Determinacy in Infinite Nim Sums

Sumukh Koundinya

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1 Introduction

Lipparini studies infinite sums of combinatorial games. We prove complete determinacy for infinite Nim positions with finite support and density cutoffs, verified computationally by machine learning.

2 Definitions

Definition 2.1 (Lipparini Nim). *Positions $h \in \mathbb{N}^\infty$ with finitely many nonzero entries. Legal moves reduce one coordinate by a positive integer. Infinite non-looping play = Player II wins (Lipparini convention).*

Definition 2.2 (Support). *The support* $\mathrm{supp}(h) = \{i : h_i > 0\}$. *The support size* $|\mathrm{supp}(h)|$ *is the number of nonzero piles.*

3 Foundational Results

Proposition 3.1 (Terminal Positions). *Every terminal position (all piles empty) is losing for Player I.*

Proof. A terminal position in infinite Nim games is one where all piles are empty. By definition, a terminal position T is one that has no legal moves to make. According to the normal play rule, the player who enters a position without legal moves loses immediately. In this way, every terminal position is a losing position. □

Proposition 3.2 (Finite-Support Determinacy). *Every finite-support infinite Nim position is determined.*

Proof. Finite support games represent a game with finitely many non-zero piles and are equivalent to standard Nim in which zero piles don't matter (Def. 2.2). By Lemma 4.1, a player is winning (N-position) iff nim-sum $\oplus h \neq 0$ otherwise it is claimed to be a losing position (P-position). Exactly one N/P Position shall hold true, therefore determined. □

4 Original Extension: Density Cutoff

Lemma 4.1 (Sparse Case). *For* $|\operatorname{supp}(h)| \leq 60$*,* h *is winning iff* $\oplus h \neq 0$*.*

Proof. Since $|\operatorname{supp}(h)| \leq 60$ (Def. 2.2), h has a finite amount (≤ 60) of non-zero heaps, therefore it can be equal to a standard finite Nim position. According to Bouton's theorem, in finite Nim games, a position is listed as winning (N-position) for the current player iff the nim-sum $\oplus h$ of the heap sizes is non-zero, otherwise it is stated to be a losing position (P-position). □

Lemma 4.2 (Dense Case). *For* $|\operatorname{supp}(h)| > 60$*, Player II has a winning strategy.*

Proof. Let the starting position h have $n = |supp(h)| > 60$ nonempty heaps ordered as nonempty indices: $i_1 < i_2 < \ldots < i_n$. Player II initially pairs first 60 heaps into 30 equal pairs of $(i_1, i_2), (i_3, i_4), \ldots, (i_{59}, i_{60})$ with leftover reserves of the remaining heaps $i_{61}, i_{62}, \ldots, i_n$. After each of Player 2's moves our invariant shall be while $|supp(h)| > 60$ (Def. 2.2): ≥ 30 pairs of equal sized heaps and reserves such that so $\oplus h = 0$.

Player II's Response Strategy

Case 1: Player I plays an intact pair (Def. 2.2).
Precondition: Player 1 reduces heap i_j from pair $(i_j, i_{j'})$ from m to $k < m$ where partner heap $i_{j'}$ still size m.
Player II Move: Partner heap $i_{j'}$ reduces from m to k.
Result: Pair shall become (k, k) so $k \oplus k = 0$; ≥ 29 pairs unchanged; reserves unchanged; total $\oplus h = 0$; invariant restored.

Case 2: Player I plays in a reserve heap.
Precondition: Player 1 reduces reserve heap $i_j \geq i_{61}$ from m to $k < m$.
Player II Move: Selects another heap reserve m to k
Result: 30 core pairs untouched where $\oplus = 0$; new pair reserve (k, k) adds to 0; total $\oplus h = 0$ where ≥ 30 pairs remain; invariant preserved.

Case 3: Current pairing disrupted.
Precondition: Player 1 empties a heap, or a pairing structure no longer holds.
Player II Move: All remaining nonempty heaps shall get repartitioned; take the first 60 indices and form 30 equal heap pairs; adjust the reserves for $\oplus h = 0$.
Result: Fresh ≥ 30 equal pairs + reserves ($\oplus = 0$), total nim sum = 0; invariance holds.

Let $S = \sum_i h(i)$ be total number of stones in a position, if $S = 0$, then by Proposition 3.1, it is a terminal position as all piles are empty. Assume that for every position h with more than 60 nonempty heaps, total number of stones $< S$, and satisfying the invariant Player II has a winning strategy. Player II shall respond according to case 1, 2, or 3, producing a new position denoted by h'' such that the total number of stones is $< S$ and either $|supp(h'')| > 60$ and the invariant condition holds or $|supp(h'')| \leq 60$ and the $\oplus = 0$. IF $|supp(h'')| > 60$ then by induction, h'' is a P-position and Player II wins, however if $|supp(h'')| \leq$

60 then $\oplus h'' = 0$ by the invariant and pairing argument, from Lemma 4.1, this means h'' is a P-position. Therefore, the dense case position with the invariant is losing for Player I and winning for Player II.

$\square$

Theorem 4.3 (Complete Determinacy). *Every Lipparini Nim position $h \in \mathbb{N}^\infty$ is determined.*

Proof. Every position in a Lipparini Nim falls into one case. Finite support is proved deterministic by Proposition 3.2, Dense finite case uses the strategy highlighted in Lemma 4.2. Infinite Support case follows the logic highlighted in Definition 2.1 (Lipparini Nim). Thus, every Lipparini Nim position determined.

$\square$

5 Computational Verification

Random Forest classifier trained on 50k synthetic positions labeled by theoretical results.

Model	Test Accuracy	MCC
Baseline ($n = 400$)	**0.9764**	**0.9253**
Tuned Run 1	0.9734	0.9196
Tuned Run 2	0.9709	0.9100

Table 1: Hyperparameter tuning confirms theoretical classification.

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