Invariant Ring Theory in M2

St. Olaf textbook edition

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Abstract

This is an introductory textbook into invariants with a special focus on the InvariantRings Package in Macaulay 2.

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[3] `aimath.org`

Introduction

Invariant Theory is the study of algebreic strucures like *group* and *rings* that remain unchanged under some action, namely **inveriants**. a common algebreic stucure we study is **polynomial rings**. For example take some trasformation T that swaps x and y. So we have

$$T(p(x,y)) = p(y,x).$$

The question invariant theory asks is what polynomials are unchaged by T We can find some simple examples easily such as

$$\begin{aligned} p_1(x,y) &= x + y \\ p_2(x,y) &= x^2 + xy + y^2 \\ p_3(x,y) &= xy \end{aligned}$$

In fact the set of all these polynomials form a subring! which there special subrings can be hard to compute by hand (thats where our M2 package will come in). There are a lof of questions that arise from these rings these rings have a lot of unique proporties, classifying them is a challenge.

The study of invarient rings started in the 19th century from algebreists such as Cayley. Who studyed linear transformations and in his paper "On the Theory of Linear Transformations (1845)" who established the first invariant theory. Furthur David hilburts work and his *Finiteness Theorm* revolutionised invarient theory saying special invarient rings are finitely generated. So if we want to find the ring we must find all the generators. Most studies of invarient theory study linear transformations over rings of polynomials, like we will focus on in this book. To begin studying invarient theory we must take a detour first to **Represenation Theory.** Which allows us to concretly represent these group actions that act on on our rings as linear transformations.

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Chapter 1

Representation theory

1.1 Introduction

Group theory has become one of the most important aspects in mathematics. Groups are a very hard object to study because sometimes they can be very abstractly described and hard to visualize. Representation Theory allows us to study groups as linear transformations and use linear algebra. This allows us to study groups less abstractly, allowing for us to learn more about groups and apply our group theory knowledge to applications. This method of representing groups as something simpler dates all the way back to Gauss we used the characters of abelian groups. It was expanded and formalized by Frobenius into a subject called Representation Theory.[2] Representation Theory is a power tool that allows us to understand abstract structures like groups, by expressing them in more familiar terms like matrices. Which allows us to use different tools like linear algebra. Linear algebra is a primary tool because it is well understood from a mathematical and geometric sense. Thus when we look at structures in linear algebra we can almost always gain some form of insight into some of the behaviors of that group. With representation theory we can build isomorphisms between some very abstract and not well understood structures such as groups or rings to gain insight. For the purpose of this paper we will define multiple representations. We will define these representations by "mapping" some structure to its representation. This is given in technical language by

$$\rho : G \mapsto \rho(G) \ | \ \rho(g) \mapsto A.$$

This paper we will use a linear algebra to represent our structures. In this paper all representations we are examining group actions which will be described more in depth and rigorously later. There are many other prevalent representations of structures. Some of the most fascinating and useful representations are Lie algebras and Lie groups. However, for now we will stick to examining representations in the forms of matrices and later in the paper we will examine one dimensional representations.

Representation theory allows us to bridge a group theory to application like geometry, physics, and number theory. As a refresher lets define groups and mappings between groups namely, ***group homomorphisms***.

1.2 Intro To Represenations

$groupG* : G \times G \to G$