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MATH 155 HW #2
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Problem 1

Assume that the original image is f, and the intensities of f is represented as $f(x, y)$ for each pixel (x, y). Also, assume that the minimum intensity of f is f_{min} and the maximum intensity of f is f_{max}.

Since the minimum and the maximum intensities for the new image is 0 and $L - 1$ respectively, the transformation is:

$$T(f(x,y)) = \frac{L-1}{f_{max} - f_{min}}[f(x,y) - f_{min}]$$

Problem 2

(a)
Since $L - 1 > 0$, $3(L-1)^2 + 2(L-1) > 0$.
Since $r \geq 0$, $6r + 2 > 0$.
Since $3(L-1)^2 + 2(L-1) > 0$ and $6r + 2 > 0$, we have

$$p_r(r) = \frac{6r+2}{3(L-1)^2 + 2(L-1)} > 0 \quad if \quad r \geq 0$$
$$p_r(r) = 0 \quad if \quad r < 0$$

Thus,$p_r(r) \geq 0$ for all $r \in (-\infty, +\infty)$. Also,

$$\int_{-\infty}^{+\infty} p_r(r)\ \mathrm{d}r = \int_0^{L-1} \frac{6r+2}{3(L-1)^2 + 2(L-1)}\ \mathrm{d}r = \frac{3r^2+2r}{3(L-1)^2+2(L-1)}\Big|_0^{L-1} = 1$$

Thus, $\int_{-\infty}^{+\infty} p_r(r)\ \mathrm{d}r = 1$
(b)

$$s = T(r) = (L-1)\int_0^r p_r(w)\ \mathrm{d}w = (L-1)\int_0^r \frac{6w+2}{3(L-1)^2+2(L-1)}\ \mathrm{d}w$$
$$= (L-1)\frac{3w^2+2w}{3(L-1)^2+2(L-1)}\Big|_0^r = \frac{3r^2+2r}{3(L-1)+2}$$

(c)

$$
\begin{aligned}
p_s(s) = p_r(r)\left|\frac{dr}{ds}\right| &= \frac{6r+2}{3(L-1)^2+2(L-1)}\left|\left[\frac{ds}{dr}\right]^{-1}\right| \\
&= \frac{6r+2}{3(L-1)^2+2(L-1)}\cdot\frac{1}{\frac{6r+2}{3(L-1)+2}} \\
&= \frac{1}{L-1}
\end{aligned}
$$

Thus, $p_s(s)$ is a uniform "flat" distribution for $s \in [0,1]$

Problem 3

$$
\begin{aligned}
s = T(r) = (L-1)\int_0^r p_r(w)\,\mathrm{d}w &= \int_0^r -2w+2\,\mathrm{d}w = -w^2+2w\Big|_0^r = -r^2+2r \\
s = G(z) = (L-1)\int_0^z p_z(w)\,\mathrm{d}w &= \int_0^z 2w\,\mathrm{d}w = w^2\Big|_0^z = z^2
\end{aligned}
$$

By taking the inverse function of G, we have $z = G^{-1}(s) = \sqrt{s}$. Thus,

$$
z = G^{-1}(T(r)) = G^{-1}(-r^2+2r) = \sqrt{-r^2+2r}
$$

Problem 4

(a) $\forall a, b \in \mathbb{R}$ and $\forall x, y \in V$,H is a linear transformation if $H(ax+by) = aH(x)+bH(y)$

(b) Suppose we have $p, q \in \mathbb{R}$ and 2 images f and m, we want to show that $\forall x, y$

$$
H(pf(x,y)+qm(x,y)) = pH(f(x,y))+qH(m(x,y))
$$

By the property of matrix addition and scalar multiplication, we can create a matrix (image) $u(x,y) = pf(x,y)+qm(x,y)$.Thus:

$$
\begin{aligned}
H(pf(x,y)+qm(x,y)) = H(u(x,y)) = g(x,y) &= \sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)u(x+s,y+t) \\
&= \sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)\left[pf(x+s,y+t)+qm(x+s,y+t)\right] \\
&= p\sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)f(x+s,y+t) + q\sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)m(x+s,y+t) \\
&= pH(f(x,y))+qH(m(x,y)) ==> H\ is\ a\ linear\ transformation.
\end{aligned}
$$