Mathematical Statistics Homework 3

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A Sufficient Statistic for a Parameter 7.2

Exercise 2.1 $X \sim N(0,\theta)$

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{\frac{-x^2}{2\theta}} \qquad , -\infty < x < \infty, \quad 0 < \theta < \infty$$

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} f(x_i;\theta) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e^{\frac{-x_i^2}{2\theta}} \\ &= (2\pi\theta)^{-\frac{n}{2}} e^{-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2} \\ &= k_1(\sum_{i=1}^{n} x_i^2;\theta) k_2(x_1, x_2 \ldots x_n) \\ &= k_1(\sum_{i=1}^{n} x_i^2;\theta) \cdot 1 \end{aligned}$$

thus, according to the factorization theorem of Neyman, $\sum_{i=1}^{n} x_i^2$ is a sufficient statistics for $\theta\sharp$

Exercise 2.2 $X \sim Poisson(\theta)$

$$f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}, \qquad 0 < \theta < \infty, \quad x > 0$$

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!} \\ &= \left(\frac{1}{\prod_{i=1}^{n} x_i!}\right)\left(\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}\right) \\ &= k_2(x_1, x_2 \ldots x_n) k_1(\sum_{i=1}^{n} x_i;\theta) \end{aligned}$$

thus, according to the factorization theorem of Neyman, $\sum_{i=1}^{n} x_i$ is a sufficient statistics for $\theta\sharp$

Exercise 2.7 $X \sim \Gamma(\theta, 6)$

$$f(x;\theta) = \frac{1}{\Gamma(\theta)6^\theta} x^{\theta-1} e^{\frac{-x}{6}} \qquad 0 < x < \infty, \quad \theta > 0$$

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \frac{1}{\Gamma(\theta)6^\theta} x_i^{\theta-1} e^{\frac{-x_i}{6}} \\ &= \left(\frac{1}{\Gamma(\theta)6^\theta}(\prod_{i=1}^{n} x_i)^{\theta-1}\right) e^{\frac{-1}{6}\sum_{i=1}^{n} x_i} \\ &= k_1(\prod_{i=1}^{n} x_i;\theta) k_2(x_1, x_2 \ldots x_n) \end{aligned}$$

thus, according to the factorization theorem of Neyman, $\prod_{i=1}^{n} x_i$ is a sufficient statistics for $\theta\sharp$

Exercise 2.8 $X \sim Beta(\alpha, \beta)$

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

given $\alpha = \beta = \theta > 0$, and a random sample of size n

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \frac{\Gamma(2\theta)}{\Gamma(\theta)^2}[x_i(1-x_i)]^{\theta-1} \\ &= (\frac{\Gamma(2\theta)}{\Gamma(\theta)^2})^n \left(\prod_{i=1}^{n} x_i(1-x_i)\right)^{\theta-1} \\ &= k_1(\prod_{i=1}^{n} x_i(1-x_i);\theta)k_2(x_1, x_2 \ldots, x_n) \end{aligned}$$

thus, according to the factorization theorem of Neyman, $\prod_{i=1}^{n} x_i(1-x_i)$ is a sufficient statistics for $\theta\sharp$

Properties of a Sufficient Statistic 7.3

Exercise 3.3

Given a random sample $X \sim \Gamma(1,\theta)$ of sample size 2

$$f(x;\theta) = \frac{1}{\theta}e^{-x/\theta} \qquad 0 < x < \infty, \quad 0 < \theta < \infty$$

The joint PDF is given by

$$f(x_1,x_2;\theta) = \frac{1}{\theta^2}e^{\frac{-1}{\theta}(x_1+x_2)} \qquad 0 < x_1, x_2 < \infty, \quad 0 < \theta < \infty\sharp$$

$$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_2 \end{cases} \rightarrow \begin{cases} X_1 = Y_1 - Y_2 \\ X_2 = Y_2, \qquad 0 < Y_2 < Y_1 < \infty \end{cases}$$

$$|J| = |\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}| = 1$$

$$f(y_1,y_2;\theta) = \frac{1}{\theta^2}e^{\frac{-1}{\theta}(y_1)} \qquad 0 < y_2 < y_1 < \infty, \quad 0 < \theta < \infty$$

marginal of f_{Y_1} is

$$\begin{aligned} f(y_1) &= \int_0^{y_1} \frac{1}{\theta^2}e^{\frac{-1}{\theta}(y_1)}\mathrm{d}y_2 \\ &= \frac{1}{\theta^2}e^{\frac{-1}{\theta}(y_1)}y_1, \qquad 0 < y_1 < \infty, \quad 0 < \theta < \infty \end{aligned}$$

the conditional pdf of Y_2 on Y_1 is

$$\begin{aligned} f(Y_2|y_1;\theta) &= \frac{f(y_1,y_2;\theta)}{f(y_1;\theta)} \\ &= \frac{\frac{1}{\theta^2}e^{\frac{-1}{\theta}(y_1)}}{\frac{1}{\theta^2}e^{\frac{-1}{\theta}(y_1)}y_1} \\ &= \frac{1}{y_1} \qquad ,0 < y_2 < y_1 < \infty \end{aligned}$$

Since $Y_2 = X_2$, and $X_2 \sim \Gamma(1,\theta)$,
thus $E(Y_2) = \theta \cdot 1 = \theta$ $\sharp$
$Var(Y_2) = 1 \cdot \theta^2 = \theta^2$ $\sharp$

$$\begin{aligned} E(Y_2|y_1) &= \int_0^{y_1} y_2\frac{1}{y_1}\mathrm{d}y_2 \\ &= \frac{1}{2}y_1 = \varphi(y_1) \ \sharp \end{aligned}$$

$$\begin{aligned} Var(\varphi(y_1)) &= Var(\frac{1}{2}Y_1), \qquad \text{since } Y_1 \sim \Gamma(2,\theta) \\ &= \frac{1}{4}2\theta^2 \\ &= \frac{\theta^2}{2} \ \sharp \end{aligned}$$