Notes from Real and Functional Analysis

Andrea Bonifacio & Ilaria Bonfanti

October 19, 2022

1 Lesson 12/09/2022

Element of set theory

Let X be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \qquad \textbf{(Power Set)}$$

Let $I \subseteq \mathbb{R}$ be a set of indexes. A family of sets induced by I is

$$\{E_i\}_{i\in I}, \quad E_i \subseteq X \qquad \textbf{(Family/Collection)}$$

If $I = \mathbb{N}$ is called a

$$\{E_n\}_{n\in\mathbb{N}} \qquad \textbf{(Sequence)}$$

Definition 1.1
$\{E_n\} \subseteq \mathcal{P}(X)$ is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1}\,\forall n \qquad (\text{resp. } E_n \supseteq E_{n+1}\,\forall n)$$

and is written as

$$\{E_n\} \nearrow \qquad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets $\{E_i\}_{i\in I} \subseteq \mathcal{P}(X)$, will be often considered

$$\bigcup_{i\in I} E_i = \{x \in X : \exists\, i \in I\; s.t.\, x \in E_i\}$$

$$\bigcap_{i\in I} E_i = \{x \in X : x \in E_i,\ \forall i \in I\}$$

$\{E_i\}$ is said to be **disjoint** if $E_i \cap E_j = \varnothing\ \forall i \neq j$.

Examples:

$$[a, b] = \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$$

$$(a, b) = \bigcup_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$$

Definition 1.2
$\{E_n\} \subseteq \mathcal{P}(X)$. We define

$$\limsup_n E_n := \bigcap_{k=1}^{\infty}(\bigcup_{n=k}^{\infty} E_n)$$

$$\liminf_n E_n := \bigcup_{k=1}^{\infty}(\bigcap_{n=k}^{\infty} E_n)$$

If these two sets are equal, then

$$\lim_n E_n = \limsup_n E_n = \liminf_n E_n$$

Proposition 1.1
Some limits are:

- $\limsup_n E_n = \{x \in X :\ x \in E_n \text{ for } \infty - \text{many indexes } n\}$
- $\liminf_n E_n = \{x \in X :\ x \in E_n \text{ for all but finitely many indexes } n\}$

- $\liminf_n E_n \subseteq \limsup_n E_n$
- $(\liminf_n E_n)^C = \limsup_n E_n^C$

Definition 1.3
We can define:

$$\begin{aligned} x \in \limsup_n E_n &\iff & x \in \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} E_n \right) \\ &\iff & \forall k \in \mathbb{N} \, : \, \bigcup_{n=k}^{\infty} E_n \\ &\iff & \forall k \in \mathbb{N} \, \exists n_k \geq k \, s.t. \, x \in E_{n_k} \end{aligned}$$

So $x \in \limsup_n E_n \implies \exists m_1 = n_1 \, s.t. \, x \in E_{n_1}$
$\exists m_2 := n_{m_1+1} \geq m_1 + 1 \, s.t. \, x \in E_{n_2}$
$\vdots$
$\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1 \, s.t. \, x \in E_{n_k}$
$\vdots$
$x \in E_{m_1}, \dots, E_{m_k}, \dots$

On the other hand, assume that $x \in E_n$ for ∞-many indexes. We claim that $\forall k \in \mathbb{N} \, \exists n_k \geq k \, s.t. \, x \in E_{n_k} \iff x \in \limsup_n E_n$. If that claim is not true, then $\exists \bar{k} \, s.t. \, x \notin E_n \, \forall n > \bar{k} \implies x$ belongs at most to $E_1, \dots, E_{\bar{k}}$, a contradiction. ★

Definition 1.4
$\{E_i\}_{i \in I}$ is a **covering** of X if

$$X \subseteq \bigcup_{i \in I} E_i$$

A subfamily of E_i that is still a covering is called a **subcovering**

Definition 1.5
Let $E \subseteq X$. The function $\chi_E \, : X \to \mathbb{R}$

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \setminus E \end{cases}$$

is called **characteristic function** of E

Let E_1, E_2 be sets:

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

$$\{E_n\} \subseteq \mathcal{P}(X), \text{ disjoint}, E = \bigcup_{n=1}^{\infty} E_n \implies \mathcal{X}_{\mathcal{E}} = \sum_{n=1}^{\infty} \chi_{E_n}$$

$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \implies \chi_P = \liminf \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that $\limsup_n a_n = \lim_{k \to \infty} \sup_{n \geq k} a_n$ and $\liminf_n a_n = \lim_{k \to \infty} \inf_{n \geq k} a_n$

Let's also check that $\chi_Q = \limsup_n \chi_{E_n}$

$$\begin{aligned} x \in \limsup_n E_n &\iff & \chi_Q(x) = 1 \\ &\iff & \forall \, k \in \mathbb{N} \, \exists \, n_k \geq k \; s.t. \; x \in E_{n_k} \end{aligned}$$

If we fix k then

$$\sup_{n\geq k}\chi_{E_n}(x) = \chi_{E_{n_k}}(x) = 1$$
$$\lim_{k\to\infty}\sup_{n\geq k}\chi_{E_n}(x) = \limsup_{n}\chi_{E_n}(x) = 1$$

Let now $x \notin \limsup E_n \iff \chi_Q(x) = 0$. Then x belongs at most to finitely many $E_n \implies \exists\,\bar{k}$ *s.t.* $x \notin E_n, \forall n \geq \bar{k}$

If $k \geq \bar{k}$, then $\sup_{n\geq k}\chi_{E_n}(x) = 0 \implies \lim_{k\to\infty}\sup_{n\geq k}\chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 0$

Relations

Given X, Y sets, is called a **relation** of X and Y a subset of $X \times Y$

$$R \subseteq X + Y \quad R = \{(x,y)\,:\,x \in X, y \in Y\}$$

$$(x,y) \in R \iff xRy$$

$$X = \{0,1,2,3\} \quad R = \{(0,1),(1,2),(2,1)\} \text{ is a relation in } X$$

Definition 1.6
A **function** from X to Y is a relation R s.t. for any element x of X $\exists!$ element y of Y s.t. xRy

Definition 1.7
R on X is an **equivalence relation** if

(1) $xRx\ \forall\, x \in X$ (R is **reflexive**)

(2) $xRy \implies yRx$ (R is **symmetric**)

(3) $xRy, yRz \implies xRz$ (R is **transitive**)

If R is an equivalence relation, the set $E_X := \{y \in X\,:\,yRx\}$, $x \in X$ is called the **equivalence class** of X

Definition 1.8
$\frac{X}{R} := \{E_X\,:\,x \in X\}$ is the **quotient set**

Ex: $X = \mathbb{Z}$, let's say that nRm if $n-m$ is even. This is an equivalence relation.

$$E_n = \{\ldots, n-4, n-2, n, n+2, n+4, \ldots\}$$

in this case if n is even, $E_n = \{\text{even numbers}\}$ and if n is odd, $E_n = \{\text{odd numbers}\}$

Measure theory

Definition 1.9
A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is called a **σ-algebra** if

(1) $X \in \mathcal{M}$

(2) $E \in \mathcal{M} \implies E^C = X \backslash E \in \mathcal{M}$

(3) If $E = \bigcup_{n\in\mathbb{N}}$ and $E_n \in \mathcal{M}\ \forall\, n$, then $E \in \mathcal{M}$

If $\mathcal{M}$ is a σ-algebra, $(X, \mathcal{M})$ is called **measurable space** and the sets in $\mathcal{M}$ are called **measurable**. Ex:

- $(X, \mathcal{P}(X))$ is a measurable space

- Let X be a sets, then $\{\varnothing, X\}$ is a σ-algebra

Remark 1.1
σ is often used to denote the closure w.r.t. countably many operators. If we replace the countable unions with finite unions in the definition of σ-algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space $(X, \mathcal{M})$:

(1) $\varnothing \in \mathcal{M}$: $\varnothing = X^C$ and $X \in \mathcal{M}$

(2) $\mathcal{M}$ is an algebra, and $E_1, \dots, E_n \in \mathcal{M}$

$$E_1 \cup \ldots \cup E_n = E_1 \cup \ldots \cup E_n \cup \underbrace{\varnothing}_{\in \mathcal{M}} \cup \varnothing \ldots \in \mathcal{M}$$

(3) $E_n \in \mathcal{M}$, $\bigcap_{n \in \mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n \in \mathbb{N}} E_n = \underbrace{\left(\bigcup_{n \in \mathbb{N}} \underbrace{E_n^C}_{\in \mathcal{M}} \right)}_{\in \mathcal{M}}^C \qquad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \implies E \backslash F \in \mathcal{M} = E \backslash F = E \cap F^C \in \mathcal{M}$
- If $\Omega \subset X$, then the **restriction** of $\mathcal{M}$ to Ω, written as

$$\mathcal{M}|_\Omega := \{F \subseteq \Omega : F = E \cap \Omega, \text{ with } E \in \mathcal{M}\}$$

  is a σ-algebra on Ω

Theorem 1.1
$\mathcal{S} \subseteq \mathcal{P}(X)$. Then it is well defined the smallest σ-algebra containing $\mathcal{S}$, the σ-algebra generated by $\mathcal{S} := \sigma_0(\mathcal{S})$:

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$ and thus is a σ-algebra
- $\forall \sigma(\mathcal{M})$ s.t. $\mathcal{M} \supseteq \mathcal{S}$, we have $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

Proof idea.

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \varnothing \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} \ : \ \mathcal{M} \in \mathcal{V}\}$, so it can be proved that this is the desired σ-algebra ★

Borel sets

Given (X, d) metric space, the σ-algebra generated by the open sets is called **Borel** σ-algebra, written as $\mathcal{B}(X)$. The sets in $\mathcal{B}(X)$ are called **Borel sets**. The following sets are Borel sets:

- open sets
- closed sets
- countable intersections of open sets: G_σ sets
- countable unions of closed sets: F_σ sets

Remark 1.2
$\mathcal{B}(\mathbb{R})$ can be equivalently defined as the σ-algebra generated by

$$\{(a,b):\ a,b\in\mathcal{R}, a<b\}$$

$$\{(-\infty,b):\ b\in\mathcal{R}\}$$
$$\{(a,+\infty):\ a\in\mathcal{R}\}$$
$$\{[a,b):\ a,b\in\mathcal{R}, a<b\}$$
$$\vdots$$

2 Lesson 14/09/2022

Question: What is $\mathcal{B}(\mathbb{R})$? Is $\mathcal{B}(\mathbb{R})\neq\mathcal{P}(\mathbb{R})$? No.

Definition 2.1
$(X,\mathcal{M})$ measurable space. A function $\mu:\ \mathcal{M}\to[0,+\infty]$ is called a **positive measure** if $\mu(\varnothing)=0$ and if μ is countably additive, that is

$$\forall\,\{E_n\}\subseteq\mathcal{M}\quad\text{disjoint}$$

we have that
$$\mu\left(\bigcup_{n=1}^{\infty}\right)=\sum_{n=1}^{\infty}\mu(E_n)\qquad\sigma\text{-additivity}$$

Remark 2.1
a set A is countable if $\exists\ f$ $1-1$ s.t. $f:A\to\mathbb{N}$ Examples: $\mathbb{Z},\mathbb{Q}$ are countable, while $\mathbb{R}$ is not, also $(0,1)$ is uncountable.

We always assume that $\exists\ E\neq\varnothing, E\in\mathcal{M}$ s.t. $\mu(E)\neq\infty$.

If $(X,\mathcal{M})$ is a measurable space, and μ is a measure on it, then $(X,\mathcal{M},\mu)$ is a measure space.

Then:

(1) μ is **finitely additive**:

$$\forall\ E,F\in\mathcal{M},\ \text{with } E\cap F\neq\varnothing\Longrightarrow\mu(E\cup F)=\mu(E)+\mu(F)$$

(2) the **excision property**

$$\forall\ E,f\in\mathcal{M},\ \text{with } E\subset F \text{ and } \mu(E)<+\infty\Longrightarrow\mu(F\backslash E)=\mu(F)-\mu(E)$$

(3) **monotonicity**
$$\forall\ E,F\in\mathcal{M},\ \text{with } E\subset F\Longrightarrow\mu(E)\leq\mu(F)$$

(4) if $\Omega\in\mathcal{M}$ then $(\Omega,\mathcal{M}|_\Omega,\mu|_{\mathcal{M}|_\Omega})$ is a measure space

Proof. (1) $E_1=E, E_2=F, E_3=\ldots=E_n=\ldots=\varnothing$ This is a disjoint sequence $\Longrightarrow$ by σ-additivity.

$$\mu(E\cup F)=\mu\left(\bigcup_n E_n\right)=\sum_n\mu(E_n)=\mu(E)+\mu(F)+\underbrace{\mu(E_k)}_{=\mu(\varnothing)}$$

(2) $E \subset F$, so $F = E \cup (F \backslash E)$ and this is disjoint $\overset{(i)}{\Longrightarrow} \mu(F) = \mu(E) + \mu(F\backslash E)$, and since $\mu(E) < \infty$, the property follows.

(3) $E \subset F \Longrightarrow \mu(F) = \mu(E) + \underbrace{\mu(F\backslash E)}_{\geq 0} \geq \mu(E)$

(4)

★

Definition 2.2
$(X, \mathcal{M}, \mu)$ measure space.

- If $\mu(X) < +\infty$, we say that μ is **finite**.
- If $\mu(X) = +\infty$, and $\exists\ \{E_n\} \subset \mathcal{M}$ s.t. $X = \bigcup_n E_n$ and each E_n has finite measure, then we say that μ is σ-finite.
- If $\mu(X) = 1$ we say that μ is a **probability measure**.

Some examples:

- Trivial Measure: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(E) = 0 \quad \forall\ E \in \mathcal{M}$
- Counting Measure: $(X, \mathcal{P}(X))$ measurable space. We define

$$\mu_C : \mathcal{P}(X) \to [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure: $(X, \mathcal{P}(X))$ measurable space, $t \in X$. We define

$$\delta_t : \mathcal{P}(X) \to [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

Continuity of the measure along monotone sequences
$(X, \mathcal{M}, \mu)$ measure space

(1) $\{E_i\} \subset \mathcal{M},\ E_i \subseteq E_{i+1}\ \forall i$ and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:

$$\mu(E) = \lim_i \mu(E_i)$$

(2) $\{E_i\} \subset \mathcal{M},\ E_{i+1} \subseteq E_i\ \forall i$ and let $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$.

Proof. (1) if $\exists\ i$ s.t. $\mu(E_i) = +\infty$, then is trivial. Assume then that every E_i has a finite measure, so that $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty}(E_{i+1}\backslash E_i)$ with $E_0 = \varnothing$.

So, by σ-additivity

$$\mu(E) = \mu\left(\bigcup_{i=0}^{\infty}(E_{i+1}\backslash E_i)\right) =$$

$$= \sum_{i=0}^{\infty} \mu(E_{i+1} \backslash E_i) \overset{(excision)}{=} \sum_{i=0}^{\infty} (\mu(E_{i+1} - \mu(E_i))) =$$

$$\overset{(telescopic\ series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n)$$

(2) For simplicity, suppose $\tau = 1$, and define $F_k = E_i \backslash E_k$

$$\{E_k\} \searrow \Longrightarrow \{F_k\} \nearrow$$

$$\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \backslash (\bigcap_k E_k)$$

$$\mu(E_i) = \mu(\bigcup_k F_k) + \underbrace{\mu(\bigcap_k E_k)}_{\mu(E)} =$$

$$\overset{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k (\mu(E_i) - \mu(E_k)) + \mu(E)$$

Since $\mu(E_i) < \infty$ we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$

★

Counterexample: given $(\mathcal{N}, \mathcal{P}(\mathbb{N}), \mu_C)$ measure space. Let $E_n = \{n, n+1, n+2, \ldots\}$. In this case $\mu_C(E_n) = +\infty, E_{n+1} \subseteq E_n \forall\, n$, but $\bigcap_n E_n = \varnothing \Longrightarrow \mu\left(\bigcap_n E_n\right) = 0$

Theorem 2.1 (σ-subadditivity of the measure)
$(X, \mathcal{M}, \mu)$ is a measure space. $\forall \{E_n\} \subseteq \mathcal{M}$ (not necessarily disjoint): $\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n)$

Proof. $E_1, E_2 \in \mathcal{M}$ and also $E_1 \cup E_2 = E_1 \cup (E_2 \backslash E_1)$ disjoint sets.

$$\mu(E_1 \cup E_2) = \mu(\underbrace{E_2 \backslash E_1}_{\subseteq E_2}) \overset{(monotonicity)}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.

$$A = \bigcup_{n=1}^{\infty} E_n, \quad A_k = \bigcup_{n=1}^{k} E_n$$

$$\{A_k\} \nearrow,\ A_{k+1} \supseteq A_k,\ \lim_k A_k = A$$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{(continuity)}{=} \lim_k \mu(A_k) = \lim_k \mu\left(\bigcup_{n=1}^{k} E_n\right) \leq$$

$$\leq \lim_k \sum_{n=1}^{k} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

★

Exercise: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, +\infty]$ s.t. μ is finitely additive, σ-subadditive and $\mu(\varnothing) = 0 \Longrightarrow \mu$ is σ-additive, and hence is a measure.

Exercise: the Borel-Cantelli lemma states that, given $(X, \mathcal{M}, \mu)$ and $\{E_n\} \subseteq \mathcal{M}$. Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \Longrightarrow \mu(\limsup_n E_n) = 0$$

It can be phrased as:

> If the series of the probability of the events E_n is convergent, then the probability that ∞-many events occur is 0

Proof. The thesis is:

$$\mu(\limsup_n E_n) = \mu\left(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k \geq n} E_k}_{A_n := \bigcup_{k \geq n} E_k}\right)$$

Is it true that $\{A_n\} \searrow$? Yes.

$$A_{n+1} = \bigcup_{k \geq n+1} E_k \subseteq \bigcup_{k \geq n} E_k = A_n$$

Does some A_n have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k \geq n} E_k\right) \overset{\sigma-sub.}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

★

Sets of 0 measure

$(X, \mathcal{M}, \mu)$ measure space.

- $N \subseteq X$ is a set of 0 measure if $N \in \mathcal{M}$ and $\mu(N) = 0$
- $E \subseteq X$ is called **negligible set** if $\exists\ N \in \mathcal{M}$ with 0 measure s.t. $E \subseteq N$ (E does not necessarily stays in $\mathcal{M}$)

Definition 2.3
$(X, \mathcal{M}, \mu)$ measure space s.t. every negligible set is measurable (and hence of 0 measure), then $(X, \mathcal{M}, \mu)$ is said to be a **complete measure space**.

A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists\ F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \backslash F) = 0\}$$

Clearly $\mathcal{M} \subseteq \overline{\mathcal{M}}$. For $E \in \overline{\mathcal{M}}$, take F and G as above and let $\bar{\mu}(E) = \bar{\mu}(F)$ then $\bar{\mu}|_{\mathcal{M}} = \mu$, and moreover:

Theorem 2.2
$(X, \mathcal{M}, \mu)$ is a complete measure space. Let's observe that $\bar{\mu}$ is well defined: let $E \subseteq X$ and F_1, F_2, G_1, G_2 s.t. $F_i \subset E \subset G_i \quad i = 1, 2$. Then $\mu(G_i \backslash F_i) = 0$. Now we have to check that $\mu(F_1) = \mu(F_2)$.

Since

$$F_1 \backslash F_2 \subseteq E \backslash F_2 \subseteq G_2 \backslash F_2$$

and $G_2 \backslash F_2$ has 0 measure $\implies \mu(F_1 \backslash F_2) = 0$. Then $F_1 = (F_1 \backslash F_2) \cup (F_1 \cap F_2) \implies \mu(F_1) = \mu(F_1 \cap F_2)$. In the same way, $\mu(F_2) = \mu(F_1 \cap F_2)$

3 Lesson 15/09/2022

The elements of $\overline{\mathcal{M}}$ are sets of the type $E \cup N$, with $E \in \mathcal{M}$ and $\bar{\mu}(N) = 0$.

Outer measure

We wish to define a measure λ "on $\mathcal{R}$" with the following properties:

(1) $\lambda((a, b)) = b - a$

(2) $\lambda(E + t)^{\dagger} = \lambda(E)$ for every measurable set $E \subset \mathbb{R}$ and $t \in \mathbb{R}$

It would be nice to define such a measure on $\mathcal{P}(\mathbb{R})$. In such case, note that $\lambda(\{x\}) = 0, \forall\, x \in \mathbb{R}$
But then

Theorem 3.1 (Ulam)
The only measure on $\mathcal{P}(\mathbb{R})$ s.t. $\lambda(\{x\}) = 0 \quad \forall\, x$ is the trivial measure. Thus, a measure satisfying the two properties of the outer measure cannot be defined on $\mathcal{P}(\mathcal{R})$

We'll learn in what follows how to create a measure space on $\mathcal{R}$, with a σ-algebra including all the Borel sets, and a measure satisfying properties of the outer measure. This is the so called **Lebesgue measure**.

Definition 3.1
Given a set X. An **outer measure** is a function $\mu^* : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ s.t.

- $\mu^*(\varnothing) = 0$
- $\mu^*(A) \le \mu^*(B)$ if $A \subseteq B$ (Monotonicity)
- $\mu^*(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} \mu^*(E_n)$ (σ-subadditivity)

The common way to define an outer measure is to start with a family of elementary sets $\mathcal{E}$ on which a notion of measure is defined (e.g. intervals on $\mathcal{R}$, rectangles on $\mathcal{R}^2, \ldots$) and then to approximate arbitrary sets from outside by **countable** unions of members of $\mathcal{E}$.

Proposition 3.1
Let $\mathcal{E} \subset \mathcal{P}(\mathbb{R})$ and $\rho : \mathcal{E} \to [0, +\infty]$ be such that $\varnothing \in \mathcal{E}, X \in \mathcal{E}$ and $\rho(\varnothing) = 0$. For any $A \in \mathcal{P}(X)$, let

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) : E_n \in \mathcal{E} \text{ and } A \subset \bigcup_{n=1}^{\infty} E_n \right\}$$

Then μ^* is an outer measure, the outer measure generated by $(\mathcal{E}, \rho)$.

† $\{x \in \mathbb{R} : x = y + t, \text{ with } y \in E\}$

Proof. $\forall\ A \subset X\ \exists \{E_n\} \subset \mathcal{E}$ s.t. $A \subset \bigcup_n E_n$: take $E_n = X \forall\ n$ then μ^* is well defined. Obviously, $\mu^*(\varnothing) = 0$ (with $E_n = \varnothing \quad \forall\ n$), and $\mu^*(A) \leq \mu^*(B)$ for $A \subset B$ (any covering of B with elements of $\mathcal{E}$ is also a covering of A.)

We have to prove the σ-subadditivity. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ and $\varepsilon > 0$. For each $n, \exists \left\{E_{n_j}\right\}_{j\in\mathbb{N}} \in \mathcal{E}$ s.t. $A_n \subset \bigcup_{i=1}^{\infty} E_{n_j}$ and $\sum_{j=1}^{\infty} \rho(E_{n_j}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$. But then, if $A = \bigcup_{n=1}^{\infty} A_n$, we have that $A \subset \bigcup_{n,j\in\mathbb{N}^2} E_{n_j}$ and

$$\mu^*(A) \leq \sum_{n,j} \rho(E_{n_j}) \leq \sum_n \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_n \mu^*(A_n) + \varepsilon$$

Since ε is arbitrary, we are done. ★

Ex:

(1) $X \in \mathbb{R}, \mathcal{E} = \{(a,b) : a \leq b, a, b \in \mathbb{R}\}$ family of open intervals:

$$\rho((a,b)) = b - a$$

(2) $X = \mathbb{R}^n, \mathcal{E} = \{(a_1, b_1) \times \ldots \times (a_n, b_n) : a_i \leq b_i, a_i, b_i \in \mathbb{R}\}$:

$$\rho((a_1, b_1) \times \ldots \times (a_n, b_n)) = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$$

Remark 3.1
$E \in \mathcal{E} \Longrightarrow \mu^*(E) = \rho(E)$.

In examples 1 and 2, we have in fact $\mu^*((a,b)) = b - a, \mu^*\left((a_1, b_1) \times \ldots \times (a_n, b_n)\right) = \prod_{i=1}^{n}(b_i - a_i)$

To pass from the outer measure to a measure there is a condition
Definition 3.2 (Caratheodory condition)
If μ^* is an outer measure on X, a set $A \subset X$ is called μ^*-**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall\ E \subset X$$

Remark 3.2
If E is a "nice" set containing A, then the above equality says that the outer measure of A, $\mu^*(E \cap A)$, is equal to $\mu^*(E) - \mu^*(E \cap A^C)$, which can be thought as an "inner measure". So basically we are saying that A is measurable if the outer and inner measure coincide. (Like the definition of Riemann integration with lower and upper sum)

Remark 3.3
μ^* is subadditive by def $\Longrightarrow \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall\ E, A \subset X$. So, to prove that a set is μ^*-measurable it is enough to prove the reverse inequality, $\forall\ E \subset X$. In fact, if $\mu^*(E) = +\infty$, then $+\infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$, and hence A is μ^*-measurable iff

$$\mu^*(E) \geq \mu(E \cap A) + \mu^*(E \cap A^C) \quad \forall\ E \subset X \text{ with } \mu^*(E) < +\infty$$

Their relevance to the notion of μ^*-measurability is clarified by the following
Theorem 3.2 (Caratheodory)
If μ^* is an outer measure on X, the family

$$\mathcal{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$$

is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

Lemma 3.1
If $A \subset X$ and $\mu^*(A) = 0$, then A is μ^*-measurable.

Proof. Let $E \subset X$ with $\mu^*(E) < +\infty$. Then

$$\mu^*(E) \geq \mu^*(E) + \mu^*(A) \overset{\ddagger}{\geq} \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

This implies that A is μ^*-measurable. ★

To sum up: X set, $(\mathcal{E}, \rho)$elementary and measurable sets, so μ^* is an outer measure. Then given μ^* and the Caratheodory condition, we have $(X, \mathcal{M}, \mu)$ that is a complete measure space.

Remark 3.4
So far we did not prove that $\mathcal{E} \subseteq \mathcal{M}$. We will do it in a particular case.

Lebesgue measure

- $X = \mathbb{R}$, $\mathcal{E}$ family of open intervals, $\rho((a, b)) = b - a = \lambda((a, b))$, the complete measure space is $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ with $\mathcal{L}(\mathbb{R})$ the Lebesgue-measurable sets on $\mathbb{R}$ and λ the Lebesgue measure on $\mathbb{R}$.
- $X = \mathbb{R}^n$, $\mathcal{E} = \{\prod_{k=1}^n (a_k, b_k) : a_k \leq b_k \quad \forall\, k = 1, \ldots, n\}$, $\rho\left(\prod_{k=1}^n (a_k, b_k)\right) = \prod_{k=1}^n (b_k - a_k)$ and this is a complete measure space $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$

4 Lesson 21/09/2022

Lebesgue measure

$\mathcal{E}$ = family of open intervals (a,b), $a, b \in \mathbb{R}^*, a < b$. ρ = lenght l. $\rho((a, b)) = b - a$.
<u>Notations:</u> open interval I with lenght $l(I)$

Outer measure

$E \subset \mathbb{R}$. The outer measure of E is

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{+\infty} l(I_n) | I_n \text{ is an open interval, } E \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

Caratheodory condition (CC)

$A \subset \mathbb{R}$ is λ^*-measurable if

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^C) \qquad \forall\, E \subset \mathbb{R}$$

$$\{A \subset \mathbb{R} : A \text{ is } \lambda^*\text{-measurable}\} =: \mathcal{L}(\mathbb{R}) \qquad \text{(Lebesgue } \sigma\text{-algebra)}$$

$$\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})} \qquad \text{(Lebesgue measure on } \mathbb{R})$$

Then, $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is a complete measure space. In particular, $\lambda^*(A) = 0 \implies A \in \mathcal{L}(\mathbb{R})$ and $\lambda(A) = 0$.

‡ $E \cap A^C \subseteq E$ and $E \cap A \subseteq A$ + monotonicity

Remark 4.1 (CC-Criterion for measurability)
To check that A is λ^*-measurable, it is sufficient to check that

$$\lambda^* \geq \lambda^*(E \cap A) + \lambda^*(E \cap A^C)$$

for every $E \subset \mathbb{R}$ rith $\lambda^*(E) < +\infty$

Proposition 4.1
Any countable set is measurable, with 0 Lebesgue measure.

Proof. Let $a \in \mathbb{R}$,

$$\{a\} \subseteq (a-\varepsilon, a+\varepsilon), \forall\, \varepsilon > 0 \overset{\text{by def.}}{\Longrightarrow} \lambda^*(\{a\}) \leq 2\varepsilon \overset{\lim}{\Longrightarrow} \lambda^*(\{a\}) = 0$$

$\{a\}$ is measurable with $\lambda(\{a\}) = 0, \forall\ a \in \mathbb{R}$. Now if a set A is countable, $A = \{a_n\}_{n\in\mathbb{N}} = \bigcup_n \{a_n\}$ (disjoint) $\Longrightarrow \lambda(A) \underset{\sigma-add}{=} \sum_n \lambda(\{a_n\}) = 0$ ★

Remark 4.2
$\lambda(\mathbb{Q} = 0)$. $\mathbb{Q}$ is dense on $\mathbb{R}$, $\bar{\mathbb{Q}} = \mathbb{R}$. In general, measure theoretical info and topological info cannot be compared.

Proposition 4.2
$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$

Remark 4.3
So far we didn't prove the fact that open intervals are $\mathcal{L}$-measurable.

Proof. We know that $\mathcal{B}(\mathbb{R})$ is generated by $\{(a, +\infty) : a \in \mathbb{R}\}$. Then, we can directly show that $(a, +\infty) \in \mathcal{L}(\mathbb{R}) \quad \forall\ a \in \mathbb{R}$. Let $a \in \mathbb{R}$ be fixed. We use the criterion for measurability and we check that

$$\lambda^*(E) \geq \lambda^* \underbrace{(E \cap (a, +\infty)}_{=:E_1} + \lambda^* \underbrace{(E \cap (-\infty, a])}_{=:E_2} \quad \forall\ E \subset \mathbb{R},\ \lambda^* < +\infty$$

Since $\lambda^*(E) < +\infty$, $\exists$ a countable union $\bigcup_n I_n \supset E$, where I_n is an open interval $\forall\ n$ and

$$\sum_n l(I_n) \leq \lambda^*(E) + \varepsilon$$

Let $I_n^1 := I_n \cap E_1, I_n^2 := I_n \cap (-\infty, a + \frac{\varepsilon}{2^n})$. These are open intervals:

$$E_1 \subset \bigcup_n I_n^1 \qquad E_2 \subset_n I_n^2 \qquad\qquad \text{countable unions}$$

and moreover

$$l(I_n) \geq l(I_n^1) + l(I_n^2) - \frac{\varepsilon}{2^n}$$

By definition of λ^*, $\lambda^*(E_1) \leq \sum_n l(I_n^1)$ and $\lambda^*(E_2) \leq \sum_n l(I_n^2)$, therefore

$$\lambda^*(E_1) + \lambda^*(E_2) \leq \sum_n l(I_n^1) + \sum_n l(I_n^2) \leq \sum_n \left(l(I_n) + \frac{\varepsilon}{2^n}\right) = \left(\sum_n l(I_n)\right) + \varepsilon \leq \lambda^*(E) + 2\varepsilon$$

Since ε was arbitrarily chosen, we have

$$\lambda^*(E) \geq \lambda^*(E_1) + \lambda^*(E_2)$$

which is the thesis. ★

So, the Lebesgue measure measures all the open, closed G_δ, F_δ sets. Clearly

$$\lambda((a,b)) = b - a$$

One can also show that λ is invariant under translation.

Questions: $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$, is it a strict inclusion or not?

- By Ulam's theorem, if a measure is such that $\lambda(\{a\}) = 0, \forall\, a$ and all the sets in $\mathcal{P}(\mathbb{R})$ are measurable, then $\lambda \equiv 0$. This and the fact that $\lambda((a,b)) \neq 0$ simply that $\mathcal{L}(\mathbb{R}) \subsetneqq$ [‡] $\mathcal{P}(\mathbb{R})$: $\exists$ non-measurable sets called Vitali sets. Every measurable set with positive measure contains a Vitali set. (Explanation)
- $\mathcal{B}(\mathbb{R}) \subsetneqq \mathcal{L}(\mathbb{R})$. The construction of a $\mathcal{L}$-measurable se which is not a Borel set will be done during exercise classes.

The relation between $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ is clarified by

Theorem 4.1 (Regularity of λ)
The following sentences are equivalent:

(1) $E \in \mathcal{L}(\mathbb{R})$

(2) $\forall\, \varepsilon > 0 \exists\, A \supset E$, A open s.t.
$$\lambda(A \backslash E) < \varepsilon$$

(3) $\exists\, G \supset E$, G of class G_δ, s.t.
$$\lambda(G \backslash E) = 0$$

(4) $\exists\, C \subset E$, C closed, s.t.
$$\lambda(E \backslash C) = 0$$

(5) $\exists\, F \subset E$, F of class F_δ, s.t.
$$\lambda(E \backslash F) = 0$$

Consequence: $E \in \mathcal{L}(\mathbb{R}) \Longrightarrow E = F \cup N$, where F is of class F_δ, and $\lambda(N) = 0$.

Partial proof. For simplicity, we will consider only sets with finite measure.

(1) $\Rightarrow$ (2) $E \in \mathcal{L}(\mathbb{R})$. By definition of λ^*, $\forall\, \varepsilon > 0 \exists \bigcup_n I_n \supset E$ s.t. each I_n is an open interval, and
$$\lambda(E) = \lambda^*(E) \geq \sum_n l(I_n) - \varepsilon$$
We define $A = \bigcup_n I_n$, which is open. Also $A \supset E$ and
$$\lambda(A) = \lambda\left(\bigcup_n I_n\right) \overset{\sigma\text{-sub.}}{\leq} \sum_n l(I_n) \leq \lambda(E) + \varepsilon$$
Then, by excision
$$\lambda(A \backslash E) = \lambda(A) - \lambda(E) \leq \varepsilon$$

(2) $\Rightarrow$ (3) Define, for every $K \in \mathbb{N}$, an open set A_k s.t. $A_k \supset E$ and $\lambda(A_k \backslash E) < \frac{1}{k}$. Let $A = \bigcap_k A_k$. This is a G_δ set, it contains E (since each A_k contains E) and
$$\lambda(A \backslash E) \underset{(A \subset A_k\ \forall\, k)}{\leq} \lambda(A_k \backslash E) < \frac{1}{k} \Longrightarrow \lambda(A \backslash E) = 0 \quad \forall\, k$$

[‡]I had no choice

(3) $\Rightarrow$ (1)] If $E \subset \mathbb{R}$ and $\exists\, G \supset E$, with G of class G_δ, s.t. $\lambda(G\backslash E) = 0$, then

$$E = G\backslash(G\backslash E) \text{ is measurable}$$

since G is a Borel set and $(G\backslash E)$ has 0 measure, then both are in $\mathcal{L}$

★

Remark 4.4
Any countable set has 0 measure. he inverse is false. An example is given by the **Cantor set**.

Let $T_0 = [0,1]$. Then we define T_{n+1} stating from T_n in the following way: given T_n, finite union of closed disjoint intervals of lenght $l_n(\frac{1}{3})^n$, T_{n+1} is obtained by removing from each interval of T_n, the open central subinterval of lenght $\frac{l_n}{3}$.

The Cantor set is $T := \bigcap_{k=0}^{+\infty}$. It can be proved that T is compact, $\lambda(T) = 0$ and T is uncountable.

If, instead of removing intervals of size $\frac{1}{3}, \frac{1}{9}, \dots, \frac{1}{3^k}$, we remove sets of size $\left(\frac{\varepsilon}{3}\right)^k$, with $\varepsilon \in (0,1)$, we obtain the **generalized Cantor set** (or **fat Cantor set**) T_ε. T_ε is uncountable, compact and has no interior points (it contains no intervals). However, $\lambda(T_\varepsilon) = \frac{3(1-\varepsilon)}{3-2\varepsilon} > 0$

Remark 4.5
We worked on $\mathbb{R}$, but everything can be adapted to $\mathbb{R}^n$

Measurable functions and integration

Definition 4.1
$f : X \to Y$, then it is well defined the counterimage

$$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(Y)$$

$$E \to f^{-1}(E) = \{x \in X : f(x) \in E\}$$

Definition 4.2
$(X,\mathcal{M}), (Y,\mathcal{N})$ measurable spaces. $f : X \to Y$ is called **measurable** or $(\mathcal{M},\mathcal{N})$-measurable if

$$f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{M}$$

so, the counterimage of measurable sets in Y is a measurable set on X.

5 Lesson 22/09/2022

To check if a function is measurable or not, it is often sed the following proposition

Proposition 5.1
$(X,\mathcal{M}), (Y,\mathcal{N})$ measurable spaces. Let $\mathcal{F} \subseteq \mathcal{P}(Y)$ be s.t. $\mathcal{N} = \sigma_0(\mathcal{F})$. Then

$$f : X \to Y \text{ is } (\mathcal{M},\mathcal{N}) - \text{measurable} \iff f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{F}$$

We will mainly focus on 2 situations:

(1) $((X,\mathcal{M}))$ is a measurable space obtained by means of an outer measure. Ex: $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$, (Y, d_y) metric space $\to (Y, \mathcal{B}(Y))$.

If $X \to Y$ is (Lebesgue) measurable $\iff (\mathcal{M}, \mathcal{B}(Y))$ is measurable

(2) $(X, d_X), (Y, d_Y)$ are metric spaces $\longrightarrow (X, \mathcal{B}(X)), (Y, \mathcal{B}(Y))$ $f : X \to Y$ is Borel measurable $\iff (\mathcal{M}(X), \mathcal{B}(Y))$-measurable.

Remark 5.1
f is Lebesgue measurable if the continuity of the borel set is a Lebesgue-measurable set.

Proposition 5.2
There are two parts:

(1) $(X, d_X), (Y, d_Y)$ metric spaces. If $f : X \to Y$ is continuous, then is Borel measurable

(2) (Y, d_Y) metric space. If $f : \mathbb{R}^n \to Y$ is continuous, then it is a Lebesgue measure.

Proof. The proof is divided in:

(1) f is continuous $\iff f^{-1}(A)$ is open $\forall\ A \subset Y$ open $\implies f^{-1}(A) \in \mathcal{B}(Y)\ \forall\ A \subset Y$ open
Since $\mathcal{B}(Y) = \sigma_0(\text{open sets})$ by proposition (1) this implies that f is Borel measurable

(2) f is continuous $\overset{(1)}{\implies}$ f is Borel measurable. $f^{-1}(A) \in \mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{L}(\mathbb{R}^n) \forall\ A \in \mathcal{B}(Y)$.
Namely f is Lebesgue measurable

★

Proposition 5.3
$(X, \mathcal{M})$ measurable space, $(X, d_Y), (Y, d_Y)$ metric spaces. If $f : X \to Y$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable and $g : Y \to Z$ is continuous $\implies g \circ f : x \to Z$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable

Proposition 5.4
$(X, \mathcal{M})$ measurable space , $u, v : X \to \mathbb{R}$ measurable functions. Let $\Phi : \mathbb{R}^2 \to Y$ be continuous where (Y, d_Y) is a metric space. Then $h : X \to Y$ defined by $h(x) = \Phi(u(x), v(x))$ is $\mathcal{M}, \mathcal{B}(Y)$-measurable.

Consequence: u, v measurable $\Rightarrow u + v$ is measurable.

Proof. Define $f : X \to \mathbb{R}^2$, $f(x) = u(x), v(x)$. By definition $h = \Phi \circ f$ by proposition (3) if we show that f is $(\mathcal{M}, \mathcal{B}(\mathbb{R}^2))$-measurable, then h is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0(\{\underbrace{(a_1, b_1) \times (a_2, b_2)}_{\text{open rectangle}} : a, b \in \mathbb{R}\})$$

Thanks to proposition (1), to check that f is measurable. We can simply check that $f^{-1}(\mathcal{R} \in \mathcal{M})\quad \forall$ open rectangle in $\mathcal{R}^2$ and $R = I \times J$, with I and J open intervals:

$$\begin{gathered} F^{-1}(\mathbb{R}) = \{x \in X : (u(x), v(x)) \in \mathbb{R}\} \\ \Updownarrow \\ u(x) \in I \text{ and } v(x) \in J \\ = \{x \in X : u(x) \in I\} \cap \{x \in X : v(x) \in J\} \\ = \underbrace{u^{-1}(I)}_{\in \mathcal{M}} \cap \underbrace{v^{-1}(J)}_{\in \mathcal{M}} \in \mathcal{M} \\ \text{since both } u, v \text{ are measurable} \end{gathered}$$

This completes the proof ★

Consequences: by proposition 3 and 4, if u and v are measurable, then also $u + v$, $u \cdot v$. Other measurable functions include $u^+ = \max\{u, 0\}, u^- = -\min\{u, 0\}, |u| = u^+ + u^-, u^2, \ldots$
Recall that $u = u^+ - u^-$

Remark 5.2
u^+ is measurable since $u^+ = g \circ u$, where:

$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

Most of the times we will work with functions $f : X \to \mathbb{R}$ or $f : X \to \underbrace{\overline{\mathbb{R}}}_{\mathbb{R}\cup\{\pm\infty\}}$ $(X, \mathcal{M})$ measurable space, then such a function f is measurable iff

$$f^{-1}((a, +\infty)]^{\dagger}) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

or equivalently

$$f^{-1}([a, +\infty)]) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

Let now $\{f_n\}$ be a sequence of measurable functions from X to $\overline{\mathbb{R}}$. Then we define

$$(\inf_n f_n)(x) = \inf_n f_n(x)$$

$$(\sup_n f_n)(x) = \sup_n f_n(x)$$

$$(\liminf_n f_n)(x) = \liminf_n f_n(x)$$

$$(\limsup_n f_n)(x) = \limsup_n f_n(x)$$

$$(\lim_n f_n)(x) = \lim_n f_n(x) \quad \text{if the limit exists}$$

Proposition 5.5
$(X, \mathcal{M})$ measurable space, $f_n : X \to \overline{\mathbb{R}}$ measurable, then

$$\sup_n f_n \;\; \inf_n f_n \;\; \liminf_n f_n \;\; \limsup_n f_n$$

are measurable, in particular if $\lim_n f_n$ is well defined, then f is measurable

***Proof*.** $(\sup f_n)^{-1}((a, \infty]) = \{x \in X : \quad \sup f_n(x) > a \quad \}$

$$\Updownarrow$$

$$\exists \text{ some indexes } n \text{ s.t. } f_n(x) > a$$

$$\bigcup_n \{x \in X : f_n(x) > a\} = \bigcup_n \underbrace{f_n^{-1}((a, +\infty])}_{\in \mathcal{M}}$$

Then $(\sup f_n)^{-1}((a, \infty])$ is measurable, since it is the countable union of measurable sets.

Now we check that the $\limsup_n f_n$ is measurable

$$\limsup_n f_n(x) = \lim_n \underbrace{(\sup_{k>n} f_k(x))}_{\text{is decreasing on } n} = \inf_n(\sup_{k\geq n} f_k(x))$$

If we write $g_n(x) = \sup_{k\geq n} f_k(x)$, then

- g_n is measurable, by what we proved previously
- $\limsup_n f_n = \inf_n g_n$ is measurable

★

[†]We use) if f takes values in $\mathbb{R}$ and] if f takes values in $\overline{\mathbb{R}}$

Simple functions

Definition 5.1
$(X, \mathcal{M})$ measurable space. A measurable function $s : X \to \overline{\mathbb{R}}$ is said to be simple if $s(X)$ is a finite set.

$$s(X) = \{a_1 \ldots, a_n\} \text{ for some } n \in \mathbb{N}, a_i \neq a_j$$

Then $s(x) = \sum_{n=1} a_n \chi_{E_n}(x)$, where E_n is a measurable set, $E_n = \{x \in X : s(X) = a_n\}$, and $E_i \cap E_j = \varnothing$ for $i \neq j$, and $\bigcup_{n=1}^{N} E_n = X$.

Particular case: if s:$\mathbb{R} \to \overline{\mathbb{R}}$, and each E_n is a finite union of intervals, then s is said to be a STEP FUNCTION.

Goal: to approximate arbitrary measurable functions with simple functions.

Theorem 5.1
$(X, \mathcal{M})$ measurable space, $f : X \to [0, \infty]$ measurable. Then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq f \quad \underset{\forall\, x \in X}{\text{(pointwise)}}$$

and $s_n(x) \to f(x) \forall\, x \in X$ as $n \to \infty$ Moreover if f is bounded then $s_n \to f$ uniformly on X as $n \to \infty$

for f bounded. Fix $n \in \mathbb{N}$ and divide $[0, n)$ in $n \cdot 2^n$ intervals called $I_j = [a_j, b_j)$ with lenght $\frac{1}{2^n}$

Let $E_0 = f^{-1}([n, +\infty)), E_j = f^{-1}([a_j, b_j))$ for $j = 1, \ldots, n \cdot 2^n$

We let $\begin{array}{ll} s_n(x) = a_j & \text{for } x \in E_j \\ s_n(x) = n & \text{for } x \in E_0 \end{array}$

Namely we define the simple function s_n as

$$s_n(x) = n\chi_{E_0}(X) + \sum_{j=1}^{n \cdot 2^n} a_j \chi_{E_j}(x)$$

Then $s_n \leq s_{n+1}$ by contradiction, and, since f is bounded, $E_0 = \varnothing$ for n sufficiently large ($n > \sup f$).

Then any $x \in X$ stays in $f^{-1}([a_j, b_j))$ for some j

$$\begin{aligned} &\Longrightarrow \underset{s_n(x)}{\overset{a_j}{\shortparallel}} \leq f(x) < b_j \\ &\Longrightarrow 0 \leq f(x) - s_n(x) < b_j - a_j = \frac{1}{2^n} \\ &\Longrightarrow \sup_{x \in X} |f(x) - s_n x| < \frac{1}{2^n} \to 0 \text{ as } n \to \infty \end{aligned}$$

Namely, $s_n \to f$ uniformly on X. ★

6 Lesson 29/09/2022

Remark 6.1
On the relation between $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda) = (\lambda = \text{Lebesgue measure})$

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is not complete. In fact, $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

Note that, $\forall\, E \in \mathcal{L}(\mathbb{R}) \exists$ a $G_\delta -$ set A and an $F_\delta -$ set B s.t.

$$\begin{array}{l} A \supset E \text{ and } \lambda(A \backslash E) = 0 \\ B \subset E \text{ and } \lambda(E \backslash B) = 0 \end{array}$$

$(X, \mathcal{M}, \mu)$ a complete measure space. Let $P(x)$ be a proposition depending on $x \in X$. We say that $P(x)$ is true $(\mu-)$almost everywhere if

$$\mu\left(\{x \in X : P(x) \text{ is false }\}\right) = 0$$

$P(x)$ is true $\underset{(\mu-\text{a.e.})}{\text{a.e.}}$ on X.

Ex: $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$, $f(x) = x^2$. Then $f(x) > 0$ a.e. on $\mathbb{R}$ (for a.e. x):

$$\{f(x) \leq 0\} = \{0\}, \text{ and } \lambda(\{0\}) = 0$$

$(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu_C)$ with μ_C counting measure. Then it is not true that $f(x) > 0$ μ_C-a.e.

$$\mu_C\left(\{0\}\right) = 1$$

It will be useful to consider sequences converging a.e.:

$$f_n \to f \qquad \text{a.e. on } X_{\text{mbox}_{\text{text}}}$$

if $\mu\left(\{x \in X : \lim_n f_n(x) \neq f(x), \text{ or does not exist }\}\right) = 0$

Proposition 6.1
$(X, \mathcal{M}, \mu)$ complete measure space.

(1) $f : X \to \mathbb{R}$ is measurable, and $g = f$ a.e. on X, then g is measurable

(2) $f_n \to f$ a.e. on X, $f_n : X \to \mathbb{R}$ measurable for all n, then f is measurable

Integration of non-negative functions

Notation:

$$\{x \in X : f(x) \geq 0\} = \{f \geq 0\}$$
$$\{x \in X : f(x) > 0\} = \{f > 0\}$$
$$\vdots$$

$(X, \mathcal{M}, \mu)$ complete measure space. We consider measurable functions $f : X \to [0, +\infty]$

Convention: we define

$$a + \infty = +\infty \quad \forall\, a \in \mathbb{R}$$
$$a \cdot (+\infty) = \begin{cases} +\infty & \text{if } a \neq 0, a > 0 \\ 0 & \text{if } a = 0 \end{cases}$$

With this convention, $+ and \cdot$ of measurable functions are measurable functions.

Definition 6.1
Let $s : X \to [0, +\infty]$ be a measurable simple function,

$$s(x) = \sum_{n=1}^{m} a_n \chi_{D_n}(\bar{x})$$

where $D_1, \ldots, D_m$ are measurable, disjoint, and $\bigcup_{n=1}^m D_n = X$. Let also $E \in \mathcal{M}$. Then we define

$$\int_E s\, d\mu := \sum_{n=1}^{m} a_n \mu(D_n \cap E)$$

Remark 6.2
Given a simple function s:

$$s : [a,b] \to \mathbb{R}, \lambda = \mu \Longrightarrow \int_E s\, d\mu \text{ is the area under the curve}$$

Remark 6.3
There are several points:

- In the definition we have already used the convention $\mu(D_n \cap E = +\infty) \quad$ for some n
- $E \in \mathcal{M} \Longrightarrow \chi_E$ is a simple function

  $$\chi_E(x) = 1 \cdot \chi_E + 0 \cdot \chi_{X \setminus E}(x)$$

  In this case
  $$\int_X \chi_E\, d\mu = 1 \cdot \mu(E) + 0 \cdot \mu(X \setminus E) = \mu(E)$$
- $s\chi_E = \sum_{n=1}^m a_n \chi_{D_n \cap E} \Longrightarrow \int_E s\, d\mu = \int_X s\chi_E\, d\mu$

Definition 6.2
$f : X \to [0, +\infty]$ measurable, $E \in \mathcal{M}$. The **Lebesgue integral** of f on E, with respect to (w.r.t.) μ, is

$$\int_E f\, d\mu = \sup \left\{ \int_E d\mu| \begin{array}{c} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

(1) If f is simple, the definitions are consistent

(2) Also for f measurable: $\int_E f\, d\mu = \int_X f\chi_E\, d\mu$

(3) $(\mathbb{N}, \mathbb{N}, \mu_c)$. $f : \mathbb{N} \to \mathbb{R}$ is a sequence $\{a_n\}_{n \in \mathbb{N}}$

$$\int_{\mathbb{N}} \{a_n\}\, d\mu_c = \sum_{n=0}^{\infty} a_n$$

Basic Properties.
Let $f, g : X \to [0, \infty]$ measurable. $E, F \in \mathcal{M}$, $\alpha \geq 0$. Then:

(1) $\mu(E) = 0 \Rightarrow \int_E f\, d\mu = 0$

(2) $f \leq g$ on $E \Rightarrow \int_E f\, d\mu \leq \int_E g\, d\mu$

(3) $E \subset F \Rightarrow \int_E f\, d\mu \leq \int_F f\, d\mu$

(4) $\alpha \geq o \Rightarrow \int_E \alpha f\, d\mu = \alpha \int_E d\, d\mu$

Remark 6.4
$([0,1], \mathcal{L}([0,1]), \lambda)$
Consider $\chi_{\mathbb{Q}}$, it is the Dirichlet function on $[0,1]$. This is not Riemann integrable. However, $\int_{[0,1]} \chi_{mathbbQ}\, d\lambda = \lambda(\mathbb{Q} \cap [0,1]) = 0$

Theorem 6.1 (Chebychev's inequality)
$f : X \to [0, \infty]$ measurable, $c > 0$. Then

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int \{f \geq c\} f\, d\mu \leq \frac{1}{c} \int_X f\, d\mu$$

Proof. $\int_X f\,d\mu \overset{X\supset\{f\geq c\}}{\geq} \int_{\{f\geq c\}} f\,d\mu \geq \int_{\{f\geq c\}} c\,d\mu = c\int_{\{f\geq c\}} d\mu = c\mu(\{f\geq c\})$ ★

Theorem 6.2
$s : X \to [0,\infty]$ simple. Define $\varphi : \mathbb{M} \to [0,\infty]$
$\varphi(E) = \int_E s\,d\mu$
$\Rightarrow \varphi$ is a measure.

Proof. $\mu(\varnothing) = 0 \Rightarrow \varphi(\varnothing) = 0$ by definition.
Definition 6.3 (sigma additivity)
$\{E_n \subset \mathbb{M}\}$ disjoint, and let $E = \bigcup_{n=1}^\infty E_n \Rightarrow s = \sum_{k=1}^m a_k\chi_{D_k}\ D_k \in \mathbb{M}$

Then, by definition and since μ is a measure and $E \cap D_k = \bigcup_n(E_n \cap D_k)$
$\varphi(E) = \sum_{k=1}^m a_k\mu(D_k \cap E) = \sum_{k=1}^\infty a_k \sum_{n=1}^\infty \mu(E_n \cap D_k = \sum_{n=1}^\infty(\sum_{k=1}^m a_k\mu(E_n \cap D_k))) = \sum_{n=1}^\infty \int_{E_n} s\,d\mu = \sum_{n=1}^\infty \varphi(E_n)$ ★

Theorem 6.3 (Vanishing Lemma)
$f : X \to [0,\infty]$ measurable. $E \subset X$ measurable

$$\int_E f\,d\mu = 0 \iff f = 0 \text{ a.e. on } E$$

Proof. $\Leftarrow$ easy // $\Rightarrow$ Consider $E\cap\{f>0\} = \bigcup_{n=1}^\infty \left(\underbrace{E_n\{f \geq \frac{1}{n}\}}_{=:E_n}\right)$ Then $\{E_n\}$ is an increasing sequence. By Chebyshev

$$\mu(E_n) \leq \frac{1}{\frac{1}{n}}\int_E f\,d\mu = 0\ \forall n \Rightarrow \mu(E_n) = 0\ \forall n$$

$\mu(E) \cup \{f>0\} \overset{\text{continuity}}{=} \lim_n \mu(E_n) = 0$, namely $f = 0$ a.e. on E ★

The $\int$ does not see sets with 0 measure.
Definition 6.4
If $f : X \to [0,\infty]$ is measurable, and $\int_X f\,d\mu < \infty$ then we say that f is integrable

Theorem 6.4 (Monotone Convergence - Beppo Levi)
$f_n : X \to [0,\infty]$ measurable. Suppose that

(1) $f_n(x) \leq f_{n+1}(x)$ for a.e. $x \in X$ for every n

(2) $f_n \to f$ a.e. on X

Then

$$\int_X f\,d\mu = \lim_n \int_X f_n\,d\mu$$

Proof. Part 1. Assume that 1) and 2) hold everywhere. First, if f is measurable $\int_X f_n\,d\mu \nearrow \Rightarrow$ $\exists\alpha = \lim_n \int_X f_n\,d\mu$ Also, $f_n \leq f$ everywhere $\Rightarrow \int_X f_n\,d\mu \leq \int_X f\,d\mu\ \forall n$
$\Rightarrow \alpha \leq \int_X f\,d\mu$ We want to show that also $\geq$ is true. Let s be a simple function s.t. $0 \leq s \leq f$ and $c \in (0,1)$ Let $E_n = \{f_n \geq cs\} \in \mathbb{M}$

(1) $E_n \in E_{n+1}\ \forall n$:
if $x \in E_n$, then $f_n(x) \geq cs(x) \Rightarrow f_{n+1}(x) \geq cs(x)$
$\Rightarrow f_{n+1}(x) \geq f_n(x) \geq cs(x) \Rightarrow x \in E_{n+1}$

(2) Moreover, $X = \bigcup_{n=1}^{\infty} E_n$. Indeed:
- if $f(x) = 0$, then $s(x) = 0 \Rightarrow f_1(x) = 0 = cs(x),\ x \in E_1$
- if $f(x) > 0$, then $cs(x) < f(x) = \lim_n f_n(x)$ since $s \leq f$ and $c < 1$
$\Rightarrow cs(x) < f_n(x)$ for n sufficiently large, namely $x \in E_n$ for n sufficiently large.

Therefore, $\alpha \geq \int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq c \int_{E_n} s \, d\mu = c\varphi(E_n)\ \forall n, \forall 0 \leq s \leq f, \forall c \in [0,1]\ \varphi(E_n) = \int_{E_n} s \, d\mu$. φ is a measure, and $\{E_n\} \nearrow$ Therefore, taking the lim when $n \to \infty$ by continuity $\alpha \geq \lim_n c \int_{E_n} s \, d\mu = c \int_X s \, d\mu\ \forall c \in [0,1]$ Take the limit when $c \to 1^- :\ \alpha \geq \int_X s \, d\mu\ \forall o \leq s \leq f$ Take the sup over s: $\alpha \geq \int_X f \, d\mu$ We proved both inequalities, so the tesis holds.
Part 2. Note that $\{x \in X :$ assumpions (1) and (2) of the theorem do not hold$\}$ is a set of zero measure. Take F. $X = E \cup F$ since we have the assumpion on E and $\mu(F) = 0$ Then, by the Vanishing Lemma, since $(f - f\chi_E) = 0$ a.e. and $(f_n - f_n\chi_E) = 0$ we have that

$$\int_X f \, d\mu = \int_E d \, d\mu = \lim_n \int_E f_n \, d\mu = \lim_n \int_X f_n \, d\mu$$

★

7 Lesson 05/10/2022

Theorem 7.1 (Monotone Convergence (or Beppo Levi's theorem))
$f_n : X \to [0, +\infty]$ measurable. Suppose that

(1) $f_n(x) \leq f_{n+1}(x)$ for a.e. $x \in X$, for every n

(2) $f_n \to f$ a.e. on X

Then

$$\int_X f \, d\mu = \lim_n \int_X f_n \, d\mu$$

Corollary 7.1
$f_n : X \to [0, +\infty]$ measurable, then

$$\int_X \left(\sum_{n=0}^{\infty} f_n \right) d\mu = \sum_{n=0}^{\infty} \int_X f_n \, d\mu$$

Theorem 7.2 (Approximation with simple functions)
Given $(X, \mathcal{M})$ measure space, $f : X \to [0, +\infty]$ measurable, then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq f \qquad \text{pointwise } \forall\, x \in X$$

and

$$s_n(x) \to f(x) \qquad \forall\, x \in X \text{as } n \to \infty$$

Moreover, if f is bounded, then $s_n \to f$ uniformly on X as $n \to \infty$.

Remark 7.1
There is also

$$\int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu \;\middle|\; \begin{array}{l} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

But let $\{s_n\}$ be the sequence given by the simple approximation theorem. By monotone convergence

$$\int_X f\,d\mu = \lim_n \int_X s_n\,d\mu$$

Ex: $f, g : X \to [0, +\infty]$. Then

$$\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$$

Lemma 7.1 (Fatou's Lemma)
Given $f_n \to [0, +\infty]$ measurable $\forall\, n$. Then

$$\int_X (\liminf_n f_n)\,d\mu \le \liminf_n \int_X f_n\,d\mu$$

In particular, if $f_n \to f$ a.e. on X.

Proof. Given that $(\liminf_n f_n)(x) = \lim_n(\underbrace{\inf_{k\ge n} f_k(x)}_{=g_n(x)})$. Now, for every $x \in X$, $\{g_n(x)\} \nearrow$

$$g_{n+1}(x) = \inf_{k\ge n+1} f_k(x) \ge \inf_{k\ge n} f_k(x) = g_n(x)$$

Also, $g_n \ge 0$ on X. Thus, by monotone convergence

$$\int_X \liminf_n f_n\,d\mu = \int_X \lim_n g_n\,d\mu = \lim_n \int_X g_n\,d\mu = \liminf \int_X g_n\,d\mu$$

Now, note that $g_n(x) = \inf_{k\ge n} f_k(x) \le f_n(x) \le \liminf_n \int_X f_n\,d\mu$ ★

Theorem 7.3 (σ-additivity of $\int$)
Given $(X, \mathcal{M}, \mu)$ measurable space, $\Phi : X \to [0, +\infty]$. Define $\nu(E) = \int_E \Phi\,d\mu$, with $E \in \mathcal{M}$. $\nu : \mathcal{M} \to [0, +\infty]$ is a measure. Moreover, let $f : X \to [0, +\infty]$ measurable

$$\int_X f\,d\nu = \int_X f\Phi\,d\nu \qquad *$$

Proof. <u>ν is a measure</u>:
$\nu(\varnothing) = 0$, since $\mu(\varnothing) = 0$. Now, let $E = \bigcup_{n=1}^{\infty} E_k$, $\{E_k\}$ disjoint. Then

$$\nu(E) = \int_X \Phi\chi_E\,d\mu = \int_X \Phi \sum_n \chi_{E_n}\,d\mu \underset{\substack{\text{monot. conv.}\\ \text{for } \sum}}{=} \sum_n \int_X \Phi\chi_{E_n}\,d\mu = \sum_n \int_X \Phi\,d\mu$$

★

8 Lesson 06/10/2022

$f \notin R(I)$. Is it true that $\exists g \in R(I)$ s.t. $g = f$ almost everywhere (a.e.) on I? No.

For instance, consider $T_\mathcal{E}$, the generalized Cantor set $(\lambda(T_\mathcal{E}))$. Consider $\chi_\mathcal{E}$. In general, χ_A is discontinuous on δA. But $T_\mathcal{E}$ has no interior parts $\Longrightarrow T_\mathcal{E} = \delta T_\mathcal{E} \Longrightarrow \chi_{T_\mathcal{E}}$ is discontinuous on $T_\mathcal{E}$, which has positive measure $\Rightarrow$ by theorem 2, χ_{T_ε} is not $R(I)$

Clearly

$$\int_{[0,1]} \chi_{T_\mathcal{E}} d\lambda = \lambda(T_\mathcal{E})$$

so $\chi_{T_\mathcal{E}} \in \mathcal{L}^1([0,1])$. If $g = \chi_{T_\mathcal{E}}$ a.e., then g is discontinuous at almost every part of $T_\mathcal{E} \Longrightarrow g$ is discontinuous on a set of positive measure $\Longrightarrow g \notin R(I)$. So, the Lebesgue integral is a true extension of the Riemann one.

Regarding generalized integrals we have

Theorem 8.1
$-\infty \leq a < b \leq +\infty, \quad f \in R^g([a,b])$ where

$$R^g([a,b]) = \{\text{Riemann-int functions on } [a,b] \text{ in the generalized sense}\}$$

Then, f is $([a,b], \mathcal{L}([a,b]))$-measurable. Moreover

(1) $f \geq 0$ on $[a,b] \Longrightarrow f \in \mathcal{L}^1([a,b])$

(2) $|f| \in R^g([a,b]) \Longrightarrow f \in \mathcal{L}^1([a,b])$

and in both cases

$$\int_{[a,b]} f d\lambda = \int_a^b f(x)dx$$

If f is in $R^g([a,b])$, but $|f| \notin R^g([a,b])$,then the two notions of $\int$ are not really related

Ex: $f(x) = \frac{\sin x}{x}, \quad x \in [1,\infty]$

$$\int_1^\infty |f(x)|dx = +\infty \Longrightarrow f \notin \mathcal{L}^1([1,+\infty])$$

. But on the other hand

$$\int_1^\infty \frac{\sin x}{x} dx = \lim_{\omega\to\infty} \int_1^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Proposition 8.1
$(X, \mathcal{M}, \mu)$ complete measure space. Let $\{f_n\} \subseteq \mathcal{L}'(X, \mathcal{M}, \mu)$. Suppose that $\sum_{n=1}^\infty \int_X |f_n|\, d\mu < \infty$ Then the series $\sum_{n=1}^\infty f_n$ converges a.e. on X, it is in $\mathcal{L}'(X)$ and

$$\int_X \left(\sum_{n=1}^\infty f_n \right) d\mu = \sum_{n=1}^\infty \int_X f_n \, d\mu$$

Spaces of integrable functions

$(X, \mathcal{M}, \mu)$ complete measure space.

$$\mathcal{L}^1 = \left\{ f : X \to \overline{\mathbb{R}} : \text{ f is integrable} \right\}$$

$\mathcal{L}^1$ is a vector space. On $\mathcal{L}^1$ we can introduce $d : \mathcal{L}^1 \times \mathcal{L}^1 \to [0,+\infty)$ defined by

$$d_1(f,g) = \int_X |f-g|$$

It is immediate to check that $d_1(f,g) = d_1(g,f)$ (symmetry)
$d_1(f,g) \leq d_1(f,h) + d_1(h,g)$ $\forall f,g,h \in \mathcal{L}'(X)$ (triangular inequality)

However, d_1 is not a distance on $\mathcal{L}^1(X)$, since

$$d_1(f,g) = 0 \Longrightarrow f = g \quad \text{a.e on } X \qquad \text{(Pseudo-distance)}$$

To overcome this problem, we introduce an equivalent relation in $\mathcal{L}^1(X)$: we say that

$$f \;\; g \Longleftrightarrow f = g \quad \text{a.e. on } X$$

If $f \in \mathcal{L}^1(X)$, we can consider the equivalence class

$$[f] = \{g \in \mathcal{L}^1(X) : g = f \text{ a.e on } X\}$$

We define

$$L^1(X) = \frac{\mathcal{L}^1(X)}{\quad} = \{[f] : f \in \mathcal{L}'(X)\}$$

$L^1(X)$ is a vector space, and on $L^1(X)$ the function d_1 is a distance:

$$d_1([f],[g]) = 0 \iff \int_X |[f]-[g]|\, d\mu = 0 \iff [f] = [g] \text{ a.e.} \quad \iff f = g \text{ a.e.}$$

To simplify the notations, the elements of $L^1(X)$ are called functions, and one writes $f \in L^1(X)$. With this, we means that we choose a representative in $[f]$, and f denotes both the representative and the equivalence class. The representative can be arbitrarily modified on any set with 0 measure.

Another relevant space of measurable functions is the space of **essentially bounded** functions

Definition 8.1
$f : X \to \overline{\mathbb{R}}$ measurable is called essentially bounded if $\exists M > 0$ s.t.

$$\mu(\{x \in X : |f(x)| \geq M\}) = 0$$

Ex:

$$f(x) = \begin{cases} 1 & x > 0 \\ +\infty & x = 0 \\ 0 & x < 0 \end{cases}$$

For $M > 1$, $\lambda(\{x \in \mathbb{R} : |f(x)| > M\}) = \lambda(\{0\}) = 0 \Longrightarrow f$ is essentially bounded.

If f is essentially bounded, it is well defined the **essential supremum** of f.

$$\underset{X}{esssup} f := \inf\{M > 0 \text{ s.t. } f \leq M \text{ a.e. on } X\} = \inf\{M > 0 \text{ s.t. } \mu(\{f \geq M\}) = 0\}$$

It can also be defined an essential inf.

Remark 8.1
Note that, by def of inf, $\forall\, \varepsilon > 0$ we have

$$f \leq (\underset{X}{esssup} f) + \varepsilon$$

We define

$$L^\infty(X, \mathcal{M}, \mu) = \frac{\mathcal{L}^\infty(X, \mathcal{M}, \mu)}{\quad}$$

$L^\infty(X)$ is a vector space, and it is also a metric space for $d_\infty(f,g) = \underset{X}{essup}|f-g|$

Relation between different types of convergence

$\{f_n\}$ sequence of measurable functions $X \to \overline{\mathbb{R}}$

- recupera
- $f_n \to f$ pointwise
- $f_n \to f$ uniformly
- $f_n \to f$
- <u>Convergence in $L^1(X)$</u>
- <u>Convergence in measure/probability</u>

cose cose parlavo con ila

Theorem 8.2 (Egorov)
Let $\mu(X) < +\infty$, and suppose that $f_n \to f$ a.e. on X. Then, $\forall\, \varepsilon > 0, \exists X_\varepsilon \subset X$, measurable, s.t.

$$\mu(X \backslash X_\varepsilon) < \varepsilon$$

and $f_n \to f$ uniformly on X_ε

Theorem 8.3
If $\mu(X) < +\infty$ and $f_n \to f$ a.e. on $X \implies f_n \to f$ is measure on X

Proof. Let $\alpha > 0$. We want to show that $\forall\, \varepsilon > 0\ \exists \bar{n} \in \mathbb{N}$ s.t.

$$n > \bar{n} \implies \mu(\{\})$$

altre cosette ★

Remark 8.2
$\mu(X) < +\infty$ is essential

For example, in $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ consider

$$f_n(x) = \chi_{[n,n+1)}(x)$$

$f_n(x) \to 0$ for every $x \in \mathbb{R}$. However, $\lambda(\left\{|f_n| \geq \frac{1}{2}\right\}) = \lambda([n, n+1)) = 1$ not 0

9 Lesson 12/10/2022

Typewriter sequence che però aveva iniziato la lezione scorsa

Remark 9.1
$f_p \nrightarrow 0$ a.e. on $[0,1]$. But consider $\{f_{p(n,1)} : n \in \mathbb{N}\}$. This is a subsequence and, by definition $f_{p(n,1)}(x) = \chi_{n,1}(x) = \chi_{\left[0,\frac{0}{n}\right]}(x)$. For this subsequence, we have $f_{p(n,1)}(x) \to 0$ as $n \to \infty$ $\forall x \in (0,1]$, then a.e. on $[0,1]$

This is not random!

Proposition 9.1
If $\mu(X) < \infty$ and $f_n \to f$ in measure, then $\exists$ a subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ a.e. on X.

Now we analize the relation between convergence in $L^1(X)$ and the other convergences.

Theorem 9.1
$\{f_n\} \subset L^1(X), f \in L^1(X)$. If $f_n \to f$ in $L^1(X)$ then $f_n \to f$ in measure on X

Proof. By contradiction. Suppose that $f_n \nrightarrow f$ in measure on X: $\exists \bar{\alpha} > 0$ s.t.

$$\limsup_{n\to\infty} \mu(\{|f_n - f| \geq \bar{\alpha}\}) > 0$$

$\Rightarrow \exists \bar{\varepsilon}$ and a subsequence $\{f_{n_k}\}$ s.t.

$$\mu(\{|f_{n_k} - f| \geq \bar{\alpha}\}) > \bar{\varepsilon}$$

Consider then $d_1(f_{n_k}, f) = \int_X |f_{n_k} - f|\, d\mu \geq \int_{\{|f_{n_k}-f|\geq\bar{\alpha}\}} 1\, d\mu = \bar{\alpha}\mu(\{|f_{n_k} - f| \geq \bar{\alpha}\}) > \bar{\alpha}\bar{\varepsilon}$ But, by assumption, $d_1(f_n, f) \to 0$

$$\Rightarrow d_1(f_{n_k}, f) \to 0$$

contradiction. ★

Remark 9.2
the convergence in measure doesn't imply the convergence in L^1. For example, consider $f_n(x) = n\chi_{[0,\frac{1}{n}]}(x)$ $\mu(\{|f_n| \geq \alpha\}) \to 0$ for every α
On the other hand $\int_{[0,1]} n\chi_{[0,\frac{1}{n}]}\, d\lambda = \int_{[0,\frac{1}{n}]} n\, d\lambda = n\frac{1}{n} = 1$ $f_n \nrightarrow 0$ in L^1

Convergence a.e. $\nRightarrow$ convergence in L^1:
use the same example above, $f_n \to 0$ a.e. on $[0,1] \nRightarrow f_n \to 0$ in L^1
Convergence in $L^1 \nRightarrow$ convergence a.e. Consider the typewriter sequence: $d_1(f_{p(n,k)}, 0) \to 0$ when $p \to \infty$
But we don't have a.e. convergence. However, recall the dominated convergence theorem: (DOM)

$$f_n \to f \text{ a.e. } + \exists \text{ of a dom function } \Rightarrow d(f_n, f) \to 0$$

It is also possible to show a reverse DOM: if $f_n \to f$ in $L^1(X)$, then $\exists$ a subsequence $\{f_{n_k}\}$ and $w \in L^1(X)$ s.t.

(1) $f_{n_k} \to f$ a.e. on X

(2) $\|f_{n_k}\| \leq w(x)$ for a.e. x $\in X$

Derivatives of measures

$(X, \mathcal{M}, \mu)$ measure space. $\phi : X \to [0,\infty]$ measurable. We learned that $\nu : \mathcal{M} \to [0,\infty]$ by

$$\nu(E) = \int_E \phi\, d\mu$$

is a measure on $X, \mathcal{M}$.

If the equation above holds, then we say that ϕ is the Radon Nykodym derivative of ν with respect to μ and we write

$$\phi = \frac{d\nu}{d\mu}$$

Definition 9.1
μ, ν measures on $(x, \mathcal{M})$. We say that ν is absolutely continuous with respect to μ, $\nu << \mu$ if

$$\mu(E) = 0 \Rightarrow \nu(E) = 0$$

Lemma 9.1
There is a necessary condition:

$$\exists \frac{d\nu}{d\mu} \Rightarrow \nu << \mu$$

Proof. $\nu(E) = \int_E (\frac{d\nu}{d\mu})\, d\mu = 0$ if $\mu(E) = 0$ by basic properties of $\int$ ★

Theorem 9.2 (Radon Nykodim Theorem)
$(X, \mathcal{M})$ measurable space, μ, ν measures. If $\nu << \mu$ and moreover μ is σ finite, then $\phi :\to [0, \infty]$ measurable s.t. $\phi = \frac{d\nu}{d\mu}$ namely $\nu(E) = \int_E \phi\, d\mu\ \forall E \in \mathcal{M}$

Remark 9.3
if μ is not sigma finite the theorem may fail. In $([0,1], \mathcal{L}([0,1]))$ consider the counting measure $\mu = \mu_c$ and the lebesque measure $\nu = \lambda$ $\nu << \mu$ since $\mu(E) = 0 \iff E = \varnothing \Rightarrow \lambda(E) = \nu(E) = 0$
But we can check that $\nexists \phi : [0,1] \to [0, \infty]$ measurable s.t. $\lambda(E) = \int_E \phi\, d\mu_c$

Check by contradiction: assume that ϕ does exist, and take $x_0 \in [0,1]$

$$0 = \lambda(\{x_0\}) = \int_{\{x_0\}} \phi\, d\mu_c = \phi(x_0)\mu_c(\{x_0\}) = \phi(x_0)$$

$\Rightarrow \phi(x_0) = 0\ \forall x_0 \in [0,1]$. But then $1 = \lambda([0,1]) = \int_{[0,1]} 0\, d\mu_c = 0$. Contradiction Note that $\mu_c([0,1]) = \infty$ and $([0,1], \mathcal{L}([0,1]), \mu_c)$ is not σ - finite ($[0,1]$ is uncountable)

Product Measure

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. The goal is to define a measure space on $X \times Y$

Definition 9.2
we call measurable rectangle in $X \times Y$ a set of type $A \times B$ where $A \in \mathcal{M}, B \in \mathcal{N}$

$$R = \{A \times B \subset X \times Y \text{ s.t. } A \in \mathcal{M}, b \in \mathcal{N}\}$$

We define the product σ algebra $\mathcal{M} \otimes \mathcal{N}$ as $\sigma_0(R)$.
This is a σ algebra in $X \times Y$

Definition 9.3
let $E \subset X \times Y$ For $\bar{x} \in X$ and $\bar{y} \in Y$ we define

$$\begin{aligned} E_{\bar{x}} &= \{y \in Y : (\bar{x}, y) \in E\} \subseteq Y && \bar{x}\text{-section of } E \\ E_{\bar{y}} &= \{x \in X : (x, \bar{y}) \in E\} \subseteq X && \bar{y}\text{-section of } E \end{aligned}$$

Proposition 9.2
$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. $E \in \mathcal{M} \otimes \mathcal{N}$
Then $E_x \in \mathcal{M}$ and $E_y \in \mathcal{N} \Rightarrow$ we can define

$$\begin{aligned} \varphi :\ & X \to [0, \infty] & \qquad \psi :\ & Y \to [0, \infty] \\ & x \mapsto \nu(E_x) & & y \mapsto \mu(E_y) \end{aligned}$$

Theorem 9.3
If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ finite spaces, then:

(1) φ is $\mathcal{M}$ measurable and ψ is $\mathcal{N}$ meas

(2) we have that $\int_X \nu(E_x)\, d\mu = \int_Y \mu(E_y)\, d\nu$

Using the fact that μ and ν are measures, and that $\int$ of non negative function is a measure, we deduce the following

Theorem 9.4 (Iterated integrals for characteristic functions)
$\mu \otimes \nu : \mathcal{M} \otimes \mathcal{N} \to \mathbb{R}$ defined by

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x)\, d\mu = \int_Y \mu(E_y)\, d\nu$$

is a measure, the product measure.

Remark 9.4
On the complection of product measure spaces:
$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ complete measures spaces. In general it is not true that $(X \times Y, (M) \otimes)$ ...

Theorem 9.5
Let λ_n be the Lebesgue measure in $\mathbb{R}^n$. If $n = K + m$, then $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$ is the complection of $(\mathbb{R}^k \times \mathbb{R}^m, \mathcal{L}(\mathbb{R}^k) \otimes \mathcal{L}(\mathbb{R}^m), \lambda_k \otimes \lambda_m)$

10 Lesson 13/10/2022

Integration on product spaces

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. $f : X \times Y \to \overline{\mathbb{R}}$ measurable.

If $f \geq 0$, then

$$\iint_{X \times Y} f d\mu \otimes d\nu$$

Goal: obtain a formula of iterated integral like the one in Analysis 2.

$\forall \bar{x} \in X$ and $\bar{y} \in Y$

cose

Proposition 10.1
If f is measurable $\Rightarrow$ $f_{\bar{x}}$ is $(\mathcal{N}, \mathcal{B}(\mathbb{R}))$-measurable and $f_{\bar{y}}$ is $(\mathcal{M}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable. Then we can conclude $\varphi : X \to \bar{\mathbb{R}}$:

$$\varphi(x) = \int_Y f_x d\nu = \int_Y f(x, y) d\nu(y)$$

and $\psi : Y \to \overline{\mathbb{R}}$

$$\psi(y) = \int_X f_y d\mu = \int_X f(x, y) d\mu(x)$$

<u>Questions:</u> what is the solution of $\iint_{X \times Y}$ cose cose

Theorem 10.1 (Tonelli's theorem)
$(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ complete measure spaces and σ-finite. Suppose that f is $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable and that $f > 0$ a.e. on $X \times Y$. Then ψ and φ are measurable and

$$\iint_{X \times Y} f d\mu \otimes d\nu = cose$$

Equally holds also if one of the integrals is ∞.

Remark 10.1
The double integral can be reduced to single integrals, iterated. Moreover we can always change the order of the integrals For sign changing functions the situation is more involved.

Theorem 10.2 (Fubini's theorem)
$(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ complete measure spaces and σ-finite. If $f \in L^1(X \times Y)$, then ψ and φ defined above are measurable, and cose holds, and all the integrals are finite.

Question: how to check if $f \in L^1(X \times Y)$? Typically, to check cosette
If $\iiiint_{X\times Y} |f| d\mu \otimes d\nu < \infty$ then we can apply Fubini for $\iint_{X\times Y} f d\mu \otimes d\nu$

Remark 10.2
the proof of Fubini's and Tonelli's theorems is based for the iterated integrals for characteristic functions. (Note that $(\mu \otimes \nu)(E) = \int_X()$ e altre cosette)

Remark 10.3
Sometimes double integrals are very useful to compute single integrals.

Ex: $\int_{-\infty}^{+\infty} \exp -x^2 = \sqrt{\pi}$

The first fundamental theorem of calculus

Consider $f \in L^1\left([a, b]\right)$nWe can define the **integral function**

$$F(x) = \int_{[a,b]} f d\lambda = \int_a^b f(t)dt,$$

If the function cose
What happens if ?