Algo : given $|H| = |S| = n$

while ($\exists h$ unmatched AND $\exists s$ in order of h's list
st. h hasn't proposed s yet)

h proposes s

If s unmatched
add $h - s$
else If s prefers h to current h'
Sub $h' - s$ // s rejects h'
add $h - s$
else
s rejects h

0. Loop terminates in atmost $O(n^2)$ iterations

Proof. Each iteration,
some h proposes to exactly one s
st. proposal $(h, s) \in H \times S$ has
not happened before

$\therefore$ Iterations cannot exceed $|H \times S| = n^2$
$= O(n^2)$

1. Algo produces a matching

Let $e(x)$ denote number of edges incident on x

1.1 $\forall h,\ e(h) \leq 1$

Proof. h forms a new edge $\Rightarrow$ h proposed
$\Rightarrow$ h was unmatched

$\therefore$ h was matched $\Rightarrow$ h does not form a new edge

Since, in Algo $e(h)$ can increase by atmost one in each iteration and initially $e(h) = 0$

$\therefore$ $\boxed{\forall h \quad e(h) \leq 1}$

1.2 $\forall s,\ e(s) \leq 1$

Proof. $e(s)$ increases from 0 to 1 when s forms its first edge. For each edge added to s afterwards exactly one previously incident edge is deleted.

$\therefore$ $\boxed{\forall s \quad e(s) \leq 1}$

From 1.1, 1.2,

$\boxed{\text{Algo produces a matching}}$

2. Algo produces a perfect matching

Let M_0 be the matching produced

2.1 M_0 covers all h

Proof: Suppose M_0 didn't cover some h_0
$\therefore$ h_0 must have proposed all s
and gotten rejected by all s

From code, • s rejects only if s is matched
• Once s is matched, it never becomes unmatched

$\therefore$ In M_0, all s are matched
Since $|H| = |S| = n$, but h_0 is unmatched
$\exists\ h \neq h_0 \mid e(h) \geq 2$ (pigeonhole)
which contradicts 1.1

$\therefore$ $\boxed{M_0 \text{ covers all } h}$

2.2 M_0 covers all s

Proof: M_0 is a matching (1)
M_0 covers all h (2.1)
$|H| = |S| = n$
$\therefore$ $\boxed{M_0 \text{ covers all } s}$

From 2.1, 2.2
$\boxed{\text{Algo produces a perfect matching}}$

Def. h, s form a blocking pair iff $h \not\to s$ AND both prefer each other more than their current partners in a given matching

3. $\exists$ no blocking pair in M_0 (M_0 is stable)

Proof. Suppose h, s form a blocking pair

$$h' \to s$$
$$h \leftarrow s'$$

pref (s) : $h' < h$

pref (h) : $s' < s$

Since h proposes in order of preference h must have proposed s before s'

but since $h \not\to s$

s must have rejected h for some h_1

if $h_1 \neq h'$, s must have rejected h_1 for some h_2

⋮

so on (Terminates because $|H| = n$)

⋮

$\therefore$ pref (s) : $h < h_1 < h_2 \ldots < h'$

which contradicts

pref (s) : $h' < h$

$\therefore$ $\boxed{\exists \text{ no blocking pair in } M_0}$

Def. Stable Matching M is h optimal iff
for any h, for any other stable matching M'
h prefers its partner in M to its partner in M'.

4. M_0 is h optimal

Proof: Suppose not, then $\exists M$ s.t.

$\underline{M_0}$ $\quad h - s \qquad\qquad \underline{M} \quad h - s'$

$pref(h) : s' > s$

For M_0, since h proposes in order of preference
h proposed s' before but was rejected for h'
s.t. $pref(s') : h < h'$

$\underline{M_0}$ $\quad h' \rightarrow s' \qquad\qquad \underline{M} \quad h - s'$

$\qquad\; h - s \qquad\qquad\qquad h' \leftarrow s''$

Since M is stable, $\exists s''$ s.t.
$h' - s''$ AND $pref(h') : s'' > s'$

Repeating above reasoning with
h' as h, s' as s, s'' as s'
we get, $\exists$ new (h'', s''') ($\because$ No vertex has 2 edges incident on it)
s.t. $h'' - s'''$ in M

Above reasoning can be repeated indefinitely
which contradicts $|H| = |S| = n$

$\therefore$ $\boxed{M_0 \text{ is } h \text{ optimal}}$

Def. Stable Matching M is s pessimal iff for any s, for any other stable matching M′

s prefers its partner in M′ to its partner in M

S: M_0 is s pessimal

Proof. Suppose not, then $\exists$ M s.t.

M_0 : h — s   M : h′ — s

pref (s): h > h′

In M, s ≠ h

s prefers h to its partner h′

Since M is stable,

$\exists$ s′ ≠ s s.t. h — s′ in M

AND pref (h): s′ > s

which contradicts h optimality

∴ $\boxed{M_0 \text{ is s pessimal}}$