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MATH 155 HW #3
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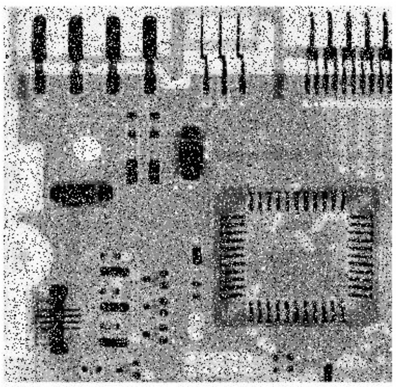
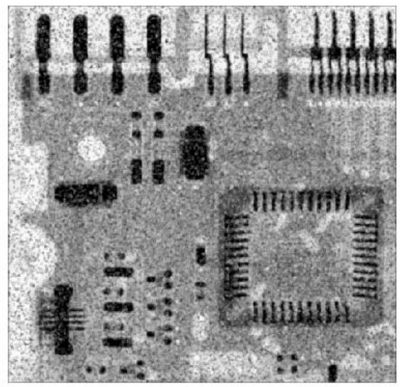
Problem 1

The original and enhanced pictures are shown below:





```
I=imread('Fig3.37(a).jpg');
w=[1/9 1/9 1/9
   1/9 1/9 1/9
   1/9 1/9 1/9];
img = conv2(double(I),w,'same');
subplot(1,2,1)
imshow(uint8(I))
subplot(1,2,2)
imshow(uint8(img))
```

Problem 2

Laplacian operator is defined as:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

To show that the Laplacian operator is isotropic, we want to show that:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2},\ \textit{given that}$$
$$x = x' \cos\theta - y' \sin\theta \textit{ and } y = x' \sin\theta + y' \cos\theta$$

Take the first derivative of f with respect to x', we have:

$$\begin{aligned}\frac{\partial f}{\partial x'} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} \\ &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\end{aligned}$$

Take the second derivative of f with respect to x', we have:

$$\frac{\partial^2 f}{\partial x'^2} = \frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\sin\theta\cos\theta + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta$$

Similarly, for y' we have:

$$\begin{aligned}\frac{\partial f}{\partial y'} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'} \\ &= -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \\ \frac{\partial^2 f}{\partial y'^2} &= \frac{\partial^2 f}{\partial x^2}\sin^2\theta - \frac{\partial}{\partial x}\frac{\partial f}{\partial y}\cos\theta\sin\theta - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}\cos^2\theta\end{aligned}$$

Adding the 2 expressions together we have:

$$\frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f$$

Thus, the Laplacian operator is isotropic.

Problem 3

We want show that for $f, g \in C^2$, and any $a \in \mathbb{R}$, we hvae:

$$\begin{aligned}\nabla^2(f+g) &= \nabla^2 f + \nabla^2 g \\ \nabla^2(af) &= a\nabla^2 f\end{aligned}$$

By the property of partial derivative, we have:

$$\begin{aligned}\nabla^2(f+g) &= \frac{\partial^2(f+g)}{\partial x^2} + \frac{\partial^2(f+g)}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 g}{\partial y^2} \\ &= \nabla^2 f + \nabla^2 g \\ \nabla^2(af) &= \frac{\partial^2(af)}{\partial x^2} + \frac{\partial^2(af)}{\partial y^2} = a\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = a\nabla f\end{aligned}$$

Thus, the continuous Laplacian is a linear operator.

Problem 4

(a) Similar as Problem 2, we want to show that:

$$|\nabla f| = \sqrt{\left(\frac{\partial f}{\partial x'}\right)^2 + \left(\frac{\partial f}{\partial y'}\right)^2}$$

By Problem 2, we can show that:

$$\begin{aligned}
\left(\frac{\partial f}{\partial x'}\right)^2 + \left(\frac{\partial f}{\partial y'}\right)^2 &= \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)^2 + \left(-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right)^2 \\
&= \left(\frac{\partial f}{\partial x}\right)^2 \cos^2\theta + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial f}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial f}{\partial x}\right)^2 \sin^2\theta \\
&- 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial f}{\partial x}\right)^2 \sin^2\theta \\
&= \left(\frac{\partial f}{\partial x}\right)^2 (\cos^2\theta + \sin^2\theta) + \left(\frac{\partial f}{\partial x}\right)^2 (\cos^2\theta + \sin^2\theta) \\
&= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2
\end{aligned}$$

Take the square root of both sides, we get:

$$\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2} = \sqrt{\left(\frac{\partial f}{\partial x'}\right)^2 + \left(\frac{\partial f}{\partial y'}\right)^2} = |\nabla f|$$

Thus,the magnitude of the gradient is an isotropic operation.

(b)

$$\left|\frac{\partial f}{\partial x'}\right| + \left|\frac{\partial f}{\partial y'}\right| = \left|\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right| + \left|-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right|$$

Suppose we have $\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \geq 0$ and $-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \geq 0$, we have

$$\begin{aligned}
\left|\frac{\partial f}{\partial x'}\right| + \left|\frac{\partial f}{\partial y'}\right| &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta - \frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \\
&= \frac{\partial f}{\partial x}(\cos\theta - \sin\theta) + \frac{\partial f}{\partial y}(\cos\theta + \sin\theta)
\end{aligned}$$

The isotropic property is achieved only when $\sin\theta = 0, \cos\theta = 1$

Suppose we have $\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta < 0$ and $-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \geq 0$, we have

$$\begin{aligned}
\left|\frac{\partial f}{\partial x'}\right| + \left|\frac{\partial f}{\partial y'}\right| &= -\frac{\partial f}{\partial x}\cos\theta - \frac{\partial f}{\partial y}\sin\theta - \frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \\
&= \frac{\partial f}{\partial x}(-\cos\theta - \sin\theta) + \frac{\partial f}{\partial y}(\cos\theta - \sin\theta)
\end{aligned}$$

The isotropic property is achieved only when $\sin\theta = -1, \cos\theta = 0$

Suppose we have $\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \geq 0$ and $-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta < 0$, we have

$$\begin{aligned}
\left|\frac{\partial f}{\partial x'}\right| + \left|\frac{\partial f}{\partial y'}\right| &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta + \frac{\partial f}{\partial x}\sin\theta - \frac{\partial f}{\partial y}\cos\theta \\
&= \frac{\partial f}{\partial x}(\cos\theta + \sin\theta) + \frac{\partial f}{\partial y}(-\cos\theta + \sin\theta)
\end{aligned}$$

The isotropic property is achieved only when $\sin\theta = 1, \cos\theta = 0$

Suppose we have $\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta < 0$ and $-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta < 0$, we have

$$\begin{aligned}
\left|\frac{\partial f}{\partial x'}\right| + \left|\frac{\partial f}{\partial y'}\right| &= -\frac{\partial f}{\partial x}\cos\theta - \frac{\partial f}{\partial y}\sin\theta + \frac{\partial f}{\partial x}\sin\theta - \frac{\partial f}{\partial y}\cos\theta \\
&= \frac{\partial f}{\partial x}(-\cos\theta + \sin\theta) + \frac{\partial f}{\partial y}(-\cos\theta - \sin\theta)
\end{aligned}$$

The isotropic property is achieved only when $\sin\theta = 0, \cos\theta = -1$

Thus, in general, the isotropic property is only achieved when satisfying the above conditions and we can say that the isotropic property is lost in general.