Mathematical Statitstics Homework 1

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1 Maximum Likelihood Estimation 6.1

Problem 1.4.

(a) given the pdf

$$f(x;\theta) = \begin{cases} \frac{2x}{\theta^2} & \text{if } 0 < x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \frac{2x_i}{\theta^2} \\ &= \left(\frac{2}{\theta^2}\right)^n \prod_{i=1}^{n} x_i \\ &= \left(\frac{2}{\theta^2}\right)^n \prod_{i=1}^{n} x_i I(X_{(n)}, \infty) \end{aligned}$$

for $L(\theta)$ to be maximized,
$\hat{\theta}$=max$(X_1, X_2 \ldots, X_n)$

(b) let $Y = \max(X_1, X_2 \ldots, X_n)$, the CDF of Y

$$\begin{aligned} F_Y(x) = P(Y \le x) &= P(X_1 \le x, X_2 \le x \ldots X_n \le x) \\ &= P(X \le x)^n \\ &= \left(\int_0^x \frac{2x}{\theta^2}\mathrm{d}x\right)^n \\ &= \left(\frac{1}{\theta^2}x^2|_0^x\right)^n \\ &= \left(\frac{x}{\theta}\right)^{2n} \end{aligned}$$

the pdf of Y is thus

$$\begin{aligned} f_Y(x) &= \frac{\mathrm{d}F(x)}{\mathrm{d}x} \\ &= 2n\left(\frac{x}{\theta}\right)^{2n-1}\left(\frac{1}{\theta}\right) \end{aligned}$$

the expected value of Y is,

$$\begin{aligned} E(Y) &= \int_0^\theta x f_Y(x)\mathrm{d}x \\ &= \int_0^\theta x2n\left(\frac{x}{\theta}\right)^{2n-1}\left(\frac{1}{\theta}\right)\mathrm{d}x \\ &= \frac{2n}{\theta^{2n}}\int_0^\theta x^{2n}\mathrm{d}x \\ &= \frac{2n}{\theta^{2n}}\frac{1}{2n+1}x^{2n+1}|_0^\theta \\ &= \frac{2n}{2n+1}\theta \end{aligned}$$

thus, for $E(c \cdot \hat{\theta}) = \theta$, $c = \frac{2n+1}{2n}$

(c) let y be the median of $f(x;\theta)$, thus it follows that the MLE of median is

$$\begin{aligned}
F(y) &= \int_0^y \frac{2x}{\hat{\theta}^2}\mathrm{d}x \\
&= \frac{2}{\hat{\theta}^2}\int_0^y x \ \mathrm{d}x \\
&= \frac{2}{\hat{\theta}^2}\frac{x^2}{2}|_0^y \\
&= \frac{2}{\hat{\theta}^2}\frac{y^2}{2} = \frac{y^2}{\hat{\theta}^2} = 0.5 \\
y &= \sqrt{\frac{1}{2}}\hat{\theta}
\end{aligned}$$

thus the MLE of median is $\sqrt{\frac{1}{2}}\max(X_1, X_2, \ldots, X_n)$

Problem 1.5.

the likelihood function of $f(\vec{x};\theta)$ is

$$\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} 1/\theta e^{-x_i/\theta} \\
&= \theta^{-n} e^{-1/\theta \sum_{i=1}^{n} x_i}
\end{aligned}$$

and thus, the log likelihood of which is

$$l(\theta) = -ln\theta + -1/\theta \sum_{i=1}^{n} x_i$$

and by taking the derivative of log likelihood function

$$l'(\theta) = \frac{-n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i$$

when the equation equals to zero, it follows that,

$$\begin{aligned}
0 &= \frac{-n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i \\
0 &= \frac{-n\theta + \sum_{i=1}^{n} x_i}{\theta^2} \\
n\theta &= \sum_{i=1}^{n} x_i \\
\hat{\theta} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\
&= \bar{x}
\end{aligned}$$

taking the second derivative of $l(\theta)$

$$\begin{aligned}
l''(\theta) &= \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i \\
&= \frac{n\theta - 2\sum_{i=1}^{n} x_i}{\theta^3}
\end{aligned}$$

and plugging in $\theta = \hat{\theta} = \bar{x}$

thus $\bar{x}$ is the the MLE for $f(x;\theta)$

$$\begin{aligned}
l''(\bar{x}) &= \frac{n\bar{x} - 2\sum_{i=1}^{n} x_i}{\bar{x}^3} \\
&= \frac{n\bar{x} - 2n\bar{x}}{\bar{x}^3}, \qquad \text{which is smaller than 0}
\end{aligned}$$

thus the MLE for $P(X \leq 2)$ is by plugging $\theta = \bar{x}$ into the integration,

$$
\begin{aligned}
P(X \leq 2; \bar{x}) &= \int_0^2 f(x; \bar{x}) dx \\
&= \int_0^2 \bar{x} e^{-x/\bar{x}} dx \\
&= \frac{-\bar{x}}{\bar{x}} e^{-x/\bar{x}} |_0^2 \\
&= 1 - e^{-2/\bar{x}}
\end{aligned}
$$

thus the MLE for $P(X \leq 2)$ is $1 - e^{-2/\bar{x}}$

Problem 1.11.

let r.v. X follow a Poisson distribution, which means,

$$P(X) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad \text{where } x = 0, 1, 2 \ldots, \quad m > 0$$

the likelihood function has a form of

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!} \\
&= \frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!}
\end{aligned}
$$

the log likelihood function thus looks like

$$l(\theta) = (\sum_{i=1}^{n} x_i) ln\theta + -n\theta - \sum_{i=1}^{n} lnx_i$$

taking the derivative with respect to θ

$$l'(\theta) = \frac{1}{\theta} \sum_{i=1}^{n} x_i - n$$

when $l'(\theta) = 0$

$$
\begin{aligned}
0 &= \frac{1}{\theta} \sum_{i=1}^{n} x_i - n \\
n\theta &= \sum_{i=1}^{n} x_i \\
\hat{\theta} &= \frac{1}{n} \sum_{i=1}^{n} x_i \\
&= \bar{x}
\end{aligned}
$$

taking the double derivative of $l(\theta)$ thus the MLE of the θ is $\bar{x}$

$$l''(\theta) = \frac{-1}{\theta^2} \sum_{i=1}^{n} x_i$$

which is less than zero, for all values of θ, thus $\hat{\theta}$ is a maximum point.
since $0 < \theta \leq 2$,
if $\bar{x} > 2$, then 2 is the MLE
if $\bar{x} < 2$, then $\bar{x}$ is the MLE,
thus the MLE of θ is $\min\{\bar{x}, 2\}$