Notes from Real and Functional Analysis

Andrea Bonifacio

October 6, 2022

1 Lesson 12/09/2022

Element of set theory

Let X be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \qquad \textbf{(Power Set)}$$

Let $I \subseteq \mathbb{R}$ be a set of indexes. A family of sets induced by I is

$$\{E_i\}_{i\in I}\,, \quad E_i \subseteq X \qquad \textbf{(Family/Collection)}$$

If $I = \mathbb{N}$ is called a

$$\{E_n\}_{n\in\mathbb{N}} \qquad \textbf{(Sequence)}$$

Definition 1.1
$\{E_n\} \subseteq \mathcal{P}(X)$ is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1}\, \forall n \qquad (\text{resp. } E_n \supseteq E_{n+1}\, \forall n)$$

and is written as

$$\{E_n\} \nearrow \qquad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets $\{E_i\}_{i\in I} \subseteq \mathcal{P}(X)$, will be often considered

$$\bigcup_{i\in I} E_i = \{x \in X : \exists\, i \in I\, s.t.\, x \in E_i\}$$

$$\bigcap_{i\in I} E_i = \{x \in X : x \in E_i,\, \forall i \in I\}$$

$\{E_i\}$ is said to be **disjoint** if $E_i \cap E_j = \varnothing\ \forall i \neq j$.

Examples:

$$[a,b] = \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$$

$$(a,b) = \bigcup_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$$

Definition 1.2
$\{E_n\} \subseteq \mathcal{P}(X)$. We define

$$\limsup_n E_n := \bigcap_{k=1}^{\infty}(\bigcup_{n=k}^{\infty} E_n)$$

$$\liminf_n E_n := \bigcup_{k=1}^{\infty}(\bigcap_{n=k}^{\infty} E_n)$$

If these two sets are equal, then

$$\lim_n E_n = \limsup_n E_n = \liminf_n E_n$$

Proposition 1.1
Some limits are:

- $\limsup_n E_n = \{x \in X:\ x \in E_n \text{ for } \infty - \text{many indexes } n\}$
- $\liminf_n E_n = \{x \in X:\ x \in E_n \text{ for all but finitely many indexes } n\}$

- $\liminf_n E_n \subseteq \limsup_n E_n$
- $(\liminf_n E_n)^C = \limsup_n E_n^C$

Definition 1.3

$$\begin{aligned} x \in \limsup_n E_n \quad &\iff \quad x \in \bigcap_{k=1}^{\infty}\left(\bigcup_{n=k}^{\infty} E_n\right) \\ &\iff \quad \forall k \in \mathbb{N} \,:\, \bigcup_{n=k}^{\infty} E_n \\ &\iff \quad \forall k \in \mathbb{N}\, \exists n_k \geq k \, s.t.\, x \in E_{n_k} \end{aligned}$$

So $x \in \limsup_n E_n \implies \exists m_1 = n_1 \, s.t.\, x \in E_{n_1}$

$$\begin{aligned} &\exists m_2 := n_{m_1+1} \geq m_1 + 1 \, s.t.\, x \in E_{n_2} \\ &\vdots \\ &\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1 \, s.t.\, x \in E_{n_k} \\ &\vdots \\ &x \in E_{m_1}, \dots, E_{m_k}, \dots \end{aligned}$$

On the other hand, assume that $x \in E_n$ for ∞-many indexes. We claim that $\forall k \in \mathbb{N}\, \exists n_k \geq k \, s.t.\, x \in E_{n_k} \iff x \in \limsup_n E_n$. If that claim is not true, then $\exists \bar{k} \, s.t.\, x \notin E_n \, \forall n > \bar{k} \implies x$ belongs at most to $E_1, \dots, E_{\bar{k}}$, a contradiction. ★

Definition 1.4
$\{E_i\}_{i \in I}$ is a **covering** of X if

$$X \subseteq \bigcup_{i \in I} E_i$$

A subfamily of E_i that is still a covering is called a **subcovering**

Definition 1.5
Let $E \subseteq X$. The function $\chi_E : X \to \mathbb{R}$

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \backslash E \end{cases}$$

is called **characteristic function** of E

Let E_1, E_2 be sets:

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

$$\{E_n\} \subseteq \mathcal{P}(X), \text{ disjoint}, E = \bigcup_{n=1}^{\infty} E_n \Longrightarrow \mathcal{X}_{\mathcal{E}} = \sum_{n=1}^{\infty} \chi_{E_n}$$

$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \Longrightarrow \chi_P = \liminf \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that $\limsup_n a_n = \lim_{k \to \infty} \sup_{n \geq k} a_n$ and $\liminf_n a_n = \lim_{k \to \infty} \inf_{n \geq k} a_n$

Let's also check that $\chi_Q = \limsup_n \chi_{E_n}$

$$\begin{aligned} x \in \limsup_n E_n \quad &\iff \quad \chi_Q(x) = 1 \\ &\iff \quad \forall\, k \in \mathbb{N}\, \exists\, n_k \geq k \; s.t. \; x \in E_{n_k} \end{aligned}$$

If we fix k then

$$\sup_{n\geq k}\chi_{E_n}(x)=\chi_{E_{n_k}}(x)=1$$
$$\lim_{k\to\infty}\sup_{n\geq k}\chi_{E_n}(x)=\limsup_{n}\chi_{E_n}(x)=1$$

Let now $x\notin\limsup E_n \iff \chi_Q(x)=0$. Then x belongs at most to finitely many $E_n \implies \exists\,\bar{k}$ *s.t.* $x\notin E_n,\forall n\geq\bar{k}$

If $k\geq\bar{k}$, then $\sup_{n\geq k}\chi_{E_n}(x)=0\implies\lim_{k\to\infty}\sup_{n\geq k}\chi_{E_n}(x)=\limsup_n\chi_{E_n}(x)=0$

Relations

Given X,Y sets, is called a **relation** of X and Y a subset of $X\times Y$

$$R\subseteq X+Y\quad R=\{(x,y)\,:\,x\in X,y\in Y\}$$

$$(x,y)\in R\iff xRy$$

$$X=\{0,1,2,3\}\quad R=\{(0,1),(1,2),(2,1)\}\text{ is a relation in }X$$

Definition 1.6
A **function** from X to Y is a relation R s.t. for any element x of X $\exists!$ element y of Y s.t. xRy

Definition 1.7
R on X is an **equivalence relation** if

(1) $xRx\ \forall\, x\in X$ (R is **reflexive**)

(2) $xRy\implies yRx$ (R is **symmetric**)

(3) $xRy, yRz\implies xRz$ (R is **transitive**)

If R is an equivalence relation, the set $E_X:=\{y\in X\,:\,yRx\}$, $x\in X$ is called the **equivalence class** of X

Definition 1.8
$\frac{X}{R}:=\{E_X\,:\,x\in X\}$ is the **quotient set**

Ex: $X=\mathbb{Z}$, let's say that nRm if $n-m$ is even. This is an equivalence relation.

$$E_n=\{\ldots,n-4,n-2,n,n+2,n+4,\ldots\}$$

in this case if n is even, $E_n=\{\text{even numbers}\}$ and if n is odd, $E_n=\{\text{odd numbers}\}$

Measure theory

Definition 1.9
A family $\mathcal{M}\subseteq\mathcal{P}(X)$ is called a **σ-algebra** if

(1) $X\in\mathcal{M}$

(2) $E\in\mathcal{M}\implies E^C=X\backslash E\in\mathcal{M}$

(3) If $E=\bigcup_{n\in\mathbb{N}}$ and $E_n\in\mathcal{M}\ \forall\, n$, then $E\in\mathcal{M}$

If $\mathcal{M}$ is a σ-algebra, $(X,\mathcal{M})$ is called **measurable space** and the sets in $\mathcal{M}$ are called **measurable**. Ex:

- $(X,\mathcal{P}(X))$ is a measurable space

- Let X be a sets, then $\{\varnothing, X\}$ is a σ-algebra

Remark 2
σ is often used to denote the closure w.r.t. countably many operators. If we replace the countable unions with finite unions in the definition of σ-algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space $(X, \mathcal{M})$:

(1) $\varnothing \in \mathcal{M}$: $\varnothing = X^C$ and $X \in \mathcal{M}$

(2) $\mathcal{M}$ is an algebra, and $E_1, \ldots, E_n \in \mathcal{M}$

$$E_1 \cup \ldots \cup E_n = E_1 \cup \ldots \cup E_n \cup \underbrace{\varnothing}_{\in \mathcal{M}} \cup \varnothing \ldots \in \mathcal{M}$$

(3) $E_n \in \mathcal{M}$, $\bigcap_{n \in \mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n \in \mathbb{N}} E_n = \underbrace{\left(\bigcup_{n \in \mathbb{N}} \underbrace{E_n^C}_{\in \mathcal{M}} \right)}_{\in \mathcal{M}}^C \qquad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \implies E \backslash F \in \mathcal{M} = E \backslash F = E \cap F^C \in \mathcal{M}$
- If $\Omega \subset X$, then the **restriction** of $\mathcal{M}$ to Ω, written as

$$\mathcal{M}|_\Omega := \{F \subseteq \Omega : F = E \cap \Omega, \text{ with } E \in \mathcal{M}\}$$

is a σ-algebra on Ω

Theorem 2.1
$\mathcal{S} \subseteq \mathcal{P}(X)$. Then it is well defined the smallest σ-algebra containing $\mathcal{S}$, the σ-algebra generated by $\mathcal{S} := \sigma_0(\mathcal{S})$:

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$ and thus is a σ-algebra
- $\forall \sigma(\mathcal{M})$ s.t. $\mathcal{M} \supseteq \mathcal{S}$, we have $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

Proof idea.

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \varnothing \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} \ : \ \mathcal{M} \in \mathcal{V}\}$, so it can be proved that this is the desired σ-algebra ★

Borel sets

Given (X, d) metric space, the σ-algebra generated by the open sets is called **Borel** σ-algebra, written as $\mathcal{B}(X)$. The sets in $\mathcal{B}(X)$ are called **Borel sets**. The following sets are Borel sets:

- open sets
- closed sets
- countable intersections of open sets: G_σ sets
- countable unions of closed sets: F_σ sets

Remark 3
$\mathcal{B}(\mathbb{R})$ can be equivalently defined as the σ-algebra generated by

$$\{(a,b):\ a,b\in\mathcal{R}, a<b\}$$

$$\{(-\infty,b):\ b\in\mathcal{R}\}$$
$$\{(a,+\infty):\ a\in\mathcal{R}\}$$
$$\{[a,b):\ a,b\in\mathcal{R}, a<b\}$$
$$\vdots$$

4 Lesson 14/09/2022

Question: What is $\mathcal{B}(\mathbb{R})$? Is $\mathcal{B}(\mathbb{R})\neq\mathcal{P}(\mathbb{R})$? No.

Definition 4.1
$(X,\mathcal{M})$ measurable space. A function $\mu:\ \mathcal{M}\to[0,+\infty]$ is called a **positive measure** if $\mu(\varnothing)=0$ and if μ is countably additive, that is

$$\forall\,\{E_n\}\subseteq\mathcal{M}\quad\text{disjoint}$$

we have that

$$\mu\left(\bigcup_{n=1}^{\infty}\right)=\sum_{n=1}^{\infty}\mu(E_n)\qquad\sigma\text{-additivity}$$

Remark 5
a set A is countable if $\exists\ f$ $1-1$ s.t. $f:A\to\mathbb{N}$ Examples: $\mathbb{Z},\mathbb{Q}$ are countable, while $\mathbb{R}$ is not, also $(0,1)$ is uncountable.

We always assume that $\exists\ E\neq\varnothing, E\in\mathcal{M}$ s.t. $\mu(E)\neq\infty$.

If $(X,\mathcal{M})$ is a measurable space, and μ is a measure on it, then $(X,\mathcal{M},\mu)$ is a measure space.

Then:

(1) μ is **finitely additive**:

$$\forall\ E,F\in\mathcal{M},\ \text{with}\ E\cap F\neq\varnothing\implies\mu(E\cup F)=\mu(E)+\mu(F)$$

(2) the **excision property**

$$\forall\ E,f\in\mathcal{M},\ \text{with}\ E\subset F\ \text{and}\ \mu(E)<+\infty\implies\mu(F\backslash E)=\mu(F)-\mu(E)$$

(3) **monotonicity**

$$\forall\ E,F\in\mathcal{M},\ \text{with}\ E\subset F\implies\mu(E)\leq\mu(F)$$

(4) if $\Omega\in\mathcal{M}$ then $(\Omega,\mathcal{M}|_\Omega,\mu|_{\mathcal{M}|_\Omega})$ is a measure space

Proof. (1) $E_1=E,E_2=F,E_3=\ldots=E_n=\ldots=\varnothing$ This is a disjoint sequence $\implies$ by σ-additivity.

$$\mu(E\cup F)=\mu\left(\bigcup_n E_n\right)=\sum_n\mu(E_n)=\mu(E)+\mu(F)+\underbrace{\mu(E_k)}_{=\mu(\varnothing)}$$

(2) $E \subset F$, so $F = E \cup (F \backslash E)$ and this is disjoint $\overset{(i)}{\Longrightarrow} \mu(F) = \mu(E) + \mu(F \backslash E)$, and since $\mu(E) < \infty$, the property follows.

(3) $E \subset F \Longrightarrow \mu(F) = \mu(E) + \underbrace{\mu(F \backslash E)}_{\geq 0} \geq \mu(E)$

(4)

★

Definition 5.1
$(X, \mathcal{M}, \mu)$ measure space.

- If $\mu(X) < +\infty$, we say that μ is **finite**.
- If $\mu(X) = +\infty$, and $\exists\ \{E_n\} \subset \mathcal{M}$ s.t. $X = \bigcup_n E_n$ and each E_n has finite measure, then we say that μ is σ-finite.
- If $\mu(X) = 1$ we say that μ is a **probability measure**.

Some examples:

- Trivial Measure: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(E) = 0 \quad \forall\ E \in \mathcal{M}$
- Counting Measure: $(X, \mathcal{P}(X))$ measurable space. We define

$$\mu_C : \mathcal{P}(X) \to [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure: $(X, \mathcal{P}(X))$ measurable space, $t \in X$. We define

$$\delta_t : \mathcal{P}(X) \to [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

<u>Continuity of the measure along monotone sequences</u>
$(X, \mathcal{M}, \mu)$ measure space

(1) $\{E_i\} \subset \mathcal{M},\ E_i \subseteq E_{i+1}\ \forall i$ and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:

$$\mu(E) = \lim_i \mu(E_i)$$

(2) $\{E_i\} \subset \mathcal{M},\ E_{i+1} \subseteq E_i\ \forall i$ and let $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$.

Proof. (1) if $\exists\ i$ s.t. $\mu(E_i) = +\infty$, then is trivial. Assume then that every E_i has a finite measure, so that $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty}(E_{i+1} \backslash E_i)$ with $E_0 = \varnothing$.

So, by σ-additivity

$$\mu(E) = \mu\left(\bigcup_{i=0}^{\infty}(E_{i+1} \backslash E_i)\right) =$$

$$= \sum_{i=0}^{\infty} \mu(E_{i+1} \backslash E_i) \overset{(excision)}{=} \sum_{i=0}^{\infty} (\mu(E_{i+1} - \mu(E_i))) =$$

$$\overset{(telescopic\ series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n)$$

(2) For simplicity, suppose $\tau = 1$, and define $F_k = E_i \backslash E_k$

$$\{E_k\} \searrow \Longrightarrow \{F_k\} \nearrow$$

$$\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \backslash (\bigcap_k E_k)$$

$$\mu(E_i) = \mu(\bigcup_k F_k) + \underbrace{\mu(\bigcap_k E_k)}_{\mu(E)} =$$

$$\overset{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k (\mu(E_i) - \mu(E_k)) + \mu(E)$$

Since $\mu(E_i) < \infty$ we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$

★

Counterexample: given $(\mathcal{N}, \mathcal{P}(\mathbb{N}), \mu_C)$ measure space. Let $E_n = \{n, n+1, n+2, \ldots\}$. In this case $\mu_C(E_n) = +\infty, E_{n+1} \subseteq E_n \forall\ n$, but $\bigcap_n E_n = \varnothing \Longrightarrow \mu\left(\bigcap_n E_n\right) = 0$

Theorem 5.1 (σ-subadditivity of the measure)
$(X, \mathcal{M}, \mu)$ is a measure space. $\forall \{E_n\} \subseteq \mathcal{M}$ (not necessarily disjoint): $\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n)$

Proof. $E_1, E_2 \in \mathcal{M}$ and also $E_1 \cup E_2 = E_1 \cup (E_2 \backslash E_1)$ disjoint sets.

$$\mu(E_1 \cup E_2) = \mu(\underbrace{E_2 \backslash E_1}_{\subseteq E_2}) \overset{(monotonicity)}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.

$$A = \bigcup_{n=1}^{\infty} E_n, \quad A_k = \bigcup_{n=1}^{k} E_n$$

$$\{A_k\} \nearrow,\ A_{k+1} \supseteq A_k,\ \lim_k A_k = A$$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{(continuity)}{=} \lim_k \mu(A_k) = \lim_k \mu\left(\bigcup_{n=1}^{k} E_n\right) \leq$$

$$\leq \lim_k \sum_{n=1}^{k} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

★

Exercise: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, +\infty]$ s.t. μ is finitely additive, σ-subadditive and $\mu(\varnothing) = 0 \Longrightarrow \mu$ is σ-additive, and hence is a measure.

Exercise: the Borel-Cantelli lemma states that, given $(X, \mathcal{M}, \mu)$ and $\{E_n\} \subseteq \mathcal{M}$. Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \Longrightarrow \mu(\limsup_n E_n) = 0$$

It can be phrased as:

> If the series of the probability of the events E_n is convergent, then the probability that ∞-many events occur is 0

Proof. The thesis is:

$$\mu(\limsup_n E_n) = \mu\left(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k \geq n} E_k}_{A_n := \bigcup_{k\geq n} E_k}\right)$$

Is it true that $\{A_n\} \searrow$? Yes.

$$A_{n+1} = \bigcup_{k \geq n+1} E_k \subseteq \bigcup_{k \geq n} E_k = A_n$$

Does some A_n have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k \geq n} E_k\right) \overset{\sigma-sub.}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

★

Sets of 0 measure

$(X, \mathcal{M}, \mu)$ measure space.

- $N \subseteq X$ is a set of 0 measure if $N \in \mathcal{M}$ and $\mu(N) = 0$
- $E \subseteq X$ is called **negligible set** if $\exists\, N \in \mathcal{M}$ with 0 measure s.t. $E \subseteq N$ (E does not necessarily stays in $\mathcal{M}$)

Definition 5.2
$(X, \mathcal{M}, \mu)$ measure space s.t. every negligible set is measurable (and hence of 0 measure), then $(X, \mathcal{M}, \mu)$ is said to be a **complete measure space**.

A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists\, F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \backslash F) = 0\}$$

Clearly $\mathcal{M} \subseteq \overline{\mathcal{M}}$. For $E \in \overline{\mathcal{M}}$, take F and G as above and let $\bar{\mu}(E) = \bar{\mu}(F)$ then $\bar{\mu}|_{\mathcal{M}} = \mu$, and moreover:

Theorem 5.2
$(X, \mathcal{M}, \mu)$ is a complete measure space. Let's observe that $\bar{\mu}$ is well defined: let $E \subseteq X$ and F_1, F_2, G_1, G_2 s.t. $F_i \subset E \subset G_i \quad i = 1, 2$. Then $\mu(G_i \backslash F_i) = 0$. Now we have to check that $\mu(F_1) = \mu(F_2)$.

Since

$$F_1 \backslash F_2 \subseteq E \backslash F_2 \subseteq G_2 \backslash F_2$$

and $G_2 \backslash F_2$ has 0 measure $\Longrightarrow \mu(F_1 \backslash F_2) = 0$. Then $F_1 = (F_1 \backslash F_2) \cup (F_1 \cap F_2) \Longrightarrow \mu(F_1) = \mu(F_1 \cap F_2)$. In the same way, $\mu(F_2) = \mu(F_1 \cap F_2)$

6 Lesson 15/09/2022

The elements of $\overline{\mathcal{M}}$ are sets of the type $E \cup N$, with $E \in \mathcal{M}$ and $\bar{\mu}(N) = 0$.

Outer measure

We wish to define a measure λ "on $\mathcal{R}$" with the following properties:

(1) $\lambda((a,b)) = b - a$

(2) $\lambda(E+t)^{\dagger} = \lambda(E)$ for every measurable set $E \subset \mathbb{R}$ and $t \in \mathbb{R}$

It would be nice to define such a measure on $\mathcal{P}(\mathbb{R})$. In such case, note that $\lambda(\{x\}) = 0, \forall\, x \in \mathbb{R}$
But then

Theorem 6.1 (Ulam)
The only measure on $\mathcal{P}(\mathbb{R})$ s.t. $\lambda(\{x\}) = 0 \quad \forall\, x$ is the trivial measure. Thus, a measure satisfying the two properties of the outer measure cannot be defined on $\mathcal{P}(\mathcal{R})$

We'll learn in what follows how to create a measure space on $\mathcal{R}$, with a σ-algebra including all the Borel sets, and a measure satisfying properties of the outer measure. This is the so called **Lebesgue measure**.

Definition 6.1
Given a set X. An **outer measure** is a function $\mu^* : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ s.t.

- $\mu^*(\varnothing) = 0$
- $\mu^*(A) \le \mu^*(B)$ if $A \subseteq B$ (Monotonicity)
- $\mu^*(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} \mu^*(E_n)$ (σ-subadditivity)

The common way to define an outer measure is to start with a family of elementary sets $\mathcal{E}$ on which a notion of measure is defined (e.g. intervals on $\mathcal{R}$, rectangles on $\mathcal{R}^2, \ldots$) and then to approximate arbitrary sets from outside by **countable** unions of members of $\mathcal{E}$.

Proposition 6.1
Let $\mathcal{E} \subset \mathcal{P}(\mathbb{R})$ and $\rho : \mathcal{E} \to [0, +\infty]$ be such that $\varnothing \in \mathcal{E}, X \in \mathcal{E}$ and $\rho(\varnothing) = 0$. For any $A \in \mathcal{P}(X)$, let

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) : E_n \in \mathcal{E} \text{ and } A \subset \bigcup_{n=1}^{\infty} E_n \right\}$$

Then μ^* is an outer measure, the outer measure generated by $(\mathcal{E}, \rho)$.

Proof. $\forall\, A \subset X\ \exists \{E_n\} \subset \mathcal{E}$ s.t. $A \subset \bigcup_n E_n$: take $E_n = X \forall\, n$ then μ^* is well defined. Obviously, $\mu^*(\varnothing) = 0$ (with $E_n = \varnothing \quad \forall\, n$), and $\mu^*(A) \le \mu^*(B)$ for $A \subset B$ (any covering of B with elements of $\mathcal{E}$ is also a covering of A.)

We have to prove the σ-subadditivity. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ and $\epsilon > 0$. For each $n, \exists \left\{E_{n_j}\right\}_{j\in\mathbb{N}} \in \mathcal{E}$ s.t. $A_n \subset \bigcup_{i=1}^{\infty} E_{n_j}$ and $\sum_{j=1}^{\infty} \rho(E_{n_j}) \le \mu^*(A_n) + \frac{\epsilon}{2^n}$. But then, if $A = \bigcup_{n=1}^{\infty} A_n$, we have that $A \subset \bigcup_{n,j\in\mathbb{N}^2} E_{n_j}$ and

$$\mu^*(A) \le \sum_{n,j} \rho(E_{n_j}) \le \sum_n \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) = \sum_n \mu^*(A_n) + \epsilon$$

Since ϵ is arbitrary, we are done. ★

---

† $\{x \in \mathbb{R} : x = y + t,\ \text{with } y \in E\}$

Ex:

(1) $X \in \mathbb{R}, \mathcal{E} = \{(a,b) : a \leq b, a, b \in \mathbb{R}\}$ family of open intervals:

$$\rho((a,b)) = b - a$$

(2) $X = \mathbb{R}^n, \mathcal{E} = \{(a_1,b_1) \times \ldots \times (a_n,b_n) : a_i \leq b_i, a_i, b_i \in \mathbb{R}\}$:

$$\rho((a_1,b_1) \times \ldots \times (a_n,b_n)) = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$$

Remark 7
$E \in \mathcal{E} \Longrightarrow \mu^*(E) = \rho(E)$.

In examples 1 and 2, we have in fact $\mu^*((a,b)) = b - a, \mu^*\left((a_1,b_1) \times \ldots \times (a_n,b_n)\right) = \prod_{i=1}^n (b_i - a_i)$

To pass from the outer measure to a measure there is a condition
Definition 7.1 (Caratheodory condition)
If μ^* is an outer measure on X, a set $A \subset X$ is called μ^*-**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall\, E \subset X$$

Remark 8
If E is a "nice" set containing A, then the above equality says that the outer measure of A, $\mu^*(E \cap A)$, is equal to $\mu^*(E) - \mu^*(E \cap A^C)$, which can be thought as an "inner measure". So basically we are saying that A is measurable if the outer and inner measure coincide. (Like the definition of Riemann integration with lower and upper sum)

Remark 9
μ^* is subadditive by def $\Longrightarrow \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall\, E, A \subset X$. So, to prove that a set is μ^*-measurable it is enough to prove the reverse inequality, $\forall\, E \subset X$. In fact, if $\mu^*(E) = +\infty$, then $+\infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$, and hence A is μ^*-measurable iff

$$\mu^*(E) \geq \mu(E \cap A) + \mu^*(E \cap A^C) \quad \forall\, E \subset X \text{ with } \mu^*(E) < +\infty$$

Their relevance to the notion of μ^*-measurability is clarified by the following
Theorem 9.1 (Caratheodory)
If μ^* is an outer measure on X, the family

$$\mathcal{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$$

is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

Lemma 9.1
If $A \subset X$ and $\mu^*(A) = 0$, then A is μ^*-measurable.

Proof. Let $E \subset X$ with $\mu^*(E) < +\infty$. Then

$$\mu^*(E) \geq \mu^*(E) + \mu^*(A) \overset{\ddagger}{\geq} \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

This implies that A is μ^*-measurable. ★

To sum up: X set, $(\mathcal{E}, \rho)$elementary and measurable sets, so μ^* is an outer measure. Then given μ^* and the Caratheodory condition, we have $(X, \mathcal{M}, \mu)$ that is a complete measure space.
Remark 10
So far we did not prove that $\mathcal{E} \subseteq \mathcal{M}$. We will do it in a particular case.

‡ $E \cap A^C \subseteq E$ and $E \cap A \subseteq A$ + monotonicity

Lebesgue measure

- $X = \mathbb{R}$, $\mathcal{E}$ family of open intervals, $\rho((a, b)) = b - a = \lambda((a, b))$, the complete measure space is $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ with $\mathcal{L}(\mathbb{R})$ the Lebesgue-measurable sets on $\mathbb{R}$ and λ the Lebesgue measure on $\mathbb{R}$.

- $X = \mathbb{R}^n$, $\mathcal{E} = \{\prod_{k=1}^{n}(a_k, b_k) : a_k \leq b_k \quad \forall\, k = 1, \dots, n\}$, $\rho\left(\prod_{k=1}^{n}(a_k, b_k)\right) = \prod_{k=1}^{n}(b_k - a_k)$ and this is a complete measure space $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$

11 Lesson 21/09/2022

Lebesgue measure

12 Lesson 22/09/2022

(1) $((X, \mathcal{M}))$ is a measurable space obtained by means of an outer measure. Ex: $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$, (Y, d_y) metric space If $X \to Y$ is (Lebesgue) measurable $\iff$ $(\mathcal{M}, \mathcal{B}(Y))$ is measurable

(2) $(X, d_X), (Y, d_Y)$ are metric spaces $\longrightarrow$ $(X, \mathcal{B}(X))$ If $X \to Y$ are borel measurable $\iff$ $(\mathcal{B}(X), \mathcal{B}(Y))$measurable

Remark 13
f is Lebesgue measurable if the continuity of the borel set is a Lebesgue-measurable set.

Proposition 13.1 (1) $(X, d_X), (Y, d_Y)$ metric spaces. If $X \to Y$ is continuous, then is Borel measurable

(2) (Y, d_Y) metric space. If $\mathbb{R}^n \to Y$ is continuous, then it is a Lebesgue measure.

Proof. (1) f is continuous $\iff$ $f^{-1}(A)$ is open $\forall\, A \in Y$ open $\implies$ $f^{-1}(A) \in \mathcal{B}(Y)$ $\forall\, A \in Y$ open Since $\mathcal{B}(Y) = \sigma_0$(open sets) by proposition 1 thus implies that f is Borel measurable

(2) f is continuous $\implies$ f is Borel measurable mancano pezzi namely f is Lebesgue measurable ★

Proposition 13.2
$(X, \mathcal{M})$ measurable space, $(X, d_Y), (Y, d_Y)$ metric spaces. if $f : X \to Y$ is $\mathcal{M}, \mathcal{B}(Y)$-measurable and $g : Y \to Z$ is continuous $\implies$ $g \circ f : x \to Z$ is $\mathcal{M}, \mathcal{B}(Y)$-measurable

Proposition 13.3
$(X, \mathcal{M})$ measurable space Let $\Phi : \mathbb{R}^n \to Y$ be continuous where (Y, d_Y) is a metric space. Then $h : X \to Y$ defined by $h(x) = \Phi(u(x), boh)$ is $\mathcal{M}, \mathcal{B}(Y)$-measurable.

Proof. Define $f : X \to \mathbb{R}^n$, $f(x) = u(x), v(x)$. By def $h = \Phi \circ f$ by prop 3 if we show that f is measurable, then h is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0\left(\{(a_1, b_1) \times (a_2, b_2) : a, b \in \mathbb{R}\}\right)$$

pezzi $f^{-1}(\mathcal{R} \in \mathcal{M}) \quad \forall$open rectangle in $\mathcal{R}^2$ $R = I \times J$ $F^{-1} = \{x \in X\}$ ★

Remark 14
roba

$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

cosine $(X, \mathcal{M})$ measurable space, then such a function f is measurable iff

$$f^{-1}(a, +\infty)] \in \mathcal{M} \quad \forall a \in \mathcal{R}$$

LEt now $\{f_n\}$ be a Sequence of measurable functions from X to $\bar{\mathcal{R}}$. Then we define

$$(\inf_n f_n)(x) = \inf_n f_n(x)$$

$$(\sup_n f_n)(x) = \sup_n f_n(x)$$

$$(\liminf_n f_n)(x) = \liminf_n f_n(x)$$

$$(\limsup_n f_n)(x) = \limsup_n f_n(x)$$

$$(\lim_n f_n)(x) = \lim_n f_n(x) \quad \text{if the limit exists}$$

Proposition 14.1
$(X, \mathcal{M})$ measurable space, $f_n : X \to \bar{\mathcal{R}}$ measurable, then sup inf lim inf lim sup of f_n are measurable, in particular if $\lim f_n$ exists, then f is measurable

Proof. $(\sup f_n)^{-1}((a, \infty]) = \{x \in X : \sup f_n(x) > a\}$ (manca pezzi)

$$\bigcup \{x \in X : f_n(x) > a\}$$

Then $(\sup f_n)^{-1}((a, \infty])$ is measurable, cose da aggiungere Noe the limsup

$$\limsup_n f_n = \lim_n (\sup_{k>n} f_n(x))$$

cose cose ★

Simple functions

Definition 14.1
$(X, \mathcal{M})$ measurable space. A measurable function s: $X \to \bar{\mathcal{R}}$ is said to be simple if $s(X)$ is a finite set altre cose Then $s(x) = \sum_{n=1} a_n \chi_{E_n}(x)$ where E_n is a measurable set sistemare.

Particular case: if s:$\mathbb{R} \to \bar{\mathbb{R}}$ and each E_n is a finite union of intervals, then s is said to be a STEP FUNCTION.

The idea is to approximate functions with simple functions.

Theorem 14.1
$(X, \mathcal{M})$ measurable space, $f : X \to [0, \infty]$ measurable. Then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq f \quad \text{pointwise}$$

and $s_n(x) \to f(x)$ Moreover if f is bounded then $s_n \to f$ uniformly on X as $n \to \infty$

f is bounded. Fix $n \in \mathbb{N}$ and divide $[0, n)$ in $n \cdot 2^n$ intervals called $I_j = [a_j, b_j)$ with lenght $\frac{1}{2^n}$

Let $E_0 = f^{-1}([n, \infty)), E_j = f^{-1}([a_j, b_j))$ for $j = 1, \ldots, n \cdot 2^n$

We let Array

Namely we define

$$s_n(x) = n\chi_{E_0}(X) + \sum_{j=1}^{n \cdot 2^n} a_j \chi_{E_j}(x)$$

Then $s_n \leq s_{n+1}$ by contradiction

Then any $x \in X$ stays in $f^{-1}([a_j, b_j))$ for some j $\Longrightarrow$ ★

15 06/10/2022

$f \notin R(I)$. Is it true that $\exists g \in R(I)$ s.t. $g = f$ almost everywhere (a.e.) on I? No.

For instance, consider $T_{\mathcal{E}}$, the generalized Cantor set $(\lambda(T_{\mathcal{E}}))$. Consider $\chi_{\mathcal{E}}$. In general, χ_A is discontinuous on δA. But $T_{\mathcal{E}}$ has no interior parts $\implies T_{\mathcal{E}} = \delta T_{\mathcal{E}} \implies \chi_{T_{\mathcal{E}}}$ is discontinuous on $T_{\mathcal{E}}$. cosine

Clearly

$$\int_{[0,1]} \chi_{T_{\mathcal{E}}} d\lambda = \lambda(T_{\mathcal{E}})$$

so $\chi_{T_{\mathcal{E}}} \in \mathcal{L}^1([0,1])$. If $g = \chi_{T_{\mathcal{E}}}$ a.e., then g is discontinuous at almost every part of $T_{\mathcal{E}} \implies g$ is discontinuous on a set of positive measure $\implies g \notin R(I)$. So, the Lebesgue integral is a true extension of the Riemann one.

Regarding generalized integrals we have

Theorem 15.1
$-\infty \leq a < b \leq +\infty, \quad f \in R^g([a,b])$ where

$$R^g([a,b]) = \{\text{Riemann-int functions on } [a,b] \text{ in the generalized sense}\}$$

Then, f is $([a,b], \mathcal{L}([a,b]))$-measurable. Moreover

(1) $f \geq 0$ on $[a,b] \implies f \in \mathcal{L}^1([a,b])$

(2) $|f| \in R^g([a,b]) \implies f \in \mathcal{L}^1([a,b])$

and in both cases

$$\int_{[a,b]} f d\lambda = \int_a^b f(x)dx$$

If f is in $R^g([a,b])$, but $|f| \notin R^g([a,b])$,then the two notions of $\int$ are not really related

Ex: $f(x) = \frac{\sin x}{x}, \quad x \in [1, \infty]$

$$\int_1^\infty |f(x)|dx = +\infty \implies f \notin \mathcal{L}^1([1, +\infty])$$

. But on the other hand

$$\int_1^\infty \frac{\sin x}{x} dx = \lim_{\omega \to \infty} \int_1^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Spaces of integrable functions

$(X, \mathcal{M}, \mu)$ complete measure space.

$$\mathcal{L}^1 = \left\{ f : X \to \bar{\mathbb{R}} : \text{ f is integrable} \right\}$$

$\mathcal{L}^1$ is a vector space. On $\mathcal{L}^1$ we can introduce $d : \mathcal{L}^1 \times \mathcal{L}^1 \to [0, +\infty)$ defined by

$$d_1(f,g) = \int_X |f - g|$$

cose

However, d_1 is not a distance on $\mathcal{L}^1(X)$, since

$$d_1(f,g) = 0 \implies f = g \quad \text{a.e on } X \qquad \text{(Pseudo-distance)}$$

To overcome this problem, we introduce an equivalent relation in $\mathcal{L}^1(X)$: we say that

$$f\ g \Longleftrightarrow f = g \quad \text{a.e. on } X$$

If $f \in \mathcal{L}^1(X)$, we can consider the equivalence class

$$[f] = \{g \in \mathcal{L}^1(X) : g = f \text{ a.e on } X\}$$

We define

$$L^1(X) = \frac{\mathcal{L}^1(X)}{}$$

$L^1(X)$ is a vector space, and on $L^1(X)$ the function d_1 is a distance:

$$d_1([f],[g]) cosoeocoeoce$$

To simplify the notations, the elements of $L^1(X)$ are called functions, and one writes $f \in L^1(X)$. With this, we means that we choose a representative in $[f]$, and f denotes both the representative and the equivalence class. The representative can be arbitrarily modified on any set with 0 measure.

Another relevant space of measurable functions is the space of **essentially bounded** functions

Definition 15.1
$f : X \to \bar{\mathbb{R}}$ measurable is called essentially bounded if $\exists M > 0$ s.t.

$$\mu(\{x \in X : |f(x)| \geq M\}) = 0$$

Ex:

$$f(x) = \begin{cases} 1 & x > 0 \\ +\infty & x = 0 \\ 0 & x < 0 \end{cases}$$

For $M > 1$, $\lambda(\{x \in \mathbb{R} : |f(x)| > M\}) = \lambda(\{0\}) = 0 \implies f$ is essentially bounded. If f is essentially bounded, it is well defined the **essential supremum** of f.

$$\underset{X}{esssup} f := \inf\{M > 0 \text{ s.t. } f \leq M \text{ a.e. on } X\} = pezzi$$

It can also be defined on essential inf.

Remark 16
Note that, by def of inf, $\forall\ \epsilon > 0$ we have

$$f \leq (\underset{X}{esssup} f) + \epsilon$$

We define

$$L^\infty(X, \mathcal{M}, \mu) = \frac{\mathcal{L}^\infty(X, \mathcal{M}, \mu)}{}$$

$L^\infty(X)$ is a vector space, and it is also a metric space for $d_\infty(f,g) = \underset{X}{essup}|f-g|$

Relation between different types of convergence

$\{f_n\}$ sequence of measurable functions $X \to \bar{\mathbb{R}}$

- recupera
- $f_n \to f$ pointwise
- $f_n \to f$ uniformly