Notes from Real and Functional Analysis

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November 23, 2022

1 Lecture 12/09/2022

Element of set theory

Let X be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \tag{\textbf{Power Set}}$$

Let $I \subseteq \mathbb{R}$ be a set of indexes. A family of sets induced by I is

$$\{E_i\}_{i\in I}, \quad E_i \subseteq X \tag{\textbf{Family/Collection}}$$

If $I = \mathbb{N}$ is called a

$$\{E_n\}_{n\in\mathbb{N}} \tag{\textbf{Sequence}}$$

Definition 1.1
$\{E_n\} \subseteq \mathcal{P}(X)$ is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1}\,\forall n \qquad (\text{resp. } E_n \supseteq E_{n+1}\,\forall n)$$

and is written as

$$\{E_n\} \nearrow \qquad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets $\{E_i\}_{i\in I} \subseteq \mathcal{P}(X)$, will be often considered

$$\bigcup_{i\in I} E_i = \{x \in X : \exists\, i \in I\; s.t.\; x \in E_i\}$$

$$\bigcap_{i\in I} E_i = \{x \in X : x \in E_i,\ \forall i \in I\}$$

$\{E_i\}$ is said to be **disjoint** if $E_i \cap E_j = \varnothing\ \forall i \neq j$.

Examples:

$$[a,b] = \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$$

$$(a,b) = \bigcup_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$$

Definition 1.2
$\{E_n\} \subseteq \mathcal{P}(X)$. We define

$$\limsup_n E_n := \bigcap_{k=1}^{\infty}(\bigcup_{n=k}^{\infty} E_n)$$

$$\liminf_n E_n := \bigcup_{k=1}^{\infty}(\bigcap_{n=k}^{\infty} E_n)$$

If these two sets are equal, then

$$\lim_n E_n = \limsup_n E_n = \liminf_n E_n$$

Proposition 1.1
Some limits are:

- $\limsup_n E_n = \{x \in X :\ x \in E_n \text{ for } \infty - \text{many indexes } n\}$
- $\liminf_n E_n = \{x \in X :\ x \in E_n \text{ for all but finitely many indexes } n\}$

- $\liminf_n E_n \subseteq \limsup_n E_n$
- $(\liminf_n E_n)^C = \limsup_n E_n^C$

Definition 1.3
We can define:

$$\begin{aligned} x \in \limsup_n E_n \quad &\Leftrightarrow \qquad x \in \bigcap_{k=1}^{\infty}\left(\bigcup_{n=k}^{\infty} E_n\right) \\ &\Leftrightarrow \qquad \forall k \in \mathbb{N} : \bigcup_{n=k}^{\infty} E_n \\ &\Leftrightarrow \quad \forall k \in \mathbb{N}\, \exists n_k \geq k\, s.t.\, x \in E_{n_k} \end{aligned}$$

So $x \in \limsup_n E_n \;\Rightarrow\;$
$$\begin{aligned} &\exists m_1 = n_1\, s.t.\, x \in E_{n_1} \\ &\exists m_2 := n_{m_1+1} \geq m_1 + 1\, s.t.\, x \in E_{n_2} \\ &\vdots \\ &\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1\, s.t.\, x \in E_{n_k} \\ &\vdots \\ &x \in E_{m_1}, \dots, E_{m_k}, \dots \end{aligned}$$

On the other hand, assume that $x \in E_n$ for ∞-many indexes. We claim that $\forall k \in \mathbb{N}\, \exists n_k \geq k\, s.t.\, x \in E_{n_k} \;\Leftrightarrow\; x \in \limsup_n E_n$. If that claim is not true, then $\exists \bar{k}\, s.t.\, x \notin E_n\, \forall n > \bar{k} \Rightarrow x$ belongs at most to $E_1, \dots, E_{\bar{k}}$, a contradiction. ★

Definition 1.4
$\{E_i\}_{i \in I}$ is a **covering** of X if

$$X \subseteq \bigcup_{i \in I} E_i$$

A subfamily of E_i that is still a covering is called a **subcovering**

Definition 1.5
Let $E \subseteq X$. The function $\chi_E : X \to \mathbb{R}$

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \setminus E \end{cases}$$

is called **characteristic function** of E

Let E_1, E_2 be sets:

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

$$\{E_n\} \subseteq \mathcal{P}(X),\ \text{disjoint}, E = \bigcup_{n=1}^{\infty} E_n \Rightarrow \mathcal{X}_{\mathcal{E}} = \sum_{n=1}^{\infty} \chi_{E_n}$$

$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \Rightarrow \chi_P = \liminf \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that $\limsup_n a_n = \lim_{k \to \infty} \sup_{n \geq k} a_n$ and $\liminf_n a_n = \lim_{k \to \infty} \inf_{n \geq k} a_n$

Let's also check that $\chi_Q = \limsup_n \chi_{E_n}$

$$\begin{aligned} x \in \limsup_n E_n \quad &\Leftrightarrow \qquad \chi_Q(x) = 1 \\ &\Leftrightarrow \quad \forall\, k \in \mathbb{N}\, \exists\, n_k \geq k\ s.t.\ x \in E_{n_k} \end{aligned}$$

If we fix k then

$$\sup_{n\geq k} \chi_{E_n}(x) = \chi_{E_{n_k}}(x) = 1$$
$$\lim_{k\to\infty} \sup_{n\geq k} \chi_{E_n}(x) = \limsup_{n} \chi_{E_n}(x) = 1$$

Let now $x \notin \limsup E_n \Leftrightarrow \chi_Q(x) = 0$. Then x belongs at most to finitely many $E_n \Rightarrow \exists\, \bar{k}\ s.t.\ x \notin E_n, \forall n \geq \bar{k}$

If $k \geq \bar{k}$, then $\sup_{n\geq k} \chi_{E_n}(x) = 0 \Rightarrow \lim_{k\to\infty} \sup_{n\geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 0$

Relations

Given X, Y sets, is called a **relation** of X and Y a subset of $X \times Y$

$$R \subseteq X + Y \quad R = \{(x, y) \,:\, x \in X, y \in Y\}$$

$$(x, y) \in R \Leftrightarrow xRy$$

$$X = \{0, 1, 2, 3\} \quad R = \{(0, 1), (1, 2), (2, 1)\} \text{ is a relation in } X$$

Definition 1.6
A **function** from X to Y is a relation R s.t. for any element x of X $\exists!$ element y of Y s.t. xRy

Definition 1.7
R on X is an **equivalence relation** if

(1) $xRx\ \forall\, x \in X$ (R is **reflexive**)

(2) $xRy \Rightarrow yRx$ (R is **symmetric**)

(3) $xRy, yRz \Rightarrow xRz$ (R is **transitive**)

If R is an equivalence relation, the set $E_X := \{y \in X \,:\, yRx\}$, $x \in X$ is called the **equivalence class** of X

Definition 1.8
$\frac{X}{R} := \{E_X \,:\, x \in X\}$ is the **quotient set**

Ex: $X = \mathbb{Z}$, let's say that nRm if $n - m$ is even. This is an equivalence relation.

$$E_n = \{\ldots, n - 4, n - 2, n, n + 2, n + 4, \ldots\}$$

in this case if n is even, $E_n = \{\text{even numbers}\}$ and if n is odd, $E_n = \{\text{odd numbers}\}$

Measure theory

Definition 1.9
A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is called a **σ-algebra** if

(1) $X \in \mathcal{M}$

(2) $E \in \mathcal{M} \Rightarrow E^C = X \setminus E \in \mathcal{M}$

(3) If $E = \bigcup_{n\in\mathbb{N}}$ and $E_n \in \mathcal{M}\ \forall\, n$, then $E \in \mathcal{M}$

If $\mathcal{M}$ is a σ-algebra, $(X, \mathcal{M})$ is called **measurable space** and the sets in $\mathcal{M}$ are called **measurable**. Ex:

- $(X, \mathcal{P}(X))$ is a measurable space

- Let X be a sets, then $\{\varnothing, X\}$ is a σ-algebra

Remark 1.1
σ is often used to denote the closure w.r.t. countably many operators. If we replace the countable unions with finite unions in the definition of σ-algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space $(X, \mathcal{M})$:

(1) $\varnothing \in \mathcal{M}$: $\varnothing = X^C$ and $X \in \mathcal{M}$

(2) $\mathcal{M}$ is an algebra, and $E_1, \ldots, E_n \in \mathcal{M}$

$$E_1 \cup \ldots \cup E_n = E_1 \cup \ldots \cup E_n \cup \underbrace{\varnothing}_{\in \mathcal{M}} \cup \varnothing \ldots \in \mathcal{M}$$

(3) $E_n \in \mathcal{M}$, $\bigcap_{n\in\mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n\in\mathbb{N}} E_n = \underbrace{\left(\bigcup_{n\in\mathbb{N}} \underbrace{E_n^C}_{\in \mathcal{M}}\right)^C}_{\in \mathcal{M}} \qquad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \Rightarrow E \setminus F \in \mathcal{M} = E \setminus F = E \cap F^C \in \mathcal{M}$
- If $\Omega \subset X$, then the **restriction** of $\mathcal{M}$ to Ω, written as

$$\mathcal{M}|_\Omega := \{F \subseteq \Omega : F = E \cap \Omega, \text{ with } E \in \mathcal{M}\}$$

  is a σ-algebra on Ω

Theorem 1.1
$\mathcal{S} \subseteq \mathcal{P}(X)$. Then it is well defined the smallest σ-algebra containing $\mathcal{S}$, the σ-algebra generated by $\mathcal{S} := \sigma_0(\mathcal{S})$:

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$ and thus is a σ-algebra
- $\forall \sigma(\mathcal{M})$ s.t. $\mathcal{M} \supseteq \mathcal{S}$, we have $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

Proof idea.

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \varnothing \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} \ : \ \mathcal{M} \in \mathcal{V}\}$, so it can be proved that this is the desired σ-algebra ★

Borel sets

Given (X, d) metric space, the σ-algebra generated by the open sets is called **Borel** σ-algebra, written as $\mathcal{B}(X)$. The sets in $\mathcal{B}(X)$ are called **Borel sets**. The following sets are Borel sets:

- open sets
- closed sets
- countable intersections of open sets: G_σ sets
- countable unions of closed sets: F_σ sets

Remark 1.2
$\mathcal{B}(\mathbb{R})$ can be equivalently defined as the σ-algebra generated by

$$\{(a,b):\ a,b\in\mathbb{R}, a<b\}$$

$$\{(-\infty,b):\ b\in\mathbb{R}\}$$
$$\{(a,+\infty):\ a\in\mathbb{R}\}$$
$$\{[a,b):\ a,b\in\mathbb{R}, a<b\}$$
$$\vdots$$

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Question: What is $\mathcal{B}(\mathbb{R})$? Is $\mathcal{B}(\mathbb{R})\neq\mathcal{P}(\mathbb{R})$? No.

Definition 2.1
$(X,\mathcal{M})$ measurable space. A function $\mu:\ \mathcal{M}\to[0,+\infty]$ is called a **positive measure** if $\mu(\varnothing)=0$ and if μ is countably additive, that is

$$\forall\,\{E_n\}\subseteq\mathcal{M}\quad\text{disjoint}$$

we have that

$$\mu\left(\bigcup_{n=1}^{\infty}\right)=\sum_{n=1}^{\infty}\mu(E_n)\qquad\sigma\text{-additivity}$$

Remark 2.1
a set A is countable if $\exists\ f$ $1-1$ s.t. $f:A\to\mathbb{N}$ Examples: $\mathbb{Z},\mathbb{Q}$ are countable, while $\mathbb{R}$ is not, also $(0,1)$ is uncountable.

We always assume that $\exists\ E\neq\varnothing, E\in\mathcal{M}$ s.t. $\mu(E)\neq\infty$.

If $(X,\mathcal{M})$ is a measurable space, and μ is a measure on it, then $(X,\mathcal{M},\mu)$ is a measure space.

Then:

(1) μ is **finitely additive**:

$$\forall\ E,F\in\mathcal{M},\ \text{with } E\cap F\neq\varnothing\Rightarrow\mu(E\cup F)=\mu(E)+\mu(F)$$

(2) the **excision property**

$$\forall\ E,f\in\mathcal{M},\ \text{with } E\subset F \text{ and } \mu(E)<+\infty\Rightarrow\mu(F\setminus E)=\mu(F)-\mu(E)$$

(3) **monotonicity**

$$\forall\ E,F\in\mathcal{M},\ \text{with } E\subset F\Rightarrow\mu(E)\leq\mu(F)$$

(4) if $\Omega\in\mathcal{M}$ then $(\Omega,\mathcal{M}|_\Omega,\mu|_{\mathcal{M}|_\Omega})$ is a measure space

Proof. (1) $E_1=E, E_2=F, E_3=\ldots=E_n=\ldots=\varnothing$ This is a disjoint sequence $\Rightarrow$ by σ-additivity.

$$\mu(E\cup F)=\mu\left(\bigcup_n E_n\right)=\sum_n\mu(E_n)=\mu(E)+\mu(F)+\underbrace{\mu(E_k)}_{=\mu(\varnothing)}$$

(2) $E \subset F$, so $F = E \cup (F \setminus E)$ and this is disjoint $\overset{(i)}{\Rightarrow} \mu(F) = \mu(E) + \mu(F \setminus E)$, and since $\mu(E) < \infty$, the property follows.

(3) $E \subset F \Rightarrow \mu(F) = \mu(E) + \underbrace{\mu(F \setminus E)}_{\geq 0} \geq \mu(E)$

(4)

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Definition 2.2
$(X, \mathcal{M}, \mu)$ measure space.

- If $\mu(X) < +\infty$, we say that μ is **finite**.
- If $\mu(X) = +\infty$, and $\exists\ \{E_n\} \subset \mathcal{M}$ s.t. $X = \bigcup_n E_n$ and each E_n has finite measure, then we say that μ is σ-finite.
- If $\mu(X) = 1$ we say that μ is a **probability measure**.

Some examples:

- Trivial Measure: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(E) = 0 \quad \forall\ E \in \mathcal{M}$
- Counting Measure: $(X, \mathcal{P}(X))$ measurable space. We define

$$\mu_C : \mathcal{P}(X) \to [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure: $(X, \mathcal{P}(X))$ measurable space, $t \in X$. We define

$$\delta_t : \mathcal{P}(X) \to [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

<u>Continuity of the measure along monotone sequences</u>
$(X, \mathcal{M}, \mu)$ measure space

(1) $\{E_i\} \subset \mathcal{M},\ E_i \subseteq E_{i+1}\ \forall i$ and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:

$$\mu(E) = \lim_i \mu(E_i)$$

(2) $\{E_i\} \subset \mathcal{M},\ E_{i+1} \subseteq E_i\ \forall i$ and let $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$.

Proof. (1) if $\exists\ i$ s.t. $\mu(E_i) = +\infty$, then is trivial. Assume then that every E_i has a finite measure, so that $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i)$ with $E_0 = \varnothing$.

So, by σ-additivity

$$\mu(E) = \mu\left(\bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i)\right) =$$

$$= \sum_{i=0}^{\infty} \mu(E_{i+1} \setminus E_i) \overset{(excision)}{=} \sum_{i=0}^{\infty} (\mu(E_{i+1} - \mu(E_i))) =$$

$$\overset{(telescopic\ series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n)$$

(2) For simplicity, suppose $\tau = 1$, and define $F_k = E_i \setminus E_k$

$$\{E_k\} \searrow \Rightarrow \{F_k\} \nearrow$$

$$\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \setminus (\bigcap_k E_k)$$

$$\mu(E_i) = \mu(\bigcup_k F_k) + \underbrace{\mu(\bigcap_k E_k)}_{\mu(E)} =$$

$$\overset{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k (\mu(E_i) - \mu(E_k)) + \mu(E)$$

Since $\mu(E_i) < \infty$ we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$

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Counterexample: given $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_C)$ measure space. Let $E_n = \{n, n+1, n+2, \ldots\}$. In this case $\mu_C(E_n) = +\infty, E_{n+1} \subseteq E_n \forall\, n$, but $\bigcap_n E_n = \varnothing \Rightarrow \mu\left(\bigcap_n E_n\right) = 0$

Theorem 2.1 (σ-subadditivity of the measure)
$(X, \mathcal{M}, \mu)$ is a measure space. $\forall \{E_n\} \subseteq \mathcal{M}$ (not necessarily disjoint): $\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n)$

Proof. $E_1, E_2 \in \mathcal{M}$ and also $E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1)$ disjoint sets.

$$\mu(E_1 \cup E_2) = \mu(\underbrace{E_2 \setminus E_1}_{\subseteq E_2}) \overset{(monotonicity)}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.

$$A = \bigcup_{n=1}^{\infty} E_n, \quad A_k = \bigcup_{n=1}^{k} E_n$$

$$\{A_k\} \nearrow,\ A_{k+1} \supseteq A_k,\ \lim_k A_k = A$$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{(continuity)}{=} \lim_k \mu(A_k) = \lim_k \mu\left(\bigcup_{n=1}^{k} E_n\right) \leq$$

$$\leq \lim_k \sum_{n=1}^{k} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

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Exercise: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, +\infty]$ s.t. μ is finitely additive, σ-subadditive and $\mu(\varnothing) = 0 \Rightarrow \mu$ is σ-additive, and hence is a measure.

Exercise: the Borel-Cantelli lemma states that, given $(X, \mathcal{M}, \mu)$ and $\{E_n\} \subseteq \mathcal{M}$. Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \Rightarrow \mu(\limsup_n E_n) = 0$$

It can be phrased as:

> If the series of the probability of the events E_n is convergent, then the probability that ∞-many events occur is 0

Proof. The thesis is:

$$\mu(\limsup_n E_n) = \mu\left(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k \geq n} E_k}_{A_n := \bigcup_{k \geq n} E_k}\right)$$

Is it true that $\{A_n\} \searrow$? Yes.

$$A_{n+1} = \bigcup_{k \geq n+1} E_k \subseteq \bigcup_{k \geq n} E_k = A_n$$

Does some A_n have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k \geq n} E_k\right) \overset{\sigma-sub.}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

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Sets of 0 measure

$(X, \mathcal{M}, \mu)$ measure space.

- $N \subseteq X$ is a set of 0 measure if $N \in \mathcal{M}$ and $\mu(N) = 0$
- $E \subseteq X$ is called **negligible set** if $\exists\ N \in \mathcal{M}$ with 0 measure s.t. $E \subseteq N$ (E does not necessarily stays in $\mathcal{M}$)

Definition 2.3
$(X, \mathcal{M}, \mu)$ measure space s.t. every negligible set is measurable (and hence of 0 measure), then $(X, \mathcal{M}, \mu)$ is said to be a **complete measure space**.

A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists\ F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \setminus F) = 0\}$$

Clearly $\mathcal{M} \subseteq \overline{\mathcal{M}}$. For $E \in \overline{\mathcal{M}}$, take F and G as above and let $\bar{\mu}(E) = \bar{\mu}(F)$ then $\bar{\mu}|_{\mathcal{M}} = \mu$, and moreover:

Theorem 2.2
$(X, \mathcal{M}, \mu)$ is a complete measure space. Let's observe that $\bar{\mu}$ is well defined: let $E \subseteq X$ and F_1, F_2, G_1, G_2 s.t. $F_i \subset E \subset G_i \quad i = 1, 2$. Then $\mu(G_i \setminus F_i) = 0$. Now we have to check that $\mu(F_1) = \mu(F_2)$.

Since

$$F_1 \setminus F_2 \subseteq E \setminus F_2 \subseteq G_2 \setminus F_2$$

and $G_2 \setminus F_2$ has 0 measure $\Rightarrow \mu(F_1 \setminus F_2) = 0$. Then $F_1 = (F_1 \setminus F_2) \cup (F_1 \cap F_2) \Rightarrow \mu(F_1) = \mu(F_1 \cap F_2)$. In the same way, $\mu(F_2) = \mu(F_1 \cap F_2)$

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The elements of $\overline{\mathcal{M}}$ are sets of the type $E \cup N$, with $E \in \mathcal{M}$ and $\bar{\mu}(N) = 0$.

Outer measure

We wish to define a measure λ "on $\mathbb{R}$" with the following properties:

(1) $\lambda((a, b)) = b - a$

(2) $\lambda(E + t)^{\dagger} = \lambda(E)$ for every measurable set $E \subset \mathbb{R}$ and $t \in \mathbb{R}$

It would be nice to define such a measure on $\mathcal{P}(\mathbb{R})$. In such case, note that $\lambda(\{x\}) = 0, \forall\, x \in \mathbb{R}$
But then

Theorem 3.1 (Ulam)
The only measure on $\mathcal{P}(\mathbb{R})$ s.t. $\lambda(\{x\}) = 0 \quad \forall\, x$ is the trivial measure. Thus, a measure satisfying the two properties of the outer measure cannot be defined on $\mathcal{P}(\mathbb{R})$

We'll learn in what follows how to create a measure space on $\mathbb{R}$, with a σ-algebra including all the Borel sets, and a measure satisfying properties of the outer measure. This is the so called **Lebesgue measure**.

Definition 3.1
Given a set X. An **outer measure** is a function $\mu^* : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ s.t.

- $\mu^*(\varnothing) = 0$
- $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$ (Monotonicity)
- $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$ (σ-subadditivity)

The common way to define an outer measure is to start with a family of elementary sets $\mathcal{E}$ on which a notion of measure is defined (e.g. intervals on $\mathbb{R}$, rectangles on $\mathbb{R}^2, \ldots$) and then to approximate arbitrary sets from outside by **countable** unions of members of $\mathcal{E}$.

Proposition 3.1
Let $\mathcal{E} \subset \mathcal{P}(\mathbb{R})$ and $\rho : \mathcal{E} \to [0, +\infty]$ be such that $\varnothing \in \mathcal{E}, X \in \mathcal{E}$ and $\rho(\varnothing) = 0$. For any $A \in \mathcal{P}(X)$, let

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) : E_n \in \mathcal{E} \text{ and } A \subset \bigcup_{n=1}^{\infty} E_n \right\}$$

Then μ^* is an outer measure, the outer measure generated by $(\mathcal{E}, \rho)$.

$^{\dagger}\{x \in \mathbb{R} : x = y + t, \text{ with } y \in E\}$

Proof. $\forall\ A \subset X\ \exists \{E_n\} \subset \mathcal{E}$ s.t. $A \subset \bigcup_n E_n$: take $E_n = X \forall\ n$ then μ^* is well defined. Obviously, $\mu^*(\varnothing) = 0$ (with $E_n = \varnothing \quad \forall\ n$), and $\mu^*(A) \leq \mu^*(B)$ for $A \subset B$ (any covering of B with elements of $\mathcal{E}$ is also a covering of A.)

We have to prove the σ-subadditivity. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ and $\varepsilon > 0$. For each $n, \exists \left\{E_{n_j}\right\}_{j\in\mathbb{N}} \in \mathcal{E}$ s.t. $A_n \subset \bigcup_{i=1}^{\infty} E_{n_j}$ and $\sum_{j=1}^{\infty} \rho(E_{n_j}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$. But then, if $A = \bigcup_{n=1}^{\infty} A_n$, we have that $A \subset \bigcup_{n,j\in\mathbb{N}^2} E_{n_j}$ and

$$\mu^*(A) \leq \sum_{n,j} \rho(E_{n_j}) \leq \sum_n \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_n \mu^*(A_n) + \varepsilon$$

Since ε is arbitrary, we are done. ★

Ex:

(1) $X \in \mathbb{R}, \mathcal{E} = \{(a,b) : a \leq b, a, b \in \mathbb{R}\}$ family of open intervals:

$$\rho((a,b)) = b - a$$

(2) $X = \mathbb{R}^n, \mathcal{E} = \{(a_1,b_1) \times \ldots \times (a_n,b_n) : a_i \leq b_i, a_i, b_i \in \mathbb{R}\}$:

$$\rho((a_1,b_1) \times \ldots \times (a_n,b_n)) = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$$

Remark 3.1
$E \in \mathcal{E} \Rightarrow \mu^*(E) = \rho(E)$.

In examples 1 and 2, we have in fact $\mu^*((a,b)) = b - a, \mu^*\left((a_1,b_1) \times \ldots \times (a_n,b_n)\right) = \prod_{i=1}^{n}(b_i - a_i)$

To pass from the outer measure to a measure there is a condition
Definition 3.2 (Caratheodory condition)
If μ^* is an outer measure on X, a set $A \subset X$ is called μ^*-**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall\ E \subset X$$

Remark 3.2
If E is a "nice" set containing A, then the above equality says that the outer measure of A, $\mu^*(E \cap A)$, is equal to $\mu^*(E) - \mu^*(E \cap A^C)$, which can be thought as an "inner measure". So basically we are saying that A is measurable if the outer and inner measure coincide. (Like the definition of Riemann integration with lower and upper sum)

Remark 3.3
μ^* is subadditive by def $\Rightarrow \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall\ E, A \subset X$. So, to prove that a set is μ^*-measurable it is enough to prove the reverse inequality, $\forall\ E \subset X$. In fact, if $\mu^*(E) = +\infty$, then $+\infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$, and hence A is μ^*-measurable iff

$$\mu^*(E) \geq \mu(E \cap A) + \mu^*(E \cap A^C) \quad \forall\ E \subset X \text{ with } \mu^*(E) < +\infty$$

Their relevance to the notion of μ^*-measurability is clarified by the following
Theorem 3.2 (Caratheodory)
If μ^* is an outer measure on X, the family

$$\mathcal{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$$

is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

Lemma 3.1
If $A \subset X$ and $\mu^*(A) = 0$, then A is μ^*-measurable.

Proof. Let $E \subset X$ with $\mu^*(E) < +\infty$. Then

$$\mu^*(E) \geq \mu^*(E) + \mu^*(A) \overset{\ddagger}{\geq} \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

This implies that A is μ^*-measurable. ★

To sum up: X set, $(\mathcal{E}, \rho)$elementary and measurable sets, so μ^* is an outer measure. Then given μ^* and the Caratheodory condition, we have $(X, \mathcal{M}, \mu)$ that is a complete measure space.

Remark 3.4
So far we did not prove that $\mathcal{E} \subseteq \mathcal{M}$. We will do it in a particular case.

Lebesgue measure

- $X = \mathbb{R}$, $\mathcal{E}$ family of open intervals, $\rho((a,b)) = b - a = \lambda((a,b))$, the complete measure space is $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ with $\mathcal{L}(\mathbb{R})$ the Lebesgue-measurable sets on $\mathbb{R}$ and λ the Lebesgue measure on $\mathbb{R}$.
- $X = \mathbb{R}^n$, $\mathcal{E} = \{\prod_{k=1}^n (a_k, b_k) : a_k \leq b_k \quad \forall\, k = 1, \dots, n\}$, $\rho\left(\prod_{k=1}^n (a_k, b_k)\right) = \prod_{k=1}^n (b_k - a_k)$ and this is a complete measure space $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$

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Lebesgue measure

$\mathcal{E}$ = family of open intervals (a,b), $a, b \in \mathbb{R}^*, a < b$. ρ = lenght l. $\rho((a,b)) = b - a$.
Notations: open interval I with lenght $l(I)$

Outer measure

$E \subset \mathbb{R}$. The outer measure of E is

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{+\infty} l(I_n) | I_n \text{ is an open interval, } E \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

Caratheodory condition (CC)

$A \subset \mathbb{R}$ is λ^*-measurable if

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^C) \qquad \forall\, E \subset \mathbb{R}$$

$$\{A \subset \mathbb{R} : A \text{ is } \lambda^*\text{-measurable}\} =: \mathcal{L}(\mathbb{R}) \qquad \text{(Lebesgue } \sigma\text{-algebra)}$$

$$\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})} \qquad \text{(Lebesgue measure on } \mathbb{R}\text{)}$$

Then, $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is a complete measure space. In particular, $\lambda^*(A) = 0 \Rightarrow A \in \mathcal{L}(\mathbb{R})$ and $\lambda(A) = 0$.

‡ $E \cap A^C \subseteq E$ and $E \cap A \subseteq A$ + monotonicity

Remark 4.1 (CC-Criterion for measurability)
To check that A is λ^*-measurable, it is sufficient to check that

$$\lambda^* \geq \lambda^*(E \cap A) + \lambda^*(E \cap A^C)$$

for every $E \subset \mathbb{R}$ rith $\lambda^*(E) < +\infty$

Proposition 4.1
Any countable set is measurable, with 0 Lebesgue measure.

Proof. Let $a \in \mathbb{R}$,

$$\{a\} \subseteq (a-\varepsilon, a+\varepsilon), \forall\, \varepsilon > 0 \overset{\text{by def.}}{\Rightarrow} \lambda^*(\{a\}) \leq 2\varepsilon \overset{\lim}{\Rightarrow} \lambda^*(\{a\}) = 0$$

$\{a\}$ is measurable with $\lambda(\{a\}) = 0, \forall\, a \in \mathbb{R}$. Now if a set A is countable, $A = \{a_n\}_{n\in\mathbb{N}} = \bigcup_n \{a_n\}$ (disjoint) $\Rightarrow \lambda(A) \underset{\sigma-add}{=} \sum_n \lambda(\{a_n\}) = 0$ ★

Remark 4.2
$\lambda(\mathbb{Q} = 0)$. $\mathbb{Q}$ is dense on $\mathbb{R}$, $\bar{\mathbb{Q}} = \mathbb{R}$. In general, measure theoretical info and topological info cannot be compared.

Proposition 4.2
$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$

Remark 4.3
So far we didn't prove the fact that open intervals are $\mathcal{L}$-measurable.

Proof. We know that $\mathcal{B}(\mathbb{R})$ is generated by $\{(a, +\infty) : a \in \mathbb{R}\}$. Then, we can directly show that $(a, +\infty) \in \mathcal{L}(\mathbb{R}) \quad \forall\, a \in \mathbb{R}$. Let $a \in \mathbb{R}$ be fixed. We use the criterion for measurability and we check that

$$\lambda^*(E) \geq \lambda^* \underbrace{(E \cap (a, +\infty)}_{=:E_1} + \lambda^* \underbrace{(E \cap (-\infty, a])}_{=:E_2} \quad \forall\, E \subset \mathbb{R},\, \lambda^* < +\infty$$

Since $\lambda^*(E) < +\infty$, $\exists$ a countable union $\bigcup_n I_n \supset E$, where I_n is an open interval $\forall\, n$ and

$$\sum_n l(I_n) \leq \lambda^*(E) + \varepsilon$$

Let $I_n^1 := I_n \cap E_1, I_n^2 := I_n \cap (-\infty, a + \frac{\varepsilon}{2^n})$. These are open intervals:

$$E_1 \subset \bigcup_n I_n^1 \qquad E_2 \subset_n I_n^2 \qquad\qquad \text{countable unions}$$

and moreover

$$l(I_n) \geq l(I_n^1) + l(I_n^2) - \frac{\varepsilon}{2^n}$$

By definition of λ^*, $\lambda^*(E_1) \leq \sum_n l(I_n^1)$ and $\lambda^*(E_2) \leq \sum_n l(I_n^2)$, therefore

$$\lambda^*(E_1) + \lambda^*(E_2) \leq \sum_n l(I_n^1) + \sum_n l(I_n^2) \leq \sum_n \left(l(I_n) + \frac{\varepsilon}{2^n}\right) = \left(\sum_n l(I_n)\right) + \varepsilon \leq \lambda^*(E) + 2\varepsilon$$

Since ε was arbitrarily chosen, we have

$$\lambda^*(E) \geq \lambda^*(E_1) + \lambda^*(E_2)$$

which is the thesis. ★

So, the Lebesgue measure measures all the open, closed G_δ, F_δ sets. Clearly

$$\lambda((a,b)) = b - a$$

One can also show that λ is invariant under translation.

Questions: $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$, is it a strict inclusion or not?

- By Ulam's theorem, if a measure is such that $\lambda(\{a\}) = 0, \forall\, a$ and all the sets in $\mathcal{P}(\mathbb{R})$ are measurable, then $\lambda \equiv 0$. This and the fact that $\lambda((a,b)) \neq 0$ simply that $\mathcal{L}(\mathbb{R}) \subsetneqq$ [‡]$\mathcal{P}(\mathbb{R})$: $\exists$ non-measurable sets called Vitali sets. Every measurable set with positive measure contains a Vitali set. (Explanation)
- $\mathcal{B}(\mathbb{R}) \subsetneqq \mathcal{L}(\mathbb{R})$. The construction of a $\mathcal{L}$-measurable se which is not a Borel set will be done during exercise classes.

The relation between $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ is clarified by

Theorem 4.1 (Regularity of λ)
The following sentences are equivalent:

(1) $E \in \mathcal{L}(\mathbb{R})$

(2) $\forall\, \varepsilon > 0 \exists\, A \supset E$, A open s.t.
$$\lambda(A \setminus E) < \varepsilon$$

(3) $\exists\, G \supset E$, G of class G_δ, s.t.
$$\lambda(G \setminus E) = 0$$

(4) $\exists\, C \subset E$, C closed, s.t.
$$\lambda(E \setminus C) = 0$$

(5) $\exists\, F \subset E$, F of class F_δ, s.t.
$$\lambda(E \setminus F) = 0$$

Consequence: $E \in \mathcal{L}(\mathbb{R}) \Rightarrow E = F \cup N$, where F is of class F_δ, and $\lambda(N) = 0$.

Partial proof. For simplicity, we will consider only sets with finite measure.

(1) $\Rightarrow$ (2) $E \in \mathcal{L}(\mathbb{R})$. By definition of λ^*, $\forall\, \varepsilon > 0 \exists \bigcup_n I_n \supset E$ s.t. each I_n is an open interval, and
$$\lambda(E) = \lambda^*(E) \geq \sum_n l(I_n) - \varepsilon$$
We define $A = \bigcup_n I_n$, which is open. Also $A \supset E$ and
$$\lambda(A) = \lambda\left(\bigcup_n I_n\right) \overset{\sigma-\text{sub.}}{\leq} \sum_n l(I_n) \leq \lambda(E) + \varepsilon$$
Then, by excision
$$\lambda(A \setminus E) = \lambda(A) - \lambda(E) \leq \varepsilon$$

(2) $\Rightarrow$ (3) Define, for every $K \in \mathbb{N}$, an open set A_k s.t. $A_k \supset E$ and $\lambda(A_k \setminus E) < \frac{1}{k}$. Let $A = \bigcap_k A_k$. This is a G_δ set, it contains E (since each A_k contains E) and
$$\lambda(A \setminus E) \underset{(A \subset A_k\ \forall\, k)}{\leq} \lambda(A_k \setminus E) < \frac{1}{k} \Rightarrow \lambda(A \setminus E) = 0 \quad \forall\, k$$

[‡]I had no choice

(3) $\Rightarrow$ (1)] If $E \subset \mathbb{R}$ and $\exists\, G \supset E$, with G of class G_δ, s.t. $\lambda(G \setminus E) = 0$, then

$$E = G \setminus (G \setminus E) \text{ is measurable}$$

since G is a Borel set and $(G \setminus E)$ has 0 measure, then both are in $\mathcal{L}$

★

Remark 4.4
Any countable set has 0 measure. he inverse is false. An example is given by the **Cantor set**.

Let $T_0 = [0,1]$. Then we define T_{n+1} stating from T_n in the following way: given T_n, finite union of closed disjoint intervals of lenght $l_n(\frac{1}{3})^n$, T_{n+1} is obtained by removing from each interval of T_n, the open central subinterval of lenght $\frac{l_n}{3}$.

The Cantor set is $T := \bigcap_{k=0}^{+\infty}$. It can be proved that T is compact, $\lambda(T) = 0$ and T is uncountable.

If, instead of removing intervals of size $\frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^k}$, we remove sets of size $\left(\frac{\varepsilon}{3}\right)^k$, with $\varepsilon \in (0,1)$, we obtain the **generalized Cantor set** (or **fat Cantor set**) T_ε. T_ε is uncountable, compact and has no interior points (it contains no intervals). However, $\lambda(T_\varepsilon) = \frac{3(1-\varepsilon)}{3-2\varepsilon} > 0$

Remark 4.5
We worked on $\mathbb{R}$, but everything can be adapted to $\mathbb{R}^n$

Measurable functions and integration

Definition 4.1
$f : X \to Y$, then it is well defined the counterimage

$$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(Y)$$

$$E \to f^{-1}(E) = \{x \in X : f(x) \in E\}$$

Definition 4.2
$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. $f : X \to Y$ is called **measurable** or $(\mathcal{M}, \mathcal{N})$-measurable if

$$f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{M}$$

so, the counterimage of measurable sets in Y is a measurable set on X.

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To check if a function is measurable or not, it is often sed the following proposition

Proposition 5.1
$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. Let $\mathcal{F} \subseteq \mathcal{P}(Y)$ be s.t. $\mathcal{N} = \sigma_0(\mathcal{F})$. Then

$$f : X \to Y \text{ is } (\mathcal{M}, \mathcal{N}) - \text{measurable} \Leftrightarrow f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{F}$$

We will mainly focus on 2 situations:

(1) $((X, \mathcal{M}))$ is a measurable space obtained by means of an outer measure.
Ex: $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$, (Y, d_y) metric space $\to (Y, \mathcal{B}(Y))$.

If $X \to Y$ is (Lebesgue) measurable $\Leftrightarrow (\mathcal{M}, \mathcal{B}(Y))$ is measurable

(2) $(X, d_X), (Y, d_Y)$ are metric spaces $\longrightarrow (X, \mathcal{B}(X)), (Y, \mathcal{B}(Y))$
$f : X \to Y$ is Borel measurable $\Leftrightarrow (\mathcal{M}(X), \mathcal{B}(Y))$-measurable.

Remark 5.1
f is Lebesgue measurable if the continuity of the Borel set is a Lebesgue-measurable set.

Proposition 5.2
There are two parts:

(1) $(X, d_X), (Y, d_Y)$ metric spaces. If $f : X \to Y$ is continuous, then is Borel measurable

(2) (Y, d_Y) metric space. If $f : \mathbb{R}^n \to Y$ is continuous, then it is a Lebesgue measure.

Proof. The proof is divided in:

(1) f is continuous $\Leftrightarrow f^{-1}(A)$ is open $\forall\ A \subset Y$ open $\Rightarrow f^{-1}(A) \in \mathcal{B}(Y)\ \forall\ A \subset Y$ open Since $\mathcal{B}(Y) = \sigma_0$ (open sets) by proposition (1) this implies that f is Borel measurable

(2) f is continuous $\overset{(1)}{\Rightarrow}$ f is Borel measurable. $f^{-1}(A) \in \mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{L}(\mathbb{R}^n) \forall\ A \in \mathcal{B}(Y)$. Namely f is Lebesgue measurable

★

Proposition 5.3
$(X, \mathcal{M})$ measurable space, $(X, d_Y), (Y, d_Y)$ metric spaces. If $f : X \to Y$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable and $g : Y \to Z$ is continuous $\Rightarrow g \circ f : x \to Z$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable

Proposition 5.4
$(X, \mathcal{M})$ measurable space , $u, v : X \to \mathbb{R}$ measurable functions. Let $\Phi : \mathbb{R}^2 \to Y$ be continuous where (Y, d_Y) is a metric space. Then $h : X \to Y$ defined by $h(x) = \Phi(u(x), v(x))$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable.

Consequence: u, v measurable $\Rightarrow u + v$ is measurable.

Proof. Define $f : X \to \mathbb{R}^2$, $f(x) = u(x), v(x)$. By definition $h = \Phi \circ f$ by proposition (3) if we show that f is $(\mathcal{M}, \mathcal{B}(\mathbb{R}^2))$-measurable, then h is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0(\{\underbrace{(a_1, b_1) \times (a_2, b_2)}_{\text{open rectangle}} : a, b \in \mathbb{R}\})$$

Thanks to proposition (1), to check that f is measurable. We can simply check that $f^{-1}(\mathbb{R} \in \mathcal{M})\quad \forall$ open rectangle in $\mathbb{R}^2$ and $R = I \times J$, with I and J open intervals:

$$\begin{gathered} F^{-1}(\mathbb{R}) = \{x \in X : (u(x), v(x)) \in \mathbb{R}\} \\ \Updownarrow \\ u(x) \in I \text{ and } v(x) \in J \\ = \{x \in X : u(x) \in I\} \cap \{x \in X : v(x) \in J\} \\ = \underbrace{u^{-1}(I)}_{\in \mathcal{M}} \cap \underbrace{v^{-1}(J)}_{\in \mathcal{M}} \in \mathcal{M} \\ \text{since both } u, v \text{ are measurable} \end{gathered}$$

This completes the proof ★

Consequences: by proposition 3 and 4, if u and v are measurable, then also $u + v$, $u \cdot v$. Other measurable functions include $u^+ = \max\{u, 0\}, u^- = -\min\{u, 0\}, |u| = u^+ + u^-, u^2, \ldots$

Recall that $u = u^+ - u^-$

Remark 5.2
u^+ is measurable since $u^+ = g \circ u$, where:

$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

Most of the times we will work with functions $f : X \to \mathbb{R}$ or $f : X \to \underbrace{\overline{\mathbb{R}}}_{\mathbb{R}\cup\{\pm\infty\}}$ $(X, \mathcal{M})$ measurable space, then such a function f is measurable iff

$$f^{-1}((a, +\infty)]^{\dagger}) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

or equivalently

$$f^{-1}([a, +\infty)]) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

Let now $\{f_n\}$ be a sequence of measurable functions from X to $\overline{\mathbb{R}}$. Then we define

$$(\inf_n f_n)(x) = \inf_n f_n(x)$$

$$(\sup_n f_n)(x) = \sup_n f_n(x)$$

$$(\liminf_n f_n)(x) = \liminf_n f_n(x)$$

$$(\limsup_n f_n)(x) = \limsup_n f_n(x)$$

$$(\lim_n f_n)(x) = \lim_n f_n(x) \quad \text{if the limit exists}$$

Proposition 5.5
$(X, \mathcal{M})$ measurable space, $f_n : X \to \overline{\mathbb{R}}$ measurable, then

$$\sup_n f_n \ \ \inf_n f_n \ \ \liminf_n f_n \ \ \limsup_n f_n$$

are measurable, in particular if $\lim_n f_n$ is well defined, then f is measurable

Proof. $(\sup f_n)^{-1}((a, \infty]) = \{x \in X : \quad \sup f_n(x) > a \quad \}$

$$\Updownarrow$$

$$\exists \text{ some indexes } n \text{ s.t. } f_n(x) > a$$

$$\bigcup_n \{x \in X : f_n(x) > a\} = \bigcup_n \underbrace{f_n^{-1}((a, +\infty])}_{\in\mathcal{M}}$$

Then $(\sup f_n)^{-1}((a, \infty])$ is measurable, since it is the countable union of measurable sets.

Now we check that the $\limsup_n f_n$ is measurable

$$\limsup_n f_n(x) = \lim_n \underbrace{(\sup_{k>n} f_k(x))}_{\text{is decreasing on } n} = \inf_n (\sup_{k\geq n} f_k(x))$$

If we write $g_n(x) = \sup_{k\geq n} f_k(x)$, then

- g_n is measurable, by what we proved previously
- $\limsup_n f_n = \inf_n g_n$ is measurable

★

†We use) if f takes values in $\mathbb{R}$ and] if f takes values in $\overline{\mathbb{R}}$

Simple functions

Definition 5.1
$(X, \mathcal{M})$ measurable space. A measurable function $s : X \to \overline{\mathbb{R}}$ is said to be simple if $s(X)$ is a finite set.

$$s(X) = \{a_1 \dots, a_n\} \text{ for some } n \in \mathbb{N}, a_i \neq a_j$$

Then

$$s(x) = \sum_{n=1} a_n \chi_{E_n}(x)$$

where E_n is a measurable set, $E_n = \{x \in X : s(X) = a_n\}$, and $E_i \cap E_j = \varnothing$ for $i \neq j$, and $\bigcup_{n=1}^{N} E_n = X$.

Particular case: if s:$\mathbb{R} \to \overline{\mathbb{R}}$, and each E_n is a finite union of intervals, then s is said to be a STEP function.

Goal: to approximate arbitrary measurable functions with simple functions.

Theorem 5.1
$(X, \mathcal{M})$ measurable space, $f : X \to [0, \infty]$ measurable. Then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \dots \leq s_n \leq \dots \leq f \quad \underset{\forall\, x \in X}{\text{(pointwise)}}$$

and $s_n(x) \to f(x) \quad \forall\, x \in X$ as $n \to \infty$.

Moreover if f is bounded then $s_n \to f$ uniformly on X as $n \to \infty$

proof - for f bounded. Fix $n \in \mathbb{N}$ and divide $[0, n)$ in $n \cdot 2^n$ intervals called $I_j = [a_j, b_j)$ with lenght $\frac{1}{2^n}$

Let $E_0 = f^{-1}([n, +\infty)), E_j = f^{-1}([a_j, b_j))$ for $j = 1, \dots, n \cdot 2^n$

We let $\begin{array}{ll} s_n(x) = a_j & \text{for } x \in E_j \\ s_n(x) = n & \text{for } x \in E_0 \end{array}$

Namely we define the simple function s_n as

$$s_n(x) = n\chi_{E_0}(X) + \sum_{j=1}^{n \cdot 2^n} a_j \chi_{E_j}(x)$$

Then $s_n \leq s_{n+1}$ by contradiction, and, since f is bounded, $E_0 = \varnothing$ for n sufficiently large ($n > \sup f$).

Then any $x \in X$ stays in $f^{-1}([a_j, b_j))$ for some j

$$\begin{aligned} &\Rightarrow \underset{\substack{\shortparallel \\ s_n(x)}}{a_j} \leq f(x) < b_j \\ &\Rightarrow 0 \leq f(x) - s_n(x) < b_j - a_j = \frac{1}{2^n} \\ &\Rightarrow \sup_{x \in X} |f(x) - s_n x| < \frac{1}{2^n} \to 0 \text{ as } n \to \infty \end{aligned}$$

Namely, $s_n \to f$ uniformly on X. ★

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Remark 6.1
On the relation between $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ $\qquad (\lambda = \text{Lebesgue measure})$

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is not complete. In fact, $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.
Note that, $\forall\, E \in \mathcal{L}(\mathbb{R}) \exists$ a $G_\delta -$ set A and an $F_\delta -$ set B s.t.

$$\begin{aligned} A \supset E &\text{ and } \lambda(A \setminus E) = 0 \\ B \subset E &\text{ and } \lambda(E \setminus B) = 0 \end{aligned}$$

$(X, \mathcal{M}, \mu)$ complete measure space.
Let $P(x)$ be a proposition depending on $x \in X$. We say that $P(x)$ is true ($\mu-$) almost everywhere if

$$\mu\left(\{x \in X : P(x) \text{ is false}\}\right) = 0$$

$P(x)$ is true $\underset{(\mu-\text{a.e.})}{\text{a.e.}}$ on X.

Ex: $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$, $f(x) = x^2$. Then $f(x) > 0$ a.e. on $\mathbb{R}$ (for a.e. x):

$$\{f(x) \leq 0\} = \{0\}\,, \text{ and } \lambda(\{0\}) = 0$$

$(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu_C)$ with μ_C counting measure. Then it is not true that $f(x) > 0$ μ_C-a.e.

$$\mu_C\left(\{0\}\right) = 1$$

It will be useful to consider sequences converging a.e.:

$$f_n \to f \qquad \text{a.e. on } X$$

if $\mu\left(\{x \in X : \lim_n f_n(x) \neq f(x), \text{ or does not exist}\}\right) = 0$

Proposition 6.1
$(X, \mathcal{M}, \mu)$ complete measure space.

(1) $f : X \to \mathbb{R}$ is measurable, and $g = f$ a.e. on X, then g is measurable

(2) $f_n \to f$ a.e. on X, $f_n : X \to \mathbb{R}$ measurable for all n, then f is measurable

Integration of non-negative functions

Notation:

$$\begin{gathered} \{x \in X : f(x) \geq 0\} = \{f \geq 0\} \\ \{x \in X : f(x) > 0\} = \{f > 0\} \\ \vdots \end{gathered}$$

$(X, \mathcal{M}, \mu)$ complete measure space. We consider measurable functions $f : X \to [0, +\infty]$

Convention: we define

$$\begin{aligned} &a + \infty = +\infty \quad \forall\, a \in \mathbb{R} \\ &a \cdot (+\infty) = \begin{cases} +\infty & \text{if } a \neq 0, a > 0 \\ 0 & \text{if } a = 0 \end{cases} \end{aligned}$$

With this convention, $+$ and $\cdot$ of measurable functions are measurable functions.

Definition 6.1
Let $s : X \to [0, +\infty]$ be a measurable simple function,

$$s(x) = \sum_{n=1}^{m} a_n \chi_{D_n}(\bar{x})$$

where $D_1, \ldots, D_m$ are measurable, disjoint, and $\bigcup_{n=1}^m D_n = X$. Let also $E \in \mathcal{M}$. Then we define

$$\int_E s \, d\mu := \sum_{n=1}^{m} a_n \mu(D_n \cap E)$$

Remark 6.2
Given a simple function s:

$$s : [a, b] \to \mathbb{R}, \lambda = \mu \Rightarrow \int_E s \, d\mu \text{ is the area under the curve}$$

Remark 6.3
There are several points:

- In the definition we have already used the convention $\mu(D_n \cap E = +\infty) \quad$ for some n
- $E \in \mathcal{M} \Rightarrow \chi_E$ is a simple function

  $$\chi_E(x) = 1 \cdot \chi_E + 0 \cdot \chi_{X \setminus E}(x)$$

  In this case

  $$\int_X \chi_E \, d\mu = 1 \cdot \mu(E) + 0 \cdot \mu(X \setminus E) = \mu(E)$$

- $s\chi_E = \sum_{n=1}^m a_n \chi_{D_n \cap E} \Rightarrow \int_E s \, d\mu = \int_X s\chi_E \, d\mu$

Definition 6.2
$f : X \to [0, +\infty]$ measurable, $E \in \mathcal{M}$. The **Lebesgue integral** of f on E, with respect to (w.r.t.) μ, is

$$\int_E f \, d\mu = \sup \left\{ \int_E d\mu | \begin{array}{c} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

(1) If f is simple, the definitions are consistent

(2) Also for f measurable: $\int_E f \, d\mu = \int_X f\chi_E \, d\mu$

(3) $(\mathbb{N}, \mathbb{N}, \mu_C)$. $f : \mathbb{N} \to \mathbb{R}$ is a sequence $\{a_n\}_{n \in \mathbb{N}}$

$$\int_{\mathbb{N}} \{a_n\} \, d\mu_C = \sum_{n=0}^{\infty} a_n$$

Basic Properties.
Let $f, g : X \to [0, \infty]$ measurable. $E, F \in \mathcal{M}$, $\alpha \geq 0$. Then:

(1) $\mu(E) = 0 \Rightarrow \int_E f \, d\mu = 0$

(2) $f \leq g$ on $E \Rightarrow \int_E f \, d\mu \leq \int_E g \, d\mu$

(3) $E \subset F \Rightarrow \int_E f \, d\mu \leq \int_F f \, d\mu$

(4) $\alpha \geq o \Rightarrow \int_E \alpha f \, d\mu = \alpha \int_E d \, d\mu$

Remark 6.4
$([0, 1], \mathcal{L}([0, 1]), \lambda)$
Consider $\chi_{\mathbb{Q}}$, it is the Dirichlet function on $[0, 1]$. This is not Riemann integrable.
However, $\int_{[0,1]} \chi_{\mathbb{Q}} \, d\lambda = \lambda (\mathbb{Q} \cap [0, 1]) = 0$

Theorem 6.1 (Chebychev's inequality)
$f : X \to [0, \infty]$ measurable, $c > 0$. Then

$$\mu (\{f \geq c\}) \leq \frac{1}{c} \int \{f \geq c\} f \, d\mu \leq \frac{1}{c} \int_X f \, d\mu$$

Proof.

$$\int_X f\,d\mu \overset{X\supset\{f\geq c\}}{\geq} \int_{\{f\geq c\}} f\,d\mu \geq \int_{\{f\geq c\}} c\,d\mu = c\int_{\{f\geq c\}} d\mu = c\mu\left(\{f\geq c\}\right)$$

★

Theorem 6.2
$s : X \to [0,\infty]$ simple. Define $\varphi : \mathcal{M} \to [0,\infty]$
$\varphi(E) = \int_E s\,d\mu$
$\Rightarrow \varphi$ is a measure.

Proof. $\mu(\varnothing) = 0 \Rightarrow \varphi(\varnothing) = 0$ by definition.
Definition 6.3 (sigma additivity)
$\{E_n \subset \mathcal{M}\}$ disjoint, and let $E = \bigcup_{n=1}^{\infty} E_n \Rightarrow s = \sum_{k=1}^{m} a_k\chi_{D_k}\ D_k \in \mathcal{M}$

Then, by definition and since μ is a measure and $E \cap D_k = \bigcup_n (E_n \cap D_k)$

$$\varphi(E) = \sum_{k=1}^{m} a_k\mu(D_k \cap E) = \sum_{k=1}^{\infty} a_k \sum_{n=1}^{\infty} \mu(E_n \cap D_k) =$$

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{m} a_k\mu(E_n \cap D_k)\right) = \sum_{n=1}^{\infty}\int_{E_n} s\,d\mu = \sum_{n=1}^{\infty}\varphi(E_n)$$

★

Theorem 6.3 (Vanishing Lemma)
$f : X \to [0,\infty]$ measurable. $E \subset X$ measurable

$$\int_E f\,d\mu = 0 \Leftrightarrow f = 0 \text{ a.e. on } E$$

Proof. $\Leftarrow$ easy
$\Rightarrow$ Consider $E \cap \{f > 0\} = \bigcup_{n=1}^{\infty} \underbrace{\left(E_n\{f \geq \frac{1}{n}\}\right)}_{=:E_n}$
Then $\{E_n\}$ is an increasing sequence. By Chebyshev

$$\mu(E_n) \leq \frac{1}{\frac{1}{n}}\int_E f\,d\mu = 0\ \forall n \Rightarrow \mu(E_n) = 0 \quad \forall\, n$$

$\mu(E) \cup \{f > 0\} \overset{\text{continuity}}{=} \lim_n \mu(E_n) = 0$, namely $f = 0$ a.e. on E ★

The $\int$ does not see sets with 0 measure.
Definition 6.4
If $f : X \to [0,\infty]$ is measurable, and $\int_X f\,d\mu < \infty$ then we say that f is integrable.

Theorem 6.4 (Monotone Convergence - Beppo Levi)
$f_n : X \to [0,\infty]$ measurable. Suppose that

- $f_n(x) \leq f_{n+1}(x)$ for a.e. $x \in X$ for every n
- $f_n \to f$ a.e. on X

Then

$$\int_X f\,d\mu = \lim_n \int_X f_n\,d\mu$$

Proof. Part 1.
Assume that the two hypotesis hold everywhere. First, if f is measurable

$$\int_X f_n\,d\mu \nearrow \quad \Rightarrow \exists\ \alpha = \lim_n \int_X f_n\,d\mu$$

Also, $f_n \leq f$ everywhere $\Rightarrow \int_X f_n\,d\mu \leq \int_X f\,d\mu \quad \forall\ n$

$$\Rightarrow \alpha \leq \int_X f\,d\mu$$

We want to show that also $\geq$ is true. Let s be a simple function s.t. $0 \leq s \leq f$ and $c \in (0,1)$
Let $E_n = \{f_n \geq cs\} \in \mathcal{M}$

- $E_n \in E_{n+1}\ \forall\ n$:
  if $x \in E_n$, then $f_n(x) \geq cs(x) \Rightarrow f_{n+1}(x) \geq cs(x)$
  $\Rightarrow f_{n+1}(x) \geq f_n(x) \geq cs(x) \Rightarrow x \in E_{n+1}$
- Moreover, $X = \bigcup_{n=1}^{\infty} E_n$. Indeed:
  - if $f(x) = 0$, then $s(x) = 0 \Rightarrow f_1(x) = 0 = cs(x),\ x \in E_1$
  - if $f(x) > 0$, then $cs(x) < f(x) = \lim_n f_n(x)$ since $s \leq f$ and $c < 1$
  $\Rightarrow cs(x) < f_n(x)$ for n sufficiently large, namely $x \in E_n$ for n sufficiently large.

Therefore,

$$\alpha \geq \int_X f_n\,d\mu \geq \int_{E_n} f_n\,d\mu \geq c\int_{E_n} s\,d\mu = c\varphi(E_n)$$

$\forall n,\ \forall 0 \leq s \leq f, \forall c \in [0,1] \quad \varphi(E_n) = \int_{E_n} s\,d\mu$. φ is a measure, and $\{E_n\} \nearrow$
Therefore, taking the lim when $n \to \infty$ by continuity

$$\alpha \geq \lim_n c\int_{E_n} s\,d\mu = c\int_X s\,d\mu \quad \forall c \in [0,1]$$

Take the limit when $c \to 1^-$: $\alpha \geq \int_X s\,d\mu \quad \forall\ 0 \leq s \leq f$
Take the sup over s: $\alpha \geq \int_X f\,d\mu$. We proved both inequalities, so the thesis holds.
Part 2.
Note that $\{x \in X :$ assumpions of the theorem do not hold$\}$ is a set of zero measure. Take F. $X = E \cup F$ since we have the assumpion on E and $\mu(F) = 0$.
Then, by the Vanishing Lemma, since $(f - f\chi_E) = 0$ a.e. and $(f_n - f_n\chi_E) = 0$ we have that

$$\int_X f\,d\mu = \int_E d\,d\mu = \lim_n \int_E f_n\,d\mu = \lim_n \int_X f_n\,d\mu$$

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7 Lecture 05/10/2022

Theorem 7.1 (Monotone Convergence or Beppo Levi's theorem)
$f_n : X \to [0,+\infty]$ measurable. Suppose that

(1) $f_n(x) \leq f_{n+1}(x)$ for a.e. $x \in X$, for every n

(2) $f_n \to f$ a.e. on X

Then

$$\int_X f\,d\mu = \lim_n \int_X f_n\,d\mu$$

Corollary 7.1 (Monotone convergence for series)
$f_n : X \to [0, +\infty]$ measurable, then

$$\int_X \left(\sum_{n=0}^{\infty} f_n \right) d\mu = \sum_{n=0}^{\infty} \int_X f_n\,d\mu$$

Theorem 7.2 (Approximation with simple functions)
Given $(X, \mathcal{M})$ measure space, $f : X \to [0, +\infty]$ measurable, then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \le s_1 \le \ldots \le s_n \le \ldots \le f \qquad \text{pointwise } \forall\, x \in X$$

and

$$s_n(x) \to f(x) \qquad \forall\, x \in X \text{as } n \to \infty$$

Moreover, if f is bounded, then $s_n \to f$ uniformly on X as $n \to \infty$.

Remark 7.1
There is also

$$\int_X f\,d\mu = \sup \left\{ \int_X s\,d\mu \;\middle|\; \begin{array}{c} s \text{ is simple} \\ 0 \le s \le f \end{array} \right\}$$

But let $\{s_n\}$ be the sequence given by the simple approximation theorem. By monotone convergence

$$\int_X f\,d\mu = \lim_n \int_X s_n\,d\mu$$

Ex: $f, g : X \to [0, +\infty]$. Then

$$\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$$

Lemma 7.1 (Fatou's Lemma)
Given $f_n \to [0, +\infty]$ measurable $\forall\, n$. Then

$$\int_X (\liminf_n f_n)\,d\mu \le \liminf_n \int_X f_n\,d\mu$$

In particular, if $f_n \to f$ a.e. on X, then

$$\int_X f\,d\mu \le \liminf_n \int_X f_n d\mu$$

Proof. Given that $(\liminf_n f_n)(x) = \lim_n (\underbrace{\inf_{k \ge n} f_k(x)}_{=g_n(x)})$. Now, for every $x \in X$, $\{g_n(x)\} \nearrow$

$$g_{n+1}(x) = \inf_{k \ge n+1} f_k(x) \ge \inf_{k \ge n} f_k(x) = g_n(x)$$

Also, $g_n \ge 0$ on X. Thus, by monotone convergence

$$\int_X \liminf_n f_n\,d\mu = \int_X \lim_n g_n\,d\mu = \lim_n \int_X g_n\,d\mu = \liminf \int_X g_n\,d\mu$$

Now, note that $g_n(x) = \inf_{k \ge n} f_k(x) \le f_n(x) \le \liminf_n \int_X f_n\,d\mu$ ★

Theorem 7.3 (σ-additivity of $\int$)
Given $(X, \mathcal{M}, \mu)$ measurable space, $\phi : X \to [0, +\infty]$. Define $\nu(E) = \int_E \phi \, d\mu$, with $E \in \mathcal{M}$. $\nu : \mathcal{M} \to [0, +\infty]$ is a measure. Moreover, let $f : X \to [0, +\infty]$ measurable

$$\int_X f \, d\nu = \int_X f\phi \, d\nu \qquad *$$

Proof. $\underline{\nu \text{ is a measure}}$:

$\nu(\varnothing) = 0$, since $\mu(\varnothing) = 0$. Now, let $E = \bigcup_{n=1}^{\infty} E_k$, $\{E_k\}$ disjoint. Then

$$\nu(E) = \int_X \phi\chi_E \, d\mu = \int_X \phi \sum_n \chi_{E_n} \, d\mu \underset{\substack{\text{monot. conv.} \\ \text{for } \sum}}{=} \sum_n \int_X \phi\chi_{E_n} \, d\mu = \sum_n \int_X \phi \, d\mu = \sum_n \nu(E_n)$$

We have proven σ additivity, so ν is a measure.

$\underline{\text{(*) holds}}$: Let $E \in \mathcal{M}$. Then

$$\int_X \chi_E \, d\nu = \int_E 1 \, d\nu = \nu(E) = \int_E = \int_E \phi \, d\mu = \int_X \phi\chi_E \, d\mu$$

This shows that $(*)$ holds for χ_E, $\forall\, E$. Then it holds for simple functions.
Let now f be any measurable function, positive. Then we can take $\{s_n\}$ given by the simple approximation theorem

$$\int_X f \, d\nu \overset{\text{monot}}{=} \lim_n \int_X s_n \, d\nu = \lim_n \int_X s_n\phi \, d\mu \overset{\text{monot}}{=} \int_X f\phi \, d\mu$$

which is $(*)$ ★

Remark 7.2
$X, \mathcal{M}, \mu$ complete measure space. Then, by the vanishing lemma, it is not difficult to deduce that

$$f = g \text{ a.e. on } X \Leftrightarrow \int_E d \, d\mu = \int_E g \, d\mu \qquad \forall\, E \in \mathcal{M}$$

The $\int$ does not see differences of sets with 0 measure. As a consequence, in all the theorems, it is sufficient to assume that the assumptions are satisfid a.e.

Integrals for real valued functions

$X, \mathcal{M}, \mu$ complete measure space.
$f : X \to \overline{\mathbb{R}} = [-\infty, \infty]$ measurable. Recall $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$ and $|f| = f^+ + f^-$. Note that both are positive and measurable.
Definition 7.1
we say that $f : X \to \overline{\mathbb{R}}$ measurable is integrable on X if

$$\int_X |f| \, d\mu < \infty$$

If f is integrable, we define $\int_X f \, d\mu = \int_X f^+ \, d\mu + \int_X f^- \, d\mu$
The set of integrable functions is denoted by

$$\mathcal{L}^1(X, \mathcal{M}, \mu) := \{f : X \to \overline{\mathbb{R}} \text{ integrable}\}$$

$$\mathcal{L}^1(X, \mathcal{M}, \mu) = \mathcal{L}^1(X) = \mathcal{L}^1$$

If $E \in \mathcal{M}$, we define

$$\int_E f \, d\mu = \int_X f\chi_E \, d\mu$$

Remark 7.3
$f \in \mathcal{L}^1(X) \Rightarrow \int_X f\,d\mu \in \mathbb{R}$. Indeed $0 \le f^{\pm} \le |f|$

$$\Rightarrow 0 \le \int_X f^+\,d\mu,\ \int_X f^-\,d\mu \le \int_X |f|\,d\mu < \infty$$

$$\Rightarrow \int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu \in \mathbb{R}$$

Proposition 7.1
$f : X \to \overline{\mathbb{R}}$ measurable. Then

(1) $f \in \mathcal{L}^1 \Leftrightarrow |f| \in \mathcal{L}^1 \Leftrightarrow$ both $f^+,\ f^- \in \mathcal{L}^1$

(2) $f \in \mathcal{L}^1$, then
$$\left|\int_X f\,d\mu\right| \le \int_X |f|\,d\mu \qquad \text{(triangle inequality)}$$

Proof. Of the second part.

$$\left|\int_X f\,d\mu\right| = \left|\int_X f^+\,d\mu + \int_X f^-\,d\mu\right| \le \int_X f^+\,d\mu + \int_X f^-\,d\mu = \int_X |f|\,d\mu$$

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Proposition 7.2
$\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a vector space, and $f,\ g \in \mathcal{L}^1$, $\alpha \in \mathbb{R}$

$$\Rightarrow \int_X (\alpha f + g)\ d\mu = \alpha \int_X f\,d\mu + \int_X g\,d\mu$$

by linearity of the integrals.

Many results can be extended from non negative functions to general functions.

Theorem 7.4
$(X, \mathcal{M}, \mu)$ complete measure space. $f,\ g \in \mathcal{L}^1$. Then

$$f = g \text{ a.e. on } X \Leftrightarrow \int_X |f - g|\,d\mu = 0 \Leftrightarrow \int_E f\,d\mu = \int_E g\,d\mu \qquad \forall\, E \in \mathcal{M}$$

The most relevant theorem for convergence is the following

Theorem 7.5 (Dominated convergence theorem)
$\{f_n\}$ sequence of measurable functions $X \to \overline{\mathbb{R}}$. Suppose that

(1) $f_n \to f$ a.e. on X

(2) $\exists g : X \to \overline{\mathbb{R}}$, $g \in \mathcal{L}^1(X)$, such that $|f_n(x) \le g(x)|$ a.e. on X $\forall\, n \in \mathbb{N}$

Then $f \in \mathcal{L}^1$ and

$$\lim_n \int_X |f_n - f|\,d\mu = 0 \qquad \left(\Rightarrow \int_X f\,d\mu = \lim_n \int_X f_n\,d\mu\right)$$

Proof. Note that $f_n \in \mathcal{L}^1\ \forall\ n$, since $|f_n| \le g$ and we have the monotonicity of $\int$ for non negative functions

$$|f_n(x)| \le g(x) \quad n \to \infty \qquad |f(x)| \le g(x) \text{ a.e. on } X$$

$$\Rightarrow f \in \mathcal{L}^1(X)$$

Now, consider $\phi_n = 2g - |f_n - f|$. We have

$$|f_n - f| \le |f_n| + |f| \le 2g \quad \text{ a.e. on } X \quad \phi_n \ge 0 \quad \text{ a.e. on } X$$

We can use Fatou's lemma:

$$\int_X \underbrace{(\liminf_n \phi_n)}_{= 2\text{g a.e.}} d\mu \le \liminf_n \int_X \phi_n \, d\mu = \liminf_n \int_X (2g - |f_n - f|)\, d\mu =$$

$$\int_X 2g\, d\mu$$

$$= \int_X 2g\, d\mu + \liminf_n (- \int_X |f_n - f|\, d\mu) = \int_X 2g\, d\mu - \limsup_n \int_X |f_n - f|\, d\mu$$

Subtracting $\int_X 2g\, d\mu$ from both sides

$$0 \le - \limsup_n \int_X |f_n - f|\, d\mu \Rightarrow 0 \le \liminf_n \int_X |f_n - f|\, d\mu \le \limsup_n \int_X |f_n - f|\, d\mu \le 0$$

★

Remark 7.4
If $\mu(X) < +\infty$, and $\exists M > 0$ s.t. $|f_n| \le M$ a.e. on $X, \forall\ n$, then we can take $g = M$ as dominating function.

Comments on the relation between Riemann and Lebesgue integrals
Let $f : I \subset \mathbb{R} \to \mathbb{R}$, I interval, be bounded. Assume also that I is closed and bounded.

Theorem 7.6
Let f be Riemann-integrable on I ($f \in R(I)$). Then

$$f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$$

and

$$\int_I f\, d\lambda = \int_I f(x)\, dx$$

Theorem 7.7
$f \in R(I) \Leftrightarrow f$ is continuous on x, for a.e. $x \in I$.

Ex: $\chi_{\mathbb{Q}}$ on $[0, 1]$ is not Riemann integrable, because it is discontinuous at any point. Note that, instead, $\chi_{\mathbb{Q}} = 0$ a.e. on $[0, 1] \Rightarrow \int_{[0,1]} \chi_{\mathbb{Q}}\, d\lambda = 0$

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Let $f \notin R(I)$. Is it true that $\exists\ g \in R(I)$ s.t. $g = f$ a.e. on I? No.

For instance, consider $T_{\mathcal{E}}$, the generalized Cantor set ($\lambda(T_{\mathcal{E}}) = 0$) and then consider $\chi_{T_{\mathcal{E}}}$. In general, χ_A is discontinuous on δA. But $T_{\mathcal{E}}$ has no interior parts $\Rightarrow T_{\mathcal{E}} = \delta T_{\mathcal{E}} \Rightarrow \chi_{T_{\mathcal{E}}}$ is discontinuous on $T_{\mathcal{E}}$, which has positive measure $\Rightarrow$ by the last theorem, χ_{T_ε} is not $R(I)$

Clearly

$$\int_{[0,1]} \chi_{T_\mathcal{E}} d\lambda = \lambda(T_\mathcal{E})$$

so $\chi_{T_\mathcal{E}} \in \mathcal{L}^1([0,1])$.

If $g = \chi_{T_\mathcal{E}}$ a.e., then g is discontinuous at almost every part of $T_\mathcal{E} \Rightarrow g$ is discontinuous on a set of positive measure $\Rightarrow g \notin R(I)$. So, the Lebesgue integral is a true extension of the Riemann one.

Regarding generalized integrals we have

Theorem 8.1
$-\infty \leq a < b \leq +\infty, \quad f \in R^g([a,b])$ where

$$R^g([a,b]) = \{\text{Riemann-int functions on } [a,b] \text{ in the generalized sense}\}$$

Then, f is $([a,b], \mathcal{L}([a,b]))$-measurable. Moreover

(1) $f \geq 0$ on $[a,b] \Rightarrow f \in \mathcal{L}^1([a,b])$

(2) $|f| \in R^g([a,b]) \Rightarrow f \in \mathcal{L}^1([a,b])$

and in both cases

$$\int_{[a,b]} f d\lambda = \int_a^b f(x)dx$$

If f is in $R^g([a,b])$, but $|f| \notin R^g([a,b])$, then the two notions of $\int$ are not really related

Ex: $f(x) = \frac{\sin x}{x}, x \in [1,\infty]$

$$\int_1^\infty |f(x)|dx = +\infty \Rightarrow f \notin \mathcal{L}^1([1,+\infty])$$

. But on the other hand

$$\int_1^\infty \frac{\sin x}{x}dx = \lim_{\omega\to\infty}\int_1^\omega \frac{\sin x}{x}dx = \frac{\pi}{2}$$

Proposition 8.1
$(X, \mathcal{M}, \mu)$ complete measure space. Let $\{f_n\} \subseteq \mathcal{L}^1(X, \mathcal{M}, \mu)$.

Suppose that $\sum_{n=1}^\infty \int_X |f_n|\, d\mu < \infty$ Then the series $\sum_{n=1}^\infty f_n$ converges a.e. on X, it is in $\mathcal{L}^1(X)$ and

$$\int_X \left(\sum_{n=1}^\infty f_n\right) d\mu = \sum_{n=1}^\infty \int_X f_n\, d\mu$$

Spaces of integrable functions

$(X, \mathcal{M}, \mu)$ complete measure space.

$$\mathcal{L}^1 = \left\{f : X \to \overline{\mathbb{R}} : \text{ f is integrable}\right\}$$

$\mathcal{L}^1$ is a vector space. On $\mathcal{L}^1$ we can introduce $d : \mathcal{L}^1 \times \mathcal{L}^1 \to [0,+\infty)$ defined by

$$d_1(f,g) = \int_X |f-g|$$

It is immediate to check that

$$d_1(f,g) = d_1(g,f) \qquad \text{(symmetry)}$$

$$d_1(f,g) \leq d_1(f,h) + d_1(h,g) \;\; \forall f,g,h \in \mathcal{L}^1(X) \qquad \text{(triangular inequality)}$$

However, d_1 is not a distance on $\mathcal{L}^1(X)$, since

$$d_1(f,g) = 0 \Rightarrow f = g \quad \text{a.e on } X \qquad \text{(pseudo-distance)}$$

To overcome this problem, we introduce an equivalent relation in $\mathcal{L}^1(X)$: we say that

$$f \sim g \Leftrightarrow f = g \quad \text{a.e. on } X$$

If $f \in \mathcal{L}^1(X)$, we can consider the equivalence class

$$[f] = \big\{ g \in \mathcal{L}^1(X) : g = f \text{ a.e on } X \big\}$$

We define

$$L^1(X) = \frac{\mathcal{L}^1(X)}{\sim} = \{[f] : f \in \mathcal{L}^1(X)\}$$

$L^1(X)$ is a vector space, and on $L^1(X)$ the function d_1 is a distance:

$$d_1([f],[g]) = 0 \Leftrightarrow \int_X |[f]-[g]|\, d\mu = 0 \Leftrightarrow [f] = [g] \text{ a.e.} \;\; \Leftrightarrow f = g \text{ a.e.}$$

To simplify the notations, the elements of $L^1(X)$ are called functions, and one writes $f \in L^1(X)$. With this, we means that we choose a representative in $[f]$, and f denotes both the representative and the equivalence class. The representative can be arbitrarily modified on any set with 0 measure.

Another relevant space of measurable functions is the space of **essentially bounded** functions.

Definition 8.1
$f : X \to \overline{\mathbb{R}}$ measurable is called essentially bounded if $\exists\, M > 0$ s.t.

$$\mu(\{x \in X : |f(x)| \geq M\}) = 0$$

Ex:

$$f(x) = \begin{cases} 1 & x > 0 \\ +\infty & x = 0 \\ 0 & x < 0 \end{cases}$$

For $M > 1$, $\lambda(\{x \in \mathbb{R} : |f(x)| > M\}) = \lambda(\{0\}) = 0 \Rightarrow f$ is essentially bounded.

If f is essentially bounded, it is well defined the **essential supremum** of f.

$$\operatorname*{ess\,sup}_X f := \inf\{M > 0 \text{ s.t. } f \leq M \text{ a.e. on } X\} = \inf\{M > 0 \text{ s.t. } \mu(\{f \geq M\}) = 0\}$$

It can also be defined an essential inf.

Remark 8.1
Note that, by def of inf, $\forall\, \varepsilon > 0$ we have

$$f \leq (\operatorname*{ess\,sup}_X f) + \varepsilon \qquad \text{a.e. on } X$$

We define

$$L^\infty(X, \mathcal{M}, \mu) = \frac{\mathcal{L}^\infty(X, \mathcal{M}, \mu)}{\sim}$$

$L^\infty(X)$ is a vector space, and it is also a metric space for $d_\infty(f,g) = \operatorname*{ess\,sup}_X |f-g|$

Relation between different types of convergence

$\{f_n\}$ sequence of measurable functions $X \to \overline{\mathbb{R}}$

- $f_n \to f$ pointwise (everywhere) on X if $f_n(x) \overset{n\to\infty}{\to} f(x)\ \forall\, x \in X$
- $f_n \to f$ uniformly on X if $\sup_{x\in X} |f_n(x) - f(x)| \overset{n\to\infty}{\to} 0$
- $f_n \to f$ a.e. on X if

$$\mu\left(\left\{x \in X : \lim_n f_n(x) \neq f(x) \text{ or does not exist}\right\}\right) = 0$$
$$\Updownarrow$$
$$f_n(x) \to f(x) \text{ for a.e } x \in X$$

- Convergence in $L^1(X)$: $f_n \to f$ in $L^1(X)$ if

$$\underbrace{\int_X |f_n - f|\, d\mu}_{= d_1(f_n, f)} \overset{n\to\infty}{\to} 0$$

- Convergence in measure/probability: $f_n \to f$ in measure if $\forall\, \alpha 0$

$$\lim_{n\to\infty} \mu\left(\{|f_n - f| \geq \alpha\}\right) = 0$$

Basic facts: uniformly convergence $\underset{\not\Leftarrow}{\Rightarrow}$ pointwise $\underset{\not\Leftarrow}{\Rightarrow}$ a.e. convergence.

Ex: $f_n(x) = \exp\{-nx\}, x \in [0,1]$

$$f(x) = 0, \quad g(x) = \begin{cases} 0 & x \in (0,1] \\ 1 & x = 0 \end{cases}$$

Then $f_n \to g$ pointwise, $g = f$ a.e. $\Rightarrow f_n \to f$ a.e. on $[0,1]$. But $f(0) \neq g(0) \Rightarrow f_n \to f$ pointwise.

$$f_n \nrightarrow g \text{ uniformly on } [0,1] \left|\ \begin{array}{l} f_n \in \mathcal{C}([0,1]) \\ f_n \to g \Rightarrow g \in \mathcal{C}([0,1]) \end{array}\right.$$

a.e. $\not\Rightarrow$ uniform, but not all is lost...

Theorem 8.2 (Egorov)
Let $\mu(X) < +\infty$, and suppose that $f_n \to f$ a.e. on X. Then, $\forall\, \varepsilon > 0, \exists X_\varepsilon \subset X$, measurable, s.t.

$$\mu(X \setminus X_\varepsilon) < \varepsilon$$

and $f_n \to f$ uniformly on X_ε

Ex: in an example $f_n \to 0$ a.e., $f_n \to 0$ uniformly on $[0,1]$, but $f_n \to 0$ uniformly on $[\varepsilon, 1]$.
Regarding a.e. convergence and in measure convergence there is the following theorem

Theorem 8.3
If $\mu(X) < +\infty$ and $f_n \to f$ a.e. on $X \Rightarrow f_n \to f$ is measure on X

Proof. Let $\alpha > 0$. We want to show that $\forall\, \varepsilon > 0\ \exists \bar{n} \in \mathbb{N}$ s.t.

$$n > \bar{n} \Rightarrow \mu(\{|f_n - f| \geq \alpha\}) < \varepsilon$$

$f_n \to f$ a.e. on X, $\mu(X) < +\infty \overset{\text{Egorov}}{\Rightarrow} \exists\, X_\varepsilon \subseteq X$ s.t. $\mu(X \setminus X_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on $X_\varepsilon \Leftrightarrow \sup_{X_\varepsilon} |f_n - f| \overset{n\to\infty}{\to} 0$.

In particular, this means that $\exists\, \bar{n} \in \mathbb{N}$ s.t. $n > \bar{n} \Rightarrow |f_n - f| < \alpha$ on X_ε.
Therefore

$$\{|f_n - f| \geq \alpha\} \cap X_\varepsilon = \varnothing \Rightarrow \{|f_n - f| \geq \alpha\} \subseteq X \setminus X_\varepsilon \qquad \text{for } n > \bar{n}$$

By monotonicity of μ, we deduce that

$$\mu\left(\{|f_n - f| \geq \alpha\}\right) \leq \mu(X \setminus X_\varepsilon) < \varepsilon \quad \text{for } n > \bar{n}$$

Namely, $f_n \to f$ in measure. ★

Remark 8.2
$\mu(X) < +\infty$ is essential

For example, in $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ consider

$$f_n(x) = \chi_{[n,n+1)}(x)$$

$f_n(x) \to 0$ for every $x \in \mathbb{R}$. However, $\lambda(\left\{|f_n| \geq \frac{1}{2}\right\}) = \lambda([n, n+1)) = 1$ not 0

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Remark 9.1
Convergence in measure $\Rightarrow$ a.e convergence?
No, not even if $\mu(X) < +\infty$.
Consider $\chi_{n,k} = \chi_{[\frac{k-1}{n}, \frac{k}{n}]}$ with $n \in \mathbb{N}, k = 1, \ldots, n$

$$\begin{aligned}
&\chi_{1,1}(x) = \chi_{[0,1]}(x) \\
&\chi_{2,1}(x) = \chi_{[0,\frac{1}{2}]}(x) \quad \chi_{2,2}(x) = \chi_{[\frac{1}{2},1]}(x) \\
&\chi_{3,1}(x) = \chi_{[0,\frac{1}{3}]}(x) \quad \chi_{3,2}(x) = \chi_{[\frac{1}{3},\frac{2}{3}]}(x) \quad \chi_{3,3}(x) = \chi_{[\frac{2}{3},1]}(x)
\end{aligned}$$

For n fixed and k variable, we move the χ from the left to right. When the χ reaches 1, we switch n, and χ reappear from the left, being thinner.

$$\int_{[0,1]} \chi_{n,k}\, d\lambda = \frac{1}{n} \quad \int_{[0,1]} \chi_{n+1,k}\, d\lambda = \frac{1}{n+1}$$

We can order the elements of $\chi_{n,k}$ in a sequence, letting $f_p = \chi_{n,k}$ for $p = 1+2+\ldots+(n-1)+k$. We will prove that $\{f_p\}$ converges in measure, but not a.e.

This is the **typewriter sequence** $(p(n,k))$. For every $x \in [0,1]$ there are ∞ many indexes s.t. $f_p(x) = 1$ and ∞ many indexes s.t. $f_p(x) = 0$, meaning that $\nexists \lim_{p\to\infty} f_p(x)$ $f_p \nrightarrow 0$ a.e. on $[0,1]$.

But we do have convergence in measure to 0: $\alpha \in (0,1)$

$$\lambda\left(\left\{\left|f_{p(n,k)}\right| \geq \alpha\right\}\right) = \lambda\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right) = \frac{1}{n} \to 0 \text{ as } \begin{array}{c} n \to \infty \\ \Updownarrow \\ p \to \infty \end{array}$$

Remark 9.2
So, $f_p \nrightarrow 0$ a.e. on $[0,1]$. But consider $\{f_{p(n,1)} : n \in \mathbb{N}\}$. This is a subsequence and, by definition

$$f_{p(n,1)}(x) = \chi_{n,1}(x) = \chi_{\left[0,\frac{0}{n}\right]}(x)$$

For this subsequence, we have $f_{p(n,1)}(x) \to 0$ as $n \to \infty$ $\forall x \in (0,1]$, then a.e. on $[0,1]$
This is not random!

Proposition 9.1
If $\mu(X) < \infty$ and $f_n \to f$ in measure, then $\exists$ a subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ a.e. on X.

Now we analize the relation between convergence in $L^1(X)$ and the other convergences.

Theorem 9.1
$\{f_n\} \subset L^1(X), f \in L^1(X)$. If $f_n \to f$ in $L^1(X)$ then $f_n \to f$ in measure on X

Proof. By contradiction. Suppose that $f_n \nrightarrow f$ in measure on X: $\exists\, \bar{\alpha} > 0$ s.t.

$$\limsup_{n\to\infty} \mu(\{|f_n - f| \geq \bar{\alpha}\}) > 0$$

$\Rightarrow \exists\, \bar{\varepsilon}$ and a subsequence $\{f_{n_k}\}$ s.t.

$$\mu(\{|f_{n_k} - f| \geq \bar{\alpha}\}) > \bar{\varepsilon}$$

Consider then

$$d_1(f_{n_k}, f) = \int_X |f_{n_k} - f|\, d\mu \overset{\text{monot. } \int}{\geq} \int_{\{|f_{n_k} - f| \geq \bar{\alpha}\}} |f_{n_k} - f|\, d\mu \geq$$
$$\geq \int_{\{|f_{n_k} - f| \geq \bar{\alpha}\}} 1\, d\mu = \bar{\alpha}\; \mu(\{|f_{n_k} - f| \geq \bar{\alpha}\}) > \bar{\alpha}\, \bar{\varepsilon}$$

But, by assumption, $d_1(f_n, f) \to 0$

$$\Rightarrow d_1(f_{n_k}, f) \to 0$$

Contradiction. ★

Remark 9.3
The convergence in measure doesn't imply the convergence in L^1.
For example, consider

$$f_n(x) = n\chi_{[0,\frac{1}{n}]}(x)$$

$\underbrace{\mu\left(\{|f_n| \geq \alpha\}\right)}_{=\frac{1}{n}} \to 0$ for every α
On the other hand

$$\int_{[0,1]} n\chi_{[0,\frac{1}{n}]}\, d\lambda = \int_{[0,\frac{1}{n}]} n\, d\lambda = n\frac{1}{n} = 1$$

$f_n \nrightarrow 0$ in L^1

Convergence a.e. $\nRightarrow$ convergence in L^1:
Use the same example above, $f_n \to 0$ a.e. on $[0,1] \nRightarrow f_n \to 0$ in L^1

Convergence in $L^1 \nRightarrow$ convergence a.e.:
Consider the typewriter sequence: $d_1(f_{p(n,k)}, 0) \to 0$ when $p \to \infty$
But we don't have a.e. convergence.
However, recall the dominated convergence theorem: (DOM)

$$f_n \to f \text{ a.e. } + \exists \text{ of a dominating function } \Rightarrow d(f_n, f) \to 0$$

It is also possible to show a reverse DOM:
If $f_n \to f$ in $L^1(X)$, then $\exists$ a subsequence $\{f_{n_k}\}$ and $w \in L^1(X)$ s.t.

(1) $f_{n_k} \to f$ a.e. on X

(2) $|f_{n_k}(x)| \leq w(x)$ for a.e. $x \in X$

Derivatives of measures

$(X, \mathcal{M}, \mu)$ measure space, $\phi : X \to [0, \infty]$ measurable.

We learned that $\nu : \mathcal{M} \to [0, \infty]$ by

$$\nu(E) = \int_E \phi \, d\mu \text{ is a measure on } (X, \mathcal{M})$$

If the equation above holds, then we say that ϕ is the Radon Nykodym derivative of ν with respect to μ and we write

$$\phi = \frac{d\nu}{d\mu}$$

Definition 9.1
μ, ν measures on $(X, \mathcal{M})$. We say that ν is absolutely continuous with respect to μ, $\nu << \mu$ if

$$\mu(E) = 0 \Rightarrow \nu(E) = 0$$

Lemma 9.1
There is a necessary condition:

$$\exists \, \frac{d\nu}{d\mu} \Rightarrow \nu << \mu$$

Proof.

$$\nu(E) = \int_E \left(\frac{d\nu}{d\mu} \right) \, d\mu = 0$$

if $\mu(E) = 0$ by basic properties of $\int$ ★

Theorem 9.2 (Radon Nykodim Theorem)
$(X, \mathcal{M})$ measurable space, μ, ν measures.
If $\nu << \mu$ and moreover μ is σ-finite, then $\phi : X \to [0, \infty]$ measurable s.t.

$$\phi = \frac{d\nu}{d\mu} \quad \text{namely } \nu(E) = \int_E \phi \, d\mu \quad \forall \, E \in \mathcal{M}$$

Remark 9.4
If μ is not sigma finite the theorem may fail.
In $([0, 1], \mathcal{L}([0, 1]))$ consider the counting measure $\mu = \mu_C$ and the Lebesgue measure $\nu = \lambda$
$\nu << \mu$ since $\mu(E) = 0 \Leftrightarrow E = \varnothing \Rightarrow \lambda(E) = \nu(E) = 0$
But we can check that $\nexists \, \phi : [0, 1] \to [0, \infty]$ measurable s.t. $\lambda(E) = \int_E \phi \, d\mu_C$

Check by contradiction: assume that ϕ does exist, and take $x_0 \in [0, 1]$

$$0 = \lambda(\{x_0\}) = \int_{\{x_0\}} \phi \, d\mu_C = \phi(x_0) \, \overbrace{\mu_C(\{x_0\})}^{=1} = \phi(x_0)$$

$\Rightarrow \phi(x_0) = 0 \; \forall \, x_0 \in [0, 1]$.

But then $1 = \lambda([0, 1]) = \int_{[0,1]} 0 \, d\mu_C = 0$. Contradiction

Note that $\mu_C([0, 1]) = \infty$ and $([0, 1], \mathcal{L}([0, 1]), \mu_C)$ is not σ-finite ($[0, 1]$ is uncountable)

Product Measure

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. The goal is to define a measure space on $X \times Y$

Definition 9.2

We call **measurable rectangle** in $X \times Y$ a set of type $A \times B$ where $A \in \mathcal{M}, B \in \mathcal{N}$

$$R = \{A \times B \subset X \times Y \text{ s.t. } A \in \mathcal{M}, B \in \mathcal{N}\}$$

We define the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$ as $\sigma_0(R)$.
This is a σ-algebra in $X \times Y$

Definition 9.3

Let $E \subset X \times Y$. For $\bar{x} \in X$ and $\bar{y} \in Y$ we define

$$\begin{aligned} E_{\bar{x}} &= \{y \in Y : (\bar{x}, y) \in E\} \subseteq Y \qquad && \bar{x}\text{-section of } E \\ E_{\bar{y}} &= \{x \in X : (x, \bar{y}) \in E\} \subseteq X \qquad && \bar{y}\text{-section of } E \end{aligned}$$

Proposition 9.2

$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. $E \in \mathcal{M} \otimes \mathcal{N}$
Then $E_x \in \mathcal{M}$ and $E_y \in \mathcal{N} \Rightarrow$ we can define

$$\begin{aligned} \varphi : \ & X \to [0, \infty] \qquad & \psi : \ & Y \to [0, \infty] \\ & x \mapsto \nu(E_x) & & y \mapsto \mu(E_y) \end{aligned}$$

Theorem 9.3

If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ finite spaces, then:

(1) φ is $\mathcal{M}$-measurable and ψ is $\mathcal{N}$-measurable

(2) we have that $\int_X \nu(E_x)\, d\mu = \int_Y \mu(E_y)\, d\nu$

Using the fact that μ and ν are measures, and that $\int$ of non negative function is a measure, we deduce the following

Theorem 9.4 (Iterated integrals for characteristic functions)

$\mu \otimes \nu : \mathcal{M} \otimes \mathcal{N} \to \mathbb{R}$ defined by

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x)\, d\mu = \int_Y \mu(E_y)\, d\nu$$

is a measure, the product measure.

Remark 9.5 (On the complection of product measure spaces)

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ complete measures spaces.

In general it is not true that $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \otimes \nu)$ is complete.

Example: $X = Y = \mathbb{R}$, $\mathcal{M} = \mathcal{N} = \mathcal{L}(\mathbb{R}), \mu = \nu = \lambda$.

Given A non meas. set , $A \subseteq [0, 1], B = \{y_0\}, E = A \times B$. If E were measurable, then its sections must be measurable. But $E_{y_0} = A$ which is not measurable.

However, E is negligible:

$$E \subseteq [0, 1] \times \{y_0\}, \text{ and } (\lambda \otimes \lambda)([0, 1] \times \{y_0\}) = 0$$

Then $(\lambda \otimes \lambda)$ is not complete

$$\Rightarrow (\mathbb{R}^2, \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}), \lambda \otimes \lambda) \neq (\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2), \lambda_2)$$

Theorem 9.5

Let λ_n be the Lebesgue measure in $\mathbb{R}^n$. If $n = K + m$, then $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$ is the complection of $(\mathbb{R}^k \times \mathbb{R}^m, \mathcal{L}(\mathbb{R}^k) \otimes \mathcal{L}(\mathbb{R}^m), \lambda_k \otimes \lambda_m)$

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Integration on product spaces

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. $f : X \times Y \to \overline{\mathbb{R}}$ measurable.

If $f \geq 0$, then

$$\iint_{X\times Y} f d\mu \otimes d\nu$$

Goal: obtain a formula of iterated integral like the one in Analysis 2.

$\forall\, \bar{x} \in X$ and $\bar{y} \in Y$, we define

$$f_{\bar{x}} : \begin{array}{l} Y \to \overline{\mathbb{R}} \\ y \mapsto f(\bar{x}, y) \end{array} \qquad f_{\bar{y}} : \begin{array}{l} X \to \overline{\mathbb{R}} \\ x \mapsto f(x, \bar{y}) \end{array}$$

Proposition 10.1
If f is measurable $\Rightarrow$ $f_{\bar{x}}$ is $(\mathcal{N}, \mathcal{B}(\mathbb{R}))$-measurable and $f_{\bar{y}}$ is $(\mathcal{M}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable. Then we can consider

$$\varphi : X \to \overline{\mathbb{R}} \quad \varphi(x) = \int_Y f_x d\nu = \int_Y f(x, y) \underbrace{d\nu(y)}_{dy}$$

$$\psi : Y \to \overline{\mathbb{R}} \quad \psi(y) = \int_X f_y d\mu = \int_X f(x, y) d\mu(x)$$

Questions: what is the solution of $\iint_{X\times Y}$, φ and ψ?

Theorem 10.1 (Tonelli's theorem)
$(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ complete measure spaces and σ-finite.
Suppose that f is $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable and that $f > 0$ a.e. on $X \times Y$. Then ψ and φ are measurable and

$$\iint_{X\times Y} f d\mu \otimes d\nu = \int_X \varphi(x)\, d\mu(x) = \int_Y \psi(y)\, d\nu(y) \qquad \text{Integration formula}$$

Equally holds also if one of the integrals is ∞.

$$\int_X \varphi(x)\, d\mu(x) = \int_X \left(\int_Y f(x, y)\, d\nu(y) \right) d\mu(x)$$
$$\int_Y \psi(y)\, d\nu(y) = \int_Y \left(\int_X f(x, y)\, d\mu(x) \right) d\nu(y)$$

Remark 10.1
The double integral can be reduced to single integrals, iterated. Moreover we can always change the order of the integrals For sign changing functions the situation is more involved.

Theorem 10.2 (Fubini's theorem)
$(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ complete measure spaces and σ-finite. If $f \in L^1(X \times Y)$, then ψ and φ defined above are measurable, the integration formula holds, and all the integrals are finite.

Question: how to check if $f \in L^1(X \times Y)$? Typically, to check that $f \in L^1(X \times Y)$ one uses Tonelli:

$$f \in L^1(X \times Y) \Leftrightarrow \iint_{X\times Y} |f|\, d\mu \otimes d\nu$$

We use Tonelli to check that this is finite. If $\iint_{X\times Y} |f| d\mu \otimes d\nu < \infty$ then we can apply Fubini for $\iint_{X\times Y} f d\mu \otimes d\nu$

Remark 10.2
the proof of Fubini's and Tonelli's theorems is based for the iterated integrals for characteristic functions. Note that

$$(\mu \otimes \nu)(E) = \begin{array}{l} \displaystyle\int_X \varphi(x)\, d\mu(x) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x) \\ \displaystyle\int_Y \psi(y)\, d\nu(y) = \int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y) \end{array}$$

Remark 10.3
Sometimes double integrals are very useful to compute single integrals.

Ex: $\int_{-\infty}^{+\infty} e^{-x^2} = \sqrt{\pi}$

11 Lecture 19/10/2022

The first fundamental theorem of calculus

Consider $f \in L^1([a,b])$. We can define the **integral function**

$$F(x) = \int_{[a,b]} f d\lambda = \int_a^b f(t)dt, \quad x \in [a,b]$$

If $f \in \mathcal{C}([a,b])$, then F is differentiable on $[a,b]$, and $F'(x) = f(x)$

What happens if $f \in L^1([a,b])$?

Definition 11.1
Given $f \in L^1([a,b])$. We say that $x \in [a,b]$ is a **Lebesgue point** for f if

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt = 0$$

If $x = a$ or $x = b$, this is the left/right lim.

Remark 11.1
A point x is called a Lebesgue point for f if f 'does not oscillate too much' close to x:

- f $\mathcal{C}([a,b]) \to$ every $x \in [a,b]$ is a Lebesgue point.
-

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\lim_{h \to 0} \frac{1}{h} \int_0^h |f(t) - f(0)|\, dt = \lim_{h \to 0} \frac{1}{|h|} \int_0^h |0 - 1|\, dt = 0$$

Theorem 11.1 (Lebesgue)
If $f \in L^1([a.b])$ then a.e. $x \in [a,b]$ is a Lebesgue point for f

Remark 11.2
In the definition of Lebesgue point, the pointwise values of f are relevant

$$f = g \in L^1 \Leftrightarrow f = g \text{ a.e.}$$

Then the Lebesgue point of f could be different from the one of g. This is not a big problem if $f = g$ a.e. on $[a,b] \Rightarrow f = g \in [a,b] \setminus N$ where $\lambda(N) = 0$; x is a Lebesgue point for f, $\forall\, x \in [a,b] \setminus M,\ \lambda(M) = 0$
$\Rightarrow x$ is a Lebesgue point for g, $\forall x \in [a,b] \setminus (M \cup N)$
$[a,b] \setminus (M \cup N)$ is a set of full measure of Lebesgue points for f and g.

To speak about Lebesgue points, one has to choose a specific representative $f \in L^1([a,b])$. If you change representative, you obtain the same set of Lebesgue points up to sets with 0-measure.

Theorem 11.2 (First fundamental theorem of calculus)
Given $f \in L^1([a,b]),\ F(x) = \int_a^x f(t)\,dt$
Then f is differentiable a.e. on $[a,b]$ and $F'(x) = f(x)$ a.e. in $[a,b]$

Proof. Let $x \in [a,b]$ for any Lebesgue point for f (a.e. $x \in [a,b]$ is fine). Consider

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} (f(t)-f(x))\,dt\right| \leq \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,dt \to 0$$

Since x is a Lebesgue point. ★

Definition 11.2
Given $f : I \to \mathbb{R}$ is called **absolutely continuous** in I, $f \in AC(I)$, if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$$\bigcup_{k=1}^{n} [a_x, b_x] \in I \text{ with disjoint interiors}$$

$$\lambda(\bigcup_{k=1}^{n} [a_x, b_x]) = \sum_{k=1}^{n} (b_x - a_x) < \delta$$

$$\Rightarrow \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

Remark 11.3
f is uniformly continuous on $[a,b]$ if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$$|t - \tau| < \delta \Rightarrow |f(t) - f(\tau)| < \varepsilon$$

An absolutely continuous function is also uniformly continuous.
But the converse is false.

- If f is Lipschitz on $[a,b] \Rightarrow f \in \mathrm{AC}([a,b])$

Recall that $f \in \mathrm{Lip}([a,b])$ if $\exists\, L > 0$ s.t.

$$|f(x) - f(y)| \leq L|x-y| \qquad \forall x, y \in [a,b]$$

<u>Check</u>: For any $\varepsilon > 0$, and consider

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \sum_{k=1}^{n} L(b_k - a_k) = L\sum_{k=1}^{n} (b_k - a_k)$$

If we take $\delta = \delta(\varepsilon) = \frac{\varepsilon}{L}$, then

$$\sum_{k=1}^{n} (b_x, a_x) < \delta \Rightarrow \sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq L\sum_{k=1}^{n} (b_k - a_k)$$

★

$$\mathrm{Lip}([a,b]) \subsetneqq \mathrm{AC}([a,b]) \subsetneqq \mathrm{UC}([a,b])$$

Theorem 11.3 (Regularity of integral functions)
Given $f \in L^1([a,b]), F(x) = \int_a^x f(t)\,dt$, then $F \in \mathrm{AC}([a,b])$

To prove the theorem we need the

Theorem 11.4 (Absolute continuity of the integral)
Given $f \in L^1(X, \mathcal{M}, \mu)$. Then $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$$\begin{array}{l} E \in \mathcal{M} \\ \mu(E) < \delta \end{array} \Rightarrow \int_E |f|\,d\mu < \varepsilon$$

Proof. We fix $\varepsilon > 0$. Let $F_n := \{|f| < n\}$, $n \in \mathbb{N}$. Also $F_n \in \mathcal{M} \forall\, n$, $F_n \subseteq F_{n+1}$ and

$$\bigcup_{n=1}^{\infty} F_n = \{|f| < \infty\} =: F$$

$f \in L^1 \Rightarrow |f|$ is finite a.e.: $\mu(X \setminus F) = 0$. Therefore:

$$\int_X |f|\,d\mu = \int_{X\setminus F} |f|\,d\mu + \int_F |f|\,d\mu = \lim_{n\to\infty} \int_{F_n} |f|\,d\mu$$

$$\lim_{n\to\infty} \int_X |f|\,(\chi_{F_n})\;d\mu = 0$$

$\forall\, \varepsilon > 0\ \exists\, \bar{n} \in \mathbb{N}$ s.t.

$$n > \bar{n} \Rightarrow \left|\int_X |f|\chi_{F_n^C}\,d\mu\right| < \frac{\varepsilon}{2}$$

Now, fix $\varepsilon > 0$, and take $n > \bar{n}(\varepsilon)$. If $E \in \mathcal{M}$, then

$$\int_E |f|\,d\mu = \int_{E\cap F_n} |f|\,d\mu + \int_{E\cap F_n^C} |f|\,d\mu \leq n \int_E 1\,d\mu + \int_{F_n^C} |f|\,d\mu$$

If we suppose that $\mu(E) < \frac{\varepsilon}{2n} =: \delta(\varepsilon)$, we deduce that

$$n \int_E 1\,d\mu = n\mu(E) < \frac{\varepsilon}{2}$$

Also, since $n > \bar{n}$

$$\int_{F_n^C} |f|\,d\mu < \frac{\varepsilon}{2}$$

$$\Rightarrow \int_E |f|\,d\mu < \varepsilon$$

★

Regularity of integral functions. Let $\varepsilon > 0$, and $\delta = \delta(\varepsilon) > 0$ be the value given by the absolute continuity of $\int |f|\,d\mu$. Take

$$E = \bigcup_{k=1}^{n} [a_k, b_k] \qquad E \subseteq [a,b]$$

If $\lambda(E) < \delta$, then

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| = \sum_{k=1}^{n} \left|\int_{a_k}^{b_k} f(t)\,dt\right| \leq \sum_{k=1}^{n} \int_{a_k}^{b_k} |f(t)|\,dt = \int_E |f|\,d\lambda < \varepsilon$$

by absolute continuity of $\int$ ★

Remark 11.4
$\sqrt{x}$ is AC($[0,1]$), but is not Lip($[0,1]$).

$$\sqrt{x} = \int_0^x \frac{1}{2\sqrt{t}}\, dt$$

$\Rightarrow \sqrt{x}$ is the $\int$ function of a L^1 function
$\Rightarrow \sqrt{x} \in \mathrm{AC}([0,1])$

To sum up: the $\int$ function of a (L^1 function is AC, it is differentiable a.e., and

$$F(x) - F(a) = \int_a^x F'(t)\, dt \qquad \text{FC}$$

Suppose G is differentiable a.e. on $[a,b]$ and FC holds for G:

$$G(x) - G(a) = \int_a^x G'(t)\, dt$$

What can we say about G?
Remark 11.5
If $G \in \mathcal{C}^1([a,b]) \Rightarrow$ FC holds.
If FC holds, then $G' \in L^1([a,b])$ (necessary condition)
Is the necessary condition also sufficient? In general not. Take $v(x)$, the Vital Cantor function: $v \in \mathcal{C}([0,1]), v(0) = 0, v(1) = 1$. v is differentiable a.e. on $[0,1]$ but the calculus formula doesn't hold!

Remark 11.6
A function which is differentiable a.e. on an interval can behave very badly

Theorem 11.5
$G \in \mathrm{AC}([a,b])$. Then G is differentiable a.e. on $[a,b]$, $G' \in L^1([a,b])$, and FC holds.

Remark 11.7
These theorems say that AC function are precisely the ones for which FC holds:

- $G \in \mathrm{AC} \Rightarrow$ FC holds.
- If FC holds, then $G' \in L^1([a,b])$

$\Rightarrow \int_a^x G'(t)\, dt \in \mathrm{AC}$
$\Rightarrow G(x) - G(a) = \int_a^x G'(t)\, dt \in \mathrm{AC}$

Remark 11.8
$v \in \mathrm{UC}([0,1])$ by continuity and Heine Cantor, but $v \notin \mathrm{AC}([0,1])$ because FC does not hold.

The proof of the second fundamental theorem of calculus is divided into two steps.
Lemma 11.1
The second fundamental theorem hold under the additional assumption that G is monotone.

Second step: to get rid of the monotonicity.
For step 2, is it useful to give the

Definition 11.3
$[a,b] \subset \mathbb{R}$. Let

$$\mathcal{P}_{[a,b]} := \{(x_0, x_1, \ldots, x_n) : n \in \mathbb{N} \text{ and } a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

For $P \in \mathcal{P}_{[a,b]}$ and $f : [a,b] \to \overline{\mathbb{R}}$, define

$$v_a^b(f, P) := \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

The total variation of f on $[a,b]$ is

$$V_a^b(f) := \sup_{P \in \mathcal{P}_{[a,b]}} v_a^b(f,P) = \sup\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b\}$$

If $V_a^b(f) < \infty$, we say that f is a function with bounded variation, $f \in \mathrm{BV}\ ([a,b])$

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Goal: The 2^{nd} fundamental theorem of calculus. Given $G \in \mathrm{AC}([a,b])$. Then G is differentiable a.e. on $[a,b]$, $G' \in L^1([a,b])$, and (FC) holds. Example and comments:

- If f is bounded and monotone $\Rightarrow f \in \mathrm{BV}$

$$V_a^b(f) = |f(b) - f(a)|$$

  Note that f may not be continuous

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \Rightarrow f \in \mathrm{BV}([-1,1])$$

- $f \in \mathrm{BV}([a,b]) \Rightarrow f$ is bounded. Indeed

$$\sup_{x \in [a,b]} |f(x)| \leq |f(x)| + V_a^b(f) \overset{f \in \mathrm{BV}}{<} +\infty$$

- f is continuous on $[a,b]$, or even if f is differentiable everywhere in $[a,b] \not\Rightarrow f \in \mathrm{BV}([a,b])$

$$f(x) = \begin{cases} x^2 \cos \frac{2\pi}{x^2} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

  It is continuous in $[0,1]$, but $f \notin \mathrm{BV}([0,1])$

- $f \in \mathrm{BV}([a,b]) \cap \mathrm{UC}([a,b]) \not\Rightarrow f \in \mathrm{AC}([a,b])$

  v a Vitali-Cantor function
  v is bounded and monotone $\Rightarrow v \in \mathrm{BV}([0,1])$
  $v \in \mathrm{UC}([0,1])$

  But $v \notin \mathrm{AC}([0,1])$

- If $f \in \mathrm{BV}([a,b]) \Rightarrow f$ is differentiable a.e. on $[a,b]$, and $f' \in L^1([a,b])$

We can now come back to the proof of Lemma 1 of the last lesson.

Preliminary result: $A \in \mathbb{R}$ open. Then

$$A = \bigcup_{n=1}^{\infty} (a_n, b_n) \text{ disjoint}$$

any open set of $\mathbb{R}$ is the (at most) countable union of open disjoint intervals.

Preliminary result (equivalent definition for AC): $f \in \mathrm{AC}([a,b]) \Leftrightarrow \forall\, \varepsilon > 0 \exists\, \delta > 0$ depending on ε s.t.

$$\forall \bigcup_{n=1}^{\infty} [a_n, b_n], \quad [a_n, b_n] \text{ have disjoint interiors}$$

$$\sum_{n=1}^{\infty} (b_n - a_n) < \delta \Rightarrow \sum_{n=1}^{\infty} |f(b_n) - f(a_n)| < \varepsilon$$

Proof. We defined λ starting from two properties

- invariance under translations
- $\lambda((x,y)) = y - x \; \forall\, a \leq < y \leq b$

Now, G is monotone, say G increasing (if $G \searrow$, take $-G$). We can repeat the construction of λ in order to obtain a measure μ s.t.

- μ is invariant under translations
- $\mu((x,y)) = \underbrace{G(y) - G(x)}_{\geq 0} \; \forall\, a \leq x < y \leq b$ (for λ, take $G(t) = t$)

It can be proved that we obtain a measure on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$, complete.

On $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$ we have two measures: λ and μ.

Idea: We take these measures on $([a,b], \mathcal{L}([a,b]))$, and we want to show that $\exists\, \frac{d\mu}{d\Lambda}$ (Radon-Nikodym)

We can check the hypothesis of the Radon-Nikodym theorem:

- λ is σ-finite: $\lambda([a,b]) = b - a < +\infty$
- $\mu << \lambda$: $E \in \mathcal{L}([a,b])$, $\lambda(E) = 0 \Rightarrow \mu(E) = 0$

Assume $\lambda(E) = 0$. G is $\mathrm{AC}([a,b])$: then $\forall\, \varepsilon > 0\; \exists\, \delta = \delta(\varepsilon) > 0$ s.t.

$$\forall \bigcup_{n=1}^{\infty} [a_n, b_n], \quad [a_n, b_n] \text{ have disjoint interiors}$$

$$\lambda\left(\bigcup_{n=1}^{\infty} [a_n, b_n]\right) < \delta \Rightarrow \sum_{n=1}^{\infty} |G(b_n) - G(a_n)| < \varepsilon$$

Take this δ. By regularity of λ, $\exists\, A$ open set of $[a,b]$s.t. $A \supset E$ and $\lambda(A) < \delta$

$$A \text{ is open} \Rightarrow A = \left(\bigcup_{n=1}^{\infty} I_n{}^{\dagger}\right), \text{ disjoint}$$

†open intervals $= (x_n, y_n)$

it is a countable union of open intervals (maybe two of them contains a or b)

$$\lambda(A) < \delta \Leftrightarrow \sum_{n=1}^{\infty}(y_n - x_n) < \delta$$

But then, since μ is a measure it is countably additive

$$\mu(E) \leq \mu(A) = \sum_n \mu(I_n) = \sum_n G(y_n) - G(x_n) \underset{\substack{\downarrow \\ \text{by the choice of } \delta \\ \text{and the fact that} \\ G \in \text{AC}}}{<} \varepsilon$$

We proved that

$$\lambda(E) = 0 \Rightarrow \forall\, \varepsilon > 0:\ \mu(E) < \varepsilon \Rightarrow \mu(E) = 0$$

So $\mu << \lambda$. We can apply Radon Nikodym $\exists\ \phi : [a,b] \to [0,\infty]$ s.t.

$$G(x) - G(a) = \int_a^x \phi\, d\lambda$$

Since G is bounded, then $\phi \in L^1([a,b])$

$$G(x) = G(a) + \int_a^x \phi(t)\, dt$$

By the first fundamental theorem of calculus, this is differentiable a.e.

$$\Rightarrow G'(x) = \phi(x) \text{ a.e. on } [a,b]$$

$$\Rightarrow G'(x) = G(a) + \int_a^x G'(t)\, dt$$

★

Now we want to get rid of the additional assumption (monotonicity).
Preliminary result: $f \in \text{BV}([a,b])$. Then

$$\varphi(x) = V_a^x(f), \quad \forall\, x \in [a,b]$$

is an increasing function.

Proof. By $a \leq x < y \leq b$. Then

$$V_a^y(f) = V_a^x(f) + \underbrace{V_x^y(f)}_{\geq 0} \geq V_a^x(f)$$

★

Preliminary result: If $G \in \text{AC}([a,b])$, then $G \in \text{BV}([a,b])$, and moreover

$$\varphi(x) = V_a^x(G) \text{ is in AC}([a,b])$$

Proof of the second fundamental theorem of calculus in the general case. $G \in \mathrm{AC}([a,b])$

We want to write $G = G_1 + G_2$ where $G_1 \nearrow$ and $G_2 \searrow$, both AC.

Then the second fundamental theorem holds for G_1 and G_2 so it holds for G by linearity of the integral.

We pose:

$$G_1(x) = \frac{G(x) + V_a^x(G)}{2}$$
$$G_2(x) = \frac{G(x) - V_a^x(G)}{2}$$

Clearly, $G_1 + G_2 = G$, G_1, G_2 are AC, by the last preliminary result.

$\underline{G_1 \nearrow}$: Let $a \leq x < y \leq b$

$$|G(y) - G(x)| \leq V_x^y(G)$$

Therefore,

$$G_1(y) - G_1(x) = \frac{1}{2}(\underbrace{G(y) - G(x)}_{\substack{\geq -|G(y)-G(x)| \\ \geq -V_x^y(G)}} + V_a^y(G) + V_a^x(G)) \geq \frac{1}{2}(-V_x^y(G) + V_x^y(G)) = 0$$

So G_1 is decreasing. In an analogue way, we can prove that G_2 is decreasing. ★

Functional analysis

Normed spaces and Banach spaces

Definition 12.1

Given X vector space, a norm on X is a function $\|\cdot\| : X \to [0, \infty)$ s.t.

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\forall\, \alpha \in \mathbb{R}, \forall\, x \in X$:
$$\|\alpha x\| = |\alpha| \|x\| \text{ (positive homogeneity)} \qquad \text{(positive homogeneity)}$$
- $\forall\, x, y \in X$:
$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{(triangle inequality)}$$

Then, $(X, \|\cdot\|)$ is called a **normed space**

Ex: $|\|x\| - \|y\|| \leq \|x - y\| \;\forall\, x, y \in X$

Proposition 12.1

$(X, \|\cdot\|)$ normed space. Then (X, d) is a metric space for

$$d(x, y) = \|x - y\|$$

Remark 12.1

Normed space $\underset{\not\leftarrow}{\rightarrow}$ metric space

Examples:

- $\mathbb{R}^N, \|x\|_p := (\sum_{i=1}^N |x_i|^p)^{\frac{1}{p}} \;\forall p \in [1, +\infty)$
$$\|x\|_\infty := \max_{i=1,\dots,N} |x_i|$$

- $\mathcal{C}^0([a,b])$
$$\|f\|_\infty := \max_{x\in[a,b]} |f(x)|$$
- $L^1(X, \mathcal{M}, \mu)$
$$\|f\|_1 := \int_X |f|\, d\mu$$
This is a norm in L^1, but not on $\mathcal{L}^1$ ($\int_x |f|\, d\mu = 0 \Rightarrow f = 0$ a.e.)
- $L^\infty(X, \mathcal{M}, \mu)$
$$\|f\|_\infty := \operatorname*{ess\,sup}_{[a,b]} |f|$$

$(X, \|\cdot\|)$ normed space $\to$ (X, d) metric space $\to$ convergent sequences on X: $\{x_n\} \subset X$ is convergent in X iff
$$d(x_n, x) \to 0 \Leftrightarrow \|x_n - x\| \to 0 \text{ as } n \to \infty$$
Ex: $x_n \to x$ in X, then $\|x_n\| \to \|x\|$ (the norm is a continuous function on X)

Definition 12.2
$\{x_n\}$ is a **Cauchy sequence** in $(X, \|\cdot\|)$ if $\forall\, \varepsilon > 0\ \exists\, \bar{n} \in \mathbb{N}$ s.t.
$$n, m > \bar{n} \Rightarrow \|n_n - x_m\| < \varepsilon$$

Definition 12.3
$(X, \|\cdot\|)$ is called a **Banach space** if (X, d) is complete, namely if any Cauchy sequence in (X, d) is convergent.

If $(X, \|\cdot\|)$ is a normed space, we can speak about series in X. Let $\{x_n\} \subset X$ and $s_n = x_0 + x_1 + \ldots + x_n$, then $\sum_{n=0}^{+\infty} x_n = \{s_n\}$.

Then $\sum x_n$ is convergent if $\{s_n\}$ is convergent. If $\sum x_n$ is convergent, we write
$$s = \sum_{n=0}^{+\infty} x_n \Leftrightarrow s_n \to s$$
For numerical series
$$\sum_{n=1}^{\infty} |a_n| < +\infty \Rightarrow \sum a_n \text{ is convergent}$$
In general, in normed spaces
$$\sum_{n=1}^{\infty} \|x_n\| < +\infty \not\Rightarrow \sum_{n=1}^{\infty} x_n \text{ is convergent}$$

Characterization

$(X, \|\cdot\|)$ is a Banach space $\Leftrightarrow$ every series s.t. $\sum \|x_n\| < +\infty$ is also s.t. $\sum x_n$ is convergent

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$(X, \|\cdot\|) \to (X, d) \to$ open sets, closed sets, bounded sets....

In $\mathbb{R}^n$ we are used to work with $\|\cdot\|_2$, but we could have many different norms.

Definition 13.1
Let $\|\cdot\|$ and $\|\cdot\|_2$ be two norms on the same vector space X. We say that these norms are **equivalent** if $\exists\, m, M > 0$ s.t.

$$m\|x\| \leq \|x\|_2 \leq M\|x\| \quad \forall\, x \in X$$

It can be proved that if two norms are equivalent they lead to different metric spaces, but to the same open sets, closed sets, convergent sequences, compact sets ...

Theorem 13.1
If X is any finite dimension vector space, then all the norms on X are equivalent.

Remark 13.1
This is why in $\mathbb{R}^n$ usually one does not specify the choice of the norm. One choose the Euclidean norm, since it comes from a scalar product. (ref. Hilbert spaces)

Preliminary fact: The set $S_1 = \{s \in \mathbb{R}^n : \|x\|_1 = 1\}$ is compact in $(\mathbb{R}^n, d)$

Proof. We show that any norm is equivalent to $\|\cdot\|_1$

$$x = \sum_{i=1}^{n} x_i e_i \qquad \{e_i\}_{i=1,\dots,n} \text{ canonical basis}$$

Let's introduce the norm star

$$\|x\|_* = \left\| \sum_{i=1}^{n} x_i e_i \right\|_* \leq \sum_{i=1}^{n} \|x_i e_i\|_* = \sum_{i=1}^{n} |x_i| \|e_i\|_* \leq \left(\max_{1 \leq i \leq n} \|e_i\|_* \right) \sum_{i=1}^{n} |x_i| = M\|x\|_1$$

We proved that $\exists\, M > 0$ s.t.

$$\|x\|_* \leq M\|x\|_1 \quad \forall\, x \in X \tag{1}$$

Note that this proves that $\varphi(x) = \|x\|_*$ is continuous in (X, d). Indeed

$$x_n \to x \Leftrightarrow d_1(x_n, x) \to 0$$

then

$$|\varphi(x_n) - \varphi(x)| = |\|x_n\|_* - \|x\|| \leq \|x_n - x\|_* \overset{(1)}{\leq} M\|x_n - x\|_1 \to 0$$

Therefore, by the Weierstrass theorem, $\exists$ a minimum point $x_0 \in S_1$ s.t.

$$\varphi(x) \geq \varphi(x_0) = m \quad \forall\, x \in S_1$$

(recall that S_1 is compact)

$$\|x\|_* \geq m \quad \forall\, x \in S_1$$

We claim that $m > 0$. If $m = 0$ then $\|x_0\|_* = 0 \Rightarrow x_0 = 0$ that is impossible, since $x_0 \in S_1$.

Thus $m > 0$. Let now $y \in \mathbb{R}^n, y \neq 0$. Then

$$\frac{y}{\|y\|_1} \in S_1 \Rightarrow \left\| \frac{y}{\|y\|_1} \right\|_* \geq m \Rightarrow \frac{1}{\|y\|_1} \|y\|_* \geq m \Rightarrow \|y\|_* m \geq m\|y\|_1 \quad \forall\, y \in \mathbb{R}^n$$

★

If $\dim X = +\infty$, then there are many non-equivalent norms.
Ex: In $\mathcal{C}^0([a,b])$, we can define $\|\cdot\|_\infty$ and $\|f\|_1 = \int_a^b |f(t)|\, dt$.
This is a norm in $\mathcal{C}^0$, but these norms are not equivalent.

Separability

(X, d) metric space.
Definition 13.2
We say that X is separable if $\exists\ A \in X$ which is dense ($\bar{A} = X$) and countable

In $\mathbb{R}^n$, $\mathbb{Q}^n$ which is dense and countable. In $\infty -$ dim we can have separable spaces or not. For instance, $(L^\infty, \|\cdot\|_\infty)$ is not separable. Instead $(\mathcal{C}^0([a,b]), \|\cdot\|_\infty)$ is a separable space.

Sketch of the proof. We will use the **Stone-Weierstrass theorem**.

The set of polynomials is dense on $\mathcal{C}^0([a,b])$ and is an uncountable set. However it can be proved that the set of polynomials with coefficients in $\mathbb{Q}$ is dense in the set of all polynomials

Moreover this set is countable. Then, by Stone-Weierstrass this is a countable dense set in $\mathcal{C}^0([a,b])$ ★

Remark 13.2
One can show that $\mathcal{C}^0(K)$ is separable whenever K is a compact set of a metric space (X, d)

Compactness

In finite dimension (in $\mathbb{R}^n$), one has that

$$E \subset X \text{ is compact } \Leftrightarrow E \text{ is closed and bounded}$$

If $\dim X = \infty$, then only '$\Rightarrow$' is true. In finite dimension, we know that the closed unit ball is compact

$$\bar{B}_1(0) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

What happens now if $(X, \|\cdot\|)$ is on $\infty -$ dim normed space?
Theorem 13.2 (Riesz's theorem)
X normed space, $\dim X = +\infty \Rightarrow \bar{B}_1(0)$ is not compact
Remark 13.3
It is well known that if $E \in \mathbb{R}^n$ is compact, then $\forall\ \{x_n\} \in E\ \exists\ \{x_{n_k}\}$ subsequence s.t. $x_{n_k} \to x \in E$. This proposition is much harder to prove in $\infty -$ dim.

The proof of the Riesz's theorem is based on the Riesz's **quasi-orthogonality lemma**.
Lemma 13.1 (Riesz Quasi-Orthogonality Lemma)
Let X be a normed space, $E \subsetneq X$ a closed subspace. Then $\forall\, \varepsilon \in (0,1)\ \exists\, x \in X$ s.t.

$$\|x\| = 1 \text{ and } \operatorname{dist}(x, E) = \inf_{y \in E} \|x - y\| \geq 1 - \varepsilon$$

Proof. Of the Riesz's Theorem. Assume that $\bar{B}_1(0)$ is compact, and X has infinite dimension.

$\exists$ a sequence $\{E_n\}$ of finite dimensional subspaces (hence closed) of X s.t.

$$E_{n-1} \subset E_n \text{ and } E_{n-1} \neq E_n$$

E_{n-1} is a proper closed subspace of E_n, $\forall\, n$

We can apply the Riesz Lemma with $X = E_n$, $E = E_{n-1}$, $\varepsilon = \frac{1}{2}$. Then $\forall\, n\ \exists\, u_n \in E_n$ s.t. $\|u_n\| = 1$ and $\operatorname{dist}(u_n, E_{n-1}) \geq \frac{1}{2} \quad \forall\, n$

Therefore, we have a sequence $\{u_n\}$ with the following properties

$$\|u_n\| = 1 \qquad \forall\, n$$

$$\|u_n - u_m\| \geq \frac{1}{2} \qquad \forall\, n \neq m$$

$\Rightarrow$ this sequence cannot have any convergent subsequence. But then $\bar{B}_1(0) \supseteq \{u_n\}$, this implies that $\bar{B}_1(0)$ is not compact. Contradiction.

(In any $(X, \|\cdot\|)$ normed space, if E is compact, then $\forall\, \{x_n\} \subset E\ \exists\, \{x_{n_k}\}$ s.t. $x_{n_k} \to x \in E$) ★

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(X, d) metric space.

Definition 14.1
$E \subset X$ is compact if for any open covering $\{A_i\}_{i \in I}$ has a finite subcover.

Definition 14.2
$E \subset X$ is sequentially compact if $\forall \ \{x_n\} \subset E$ there exists $\{x_{n_k}\}$ subsequence convergent to some limit $x \in E$

Well known fact: if (X, d) is a metric space, then E is compact $\Leftrightarrow$ E is sequentially compact.

Theorem 14.1 (Riesz Theorem)
X normed space, $\dim X = \infty \Leftrightarrow \bar{B}_1(0)$ is not compact.

Lemma 14.1 (Riesz quasi orthogonality Lemma)
X normed space, $E \subsetneq X$ closed subspace. Then $\forall \ \varepsilon \in (0,1) \ \exists \ x \in X$ s.t.

$$\|x\| = 1 \text{ and } \operatorname{dist}(x, E) = \inf_{y \in E} \|x - y\| \geq 1 - \varepsilon$$

Remark 14.1

- $E \in X$ closed. Then $\operatorname{dist}(x, E) = 0 \Leftrightarrow x \in E$
- By definition of infimum, if $d = \operatorname{dist}(x, E)$, then $\forall \rho > 0 \ \exists \ z \in E$ s.t.

$$\|x - z\| < (1 + \rho) d$$

Proof. Let $y \in X \setminus E$, and $d := \operatorname{dist}(y, E) > 0$, since E is closed.
$\forall \ \rho > 0 \ \exists z \in E$ s.t.

$$\|y - z\| \leq (1 + \rho) d = \frac{d}{1 - \varepsilon} \tag{1}$$

since we choose ρ s.t. $1 + \rho = \frac{1}{1-\varepsilon}$. Now we set $x = \frac{y-z}{\|y-z\|}$.
Clearly $\|x\| = 1$. Moreover, $\forall \ u \in E$, we have that

$$\|x - u\| = \left\| \frac{y - z}{\|y - z\|} - u \right\| = \left\| \frac{y - z - \|y - z\| u}{\|y - z\|} \right\| = \frac{1}{\|y - z\|} \|y - (z + \|y - z\| u)\| =$$

$$= \frac{1}{\|y - z\|} \|y - w\| \geq \frac{1}{\|y - z\|} \operatorname{dist}(y, E) \overset{(1)}{\geq} \frac{1 - \varepsilon}{d} d = 1 - \varepsilon$$

Since this is true $\forall \ u \in E$, we deduce that

$$\operatorname{dist}(x, E) \geq 1 - \varepsilon$$

★

Compactness on $\mathcal{C}^0([a, b])$

Definition 14.3
$\{f_n\}$ sequence in $\mathcal{C}^0([a, b])$. We say that $\{f_n\}$ is uniformly equicontinuous in $[a, b]$ if $\forall \ \varepsilon > 0 \ \exists \ \delta > 0$ depending only on ε s.t.

$$|t - \tau| < \delta \Rightarrow \|f_n(t) - f_n(\tau)\| < \varepsilon \qquad \forall \ n$$

Remark 14.2
With respect to the uniform continuity, in this case δ does not depend on f. δ is the same for all the f_n s

Theorem 14.2
$\{f_n\} \subseteq \mathcal{C}^0([a,b])$. Suppose that:

- $\{f_n\}$ is uniformly equicontinuous
- $\{f_n\}$ is bounded: $\exists\, M>0$ s.t. $\|f_n\|_\infty < M \qquad \forall\, n$

Then $\exists$ a subsequence $\{f_{n_k}\}$ and $f \in \mathcal{C}^0([a,b])$ s.t. $f_{n_k} \to f$ uniformly.

Lebesgue spaces.
$(X, \mathcal{M}, \mu)$ measure space, $p \in [1,\infty]$. We defined $L^1(X)$ and $L^\infty(X)$. In a similar way, we define $L^p(X)$ $\forall\, p \in [1,\infty]$

$$\mathcal{L}^p(X,\mathcal{M},\mu) := \{f : X \to \overline{\mathbb{R}} \text{ measurable s.t. } \int_X |f|^p \, d\mu < \infty\}$$

On $\mathcal{L}^p$ we introduce the equivalent relation

$$f \sim g \text{ in } \mathcal{L}^p \Leftrightarrow f = g \text{ a.e. on } X$$

and define
$$L^p(X,\mathcal{M},\mu) := \frac{\mathcal{L}^p(X,\mathcal{M},\mu)}{\sim}$$

We want to show that this is a normed space with

$$\|f\|_p := \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} & p \in [1,\infty) \\ \operatorname*{ess\,sup}_X |f| & p = \infty \end{cases}$$

The fact that L^p is a vector space is easy to prove. The only non trivial part is that $f, g \in L^p \Rightarrow f+g \in L^p$.
This comes directly from the

Lemma 14.2
$p \in [1,\infty)$, $a, b \geq 0$. Then
$$(a+b)^p \leq 2^{p-1}\left(a^p + b^p\right)$$

$f, g \in L^p$, $p \in [1,\infty)$

$$\int_X |f+g|^p \, d\mu \leq \int_X (|f|+|g|)^p \, d\mu \leq 2^{p-1} \int_X (|f|^p + |g|^p) \, d\mu$$
$$= 2^{p-1} \int_X |f|^p \, d\mu + 2^{p-1} \int_X |g|^p \, d\mu < \infty$$

L^p is a vector space, $\forall\, p \in [1,\infty)$.
$f, g \in L^\infty$. Then a.e.

$$\Rightarrow |f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty < \infty \Rightarrow f+g \in L^\infty$$

L^∞ is a vector space.

Remark 14.3
$l^p := L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$. l^p is a particular case of L^p

$$l^p = \{x = \left(x^{(k)}\right)_{k\in\mathbb{N}} : \sum_{k=1}^\infty |x^{(k)}|^p < \infty\} \quad \|x\|_p = \left(\sum_{k=1}^\infty |x^{(k)}|^p\right)^{\frac{1}{p}} \qquad p \in [1,\infty)$$
$$l^\infty = \{x = \left(x^{(k)}\right)_{k\in\mathbb{N}} : \sup_{k\in\mathbb{N}} |x^{(k)}| < \infty\} \quad \|x\|_\infty = \sup_{k\in\mathbb{N}} |x^{(k)}|$$

Now we prove that $\|.\|_p$ is actually a norm in L^p. We will concentrate on $p < \infty$ ($p = \infty$ is the easy case)

Properties 1 and 2 of the norm are immediate to check:

(1) $\|f\|_p = 0 \Leftrightarrow \int_X |f|^p \, d\mu = 0 \Leftrightarrow f = 0$ a.e. on $X \Leftrightarrow f = 0 \in L^p$

(2) Obvious, by linearity

(3) About triangle inequality? We need some preliminaries

Theorem 14.3 (Young's Inequality)
Let $p \in (1, \infty)$, $a, b \geq 0$. We say that q is the conjugate exponent of p if

$$\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow q = \frac{p}{p-1}$$

Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Remark 14.4
$p \in (1, \infty) \Rightarrow q \in (1, \infty)$. Moreover, we say that 1 and ∞ are conjugate

Proof. $\varphi(x) = e^x$ is convex:

$$\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y) \qquad \forall x, y \in \mathbb{R} \quad \forall\, t \in [0,1]$$

If $a = 0$ or $b = 0$, then the thesis holds.
If $a, b > 0$

$$ab = e^{\log a} e^{\log b} = e^{\log a^{\frac{p}{p}}} e^{\log b^{\frac{q}{q}}} = e^{\frac{1}{p}\log a^p} e^{\frac{1}{q}\log b^q}$$

Since φ is convex

$$\frac{1}{p} e^{\log a^p} + \frac{1}{q} e^{\log b^q} = \frac{1}{p} a^p + \frac{1}{q} b^q$$

$x = \log a^p$, $y = \log b^q$ $\qquad 1 - t = \frac{1}{p}$, $t = \frac{1}{q}$ ★

Theorem 14.4
$(X, \mathcal{M}, \mu)$ measure space. f, g measurable functions. $p, q \in [1, \infty]$ conjugate exponents.

Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Proof. Case $p, q \in (1, \infty)$. Obvious if $\|f\|_p \|g\|_q = \infty$.
If $\|f\|_p \|g\|_q = 0 \Rightarrow$ either $f = 0$ a.e. on X or $g = 0$ a.e. on X $\Rightarrow fg = 0$ a.e. on $X \Rightarrow \|fg\|_1 = 0$.
Let then $\|f\|_p$, $\|g\|_p \in (0, \infty)$.
For $x \in X$, we set

$$a := \frac{|f(x)|}{\|f\|_p},\ b := \frac{|g(x)|}{\|g\|_q}$$

and use Young:

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^q}$$

$\forall\, x \in X$. By integrating, we obtain

$$\frac{1}{\|f\|_p \|g\|_q} \int_X |fg| \, d\mu \leq \frac{1}{p\|f\|_p^p} \int_X |f|^p \, d\mu + \frac{1}{q\|g\|_q^q} \int_X |g|^q \, d\mu = \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \|fg\| \leq \|f\|_p \|g\|_q$$

Case $p = 1$, $q = \infty$. Exercise ★

Theorem 14.5 (Minkowski Inequality)
$f, g \in L^p(X, \mathcal{M}, \mu)$, $p \in [1, \infty]$. Then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. $p \in (1, \infty)$

$$\|f+g\|_p^p = \int_X |f+g|^p \, d\mu = \int_X |f+g||f+g|^{p-1} \, d\mu$$

$$\leq \int_X (|f| + |g|) \, |f+g|^{p-1} \, d\mu = \int_X |f||f+g|^{p-1} \, d\mu + \int_X |g||f+g|^{p-1} \, d\mu$$

Using Holder with p, $q = \frac{p}{p-1}$

$$\leq \|f\|_p \left(\int_X \left(|f+g|^{p-1}\right)^{\frac{p}{p-1}} \, d\mu \right)^{\frac{p-1}{p}} + \|g\|_p \left(\int_X \left(|f+g|^{p-1}\right)^{\frac{p}{p-1}} \, d\mu \right)^{\frac{p-1}{p}}$$

$$= \|f\|_p \|f+g\|_p^{p-1} + \|g\|_p \|f+g\|_p^{p-1}$$

We divide left hand side and right hand side by $\|f+g\|_p^{p-1}$:

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

★

15 Lecture 09/11/2022

We introduced $L^p(X, \mathcal{M}, \mu)$, and we proved that this is a normed space with

$$\|f\|_p := \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} & \text{if } p \in [1, +\infty) \\ \operatorname*{ess\,sup}_X |f| & \text{if } p = +\infty \end{cases}$$

Inclusion of L^p spaces

Theorem 15.1
Suppose that $\mu(X) < +\infty$. Then

$$1 \leq p \leq q \leq \infty \Rightarrow L^q(X) \subseteq L^p(X)$$

Meaning that any $f \in L^q$ is also in L^p. More precisely, $\exists \, C > 0$ depending on $\mu(X), p, q$ s.t.

$$\|f\|_p \leq \|f\|_q \quad f \in L^q(X)$$

Proof. If $q = +\infty$

$f \in L^\infty(X)$: then $|f(x)| \leq \operatorname*{ess\,sup}_X |f| = \|f\|_\infty$ for a.e. $x \in X$, say $\forall \, x \in X \setminus A$, with $\mu(A) = 0$. Then

$$\int_X |f|^p \, d\mu = \int_{X \setminus A} |f|^p \, d\mu \leq \|f\|_\infty^p \int_{X \setminus A} 1 \, d\mu = \|f\|_\infty^p \underbrace{\mu(X)}_{=\mu(X \setminus A)}$$

If $q < +\infty$

Then $\frac{q}{p} > 1$, and we can use Hölder$\left(\frac{q}{p}, \left(\frac{q}{p}\right)'\right)$, where $\left(\frac{q}{p}\right)' = \frac{\frac{q}{p}}{\frac{q}{p}-1} = \frac{q}{q-p}$

$$\|f\|_p^p = \int_X |f|^p \, d\mu \overset{\text{Hölder}}{\leq} \left(\int_X \left(|f|^{\cancel{p}}\right)^{\frac{q}{\cancel{p}}} d\mu\right)^{\frac{p}{q}} \cdot \left(\int_X 1 \, d\mu\right)^{\frac{q-p}{p}} = \left(\int_X |f|^q \, d\mu\right)^{\frac{p}{q}} \cdot (\mu(X))^{\frac{q-p}{p}}$$

$$\Rightarrow \|f\|_p \leq \mu(X)?\frac{q-p}{qp}\|f\|_q$$

★

The assumption $\mu(X) < \infty$ is essential. For example, in $X = [1, \infty]$

$$\frac{1}{x} \in L^2([1,\infty]) \Leftrightarrow \int_1^\infty \frac{dx}{x^2} < \infty$$

$$\frac{1}{x} \notin L^1([1,\infty]) \Leftrightarrow \int_1^\infty \frac{dx}{x} = \infty$$

In particular, the previous theorem is false for l^p-spaces

$$l^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_C)$$

$$1 \leq p \leq q \leq \infty \Rightarrow l^p \subseteq l^q, \text{ and } \exists\, C > 0 \text{ s.t. } \|x\|_q \leq C\|x\|_p \quad \forall\, x \in l^p$$

Without assumptions on $\mu(X)$, in general one has the interpolation inequality.

Theorem 15.2
$(X, \mathcal{M}, \mu)$ measure space. Let $1 \leq p \leq q \leq \infty$. If $f \in L^p(X) \cap L^q(X)$, then

$$f \in L^r(X) \quad \forall\, r \in (p, q)$$

and moreover

$$\|f\|_r \leq \|f\|_p^\alpha \|f\|_q^{1-\alpha}$$

where α is such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$

Proof. For exercise. Use Holder ★

Completeness and Separability

Theorem 15.3
For $1 \leq p \leq \infty$, $L^p(X, \mathcal{M}, \mu)$ is a Banach space (with reference to $\|\cdot\|_p$)

Proof.

$p < \infty$ By using the characterization of completeness with the series, we want to show that if $\{f_n\} \subseteq L^p(X)$, and $\sum_{k=1}^\infty \|f_k\|_p < \infty \Rightarrow \sum_{k=1}^\infty f_k$ is convergent in L^p, namely $s_n = \sum_{k=1}^n f_k$ has a limit in L^p: $\|s_n - s\|_p \to 0$ as $n \to \infty$.

Let then $\{f_n\} \subseteq L^p(X)$ s.t.

$$\sum_{k=1}^\infty \|f_k\|_p = M < \infty$$

Define

$$g_n(x) = \sum_{k=1}^n |f_k(x)|$$

By Minkowski, $\|g_n\|_p \leq \sum_{k=1}^n \|f_k\|_p \leq M < \infty$. Moreover, for every $x \in X$ fixed, $\{g_n(x)\}$ is increasing $\Rightarrow g_n(x) \to g(x)$ as $n \to \infty$, $\forall\, x \in X$

$$\int_X |g|^p \, d\mu \overset{\text{Monot conv}}{=} \lim_n \int_X |g_n|^p \leq M^p < \infty \Rightarrow g \in L^p(X)$$

$\Rightarrow |g|^p$ is finite a.e.:

$$\sum_{k=1}^{\infty} |f_k(x)| < \infty \text{ for a.e. } x \in X$$

$$\Rightarrow \sum_{k=1}^{\infty} f_k(x) \text{ is convergent a.e. to a limit } s(x)$$

Thus, we proved that $s_n(x) = \sum_{k=1}^n f_k(x) \to s(x)$ a.e. in X. Namely $|s_n - s|^p \to 0$ a.e. in X. To find a dominating function for $|s_n - s|^p$, we start by observing that

$$|s_n(x)| = \left| \sum_{k=1}^n f_k(x) \right| \leq \sum_{k=1}^n |f_k(x)| = g_n(x) \leq g(x) \text{ for a.e. } x \in X$$

Therefore

$$|s_n - s|^p \leq 2^{p-1}(|s_n|^p + |s|^p) \leq 2^{p-1}(g^p + g^p) = 2^p g^p \in L^1(X)$$

By the dominated convergence theorem

$$\int_X |s_n - s|^p \, d\mu \to 0 \Leftrightarrow \|s_n - s\|_p \to 0$$

Thus L^p is complete.

$p = \infty$ exercise ★

To speak about separability, we give a

Definition 15.1
$g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. The support of g is

$$\operatorname{supp} g = \overline{\{x \in \Omega : g(x) \neq 0\}}$$

Also

$$\mathcal{C}_C^0 = \left\{ f \in \mathcal{C}^0(\Omega) : \operatorname{supp} f \text{ is compact in } \Omega \right\} = \mathcal{C}_O^0(\Omega) = \mathcal{C}_C(\Omega)$$

Theorem 15.4
$\Omega \in \mathcal{L}(\mathbb{R}), \lambda(\mathbb{R}) < +\infty$. Let also $f : \mathbb{R} \to \mathbb{R}$ measurable, s.t. $f \equiv 0$ in Ω^C.

Then $\forall\, \varepsilon > 0 \exists\, g \in \mathcal{C}_C^0(\mathbb{R})$ s.t.

$$\lambda\left(\{x \in \mathbb{R} : g(x) \neq f(x)\}\right) < \varepsilon$$

and

$$\sup_{\mathbb{R}} |g| \leq \sup_{\mathbb{R}} |f|$$

Definition 15.2
Given s simple function $= \sum_{k=1}^n a_k \chi_{E_k}$, where $E_1, \ldots, E_n$ are $\mathcal{L}$-measurable sets, $a_1, \ldots, a_n \in \mathbb{R}$.

$$E_1 \cup E_2 \cup \ldots \cup E_n = \mathbb{R}$$

We consider

$$\bar{\mathcal{S}}(\mathbb{R}) = \{s \text{ simple in } \mathbb{R} \text{ s.t. } \lambda((\{s \neq 0\}) < +\infty)\}$$

What does it mean for a simple function to be in $L^p(\mathbb{R})$?

$$\int_{\mathbb{R}} |s|^p \, d\mu = \sum_{k=1}^{n} a_k^p \lambda(E_k) < +\infty \qquad\qquad 1 \le p \le +\infty$$

iff $s \equiv 0$ outside a set of finite measure $\Leftrightarrow s \in \bar{\mathcal{S}}(\mathbb{R})$.

$\bar{\mathcal{S}}(\mathbb{R})$ is the set of integrable simple functions.

Theorem 15.5
$\bar{\mathcal{S}}(\mathbb{R})$ is dense in L^p, $\forall\, p \in (1, +\infty)$

Ma posso chiederti perchè hai deciso di scrivere s di r barrato e non tilde s di r?

Proof. $f \in L^p(\mathbb{R}), f \ge 0$ a.e. in $\mathbb{R}$.

We want to show that $\exists\ \{s_n\} \subseteq \bar{\mathcal{S}}(\mathbb{R})$ s.t. $\|s_n - f\|_p \to 0$.

By the simple approximation theorem, $\exists\ \{s_n\}$ of simple functions s.t. $\{s_n(x)\}$ is increasing, for every x, and $s_n \to f$ pointwise in $\mathbb{R}$.

Since $|s_n|^p \le f^p \Rightarrow s_n \in L^p$ for every n $\Rightarrow \{s_n\} \subseteq \bar{\mathcal{S}}(\mathbb{R})$. Moreover

$$|s_n - f|^p \to 0 \qquad \text{a.e. in } \mathbb{R}$$

$$|s_n - f|^p \le 2^{p-1}(|s_n|^p + |f|^p) \le 2^p |f|^p \in L^1$$

$\Rightarrow$ by dominated convergence

$$\int_{\mathbb{R}} |s_n - f|^p \, d\lambda \to 0 \text{ , namely } \|s_n - f\|_p \to 0$$

If f is sign changing, then $f = f^+ - f^-$ and argue as before on f^+ and f^- ★

Theorem 15.6
$\forall p \in [1, \infty)$, the space $L^p(\mathbb{R})$ is separable.

Proof. sketch

- Step 1: $\mathcal{C}_C^0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, $\forall \quad \le p \le \infty$.

  Take $s \in \tilde{\mathcal{S}}(\mathbb{R})$. Then, by Lusin theorem, $\exists \{f_n\} \subseteq \mathcal{C}_C^0(\mathbb{R})$ s.t. $\|f_n - s\|_p \to 0$. Then, since any $f \in L^p$ can be approximated by simple integrable functions, we have that f can be approximated by functions in $\mathcal{C}_C^0(\mathbb{R})$.

- Step 2:

  By Stone Weierstrass, the set of polinomials $\mathcal{P}(\mathbb{R})$ is dense in $\mathcal{C}_C^0(\mathbb{R})$ with the $\|\cdot\|_\infty$ norm. Since we work with functions with compact support, this implies that $\mathcal{P}(\mathbb{R})$ is dense in $\mathcal{C}_C^0(\mathbb{R})$ also with respect to $\|\cdot\|_p$

  $$\int_{-M}^{M} |f - p_n|^p \, d\lambda \le \|f - p_n\|_\infty^p 2M \to 0$$

  if $\|f - p_n\|_\infty \to 0$, $\Rightarrow \mathcal{P}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

  $\tilde{\mathcal{P}}(\mathbb{R}) = \{$ polynomials with rational coefficients $\}$. This is countable, and is dense in $(\mathcal{P}(\mathbb{R})), \|\cdot\|_p$. $\Rightarrow$ is dense in L^p

★

What about $L^\infty(\mathbb{R})$? In this case $\mathcal{C}(\mathbb{R})$ are not dense in $L^\infty(\mathbb{R})$. For example, consider

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

If $g \in L^\infty$ s.t. $\|g - f\|_\infty < \frac{1}{3}$, then g cannot be continuous. Assume by contradiction that $\exists\, g \in \mathcal{C}(\mathbb{R})$ s.t. $\|g - f\|_\infty < \frac{1}{3}$. Then

$$\operatorname{ess\,sup}_{\mathbb{R}} |g(x) - f(x)| < \frac{1}{3}$$

In particular, $g(x) < \frac{1}{3}\ \forall\, x < 0$

$$\Rightarrow \lim_{x \to 0^-} g(x) \leq \frac{1}{3}$$

On the other hand, $g(x) > \frac{2}{3} \quad \forall\, x > 0$

$$\Rightarrow g(0) = \lim_{x \to 0^+} g(x) \geq \frac{2}{3}$$

16 Lecture 10/11/2022

Quick recap about the 'delirium' on the separability

The thing that you need to know, in $\to L^p(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$, are:

(1) L^p is separable $\forall\, p \in [1, \infty)$

(2) $\tilde{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ $\forall\, p \in [1, \infty)$, namely $\forall p \in L^p(\mathbb{R})$ and $\forall\, \varepsilon > 0\ \exists\, s \in \tilde{S}(\mathbb{R})$ s.t.

$$\|f - s\|_p < \varepsilon$$

(3) $\mathcal{C}_C^0(\mathbb{R})$ is dense in L^p, namely $\forall p \in L^p(\mathbb{R})$ and $\forall\, \varepsilon > 0\ \exists\, g \in \mathcal{C}_C^0(\mathbb{R})$ s.t.

$$\|f - g\|_p < \varepsilon$$

Everything remains true if you replace $\mathbb{R}$ with X open or closed, or with $X \in L(\mathbb{R}^n)$, and consider $(X, L(X), \lambda)$.

What happens for $L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$?

$\mathcal{C}(\mathbb{R})$ is not dense in L^∞.

By the simple approximation theorem, we have that simple functions are dense in L^∞.

Theorem 16.1
$L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is not separable.

Proof. $\{\chi_{[-\alpha,\alpha]} : \alpha > 0\} \subseteq L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ $\chi_\alpha = \chi_{[-\alpha,\alpha]}$

This is an uncountable family of functions. $\|\chi_\alpha - \chi_{\alpha'}\|\ \forall\, \alpha \neq \alpha'$

$$|\chi_\alpha(x) - \chi_{\alpha'}(x)| = \begin{cases} 0 & if x \in [-\alpha, \alpha] \cup (\alpha', \infty) \cup (-\infty, -\alpha') \\ 1 & if x \in (\alpha, \alpha'] \cup [-\alpha', \alpha) \end{cases}$$

In particular, $B_{\frac{1}{2}}(\chi_\alpha) \cap B_{\frac{1}{2}}(\chi_{\alpha'}) = \varnothing\ \forall\, \alpha \neq \alpha'$

Assume by contradiction that $L^\infty(\mathbb{R})$ is separable: $\exists Z \subset L^\infty$ which is countable and dense. In particular, $\forall\, f \subset L^\infty\ \exists\, g \in Z$ s.t.

$$\|g - f\|_\infty < \frac{1}{2}$$

Therefore, $\forall\ \alpha\ \exists g_\alpha \in B_{\frac{1}{2}}(\chi_\alpha) \cap Z$. But $B_{\frac{1}{2}(\chi_\alpha) \cap B_{\{\frac{1}{2}(\chi_{\alpha'})}} = \varnothing$

$$\Rightarrow \alpha \neq \alpha', \text{ we have } g_\alpha \neq g_{\alpha'}$$

$Z \supseteq \{g_\alpha : \alpha > 0\}$, which is uncountable. This is not possible, since Z is countable. ★

Remark 16.1
The same is true if $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is swapped with $(X, \mathcal{L}(X), \lambda)$, X is open or closed on $\mathbb{R}$ or $\mathbb{R}^n$

Linear operators

$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ normed spaces.

Definition 16.1
$T : D(T) \subseteq X \to Y$ is a **linear operator** (or map) if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall\ x_1, x_2, \in D(T) \quad \forall\ \alpha_1, \alpha_2 \in \mathbb{R}$$

$D(T)$ is a linear subspace of X, and is called the domain of T. When $D(T) = X$ and $Y = \mathbb{R}$, T is called linear functional.

Definition 16.2
A linear operator $T : D(T) \subseteq X \to Y$ is bounded if $D(T) = X$ and $\exists\ M > 0$ s.t.

$$\|T_X\|_Y \leq M|x|_X \forall\ x \in X$$

Recall that T is continuous in $x_0 \in X$ iff

$$\forall\ \{x_n\} \subset X, x_n \xrightarrow{X} x_0 \Rightarrow Tx_n \xrightarrow{Y} Tx_0$$

<u>Ex</u>:

- $L : \mathbb{R}^n \to \mathbb{R}$ is a linear functional . Then $\exists\ y \in \mathbb{R}^n$ s.t.

  $$Lx =< y, x >= (y, x) = y \cdot x$$

  In particular, then L is continuous on $\mathbb{R}^n$ and bounded:

  $$|L_X| < |< y, x >| \overset{\text{Cauchy-Schwarz}}{\leq} \|y\|\|x\| \qquad \forall\ x \in \mathbb{R}^n$$

  So L is bounded with $M = \|y\|$.

- Linear operators in ∞-dim may not be defined everywhere, and many may not be continuous: $(X, \|\cdot\|_X) = (Y, \|\cdot\|_Y) = (\mathcal{C}([0, 1]), \|\cdot\|_\infty)$.
  Consider
  $$\begin{aligned} \frac{d}{dx} : \mathcal{C}'([0, 1]) \subseteq X \to Y \quad & \frac{d}{dx}(\alpha f + \beta g) = \alpha \frac{d}{dx} f + \beta \frac{d}{dx} g \\ f \mapsto f' \quad & \end{aligned}$$

  This is not continuous or bounded. For example, take $f_n(x) = \frac{1}{n} \sin 2\pi n x$. $\|f_n\|_\infty \to 0$ but $\|f_n'\|_\infty = 1$
  In this case $f_n \to 0 \not\Rightarrow \frac{d}{dx} f_n \to 0$, then $\frac{d}{dx}$ is not bounded as well.

- Let $(X, \|\cdot\|_X)$ be a normed space. If $\dim X = 0$, is it possible to find linear functionals which are not bounded? Yes.

Definition 16.3
A subset $\{qualcosa\}$ is called **Hamel basis** of X if

$$\|e_i\|_X = 1 \ \forall\, i$$

and if every $x \in X$ can be written in a unique way as

$$x = \sum_{k=1}^{n} x_k e_{i_k}, \quad x_k \in \mathbb{R},\ n \in \mathbb{N}$$

Every x can be written uniquely as a finite linear combination of element of the basis. If $\dim X = \infty$ is not immediate that the Hamel basis exists. This can be proved using the axiom of choice. (Zorn's lemma).

Any normed space has a Hamel basis $\dim X = \infty \Rightarrow \{e_i\}_{i\in I}$ has ∞ many elements.

Let then $(X, \|\cdot\|_X)$ be ∞ dim, with Hamel basis $\{e_i\}_{i\in I}$. I is infinite $\Rightarrow I \supseteq \mathbb{N}$.

We define $L : X \to \mathbb{R}$ in the following way

$$\begin{aligned} Le_0 &= 0 \qquad Le_1 = 1 \quad \dots \quad Le_n = n \quad \dots \\ Le_i &= 0 \quad \forall\, i \in I \setminus \mathbb{N} \end{aligned}$$

Then, for $x \in X$ we set

$$Lx = L\left(\sum_{k=1}^{n} x_k e_{i_k}\right) = \sum_{k=1}^{n} x_k L e_{i_k}$$

L is linear by contradiction, and it is not bounded:

$$\begin{gathered} |Le_n| = n \to \infty \quad \|e_n\|_X = 1\ \forall\, n \\ \frac{|Le_n|}{\|e_n\|_X} \to \infty \Rightarrow L \text{ L is not bounded} \end{gathered}$$

Remark 16.2
In practice, Hamel basis are hard to use. They differ from Hilbertian basis.

For linear operators, boundedness and continuity are equivalent.

Theorem 16.2
$T : X \to Y$ linear map. Then the following are equivalent

(1) T is continuous in $0 \in X$

(2) T is continuous everywhere in X

(3) T is bounded

Remark 16.3
T linear $\Rightarrow T0 = 0$. Indeed

$$T0 = T(0x) = 0Tx = 0$$

Proof.
- (2) $\Rightarrow$ (1) obvious.
- (1) $\Rightarrow$ (3) Suppose by contradiction that T is not bounded.

  Then $\exists \{x_n\} \subset X$, $x_n \neq 0$, s.t.

$$\frac{\|Tx_n\|_Y}{\|x_n\|_X} \geq n \quad \forall\, n$$

Define

$$z_n := \frac{x_n}{n\|x_n\|_X}$$

Then $\|z_n\|_X = \frac{1}{n\|x_n\|}\|x_n\|_X \to 0$, namely $z_n \to 0$ in $X \Rightarrow$ (T is continuous in 0) $Tz_n \to T0 = 0$. However,

$$\|Tz_n\|_Y = \left\| T\left(\frac{x_n}{n\|x_n\|_X}\right)\right\| = \frac{1}{n\|x_n\|_X}\|Tx_n\|_Y \geq 1 \;\forall\, n$$

Contradiction.

- (2) $\Rightarrow$ (2) We observe that $\|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y \leq M\|x_1 - x_2\|_X \;\forall\, x_1\; x_2 \in X$ Then, let $x \in X$ and let $x_n \to x$ in X: $\|x_n - x\|_X \to 0$. But then

$$\|Tx_n - Tx\|_Y \leq M\|x_n - x\|_X \to 0$$

namely $Tx_n \to Tx$ in Y. This is the continuity.

★

Definition 16.4
The set of linear operators $T : X \to Y$ which are also bounded (continuous) is denoted by $\mathcal{L}(X, Y)$

If $Y = X$, one simply writes $\mathcal{L}(X)$

This is a vector space. $\forall T, S \in \mathcal{L}(X, Y)$, $\forall\, \alpha, \beta \in \mathbb{R}$:

$$(\alpha T + \beta S)(x) = \alpha Tx + \beta Sx \qquad \in \mathcal{L}(X, Y)$$

We can also introduce a norm:

$$\|T\|_{\mathcal{L}(X,Y)} = \|T\|_{\mathcal{L}} := \sup_{\|x\|_X \leq 1} \|Tx\|_Y$$

Also,

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X = 1} \|Tx\|_Y = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \inf M > 0 \text{ s.t. } \|Tx\|_Y \leq M\|x\|_X \quad \forall\, x \in X$$

Theorem 16.3
X normed space, Y Banach space. Then $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ is a Banach space.

Proof. Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(X, Y)$. We want to show that $\exists T \in \mathcal{L}(X, Y)$ s.t.

$$\|T_n - T\|_{\mathcal{L}} \to 0$$

$\{T_n\}$ cauchy: $\forall\, \varepsilon > 0\; \exists \bar{n} \in \mathbb{N}$ s.t.

$$n, m > \bar{n} \Rightarrow \|T_n - T_m\|_{\mathcal{L}} < \varepsilon$$

Consider then $\{T_n x\}$, $x \in X$

$$\|T_n x - T_m x\|_Y = \|(T_n - T_m)x\|_Y \leq \|T_n - T_m\|_Y \|x\|_X \leq \varepsilon\|x\|_X \tag{*}$$

This means that $\{T_n x\}$ is a Cauchy sequence in Y, which is complete: then $\forall\, x \in X\; \exists$ a vector $y_x \in Y$ s.t. $T_n x \to y_x$ in Y.

Define

$$T : X \to Y \qquad x \mapsto y_x = Tx$$

T is linear: indeed, $\forall\, x_1,\, x_2 \in X$ and $\alpha_1,\, \alpha_2 \in \mathbb{R}$:

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n\to\infty} T_n(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n\to\infty} (\alpha_1 T_n x_1 + \alpha_2 T_n x_2) = \alpha_1 T x_1 + \alpha_2 T x_2$$

So T is linear. It remains to show that T is bounded, and that $\|T_n - T\|_{\mathcal{L}} \to 0$. To show that T is bounded, note that, by (*), $\forall\, \varepsilon > 0\ \exists\, \bar{n}$ s.t.

$$n, m > \bar{n} \Rightarrow \|T_n x - T_m x\|_Y \leq \varepsilon \|x\|_X \quad \forall\, x$$

Take the limit for $m \to \infty$:

$$\|T_n x - T x\|_Y \leq \varepsilon \|x\|_X$$

But then, since T_n is bounded,

$$\|Tx\|_Y = \|Tx \pm T_n x\|_Y \leq \|T_n x\|_Y + \|Tx - T_n x\|_Y \leq M_n \|x\|_X + \varepsilon \|x\|_X = (M_n + \varepsilon)\|x\|_X$$

and T is bounded. To show that $\|T_n - T\|_{\mathcal{L}} \to 0$, observe that $\forall\, \varepsilon > 0\ \exists\, \bar{n}$ s.t. $n > \bar{n}$

$$\|T_n x - T x\|_Y \leq \varepsilon \|x\|_X \Leftrightarrow \frac{\|(T_n - T)x\|_Y}{\|x\|_X} \leq \varepsilon \quad \forall\, x \in X \setminus 0 \overset{\text{take sup over } x \neq 0}{\Rightarrow} \|T_n - T\|_{\mathcal{L}} < \varepsilon$$

namely, $T_n \to T$ in $\mathcal{L}$ ★

17 Lecture 16/11/2022

Let T be a linear operator from X to Y.

Definition 17.1
The **kernel** of T is the set

$$\ker(T) = \{x \in X : Tx = 0\} \subset X$$

This is a vector subspace of X.

T is injective $\Leftrightarrow \ker(T) = \{0\}$. If T is continuous, $\ker(T)$ is closed

$$\ker(T) = T^{-1}(\{0\})$$

Definition 17.2
X, Y normed spaces. X and Y are isomorphic if $\exists\ T \in \mathcal{L}(X, Y)$ bijective, and such that $T^{-1} \in \mathcal{L}(X, Y)$

Definition 17.3
$T \in \mathcal{L}(X, Y)$ is an isometry if

$$\|Tx\|_Y = \|x\|_X \quad \forall\, x \in X$$

Definition 17.4
If $X \subseteq Y$ is a vector subspace, and $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed space, then we can consider

$$\begin{aligned} J:\ \ & X \to Y \quad \text{(inclusion map)} \\ & x \mapsto x \end{aligned}$$

If $J \in \mathcal{L}(X, Y)$ (namely, if $\exists\, M > 0$ s.t. $\|x\|_Y \leq M \|x\|_X\ \forall\, x \in X$), then we say that J is an embedding of X into Y, and we write $X \hookrightarrow Y$

Ex: $\mu(X) < \infty$, $1 \leq p < q \leq \infty$

$$L^q(X) \hookrightarrow L^p(X) \qquad \text{(inclusion of } L^p \text{ spaces)}$$

Some fundamentals theorems on linear operators

Definition 17.5
(X, d) metric space. $A \subset X$. $x \in X$ is an **adherence point** of A if $\forall\, r > 0 : B_r(x) \cap A \neq \varnothing$

$$\bar{A} = \{x \in X : x \text{ is an adherence point of } A\} = A \cup \partial A$$

Definition 17.6
$A \subset X$ is dense in X if $\bar{A} = X$.

For example, $\mathbb{Q}$ is dense in $\mathbb{R}$, and (a, b) is dense in $[a, b]$.

Definition 17.7
$A \subset X$ is nowhere dense if the interior of the closure of A is empty, namely

$$\operatorname{int}(\bar{A}) = \bar{A}^{\mathrm{o}} = \varnothing$$

Ex: $\bar{\{x\}}^{\mathrm{o}} = \{x\}^{\mathrm{o}} = \varnothing$
$\mathbb{Z} \subset \mathbb{R}$: $\bar{\mathbb{Z}}^{\mathrm{o}} = \mathbb{Z}^{\mathrm{o}} = \varnothing$
$\mathbb{Q}$ is not nowhere dense: $(\bar{\mathbb{Q}})^{\mathrm{o}} = (\mathbb{R})^{\mathrm{o}} = \mathbb{R}$

Definition 17.8
$A \subset X$ is called **of first category** (or **meager set**) in X if A is the (at most) countable union of nowhere dense sets.

Ex: $\mathbb{Q}$ is of first category in $\mathbb{R}$: countable union of nowhere dense sets

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

Definition 17.9
$A \subset X$ is of second category if it is not of first category.

Theorem 17.1 (Baire category theory)
(X, d) complete metric space. Then

- $\{U_n\}_{n=0}^{\infty}$ is a sequence of open and dense sets in $X \Rightarrow \cap_{n=0}^{\infty} U_n$ is dense in X.
- X is of second category in itself: X cannot be the countable union of nowhere dense sets.

Preliminaries:

- $A \subset X$ is dense $\Leftrightarrow \forall\, W \subset X$, W open, $W \neq \varnothing$, we have that $A \cap W \neq \varnothing$
- A is nowhere dense $\Leftrightarrow \left(\bar{A}\right)^C$ is open and dense

Proof. Here's the proof of the two parts of the theorem:

(a) Thanks to the first preliminary, we show that $\forall W \subset X$ open and non empty we have $(\cap_n U_n) \cap W \neq \varnothing$

$$U_0 \text{ is open and dense: } \overset{1^{st}\text{prel.}}{\Rightarrow} \underbrace{U_0 \cap W}_{\text{is open}} \neq \varnothing$$
$$\Rightarrow \text{it contains an open ball}$$
$$\Rightarrow (U_0 \cap W) \supset B_{r_0}(x_0) \text{ for some } x_0 \in X \text{ and } r_0 > 0$$

For $n > 0$, we choose $x_n \in X$ and $r_n > 0$ inductively in the following way: we have

$$U_n \cap B_{r_{n-1}}(x_{n-1}) \neq \varnothing \qquad (1^{st} \text{ prel. } + U_n \text{ is dense})$$

$$\Rightarrow \overline{B_{r_n}(x_n)} \subset (U_n \cap B_{r_{n-1}}(x_{n-1}))$$

all these balls are included in $B_{r_0}(x_0)$

with $x_n \in X$ and $0 < r_n < \frac{1}{2^n}$

By the condition on r_n, we see that

$$x_n, x_m \in B_{r_N}(x_N) \quad \forall\, n, m > N$$

$\Rightarrow \{x_n\}$ is a Cauchy sequence in X

$$d(x_n, x_m) \leq \frac{1}{2^N} \quad \forall\, n, m > N$$

X is complete: $x_n \xrightarrow{d} x \in X$ Since

$$\begin{aligned}
&x_n \in B_{r_N}(x_N) && \forall\, n > N \\
&\Rightarrow x = \lim_n x_n \in \overline{B_{r_N}(x_N)} \subset (U_n \cap B_{r_0}(x_0)) \subset (U_N \cap W) && \forall\, n \in \mathbb{N} \\
&\Rightarrow x_n \in \bigcap_n (U_n \cap W) = \left(\bigcap_n U_n\right) \cap W
\end{aligned}$$

This means that $\bigcap_n U_n$ is dense.

(b) It follows from (a):

If $\{E_n\}$ is a sequence of nowhere dense sets in X, then, by the second preliminary $\{(E_n)^C\}$ is a sequence of open and dense sets. By (a)

$$\bigcap_n (\overline{E_n})^C \neq \varnothing$$

$$\Rightarrow \bigcup_n E_n \subset \bigcup_n \overline{E_n} = X \setminus \overset{=\varnothing}{\left(\bigcap_n (\overline{E_n})^C\right)} \neq X$$

★

Ex: $(X, \|\cdot\|)$ $\infty-$dim Banach space. $\{e_i\}_{i \in I}$ Hamel basis.
Then I is uncountable.

Theorem 17.2 (Banach Steinhaus)
X Banach space, Y normed space, $\mathcal{F} \subseteq \mathcal{L}(X, Y)$ family. Suppose that $\mathcal{F}$ is pointwise bounded:

$$\forall\, x \in X \quad \exists\, M_x > 0 \text{ s.t. } \sup_{T \in \mathcal{F}} \|Tx\|_Y \leq M_x \tag{PB}$$

Then $\mathcal{F}$ is uniformly bounded:

$$\exists\, M \geq 0 \text{ s.t. } \sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}(X,Y)} \leq M \tag{UB}$$

Proof. $\forall\, n \in \mathbb{N}$, let

$$C_n := \{x \in X : \|Tx\|_Y \leq n \quad \forall\, T \in \mathcal{F}\} = \cap_{T\in\mathcal{F}}\{x \in X : \|Tx\|_Y \leq n\}$$

C_n is a closed set $\forall\, n$, since T is continuous. (also $\varphi : X \to \mathbb{R}$ $\varphi(x) = \|Tx\|_Y$ is continuous)

By (PB), every $x \in X$ stays in some C_n: $X = \cup_{n=1}^{\infty} C_n$. Since X is Banach, by the Baire theorem it is necessary that $\exists n_0 \in \mathbb{N}$ s.t. ${C_{n_0}}^{\circ} \neq \varnothing \Rightarrow$ a ball $\bar{B_r}(x_0) \subset C_{n_0}$: then

$$n_0 \geq \|T(x_0 + rz)\|_Y \overset{\text{linearity}}{=} \|Tx_0 + rTz\|_Y \overset{\text{triangle ineq}}{\leq} r\|Tz\|_Y - \|Tx_0\|_Y \quad \forall\, T \in \mathcal{F}\, \forall\, z \in \bar{B_1}(0)$$

To sum up: $\forall\, T \in \mathcal{F}$, $\forall\, z \in \bar{B_1}(0)$ we have

$$r\|Tz\|_Y - \|Tx_0\|_Y \leq n_0 \Rightarrow \|Tz\|_Y \leq \frac{1}{r}(n_0 + M_{x_0})$$

We take sup over $T \in \mathcal{F}$:

$$\sup_{T\in\mathcal{F}} \|T\|_{\mathcal{L}(X,Y)} \leq \frac{1}{r}\,(n_0 + M_{x_0}) =: M$$

★

Corollary 17.1
X Banach space, Y normed space. $\{T_n\} \subseteq \mathcal{L}(X,Y)$ s.t. $\{T_n x\}$ have a limit, denoted by Tx, $\forall\, x \in X$ (pointwise convergence). Then $T \in \mathcal{L}(X,Y)$

Proof. T is linear:

$$\begin{array}{ccc} T_n(\alpha_1 x_1 + \alpha_2 x_2) & = & \alpha_1 T_n x_1 + \alpha_2 T_n x_2 \\ \downarrow & & \downarrow \\ T(\alpha_1 x_1 + \alpha_2 x_2) & = & \alpha_1 T x_1 + \alpha_2 T x_2 \end{array}$$

Now we observe that we have (PB): if $\{T_n x\}$ is convergent $\Rightarrow \{T_n x\}$ is bounded $\Rightarrow$ by Banach Steinhaus, $\{T_n\}$ is uniformly bounded:

$$\exists M > 0 \text{ s.t. } \sup_n \|T_n\|_{\mathcal{L}(X,Y)} \leq M$$

Therefore, $\forall\, x \in X$:

$$\|Tx\|_Y = \left\|\lim_n (T_n x)\right\|_Y = \lim_n \|T_n x\|_Y \leq \lim_n \|T_n\|_{\mathcal{L}} \|x\|_X \leq \lim_n M\|x\|_X = M\|x\|_X$$

Thus, T is bounded: $T \in \mathcal{L}(X,Y)$ ★

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Let X, Y be normed spaces.

Definition 18.1
$T : X \to Y$ is called **open map** if, $\forall\, A \subset X$ open ,the set $T(A) \subset Y$ is open.

Remark 18.1
Recall that T is continuous on X if $T^{-1}(O)$ is open on X, $\forall\, O$ open in Y.

<u>Ex</u>: $f(x)$: constant is continuous, but not open. $f((a,b)) = \{\text{const}\}$

Theorem 18.1 (Open map theorem)
X, Y Banach spaces. $T \in \mathcal{L}(X,Y)$ is surjective. Then T is an open map.

Corollary 18.1
X, Y Banach spaces, $T \in \mathcal{L}(X,Y)$ is bijective. Then T is an isomprphism: $T^{-1} \in \mathcal{L}(X,Y)$

Proof. • $T : Y \to X$ is linear. (Exercise. Hint: Use $T^{-1} \circ T = \text{Id}+$ linearity of T)

- We want now to check that T^{-1} is continuous on Y: $(T^{-1})^{-1}(O)$ is open in Y, $\forall\, O$ open in X. We know that T is an open map thanks to the open map theorem.

$$(T^{-1})^{-1}(O) = \big\{y \in Y, T^{-1}(y) \in O\big\} = \big\{y \in Y, T^{-1}(y) = x, \text{ for some } x \in O\big\} =$$
$$= \{y \in Y, y = Tx, \text{ for some } x \in O\} = T(O) \text{ is open}$$

Since T is an open map, $\forall\, O \subset X$, open.

★

Corollary 18.2
X vector space, $\|\cdot\|, \|\cdot\|_*$ norms on X. Assume $(X, \|\cdot\|), (X, \|\cdot\|_*)$ are Banach spaces. Assume that $\exists\, C_1 > 0$ s.t.

$$\|x\|_* \leq C_1 \|x\| \quad \forall; x \in X$$

Then $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, namely $\exists\, C_2 > 0$ s.t.

$$\|x\| \leq C_2 \|x\|_*$$

Proof. Consider

$$I : \begin{array}{cl} (X, \|\cdot\|) & \to (X, \|\cdot\|_*) \\ x & \mapsto x \end{array}$$

By assumption, I is bounded: $\exists\, C_1 > 0$ s.t.

$$\|Ix\|_* = \|x\|_* \leq C_2 \|x\|$$

I is bijective.

Thus, by the corollary before

$$I^{-1} = I \in \mathcal{L}((X, \|\cdot\|_*), (X, \|\cdot\|))$$

namely $\exists\, C_2 > 0$ s.t.

$$\underset{\substack{\shortparallel \\ \|x\|}}{\|Ix\|} \leq C_2 \|x\|_*$$

★

Definition 18.2
$T : D(T) \subset X \to Y$ linear operator. We say that T is **closed** if $\forall\ \{x_n\} \subset D(t)$.

$$\left.\begin{array}{ll} x_n \to x & \text{in X} \\ Tx_n \to y & \text{in Y} \end{array}\right\} \Rightarrow x \in D(T) \text{ and } Tx = y$$

<u>Ex</u>: $X = Y = \mathcal{C}^0([0,1])$ with the supremum norm.

$$T = \frac{d}{dx}$$

T is not continuous. But it is closed: it can be proved that if $\{f_n\} \subset \mathcal{C}^1([0,1])$ is s.t.

$$\left.\begin{array}{ll} f_n \to f & \text{uniformly} \\ f_n' \to g & \text{uniformly} \end{array}\right\} \Rightarrow f \text{ is } \mathcal{C}^1([0,1]) \text{ and } f' = g$$

<u>Ex</u>: $T \in \mathcal{L}(X,Y) \Rightarrow T$ is closed

Remark 18.2
T is a closed operator $\Leftrightarrow$ the graph of T

$$\text{graph}(T) = \{(x, Tx) : x \in X\} \text{ is closed}$$

Theorem 18.2 (Closed graph theorem)
X, Y Banach spaces.
$T : X \to Y$ linear closed operator ($D(T) = X$).
Then $T \in \mathcal{L}(X, Y)$.

Remark 18.3
In general it is easier to prove that an operator is closed, rather than it it continuous.

***Proof*.** Define on X the graph-norm of T

$$\|x\|_* = \|x\|_X + \|Tx\|_Y$$

Then is a norm on X. If $\{x_n\} \in X$ is a Cauchy sequence for $\|\cdot\|_*$, then $\{x_n\}$ is a Cauchy sequence in $(X, \|\cdot\|_X)$ and $\{Tx_n\}$ is a Cauchy sequence on $(Y, \|\cdot\|_Y)$

$$\Rightarrow \left.\begin{array}{ll} x_n \to x & \text{in X} \\ Tx_n \to y & \text{in Y} \end{array}\right\} \text{ since } T \text{ is closed, we deduce that } y = Tx$$

Thus

$$\|x_n - x\|_X + \|Tx_n - Tx\| \to 0$$

This proves that $(X, \|\cdot\|_*)$ is a Banach space. Also, we know that

$$\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_*$$

By the last corollary of the open map theorem, $\exists\ C_2$ s.t.

$$\|x\|_* \leq C_2 \|x_X\|$$

$$\|Tx\|_Y \leq \|x\|_* \leq C_2 \|x\|_X \quad \forall\, x \in X$$

This means that T is bounded. ★

Dual spaces

X normed space:

$X^* = \mathcal{L}(X, \mathbb{R})$ is called **dual space of** X

X normed space, Y Banach space $\Rightarrow \mathcal{L}(X, Y)$ is a Banach space with $\|\cdot\|_{\mathcal{L}}$.
Since $\mathbb{R}$ is a Banach space, the dual space X^* is a Banach space with

$$\|L\|_* = \sup_{\|x\|_X \leq 1} |Lx|$$

<u>Ex</u>:

- In $\mathbb{R}^n$, only linear functional is separated by a scalar product:

  $$L : \mathbb{R}^n \to \mathbb{R} \text{ is linear } \Rightarrow \exists!\ y \in \mathbb{R}^n \text{ s.t. } Lx =< y, x >$$

  It can be proved that

  $$L \subset (\mathbb{R}^n)^* \mapsto y \in \mathbb{R}^n$$

  is an isometric isomorphism

  $$(\mathbb{R}^n)^* \approxeq$$

  Then X^* is very complicated.

- mille cose

...

Proposition 18.1
If $p \in [1, \infty]$ then $L_g \in (L^p(X)^*)$. Moreover,

- if $p > 1$, then $\|L_g\|_* = \|g\|_{p'}$
- if $p = 1$ then $\|L_g\|_* = \|g\|_\infty$ with more assumptions (they are satisfied in $(X, \mathcal{L}(X), \lambda)$)

Remark 18.4
We are saying that $L^{p'}$ can be identified with a subspace of the dual space $(L^p)^*$ and this identification is an isometry.

Question: are there functional in $(L^p)^*$?

Proof. (of the proposition)

- Case $p = \infty$ ex
- Case $p = 1$ but difficult it's ok if you don't do it
- Case $p \in (1, \infty)$
  L_g is clearly linear, by linearity of $\int$, indeed: $\forall\ \alpha\ \beta \in \mathbb{R}$, $f_1\ f_2 \in L^p(X)$. Then

  ...

  We want to show now that L_g is bounded. We proved in (*) that

  $$|L_g f| \leq \|g\|_{p'} \|f\|_p \quad \forall\ f \in L^p(\Omega)$$

  This shows that L_g is bounded, with norm $\|L_g\|_* \leq \|g\|_{p'}$ (remember that $\|T\|_{\mathcal{L}} = \inf\{M > 0 : \|Tx\|_Y \leq M\|x\|_X \quad \forall x \in X\}$)
  We want to show that $\|L_g\|_* = \|g\|_{p'}$. If $\|L_g\|_* < \|g\|_{p'}$, then $\exists\ M < \|g\|_{p'}$ s.t.

  $$|L_g f| \leq M\|f\|_p \quad \forall\ f \in L^p$$

  We rule out this possibility by choosing an explicit $\tilde{f} \in L^p$ s.t.

  $$\left|L_g \tilde{f}\right| = \|g\|_{p'} \left\|\tilde{f}\right\|_*$$

  We take

  .

  Now,

  $$\left\|\tilde{f}\right\|_p^p = \int_X \left|\tilde{f}\right|^p d\mu = \int_X \frac{|g|^{p(p'-1)}}{\|g\|_p^{p(p'-1)}}\, d\mu = (*)$$

  $(p')' = p \Rightarrow p = \frac{p'}{p'-1} \Rightarrow p(p'-1) = p'$

  $$(*) = \frac{1}{\|g\|_{p'}^{p'}} \int_X |g|^{p'}\, d\mu = \frac{\|g\|_{p'}^{p'}}{\|g\|_{p'}^{p'}} = 1$$

  $$\left|L_g \tilde{f}\right| = \left|\int_X\right| \frac{|g|}{\|g\|_{p'-1}^{p'}} \cdots\cdots = \frac{1}{\|g\|_{p'-1}^{p'}} \|g\|_{p'}^{p'} = \|g\|_{p'} = \|g\|_{p'} \left\|\tilde{f}\right\|_p$$

★

Hahn Banach

Definition 18.3
X vector space. A map $p : X \to \mathbb{R}$ is called sublinear functional if

- $p(\alpha x) = \alpha p(x) \qquad \forall\, x \in X,\ \alpha > 0$
- $p(x + y) \leq p(x) + p(y) \qquad \forall x, y \in X$

Theorem 18.3 (Hahn Banach)
X real vector space, $p : X \to \mathbb{R}$ sublinear functional. Y subspace of X and suppose that $\exists\, f : Y \to \mathbb{R}$ linear on Y s.t.

$$f(y) \leq p(y) \quad \forall\, y \in Y$$

Then $\exists$ a linear functional $F : X \to \mathbb{R}$ s.t.

$$F(y) = f(y) \quad \forall\, y \in Y \qquad \qquad F \text{ is an extension of } f$$

Moreover,

$$F(x) \leq p(x) \quad \forall\, x \in X$$

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