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MATH 155 HW #5
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Problem 1

(a)

$$\begin{aligned}
\frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{2\pi iux/M} &= \frac{1}{M}\sum_{u=0}^{M-1}\left[\sum_{x=0}^{M-1} f(x)e^{-2\pi iux/M}\right]e^{2\pi iux/M} \\
&= \frac{1}{M}\sum_{u=0}^{M-1}\sum_{x=0}^{M-1} f(x)e^{-2\pi iux/M}e^{2\pi iux/M} \\
&= \frac{1}{M}f(x)\cdot M \\
&= f(x)
\end{aligned}$$

Since $RHS = LHS$, the identity relation is proved.
(b)

$$\begin{aligned}
\sum_{x=0}^{M-1} f(x)e^{-2\pi iux/M} &= \sum_{x=0}^{M-1}\left[\frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{2\pi iux/M}\right]e^{-2\pi iux/M} \\
&= \frac{1}{M}\sum_{x=0}^{M-1}\sum_{u=0}^{M-1} F(u)e^{2\pi iux/M}e^{-2\pi iux/M} \\
&= \frac{1}{M}F(u)\cdot M \\
&= F(u)
\end{aligned}$$

Since $RHS = LHS$, the identity relation is proved.

Problem 2

To show that 2D Fourier transformation is a linear process, we want to show that for two function f and g, and $a, b \in \mathbb{R}$, and function $h = af + bg$ where $h(x,y) = af(x,y) + bg(x,y)$. After performing the 2D Fourier transform on these three matrices, we have F_f, F_g, F_h respectively. We want to show that for any $u, v \in \mathbb{R}$, we have $F_h(u,v) = aF_f(u,v) + bF_g(u,v)$.

$$
\begin{aligned}
F_h(u,v) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i(ux+vy)}h(x,y) \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i(ux+vy)}[af(x,y)+bg(x,y)] \\
&= a\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i(ux+vy)}f(x,y) + b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i(ux+vy)}g(x,y) \\
&= aF_f(u,v)+bF_g(u,v)
\end{aligned}
$$

Thus, the continuous 2D Fourier transform is linear.

Problem 3

$$
\begin{aligned}
F(u) &= \mathscr{F}(f)(u) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi iux}dx = \int_0^K Ae^{-2\pi iux}dx = \frac{-A}{2\pi iu}\left[e^{-2\pi iux}\right]_0^K \\
&= \frac{-A}{2\pi iu}\left[e^{-2\pi iuK}-1\right] \\
&= \frac{A}{2\pi iu}(e^{i\pi uK}-e^{-i\pi uK})e^{-i\pi uK} \\
&= \frac{A}{2\pi iu}[\cos(\pi uK)+i\sin(\pi uK)-\cos(-\pi uK)-i\sin(-\pi uK)]e^{-i\pi uK} \\
&= \frac{A}{2\pi iu}[2i\sin(\pi uK)]e^{-i\pi uK} \\
&= \frac{A\sin(\pi uK)}{\pi u}e^{-i\pi uK} \\
F(0) &= \int_0^K f(x)dx = \int_0^k Adx = AK
\end{aligned}
$$

Problem 4

We have:

$$\begin{aligned}
H(u,v) &= \overline{H(u,v)} = \overline{H(-u,-v)} = H(-u,-v) \\
\overline{H(u,v)} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{h(x,y)}\ \overline{e^{-2\pi i(ux+vy)}}dxdy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{h(x,y)}e^{2\pi i(ux+vy)}dxdy \\
\overline{H(-u,-v)} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{h(x,y)}\ \overline{e^{-2\pi i(-ux-vy)}}dudv = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{h(x,y)}e^{-2\pi i(ux+vy)}dudv
\end{aligned}$$

We want to show that:

$$h(x,y) = \overline{h(x,y)} = \overline{h(-x,-y)} = h(-x,-y)$$

First, we want to show that $h(x,y) = \overline{h(x,y)}$

$$\begin{aligned}
h(x,y) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H(u,v)e^{2\pi i(ux+vy)}dudv = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{H(-u,-v)}e^{2\pi i(ux+vy)}dudv \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{h(x,y)}e^{-2\pi i(ux+vy)}e^{2\pi i(ux+vy)}dudv = \overline{h(x,y)}
\end{aligned}$$

Then, we want to show that $\overline{h(-x,-y)} = \overline{h(x,y)}$.

$$\begin{aligned}
\overline{h(-x,-y)} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{H(u,v)}\ \overline{e^{2\pi i(-ux-vy)}}dudv = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{H(-u,-v)}e^{2\pi i(ux+vy)}dudv \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{h(x,y)}e^{-2\pi i(ux+vy)}e^{2\pi i(ux+vy)}dudv = \overline{h(x,y)}
\end{aligned}$$

Lastly, we want to show that $h(-x,-y) = \overline{h(x,y)}$

$$\begin{aligned}
h(-x,-y) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H(u,v)e^{2\pi i(-ux-vy)}dudv = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{H(u,v)}e^{-2\pi i(ux+vy)} \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{h(x,y)}e^{2\pi i(ux+vy)}e^{-2\pi i(ux+vy)}dudv = \overline{h(x,y)}
\end{aligned}$$

Since $h(x,y) = \overline{h(x,y)}, \overline{h(-x,-y)} = \overline{h(x,y)}, h(-x,-y) = \overline{h(x,y)}$, we have $h(x,y) = \overline{h(x,y)} = \overline{h(x,y)} = h(x,y)$ and $h(x,y)$ is real and symmetric.

Problem 5

```
img = imread ('Fig5.26a.jpg');
f = fft2(img);
f = fftshift(f);
margin = log(abs(f));
phase = log(angle(f)*180/pi);
l = log(f);
subplot(2,2,1)
imshow(uint8(img))
subplot(2,2,2)
imshow(l,[])
subplot(2,2,3)
imshow(margin,[])
subplot(2,2,4)
imshow(phase,[])
g = fft2(img);
M = mean2(img);
g(1,1) / (size(img,1)*size(img,2))
% by calculating F(0,0)/MN, we know that the average of the originial
% image is 138.0044, which is the same if we use the mean2 function.
```