1 Lesson 12/09/2022

Element of set theory

Let X be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \qquad \textbf{(Power Set)}$$

Let $I \subseteq \mathbb{R}$ be a set of indexes. A family of sets induced by I is

$$\{E_i\}_{i \in I}, \quad E_i \subseteq X \qquad \textbf{(Family/Collection)}$$

If $I = \mathbb{N}$ is called a

$$\{E_n\}_{n \in \mathbb{N}} \qquad \textbf{(Sequence)}$$

Definition 1.1
$\{E_n\} \subseteq \mathcal{P}(X)$ is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1} \,\forall n \qquad (\text{resp. } E_n \supseteq E_{n+1} \,\forall n)$$

and is written as

$$\{E_n\} \nearrow \qquad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets $\{E_i\}_{i \in I} \subseteq \mathcal{P}(X)$, will be often considered

$$\bigcup_{i \in I} E_i = \{x \in X : \exists\, i \in I \; s.t. \; x \in E_i\}$$

$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i, \, \forall i \in I\}$$

$\{E_i\}$ is said to be **disjoint** if $E_i \cap E_j = \varnothing \; \forall i \neq j$.

Examples:

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

Definition 1.2
$\{E_n\} \subseteq \mathcal{P}(X)$. We define

$$\limsup_n E_n := \bigcap_{k=1}^{\infty} (\bigcup_{n=k}^{\infty} E_n)$$

$$\liminf_n E_n := \bigcup_{k=1}^{\infty} (\bigcap_{n=k}^{\infty} E_n)$$

If these two sets are equal, then

$$\lim_n E_n = \limsup_n E_n = \liminf_n E_n$$

Proposition 1.1
Some limits are:

- $\limsup_n E_n = \{x \in X : \; x \in E_n \text{ for } \infty - \text{many indexes } n\}$
- $\liminf_n E_n = \{x \in X : \; x \in E_n \text{ for all but finitely many indexes } n\}$

- $\liminf_n E_n \subseteq \limsup_n E_n$
- $(\liminf_n E_n)^C = \limsup_n E_n^C$

Definition 1.3

$$\begin{aligned} x \in \limsup_n E_n &\iff x \in \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} E_n \right) \\ &\iff \forall k \in \mathbb{N} \,:\, \bigcup_{n=k}^{\infty} E_n \\ &\iff \forall k \in \mathbb{N}\; \exists n_k \geq k \, s.t. \, x \in E_{n_k} \end{aligned}$$

So $x \in \limsup_n E_n \implies$
$$\begin{aligned} &\exists m_1 = n_1 \, s.t. \, x \in E_{n_1} \\ &\exists m_2 := n_{m_1+1} \geq m_1 + 1 \, s.t. \, x \in E_{n_2} \\ &\vdots \\ &\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1 \, s.t. \, x \in E_{n_k} \\ &\vdots \\ &x \in E_{m_1}, \ldots, E_{m_k}, \ldots \end{aligned}$$

On the other hand, assume that $x \in E_n$ for ∞-many indexes. We claim that $\forall k \in \mathbb{N}\, \exists n_k \geq k \, s.t. \, x \in E_{n_k} \iff x \in \limsup_n E_n$. If that claim is not true, then $\exists \bar{k} \, s.t. \, x \notin E_n \, \forall n > \bar{k} \implies x$ belongs at most to $E_1, \ldots, E_{\bar{k}}$, a contradiction. ★

Definition 1.4
$\{E_i\}_{i \in I}$ is a **covering** of X if

$$X \subseteq \bigcup_{i \in I} E_i$$

A subfamily of E_i that is still a covering is called a **subcovering**

Definition 1.5
Let $E \subseteq X$. The function $\chi_E : X \to \mathbb{R}$

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \backslash E \end{cases}$$

is called **characteristic function** of E

Let E_1, E_2 be sets:

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

$$\{E_n\} \subseteq \mathcal{P}(X), \text{ disjoint}, E = \bigcup_{n=1}^{\infty} E_n \implies \mathcal{X}_{\mathcal{E}} = \sum_{n=1}^{\infty} \chi_{E_n}$$

$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \implies \chi_P = \liminf \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that $\limsup_n a_n = \lim_{k \to \infty} \sup_{n \geq k} a_n$ and $\liminf_n a_n = \lim_{k \to \infty} \inf_{n \geq k} a_n$

Let's also check that $\chi_Q = \limsup_n \chi_{E_n}$

$$\begin{aligned} x \in \limsup_n E_n &\iff \chi_Q(x) = 1 \\ &\iff \forall\, k \in \mathbb{N}\, \exists\, n_k \geq k \; s.t. \; x \in E_{n_k} \end{aligned}$$

If we fix k then

$$\sup_{n\geq k} \chi_{E_n}(x) = \chi_{E_{n_k}}(x) = 1$$
$$\lim_{k\to\infty} \sup_{n\geq k} \chi_{E_n}(x) = \limsup_{n} \chi_{E_n}(x) = 1$$

Let now $x \notin \limsup E_n \iff \chi_Q(x) = 0$. Then x belongs at most to finitely many $E_n \implies \exists\, \bar{k}$ *s.t.* $x \notin E_n, \forall n \geq \bar{k}$

If $k \geq \bar{k}$, then $\sup_{n\geq k} \chi_{E_n}(x) = 0 \implies \lim_{k\to\infty} \sup_{n\geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 0$

Relations

Given X, Y sets, is called a **relation** of X and Y a subset of $X \times Y$

$$R \subseteq X + Y \quad R = \{(x, y) \,:\, x \in X, y \in Y\}$$

$$(x, y) \in R \iff xRy$$

$$X = \{0, 1, 2, 3\} \quad R = \{(0, 1), (1, 2), (2, 1)\} \text{ is a relation in } X$$

Definition 1.6
A **function** from X to Y is a relation R s.t. for any element x of X $\exists!$ element y of Y s.t. xRy

Definition 1.7
R on X is an **equivalence relation** if

(1) $xRx\ \forall\, x \in X$ (R is **reflexive**)

(2) $xRy \implies yRx$ (R is **symmetric**)

(3) $xRy, yRz \implies xRz$ (R is **transitive**)

If R is an equivalence relation, the set $E_X := \{y \in X \,:\, yRx\}$, $x \in X$ is called the **equivalence class** of X

Definition 1.8
$\frac{X}{R} := \{E_X \,:\, x \in X\}$ is the **quotient set**

Ex: $X = \mathbb{Z}$, let's say that nRm if $n - m$ is even. This is an equivalence relation.

$$E_n = \{\ldots, n-4, n-2, n, n+2, n+4, \ldots\}$$

in this case if n is even, $E_n = \{\text{even numbers}\}$ and if n is odd, $E_n = \{\text{odd numbers}\}$

Measure theory

Definition 1.9
A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is called a **σ-algebra** if

(1) $X \in \mathcal{M}$

(2) $E \in \mathcal{M} \implies E^C = X \backslash E \in \mathcal{M}$

(3) If $E = \bigcup_{n\in\mathbb{N}}$ and $E_n \in \mathcal{M}\ \forall\, n$, then $E \in \mathcal{M}$

If $\mathcal{M}$ is a σ-algebra, $(X, \mathcal{M})$ is called **measurable space** and the sets in $\mathcal{M}$ are called **measurable**. Ex:

- $(X, \mathcal{P}(X))$ is a measurable space

- Let X be a sets, then $\{\varnothing, X\}$ is a σ-algebra

Remark 2
σ is often used to denote the closure w.r.t. countably many operators. If we replace the countable unions with finite unions in the definition of σ-algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space $(X, \mathcal{M})$:

(1) $\varnothing \in \mathcal{M}$: $\varnothing = X^C$ and $X \in \mathcal{M}$

(2) $\mathcal{M}$ is an algebra, and $E_1, \dots, E_n \in \mathcal{M}$

$$E_1 \cup \ldots \cup E_n = E_1 \cup \ldots \cup E_n \cup \underbrace{\varnothing}_{\in \mathcal{M}} \cup \varnothing \ldots \in \mathcal{M}$$

(3) $E_n \in \mathcal{M}$, $\bigcap_{n\in\mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n\in\mathbb{N}} E_n = \underbrace{\left(\bigcup_{n\in\mathbb{N}} \underbrace{E_n^C}_{\in \mathcal{M}}\right)}_{\in \mathcal{M}}^C \qquad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \implies E \backslash F \in \mathcal{M} = E \backslash F = E \cap F^C \in \mathcal{M}$
- If $\Omega \subset X$, then the **restriction** of $\mathcal{M}$ to Ω, written as

$$\mathcal{M}|_\Omega := \{F \subseteq \Omega : F = E \cap \Omega, \text{ with } E \in \mathcal{M}\}$$

  is a σ-algebra on Ω

Theorem 2.1
$\mathcal{S} \subseteq \mathcal{P}(X)$. Then it is well defined the smallest σ-algebra containing $\mathcal{S}$, the σ-algebra generated by $\mathcal{S} := \sigma_0(\mathcal{S})$:

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$ and thus is a σ-algebra
- $\forall \sigma(\mathcal{M})$ s.t. $\mathcal{M} \supseteq \mathcal{S}$, we have $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

Proof idea.

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \varnothing \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} \ : \ \mathcal{M} \in \mathcal{V}\}$, so it can be proved that this is the desired σ-algebra ★

Borel sets

Given (X, d) metric space, the σ-algebra generated by the open sets is called **Borel** σ-algebra, written as $\mathcal{B}(X)$. The sets in $\mathcal{B}(X)$ are called **Borel sets**. The following sets are Borel sets:

- open sets
- closed sets
- countable intersections of open sets: G_σ sets
- countable unions of closed sets: F_σ sets

Remark 3
$\mathcal{B}(\mathbb{R})$ can be equivalently defined as the σ-algebra generated by

$$\{(a,b):\ a,b \in \mathcal{R}, a<b\}$$

$$\{(-\infty,b):\ b \in \mathcal{R}\}$$
$$\{(a,+\infty):\ a \in \mathcal{R}\}$$
$$\{[a,b):\ a,b \in \mathcal{R}, a<b\}$$
$$\vdots$$