$$\frac{dy}{dx} = x + y + xyy\ (0) = 1 \quad \textit{taking step size } h = 0.025.$$

Solution. The given differential equation is $\frac{dy}{dx} = x + y + xy,\ y(0) = 1$

Here, $f(x, y) = x + y + xy$

The initial condition is $x_0 = 0, y_0 = 1$. Also $h = 0.025$

Euler's algorithm is $y_{n+1} = y_n + hf(x_n, y_n)$ ...(1)

From (1), for $n = 0, x = x_0$ we have $y_1 = y_0 + 0.025 f(x_0, y_0)$

$= 1 + 0.025 f(0, 1) = 1 + 0.025\ (0 + 1 + 0 \times 1) = 1.025$

Next, we have $x_1 = x_0 + h = 0 + 0.025 = 0.025$

∴ From (1), for $n = 1$, we have $y_2 = y_1 + 0.025 f(x_1, y_1)$

$= 1.025 + 0.025\ f(0.025, 1.025)$

$= 1.025 + 0.025\ (0.025 + 1.025 + 0.025 \times 1.025) = 1.0518$

Next, we have $x_2 = x_1 + h = 0.025 + 0.025 = 0.05$

∴ From (1), for $n = 2$, we have $y_3 = y_2 + 0.025 f(x_2, y_2)$

$= 1.0518 + 0.025 f(0.05, 1.0518)$

$= 1.0518 + 0.025\ (0.05 + 1.0518 + 0.05 \times 1.0518) = 1.0806$

Next, we have $x_3 = x_2 + h = 0.05 + 0.025 = 0.075$

∴ From (1), for $n = 3$, we have

$y_4 = y_3 + 0.025 f(x_3, y_3)$

$= 1.0806 + 0.025 f(0.075, 1.0806) = 1.0806 + 0.025$

$= (0.075 - 1.0806 + 0.075 \times 1.0806) = 1.1115$

Next, we have $x_4 = x_3 + h = 0.075 + 0.025 = 0.1$

When, $x = x_4$, we get the value of y_5 which is not needed

$y_4 = 1.1115$, when $x_4 = 0.1$

The computation work can be conveniently carried out in the Table form as follows :

n	x_n	y_n	$f(x_n, y_n)$	$hf(x_n, y_n)$	$y_{n+1} = y_n + hf(x_n, y_n)$
0	0	1	1	0.025	1.025
1	0.025	1.025	1.0756	0.0268	1.0518
2	0.050	1.0518	1.1543	0.0288	1.0806
3	0.075	1.0806	1.2366	0.0309	1.1115
4	0.1	1.1115			

Example 2. *Find y(0.1) by Euler's method where* $\frac{dy}{dx} = y - 2x^2$ *; y (0) = 1.* **(P.T.U., May 2005)**

Solution. Given $f(x, y) = y - 2x^2$ $x_0 = 0, y_0 = 1$

Take $h = 0.025$

Euler's method is $y_{n+1} = y_n + h f(x_n, y_n)$

$$y_1 = y(0.025) = y_0 + (0.025)[y_0 - 2x_0^2] \qquad \because\ y_1 = y(x_0+h)$$

$$= 1 + (0.025)(1-0) = 1.025$$

$\therefore$

$$x_1 = x_0 + h = 0.025,\ y_1 = 1.025$$

Now.

$$y_2 = y(x_1+h) = y(0.050) = y_1 + h(y_1 - 2x_1^2) = 1.025 + (0.025)[(1.025 - 2(0.025)^2)]$$

$$= 1.0505937$$

$$y_3 = y(x_2+h) = y(0.075) = y_2 + h\left(y_2 - 2x_2^2\right) = 1.0505937 + (0.025)[1.0505937 - 2(0.050)^2]$$

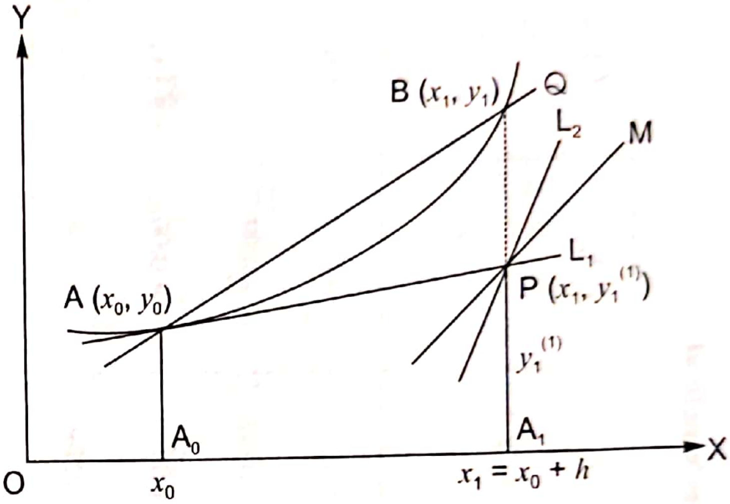
$$= 1.0767335$$

$$y_4 = y(x_3+h) = y(0.1) = y_3 + h\left(y_3 - 2x_3^2\right) = 1.0767335 + (0.025)[1.0767335 - 2(0.075)^2]$$

$$= 1.1033705$$

Hence. $y(0.1) = 1.1033705.$

6.3. IMPROVED EULER'S METHOD



Here, we consider a line passing through A (x_0, y_0) whose slope is the average of the slopes at A(x_0, y_0) and P $(x_1, y_1^{(1)})$, such that

$$y_1^{(1)} = y_0 + hf(x_0, y_0).$$

In the adjoining figure let AL_1 be the tangent to the curve at A (x_0, y_0) and PL_2 be the line through P$(x_1, y_1^{(1)})$ having the slope $f(x_1, y_1^{(1)})$. Now PM is the line having slope $\frac{1}{2}\left\{f(x_0, y_0) + f(x_1, f_1')\right\}$. *i.e.*, average of two slopes $f(x_0, y_0)$ and $f(x_1, y_1^{(1)})$.

The line AQ through (x_0, y_0) and parallel to PM is used to approximate the curve. Then, ordinate of the point B will give the value of y_1.

$\therefore$ Equation to ABQ is

$$y - y_0 = (x - x_0)\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\} \qquad ...(1)$$

As we are assuming that $A_1B = y_1$, so co-ordinates of B will be (x_1, y_1). This point will lie on AQ

$\therefore$

$$y_1 - y_0 = (x_1 - x_0)\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

or

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

$$= y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + h, f(x_0, y_0)]\} \qquad ...(2)$$

In general, we have the formula

$$y_{n+1} = y_n + \frac{h}{2}\{f(x_n, y_n) + f[x_n + h, y_n + h\, f(x_n, y_n)]\}$$

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where $x_n - x_{n-1} = h$.

$$y_{n+1} = y_n + \frac{h}{2}\{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\}, \text{ where } y^*_{n+1} = y_n + hf(x_n, y_n)$$

Note. The value of y^*_{n+1} is same as that of y_{n+1} in Euler's formula.

Example 1. *Use Euler's method and improved Euler's method to approximate y when x = 0.1 given that* $\frac{dy}{dx} = \frac{y-x}{y+x}$ *; y (0) = 1 by taking h = 0.02.*

Solution. Here, $f(x, y) = \frac{dy}{dx} = \frac{y-x}{y+x}$

$$x_0 = 0, y_0 = 1 \quad \text{and} \quad h = 0.02.$$

We can conveniently use table form to apply Euler's method and improved Euler's method.

(i) **Euler's Method:**

n	x_n	y_n	$f(x_n, y_n) = \frac{y_n - x_n}{y_n + x_n}$	$hf(x_n, y_n)$	$y_{n+1} = y_n + hf(x_n, y_n)$
0	$x_0 = 0$	$y_0 = 1$	$f(x_0, y_0) = 1$	0.02	$y_1 = 1.02$
1	$x_1 = 0.02$	$y_1 = 1.02$	$f(x_1, y_1) = 0.9615$	0.01923	$y_2 = 1.0392$
2	$x_2 = 0.04$	$y_2 = 1.0392$	$f(x_2, y_2) = 0.9175$	0.01835	$y_3 = 1.0576$
3	$x_3 = 0.06$	$y_3 = 1.0576$	$f(x_3, y_3) = 0.8926$	0.017852	$y_4 = 1.0755$
4	$x_4 = 0.08$	$y_4 = 1.0755$	$f(x_4, y_4) = 0.8606$	0.017212	$y_5 = 1.0927$
5	$x_5 = 0.1$	$y_5 = 1.0927$			

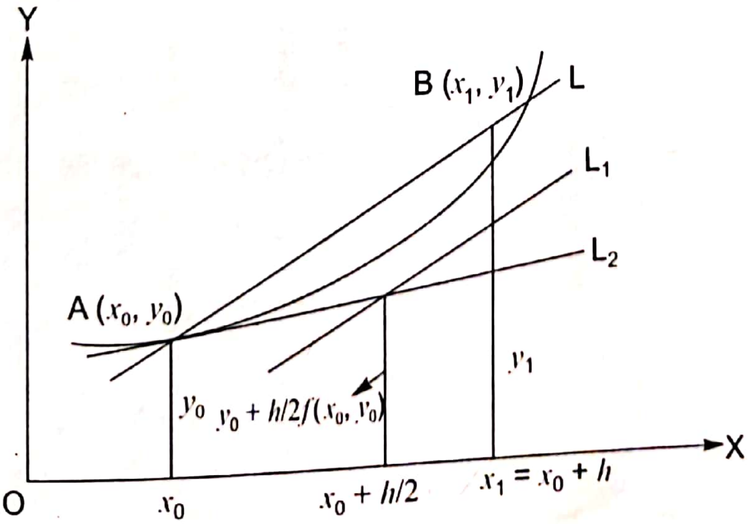
$\therefore$ By Euler's method $y(0.1) = 1.0927$

(ii) **Improved Euler's Method:** (See Tables on page 266)

6.4. MODIFIED EULER'S METHOD

In this method the curve in the interval (x_0, x_1), $x_1 = x_0 + h$, is approximated by the line though (x_0, y_0) with slope

$$f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\} \quad ...(1)$$



i.e., the slope at the middle point whose abscissa is the average of x_0 and x_1 *i.e.*, $x_0 + \frac{h}{2}$.

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Table for Modified Euler's Method

n	x_n	y_n	$f(x_n, y_n) = x_n + y_n$	$y_n + \frac{h}{2} f(x_n, y_n)$	$f\left\{x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right\}$	$y_{n+1} = y_n + hf\left\{x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right\}$
0	$x_0 = 0$	$y_0 = 1$	$f(x_0, y_0) = 1$	$y_0 + (0.05) f(x_0, y_0)$ $= 1.05$	$f\{0.05, 1.05\}$ $= 1.1$	$y_1 = y_0 + (0.1)(1.1)$ $= 1 + .11 = 1.11$
1	$x_1 = 0.1$	$y_1 = 1.11$	$f(x_1, y_1) = 1.21$	$y_1 + (0.05) f(x_1, y_1)$ $= 1.1705$	$f\{0.15, 1.1705\}$ $= 1.3205$	$y_2 = y_1 + (0.1)(1.3205)$ $= 1.11 + 0.1321 = 1.2421$
2	$x_2 = 0.2$	$y_2 = 1.2421$	$f(x_2, y_2) = 1.4421$	$y_2 + (0.05) f(x_2, y_2)$ $= 1.3142$	$f\{0.25, 1.3142\}$ $= 1.5642$	$y_3 = y_2 + (0.1)(1.5642)$ $= 1.2421 + 0.1564 = 1.3985$
3	$x_3 = 0.3$	$y_3 = 1.3985$				

Hence for $x = 0.1$; $y_1 = 1.11$

for $x = 0.2$; $y_2 = 1.2421$

for $x = 0.3$; $y_3 = 1.3985$

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Geometrically, the line L through (x_0, y_0) which is parallel to L_1, a line through $\left\{x_0 + \frac{h}{2}, \frac{h}{2} f(x_0, y_0)\right\}$ with the slope (1) approximates the curve in the interval $[x_0, x_1]$. The ordinate at $x = x_1$, meeting the line L at B will give the value of y_1.

The equation of the line L is $\quad y - y_0 = (x - x_0) f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$

Putting $x = x_1$, we get

$$y_1 = y_0 + (x_1 - x_0) f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$$

$$= y_0 + hf\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\} \qquad \text{...(2)}$$

Proceeding in the same way, it can be shown that

$$\mathbf{y_{n+1} = y_n + hf\left\{x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right\}}.$$

There is lot of confusion among the authors; some take the improved Euler's method as Modified Euler's method and improved Euler's method is not mentioned at all.

Example 1. *Using modified Euler's method solve $y' = x + y$ for $x = 0.1, 0.2, 0.3$ (assuming $h = 0.1$) given $y(0) = 1$.* **(P.T.U., May 2005, Dec. 2011)**

Solution. Here, $f(x, y) = x + y$; $x_0 = 0, y_0 = 1, h = 0.1$.

Modified Euler's method is $y_{n+1} = y_n + hf\left\{x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right\}$

Modified Euler's Method: (See Table on page 268)

Example 2. *Solve $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$ in the range $0 \le x \le 0.2$ using (i) Euler's method (ii) Improved Euler's Method and (iii) Modified Euler's method. Take $h = 0.1$.* **(P.T.U., Dec. 2013)**

Solution. Given $\frac{dy}{dx} = y - \frac{2x}{y}$; $y(0) = 1, h = 0.1$

Now we have to find out the solutions at $x = 0.1$ and $x = 0.2$.

(*i*) **Euler's Method**

The algorithm is $\quad \frac{dy}{dx} = f(x, y), y(x_0) = y_0$

then $$y_{m+1} = y_m + h f(x_m, y_m) \qquad \text{...(1)}$$

Putting $m = 0$, we get

$$y(0.1) = y_1 = y_0 + h f(x_0, y_0) = y_0 + h\left\{y_0 - \frac{2x_0}{y_0}\right\}$$

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$$= 1 + (0.1)\left\{1 - \frac{2(0)}{1}\right\} = 1.1 \qquad [\because x_0 = 0, y_0 = 1]$$

Putting $m = 1$ in (1), we get

$$y(0.2) = y_2 = y_1 + hf(x_1, y_1) = y_1 + (0.1)\left\{y_1 - \frac{2x_1}{y_1}\right\}$$

$$= 1.1 + (0.1)\left\{1.1 - \frac{2(0.1)}{1.1}\right\} \qquad \{\because x_1 = 0.1, y_1 = 1.1\}$$

$$= 1.1918182.$$

(*ii*) Improved Euler's Method

Here $$y_{m+1} = y_m + \frac{h}{2}[f(x_m, y_m) + f(x_{m+1}, y^*_{m+1})] \qquad ...(2)$$

where $$y^*_{m+1} = y_m + hf(x_m, y_m)$$

Putting $m = 0$ in (2), we get $$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^*)] \qquad ...(3)$$

where $$y_1^* = y_0 + hf(x_0, y_0) = 1 + (0.1)\left[1 - \frac{2(0)}{1}\right] = 1.1$$

∴ From (3), $$y_1 = y_0 + \frac{h}{2}\left[\left(y_0 - \frac{2x_0}{y_0}\right) + \left(y_1^{(*)} - \frac{2x_1}{y_1^{(*)}}\right)\right]$$

$$= 1 + \frac{0.1}{2}\left[\left\{1 - \frac{2(0)}{1}\right\} + \left\{1.1 - \frac{2(0 \cdot 1)}{1.1}\right\}\right] = 1.0959091$$

Putting $m = 1$ in (2), we get $$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^*)] \qquad ...(4)$$

where $$y_2^* = y_1 + hf(x_1, y_1)$$

Now, $$f(x_1, y_1) = y_1 - \frac{2x_1}{y_1} = 1.0959091 - \frac{2(0.1)}{1.0959091} = 0.9134122$$

$$y_2^* = y_1 + hf(x_1, y_1) = 1.0959091 + (0.1)(0.9134122) = 1.1872503$$

$$f(x_2, y_2^*) = y_2^* - \frac{2x_2}{y_2^*} = 1.1872503 - \frac{2(0.2)}{1.1872503} = 0.8503373$$

Substituting all the requisites in (4), we get

$$y_2 = 1.0959091 + \frac{0.1}{2}[0.9134122 + 0.8503373] = 1.1840966.$$

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(iii) **Modified Euler's Method**

The formula is

$$y_{m+1} = y_m + hf\left\{x_m + \frac{h}{2}, y_m + \frac{h}{2} f\left(x_m, y_m\right)\right\} \quad ...(5)$$

Putting $m = 0$ in above, we get

$$y_1 = y_0 + hf\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\} \quad ...(6)$$

$$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1$$

$$y_0 + \frac{h}{2} f(x_0, y_0) = 1 + \frac{0.1}{2}(1) = 1.05; x_0 + \frac{h}{2} = 0.05$$

$$\therefore f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\} = f\{0.05, 1.05\} = 1.05 - \frac{2(0.05)}{1.05} = 0.9547619$$

Hence from (6), we get $\quad y_1 = 1 + (0.1)(0.9547619) = 1.0954762$

Putting $m = 1$ in (5), we get $\quad y_2 = y_1 + hf\left\{x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right\} \quad ...(7)$

$$x_1 + \frac{h}{2} = 0.1 + \frac{0.1}{2} = 0.15$$

$$f(x_1, y_1) = y_1 - \frac{2x_1}{y_1} = 1.0954762 - \frac{2(0.1)}{1.0954762} = 0.9129071$$

$$y_1 + \frac{h}{2} f(x_1, y_1) = 1.0954762 + \frac{0.1}{2}(0.9129071) = 1.411216$$

$$\therefore f\left\{x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right\} = f\{0.15, 1.1411216\} = 1.1411216 - \frac{2(0.15)}{1.1411216} = 0.8782223$$

Now substituting the requisite in (7), we get

$$y_2 = 1.0954762 + (0.1)(0.8782223) = 1.1832984$$

Hence the values in tabular form are:

x	***Euler's Method***	***Improved Euler's method***	***Modified Euler's method***
0	1	1	1
0.1	1.1	1.0959091	1.0954762
0.2	1.1918182	1.18490966	1.1832984

Note. Students can use table forms of all the three methods.

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TEST YOUR KNOWLEDGE

1. Use improved Euler's method to approximate y when $x = 0.5$ given that $\frac{dy}{dx} = x + y^2$, $y(0) = 1$ by taking $h = 0.1$
(P.T.U., May 2010)
2. Solve $y' = 3x^2 + y$ in $0 \le x \le 1$ by Euler's method taking $h = 0.1$ given that $y(0) = 4$.
3. Solve $y' = x + y$, $y(0) = 0$ choosing the step length 0.2 for $y(1.4)$ by Euler's method.
4. Using Modified Euler's method solve $\frac{dy}{dx} = 1 - y$, $y(0) = 0$ in the range $0 \le x \le 0.3$ by taking $h = 0.1$
5. Using Euler's method find $y(0.6)$ of $y' = 1 - 2xy$ given that $y(0) = 0$ by taking $h = 0.2$.
6. Solve $y' = -y$; $y(0) = 1$ by (*i*) Euler's method for $y(0.04)$ and (*ii*) modified Euler's method for $y(0.6)$
(P.T.U., Dec. 2010)
7. Solve $y' = x + y + xy$, $y(0) = 1$ for $y(0.1)$ by taking $h = 0.025$, using Euler's method.
8. Given that $y' = \log(x + y)$ with $y(0) = 1$. Use.
   (*i*) Improved Euler's method to find $y(0.2)$, $y(0.5)$
   (*ii*) Modified Euler's method to find $y(0.2)$.
9. Use Euler's method and its modified form to obtain $y(0.2)$, $y(0.4)$ and $y(0.6)$ correct to three decimal places given that $y' = y - x^2$, $y(0) = 1$.
10. Use Euler's modified method to get $y(0.25)$ given that $y' = 2xy$, $y(0) = 1$.
11. Use improved Euler's method to solve $y' = x + |\sqrt{y}|$, $y(0) = 1$ in the range $0 \le x \le 0.6$ by taking $h = 0.2$
12. Given that $y' = 2 + \sqrt{xy}$ and $y(1) = 1$. Find $y(2)$ in steps of (0.2) using improved Euler's method.
13. Given $y' = x^2 + y^2$, $y(0) = 1$ determine $y(0.1)$ and $y(0.2)$ by modified Euler's method.
14. Solve $y' = y + e^x$, $y(0) = 0$ for $y(0.2)$, $y(0.4)$ by improved Euler's method.
15. Solve $y' = y + y^2$, $y(0) = 1$ for $y(0.02)$, $y(0.04)$ and $y(0.06)$ using Euler's modified method.
16. Apply Euler's Modified method to solve $\frac{dy}{dx} = x + 3y$ subject to $y(0) = 1$ and hence find an approximate value of y when $x = 1$. [**Hint:** Take $h = 0.2$ and find y_5]
(P.T.U., Dec. 2003)

ANSWERS

1. 2.2067
2. 4.4, 4.843, 5.3393, 5.90023, 6.538253, 7.2670786, 8.1017861, 9.0589647, 9.1039647, 10.257361
3. 1.18318
4. $y(0.1) = 0.095$, $y(0.2) = 0.18098$, $y(0.3) = 0.25878$
5. 0.52256
6. 0.9606 ; 0.551368
7. 1.1117
8. 1.0082, 1.0490 ; 1.0095
9. 1.2, 1.432, 1.686 ; 1.218, 1.467, 1.737
10. 1.0625
11. 1.2309, 1.5253, 1.8851
12. 5.051
13. 1.1105, 1.25026
14. 0.24214, 0.59116
15. 1.0202, 1.0408, 1.0619
16. 21.081.

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$f(x, y)$. They agree with Taylor's series upto the terms of h^r, where r is different for different methods and it's known as the order of that Runge-Kutta method. (P.T.U., May 2005)

(i) Runge-Kutta Method of First Order:

To solve the differential equation $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ by R.K Method of First Order:

Take $k_1 = hf(x_0, y_0)$

then, $y_1 = y_0 + k$, where $k = k_1 = hf(x_0, y_0)$ is the first approximation of y.

To compute y_2; $y_2 = y_1 + k$, where $k = hf(x_1, y_1)$ *i.e.*, to compute y_2 we have to simply replace x_0 by x_1 and y_0 by y_1 in the value of y_1. Similarly, we can continue with $y_3, y_4, \ldots\ldots$

(ii) Runge-Kutta Method of Second Order: (P.T.U., May 2014)

To compute y_1; take
$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf(x_0 + h, y_0 + k_1)$$

then,
$$k = \frac{k_1 + k_2}{2}$$

$\therefore$
$$y_1 = y_0 + k$$

To compute y_2; $y_2 = y_1 + k$, where $k = \frac{k_1 + k_2}{2}$ and

$k_1 = hf(x_1, y_1)$, $k_2 = hf(x_1 + h, y_1 + k_1)$ *i.e.*, k is obtained by replacing x_0 by x_1 and y_0 by y_1 in the values of k_1 and k_2.

Note. Here, k is the weighted mean of k_1 and k_2 that is why it is called R.K Method of second order.

(iii) Runge-Kutta Method of Third Order: (P.T.U., Dec. 2013)

To compute y_1; take
$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$k' = hf(x_0 + h, y_0 + k_1)$$
$$k_3 = hf(x_0 + h, y_0 + k')$$

and
$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$\therefore$
$$y_1 = y_0 + k$$

To compute y_2; $y_2 = y_1 + k$, where k is obtained as above by replacing x_0 to x_1 and y_0 to y_1 in k_1, k_2, k' and k_3

Similarly, we can find $y_3, y_4, \ldots\ldots$

As k is the weighted mean of k_1, k_2, k_3 therefore it is called R.K Method of 3rd order.

(iv) Runge-Kutta Method of Fourth Order: (P.T.U., May 2010)

To compute y_1; Take
$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$

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$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and

$$y_1 = y_0 + k$$

then.

To compute y_2; $y_2 = y_1 + k$, where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, where k_1, k_2, k_3, k_4 are obtained by replacing x_0 by x_1 and y_0 and y_1 in the above values *i.e.*, $k_1 = hf(x_1, y_1)$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right); k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right); k_4 = hf(x_1 + h, y_1 + k_3).$$

Similarly, we can find $y_3, y_4, \ldots\ldots$

As k is the weighted mean of $k_1, k_2, k_3, k_4, \ldots\ldots$ it is called R.K Method of 4^{th} order.

Note. The fourth order Runge-Kutta method is most commonly used in practice and is often referred to as "the Runge-Kutta Method" only without any reference to the order.

6.7. EULER'S METHOD, IMPROVED EULER'S METHOD AND RUNGE'S METHOD ARE RUNGE-KUTTA METHODS OF FIRST, SECOND AND THIRD ORDER RESPECTIVELY

(*i*) **Euler's method as R.K. Method of first order** **(P.T.U., Dec. 2006, May 2007)**

Euler's method is $\quad y_{n+1} = y_n + hf(x_n, y_n)$

i.e., $\quad y_1 = y_0 + hf(x_0, y_0)$

To convert it to R.K. Method

Take $\quad k_1 = hf(x_0, y_0)$

then, $\quad y_1 = y_0 + k$, where $k = k_1 = hf(x_0, y_0)$

Similarly, $\quad y_2 = y_1 + k$, where $k = hf(x_1, y_1)$ and so on.

So Euler's method is expressed in the form of R.K Method of first order.

(*ii*) **Improved Euler's Method as R.K. Method of second order**

Improved Euler's method is

$$y_{n+1} = y_n + \frac{h}{2}\left\{f(x_n, y_n) + f\left(x_{n+1}, y^*_{n+1}\right)\right\}, \text{ where}$$

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

i.e.,
$$y_1 = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f\left(x_1, y_1^*\right)\right\}, \text{ where } y_1^* = y_0 + hf(x_0, y_0)$$

To convert it to R.K. Method,

Take
$$k_1 = hf(x_0, y_0)$$

$\therefore$
$$y_1^* = y_0 + k_1$$

$$k_2 = hf\left(x_1, y_1^*\right) = hf(x_0 + h, y_0 + k_1)$$

and
$$k = \frac{k_1 + k_2}{2}$$

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$\therefore \qquad y_1 = y_0 + k$, where $k = \dfrac{k_1 + k_2}{2}$

To compute y_2; $\qquad y_2 = y_1 + k$, where $k = \dfrac{k_1 + k_2}{2}$ and k_1, k_2 are obtained as above by replacing x_0 by x_1 and y_0 by y_1 in the values of k_1 and k_2.

Similarly, we can find y_3, y_4 and so on.

So improved Euler's method is expressed in the form of R.K. Method of second order.

(iii) Runge's Method as R.K. Method of third order

Runge's method is $\qquad f_0 = f(x_0, y_0)$

$$f_m = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_0\right)$$

$$f_k = f\left(x_0 + h, y_0 + hf_0\right)$$

$$f_l = f(x_0 + h, y_0 + hf_k)$$

and $$y_1 = y_0 + \frac{h}{6}\left(f_0 + 4f_m + f_l\right)$$

To convert it to R.K. Method

Take $\qquad k_1 = hf(x_0, y_0) = hf_0$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf_m$$

$$k' = hf(x_0 + h, y_0 + k_1) = hf_k$$

$$k_3 = hf(x_0 + h, y_0 + k') = hf_l$$

and $$k = \frac{1}{6}\left(k_1 + 4k_2 + k_3\right)$$

and $$y_1 = y_0 + k$$

To compute y_2; $\qquad y_2 = y_1 + k$, where $k = \dfrac{1}{6}\left(k_1 + 4k_2 + k_3\right)$ and k_1, k_2, k_3 are obtained as above by replacing x_0 to x_1 and y_0 to y_1 in the values of k_1, k_2, k_3.

Similarly, we can find y_3, y_4 and so on.

So Runge's method is expressed as R.K Method of third order.

Example 1. *Apply Runge-Kutta method to find an approximate value of y when $x = 0.2$ given that $\dfrac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.* **(P.T.U., May 2007)**

Solution. Here $f(x, y) = x + y$, $y(0) = 1$, *i.e.*, $x_0 = 0, y_0 = 1$.

Since, h is not mentioned in the question

$\therefore \quad$ We take $\qquad h = 0.1$

To compute, $\qquad y_1 = y(0.1)$

$$k_1 = hf(x_0, y_0) = (0.1)(0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\, f\,(0.05, 1.05)$$

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$$= (0.1)(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1) f(0.05, 1.055)$$

$$= (0.1)(0.05 + 1.055) = (0.1)(1.105) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0.1) f(0.1, 1.1105) = (0.1)(0.1 + 1.1105)$$

$$= 0.12105$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105)$$

$$= 0.110341667 = 0.110342 \text{ (app.)}$$

$\therefore$ $$y_1 = y(0.1) = y_0 + k = 1 + 0.110342 = 1.110342$$

To compute y_2; $$x_1 = 0.1, y_1 = 1.110342$$

$$k_1 = hf(x_1, y_1) = (0.1) f(0.1, 1.110342)$$

$$= (0.1)(0.1 + 1.110342) = 0.1210342$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1\right) = (0.1) f(0.15, 1.170859)$$

$$= (0.1)(0.15 + 1.170859) = 0.1320859$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1) f(0.15, 1.1763848)$$

$$= (0.1)(0.15 + 1.1763848) = 0.13263848$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 1.24298048)$$

$$= (0.1)(0.2 + 1.24298048) = 0.144298048$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.794781008] = 0.1324635$$

$$y_2 = y(0.2) = y_1 + k = 1.110342 + 0.1324635$$

$$= 1.2428055$$

$\therefore$ $y_2 = 1.2428$ (correct to four decimal places).

Example 2. *Given $y' = x^2 - y$, $y(0) = 1$, find $y(0.1)$, $y(0.2)$ using Runge-Kutta methods of (i) Second order (ii) Third order and (iii) Fourth order.* **(P.T.U., May 2009)**

Solution. Given $y' = f(x, y) = x^2 - y$, $x_0 = 0$, $y_0 = 1$

$\therefore$ $f(x_0, y_0) = -1$. Let $h = 0.1$

Runge-Kutta method of second order

Here, $$k_1 = hf(x_0, y_0) = (0.1)(-1) = -0.1$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = hf(0.1, 0.9) = (0.1)[(0.1)^2 - 0.9] = -0.089$$

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$\therefore \quad k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[(0.1) + (-0.089)] = -0.0945$

$\therefore \quad y_1 = y(0.1) = y_0 + k = 1 - 0.0945 = 0.9055$

Again taking $x_1 = 0.1, y_1 = 0.9055$ in place of (x_0, y_0) and repeat the process.

$$k_1 = hf(x_1, y_1) = h[x_1^2 - y_1] = (0.1)[(0.1)^2 - 0.9055] = -0.08955$$

$$k_2 = hf[x_1 + h, y_1 + k_1] = hf[0.2, 0.81595] = (0.1)[(0.2)^2 - 0.81595] = -0.077595$$

$\therefore \quad k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[-0.08955 - 0.077595] = -0.0835725$

$\therefore \quad y_2 = y(0.2) = y_1 + k = 0.9055 - 0.0835725 = 0.8219275.$

Runge-Kutta method of third order

Here, $k_1 = hf(x_0, y_0) = -0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= hf[0.05, 0.95] = (0.1)\left[(0.05)^2 - (0.95)\right] = -0.09475$$

$$k' = hf(x_0 + h, y_0 + k_1) = (0.1)f(0.1, 0.9) = (0.1)\left[(0.1)^2 - (0.9)\right]$$

$$k_3 = hf(x_0 + h, y_0 + k') = hf(0.1, 0.911) = (0.1)[(0.1)^2 + 0.911] = -0.0901$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3) = \frac{1}{6}[(-0{\cdot}1) + 4(-0{\cdot}09475) + (-0{\cdot}0901)] = -0{\cdot}09485$$

$\therefore \quad y_1 = y(0.1) = y_0 + k = 0.90515$

Again taking $x_1 = 0.1, y_1 = 0.90515, h = 0.1$ in place of x_0, y_0 and repeat the process.

$$k_1 = hf(x_1, y_1) = (0.1)[(0.1)^2 - 0.90515] = -0.089515$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = hf[0.15, 0.8603925]$$

$$= (0.1)[(0.15)^2 - 0.8603925] = -0.0837892$$

$$k' = hf(x_1 + h, y_1 + k_1) = hf(0.2, 0.815635)$$

$$= (0.1)[(0.2)^2 - 0.815635] = -0.0775635$$

$$k_3 = hf(x_1 + h, y_1 + k_2) = hf(0.2, 0.8275865)$$

$$= (0.1)[(0.2)^2 - 0.8275865] = -0.0787586$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$= \frac{1}{6}[-0.089515 + 4(-0.0837892) - 0.0787586] = -0.0839051$$

$\therefore \quad y_2 = y(0.2) = y_1 + k = 0.90515 - 0.0839051 = 0.8212449.$

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Here.

$$k_1 = hf(x_0, y_0) = -0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) f[0.05, 0.95]$$

$$= (0.1)[(0.05)^2 - 0.95] = -0.09475$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf[0.05, 0.952625] = (0.1)[(0.05)^2 - 0.952625]$$

$$= 0.0950125$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = hf[0.1, 0.9049875]$$

$$= (0.1)[(0.1)^2 - 0.9049875] = -0.0894987$$

Now,

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[-0.1 + 2(-0.09475) + 2(-0.0950125) - 0.0894987]$$

$$= -0.0948372$$

$\therefore$
$$y_1 = y(0.1) = y_0 + k = 1 - 0.0948372 = 0.9051627$$

Again taking $x_1 = 0.1, y_1 = 0.9051627$ in place of x_0, y_0 and repeat the process

$$k_1 = hf(x_1, y_1) = hf[0.1, 0.9051627] = (0.1)[(0.1)^2 - 0.9051627] = -0.0895162$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = hf[0.15, 0.8604046]$$

$$= (0.1)[(0.15)^2 - 0.8604046] = -0.0837904$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = hf[0.15, 0.8632674]$$

$$= (0.1)[(0.15)^2 - 0.8632674] = -0.0840767$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = hf(0.2, 0.8210859)$$

$$= (0.1)[(0.2)^2 - 0.8210859] = -0.0781085$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[-0.0895162 + 2(-0.0837904) + 2(-0.0840767) - 0.0781085]$$

$$= -0.0838931$$

$\therefore$
$$y_2 = y(0.2) = y_1 + k = 0.9051627 - 0.0838931 = 0.8212695.$$

Example 3. *Given:* $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ *; $x_0 = 0$, $y_0 = 1$, $h = 0.2$ find y_1, y_2 using Runge-Kutta Method.*

(P.T.U., May 2005)

Solution. Here, $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$; $x_0 = 0, y_0 = 1, h = 0.2$

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To compute y_1; $\quad k_1 = hf(x_0, y_0) = (0.2)f(0.1) = (0.2)\dfrac{1-0}{1+0} = 0.2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)f(0.1, 1.1)$$

$$= (0.2)\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2} = (0.2)\frac{1.21 - 0.01}{1.21 + 0.01} = 0.1967213 = 0.1967$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k}{2}\right) = (0.2)f(0.1, 1.0983606) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)f(0.2, 1.967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.2 + 0.3934 + 0.3934 + 0.1891) = 0.19598$$

$\therefore \quad y_1 = y(0.2) = y_0 + k = 1.19598$

To compute y_2; $\quad x_1 = 0.2, y_1 = 1.19598$

$$k_1 = hf(x_1, y_1) = (0.2)\frac{(1.19598)^2 - (0.2)^2}{(1.19598)^2 + (0.2)^2} = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.2)f(0.3, 1.29055) = (0.2)\frac{(1.29055)^2 - (0.3)^2}{(1.29055)^2 + (0.3)^2} = 0.17949$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.2)f(0.3, 1.28572) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.2)f(0.4, 1.37528) = 0.1687$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1792$$

$\therefore \quad y_2 = y(0.4) = y_1 + k = 1.19598 + 0.1792 = 1.37518 = 1.3752$

$\therefore \quad y_2 = y(0.4) = 1.3752$

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TEST YOUR KNOWLEDGE

1. Using second order Runge-Kutta method find y at $x = 0.1, 0.2$ and 0.3; given $2y' = (1 + x)y^2$; $y(0) = 1$.
2. Find $y(1.2)$ by Runge-Kutta method of fourth order given $y' = x^2 + y^2$; $y(1) = 1.5$. Take $h = 0.1$.
3. If $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$; $y(1) = 0$ solve for y at $x = 1.2, 1.4$ using Runge-Kutta method of fourth order.
4. Using Runge-Kutta method of fourth order find y at $x = 1.1, 1.2$ given that $2y' = 2x^3 + y$; $y(1) = 2$.
5. Find y at $x = 0.1, 0.2$ using fourth order Runge-Kutta algorithm given that $y' - yx^2 = 0$; $y(0) = 1$.
6. Use Runge-Kutta method to evaluate y at $x = 0.2, 0.4, 0.6$ given that $\frac{dy}{dx} - xy = 1$; $y(0) = 2$.
7. Use Runge-Kutta method of fourth order to find $y(0.1)$, $y(0.2)$ given that $\frac{dy}{dx} - y = -x$; $y(0) = 2$.
8. Using Runge's formula (third order) solve $\frac{dy}{dx} = x - y$; $y(1) = 1$.

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ANSWERS

1. 1.0552, 1.1230, 1.2073
2. 2.5061
3. 0.1402, 0.2705
4. 2.2213, 2.4914
5. 1.0053, 1.0227
6. 2.243, 2.589, 2.072
7. 2.20517, 2.42139
8. 1.004833

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