Linear Algebra Homework 3

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Exercise 6.7

Problem 13

(a) find $[v]_T$

$$\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}$$
$$\to\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}-7\\4\end{bmatrix},\qquad [v]_T=\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}-7\\4\end{bmatrix}\sharp$$

find $[w]_T$

$$\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$$
$$\to\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}7\\-1\end{bmatrix},\qquad [w]_T=\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}7\\-1\end{bmatrix}\sharp$$

(b) find $P_{S\leftarrow T}$

$$P_{S\leftarrow T}=[[T_1]_S\ [T_2]_S]$$

$$\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}c_1 & c_3\\c_2 & c_4\end{bmatrix}=\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}$$
$$\to\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}c_1 & c_3\\c_2 & c_4\end{bmatrix}=\begin{bmatrix}1 & 2\\-1 & -1\end{bmatrix}$$
$$P_{S\leftarrow T}=\begin{bmatrix}1 & 2\\-1 & -1\end{bmatrix}\sharp$$

(c) find $[v]_S,\ [w]_S$

$$\begin{aligned}[v]_S&=P_{S\leftarrow T}[v]_T\\&=\begin{bmatrix}1 & 2\\-1 & -1\end{bmatrix}\begin{bmatrix}-7\\4\end{bmatrix}\\&=\begin{bmatrix}1\\3\end{bmatrix}\sharp\end{aligned}$$

$$\begin{aligned}[w]_S&=P_{S\leftarrow T}[w]_T\\&=\begin{bmatrix}1 & 2\\-1 & -1\end{bmatrix}\begin{bmatrix}7\\-1\end{bmatrix}\\&=\begin{bmatrix}5\\-6\end{bmatrix}\sharp\end{aligned}$$

(d) find $[w]_S$ and $[v]_S$

$$\begin{bmatrix}S_1 & S_2\end{bmatrix}\begin{bmatrix}[v]_S[w]_S\end{bmatrix}=\begin{bmatrix}v & w\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}c_1 & c_3\\c_2 & c_4\end{bmatrix}=\begin{bmatrix}1 & 5\\5 & 4\end{bmatrix}$$
$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}c_1 & c_3\\c_2 & c_4\end{bmatrix}=\begin{bmatrix}1 & 5\\3 & -6\end{bmatrix}\qquad ,[v]_S=\begin{bmatrix}1\\3\end{bmatrix}\quad [w]_S=\begin{bmatrix}5\\-6\end{bmatrix}\sharp$$

(e) find $Q_{T\leftarrow S}$

$$Q_{T\leftarrow S} = [[S_1]_T[S_2]_T]$$

$$\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}[[S_1]_T[S_2]_T] = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$$
$$\to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}[[S_1]_T[S_2]_T] = \begin{bmatrix}-1 & -2\\1 & 1\end{bmatrix} \qquad Q_{T\leftarrow S} = \begin{bmatrix}-1 & -2\\1 & 1\end{bmatrix}\sharp$$

(f) find $[v]_T$ and $[w]_T$

$$\begin{aligned}[[v]_T[w]_T] &= Q_{T\leftarrow S}[[v]_S[w]_S]\\ &= \begin{bmatrix}-1 & -2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 5\\3 & -6\end{bmatrix}\\ &= \begin{bmatrix}-7 & 7\\4 & -1\end{bmatrix} \qquad [v]_T = \begin{bmatrix}-7\\4\end{bmatrix}\ [w]_T = \begin{bmatrix}7\\-1\end{bmatrix}\sharp\end{aligned}$$

which corresponds to the result in (a)

Problem 15

given bases $S = \{t^2+1, t-2, t+3\}, T = \{2t^2+t, t^2+3, t\}$ for P_2

$v = 8t^2 - 4t + 6,\ w = 7t^2 - t + 9$

(a) find $[v]_T, [w]_T$

$$\begin{bmatrix}T_1 & T_2 & T_3\end{bmatrix}\begin{bmatrix}[v]_T & [w]_T\end{bmatrix} = \begin{bmatrix}v & w\end{bmatrix}$$

$$\begin{bmatrix}2 & 1 & 0\\1 & 0 & 1\\0 & 3 & 0\end{bmatrix}\begin{bmatrix}[v]_T & [w]_T\end{bmatrix} = \begin{bmatrix}8 & 7\\-4 & -1\\6 & 9\end{bmatrix}$$
$$\to \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}[v]_T & [w]_T\end{bmatrix} = \begin{bmatrix}3 & 2\\2 & 3\\-7 & -3\end{bmatrix}$$

$$[v]_T = \begin{bmatrix}3\\2\\-7\end{bmatrix}, [w]_T = \begin{bmatrix}2\\3\\-3\end{bmatrix}\sharp$$

(b)

$$P_{S\leftarrow T} = \begin{bmatrix}[W_1]_S & [W_2]_S & [W_3]_S\end{bmatrix}$$

$$M_S P_{S\leftarrow T} = M_T$$
$$\to \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & -2 & 3\end{bmatrix} P_{S\leftarrow T} = \begin{bmatrix}2 & 1 & 0\\1 & 0 & 1\\0 & 3 & 0\end{bmatrix}$$
$$\to \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} P_{S\leftarrow T} = \begin{bmatrix}2 & 1 & 0\\1 & -0.4 & 0.6\\0 & 0.4 & 0.4\end{bmatrix}$$
$$\to P_{S\leftarrow T} = \begin{bmatrix}2 & 1 & 0\\1 & -0.4 & 0.6\\0 & 0.4 & 0.4\end{bmatrix}\sharp$$

(c)

$$\begin{bmatrix}[v]_S & [w]_S\end{bmatrix} = P_{S\leftarrow T}\begin{bmatrix}[v]_T & [w]_T\end{bmatrix}$$
$$\to \begin{bmatrix}[v]_S & [w]_S\end{bmatrix} = \begin{bmatrix}2 & 1 & 0\\1 & -0.4 & 0.6\\0 & 0.4 & 0.4\end{bmatrix}\begin{bmatrix}3 & 2\\2 & 3\\-7 & -3\end{bmatrix}$$
$$= \begin{bmatrix}8 & 7\\-2 & -1\\-2 & 0\end{bmatrix} \qquad , [v]_S = \begin{bmatrix}8\\-2\\-2\end{bmatrix}\ [w]_S = \begin{bmatrix}7\\-1\\0\end{bmatrix}\sharp$$

(d)

$$\begin{bmatrix}1&0&0\\0&1&1\\1&-2&3\end{bmatrix}\begin{bmatrix}[v]_S & [w]_S\end{bmatrix}=\begin{bmatrix}8&7\\-4&-1\\6&9\end{bmatrix}$$

$$\to [v]_S=\begin{bmatrix}-8\\-2\\-2\end{bmatrix}\ [w]_S=\begin{bmatrix}7\\-1\\0\end{bmatrix}\ \sharp$$

(e)

$$M_S Q_{T\leftarrow S}=M_T$$
$$\begin{bmatrix}2&1&0\\1&0&1\\0&3&0\end{bmatrix}Q_{T\leftarrow S}=\begin{bmatrix}1&0&0\\0&1&1\\1&-2&3\end{bmatrix}$$
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}Q_{T\leftarrow S}=\begin{bmatrix}1/3&1/3&-0.5\\1/3&-2/3&1\\-1/3&2/3&1.5\end{bmatrix}\ \sharp$$

(f)

$$[[v]_T[w]_T]=P_{S\to T}[[v]_S[w]_S]$$
$$\to [[v]_T[w]_T]=\begin{bmatrix}3&2\\2&3\\-7&-3\end{bmatrix}$$

which corresponds to the result of (a)$\sharp$

Problem 17

(a)

$$\begin{bmatrix}1&0&0&1\\0&1&0&0\\1&0&0&0\\1&0&1&0\end{bmatrix}[[v]_T[w]_T]=\begin{bmatrix}1&1\\1&-2\\1&2\\1&1\end{bmatrix}$$
$$\to\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}[[v]_T[w]_T]=\begin{bmatrix}1&2\\1&-2\\1&1\\0&-1\end{bmatrix}$$

$$[v]_T=\begin{bmatrix}1\\1\\1\\0\end{bmatrix}\ [w]_T=\begin{bmatrix}2\\-2\\1\\-1\end{bmatrix}\ \sharp$$

(b)

$$\begin{bmatrix}1&0&0&0\\0&1&0&1\\0&1&2&0\\0&0&1&1\end{bmatrix}P_{S\leftarrow T}=\begin{bmatrix}1&0&0&1\\0&1&0&0\\1&0&0&0\\1&0&1&0\end{bmatrix}$$
$$\to\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}P_{S\leftarrow T}=\begin{bmatrix}1&0&0&1\\1/3&2/3&-2/3&0\\1/3&-1/3&1/3&0\\-1/3&1/3&2/3&0\end{bmatrix}$$

$$P_{S\leftarrow T}=\begin{bmatrix}1&0&0&1\\1/3&2/3&-2/3&0\\1/3&-1/3&1/3&0\\-1/3&1/3&2/3&0\end{bmatrix}\ \sharp$$

(c)

$$[[v]_S[w]_S] = P_{S\leftarrow T}\begin{bmatrix}1 & 2\\ 1 & -2\\ 1 & 1\\ 0 & -1\end{bmatrix}$$
$$= \begin{bmatrix}1 & 0 & 0 & 1\\ 1/3 & 2/3 & -2/3 & 0\\ 1/3 & -1/3 & 1/3 & 0\\ -1/3 & 1/3 & 2/3 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & -2\\ 1 & 1\\ 0 & -1\end{bmatrix}$$
$$= \begin{bmatrix}1 & 1\\ 1/3 & -4/3\\ 1/3 & 5/3\\ 2/3 & -2/3\end{bmatrix} \quad , [v]_S = \begin{bmatrix}1\\ 1/3\\ 1/3\\ 2/3\end{bmatrix} [w]_S = \begin{bmatrix}1\\ -4/3\\ 5/3\\ -2/3\end{bmatrix} \sharp$$

(d)

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 2 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}[[v]_S[w]_S] = \begin{bmatrix}1 & 1\\ 1 & -2\\ 1 & 2\\ 1 & 1\end{bmatrix}$$
$$\rightarrow \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\end{bmatrix}[[v]_S[w]_S] = \begin{bmatrix}1 & 1\\ 1/3 & -4/3\\ 1/3 & 5/3\\ 2/3 & -2/3\end{bmatrix} \quad , [v]_S = \begin{bmatrix}1\\ 1/3\\ 1/3\\ 2/3\end{bmatrix} [w]_S = \begin{bmatrix}1\\ -4/3\\ 5/3\\ -2/3\end{bmatrix} \sharp$$

(e)

$$\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\end{bmatrix} Q_{T\leftarrow S} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 2 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$
$$\rightarrow \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} Q_{T\leftarrow S} = \begin{bmatrix}0 & 1 & 2 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\\ 1 & -1 & -2 & 0\end{bmatrix}$$
$$\rightarrow Q_{T\leftarrow S} = \begin{bmatrix}0 & 1 & 2 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\\ 1 & -1 & -2 & 0\end{bmatrix} \sharp$$

(f)

$$[[v]_T[w]_T] = \begin{bmatrix}0 & 1 & 2 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\\ 1 & -1 & -2 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1/3 & -4/3\\ 1/3 & 5/3\\ 2/3 & -2/3\end{bmatrix}$$
$$= \begin{bmatrix}1 & 2\\ 1 & -2\\ 1 & 1\\ 0 & -1\end{bmatrix}$$

which corresponds to the result in (a)$\sharp$

Problem 23

$$P_{S\leftarrow T} = \begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ -1 & -1 & 1\end{bmatrix}$$
$$= [[w_1]_S[w_2]_S[w_3]_S]$$

thus it follows that

$$
\begin{aligned}
\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} [[w_1]_S[w_2]_S[w_3]_S] &= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \\
\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} &= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \\
\rightarrow \begin{bmatrix} 3 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} &= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}
\end{aligned}
$$

the set

$$
T = \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\} \sharp \tag{1}
$$

Problem 26 find S

$$
P_{S \leftarrow T} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = [[w_1]_S[w_2]_S]
$$

$$
\begin{aligned}
\begin{bmatrix} v_1 & v_2 \end{bmatrix} [[w_1]_S[w_2]_S] &= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \\
\rightarrow \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\
\rightarrow \begin{bmatrix} v_1 & v_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \\ 5 & -3 \end{bmatrix}
\end{aligned}
$$

$$
S = \{-t + 5, t - 3\} \sharp
$$

Theoretical Exercise 6.7

T.1.

Proof. suppose $v = w$, since S is a basis, there can only exist a unique set of coefficient $\{c_1, c_2 \dots c_n\}$, such that $c_1v_1 + c_2v_2 + \dots + c_nv_n = v$, and also, there can only exist a unique set of coefficient $\{b_1, b_2 \dots b_n\}$, such that $b_1v_1 + b_2v_2 + \dots + b_nv_n = w$. And since $v = w$, it follows that, $c_1 = b_1, c_2 = b_2 \dots, c_n = b_n$, adnd thus, $[v]_S = [w]_S$. On the contrary, suppose $[v]_S = [w]_S$, and thus, the coefficient that form v and w are the same, meaning $c_1v_1 + c_2v_2 + \dots + c_nv_n = v = w$, thus, $w = v$.
As a result, $v = w$ if and only if $[v]_S = [w]_S$ □

T.3.

Proof. suppose $\{w_1, w_2 \dots, w_k\}$ is a linearly independent set of vectors in V. It follows that $c_1w_1 + c_2w_2 \dots + c_kw_k = \vec{0}$, where only $c_1 = c_2 = \dots = c_k = 0$ satisfies the equation. Changing both sides to base S coordinate vector. It follows that,

$$
\begin{aligned}
[c_1w_1 + c_2w_2 \dots + c_kw_k]_S &= [\vec{0}]_S \\
\rightarrow [c_1w_1]_S + [c_2w_2]_S \dots + [c_kw_k]_S &= 0_n \\
\rightarrow c_1[w_1]_S + c_2[w_2]_S \dots + c_k[w_k]_S &= 0_n
\end{aligned}
$$

where only $c_1 = c_2 = \dots = c_k = 0$ satifies the equation, thus, $[w_1]_S, [w_2]_S, \dots [w_k]_S$ a is linearly independent set of vectors in $\mathbb{R}^n$

□

T.4.

Proof. Since S is a basis, it follows that

$$[v_1]_S = e_1, [v_2]_S = e_2, \ldots, [v_n]_S = e_n$$

Thus,

$$\{[v_1]_S, [v_2]_S \ldots [v_n]_S\} = \{e_1, e_2 \ldots, e_n\}$$

. Since every vector in $\mathbb{R}^n$ can be spanned by $\{e_1, e_2 \ldots, e_n\}$, and the vectors in $\{e_1, e_2 \ldots, e_n\}$ are linearly independent. Thus $\{[v_1], [v_2] \ldots [v_n]_S\}$ is a basis $\mathbb{R}^n$. □

T.7.

(a)

Proof. since

$$\begin{aligned} M_S &= \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \\ M_T &= \begin{bmatrix} w_1 & w_2 & \ldots & w_n \end{bmatrix} \end{aligned}$$

and it is known that,

$$\begin{aligned} P_{S \leftarrow T} &= [[v_1]_S [v_2]_S \ldots [v_3]_S] \qquad \text{and that} \\ M_S[w_1]_S &= w_1 \end{aligned}$$

thus,

$$\begin{aligned} M_S[[v_1]_S [v_2]_S \ldots [v_3]_S] &= M_T \\ \rightarrow M_S P_{S \leftarrow T} &= M_T \end{aligned}$$

and since M_T and M_S are non-singular, thus it follows that, $P_{S \leftarrow T} = M_S^{-1} M_T \sharp$ □

(b)

Proof. consider homogeneous system $P_{S \leftarrow T}\vec{x} = \vec{0}$ it means that

$$\begin{aligned} M_S^{-1} M_T \vec{x} &= \vec{0} \qquad \text{by multiplying } M_S \text{ on both sides} \\ \rightarrow M_T \vec{x} &= \vec{0} \end{aligned}$$

and since M_T is non-singular, it follows that the homogeneous system only contains trivial solutions. Thus, $P_{S \leftarrow T}$ is non-singular □

(c)

Proof.

$$\begin{aligned} M_S &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ \rightarrow M_S^{-1} &= \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ -2 & 1 & 4 \end{bmatrix} \\ M_T &= \begin{bmatrix} 6 & 4 & 5 \\ 3 & -1 & 5 \\ 3 & 3 & 2 \end{bmatrix} \\ P_{S \leftarrow T} &= M_S^{-1} M_T \\ &= \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ -2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 6 & 4 & 5 \\ 3 & -1 & 5 \\ 3 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

the result holds $\sharp$ □