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Section: 4A
MATH 151A HW #3
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Problem 1

(a) We have $f(x) = 1/x$, $x_0 = 1, x_1 = 2, x_2 = 3$, and $f(x_0) = 1, f(x_1) = 1/2, f(x_2) = 1/3$.

The Lagrange Method

$$\begin{aligned}
P(x) &= f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= 1\cdot\frac{(x-2)(x-3)}{(1-2)(1-3)} + \frac{1}{2}\cdot\frac{(x-1)(x-3)}{(2-1)(2-3)} + \frac{1}{3}\cdot\frac{(x-1)(x-2)}{(3-1)(3-2)} \\
&= \frac{x^2}{6} - x + \frac{11}{6}
\end{aligned}$$

The Divided Difference Method

Zeroth Divided Difference: $f[x_0] = 1, f[x_1] = 1/2, f[x_2] = 1/3$.
First Divided Difference:

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{\frac{1}{2} - 1}{2-1} = -\frac{1}{2}$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{\frac{1}{3} - \frac{1}{2}}{3-2} = -\frac{1}{6}$$

Second Divided Difference:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{-\frac{1}{6} + \frac{1}{2}}{3-1} = \frac{1}{6}$$

$$\begin{aligned}
P(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&= 1 - \frac{1}{2}(x-1) + \frac{1}{6}(x-1)(x-2) \\
&= \frac{x^2}{6} - x + \frac{11}{6}
\end{aligned}$$

(b) Natural cubic spline passing through $(-1, 1), (0, 1), (1, 2)$. Thus, we have $x_0 = -1, x_1 = 0, x_2 = 1$ and two subintervals $[-1, 0]$ and $[0, 1]$.

By Cubic Splines' formula (a), we have

$$S_0(x) = a_0 + b_0(x+1) + c_0(x+1)^2 + d_0(x+1)^3 \ on \ [-1, 0]$$
$$S_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3 \ on \ [0, 1]$$

By Cubic Splines' formula (b) and (c), we have

$$S_0(x_0) = S_0(-1) = 1, \ S_0(x_1) = S_0(0) = 1$$
$$S_1(x_1) = S_1(0) = 1, \ S_1(x_2) = S_1(1) = 2$$
$$a_0 = 1, \ a_1 = 1$$
$$a_0 + b_0 + c_0 + d_0 = 1$$
$$a_1 + b_1 + c_1 + d_1 = 2$$

By Cubic Splines' formula (d), we have

$$S_0'(0) = S_1'(0): \quad b_0 + 2c_0 + 3d_0 = b_1$$

By Cubic Splines' formula (e), we have

$$S_0'(0) = S_1''(0): \quad 2c_0 + 6d_0 = 2c_1$$

By Cubic Splines' formula (f), we have

$$S_0''(-1) = 0: \quad 2c_0 = 0$$

$$S_1''(1) = 0: \quad 2c_1 + 6d_1 = 0$$

Solving all the equations, we get $a_0 = 1, a_1 = 1, b_0 = \frac{3}{4}, b_1 = \frac{3}{2}, c_0 = 0, c_1 = \frac{3}{4}, d_0 = \frac{1}{4}, d_1 = -\frac{1}{4}$. Thus we have:

$$S(x) = \begin{cases} 1 + \frac{3}{4}(x+1) + \frac{1}{4}(x+1)^3, & \text{if } x \in [-1, 0] \\ 1 + \frac{3}{2}x + \frac{3}{4}x^2 - \frac{1}{4}x^3, & \text{if } x \in [0, 1] \end{cases}$$

Problem 2

$L_{n,j}(x_i)$ is defined by

$$L_{n,j}(x_i) = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Hence when $i \neq j$

$$H_{n,j}(x_i) = 0 \ and \ \hat{H}_{n,j}(x_i) = 0$$

for each i, we have

$$H_{n,i}(x_i) = [1 - 2(x_i - x_i)L_{n,i}'(x_i)] \cdot 1 = 1 \ and \ \hat{H}_{n,i}(x_i) = (x_i - x_i) \cdot 1^2 = 0$$

As a consequence, we have

$$H_{2n+1}(x_i) = \sum_{\substack{j=0 \\ j\neq i}}^{n} f(x_j)\cdot 0 + f(x_i)\cdot 1 + \sum_{j=0}^{n} f'(x_j)\cdot 0 = f(x_i)$$

Hence H_{2n+1} agrees with f at $x_0, x_1, ..., x_n$. Since $L_{n,j}(x)$ is a factor of $H'_{n,j}(x)$,so $H'_{n,j}(x) = 0$ when $i \neq j$. In addition, when $i = j$ we have $L_{n,j}(x_i) = 1$,so we have

$$H'_{n,i}(x_i) = -2L'_{n,i}(x_i) + [1 - 2(x_i - x_i)L'_{n,i}(x_i)L'_{n,i}(x_i) = -2L'_{n,i}(x_i) + 2L'_{n,i}(x_i) = 0$$

Hence, $H'_{n,j}(x_i) = 0$ for all i and j. Finally, we have

$$\hat{H}'_{n,j}(x_i) = L^2_{n,j}(x_i) + (x_i - x_j)2L_{n,j}(x_i)L'_{n,j}(x_i) = L_{n,j}(x_i)[L_{n,j}(x_i) + 2(x_i - x_j)L'_{n,j}(x_i)]$$

So $\hat{H}'_{n,j}(x_i) = 0$ if $i \neq j$ and $\hat{H}'_{n,i}(x_i) = 1$, then we have

$$H'_{2n+1}(x_i) = \sum_{j=0}^{n} f(x_j)\cdot 0 + \sum_{\substack{j=0 \\ j\neq i}}^{n} f'(x_j)\cdot 0 + f'(x_i)\cdot 1 = f'(x_i)$$

Thus, H_{2n+1} agrees with f and H'_{2n+1} agrees with f' at $x_0, x_1, ..., x_n$.

To prove the uniqueness of the polynomial of degree $2n + 1$, we assume that there is another polynomial $\widetilde{H}_{2n+1}$ of degree $2n + 1$ satisfying the constraints. Since $H_{2n+1}(x_i) = \widetilde{H}_{2n+1}(x_i) = f(x_i)$ for $i = 0, 1, ..., n$,$H_{2n+1} - \widetilde{H}_{2n+1}$ has at least $n+1$ zeros. By Rolle's Theorem, $H'_{2n+1} - \widetilde{H'}_{2n+1}$ has n zeros that lie within the intervals (x_{i-1}, x_i) for $i = 0, 1, ..., n - 1$.

Since $H'_{2n+1}(x_i) = \widetilde{H'}_{2n+1}(x_i) = f'(x_i)$, $H'_{2n+1} - \widetilde{H'}_{2n+1}$ has $n + 1$ additional zeros, for a total of at least $2n+1$ zeros. However, $H'_{2n+1} - \widetilde{H'}_{2n+1}$ is a polynomial of degree $2n+1$, and the only way that a polynomial of degree $2n + 1$ to have $2n + 1$ zeros is if it is identically zero. Therefore, $H_{2n+1} = \widetilde{H}_{2n+1}$ and $H'_{2n+1} = \widetilde{H'}_{2n+1}$. The Hermite polynomial is unique.

Problem 3

(a) The Lagrange Interpolation Polynomial $P(x)$ for $x = x_0 - h, x_0, x_0 + h$.

$$L_{2,0}(x) = \frac{(x - x_1)(x - x_2)}{(x'_0 - x_1)(x'_0 - x_2)} = \frac{(x - x_0)(x - x_0 - h)}{(x_0 - h - x_0)(x_0 - h - x_0 - h)} = \frac{(x - x_0)(x - x_0 - h)}{2h^2}$$

$$L_{2,1}(x) = \frac{(x - x'_0)(x - x_2)}{(x_1 - x'_0)(x_1 - x_2)} = \frac{(x - x_0 + h)(x - x_0 - h)}{(x_0 - x_0 + h)(x_0 - x_0 - h)} = \frac{(x - x_0 - h)(x - x_0 + h)}{-h^2}$$

$$L_{2,2}(x) = \frac{(x - x'_0)(x - x_1)}{(x_2 - x'_0)(x_2 - x_1)} = \frac{(x - x_0 + h)(x - x_0)}{(x_0 + h - x_0 + h)(x_0 + h - x_0)} = \frac{(x - x_0)(x - x_0 + h)}{2h^2}$$

$$P(x) = f(x_0 - h)\frac{(x-x_0)(x-x_0-h)}{2h^2} + f(x_0)\frac{(x-x_0-h)(x-x_0+h)}{-h^2} + f(x_0+h)\frac{(x-x_0)(x-x_0+h)}{2h^2}$$

(b) The error term

$$\begin{aligned}
E(x) &= \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0')(x-x_1)(x-x_2) \\
&= \frac{f^3(\xi(x))}{3!}(x-x_0+h)(x-x_0)(x-x_0-h) \\
&= \frac{f^3(\xi(x))}{6}(x-x_0+h)(x-x_0)(x-x_0-h)
\end{aligned}$$

(c) $f(x) = P(x) + E(x)$,hence $f'(x) = P'(x) + E'(x)$

$$f'(x_0) = P'(x_0) + E'(x_0) = \frac{1}{2h}(f(x_0+h) - f(x_0-h)) + \frac{f^3(\xi(x))}{6}(-h^2)$$

(d) Suppose the degree of f is less than or equal to 2, then $\frac{f^3(\xi(x))}{6}(-h^2) = 0$ for all x. Thus we have $f'(x) = P'(x) + E'(x) = P'(x)$.

(e) Since we have $|E'(x_0)| < \frac{|f^3(\xi(x))|}{6}|x_0-x_0+h||x_0-x_0-h| < max(x_0-h, x_0+h)\frac{|f^3(\xi(x))|}{6}h^2$, we can conclude that the error bound is $max(x_0-h, x_0+h)\frac{|f^3(\xi(x))|}{6}h^2$.

Problem 4

```
x_a = [0 1 2 ]; y_a = [2 5 9 ];
a = 2;
yInt_2 = Lagrange(x_a,y_a,a);
disp(yInt_2)
function yi = Lagrange(x,y,a)
n = length(x)-1;
L = ones(1,n+1);
for i = 1:(n+1)
    for j = 1:(n+1)
        if (i ~= j)
            L(:,j)=L(:,j).*(a'-x(i))/(x(j)-x(i));
        end
    end
end
yi=y*L';
end
%output P(a)=9
```