Weifeng Jin

UID: 904429456
MATH 155 HW #6
Date: 02/23/2017

Problem 1

(a)

$$\begin{aligned}\mathcal{F}\left(f(x,y)e^{2\pi i\left(u_0\frac{x}{M}+v_0\frac{y}{N}\right)}\right) &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} e^{-2\pi i\left(u\frac{x}{M}+v\frac{y}{N}\right)} f(x,y)e^{2\pi i\left(u_0\frac{x}{M}+v_0\frac{y}{N}\right)}\\ &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} e^{2\pi i\left((u_0-u)\frac{x}{M}+(v_0-v)\frac{y}{N}\right)} f(x,y)\\ &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} e^{-2\pi i\left((u-u_0)\frac{x}{M}+(v-v_0)\frac{y}{N}\right)} f(x,y) = F(u-u_0, v-v_0)\end{aligned}$$

(b)

$$\begin{aligned}F\left(u-\frac{M}{2}, v-\frac{N}{2}\right) &= \mathcal{F}\left(f(x,y)e^{2\pi i\left(\frac{M}{2}\cdot\frac{x}{M}+\frac{N}{2}\cdot\frac{y}{N}\right)}\right) = \mathcal{F}\left(f(x,y)e^{\pi i(x+y)}\right)\\ &= \mathcal{F}\left(f(x,y)\left[\cos(\pi(x+y)) + i\sin(\pi(x+y))\right]\right)\\ &= \mathcal{F}\left(f(x,y)\cos(\pi(x+y))\right)\\ &\textit{since } \sin(\pi(x+y)) = 0 \textit{ when } x+y\in\mathbb{Z}.\\ &= \mathcal{F}\left(f(x,y)(-1)^{x+y}\right)\\ &\textit{since } \cos(\pi(x+y)) = (-1)^{x+y}\textit{when } x+y\in\mathbb{Z}.\end{aligned}$$

Problem 2

(a)

$$\mathcal{F}\left(f(x-x_0, y-y_0)\right) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x-x_0, y-y_0)e^{-2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

If we let $\tilde{x} = x - x_0$,$\tilde{y} = y - y_0$, the above becomes

$$\begin{aligned}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(\tilde{x},\tilde{y})e^{-2\pi i\left(\frac{u(\tilde{x}+x_0)}{M}+\frac{v(\tilde{y}+y_0)}{N}\right)} &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(\tilde{x},\tilde{y})e^{-2\pi i(\frac{u\tilde{x}}{M}+\frac{v\tilde{y}}{N})}e^{-2\pi i(\frac{ux_0}{M}+\frac{vy_0}{N})}\\ &= F(u,v)e^{-2\pi i(\frac{x_0u}{M}+\frac{y_0v}{N})}\end{aligned}$$

(b)

We have $g(x,y) = f(x+1,y) - f(x,y)$,By linearity, we have
$G(u,v) = \mathcal{F}(f(x+1,y)) - \mathcal{F}(f(x,y))$. By the translation property, we have
$G(u,v) = F(u,v)e^{-2\pi i(-u/M)} - F(u,v)$. So the equivalent filter in the frequency domain is
$H(u,v) = e^{2\pi i(u/M)} - 1$

Problem 3

We want to show that for 2 functions f and g, we have $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$ Let F be $\mathcal{F}(f)$ and G be $\mathcal{F}(g)$, h be the convolution of f and g, and H be $\mathcal{F}(h)$ we have:

$$
\begin{aligned}
F(u) = \mathcal{F}(f) &= \sum_{x=0}^{M-1} f(x)e^{-2\pi i\frac{ux}{M}} \\
G(u) = \mathcal{F}(g) &= \sum_{x=0}^{M-1} g(x)e^{-2\pi i\frac{ux}{M}} \\
h(z) &= \sum_{x=0}^{M-1} f(x)g(z-x) \\
H(u) = \mathcal{F}(h) &= \sum_{z=0}^{M-1}\left(\sum_{x=0}^{M-1} f(x)g(z-x)\right)e^{-2\pi i\frac{zu}{M}}
\end{aligned}
$$

By the translation property, we have that:

$$
\begin{aligned}
H(u) &= \sum_{x=0}^{M-1} \mathcal{F}(g(z-x)) = \sum_{x=0}^{M-1} G(u)e^{-2\pi i\frac{xu}{M}} f(x) \\
&= G(u)\sum_{x=0}^{M-1} e^{-2\pi i\frac{xu}{M}} f(z) = F(u) \cdot G(u)
\end{aligned}
$$

Problem 4

$$
\begin{aligned}
f(x+kM) &= \frac{1}{M}\sum_{u=0}^{M-1} e^{2\pi i\frac{(x+kM)u}{M}} F(u) = \frac{1}{M}\sum_{u=0}^{M-1} e^{2\pi i\frac{ux}{M}} F(u)e^{2\pi iuk} \\
&= \frac{1}{M}\sum_{u=0}^{M-1} e^{2\pi i\frac{ux}{M}} F(u)(\cos(2uk\pi) + isin(2uk\pi)) = \frac{1}{M}\sum_{u=0}^{M-1} e^{2\pi i\frac{ux}{M}} F(u) \\
&= f(x)
\end{aligned}
$$

Problem 5

```
a=imread('Fig4.11(a).jpg');
[m n]=size(a);
fim=fft2(a);
fshift=fftshift(fim);
p=m/2;
q=n/2;
d0=25;
for i=1:m
    for j=1:n
        dist=sqrt((i-p)^2+(j-q)^2);
        low_filter(i,j)=exp(-(dist)^2/(2*(d0^2)));
    end
end
filtered_low=fshift.*low_filter;
image_ori_low=ifftshift(filtered_low);
image_low=real(ifft2(image_ori_low));

for i=1:m
    for j=1:n
        dist=sqrt((i-p)^2+(j-q)^2);
        high_filter(i,j)=1-exp(-(dist)^2/(2*(d0^2)));
    end
end
filtered_high=fshift.*high_filter;
image_ori_high=ifftshift(filtered_high);
image_high=real(ifft2(image_ori_high));

subplot(2,2,1)
imshow(uint8(a))
subplot(2,2,2)
imshow(image_low,[])
subplot(2,2,3)
imshow(image_high,[])
```