$$= \frac{\quad}{4} - \frac{\quad}{\pi}\left[\frac{\quad}{1^2} + \frac{\quad}{3^2} + \frac{\quad}{5^2} + \cdots\right] + \left[\frac{\quad}{1} - \frac{\quad}{2} + \frac{\quad}{3} - \cdots\right].$$

Example 2. *Find the Fourier series to represent the function $f(x)$ given by*

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } \pi \le x \le 2\pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, then

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi} + \left|2\pi x - \frac{x^2}{2}\right|_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\cdot\frac{\sin nx}{n} - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} + \left|(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right) + \left(-\frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1 - 1 + (-1)^n] = \frac{2}{\pi n^2}\,[(-1)^n - 1] = \begin{cases} -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0\quad, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\left[\int_0^{\pi} x \sin nx\, dx + \int_{\pi}^{2\pi}(2\pi - x) \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left| x\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi} + \left|(2\pi - x)\times\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi \cos n\pi}{n} + \frac{\pi \cos n\pi}{n}\right] = 0$$

$$\therefore\quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)$$

Putting $x = 0$, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) \quad\Rightarrow\quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 3. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$

Prove that $$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}.$$

Hence show that

(*i*) $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \dots = \dfrac{1}{2}$ (*ii*) $\dfrac{1}{1\cdot 3} - \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} - \dots = \dfrac{\pi - 2}{4}$.

Sol. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_0^{\pi} \sin x\, dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_0^{\pi} \sin x \cos nx\, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2 \cos nx \sin x\, dx = \frac{1}{2\pi}\int_0^{\pi} [\sin (n+1)x - \sin (n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right]_0^{\pi},\ n \ne 1$$

$$= \frac{1}{2\pi}\left[-\frac{\cos (n+1)\pi}{n+1} + \frac{\cos (n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \begin{cases} \dfrac{1}{2\pi}\left(-\dfrac{1}{n+1}+\dfrac{1}{n-1}+\dfrac{1}{n+1}-\dfrac{1}{n-1}\right), \text{ when } n \text{ is odd} \\ \dfrac{1}{2\pi}\left(\dfrac{1}{n+1}-\dfrac{1}{n-1}+\dfrac{1}{n+1}-\dfrac{1}{n-1}\right), \text{ when } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0, \text{ when } n \text{ is odd, } i.e., n = 3, 5, 7, \ldots \\ -\dfrac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even} \end{cases}$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^\pi \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^\pi \sin 2x \, dx = \frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^\pi = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \, dx + \int_0^\pi \sin x \sin nx \, dx\right]$$

$$= \frac{1}{2\pi}\int_0^\pi 2 \sin nx \sin x \, dx = \frac{1}{2\pi}\int_0^\pi [\cos (n-1)x - \cos (n+1)x]$$

$$= \frac{1}{2\pi}\left[\frac{\sin (n-1)x}{n-1} - \frac{\sin (n+1)x}{n+1}\right]_0^\pi = 0, \; n \neq 1$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^\pi (1 - \cos 2x) \, dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2}$$

$$\therefore \quad f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right] + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n)^2 - 1}$$

Putting $x = 0$ in (1), we have $0 = \dfrac{1}{\pi} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

$$\Rightarrow \quad \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$$

Putting $x = \dfrac{\pi}{2}$ in (1), we have $1 = \dfrac{1}{\pi} + \dfrac{1}{2} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{4n^2-1}$

$$\Rightarrow \quad \frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

$$\Rightarrow \quad \frac{\pi-2}{4} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)} = -\left(-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots\right)$$

$$\Rightarrow \quad \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi - 2}{4}.$$

Example 4. *Obtain Fourier series for the function f(x) given by*

$$f(x)=\begin{cases}1+\dfrac{2x}{\pi}, & -\pi\le x\le 0\\[2mm] 1-\dfrac{2x}{\pi}, & 0\le x\le \pi\end{cases}$$

Hence deduce that $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dots=\dfrac{\pi^2}{8}$. (P.T.U. Dec. 2007, May 2008)

Sol. When $-\pi\le x\le 0,\qquad 0\le -x\le \pi$

$$\therefore\qquad f(-x)=1-\frac{2(-x)}{\pi}=1+\frac{2x}{\pi}=f(x)$$

When $0\le x\le \pi,\qquad -\pi\le -x\le 0$

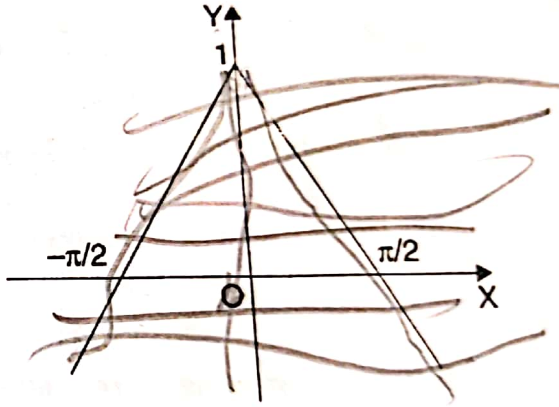
$$\therefore\qquad f(-x)=1+\frac{2(-x)}{\pi}=1-\frac{2x}{\pi}=f(x)$$

$\Rightarrow$ $f(x)$ is an even function of x in $[-\pi,\pi]$. This is also clear from its graph which is symmetrical about the y-axis.

$$\therefore\qquad b_n=0$$

Let $f(x)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty} a_n\cos nx$

R



Here, $a_0=\dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx=\dfrac{2}{\pi}\int_0^{\pi}\left(1-\dfrac{2x}{\pi}\right)dx=\dfrac{2}{x}\left[x-\dfrac{x^2}{\pi}\right]_0^{\pi}=0$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$=\frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n}-\left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[-\frac{2\cos n\pi}{\pi n^2}+\frac{2}{\pi n^2}\right]=\frac{4}{\pi^2 n^2}[1-(-1)^n]$$

$$\therefore\qquad f(x)=\frac{4}{\pi^2}\sum_{n=1}^{\infty}[1-(-1)^n]\frac{\cos nx}{n^2}$$

$$=\frac{4}{\pi^2}\left(\frac{2\cos x}{1^2}+\frac{2\cos 3x}{3^2}+\frac{2\cos 5x}{5^2}+\dots\right)=\frac{8}{\pi^2}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\dots\right)$$

Putting $x=0$, we get $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dots=\dfrac{\pi^2}{8}$ $\qquad\mid\because f(0)=1.$

ASSIGNMENT

1. Find the Fourier series to represent the function

$$f(x)=\begin{cases}-k \text{ when} & -\pi<x<0\\ k \text{ when} & 0<x<\pi\end{cases}$$

Also deduce that $\dfrac{\pi}{4}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\dots$

1.7. CHANGE OF INTERVAL

In many engineering problems, it is desired to expand a function in a Fourier ser over an interval of length $2l$ and not 2π. In order to apply foregoing theory, this interval m be transformed into an interval of length 2π. This can be achieved by a transformation of t variable.

Consider a periodic function $f(x)$ defined in the interval $c < x < c + 2l$. To change t interval into one of length 2π, we put

$$\frac{x}{l} = \frac{z}{\pi} \quad \text{or} \quad z = \frac{\pi x}{l} \text{ so that}$$

when $\quad x = c, \quad z = \frac{\pi c}{l} = d$ (say)

and $\quad$ when $\quad x = c + 2l, \quad z = \frac{\pi(c + 2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi.$

Thus the function $f(x)$ of period $2l$ in $(c, c + 2l)$ is transformed to the functi $f\left(\frac{lz}{\pi}\right) = F(z)$, say, of period 2π in $(d, d + 2\pi)$ and the latter function can be expressed as t Fourier series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \quad \ldots(1)$$

where $\quad a_0 = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\, dz; \quad a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z) \cos nz\, dz;$ and $\quad b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z) \sin nz\, dz \quad \ldots(2)$

Now making the inverse substitution $z = \frac{\pi x}{l}$, $dz = \frac{\pi}{l}\, dx$

When $\quad z = d, \quad x = c$ and when $\quad z = d + 2\pi, x = c + 2l.$

The expression (1) becomes $F(z) = F\left(\frac{\pi x}{l}\right) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}.$

and the coefficients a_0, a_n, b_n from (2) reduce to

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\, dx; \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx; \quad \text{and} \quad b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

Hence the Fourier series $f(x)$ in the interval $c < x < c + 2l$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}}$$

where $\quad a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\, dx, \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx$ and $\quad b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx.$

Cor. 1. If we put $c = 0$, the interval becomes $0 < x < 2l$ and the above results reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx; \quad a_n = \frac{1}{l}\int_0^{2l} f(x) \cos \frac{n\pi x}{l}\, dx; \quad \text{and} \quad b_n = \frac{1}{l}\int_0^{2l} f(x) \sin \frac{n\pi x}{l}\, dx.$$

Example 1. *Obtain the Fourier series expansion of*

$$f(x) = \left(\frac{\pi - x}{2}\right) \text{ for } 0 < x < 2$$

Sol. Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Here, $l = 1$

$$\therefore \quad \frac{\pi - x}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \ldots(1)$$

Here,
$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right) dx$$

$$= \frac{1}{2}\left(\pi x - \frac{x^2}{2}\right)_0^2 = \frac{1}{2}(2\pi - 2) = \pi - 1$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos n\pi x\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right)\cos n\pi x\,dx$$

$$= \frac{1}{2}\left[\left\{(\pi - x)\frac{\sin n\pi x}{n\pi}\right\}_0^2 - \int_0^2 (-1)\frac{\sin n\pi x}{n\pi}\,dx\right]$$

$$= \frac{1}{2n\pi}\left(\frac{-\cos n\pi x}{n\pi}\right)_0^2 = 0$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin n\pi x\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right)\sin n\pi x\,dx$$

$$= \frac{1}{2}\left[\left\{(\pi - x)\left(\frac{-\cos n\pi x}{n\pi}\right)\right\}_0^2 - \int_0^2 (-1)\left(\frac{-\cos n\pi x}{n\pi}\right)dx\right]$$

$$= -\frac{1}{2n\pi}[(\pi - 2) - \pi] = \frac{1}{n\pi}$$

Hence, from (1),

$$\frac{\pi - x}{2} = \frac{(\pi - 1)}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin n\pi x$$

Example 2. *Find Fourier expansion for the function $f(x) = x - x^2, -1 < x < 1$.*

(Here $l =$

Sol. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Then $$a_0 = \int_{-1}^{1} (x - x^2)\, dx = \int_{-1}^{1} x\, dx - \int_{-1}^{1} x^2\, dx = 0 - 2\int_0^1 x^2\, dx = -2\left[\frac{x^3}{3}\right]_0^1 = -$$

$$a_n = \int_{-1}^{1} (x - x^2) \cos n\pi x\, dx = \int_{-1}^{1} x \cos n\pi x\, dx - \int_{-1}^{1} x^2 \cos n\pi x\, dx$$

$$= 0 - 2\int_0^1 x^2 \cos n\pi x\, dx = -2\left[\left(x^2 \frac{\sin n\pi x}{n\pi}\right)_0^1 - \int_0^1 2x \cdot \frac{\sin n\pi x}{n\pi}\, dx\right]$$

$$= \frac{4}{n\pi}\int_0^1 x \sin n\pi x\, dx$$

$$= \frac{4}{n\pi}\left[\left\{x \cdot \left(\frac{-\cos n\pi x}{n\pi}\right)\right\}_0^1 - \int_0^1 \left(\frac{-\cos n\pi x}{n\pi}\right) dx\right]$$

$$= -\frac{4}{n^2\pi^2}\cos n\pi = \frac{-4(-1)^n}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1} (x - x^2) \sin n\pi x\, dx = \int_{-1}^{1} x \sin n\pi x\, dx - \int_{-1}^{1} x^2 \sin n\pi x\, dx$$

$$= 2\int_0^1 x \sin n\pi x\, dx - 0 = 2\left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1 \cdot \left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos n\pi}{n\pi}\right] = \frac{-2(-1)^n}{n\pi}$$

$$\therefore \quad x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \ldots\right)$$
$$+ \frac{2}{\pi}\left(\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \ldots\right)$$

Example 3. *Find the Fourier series to represent $f(x) = x^2 - 2$, when $-2 \le x \le 2$.*

Sol. Since $f(x)$ is an even function, $b_n = 0$.

Let $$f(x) = x^2 - 2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

Then $$a_0 = \frac{2}{2}\int_0^2 (x^2 - 2)\, dx = \left[\frac{x^3}{3} - 2x\right]_0^2 = \frac{8}{3} - 4 = -\frac{4}{3}$$

$$a_n = \frac{2}{2}\int_0^2 (x^2 - 2) \cos \frac{n\pi x}{2}\, dx$$

$$= \left\{(x^2 - 2)\frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}}\right\}_0^2 - \int_0^2 (2x)\frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)}\, dx = -\frac{4}{n\pi}\int_0^2 x \sin \frac{n\pi x}{2}\, dx$$

$$= -\frac{4}{n\pi}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)\right\}_0^2 - \int_0^2\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)dx\right]$$

$$= \frac{8}{n^2\pi^2}(2\cos n\pi) = \frac{16\cos n\pi}{n^2\pi^2} = \frac{16(-1)^n}{n^2\pi^2}$$

$$\therefore \quad x^2 - 2 = -\frac{2}{3} - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2} - \dots\right).$$

Example 4. *Obtain Fourier series for function* $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$. *Also deduce that* $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$. (PTU Dec. 2012)

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then $a_0 = \int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx = \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$

$$= \pi\left(\frac{1}{2}\right) + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = \pi$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx = \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \left[\pi x\cdot\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\cdot\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right] + \left[-\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right] = \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$= 0$ or $-\frac{4}{n^2\pi}$ according as n is even or odd.

$$b_n = \int_0^2 f(x)\sin n\pi x\,dx = \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots\right).$$

Now, putting $x = 0$ in above series, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$