Development of a code for solving Vorticity-Stream Function equation

MASTER OF TECHNOLOGY

by

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Problem Statement:

Solve the following partial differential equation using the finite difference method with the specified boundary conditions for the geometry with 100×100 grid size as shown in the figure.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} = -\omega$$

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial x^2}\right)$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

Convergence Criteria: Find the maximum error of stream function and vorticity and reduce that maximum error to 10-6. Apply the finite difference discretization to replace all derivatives with the corresponding central difference expressions with uniform grid and *write the discretized equations of the governing equations and boundary conditions of stream function & vorticity in the* ***report***. Write the code in such a way so that you can input the values of Re. Submit the results and discussion for **Re=100 and 400** in terms of streamlines, velocity vectors, *u* velocity along vertical centreline and *v* velocity along horizontal centreline.

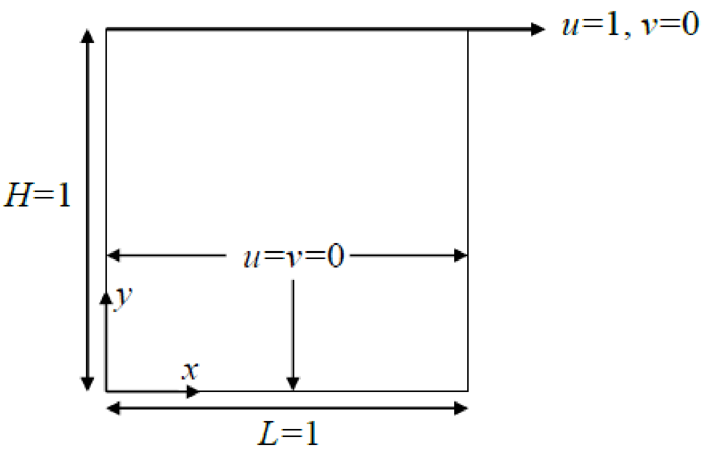


Figure: Flow inside a lid-driven cavity

Boundary conditions:

Left boundary:

$u = 0 \quad v = 0 \quad \psi = 0$

Bottom boundary:

$u = 0 \quad v = 0 \quad \psi = 0$

Top boundary:

$u = 1 \quad v = 0 \quad \psi = 0$

Right boundary:

$u = 0 \quad v = 0 \quad \psi = 0$

Vorticity boundary conditions:

Left boundary:

$$\omega_{i,j} = \frac{-2}{\Delta x^2}\left(\psi_{i+1,j} - \psi_{i,j}\right)$$

Bottom boundary:

$$\omega_{i,j} = \frac{-2}{\Delta y^2}\left(\psi_{i,j+1} - \psi_{i,j}\right)$$

Right boundary:

$$\omega_{i,j} = \frac{-2}{\Delta x^2}\left(\psi_{m-1,j} - \psi_{m,j}\right)$$

Top Boundary

$$\omega_{i,j} = \frac{-2}{\Delta y^2}\left(\psi_{i,n-1} - \psi_{i,n} + U\Delta y\right)$$

Solution of stream function:

$$\psi_{i,j} = \frac{1}{2(1+\beta^2)}\left[\Delta x^2 \omega_{i,j} + \beta^2\left(\psi_{i,j+1} + \psi_{i,j-1}\right) + \psi_{i+1,j} + \psi_{i-1,j}\right]$$

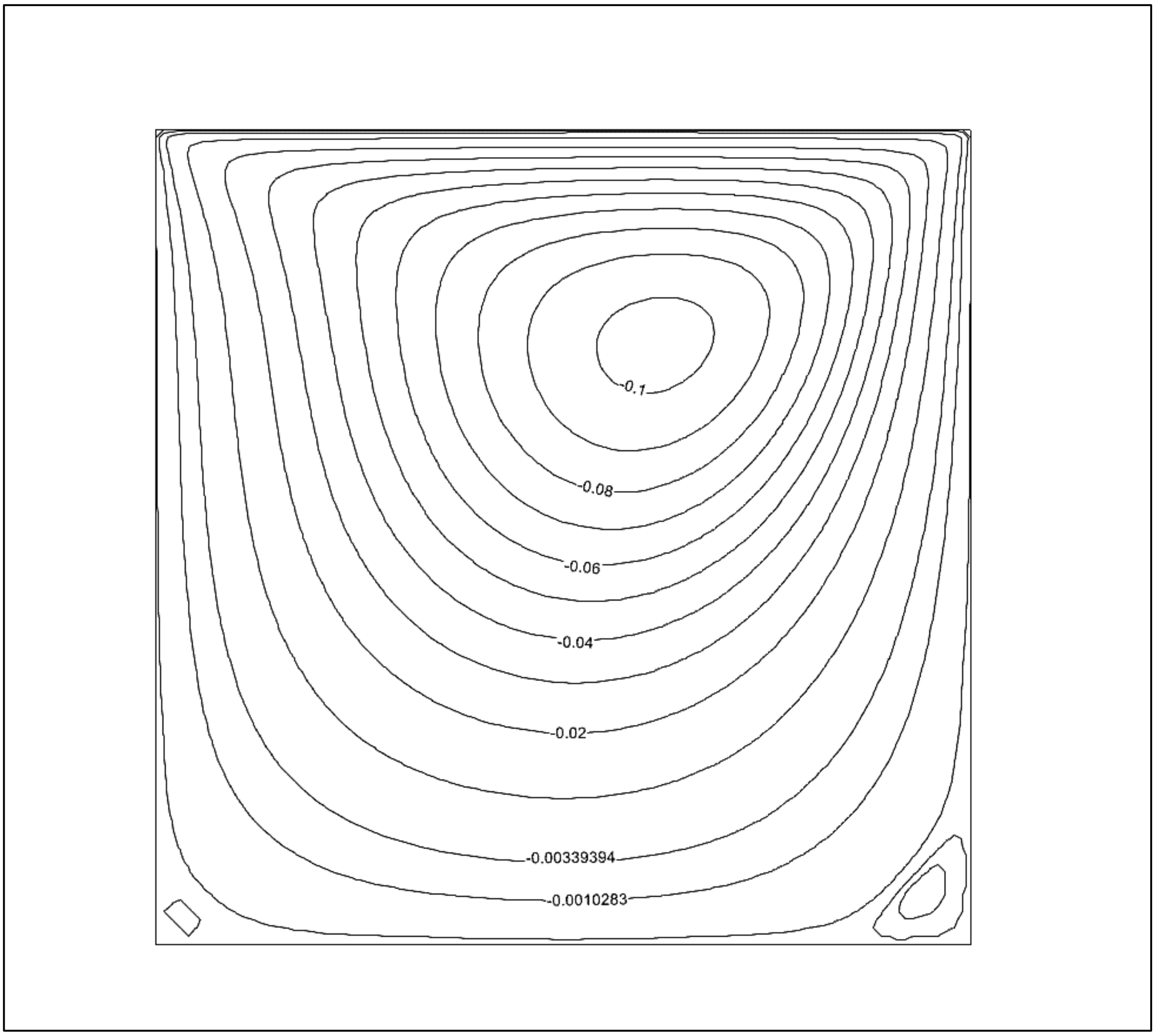
Solution of vorticity:

$$\begin{aligned}\omega_{i,j} = \frac{1}{2(1+\beta^2)}\Bigg[&\left\{1 - \left(\psi_{i,j+1} - \psi_{i,j-1}\right)\frac{\beta * Re}{4}\right\}\omega_{i+1,j} \\ &+ \left\{1 + \left(\psi_{i,j+1} - \psi_{i,j-1}\right)\frac{\beta * Re}{4}\right\}\omega_{i-1,j} \\ &+ \left\{1 + \left(\psi_{i+1,j} - \psi_{i-1,j}\right)\frac{Re}{4\beta}\right\}\beta^2\omega_{i,j+1} \\ &+ \left\{1 - \left(\psi_{i+1,j} - \psi_{i-1,j}\right)\frac{Re}{4\beta}\right\}\beta^2\omega_{i,j-1}\Bigg]\end{aligned}$$

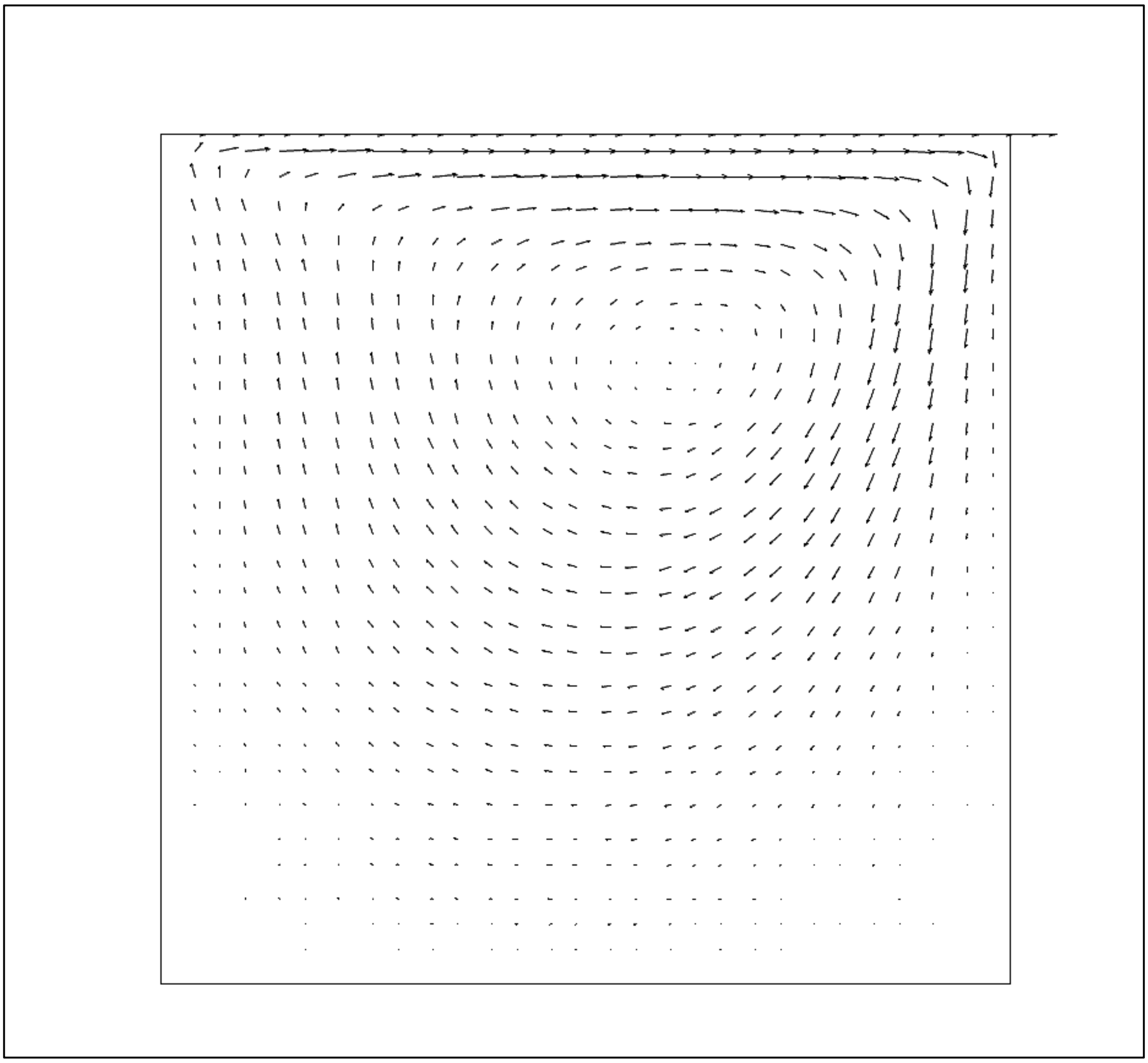
Results:

Reynolds number=100

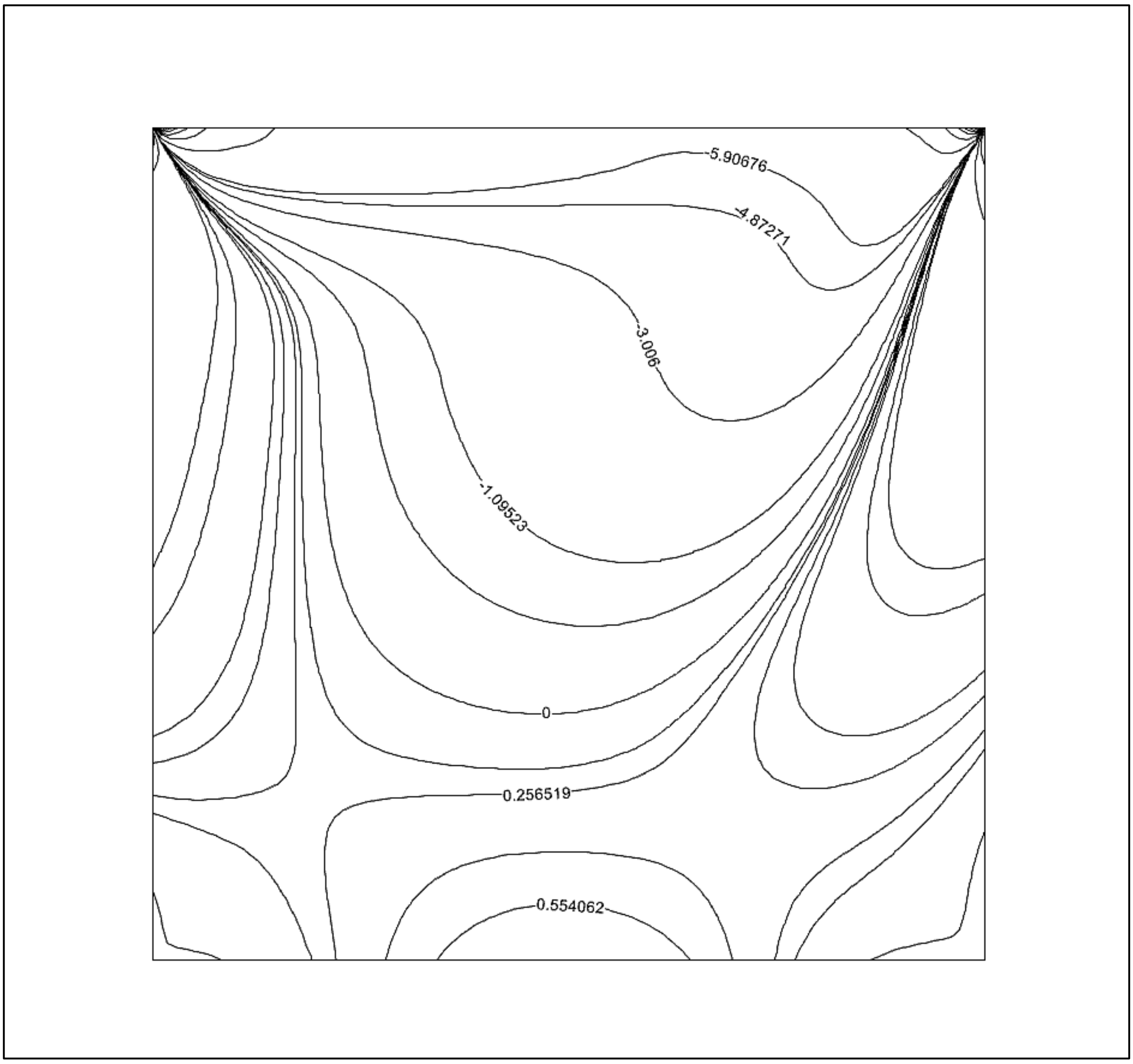
1. **Streamlines:**



2. Velocity vectors:

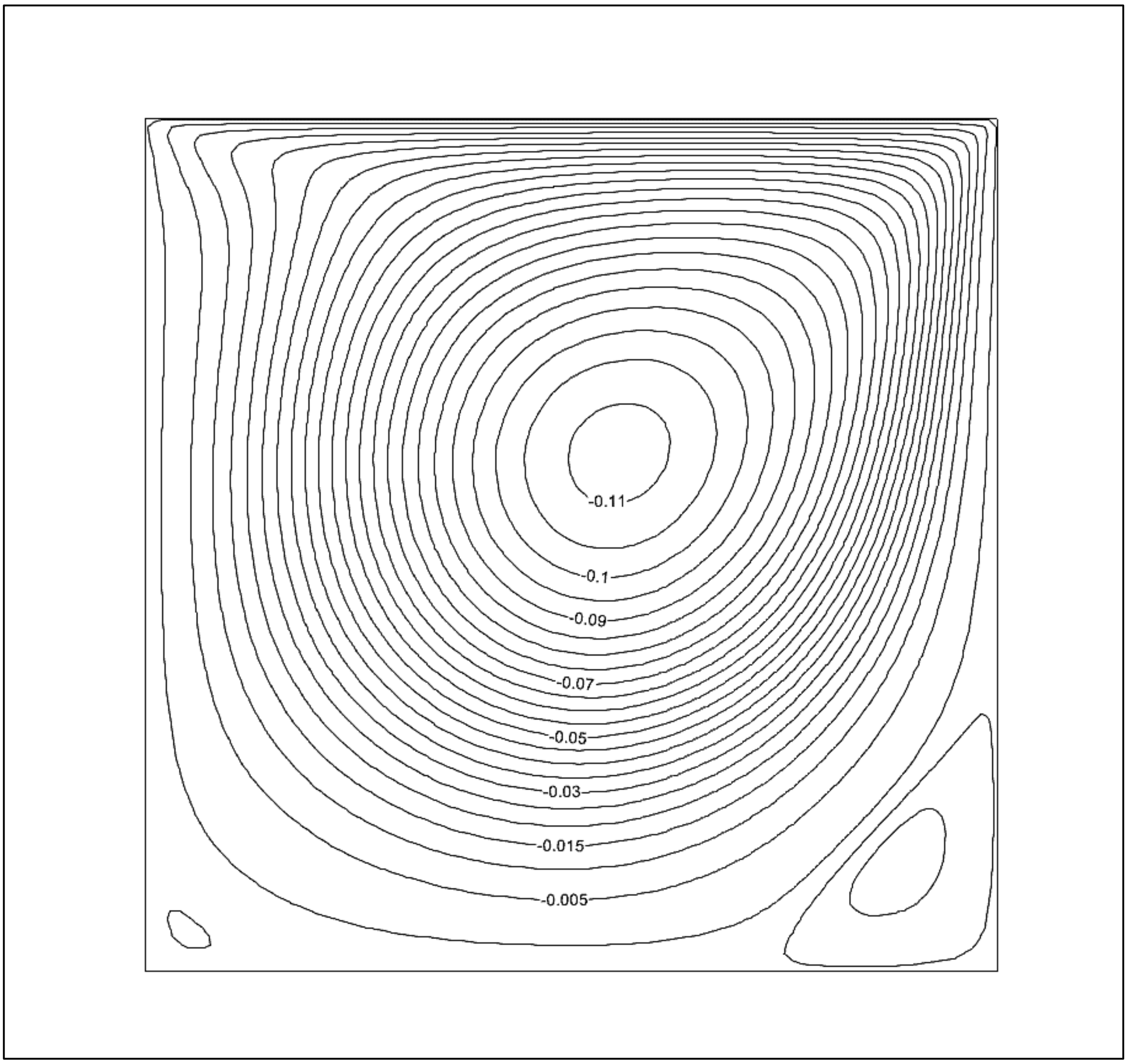


3. Vorticity:

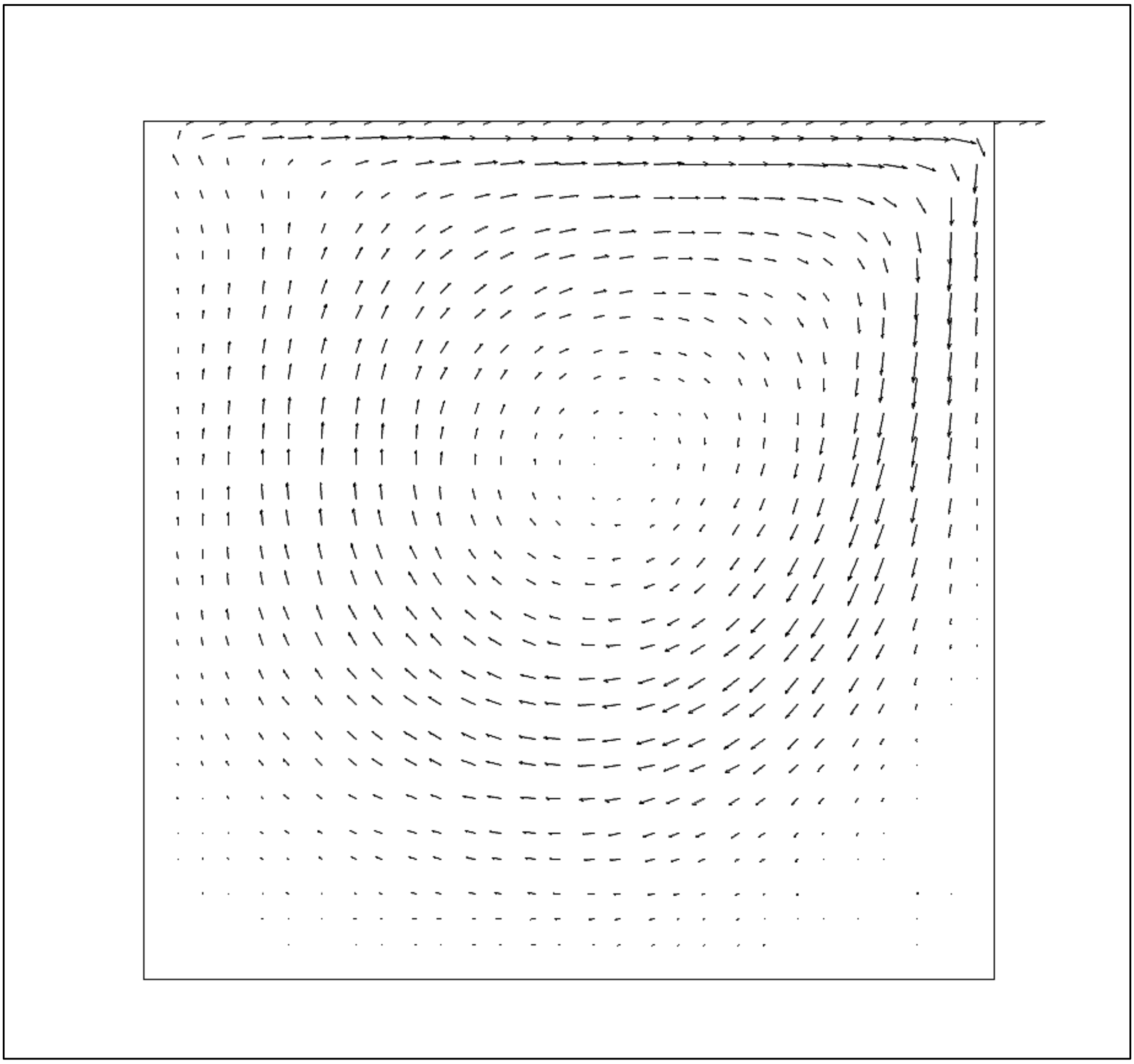


Reynolds number = 400

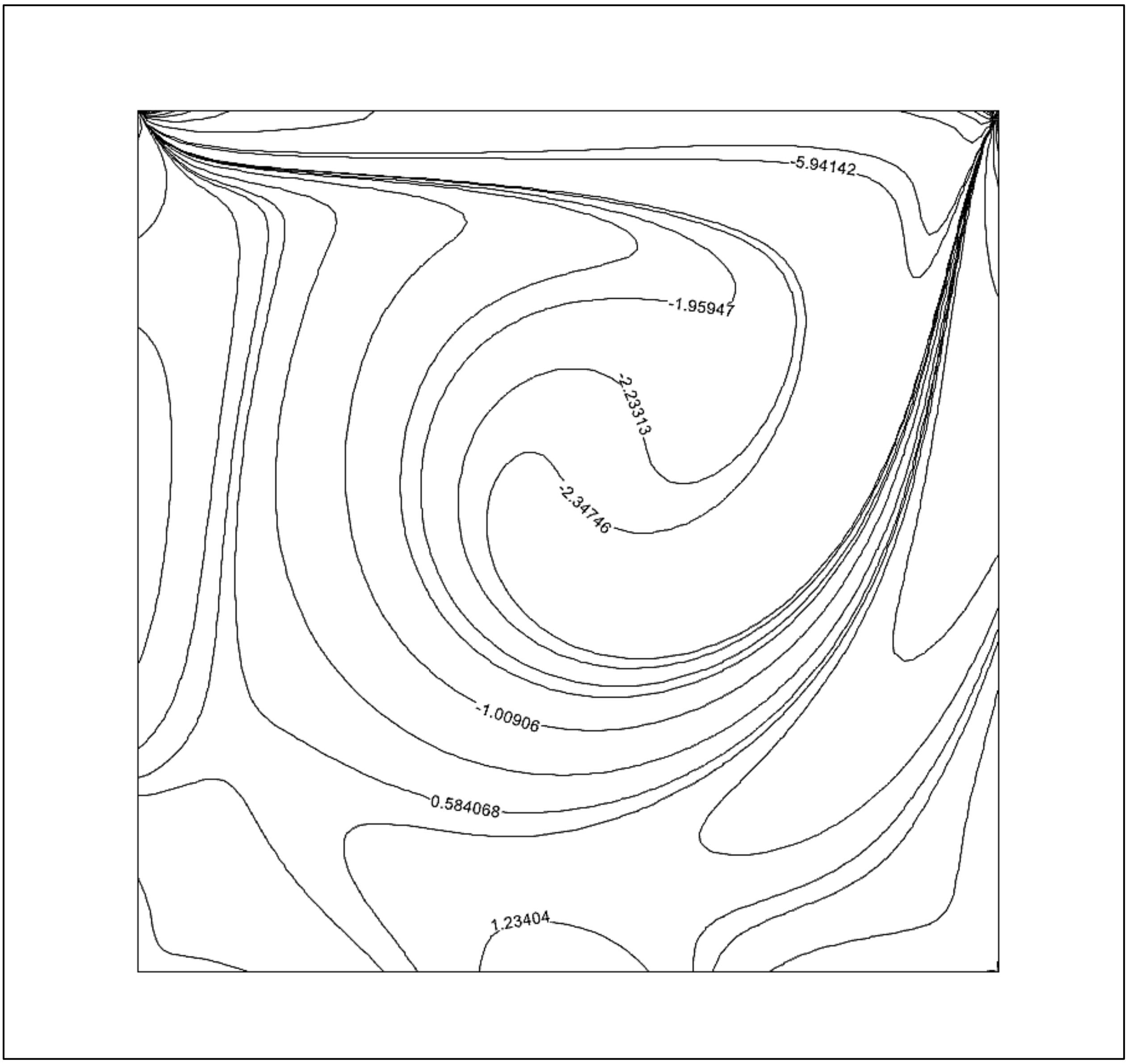
1. **Streamlines:**



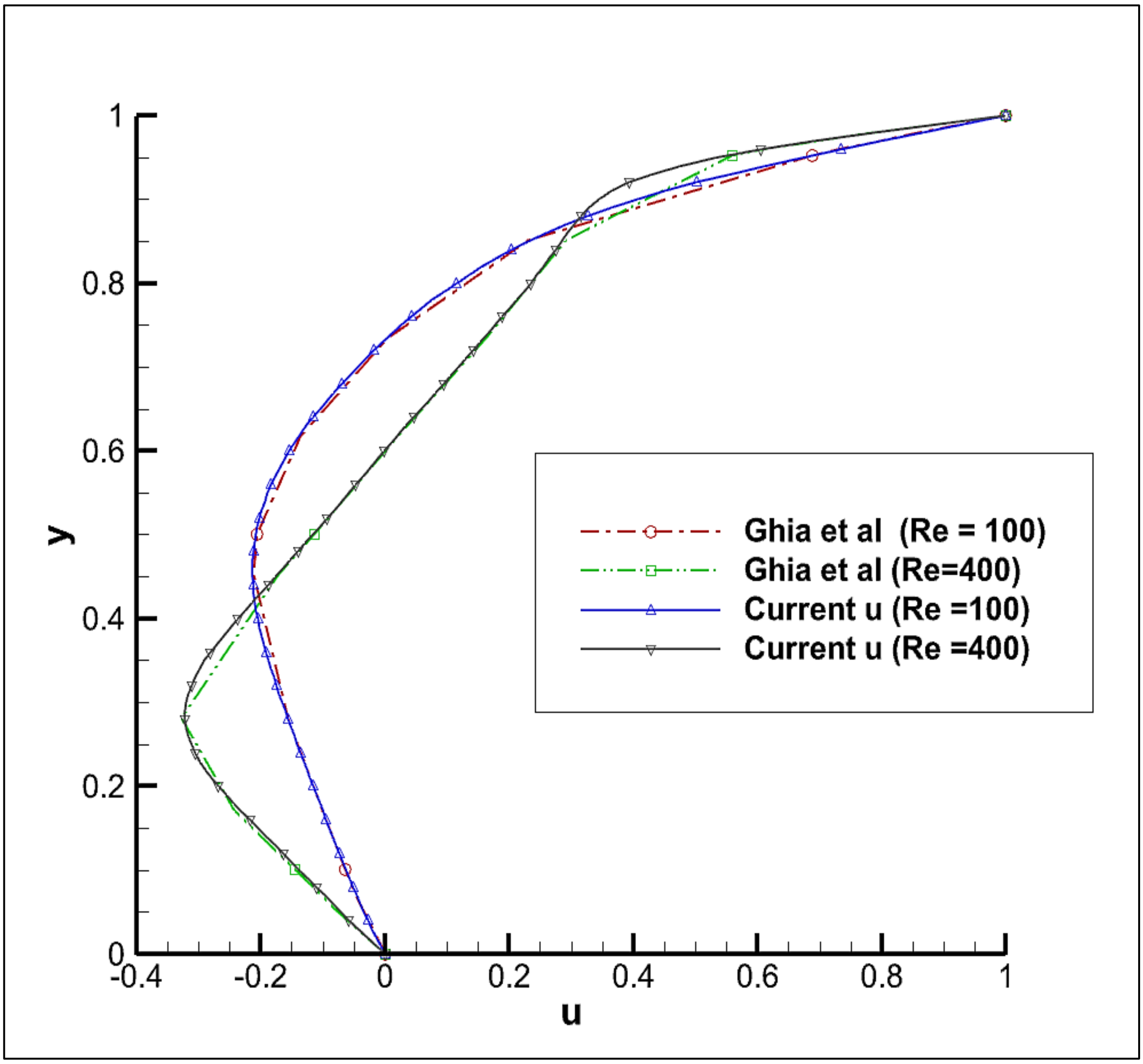
2. Velocity vectors:



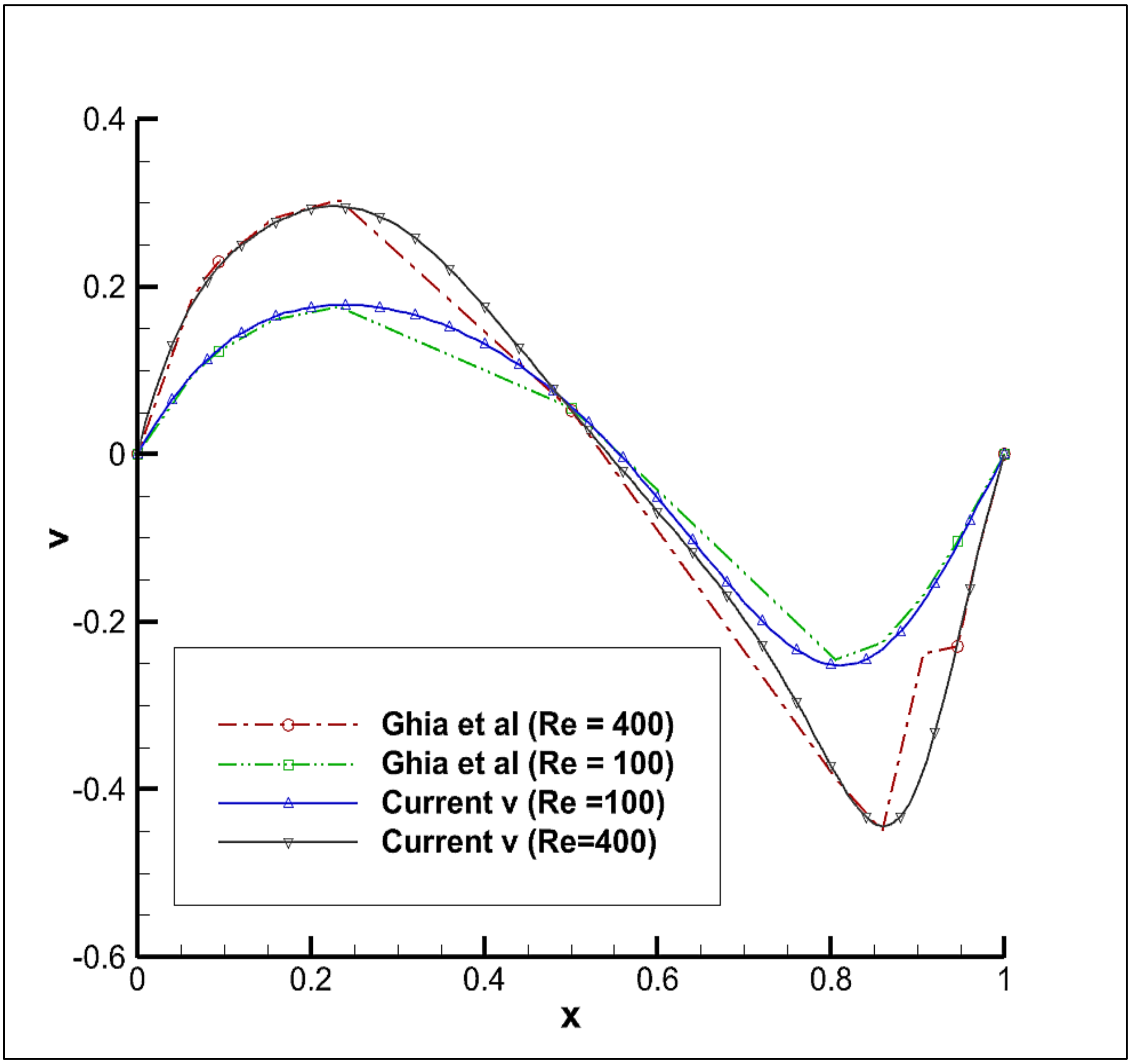
3. Vorticity:



Comparison of u velocity at Re=100 and Re=400 along vertical centreline with data from Ghia et al:



Comparison of ‘v’ velocity at Re=100 and Re=400 along horizontal centreline with data from Ghia et al:



Problem Statement:

Solve the following partial differential equation using the finite difference method with the specified boundary conditions for the geometry with 75×30 grid size as shown in the figure.

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial x^2}=-\omega$$

$$u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial x^2}\right)$$

$$u=\frac{\partial\psi}{\partial y}\qquad v=-\frac{\partial\psi}{\partial x}$$

Apply the finite difference discretization to replace all derivatives with the corresponding central difference expressions with uniform grid $N\times M$ and write the discretized equations of the governing equations and boundary conditions of stream function & vorticity in the report. Write the code in such a way so that you can input the values of Re, N, M, H, L, Δx, Δy. Submit the hard copy of the code, results, and discussion for Re=100 in terms of streamlines, velocity vectors, u velocity profile at locations x=5, 10 and 15 (in same x-y plot). Email only the soft copy of the code.

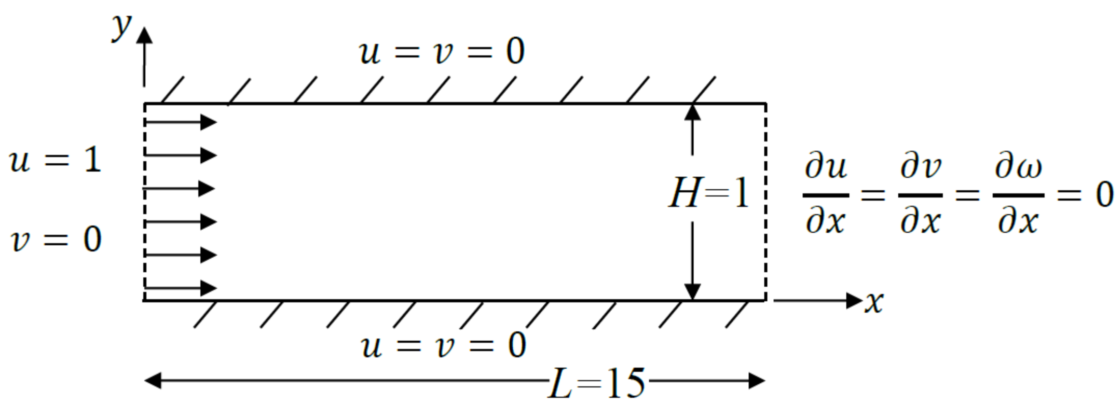


Figure: Flow inside a channel

Boundary conditions for velocity and streamline:

Left boundary:

$u = 1 \quad v = 0 \quad \psi_{i,j+1} = \psi_{i,j} + \Delta y * u_{i,j}$

Bottom boundary:

$u = 0 \quad v = 0 \quad \psi = 0$

Top boundary:

$u = 0 \quad v = 0 \quad \psi = 1$

Right boundary:

$u_{i,j} = u_{i-1,j}$; $v_{i,j} = v_{i-1,j}$; $\psi_{i,j} = \psi_{i-1,j}$

Vorticity boundary conditions:

Left boundary:

$$\omega_{i,j} = \frac{-2}{\Delta x^2}\left(\psi_{i+1,j} - \psi_{i,j}\right)$$

Bottom boundary:

$$\omega_{i,j} = \frac{-2}{\Delta y^2}\left(\psi_{i,j+1} - \psi_{i,j}\right)$$

Right boundary:

$$\omega_{i,j} = \omega_{i-1,j}$$

Top Boundary

$$\omega_{i,j} = \frac{-2}{\Delta y^2}\left(\psi_{i,j-1} - \psi_{i,j}\right)$$

Solution of stream function:

$$\psi_{i,j} = \frac{1}{2(1+\beta^2)}\left[\Delta x^2 \omega_{i,j} + \beta^2\left(\psi_{i,j+1} + \psi_{i,j-1}\right) + \psi_{i+1,j} + \psi_{i-1,j}\right]$$

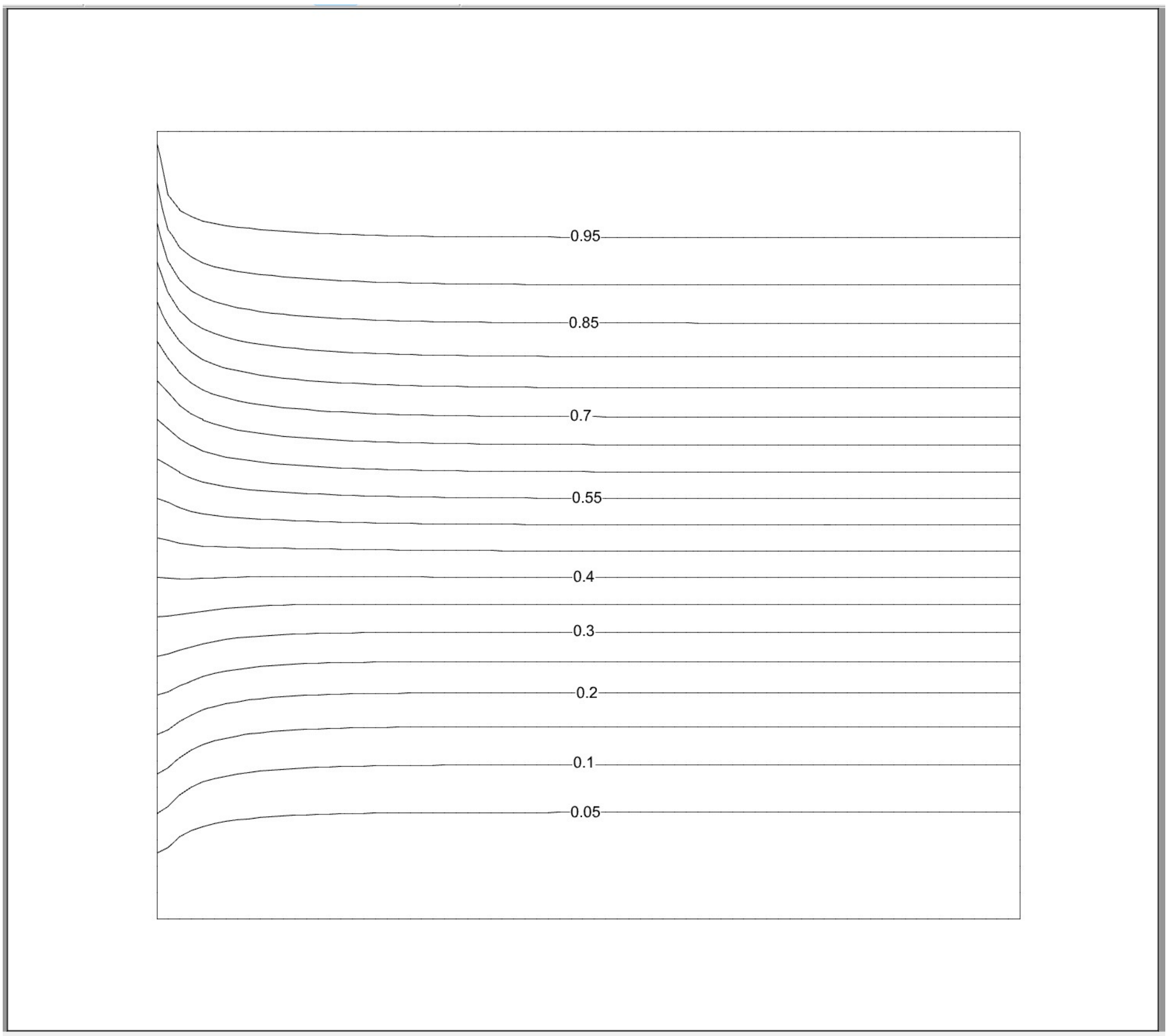
Solution of vorticity: urf – under relaxation factor; $\omega_{i,j}^{previous}$ – value at previous iteration

$$\begin{aligned}\omega_{i,j} = &\left[(1-urf) * \omega_{i,j}^{previous}\right] \\ &+ \left[urf * \frac{1}{2(1+\beta^2)}\right]\left[\left\{1 - \left(\psi_{i,j+1} - \psi_{i,j-1}\right)\frac{\beta * Re}{4}\right\}\omega_{i+1,j}\right. \\ &+ \left\{1 + \left(\psi_{i,j+1} - \psi_{i,j-1}\right)\frac{\beta * Re}{4}\right\}\omega_{i-1,j} + \left\{1 + \left(\psi_{i+1,j} - \psi_{i-1,j}\right)\frac{Re}{4\beta}\right\}\beta^2\omega_{i,j+1} \\ &\left. + \left\{1 - \left(\psi_{i+1,j} - \psi_{i-1,j}\right)\frac{Re}{4\beta}\right\}\beta^2\omega_{i,j-1}\right]\end{aligned}$$

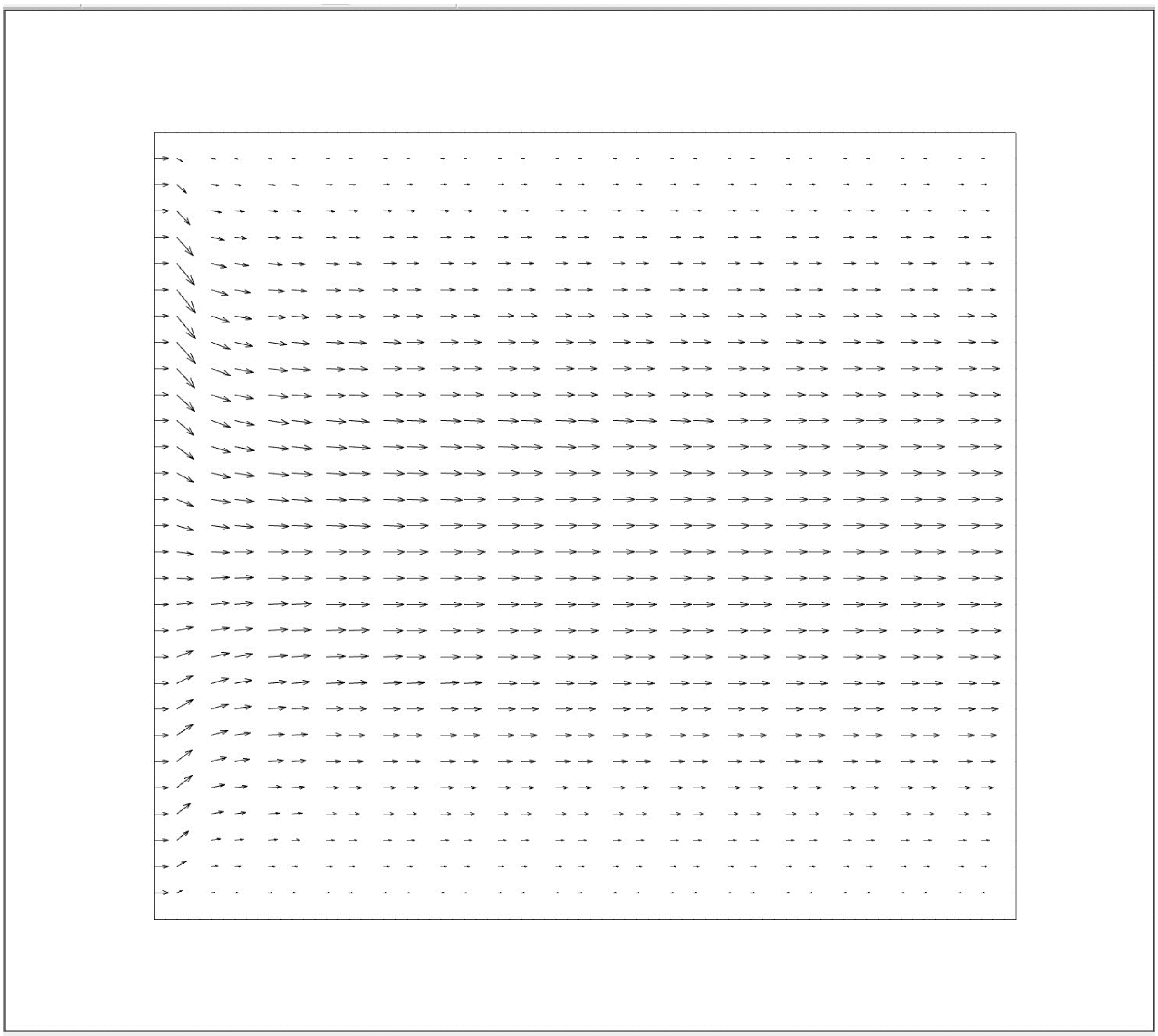
Results:

Reynolds number = 100

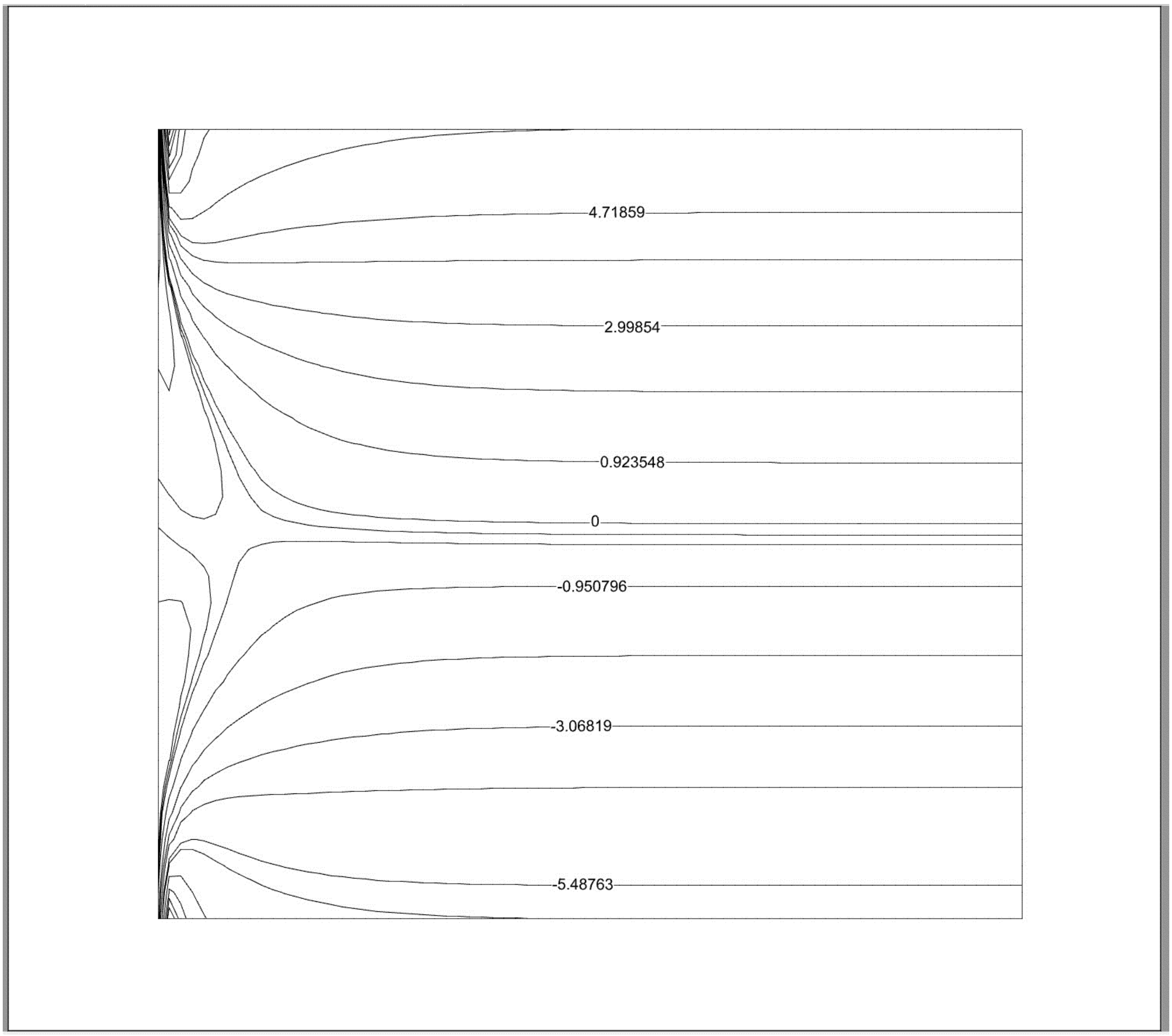
1. **Streamlines:**



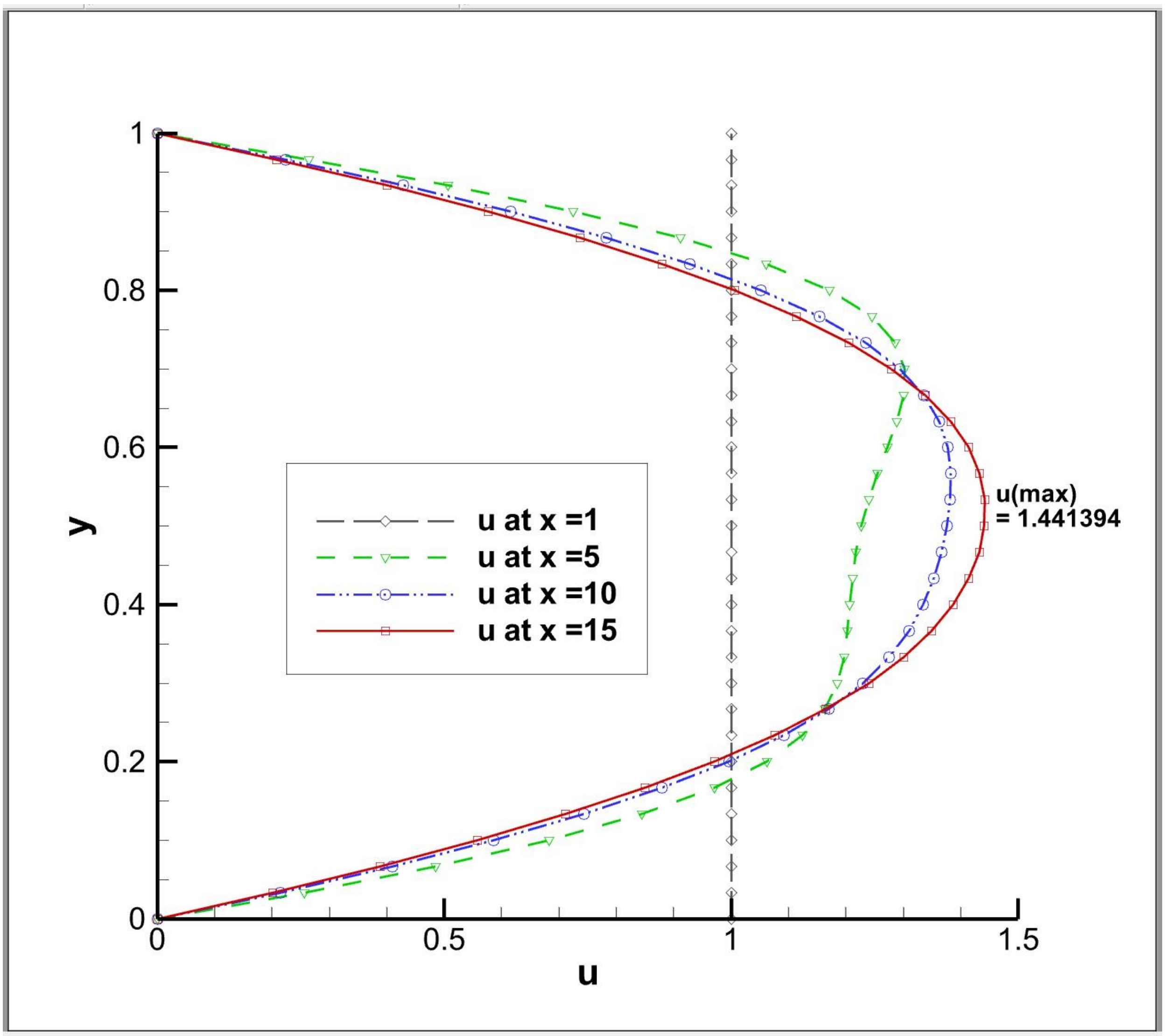
2. Velocity vectors:



3. Vorticity:



4. **u velocity profile at x = 1, 5, 10, 15**



For fully developed flow, value of u_{max} is 1.5 times average velocity. In this case u_{avg} is 1, so in theory u_{max} should be 1.5. The result obtained with the given grid size (75x30), length of the channel (L = 15), and urf (under relaxation factor) of 0.1, is:
u_{max} = 1.441394, which is close to the theoretical value.

(For urf = 1.8, L = 30, grid = 75x60, the result obtained is:
u_{max} = 1.493131 which is even closer to the expected value of 1.5)