

Mathematical Statistics Homework 3

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A Sufficient Statistic for a Parameter 7.2

Exercise 2.1 $X \sim N(0, \theta)$

$$f(x; \theta) = \frac{1}{\sqrt{2\pi\theta}} e^{\frac{-x^2}{2\theta}}, \quad -\infty < x < \infty, \quad 0 < \theta < \infty$$

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n f(x_i; \theta) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\theta}} e^{\frac{-x_i^2}{2\theta}} \\ &= (2\pi\theta)^{-\frac{n}{2}} e^{-\frac{1}{2\theta} \sum_{i=1}^n x_i^2} \\ &= k_1\left(\sum_{i=1}^n x_i^2; \theta\right) k_2(x_1, x_2 \dots x_n; \theta) \\ &= k_1\left(\sum_{i=1}^n x_i^2; \theta\right) \cdot 1 \end{aligned}$$

thus, according to the factorization theorem of Neyman, $\sum_{i=1}^n x_i^2$ is a sufficient statistics for θ

Exercise 2.2 $X \sim \text{Poisson}(\theta)$

$$f(x; \theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad 0 < \theta < \infty, \quad x > 0$$

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n \frac{\theta^{x_i} e^{-\theta}}{x_i!} \\ &= \left(\frac{1}{\prod_{i=1}^n x_i!} \right) \left(\theta^{\sum_{i=1}^n x_i} e^{-n\theta} \right) \\ &= k_2(x_1, x_2 \dots x_n; \theta) k_1\left(\sum_{i=1}^n x_i; \theta\right) \end{aligned}$$

thus, according to the factorization theorem of Neyman, $\sum_{i=1}^n x_i$ is a sufficient statistics for θ

Exercise 2.7 $X \sim \Gamma(\theta, 6)$

$$f(x; \theta) = \frac{1}{\Gamma(\theta)6^\theta} x^{\theta-1} e^{\frac{-x}{6}} \quad 0 < x < \infty, \quad \theta > 0$$

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n \frac{1}{\Gamma(\theta)6^\theta} x_i^{\theta-1} e^{\frac{-x_i}{6}} \\ &= \left(\frac{1}{\Gamma(\theta)6^\theta} \left(\prod_{i=1}^n x_i \right)^{\theta-1} \right) e^{\frac{-1}{6} \sum_{i=1}^n x_i} \\ &= k_1\left(\prod_{i=1}^n x_i; \theta\right) k_2(x_1, x_2 \dots x_n; \theta) \end{aligned}$$

thus, according to the factorization theorem of Neyman, $\prod_{i=1}^n x_i$ is a sufficient statistics for θ

Properties of a Sufficient Statistic 7.3

Exercise 3.3

Given a random sample $X \sim \Gamma(1, \theta)$ of sample size 2

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta} \quad 0 < x < \infty, \quad 0 < \theta < \infty$$

The joint PDF is given by

$$f(x_1, x_2; \theta) = \frac{1}{\theta^2} e^{\frac{-1}{\theta}(x_1+x_2)} \quad 0 < x_1, x_2 < \infty, \quad 0 < \theta < \infty$$

$$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_2 \end{cases} \rightarrow \begin{cases} X_1 = Y_1 - Y_2 \\ X_2 = Y_2, \end{cases} \quad 0 < Y_2 < Y_1 < \infty$$

$$|J| = \left| \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \right| = 1$$

$$f(y_1, y_2; \theta) = \frac{1}{\theta^2} e^{\frac{-1}{\theta}(y_1)} \quad 0 < y_2 < y_1 < \infty, \quad 0 < \theta < \infty$$

marginal of f_{Y_1} is

$$\begin{aligned} f(y_1) &= \int_0^{y_1} \frac{1}{\theta^2} e^{\frac{-1}{\theta}(y_1)} dy_2 \\ &= \frac{1}{\theta^2} e^{\frac{-1}{\theta}(y_1)} y_1, \quad 0 < y_1 < \infty, \quad 0 < \theta < \infty \end{aligned}$$

the conditional pdf of Y_2 on Y_1 is

$$\begin{aligned} f(Y_2|y_1; \theta) &= \frac{f(y_1, y_2; \theta)}{f(y_1; \theta)} \\ &= \frac{\frac{1}{\theta^2} e^{\frac{-1}{\theta}(y_1)}}{\frac{1}{\theta^2} e^{\frac{-1}{\theta}(y_1)} y_1} \\ &= \frac{1}{y_1}, \quad 0 < y_2 < y_1 < \infty \end{aligned}$$

Since $Y_2 = X_2$, and $X_2 \sim \Gamma(1, \theta)$,

thus $E(Y_2) = \theta \cdot 1 = \theta$ #

$$Var(Y_2) = 1 \cdot \theta^2 = \theta^2 \text{ #}$$

$$\begin{aligned} E(Y_2|y_1) &= \int_0^{y_1} y_2 \frac{1}{y_1} dy_2 \\ &= \frac{1}{2} y_1 = \varphi(y_1) \text{ #} \end{aligned}$$

$$\begin{aligned} Var(\varphi(y_1)) &= Var\left(\frac{1}{2}Y_1\right), \quad \text{since } Y_1 \sim \Gamma(2, \theta) \\ &= \frac{1}{4}2\theta^2 \\ &= \frac{\theta^2}{2} \text{ #} \end{aligned}$$