UNIT 1

Fourier Series

1.1. PERIODIC FUNCTIONS

2P

(PTU Dec. 2012)

A function $f(x)$ which satisfies the relation $f(x + T) = f(x)$ for all real x and some fixed T is called a periodic function. The *smallest positive number* T, for which this relation holds, is called the **period** of $f(x)$.

If T is the period of $f(x)$, then $f(x) = f(x + T) = f(x + 2T) = ... = f(x + nT) = ...$

Also $f(x) = f(x - T) = f(x - 2T) = ... = f(x - nT) = ...$

$\therefore$ $f(x) = f(x \pm nT)$, where n is a positive integer.

Thus, $f(x)$ repeats itself after periods of T.

For example, $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π.

Since
$$\tan(\theta + \pi) = \frac{\sin(\pi + \theta)}{\cos(\pi + \theta)} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$

and
$$\cot(\theta + \pi) = \frac{\cos(\pi + \theta)}{\sin(\pi + \theta)} = \frac{-\cos\theta}{-\sin\theta} = \cot\theta.$$

Therefore $\tan\theta$ and $\cot\theta$ are periodic functions with period π.

The functions $\sin nx$ and $\cos nx$ are periodic with period $\dfrac{2\pi}{n}$.

The sum of a number of periodic functions is also periodic. If T_1 and T_2 are the periods of $f(x)$ and $g(x)$, then the period of $af(x) + bg(x)$ is the least common multiple of T_1 and T_2.

For example, $\cos x$, $\cos 2x$, $\cos 3x$ are periodic functions with periods 2π, π and $\dfrac{2\pi}{3}$ respectively.

$\therefore$ $f(x) = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x$ is also periodic with period 2π, the LCM of 2π, π and $\dfrac{2\pi}{3}$.

1.2. FOURIER SERIES

(PTU May 2012)

Expansion of a function $f(x)$ in a series of sines and cosines of multiples of x was developed by French Mathematician and physicist Jacques Fourier.

We have seen how a function can be expanded in powers of x by Maclaurin's theorem but that expansion was possible only when the function and its derivatives are continuous. A need arises to expand functions which have discontinuities in their values or derivatives.

By Fourier series, we can expand both type of functions under certain conditions as an infinite series of sines and cosines of x and its integral multiples.

Fourier series for the function $f(x)$ in the interval $c < x < c + 2\pi$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx}$$

where

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Above formulae are also called Euler's formulae. Constants a_0, a_n and b_n are called Fourier coefficients of $f(x)$.

Remark. To write $\frac{a_0}{2}$ instead of a_0 is a conventional device to be able to get more symmetric formulae for the coefficients.

Note. To determine a_0, a_n and b_n, we shall need the following results: (m and n are integers)

(i) $\int_c^{c+2\pi} \sin nx\,dx = \left(\frac{-\cos nx}{n}\right)_c^{c+2\pi} = 0, n \neq 0;\ \int_c^{c+2\pi} \cos nx\,dx = \left(\frac{\sin nx}{n}\right)_c^{c+2\pi} = 0, n \neq 0$

(ii) $\int_c^{c+2\pi} \sin mx \cos nx\,dx = 0, m \neq n$

(iii) $\int_c^{c+2\pi} \cos mx \cos nx\,dx = 0, m \neq n$

(iv) $\int_c^{c+2\pi} \sin mx \sin nx\,dx = 0, m \neq n$

(v) $\int_c^{c+2\pi} \cos^2 nx\,dx = \pi, n \neq 0;\ \int_c^{c+2\pi} \sin^2 nx\,dx = \pi, n \neq 0$

(vi) $\int_c^{c+2\pi} \sin nx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} \sin 2nx\,dx = 0, n \neq 0$

(vii) $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c$

(viii) $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + c$

(ix) To integrate the product of two functions, one of which is a power of x, we apply the *generalised rule of integration by parts*. If dashes denote differentiation and suffixes denote integration w.r.t. x, the rule can be stated as follows :

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots, \text{ where } u \text{ and } v \text{ are functions of } x.$$

i.e., Integral of the product of two functions

= 1st function × integral of 2nd – go on differentiating 1st, integrating 2nd, signs alternately +ve and –ve.

[Simplification should be done only when the integration is over].

e.g., $\int x^2 \cos nx\,dx = x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)$

$$= \frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx.$$

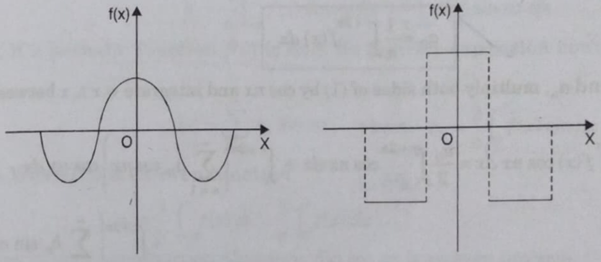
(x) $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \frac{1}{2}\right)\pi = 0$, where n is an integer.

1.3. EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be *even* if $f(-x) = f(x)$ *e.g.*, x^2, $\cos x$, $\sin^2 x$ are even functions. The graph of an even function is symmetrical about the y-axis. Here y-axis is a mirror for the reflection of the curve.

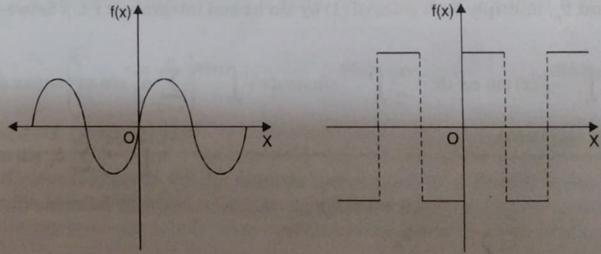
$$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_0^{\pi} f(x)\,dx$$



Graphs of even functions

A function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$ *e.g.*, x^3, $\sin x$, $\tan^3 x$ are odd functions. The graph of an odd function is symmetrical about the origin.

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$



Graphs of odd functions

The product of two even functions or two odd functions is an even function while the product of an even function and an odd function is an odd function.

(PTU May 2012)

1.4. EULER'S FORMULAE

The Fourier series for the function $f(x)$ in the interval $c < x < c + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(1)$$

In finding the co-efficients a_0, a_n and b_n, we assume that the series on the right hand side of (1) is uniformly convergent for $c < x < c + 2\pi$ and it can be integrated term by term in the given interval.

To find a_0. Integrate both sides of (1) w.r.t. x between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}(c + 2\pi - c) + 0 + 0 \quad \text{[by formula (}i\text{) above]}$$

$$= a_0\pi$$

$$\therefore \quad \boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx}$$

To find a_n, multiply both sides of (1) by $\cos nx$ and integrate w.r.t. x between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\cos nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$$= 0 + a_n\pi + 0 \quad \text{[by formulae (}i\text{), (}v\text{) and (}vi\text{)]}$$

$$= a_n\pi$$

$$\therefore \quad \boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx}$$

To find b_n, multiply both sides of (1) by $\sin nx$ and integrate w.r.t. x between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + b_n\pi \quad \text{[by formulae (}i\text{), (}vi\text{) and (}v\text{)]}$$

$$= b_n\pi$$

$$\therefore \quad \boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx}$$

These values of a_0, a_n and b_n are called **Euler's formulae.**

Cor. 1. *If $c = 0$, the interval becomes $0 < x < 2\pi$, and the formulae reduce to*

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx\,; \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx\,; \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Cor. 2. *If $c = -\pi$, the interval becomes $-\pi < x < \pi$, and the formulae reduce to*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\,; \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\,; \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Cor. 3. When $f(x)$ is an odd function 2 Marks

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

Since $\cos nx$ is an even function, therefore, $f(x)\cos nx$ is an odd function.

$$\therefore \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

Since $\sin nx$ is an odd function, therefore, $f(x)\sin nx$ is an even function.

$$\therefore \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Hence, if a periodic function $f(x)$ is odd, its Fourier expansion contains only sine terms.

i.e.,
$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Cor. 4. When $f(x)$ is an even function 2 Marks.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

Since $\cos nx$ is an even function, therefore, $f(x)\cos nx$ is an even function.

$$\therefore \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Since $\sin nx$ is an odd function, therefore, $f(x)\sin nx$ is an odd function.

$$\therefore \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

Hence, if a periodic function $f(x)$ is even, its Fourier expansion contains only cosine terms.

i.e.,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

1.5. DIRICHLET'S CONDITIONS

(PTU Dec. 2006, 2008)

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions (after Dirichlet, a German mathematician). All the functions that normally arise in engineering problems satisfy these conditions and hence they can be expressed as a Fourier series.

Any function $f(x)$ can be expressed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where a_0, a_n, b_n are constants, provided

(i) $f(x)$ is periodic, single valued and finite

(ii) $f(x)$ has a finite number of finite discontinuities in any one period.

(iii) $f(x)$ has a finite number of maxima and minima.

When these conditions are satisfied, the Fourier series converges to $f(x)$ at every poi[nt] of continuity. At a point of discontinuity, the sum of the series is equal to the mean of t[he] limits on the right and left

i.e., $\frac{1}{2}[f(x+0)+f(x-0)]$

where $f(x+0)$ and $f(x-0)$ denote the limit on the right and the limit on the left respectivel[y]

EXAMPLES

Example 1. *Find the Fourier series representing* $f(x)=x,\ 0<x<2\pi$.

Sol. Let the Fourier series be

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \quad ...(1)$$

where
$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}x\,dx=\frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi}=2\pi$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}x\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\left(\frac{x\sin nx}{n}\right)_0^{2\pi}-\int_0^{2\pi}1.\frac{\sin nx}{n}dx\right]$$

$$=-\frac{1}{\pi n}\left(\frac{-\cos nx}{n}\right)_0^{2\pi}=\frac{1}{\pi n^2}(1-1)=0$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}x\sin nx\,dx$$

$$=\frac{1}{\pi}\left[\left\{x.\left(\frac{-\cos nx}{n}\right)\right\}_0^{2\pi}-\int_0^{2\pi}1.\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$=\frac{1}{\pi}\left[\frac{-2\pi}{n}+\frac{1}{n}\left(\frac{\sin nx}{n}\right)_0^{2\pi}\right]=-\frac{2}{n}.$$

$\therefore$ From (1), $$f(x)=\pi-2\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$$

Example 2. *Prove that for all values of x between* $-\pi$ *and* π,

$$\frac{1}{2}x=\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\cdots.$$

Sol. Here the given function $f(x)$ is odd in $(-\pi,\pi)$.

Hence, $a_0=0$ and $a_n=0$

Let the Fourier series be

$$f(x)=\frac{x}{2}=\sum_{n=1}^{\infty}b_n\sin nx \quad ...(1)$$

where
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x}{2}\sin nx\,dx$$

$$=\frac{1}{\pi}\int_0^{\pi}x\sin nx\,dx$$

$$=\frac{1}{\pi}\left[\left\{x.\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi}-\int_0^{\pi}\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$=\frac{1}{\pi}\left[\frac{-\pi}{n}\cos n\pi+\frac{1}{n}\left(\frac{\sin nx}{n}\right)_0^{\pi}\right]=\frac{1}{\pi}\left[\frac{-\pi}{n}(-1)^n\right]=\frac{-1}{n}(-1)^n$$

$\therefore$ From (1), $$\frac{x}{2}=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx=\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\cdots.$$

Example 3. *Obtain the Fourier series for* $f(x)=e^{-x}$ *in the interval* $0<x<2\pi$.

(PTU Dec. 2010)

Sol. Let $$f(x)=e^{-x}=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Then $$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}e^{-x}\,dx=\frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi}=\frac{1-e^{-2\pi}}{\pi}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}e^{-x}\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx+n\sin nx)\right]_0^{2\pi}=\frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}e^{-x}\sin nx\,dx$$

$$=\frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx-n\cos nx)\right]_0^{2\pi}=\frac{1-e^{-2\pi}}{\pi}\cdot\frac{n}{1+n^2}$$

$\therefore$ $$e^{-x}=\frac{1-e^{-2\pi}}{2\pi}+\frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2}+\frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}$$

Example 4. *Obtain the Fourier series to represent* $f(x)=\frac{1}{4}(\pi-x)^2$ *in the interval* $[0, 2\pi]$.
Hence obtain the following relations:

(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$ (ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$.

Sol. Let $f(x)=\frac{1}{4}(\pi-x)^2=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$ ...(1)

By Euler's formulae, we have

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\,dx=\frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}=-\frac{1}{12\pi}[-\pi^3-\pi^3]=\frac{\pi^2}{6}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\cos nx\,dx$$

$$=\frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n}-\{-2(\pi-x)\}\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{4\pi}\left[\left(0+\frac{2\pi\cos 2n\pi}{n^2}+0\right)-\left(0-\frac{2\pi\cos 0}{n^2}+0\right)\right]=\frac{1}{4\pi}\left[\frac{2\pi}{n^2}+\frac{2\pi}{n^2}\right]=\frac{1}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\sin nx\,dx$$

$$=\frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right)-\{-2(\pi-x)\}\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{4\pi}\left[\left(-\frac{\pi^2\cos 2n\pi}{n}-0+\frac{2\cos 2n\pi}{n^3}\right)-\left(-\frac{\pi^2}{n}-0+\frac{2\cos 0}{n^3}\right)\right]$$

$$=\frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)-\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)\right]=0$$

$$\therefore\quad f(x)=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{\pi^2}{12}+\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\cdots \quad ...(2)$$

(*i*) Putting $x=0$ in eqn. (2), we get

$$\frac{\pi^2}{4}=\frac{\pi^2}{12}+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)$$

$$\Rightarrow\quad \frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots \quad ...(3)$$

(*ii*) Putting $x=\pi$ in eqn. (2), we get

$$0=\frac{\pi^2}{12}+\left[\left(\frac{-1}{1^2}\right)+\frac{1}{2^2}+\left(\frac{-1}{3^2}\right)+\frac{1}{4^2}+\cdots\right]$$

$$\Rightarrow\quad \frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots \quad ...(4)$$

(*iii*) Adding equations (3) and (4), we get

$$\frac{\pi^2}{6}+\frac{\pi^2}{12}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\Rightarrow\quad \frac{\pi^2}{4}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\Rightarrow\quad \frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

Hence the results.

Example 5. *Expand* $f(x)=x\sin x,\ 0<x<2\pi$ *as a Fourier series.*

Sol. Let $f(x)=x\sin x=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n\sin nx$

By Euler's formulae, we have

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$=\frac{1}{\pi}\Big[x(-\cos x)-1\cdot(-\sin x)\Big]_0^{2\pi}=\frac{1}{\pi}[-2\pi]=-2$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx=\frac{1}{2\pi}\int_0^{2\pi} x(2\cos nx\sin x)\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x-\sin(n-1)x]\,dx$$

$$=\frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right\}-1\cdot\left\{-\frac{\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1}+\frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

$$=-\frac{1}{n+1}+\frac{1}{n-1}=\frac{2}{n^2-1},\ n\neq 1$$

When $n=1$, we have

$$\boxed{a_1}=\frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx=\frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$=\frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-1\cdot\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}=\frac{1}{2\pi}[-\pi]=-\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx \sin x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(n-1)x - \cos(n+1)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1\cdot\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0,\ n \neq 1$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\cdot\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2\pi}(2\pi^2) = \pi$$

$$\therefore \quad f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$$

Example 6. *Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$. Hence show that*

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}.$$

(PTU Dec. 2009)

Sol. Let $x - x^2 = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\, dx - \int_{-\pi}^{\pi} x^2\, dx\right]$$

$$= -\frac{2}{\pi}\int_0^{\pi} x^2\, dx = -\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = -\frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\, dx - \int_{-\pi}^{\pi} x^2 \cos nx\, dx\right]$$

$$= -\frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx = -\frac{2}{\pi}\left[\left(x^2\,\frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi} 2x\,\frac{\sin nx}{n}\, dx\right]$$

$$= \frac{4}{\pi n}\int_0^{\pi} x \sin nx\, dx = \frac{4}{\pi n}\left[\left\{x\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right) dx\right]$$

$$= -\frac{4}{\pi n^2}(\pi \cos n\pi) = -\frac{4}{n^2}(-1)^n \qquad |\because \cos n\pi = (-1)^n,\ n \in I$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \sin nx\, dx - \int_{-\pi}^{\pi} x^2 \sin nx\, dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\left\{x\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi} 1\cdot\left(\frac{-\cos nx}{n}\right) dx\right]$$

$$= -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n \qquad |\because \sin n\pi = 0,\ n \in I$$

$$\therefore\ x - x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3} - 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots\right] - 2\left[\frac{-\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \dots\right]$$

$$= -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right]$$

Putting $x = 0$, we get

$$0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right)$$

$$\Rightarrow \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}.$$

Example 7. *Express $f(x) = |\,x\,|$, $-\pi < x < \pi$, as Fourier series. Hence show that*

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

Sol. Since $\qquad f(-x) = |-x| = |\,x\,| = f(x)$

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $\qquad f(x) = |\,x\,| = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} |\,x\,|\, dx = \frac{2}{\pi}\int_0^{\pi} x\, dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} |\,x\,|\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

Putting $x = 0$ in the above result, we get $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

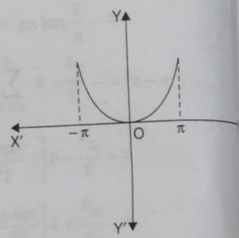
Example 8. *Obtain the Fourier series for the function $f(x) = x^2$, $-\pi \le x \le \pi$. Sketch th graph of f(x). Hence show that*

(PTU May 200

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.



Graph of f(x) = x²

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$.

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $\quad f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $\quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(x^2\frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi} 2x\cdot\frac{\sin nx}{n}\,dx\right]$$

$$= -\frac{4}{\pi n}\int_0^{\pi} x \sin nx\,dx$$

$$= -\frac{4}{\pi n}\left[\left\{x\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi} 1\cdot\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{4}{\pi n^2}(\pi\cos n\pi) = \frac{4}{n^2}(-1)^n$$

$$\therefore \quad x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right)$$

...(

Putting $x = \pi$ in (1), we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left(-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

$$\Rightarrow \quad \frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$$

0 X E → 0

$$\therefore \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \qquad \ldots(2)$$

0 X 0 → E

Putting $x = 0$ in (1), we get

E X E → E

$$0 = \frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

$$\therefore \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \qquad \ldots(3)$$

Adding (2) and (3), we get

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi^2}{4}$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

0 x 0 → E

Example 9. *Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$. Deduce that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots = \frac{\pi - 2}{4}$.*

(−x − sin x)

Sol. Since $x \sin x$ is an even function of x, $b_n = 0$

Let $\quad f(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $\quad a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x\,dx = \frac{2}{\pi}\Big[x(-\cos x) - 1\cdot(-\sin x)\Big]_0^{\pi} = \frac{2}{\pi}(-\pi\cos\pi) = 2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x(2\cos nx \sin x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\cdot\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}\right] = \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1},\ n \ne 1$$

$$\therefore \quad \sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2-a^2}\sin nx$$

$$= \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-\cdots\right)$$

ASSIGNMENT

1. Express $f(x) = \frac{1}{2}(\pi - x)$ in a Fourier series in the interval $0 < x < 2\pi$. Deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

2. (i) Obtain the Fourier series to represent e^{-x} in the interval $0 < x < 2\pi$.
   (ii) Find the Fourier series to represent e^{ax} in the interval $-\pi < x < \pi$.
3. Obtain a Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$ and hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

4. If $f(x)$ is a periodic function defined over a period $(0, 2\pi)$ as $f(x) = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}$

Prove that, $f(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ and hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

5. Find the Fourier series for the function $f(x) = x + x^2, -\pi < x < \pi$. Hence show that

(i) $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ (ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

6. Find the Fourier series to represent the function $f(x) = |\sin x|, -\pi < x < \pi$.
7. Expand $f(x) = |\cos x|$ as a Fourier series in the interval $-\pi < x < \pi$. (PTU May 2010)
8. Prove that in the interval $-\pi < x < \pi$,

$$x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{n(-1)^n}{n^2-1}\sin nx.$$

9. Prove that for $-\pi < x < \pi$, $\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + \cdots$
10. Find Fourier series of $f(x) = x^3$ in $(-\pi, \pi)$.
11. Expand the function $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in Fourier series in the interval $(-\pi, \pi)$.
12. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \frac{2a}{\pi}\sinh a\pi\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nx\right]$

When n is odd, $n \neq 1$, $n-1$ and $n+1$ are even

$$\therefore \quad a_n = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}$$

When n is even, $n-1$ and $n+1$ are odd

$$\therefore \quad a_n = \frac{-1}{n-1} + \frac{1}{n+1} = \frac{-2}{n^2-1}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\cdot\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{\pi\cos 2\pi}{2}\right] = -\frac{1}{2}$$

$$\therefore \quad x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1} + \cdots\right)$$

Putting $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 1 - 2\left(\frac{-1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} + \cdots\right)$$

$$\Rightarrow \quad \frac{\pi}{2} - 1 = 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \cdots\right)$$

$$\Rightarrow \quad \frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

Example 10. *Show that for* $-\pi < x < \pi$,

$$\sin ax = \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \cdots\right).$$

Sol. Since $\sin ax$ is an odd function of x, $\therefore \; a_0 = 0$ and $a_n = 0$

Let $\quad \sin ax = \sum_{n=1}^{\infty} b_n \sin nx$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx \, dx = \frac{1}{\pi}\int_0^{\pi}[\cos(n-a)x - \cos(n+a)x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] = -\frac{(-1)^n \sin a\pi}{\pi}\left[\frac{1}{n-a} + \frac{1}{n+a}\right]$$

$$= (-1)^{n+1}\cdot\frac{2n\sin a\pi}{\pi(n^2-a^2)}$$

$$\therefore \quad \sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,n}{n^2-a^2}\sin nx$$

$$= \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \cdots\right).$$

ASSIGNMENT

1. Express $f(x) = \frac{1}{2}(\pi - x)$ in a Fourier series in the interval $0 < x < 2\pi$. Deduce that
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$
2. (*i*) Obtain the Fourier series to represent e^x in the interval $0 < x < 2\pi$.
(*ii*) Find the Fourier series to represent e^{ax} in the interval $-\pi < x < \pi$.
3. Obtain a Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$ and hence deduce that
$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$
4. If $f(x)$ is a periodic function defined over a period $(0, 2\pi)$ as $f(x) = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}$
Prove that, $f(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ and hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.
5. Find the Fourier series for the function $f(x) = x + x^2$, $-\pi < x < \pi$. Hence show that
(*i*) $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ (*ii*) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$
6. Find the Fourier series to represent the function $f(x) = |\sin x|$, $-\pi < x < \pi$.
7. Expand $f(x) = |\cos x|$ as a Fourier series in the interval $-\pi < x < \pi$. (*PTU May 2012*)
8. Prove that in the interval $-\pi < x < \pi$,
$$x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{n(-1)^n}{n^2-1}\sin nx.$$
9. Prove that for $-\pi < x < \pi$, $\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + \cdots$
10. Find Fourier series of $f(x) = x^3$ in $(-\pi, \pi)$.
11. Expand the function $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in Fourier series in the interval $(-\pi, \pi)$.
12. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \frac{2a}{\pi}\sinh a\pi\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nx\right]$.

13. Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\frac{\pi}{\sinh \pi}$.

14. Express $f(x) = \cos wx$, $-\pi < x < \pi$, where w is a fraction, as a Fourier series. Hence prove that

$$\cot \theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2 - \pi^2} + \frac{2\theta}{\theta^2 - 4\pi^2} + \cdots$$

15. Obtain a Fourier series expansion for $\sqrt{1 - \cos x}$ in the interval $-\pi < x < \pi$.

Answers

1. $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

2. (i) $e^x = \frac{e^{2\pi} - 1}{2\pi} + \frac{e^{2\pi} - 1}{\pi} \sum_{n=1}^{\infty} \left(\frac{\cos nx}{1+n^2} - \frac{n}{1+n^2} \sin nx \right)$

(ii) $e^{ax} = \frac{\sinh a\pi}{a\pi} + \frac{2a \sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} \cos nx + \frac{2 \sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^n}{a^2+n^2} \sin nx$

3. $x^2 = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

5. $x + x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$

6. $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \cdots + \frac{\cos 2nx}{4n^2 - 1} + \cdots \right)$

7. $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi} \left(\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \cdots \right)$

10. $f(x) = 2 \sum_{n=1}^{\infty} \left(\frac{6}{n^3} - \frac{\pi^2}{n} \right) (-1)^n \sin nx$ 11. $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos nx}{n^2}$

13. $e^{-ax} = \frac{2 \sinh a\pi}{\pi} \left[\left(\frac{1}{2a} - \frac{a \cos x}{1^2 + a^2} + \frac{a \cos 2x}{2^2 + a^2} - \cdots \right) - \left(\frac{\sin x}{1^2 + a^2} - \frac{2 \sin 2x}{2^2 + a^2} + \frac{3 \sin 3x}{3^2 + a^2} - \cdots \right) \right]$

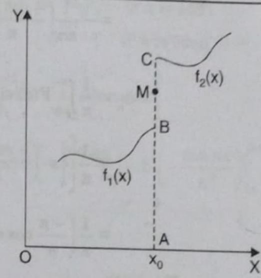
$\frac{\pi}{\sinh \pi} = 2 \left[\frac{1}{2^2 + 1} - \frac{1}{3^2 + 1} + \frac{1}{4^2 + 1} - \cdots \right]$

14. $\cos wx = \frac{2w \sin w\pi}{\pi} \left(\frac{1}{2w^2} + \frac{\cos x}{1^2 - w^2} - \frac{\cos 2x}{2^2 - w^2} + \frac{\cos 3x}{3^2 - w^2} - \cdots \right)$

15. $f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{4n^2 - 1}$

1.6. FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

In art. 1.4, we derived Euler's formulae for a_0, a_n, b_n on the assumption that $f(x)$ is continuous in $(c, c + 2\pi)$. However, if $f(x)$ has finitely many points of finite discontinuity, even then it can be expressed as a Fourier series. The integrals for a_0, a_n, b_n are to be evaluated by breaking up the range of integration.



Let $f(x)$ be defined by

$$f(x) = \begin{cases} f_1(x), & c < x < x_0 \\ f_2(x), & x_0 < x < c + 2\pi \end{cases}$$

where x_0 is the point of finite discontinuity in the interval $(c, c + 2\pi)$.

The values of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi} \left[\int_c^{x_0} f_1(x)\, dx + \int_{x_0}^{c+2\pi} f_2(x)\, dx \right]$$

$$a_n = \frac{1}{\pi} \left[\int_c^{x_0} f_1(x) \cos nx\, dx + \int_{x_0}^{c+2\pi} f_2(x) \cos nx\, dx \right]$$

$$b_n = \frac{1}{\pi} \left[\int_c^{x_0} f_1(x) \sin nx\, dx + \int_{x_0}^{c+2\pi} f_2(x) \sin nx\, dx \right]$$

At $x = x_0$, there is a finite jump in the graph of the function. Both the limits $f(x_0 - 0)$ and $f(x_0 + 0)$ exist but are unequal. The sum of the Fourier series $= \frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)]$ $= \frac{1}{2}$ (AB + AC) = AM, where M is the mid-point of BC.

EXAMPLES

Example 1. *Find the Fourier series for the periodic function*

$$F(x) = \begin{matrix} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{matrix} \Bigg\}; \; F(x + 2\pi) = F(x).$$

Sol. Let $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(1)

$\frac{1}{\pi}\left[\int_{-\pi}^{0} f(u)\,du + \int_{0}^{\pi} f(u)\,du\right]$

Here, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x)\, dx = \frac{1}{\pi} \int_0^{\pi} x\, dx = \frac{1}{\pi} \left(\frac{x^2}{2} \right)_0^{\pi} = \frac{\pi}{2}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{\pi} x \cos nx\, dx$

$x\,dx$

$= \frac{1}{\pi} \left[\left(x \frac{\sin nx}{n} \right)_0^{\pi} - \int_0^{\pi} 1 \cdot \frac{\sin nx}{n}\, dx \right]$

$$= -\frac{1}{\pi n}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{1}{\pi n^2}\,[(-1)^n - 1] = \begin{cases} \dfrac{-2}{\pi n^2}, & n \text{ is odd} \\ 0\;, & n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\left\{x\cdot\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi} 1\cdot\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left(\frac{-\pi}{n}\cos n\pi\right) = -\frac{1}{n}(-1)^n$$

$\therefore$ From (1),

$$F(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{\substack{n=1,3,5\\(n\text{ is odd})}}^{\infty}\frac{1}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right].$$

Example 2. *Find the Fourier series to represent the function $f(x)$ given by*

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } \pi \le x \le 2\pi \end{cases}$$

Deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$.

Sol. Let $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n\cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$, then

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi} + \left|2\pi x - \frac{x^2}{2}\right|_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{0}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\cdot\frac{\sin nx}{n} - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} + \left|(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right) + \left(-\frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi n^2}\,[(-1)^n - 1 - 1 + (-1)^n] = \frac{2}{\pi n^2}\,[(-1)^n - 1] = \begin{cases} -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0\;, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi} + \left|(2\pi - x)\times\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{\pi\cos n\pi}{n}\right] = 0$$

$$\therefore\; f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

Putting $x = 0$, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Example 3. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$

Prove that $f(x) = \dfrac{1}{\pi} + \dfrac{1}{2}\sin x - \dfrac{2}{\pi}\sum\limits_{n=1}^{\infty}\dfrac{\cos 2nx}{4n^2 - 1}$.

Hence show that

(i) $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \cdots = \dfrac{1}{2}$ (ii) $\dfrac{1}{1\cdot 3} - \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} - \cdots = \dfrac{\pi - 2}{4}$.

Sol. Let $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n\cos nx + \sum\limits_{n=1}^{\infty} b_n\sin nx$

Then $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\,dx\right] = \dfrac{2}{\pi}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\cos nx\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi},\; n \ne 1$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \begin{cases} \dfrac{1}{2\pi}\left(-\dfrac{1}{n+1}+\dfrac{1}{n-1}+\dfrac{1}{n+1}-\dfrac{1}{n-1}\right), \text{ when } n \text{ is odd} \\ \dfrac{1}{2\pi}\left(\dfrac{1}{n+1}-\dfrac{1}{n-1}+\dfrac{1}{n+1}-\dfrac{1}{n-1}\right), \text{ when } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0, \text{ when } n \text{ is odd, i.e., } n = 3, 5, 7, \ldots \\ -\dfrac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even} \end{cases}$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^\pi \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^\pi \sin 2x \, dx = \frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^\pi = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx = \frac{1}{\pi}\left[\int_{-\pi}^0 0 \, dx + \int_0^\pi \sin x \sin nx \, dx\right]$$

$$= \frac{1}{2\pi}\int_0^\pi 2\sin nx \sin x \, dx = \frac{1}{2\pi}\int_0^\pi [\cos(n-1)x - \cos(n+1)x] \, dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi = 0, \; n \neq 1$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^\pi (1-\cos 2x) \, dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2}$$

$$\therefore \quad f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\right] + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n)^2-1} \qquad \ldots(1)$$

Putting $x = 0$ in (1), we have $0 = \dfrac{1}{\pi} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{4n^2-1}$

$$\Rightarrow \quad \frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$$

Putting $x = \dfrac{\pi}{2}$ in (1), we have $1 = \dfrac{1}{\pi} + \dfrac{1}{2} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos n\pi}{4n^2-1}$

$$\Rightarrow \quad \frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$

$$\Rightarrow \quad \frac{\pi-2}{4} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)} = -\left(-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots\right)$$

$$\Rightarrow \quad \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi-2}{4}.$$

Example 4. *Obtain Fourier series for the function $f(x)$ given by*

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

Hence deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$. (P.T.U. Dec. 2007, May 2008)

Sol. When $-\pi \le x \le 0$, $\quad 0 \le -x \le \pi$

$$\therefore \quad f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi} = f(x)$$

When $0 \le x \le \pi$, $\quad -\pi \le -x \le 0$

$$\therefore \quad f(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = f(x)$$

$\Rightarrow$ $f(x)$ is an even function of x in $[-\pi, \pi]$. This is also clear from its graph which is symmetrical about the y-axis.

$\therefore \quad b_n = 0$

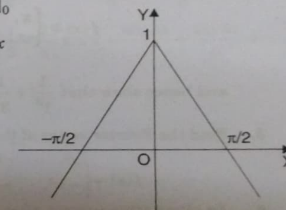
Let $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$

$$\text{Here, } a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{x}\left[x - \frac{x^2}{\pi}\right]_0^\pi = 0$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right] = \frac{4}{\pi^2 n^2}[1 - (-1)^n]$$



$$\therefore \quad f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}[1 - (-1)^n]\frac{\cos nx}{n^2}$$

$$= \frac{4}{\pi^2}\left(\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \frac{2\cos 5x}{5^2} + \cdots\right) = \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

Putting $\quad x = 0$, we get $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$ $\qquad | \because f(0) = 1.$

ASSIGNMENT

1. Find the Fourier series to represent the function

$$f(x) = \begin{cases} -k \text{ when} & -\pi < x < 0 \\ k \text{ when} & 0 < x < \pi \end{cases}$$

Also deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$

2. (i) Find the Fourier series for the function

$$f(x)=\begin{cases}-1, & -\pi<x<-\pi/2\\ 0, & -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}\\ 1, & \dfrac{\pi}{2}<x<\pi\end{cases}$$

(ii) Find a Fourier series for the function defined by:

$$f(x)=\begin{cases}-1, & \text{for } -\pi<x<0\\ 0, & \text{for } x=0\\ 1, & \text{for } 0<x<\pi\end{cases}$$

Hence prove that $\dfrac{\pi}{4}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\dfrac{1}{9}-\cdots$

3. (i) Obtain a Fourier series to represent the following periodic function

$$f(x)=\begin{cases}0, & 0<x<\pi\\ 1, & \pi<x<2\pi\end{cases}$$

(ii) Develop $f(x)$ in a Fourier series in the interval $(-\pi, \pi)$ if $f(x)=\begin{cases}0, & \text{where } -\pi<x<0\\ 1, & \text{where } 0<x<\pi.\end{cases}$

4. Obtain Fourier series for the function

$$f(x)=\begin{cases}x, & -\pi<x<0\\ -x, & 0<x<\pi\end{cases}$$

and hence show that $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\cdots=\dfrac{\pi^2}{8}$

5. Find the Fourier series of the following function:

$$f(x)=\begin{cases}x^2, & 0\le x\le\pi\\ -x^2, & -\pi\le x\le 0.\end{cases}$$

(PTU Dec. 20...

6. An alternating current after passing through a rectifier has the form

$$i=\begin{cases}I_0\sin x & \text{for } 0\le x\le\pi\\ 0 & \text{for } \pi\le x\le 2\pi\end{cases}$$

where I_0 is the maximum current and the period is 2π. Express i as a Fourier series.

7. Find the Fourier series of the function defined as

$$f(x)=\begin{cases}x+\pi, & \text{for } 0\le x\le\pi\\ -x-\pi, & \text{for } -\pi\le x<0\end{cases}\quad\text{and}\quad f(x+2\pi)=f(x).$$

8. (i) Find the Fourier series for $f(x)$ in the interval $(-\pi, \pi)$ when

$$f(x)=\begin{cases}\pi+x, & -\pi<x<0\\ \pi-x, & 0<x<\pi\end{cases}$$

(ii) Find the Fourier series for following periodic function:

$$f(x)=\begin{cases}\pi+x, & -\pi<x<0\\ -\pi-x, & 0<x<\pi\end{cases}$$

9. Find the Fourier series to represent the periodic function

$$f(x)=\begin{cases}x, & -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}\\ \pi-x, & \dfrac{\pi}{2}<x<\dfrac{3\pi}{2}\end{cases}$$

10. Find the Fourier series of

$$f(x)=\begin{cases}0, & \text{when } -\pi\le x\le 0\\ x^2, & \text{when } 0\le x\le\pi\end{cases}$$

which is assumed to be periodic with period 2π.

11. Find the Fourier series expansion for $f(x)$, if

$$f(x)=\begin{cases}-\pi, & -\pi<x<0\\ x, & 0<x<\pi\end{cases}$$

Deduce that $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\cdots=\dfrac{\pi^2}{8}$

(PTU Dec. 2010)

Answers

1. $f(x)=\dfrac{4k}{\pi}\left(\sin x+\dfrac{\sin 3x}{3}+\dfrac{\sin 5x}{5}+\cdots\right)$

2. (i) $f(x)=\dfrac{2}{\pi}\left[\sin x-\sin 2x+\dfrac{\sin 3x}{3}-\cdots\right]$ (ii) $f(x)=\dfrac{4}{\pi}\left(\sin x+\dfrac{\sin 3x}{3}+\dfrac{\sin 5x}{5}+\cdots\right)$

3. (i) $f(x)=\dfrac{1}{2}-\dfrac{2}{\pi}\left[\sin x+\dfrac{1}{3}\sin 3x+\dfrac{1}{5}\sin 5x+\cdots\right]$ (ii) $f(x)=\dfrac{1}{2}+\dfrac{2}{\pi}\left(\sin x+\dfrac{\sin 3x}{3}+\dfrac{\sin 5x}{5}+\cdots\right)$

4. $f(x)=-\dfrac{\pi}{2}+\dfrac{4}{\pi}\left[\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\cdots\right]$

5. $f(x)=2\left(\pi-\dfrac{4}{\pi}\right)\sin x-\pi\sin 2x+\dfrac{2}{3}\left(\pi-\dfrac{4}{9\pi}\right)\sin 3x-\dfrac{\pi}{2}\sin 4x+\cdots$

6. $i=\dfrac{I_0}{\pi}+\dfrac{I_0}{2}\sin x-\dfrac{2I_0}{\pi}\left(\dfrac{\cos 2x}{2^2-1}+\dfrac{\cos 4x}{4^2-1}+\dfrac{\cos 6x}{6^2-1}+\cdots\right)$

7. $f(x)=\dfrac{\pi}{2}-\dfrac{4}{\pi}\left[\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\cdots\right]+4\left[\dfrac{\sin x}{1}+\dfrac{\sin 3x}{3}+\cdots\right]$

8. (i) $f(x)=\dfrac{\pi}{2}+\dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\cdots\right)$

(ii) $f(x)=-\dfrac{\pi}{2}+\dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\cdots\right)+4\left(\dfrac{\sin x}{1}+\dfrac{\sin 3x}{3}+\cdots\right)$

9. $f(x)=\dfrac{4}{\pi}\left(\dfrac{\sin x}{1^2}-\dfrac{\sin 3x}{3^2}+\dfrac{\sin 5x}{5^2}-\cdots\right)$

10. $f(x)=\dfrac{\pi^2}{6}+2\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos nx+\sum_{n=1}^{\infty}\left[\dfrac{\pi}{n}(-1)^{n+1}+\dfrac{2\{(-1)^n-1\}}{\pi n^3}\right]\sin nx$

11. $f(x)=-\dfrac{\pi}{4}-\dfrac{2}{\pi}\left[\dfrac{1}{1^2}\cos x+\dfrac{1}{3^2}\cos 3x+\dfrac{1}{5^2}\cos 5x+\cdots\right]+\displaystyle\sum_{n=1}^{\infty}\left(\dfrac{1-2\cos n\pi}{n}\right)\sin nx$

1.7. CHANGE OF INTERVAL

In many engineering problems, it is desired to expand a function in a Fourier seri over an interval of length $2l$ and not 2π. In order to apply foregoing theory, this interval mu be transformed into an interval of length 2π. This can be achieved by a transformation of t variable.

Consider a periodic function $f(x)$ defined in the interval $c < x < c + 2l$. To change t interval into one of length 2π, we put

$$\frac{x}{l} = \frac{z}{\pi} \quad \text{or} \quad z = \frac{\pi x}{l} \text{ so that}$$

when $\quad x = c, \quad z = \dfrac{\pi c}{l} = d$ (say)

and $\quad$ when $\quad x = c + 2l, \quad z = \dfrac{\pi(c+2l)}{l} = \dfrac{\pi c}{l} + 2\pi = d + 2\pi.$

Thus the function $f(x)$ of period $2l$ in $(c, c + 2l)$ is transformed to the functi $f\left(\dfrac{lz}{\pi}\right) = F(z)$, say, of period 2π in $(d, d + 2\pi)$ and the latter function can be expressed as t Fourier series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \quad ...($$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_d^{d+2\pi} F(z)\,dz;\quad a_n = \dfrac{1}{\pi}\displaystyle\int_d^{d+2\pi} F(z)\cos nz\,dz;\quad$ and $\quad b_n = \dfrac{1}{\pi}\displaystyle\int_d^{d+2\pi} F(z)\sin nz$ ...(

Now making the inverse substitution $z = \dfrac{\pi x}{l}$, $dz = \dfrac{\pi}{l}\,dx$

When $\quad z = d,\ x = c\ $ and when $\ z = d + 2\pi, x = c + 2l.$

The expression (1) becomes $F(z) = F\left(\dfrac{\pi x}{l}\right) = f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi}{l}$

and the coefficients a_0, a_n, b_n from (2) reduce to

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx;\quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx;\quad \text{and}\quad b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,d$$

Hence the Fourier series $f(x)$ in the interval $c < x < c + 2l$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}}$$

where $a_0 = \dfrac{1}{l}\displaystyle\int_c^{c+2l} f(x)\,dx,\quad a_n = \dfrac{1}{l}\displaystyle\int_c^{c+2l} f(x)\cos\dfrac{n\pi x}{l}\,dx\quad$ and $\quad b_n = \dfrac{1}{l}\displaystyle\int_c^{c+2l} f(x)\sin\dfrac{n\pi x}{l}\,d$

Cor. 1. If we put $c = 0$, the interval becomes $0 < x < 2l$ and the above results reduce t

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx;\quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx;\quad \text{and}\quad b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Cor. 2. If we put $c = -l$, the interval becomes $-l < x < l$ and the above results reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx,\quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,\quad \text{and}\quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Cor. 3. If $f(x)$ is an even function, we have

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx,\quad a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad\text{and}\quad b_n = 0$$

Cor. 4. If $f(x)$ is an odd function, we have

$$a_0 = 0,\quad a_n = 0 \text{ and } b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

EXAMPLES

0 < x < 2
0 < x < 2l
2l = 2

Example 1. *Obtain the Fourier series expansion of*

$$f(x) = \left(\frac{\pi - x}{2}\right) \textit{ for } 0 < x < 2$$

Sol. Let $\quad f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}$

Here, $\quad l = 1$

$$\therefore \quad \frac{\pi - x}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad ...(1)$$

Here, $\quad a_0 = \dfrac{1}{l}\displaystyle\int_0^{2l} f(x)\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right) dx$

$$= \frac{1}{2}\left(\pi x - \frac{x^2}{2}\right)_0^2 = \frac{1}{2}(2\pi - 2) = \pi - 1$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos n\pi x\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right)\cos n\pi x\,dx$$

$$= \frac{1}{2}\left[\left\{(\pi - x)\frac{\sin n\pi x}{n\pi}\right\}_0^2 - \int_0^2 (-1)\frac{\sin n\pi x}{n\pi}\,dx\right]$$

$$= \frac{1}{2n\pi}\left(\frac{-\cos n\pi x}{n\pi}\right)_0^2 = 0$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin n\pi x\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right)\sin n\pi x\,dx$$

$$= \frac{1}{2}\left[\left\{(\pi - x)\left(\frac{-\cos n\pi x}{n\pi}\right)\right\}_0^2 - \int_0^2 (-1)\left(\frac{-\cos n\pi x}{n\pi}\right)dx\right]$$

$$= -\frac{1}{2n\pi}[(\pi - 2) - \pi] = \frac{1}{n\pi}$$

Hence, from (1),

$$\frac{\pi - x}{2} = \frac{(\pi - 1)}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin n\pi x$$

Example 2. *Find Fourier expansion for the function $f(x) = x - x^2, -1 < x < 1$.*

$-l<x<l$ (Here $l = 1$)

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then $a_0 = \int_{-1}^{1} (x - x^2)\, dx = \int_{-1}^{1} x\, dx - \int_{-1}^{1} x^2\, dx = 0 - 2\int_0^1 x^2\, dx = -2\left[\frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$

$$a_n = \int_{-1}^{1} (x - x^2) \cos n\pi x\, dx = \int_{-1}^{1} x \cos n\pi x\, dx - \int_{-1}^{1} x^2 \cos n\pi x\, dx$$

$$= 0 - 2\int_0^1 x^2 \cos n\pi x\, dx = -2\left[\left(x^2 \frac{\sin n\pi x}{n\pi}\right)_0^1 - \int_0^1 2x \cdot \frac{\sin n\pi x}{n\pi}\, dx\right]$$

$$= \frac{4}{n\pi}\int_0^1 x \sin n\pi x\, dx$$

$$= \frac{4}{n\pi}\left[\left\{x \cdot \left(\frac{-\cos n\pi x}{n\pi}\right)\right\}_0^1 - \int_0^1 \left(\frac{-\cos n\pi x}{n\pi}\right) dx\right]$$

$$= -\frac{4}{n^2\pi^2} \cos n\pi = \frac{-4(-1)^n}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1} (x - x^2) \sin n\pi x\, dx = \int_{-1}^{1} x \sin n\pi x\, dx - \int_{-1}^{1} x^2 \sin n\pi x\, dx$$

$$= 2\int_0^1 x \sin n\pi x\, dx - 0 = 2\left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1 \cdot \left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos n\pi}{n\pi}\right] = \frac{-2(-1)^n}{n\pi}$$

$$\therefore \quad x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \cdots\right) + \frac{2}{\pi}\left(\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \cdots\right)$$

Example 3. *Find the Fourier series to represent $f(x) = x^2 - 2$, when $-2 \le x \le 2$.*

Sol. Since $f(x)$ is an even function, $b_n = 0$.

Let $f(x) = x^2 - 2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$

Then $a_0 = \frac{2}{2}\int_0^2 (x^2 - 2)\, dx = \left[\frac{x^3}{3} - 2x\right]_0^2 = \frac{8}{3} - 4 = -\frac{4}{3}$

$$a_n = \frac{2}{2}\int_0^2 (x^2 - 2) \cos \frac{n\pi x}{2}\, dx$$

$$= \left\{(x^2 - 2)\frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}}\right\}_0^2 - \int_0^2 (2x)\frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)}\, dx = -\frac{4}{n\pi}\int_0^2 x \sin \frac{n\pi x}{2}\, dx$$

$$= -\frac{4}{n\pi}\left[\left\{x\left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)\right\}_0^2 - \int_0^2 \left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) dx\right]$$

$$= \frac{8}{n^2\pi^2}(2 \cos n\pi) = \frac{16 \cos n\pi}{n^2\pi^2} = \frac{16(-1)^n}{n^2\pi^2}$$

$$\therefore \quad x^2 - 2 = -\frac{2}{3} - \frac{16}{\pi^2}\left(\cos \frac{\pi x}{2} - \frac{1}{4}\cos \pi x + \frac{1}{9}\cos \frac{3\pi x}{2} - \cdots\right).$$

Example 4. *Obtain Fourier series for function $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$. Also deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.* (PTU Dec. 2012)

$0<x<2$
$2l$
$l=1$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then $a_0 = \int_0^2 f(x)\, dx = \int_0^1 \pi x\, dx + \int_1^2 \pi(2 - x)\, dx = \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$

$$= \pi\left(\frac{1}{2}\right) + \pi\left[(4 - 2) - \left(2 - \frac{1}{2}\right)\right] = \pi$$

$$a_n = \int_0^2 f(x) \cos n\pi x\, dx = \int_0^1 \pi x \cos n\pi x\, dx + \int_1^2 \pi(2 - x) \cos n\pi x\, dx$$

$$= \left[\pi x \cdot \frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2 - x) \cdot \frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right] + \left[-\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right] = \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$= 0$ or $-\frac{4}{n^2\pi}$ according as n is even or odd.

$$b_n = \int_0^2 f(x) \sin n\pi x\, dx = \int_0^1 \pi x \sin n\pi x\, dx + \int_1^2 \pi(2 - x) \sin n\pi x\, dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2 - x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right).$$

Now, putting $x = 0$ in above series, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

ASSIGNMENT

1. (i) Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$. *(PTU Dec. 201*

   (ii) Find the Fourier series expansion of the function *(PTU May 201*

$$f(x) = x^2; -2 \le x \le 2$$

   (iii) If $f(x) = x^2$ for $-3 \le x \le 3$, write Fourier series for $f(x)$ on $[-3, 3]$.

2. (i) Expand $f(x)$ in Fourier series in the interval $(-2, 2)$ when $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1, & 0 < x < 2. \end{cases}$

   (ii) Find the Fourier series of $f(x) = \begin{cases} 0\ , & \text{if } -2 \le x \le -1 \\ 1 + x, & \text{if } -1 \le x \le 0 \\ 1 - x, & \text{if } 0 \le x \le 1 \\ 0\ , & \text{if } 1 \le x \le 2 \end{cases}$

   [Hint. $f(x)$ is an even function]

   (iii) Find the Fourier series for the function $f(x) = \begin{cases} 0 & \text{when } -2 < x < -1 \\ k & \text{when } -1 < x < 1 \\ 0 & \text{when } 1 < x < 2 \end{cases}$.

3. (i) Develop $f(x)$ in a Fourier series in the interval $(0, 2)$ if

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$

   (ii) Find the Fourier series for the function given by $f(t) = \begin{cases} t, & 0 < t < 1 \\ 1 - t, & 1 < t < 2. \end{cases}$

4. (i) Find the Fourier expansion for $f(x) = \pi x$ from $x = -c$ to $x = c$.

   (ii) Find the Fourier expansion for the function $f(x) = x - x^3$ in the interval $-1 < x < 1$.

5. Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.

6. Find the Fourier series expansion of the following periodic function of period 4

$$f(x) = \begin{cases} 2 + x, & -2 \le x \le 0 \\ 2 - x, & 0 \le x \le 2 \end{cases} \quad \text{and} \quad f(x + 4) = f(x)$$

   Also, obtain $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

7. Obtain Fourier series of the function $F(x) = \left\{\begin{matrix} 1 + \frac{2x}{l}, & -l < x < 0 \\ 1 - \frac{2x}{l}, & 0 < x < l \end{matrix}\right\}$.

8. A sinusoidal voltage E sin ωt is passed through a half-wave rectifier which clips the negativ portion of the wave. Expand the resulting periodic function $u(t) = \begin{cases} 0 & \text{when } -\frac{T}{2} < t < 0 \\ E \sin \omega t & \text{when } 0 < t < \frac{T}{2} \end{cases}$ and $T = \frac{2\pi}{\omega}$, in a Fourier series.

Answers

1. (i) $1 - t^2 = \frac{2}{3} + \frac{4}{\pi^2}\left(\cos \pi t - \frac{\cos 2\pi t}{2^2} + \frac{\cos 3\pi t}{3^2} - \dots\right)$

   (ii) $f(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2}$

   (iii) $f(x) = 3 + \frac{36}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{3}$

2. (i) $f(x) = \frac{1}{2} + \frac{2}{\pi^2}\left(\sin \frac{\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} + \frac{1}{5} \sin \frac{5\pi x}{2} + \dots\right)$

   (ii) $f(x) = \frac{1}{4} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\left(1 - \cos \frac{n\pi}{2}\right)}{n^2} \cos \frac{n\pi x}{2}$

   (iii) $f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos \frac{\pi x}{2} - \frac{1}{3} \cos \frac{3\pi x}{2} + \frac{1}{5} \cos \frac{5\pi x}{2} - \dots\right)$

3. (i) $f(x) = \frac{1}{4} - \frac{2}{\pi^2}\left(\cos \pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots\right) + \frac{1}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \dots\right)$

   (ii) $f(t) = -\frac{4}{\pi^2}\left(\cos \pi t + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^2} + \dots\right) + \frac{2}{\pi}\left(\sin \pi t + \frac{\sin 3\pi t}{3} + \dots\right)$

4. (i) $f(x) = 2c\left[\sin\left(\frac{\pi x}{c}\right) - \frac{1}{2} \sin\left(\frac{2\pi x}{c}\right) + \frac{1}{3} \sin\left(\frac{3\pi x}{c}\right) - \dots\right]$

   (ii) $f(x) = \frac{12}{\pi^3}\left(\sin \pi x - \frac{\sin 2\pi x}{2^3} + \frac{\sin 3\pi x}{3^3} - \dots\right)$

5. $$e^{-x} = \sinh l \left[\frac{1}{l} - 2l\left\{\frac{1}{l^2 + \pi^2} \cos \frac{\pi x}{l} - \frac{1}{l^2 + 2^2\pi^2} \cos \frac{2\pi x}{l} + \frac{1}{l^2 + 3^2\pi^2} \cos \frac{3\pi x}{l} - \dots\right\}\right.$$
$$\left. -2\pi\left\{\frac{1}{l^2 + \pi^2} \sin \frac{\pi x}{l} - \frac{2}{l^2 + 2^2\pi^2} \sin \frac{2\pi x}{l} + \frac{3}{l^2 + 3^2\pi^2} \sin \frac{3\pi x}{l} - \dots\right\}\right]$$

6. $f(x) = 1 + \frac{8}{\pi^2}\left[\frac{1}{1^2} \cos \frac{\pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi x}{2} + \dots\right]$

7. $F(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \{1 - (-1)^n\} \frac{\cos n \frac{\pi x}{l}}{n^2}$

8. $u(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi}\left(\frac{\cos 2\omega t}{1 \cdot 3} + \frac{\cos 4\omega t}{3 \cdot 5} + \frac{\cos 6\omega t}{5 \cdot 7} + \dots\right)$

1.8. HALF RANGE SERIES

Sometimes it is required to expand a function $f(x)$ in the range $(0, \pi)$ in a Fourier series of period 2π or more generally in the range $(0, l)$ in a Fourier series of period $2l$.

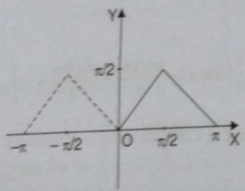
If it is required to expand $f(x)$ in the interval $(0, l)$, then it is immaterial what the function may be outside the range $0 < x < l$. We are free to choose it arbitrarily in the interval $(-l, 0)$.

If we extend the function $f(x)$ by reflecting it in the y-axis so that $f(-x) = f(x)$, then the extended function is even for which $b_n = 0$. The Fourier expansion of $f(x)$ will contain only cosine terms.

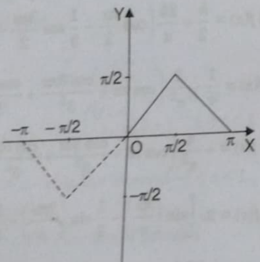
If we extend the function $f(x)$ by reflecting it in the origin so that $f(-x) = -f(x)$, then the extended function is odd for which $a_0 = a_n = 0$. The Fourier expansion of $f(x)$ will contain only sine terms.

For example, consider the function

$$f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$$



(Reflection in the y-axis)



(Reflection in the origin)

Hence a function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half-range series.

The half-range cosine series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}}$$

where $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx\,; \quad a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\,dx.$$

The half-range sine series is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}}$$

where $$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\,dx.$$

Cor. If the range is $0 < x < \pi$, then

(i) The half-range cosine series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$$

where $$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx\,; \quad a_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\,dx.$$

(ii) The half-range sine series is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

where $$b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\,dx.$$

EXAMPLES

Example 1. *Expand $\pi x - x^2$ in a half-range sine series in the interval $(0, \pi)$ upto the first three terms.*

Sol. Let $$\pi x - x^2 = \sum_{n=1}^{\infty} b_n \sin nx, \text{ then}$$

$$b_n = \frac{2}{\pi}\int_0^\pi (\pi x - x^2) \sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{n^3} + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}[1 - (-1)^n]$$

$$= 0 \text{ or } \frac{8}{\pi n^3} \text{ according as } n \text{ is even or odd.}$$

$$\therefore \quad \pi x - x^2 = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right).$$

Example 2. *Expand $f(x) = x$ as a half-range*

(i) *sine series in $0 < x < 2$* (ii) *cosine series in $0 < x < 2$.*

Sol. (i) Let $$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \quad \ldots(1)$$

where, $$b_n = \int_0^2 x \sin \frac{n\pi x}{2}\,dx$$

$$= \left\{x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)\right\}_0^2 - \int_0^2 \left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)dx$$

$$= -\frac{4}{n\pi}\cos n\pi = -\frac{4}{n\pi}(-1)^n$$

Hence from (1),

$$x = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}$$

(*ii*) Let

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \quad \ldots($$

where,

$$a_0 = \int_0^2 x\,dx = \left(\frac{x^2}{2}\right)_0^2 = 2$$

and

$$a_n = \int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left(x\,\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_0^2 - \int_0^2 \frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\,dx$$

$$= -\frac{2}{n\pi}\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_0^2 = \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

Hence,

$$x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)}{n^2}\cos\frac{n\pi x}{2}$$

Example 3. *Develop* $\sin\left(\frac{\pi x}{l}\right)$ *in half-range cosine series in the range* $0 < x < l$.

Sol. Let

$$\sin\left(\frac{\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

then

$$a_0 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\,dx = \frac{2}{l}\left[-\frac{\cos\frac{\pi x}{l}}{\frac{\pi}{l}}\right]_0^l = -\frac{2}{\pi}[\cos\pi - 1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l \left[\sin(n+1)\frac{\pi x}{l} - \sin(n-1)\frac{\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[-\frac{\cos(n+1)\frac{\pi x}{l}}{(n+1)\frac{\pi}{l}} + \frac{\cos(n-1)\frac{\pi x}{l}}{(n-1)\frac{\pi}{l}}\right]_0^l$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

When n is odd, $a_n = \frac{1}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0$

When n is even, $a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = -\frac{4}{\pi(n+1)(n-1)}$$

$$\therefore \quad \sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1.3} + \frac{\cos\frac{4\pi x}{l}}{3.5} + \frac{\cos\frac{6\pi x}{l}}{5.7} + \cdots\right].$$

Example 4. *If*

$$f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$$

show that

(*i*) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$

(*ii*) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right].$ (PTU Dec. 2007)

Sol. (*i*) For the half-range sine series,

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Then $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

When n is even, $b_n = 0$.

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$$

(*ii*) For the half-range cosine series,

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi-x)\,dx\right]$

$$= \frac{2}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi/2} + \left|\pi x - \frac{x^2}{2}\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right] = \frac{2}{\pi}\left[\frac{\pi^2}{4}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^\pi (\pi-x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x\cdot\frac{\sin nx}{n} - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi-x)\cdot\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right] + \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$\therefore$ $a_1 = 0,\ a_2 = \frac{2}{\pi\cdot 2^2}(2\cos\pi - \cos 2\pi - 1) = \frac{-2}{\pi\cdot 1^2}$,

$a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \frac{2}{\pi\cdot 6^2}(2\cos 3\pi - \cos 6\pi - 1) = \frac{-2}{\pi\cdot 3^2}$,

$a_7 = 0,\ a_8 = 0,\ a_9 = 0,\ a_{10} = \frac{2}{\pi\cdot 10^2}(2\cos 5\pi - \cos 10\pi - 1) = \frac{-2}{\pi\cdot 5^2}$, ...

Hence $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right]$.

Example 5. *Find a series of cosines of multiples of x which will represent x sin x in the interval* $(0, \pi)$ *and show that* $\frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \cdots = \frac{\pi-2}{4}$.

Sol. Let $x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$

Then $a_0 = \frac{2}{\pi}\int_0^\pi x\sin x\,dx = \frac{2}{\pi}\left[x(-\cos x) - 1\cdot(-\sin x)\right]_0^\pi = \frac{2}{\pi}[-\pi\cos\pi] = 2$

Cos n Sub $a_n = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^\pi x\,(2\cos nx\sin x)\,dx$

= Sin(A+B) − Sin(A−B)

$$= \frac{1}{\pi}\int_0^\pi x\,[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\cdot\left\{-\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos(n+1)\pi}{n+1} + \frac{\pi\cos(n-1)\pi}{n-1}\right] \quad \text{when } n \neq 1$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^{n-1}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] = \frac{2(-1)^{n-1}}{(n-1)(n+1)}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\cdot\left(-\frac{\sin 2x}{2^2}\right)\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos 2\pi}{2}\right] = -\frac{1}{2}$$

$\therefore$ $x\sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1\cdot3} - \frac{\cos 3x}{2\cdot4} + \frac{\cos 4x}{3\cdot5} - \cdots\right)$.

Putting $x = \frac{\pi}{2}$, we get $\frac{\pi}{2} = 1 - 2\left(\frac{-1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{-1}{5\cdot7} - \cdots\right)$

$\Rightarrow$ $1 + \frac{2}{1\cdot3} - \frac{2}{3\cdot5} + \frac{2}{5\cdot7} - \cdots = \frac{\pi}{2}$

$\Rightarrow$ $\frac{2}{1\cdot3} - \frac{2}{3\cdot5} + \frac{2}{5\cdot7} - \cdots = \frac{\pi}{2} - 1$

Hence $\frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \cdots = \frac{\pi-2}{4}$.

Example 6. *Obtain a half-range cosine series for*

$$f(x) = \begin{cases} kx & \text{for } 0 \le x \le \frac{l}{2} \\ k(l-x) & \text{for } \frac{l}{2} \le x \le l. \end{cases}$$

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ (PTU May 2010)

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

then $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$

$$= \frac{2}{l}\left[\left|\frac{kx^2}{2}\right|_0^{l/2} + \left|k\left(lx - \frac{x^2}{2}\right)\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{kl^2}{8} + k\left(l^2 - \frac{l^2}{2}\right) - k\left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right] = \frac{2}{l}\left(\frac{kl^2}{4}\right) = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\cdot\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cdot\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left|kx\cdot\frac{l}{n\pi}\sin\frac{n\pi x}{l} + k\cdot\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_0^{l/2}\right.$$

$$\left. + \left|k(l-x)\cdot\frac{l}{n\pi}\sin\frac{n\pi x}{l} - k\cdot\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\left|\frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right|\right.$$

$$\left. + \left|\frac{-kl^2}{n^2\pi^2}\cos n\pi - \frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right|\right]$$

$$= \frac{2}{l}\left[\frac{2kl^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{kl^2}{n^2\pi^2} - \frac{kl^2}{n^2\pi^2}\cos n\pi\right] = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

When n is odd, $\cos\frac{n\pi}{2} = 0$ and $\cos n\pi = -1$ $\quad\therefore\ a_n = 0 \Rightarrow a_1 = a_3 = a_5 = \ldots = 0$

When n is even, $a_2 = \frac{2kl}{2^2\pi^2}[2\cos\pi - 1 - \cos 2\pi] = -\frac{8kl}{2^2\pi^2}$;

$$a_4 = \frac{2kl}{4^2\pi^2}[2\cos 2\pi - 1 - \cos 4\pi] = 0$$

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$$a_6 = \frac{2kl}{6^2\pi^2}[2\cos 3\pi - 1 - \cos 6\pi] = -\frac{8kl}{6^2\pi^2} \text{ and so on.}$$

$$\therefore \quad f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \cdots\right) \quad \ldots(1)$$

Putting $\quad x = l, f(l) = 0$

$\therefore$ From (1), we have

$$0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2} + \frac{1}{6^2} + \cdots\right)$$

$$\Rightarrow \quad \frac{1}{2^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{32} \Rightarrow \frac{1}{2^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right] = \frac{\pi^2}{32}$$

Hence $\quad \frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}.$

ASSIGNMENT

1. Obtain cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

2. Find Fourier half-range even expansion of the function $f(x) = \left(-\frac{x}{l}\right) + 1;\ 0 \le x \le l.$

3. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 \le x \le 1$. Hence show that

   (i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$ $\qquad$ (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

   (iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

4. Express $\sin x$ as a cosine series in $0 < x < \pi$.

5. If

$$f(x) = \begin{cases} \frac{\pi}{3}, & 0 \le x \le \frac{\pi}{3} \\ 0, & \frac{\pi}{3} \le x \le \frac{2\pi}{3} \\ -\frac{\pi}{3}, & \frac{2\pi}{3} \le x \le \pi \end{cases}$$

   then show that $f(x) = \frac{2}{\sqrt{3}}\left[\cos x - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} - \cdots\right]$

6. (i) Find half-range sine series for the function $f(x) = e^{ax}$ for $0 < x < \pi$ where a is constant.

   (ii) Find the Fourier half-range cosine series of $f(x) = \sin 2x$ over the interval $(0, \pi)$.

   (PTU May 2006)

7. Prove that for all values of x lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, $x = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$.

8. Obtain the half-range sine series for the function $f(x) = x^2$ in the interval $0 < x < 3$.

9. Show that the series $\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}$ represents $\frac{l}{2} - x$ when $0 < x < l$.

10. Find the half-range sine series for

$$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1. \end{cases}$$

Represent the following function by Fourier sine series

$$f(x) = \begin{cases} 1 & \text{when } 0 < x < \frac{l}{2} \\ 0 & \text{when } \frac{l}{2} < x < l. \end{cases}$$

12. If $f(x)=\begin{cases}\dfrac{hx}{a}, & 0<x<a\\ \dfrac{h(l-x)}{l-a}, & a<x<l\end{cases}$, prove that for all values of x between 0 and l

$$f(x)=\frac{2hl^2}{a(l-a)\pi^2}\left[\sin\frac{\pi a}{l}\sin\frac{\pi x}{l}+\frac{1}{2^2}\sin\frac{2\pi a}{l}\sin\frac{2\pi x}{l}+\cdots\right].$$

13. (i) Show that in the range $0<x<\pi$, $c=\frac{4}{\pi}\left[\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right]c$; where c is the con function.

(ii) Show that in the interval (0, 1), $\cos \pi x=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2n\pi x$.

14. (i) Find the half-range cosine series for the function

$f(x)=x(\pi-x);\quad 0<x<\pi$

(ii) Find the half-range sine series for the function

$f(x)=2x-1;\quad 0<x<1.$

(iii) Find the half-range sine series for the function

$f(t)=t-t^2$ in the interval $0<t<1$.

Hence deduce that $\sum_{m=0}^{\infty}(-1)^m\frac{1}{(2m+1)^3}=\frac{\pi^3}{32}$.

15. Find the half-range Fourier sine series of $f(x)=\begin{cases}\sin x, & 0\le x\le\dfrac{\pi}{4}\\ \cos x, & \dfrac{\pi}{4}\le x\le\dfrac{\pi}{2}\end{cases}$.

Answers

1. $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)$; $f(x)=2\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\cdots\right.$

2. $f(x)=\frac{1}{2}+\frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\cdots\right]$

3. $f(x)=\frac{1}{3}+\frac{4}{\pi^2}\left(\cos\pi x+\frac{\cos 2\pi x}{2^2}+\frac{\cos 3\pi x}{3^2}+\cdots\right)$

4. $\sin x=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos 2x}{1.3}+\frac{\cos 4x}{3.5}+\frac{\cos 6x}{5.7}+\cdots\right]$

6. (i) $e^{ax}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{a^2+n^2}\{1-(-1)^n e^{a\pi}\}\sin nx$

(ii) $\sin 2x=\frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{4-n^2}\right)\cos nx$

8. $f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{3}$, where $b_n=\frac{18}{n\pi}(-1)^{n+1}+\frac{36}{n^3\pi^3}[(-1)^n-1]$

10. $f(x)=\left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\sin\pi x+\left(\frac{1}{3\pi}-\frac{4}{3^2\pi^2}\right)\sin 3\pi x+\left(\frac{1}{5\pi}-\frac{4}{5^2\pi^2}\right)\sin 5\pi x+\cdots$

11. $f(x)=\frac{2}{\pi}\left[\sin\frac{\pi x}{l}+\sin\frac{2\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+\cdots\right]$

14. (i) $f(x)=\frac{\pi^2}{6}-4\left(\frac{\cos 2x}{2^2}+\frac{\cos 4x}{4^2}+\cdots\right)$

(ii) $f(x)=-\frac{2}{\pi}\left(\sin 2\pi x+\frac{1}{2}\sin 4\pi x+\frac{1}{3}\sin 6\pi x+\cdots\right)$

(iii) $f(t)=\frac{8}{\pi^3}\left[\sin\pi t+\frac{1}{3^3}\sin 3\pi t+\frac{1}{5^3}\sin 5\pi t+\cdots\right]$

15. $f(x)=\frac{4\sqrt{2}}{\pi}\left[\frac{\sin 2x}{1\cdot 3}-\frac{\sin 6x}{5\cdot 7}+\frac{\sin 10x}{9\cdot 11}-\cdots\right]$.

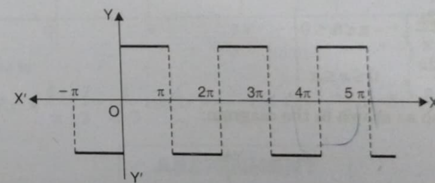
1.9. FOURIER SERIES OF DIFFERENT WAVEFORMS

Some typical waveforms usually met within communication engineering are:

(1) **Square waveform.** This waveform is the extension of the function

$$f(x)=\begin{cases}-k, & -\pi<x<0\\ k, & 0<x<\pi\end{cases}$$

where $f(x+2\pi)=f(x)$ and k is a constant. It has the graph as shown in the diagram:



Fourier series is

$$f(x)=\frac{4k}{\pi}\left(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right)$$

(2) **Saw-toothed waveform** *(PTU May 2012)*

This waveform is the extension of the function

$f(x)=x,\quad -\pi<x<\pi;\quad f(x+2\pi)=f(x)$

This function represents the discontinuous function called saw-toothed waveform. It has the graph as shown in the diagram: