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Section: 4A
MATH 151A HW #2
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Problem 1

Suppose we have

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^\alpha},\ p_n\ \textit{converges to}\ p\ \textit{with order}\ \alpha$$

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^\alpha}=\lim_{n\to\infty}\frac{|\ln(\frac{p_n}{2}+1)|}{|p_n|^\alpha}$$

Since $p_0=1$, $p_n>0$ for all $n>0$. By L'Hopital's Rule, we have

$$\lim_{n\to\infty}\frac{|\ln(\frac{p_n}{2}+1)|}{|p_n|^\alpha}=\lim_{n\to\infty}\frac{\frac{1}{p_n+2}}{\alpha|p_n|^{\alpha-1}}=\frac{1}{2}>0\ \textit{when}\ \alpha=1$$

When $\alpha\neq 1$, the limit does not exist. When $\alpha=1$, the limit would be $\frac{1}{2}$. Thus, by the order of convergence, we can conclude that the sequence converges linearly.

Problem 2

The real root is only $x=1$.
For the complex roots, suppose $x=a+bi$, so we have $(a+bi)^3=1$. Expanding the LHS,

$$a^3+3a^2bi-3ab^2-b^3i=1$$

Separating the real and imaginary parts of the euqation, we have,

$$\begin{aligned}a^3-3ab^2&=1\\3a^b-b^3&=0\end{aligned}$$

If $b=0$, we have the real roots. So suppose $b\neq 0$, we have $3a^2=b^2$, so $b=\pm a\sqrt{3}$. Plugging into the first equation, we have $a^3-9a^3=1$, so $a=-\frac{1}{2}$ and $b=\pm\frac{\sqrt{3}}{2}$. The imaginary roots are:

$$x=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}$$

By Horner's Method, we have $a_3 = 1, a_2 = 0, a_1 = 0, a_0 = -1$.
Set $b_3 = a_3 = 1$, want to compute b_0
$b_2 = a_2 + b_3 \times -2 = 0 + 1 \times -2 = -2$
$b_1 = a_1 + b_2 \times -2 = 0 + -2 \times -2 = 4$
$b_0 = a_0 + b_1 \times -2 = -1 + 4 \times -2 = -9$
Thus, by Horner's Method, we have $f(-2) = b_0 = -9$.

Problem 3

a. By Lagrange interpolation method, the polynomial f should be

$$f(x) = f(1) \cdot L_{3,0}(x) + f(2) \cdot L_{3,1}(x) + f(3) \cdot L_{3,2}(x) + f(4) \cdot L_{3,3}(x),\ where$$

$$L_{3,0}(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} = \frac{x^3 - 9x^2 + 26x - 24}{-6}$$

$$L_{3,1}(x) = \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} = \frac{x^3 - 8x^2 + 19x - 12}{2}$$

$$L_{3,2}(x) = \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} = \frac{x^3 - 7x^2 + 14x - 8}{-2}$$

$$L_{3,3}(x) = \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} = \frac{x^3 - 6x^2 + 11x - 6}{6}$$

After simplification, we have $f(x) = -\frac{4}{3}x^3 + 10x^2 - \frac{65}{3}x + 15$

b. We want to compute $P_{0,1,2}$ and $P_{1,2,3}$ for $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4$.

$$P_{0,1,2} = f(1) \cdot \frac{(x-2)(x-3)}{(1-2)(1-3)} + f(2) \cdot \frac{(x-1)(x-3)}{(2-1)(2-3)} + f(3) \cdot \frac{(x-1)(x-2)}{(3-1)(3-2)} = 2x^2 - 7x + 7$$

$$P_{1,2,3} = f(2) \cdot \frac{(x-3)(x-4)}{(2-3)(2-4)} + f(3) \cdot \frac{(x-2)(x-4)}{(3-2)(3-4)} + f(4) \cdot \frac{(x-2)(x-3)}{(4-2)(4-3)} = -2x^2 + 13x - 17$$

By Neville's Method, let j=3, i=0,$x_j = 4, x_i = 1$. So we have

$$f(x) = P(x) = \frac{(x-4)(2x^2 - 7x + 7) - (x-1)(-2x^2 + 13x - 17)}{1-4} = -\frac{4}{3}x^3 + 10x^2 - \frac{65}{3}x + 15$$

Problem 4

Theorem 1 let $\delta_n = \frac{p_{n+1}-p}{p_n-p} - \lambda$, since we have $\lim_{n\to\infty} \frac{p_{n+1}-p}{p_n-p} = \lambda$, $\delta_n = \lambda - \lambda = 0$, Then we can write

$$p_{n+1} - p = (\delta_n + \lambda)(p_n - p) \textit{ and } p_{n+2} - p = (\delta_{n+1} + \lambda)(p_{n+1} - p)$$

Also, we have

$$p_{n+1} - p_n = (p_{n+1} - p) - (p_n - p) = (\delta_n + \lambda)(p_n - p) - (p_n - p) = (\delta_n + \lambda - 1)(p_n - p)$$

We can also write

$$\begin{aligned} p_{n+2} - 2p_{n+1} + p_n &= [(p_{n+2} - p - (p_{n+1} - p)] - [(p_{n+1} - p) - (p_n - p)] \\ &= (p_{n+1} - p)(\delta_{n+1} + \lambda - 1) - (p_n - p)(\delta_n + \lambda - 1) \\ &= (p_n - p)(\delta_n + \lambda)(\delta_{n+1} + \lambda - 1) - (p_n - p)(\delta_n + \lambda - 1) \\ &= (p_n - p)[(\delta_n + \lambda)(\delta_{n+1} + \lambda - 1) - (\delta_n + \lambda - 1)] \end{aligned}$$

Using the Aitken's formula, we have

$$\begin{aligned} \hat{p}_n - p = p_n - p - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} \\ = (p_n - p) - \frac{(p_n - p)^2(\delta_n + \lambda - 1)^2}{(p_n - p)[(\delta_n + \lambda)(\delta_{n+1} + \lambda - 1) - (\delta_n + \lambda - 1)]} \\ = (p_n - p)\left(1 - \frac{(\delta_n + \lambda - 1)^2}{(\delta_n + \lambda)(\delta_{n+1} + \lambda - 1) - (\delta_n + \lambda - 1)}\right) \end{aligned}$$

Thus, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = \lim_{n\to\infty} \left(1 - \frac{(\delta_n + \lambda - 1)^2}{(\delta_n + \lambda)(\delta_{n+1} + \lambda - 1) - (\delta_n + \lambda - 1)}\right) \\ = 1 - \frac{(\lambda - 1)^2}{\lambda(\lambda - 1) - (\lambda - 1)} \\ = 1 - \frac{(\lambda - 1)^2}{(\lambda - 1)^2} = 0 \end{aligned}$$

Thus, we have proved that the Aitken's sequence converges to p faster than p_n.

Theorem 2 If $x \neq x_k$, for all $k = 0, 1, 2, ..., n$, define the function g for t in (a, b):

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{t - x_i}{x - x_i}$$

Since $f \in C^{n+1}[a,b]$, it follows that $g \in C^{n+1}[a,b]$. For $t = x_k$, we have

$$g(x_k) = f(x_k) - P(x_k) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{x_k - x_i}{x - x_i} = 0 - [f(x) - P(x)] \cdot 0 = 0$$

Also, we have

$$g(x) = f(x) - P(x) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{x - x_i}{x - x_i} = f(x) - P(x) - [f(x) - P(x)] = 0$$

Thus $g \in C^{n+1}[a,b]$ and g is zero at the $n+2$ distinct numbers $x, x_0, x_1, ..., x_n$. By Generalized Rolle's Theorem, there exists a number ξ in (a,b) for which $g^{(n+1)}(\xi) = 0$.

$$0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \prod_{i=0}^{n} \frac{t - x_i}{x - x_i}$$

Since $P(x)$ is a polynomial of degree at most n, so $P^{(n+1)}$=0. Also $\prod_{i=0}^{n} \frac{t-x_i}{x-x_i}$ is a polynomial of degree $n+1$, so we have

$$\frac{d^{n+1}}{dt^{n+1}} \prod_{i=0}^{n} \frac{t - x_i}{x - x_i} = \frac{(n+1)!}{\prod_{i=0}^{n}(x - x_i)}$$

Thus, we have

$$0 = f^{(n+1)}(\xi) - [f(x) - P(x)] \prod_{i=0}^{n}(x - x_i)$$

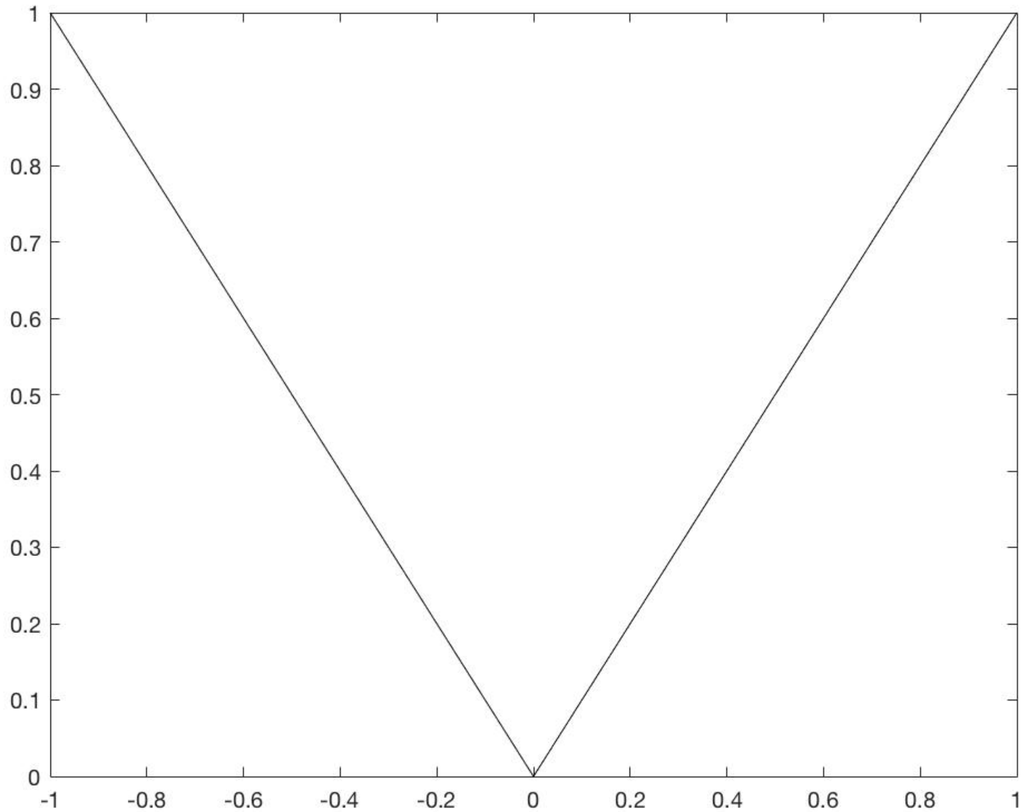
Solving for f(x), we have

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n}(x - x_i)$$

Problem 5

a.

```
fplot(@(x) abs(x),'black',[-1 1]);
```



b.

```
x_2 = [-1 0 1];
y_2 = [1 0 1];
xInt_2 = -1 : 0.01 : 1;
yInt_2 = Lagrange(x_2,y_2,xInt_2);
plot(xInt_2,yInt_2,'b',x_2,y_2,'b*');hold on
x_3 = [-1 -1/3 1/3 1];
y_3 = [1 1/3 1/3 1];
xInt_3 = -1 : 0.01 : 1;
yInt_3 = Lagrange(x_3,y_3,xInt_3);
plot(xInt_3,yInt_3,'r',x_3,y_3,'r*');hold on
x_4 = [-1 -1/2 0 1/2 1];
y_4 = [1 1/2 0 1/2 1];
xInt_4 = -1 : 0.01 : 1;
yInt_4 = Lagrange(x_4,y_4,xInt_4);
plot(xInt_4,yInt_4,'y',x_4,y_4,'y*');hold on
x_5 = [-1 -3/5 -1/5 1/5 3/5 1];
y_5 = [1 3/5 1/5 1/5 3/5 1];
```

```
xInt_5 = -1 : 0.01 : 1;
yInt_5 = Lagrange(x_5,y_5,xInt_5);
plot(xInt_5,yInt_5,'g',x_5,y_5,'g');
function [yi] = Lagrange(x,y,xi)
n = length(x)-1;
ni = length(xi);
L = ones(ni,n+1);
for i = 1:(n+1)
    for j = 1:(n+1)
        if (i ~= j)
            L(:,j)=L(:,j).*(xi'-x(i))/(x(j)-x(i));
        end
    end
end
yi=y*L';
end
```