

# Notes from Real and Functional Analysis

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# 1 Lesson 12/09/2022

## Element of set theory

Let  $X$  be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \quad (\text{Power Set})$$

Let  $I \subseteq \mathbb{R}$  be a set of indexes. A family of sets induced by  $I$  is

$$\{E_i\}_{i \in I}, \quad E_i \subseteq X \quad (\text{Family/Collection})$$

If  $I = \mathbb{N}$  is called a

$$\{E_n\}_{n \in \mathbb{N}} \quad (\text{Sequence})$$

### Definition 1.1

$\{E_n\} \subseteq \mathcal{P}(X)$  is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1} \forall n \quad (\text{resp. } E_n \supseteq E_{n+1} \forall n)$$

and is written as

$$\{E_n\} \nearrow \quad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets  $\{E_i\}_{i \in I} \subseteq \mathcal{P}(X)$ , will be often considered

$$\bigcup_{i \in I} E_i = \{x \in X : \exists i \in I \text{ s.t. } x \in E_i\}$$

$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i, \forall i \in I\}$$

$\{E_i\}$  is said to be **disjoint** if  $E_i \cap E_j = \emptyset \forall i \neq j$ .

Examples:

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

### Definition 1.2

$\{E_n\} \subseteq \mathcal{P}(X)$ . We define

$$\limsup_n E_n := \bigcap_{k=1}^{\infty} \left( \bigcup_{n=k}^{\infty} E_n \right)$$

$$\liminf_n E_n := \bigcup_{k=1}^{\infty} \left( \bigcap_{n=k}^{\infty} E_n \right)$$

If these two sets are equal, then

$$\lim_n E_n = \limsup_n E_n = \liminf_n E_n$$

### Proposition 1.1

Some limits are:

- $\limsup_n E_n = \{x \in X : x \in E_n \text{ for } \infty - \text{many indexes } n\}$
- $\liminf_n E_n = \{x \in X : x \in E_n \text{ for all but finitely many indexes } n\}$

- $\liminf_n E_n \subseteq \limsup_n E_n$
- $(\liminf_n E_n)^C = \limsup_n E_n^C$

**Definition 1.3**

$$\begin{aligned}
x \in \limsup_n E_n &\iff x \in \bigcap_{k=1}^{\infty} \left( \bigcup_{n=k}^{\infty} E_n \right) \\
&\iff \forall k \in \mathbb{N} : \bigcup_{n=k}^{\infty} E_n \\
&\iff \forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k}
\end{aligned}$$

So  $x \in \limsup_n E_n \implies \begin{aligned} &\exists m_1 = n_1 \text{ s.t. } x \in E_{n_1} \\ &\exists m_2 := n_{m_1+1} \geq m_1 + 1 \text{ s.t. } x \in E_{n_2} \\ &\vdots \\ &\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1 \text{ s.t. } x \in E_{n_k} \\ &\vdots \\ &x \in E_{m_1}, \dots, E_{m_k}, \dots \end{aligned}$

On the other hand, assume that  $x \in E_n$  for  $\infty$ -many indexes. We claim that  $\forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k} \iff x \in \limsup_n E_n$ . If that claim is not true, then  $\exists \bar{k} \text{ s.t. } x \notin E_n \forall n > \bar{k} \implies x$  belongs at most to  $E_1, \dots, E_{\bar{k}}$ , a contradiction. ★

**Definition 1.4**

$\{E_i\}_{i \in I}$  is a **covering** of  $X$  if

$$X \subseteq \bigcup_{i \in I} E_i$$

A subfamily of  $E_i$  that is still a covering is called a **subcovering**

**Definition 1.5**

Let  $E \subseteq X$ . The function  $\chi_E : X \rightarrow \mathbb{R}$

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \setminus E \end{cases}$$

is called **characteristic function** of  $E$

Let  $E_1, E_2$  be sets:

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

$$\{E_n\} \subseteq \mathcal{P}(X), \text{ disjoint, } E = \bigcup_{n=1}^{\infty} E_n \implies \chi_E = \sum_{n=1}^{\infty} \chi_{E_n}$$

$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \implies \chi_P = \liminf_n \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that  $\limsup_n a_n = \lim_{k \rightarrow \infty} \sup_{n \geq k} a_n$  and  $\liminf_n a_n = \lim_{k \rightarrow \infty} \inf_{n \geq k} a_n$

Let's also check that  $\chi_Q = \limsup_n \chi_{E_n}$

$$\begin{aligned}
x \in \limsup_n E_n &\iff \chi_Q(x) = 1 \\
&\iff \forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k}
\end{aligned}$$

If we fix  $k$  then

$$\sup_{n \geq k} \chi_{E_n}(x) = \chi_{E_{n_k}}(x) = 1$$

$$\lim_{k \rightarrow \infty} \sup_{n \geq k} \chi_{E_n}(x) = \lim_n \sup \chi_{E_n}(x) = 1$$

Let now  $x \notin \limsup E_n \iff \chi_Q(x) = 0$ . Then  $x$  belongs at most to finitely many  $E_n \implies \exists \bar{k}$  s.t.  $x \notin E_n, \forall n \geq \bar{k}$

If  $k \geq \bar{k}$ , then  $\sup_{n \geq k} \chi_{E_n}(x) = 0 \implies \lim_{k \rightarrow \infty} \sup_{n \geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 0$

## Relations

Given  $X, Y$  sets, is called a **relation** of  $X$  and  $Y$  a subset of  $X \times Y$

$$R \subseteq X \times Y \quad R = \{(x, y) : x \in X, y \in Y\}$$

$$(x, y) \in R \iff xRy$$

$$X = \{0, 1, 2, 3\} \quad R = \{(0, 1), (1, 2), (2, 1)\} \text{ is a relation in } X$$

### Definition 1.6

A **function** from  $X$  to  $Y$  is a relation  $R$  s.t. for any element  $x$  of  $X$   $\exists!$  element  $y$  of  $Y$  s.t.  $xRy$

### Definition 1.7

$R$  on  $X$  is an **equivalence relation** if

- (1)  $xRx \forall x \in X$  ( $R$  is **reflexive**)
- (2)  $xRy \implies yRx$  ( $R$  is **symmetric**)
- (3)  $xRy, yRz \implies xRz$  ( $R$  is **transitive**)

If  $R$  is an equivalence relation, the set  $E_x := \{y \in X : yRx\}$ ,  $x \in X$  is called the **equivalence class** of  $X$

### Definition 1.8

$\frac{X}{R} := \{E_x : x \in X\}$  is the **quotient set**

Ex:  $X = \mathbb{Z}$ , let's say that  $nRm$  if  $n - m$  is even. This is an equivalence relation.

$$E_n = \{\dots, n-4, n-2, n, n+2, n+4, \dots\}$$

in this case if  $n$  is even,  $E_n = \{\text{even numbers}\}$  and if  $n$  is odd,  $E_n = \{\text{odd numbers}\}$

## Measure theory

### Definition 1.9

A family  $\mathcal{M} \subseteq \mathcal{P}(X)$  is called a  **$\sigma$ -algebra** if

- (1)  $X \in \mathcal{M}$
- (2)  $E \in \mathcal{M} \implies E^C = X \setminus E \in \mathcal{M}$
- (3) If  $E = \bigcup_{n \in \mathbb{N}} E_n$  and  $E_n \in \mathcal{M} \forall n$ , then  $E \in \mathcal{M}$

If  $\mathcal{M}$  is a  $\sigma$ -algebra,  $(X, \mathcal{M})$  is called **measurable space** and the sets in  $\mathcal{M}$  are called **measurable**. Ex:

- $(X, \mathcal{P}(X))$  is a measurable space

- Let  $X$  be a set, then  $\{\emptyset, X\}$  is a  $\sigma$ -algebra

**Remark 2**

$\sigma$  is often used to denote the closure w.r.t. countably many operators. If we replace the countable unions with finite unions in the definition of  $\sigma$ -algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space  $(X, \mathcal{M})$ :

- (1)  $\emptyset \in \mathcal{M}$ :  $\emptyset = X^C$  and  $X \in \mathcal{M}$
- (2)  $\mathcal{M}$  is an algebra, and  $E_1, \dots, E_n \in \mathcal{M}$

$$E_1 \cup \dots \cup E_n = E_1 \cup \dots \cup E_n \cup \underbrace{\emptyset}_{\in \mathcal{M}} \cup \emptyset \dots \in \mathcal{M}$$

- (3)  $E_n \in \mathcal{M}$ ,  $\bigcap_{n \in \mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n \in \mathbb{N}} E_n = \left( \underbrace{\bigcup_{n \in \mathbb{N}} E_n^C}_{\in \mathcal{M}} \right)^C \quad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \implies E \setminus F \in \mathcal{M} = E \setminus F = E \cap F^C \in \mathcal{M}$
- If  $\Omega \subset X$ , then the **restriction** of  $\mathcal{M}$  to  $\Omega$ , written as

$$\mathcal{M}|_{\Omega} := \{F \subseteq \Omega : F = E \cap \Omega, \text{ with } E \in \mathcal{M}\}$$

is a  $\sigma$ -algebra on  $\Omega$

**Theorem 2.1**

$\mathcal{S} \subseteq \mathcal{P}(X)$ . Then it is well defined the smallest  $\sigma$ -algebra containing  $\mathcal{S}$ , the  $\sigma$ -algebra generated by  $\mathcal{S} := \sigma_0(\mathcal{S})$ :

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$  and thus is a  $\sigma$ -algebra
- $\forall \sigma(\mathcal{M})$  s.t.  $\mathcal{M} \supseteq \mathcal{S}$ , we have  $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

*Proof idea.*

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \emptyset \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define  $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} : \mathcal{M} \in \mathcal{V}\}$ , so it can be proved that this is the desired  $\sigma$ -algebra ★

**Borel sets**

Given  $(X, d)$  metric space, the  $\sigma$ -algebra generated by the open sets is called **Borel**  $\sigma$ -algebra, written as  $\mathcal{B}(X)$ . The sets in  $\mathcal{B}(X)$  are called **Borel sets**. The following sets are Borel sets:

- open sets
- closed sets
- countable intersections of open sets:  $G_{\sigma}$  sets
- countable unions of closed sets:  $F_{\sigma}$  sets

**Remark 3**

$\mathcal{B}(\mathbb{R})$  can be equivalently defined as the  $\sigma$ -algebra generated by

$$\begin{aligned} &\{(a, b) : a, b \in \mathcal{R}, a < b\} \\ &\{(-\infty, b) : b \in \mathcal{R}\} \\ &\{(a, +\infty) : a \in \mathcal{R}\} \\ &\{[a, b) : a, b \in \mathcal{R}, a < b\} \\ &\vdots \end{aligned}$$

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Question: What is  $\mathcal{B}(\mathbb{R})$ ? Is  $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$ ? No.

**Definition 4.1**

$(X, \mathcal{M})$  measurable space. A function  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  is called a **positive measure** if  $\mu(\emptyset) = 0$  and if  $\mu$  is countably additive, that is

$$\forall \{E_n\} \subseteq \mathcal{M} \text{ disjoint}$$

we have that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) \quad \sigma\text{-additivity}$$

**Remark 5**

a set  $A$  is countable if  $\exists f : \mathbb{N} \rightarrow A$  s.t.  $f$  is 1-1. Examples:  $\mathbb{Z}, \mathbb{Q}$  are countable, while  $\mathbb{R}$  is not, also  $(0, 1)$  is uncountable.

We always assume that  $\exists E \neq \emptyset, E \in \mathcal{M}$  s.t.  $\mu(E) < \infty$ .

If  $(X, \mathcal{M})$  is a measurable space, and  $\mu$  is a measure on it, then  $(X, \mathcal{M}, \mu)$  is a measure space.

Then:

(1)  $\mu$  is **finitely additive**:

$$\forall E, F \in \mathcal{M}, \text{ with } E \cap F = \emptyset \implies \mu(E \cup F) = \mu(E) + \mu(F)$$

(2) the **excision property**

$$\forall E, F \in \mathcal{M}, \text{ with } E \subset F \text{ and } \mu(E) < +\infty \implies \mu(F \setminus E) = \mu(F) - \mu(E)$$

(3) **monotonicity**

$$\forall E, F \in \mathcal{M}, \text{ with } E \subset F \implies \mu(E) \leq \mu(F)$$

(4) if  $\Omega \in \mathcal{M}$  then  $(\Omega, \mathcal{M}|_{\Omega}, \mu|_{\mathcal{M}|_{\Omega}})$  is a measure space

**Proof.** (1)  $E_1 = E, E_2 = F, E_3 = \dots = E_n = \dots = \emptyset$  This is a disjoint sequence  $\implies$  by  $\sigma$ -additivity.

$$\mu(E \cup F) = \mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n) = \mu(E) + \mu(F) + \underbrace{\mu(E_k)}_{=\mu(\emptyset)}$$

(2)  $E \subset F$ , so  $F = E \cup (F \setminus E)$  and this is disjoint  $\xRightarrow{(i)} \mu(F) = \mu(E) + \mu(F \setminus E)$ , and since  $\mu(E) < \infty$ , the property follows.

(3)  $E \subset F \implies \mu(F) = \mu(E) + \underbrace{\mu(F \setminus E)}_{\geq 0} \geq \mu(E)$

(4)

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### Definition 5.1

$(X, \mathcal{M}, \mu)$  measure space.

- If  $\mu(X) < +\infty$ , we say that  $\mu$  is **finite**.
- If  $\mu(X) = +\infty$ , and  $\exists \{E_n\} \subset \mathcal{M}$  s.t.  $X = \bigcup_n E_n$  and each  $E_n$  has finite measure, then we say that  $\mu$  is  $\sigma$ -finite.
- If  $\mu(X) = 1$  we say that  $\mu$  is a **probability measure**.

Some examples:

- Trivial Measure:  $(X, \mathcal{M})$  measurable space.  $\mu : \mathcal{M} \rightarrow [0, \infty]$  defined by  $\mu(E) = 0 \quad \forall E \in \mathcal{M}$
- Counting Measure:  $(X, \mathcal{P}(X))$  measurable space. We define

$$\mu_C : \mathcal{P}(X) \rightarrow [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure:  $(X, \mathcal{P}(X))$  measurable space,  $t \in X$ . We define

$$\delta_t : \mathcal{P}(X) \rightarrow [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

### Continuity of the measure along monotone sequences

$(X, \mathcal{M}, \mu)$  measure space

(1)  $\{E_i\} \subset \mathcal{M}$ ,  $E_i \subseteq E_{i+1} \forall i$  and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:

$$\mu(E) = \lim_i \mu(E_i)$$

(2)  $\{E_i\} \subset \mathcal{M}$ ,  $E_{i+1} \subseteq E_i \forall i$  and let  $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$ .

**Proof.** (1) if  $\exists i$  s.t.  $\mu(E_i) = +\infty$ , then is trivial. Assume then that every  $E_i$  has a finite measure, so that  $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i)$  with  $E_0 = \emptyset$ .

So, by  $\sigma$ -additivity

$$\mu(E) = \mu \left( \bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i) \right) =$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \mu(E_{i+1} \setminus E_i) \stackrel{(excision)}{=} \sum_{i=0}^{\infty} (\mu(E_{i+1}) - \mu(E_i)) = \\
&\stackrel{(telescopic series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n)
\end{aligned}$$

(2) For simplicity, suppose  $\tau = 1$ , and define  $F_k = E_i \setminus E_k$

$$\begin{aligned}
&\{E_k\} \searrow \implies \{F_k\} \nearrow \\
&\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \setminus \left(\bigcap_k E_k\right) \\
&\mu(E_i) = \mu\left(\bigcup_k F_k\right) + \underbrace{\mu\left(\bigcap_k E_k\right)}_{\mu(E)} = \\
&\stackrel{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k (\mu(E_i) - \mu(E_k)) + \mu(E)
\end{aligned}$$

Since  $\mu(E_i) < \infty$  we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$

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Counterexample: given  $(\mathcal{N}, \mathcal{P}(\mathbb{N}), \mu_C)$  measure space. Let  $E_n = \{n, n+1, n+2, \dots\}$ . In this case  $\mu_C(E_n) = +\infty$ ,  $E_{n+1} \subseteq E_n \forall n$ , but  $\bigcap_n E_n = \emptyset \implies \mu(\bigcap_n E_n) = 0$

**Theorem 5.1** ( $\sigma$ -subadditivity of the measure)

$(X, \mathcal{M}, \mu)$  is a measure space.  $\forall \{E_n\} \subseteq \mathcal{M}$  (not necessarily disjoint):  $\mu(\bigcup_n E_n) \leq \sum_n \mu(E_n)$

**Proof.**  $E_1, E_2 \in \mathcal{M}$  and also  $E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1)$  disjoint sets.

$$\mu(E_1 \cup E_2) = \mu\left(\underbrace{E_2 \setminus E_1}_{\subseteq E_2}\right) \stackrel{(monotonicity)}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.

$$\begin{aligned}
A &= \bigcup_{n=1}^{\infty} E_n, \quad A_k = \bigcup_{n=1}^k E_n \\
&\{A_k\} \nearrow, \quad A_{k+1} \supseteq A_k, \quad \lim_k A_k = A \\
&\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \stackrel{(continuity)}{=} \lim_k \mu(A_k) = \lim_k \mu\left(\bigcup_{n=1}^k E_n\right) \leq \\
&\leq \lim_k \sum_{n=1}^k \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)
\end{aligned}$$

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Exercise:  $(X, \mathcal{M})$  measurable space.  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  s.t.  $\mu$  is finitely additive,  $\sigma$ -subadditive and  $\mu(\emptyset) = 0 \implies \mu$  is  $\sigma$ -additive, and hence is a measure.



Exercise: the Borel-Cantelli lemma states that, given  $(X, \mathcal{M}, \mu)$  and  $\{E_n\} \subseteq \mathcal{M}$ . Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \implies \mu(\limsup_n E_n) = 0$$

It can be phrased as:

If the series of the probability of the events  $E_n$  is convergent, then the probability that  $\infty$ -many events occur is 0

**Proof.** The thesis is:

$$\mu(\limsup_n E_n) = \mu\left(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k \geq n} E_k}_{A_n := \bigcup_{k \geq n} E_k}\right)$$

Is it true that  $\{A_n\} \searrow$ ? Yes.

$$A_{n+1} = \bigcup_{k \geq n+1} E_k \subseteq \bigcup_{k \geq n} E_k = A_n$$

Does some  $A_n$  have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k \geq n} E_k\right) \stackrel{\sigma\text{-sub.}}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

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## Sets of 0 measure

$(X, \mathcal{M}, \mu)$  measure space.

- $N \subseteq X$  is a set of 0 measure if  $N \in \mathcal{M}$  and  $\mu(N) = 0$
- $E \subseteq X$  is called **negligible set** if  $\exists N \in \mathcal{M}$  with 0 measure s.t.  $E \subseteq N$  ( $E$  does not necessarily stay in  $\mathcal{M}$ )

### Definition 5.2

$(X, \mathcal{M}, \mu)$  measure space s.t. every negligible set is measurable (and hence of 0 measure), then  $(X, \mathcal{M}, \mu)$  is said to be a **complete measure space**.

A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \setminus F) = 0\}$$

Clearly  $\mathcal{M} \subseteq \overline{\mathcal{M}}$ . For  $E \in \overline{\mathcal{M}}$ , take  $F$  and  $G$  as above and let  $\bar{\mu}(E) = \mu(F)$  then  $\bar{\mu}|_{\mathcal{M}} = \mu$ , and moreover:

### Theorem 5.2

$(X, \mathcal{M}, \mu)$  is a complete measure space. Let's observe that  $\bar{\mu}$  is well defined: let  $E \subseteq X$  and  $F_1, F_2, G_1, G_2$  s.t.  $F_i \subseteq E \subseteq G_i$   $i = 1, 2$ . Then  $\mu(G_i \setminus F_i) = 0$ . Now we have to check that  $\mu(F_1) = \mu(F_2)$ .

Since

$$F_1 \setminus F_2 \subseteq E \setminus F_2 \subseteq G_2 \setminus F_2$$

and  $G_2 \setminus F_2$  has 0 measure  $\implies \mu(F_1 \setminus F_2) = 0$ . Then  $F_1 = (F_1 \setminus F_2) \cup (F_1 \cap F_2) \implies \mu(F_1) = \mu(F_1 \cap F_2)$ . In the same way,  $\mu(F_2) = \mu(F_1 \cap F_2)$

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The elements of  $\overline{\mathcal{M}}$  are sets of the type  $E \cup N$ , with  $E \in \mathcal{M}$  and  $\bar{\mu}(N) = 0$ .

### Outer measure

We wish to define a measure  $\lambda$  “on  $\mathcal{R}$ ” with the following properties:

- (1)  $\lambda((a, b)) = b - a$
- (2)  $\lambda(E + t)^\dagger = \lambda(E)$  for every measurable set  $E \subset \mathbb{R}$  and  $t \in \mathbb{R}$

It would be nice to define such a measure on  $\mathcal{P}(\mathbb{R})$ . In such case, note that  $\lambda(\{x\}) = 0, \forall x \in \mathbb{R}$ . But then

#### Theorem 6.1 (Ulam)

The only measure on  $\mathcal{P}(\mathbb{R})$  s.t.  $\lambda(\{x\}) = 0 \quad \forall x$  is the trivial measure. Thus, a measure satisfying the two properties of the outer measure cannot be defined on  $\mathcal{P}(\mathbb{R})$ .

We’ll learn in what follows how to create a measure space on  $\mathcal{R}$ , with a  $\sigma$ -algebra including all the Borel sets, and a measure satisfying properties of the outer measure. This is the so called **Lebesgue measure**.

#### Definition 6.1

Given a set  $X$ . An **outer measure** is a function  $\mu^* : \mathcal{P}(\mathbb{R}) \rightarrow [0, +\infty]$  s.t.

- $\mu^*(\emptyset) = 0$
- $\mu^*(A) \leq \mu^*(B)$  if  $A \subseteq B$  (Monotonicity)
- $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$  ( $\sigma$ -subadditivity)

The common way to define an outer measure is to start with a family of elementary sets  $\mathcal{E}$  on which a notion of measure is defined (e.g. intervals on  $\mathcal{R}$ , rectangles on  $\mathcal{R}^2, \dots$ ) and then to approximate arbitrary sets from outside by **countable** unions of members of  $\mathcal{E}$ .

#### Proposition 6.1

Let  $\mathcal{E} \subset \mathcal{P}(\mathbb{R})$  and  $\rho : \mathcal{E} \rightarrow [0, +\infty]$  be such that  $\emptyset \in \mathcal{E}, X \in \mathcal{E}$  and  $\rho(\emptyset) = 0$ . For any  $A \in \mathcal{P}(X)$ , let

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) : E_n \in \mathcal{E} \text{ and } A \subset \bigcup_{n=1}^{\infty} E_n \right\}$$

Then  $\mu^*$  is an outer measure, the outer measure generated by  $(\mathcal{E}, \rho)$ .

**Proof.**  $\forall A \subset X \exists \{E_n\} \subset \mathcal{E}$  s.t.  $A \subset \bigcup_n E_n$  : take  $E_n = X \forall n$  then  $\mu^*$  is well defined. Obviously,  $\mu^*(\emptyset) = 0$  (with  $E_n = \emptyset \quad \forall n$ ), and  $\mu^*(A) \leq \mu^*(B)$  for  $A \subset B$  (any covering of  $B$  with elements of  $\mathcal{E}$  is also a covering of  $A$ .)

We have to prove the  $\sigma$ -subadditivity. Let  $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X)$  and  $\epsilon > 0$ . For each  $n, \exists \{E_{n_j}\}_{j \in \mathbb{N}} \in \mathcal{E}$  s.t.  $A_n \subset \bigcup_{i=1}^{\infty} E_{n_j}$  and  $\sum_{j=1}^{\infty} \rho(E_{n_j}) \leq \mu^*(A_n) + \frac{\epsilon}{2^n}$ . But then, if  $A = \bigcup_{n=1}^{\infty} A_n$ , we have that  $A \subset \bigcup_{n,j \in \mathbb{N}^2} E_{n_j}$  and

$$\mu^*(A) \leq \sum_{n,j} \rho(E_{n_j}) \leq \sum_n \left( \mu^*(A_n) + \frac{\epsilon}{2^n} \right) = \sum_n \mu^*(A_n) + \epsilon$$

Since  $\epsilon$  is arbitrary, we are done. ★

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<sup>†</sup> $\{x \in \mathbb{R} : x = y + t, \text{ with } y \in E\}$

Ex:

- (1)  $X \in \mathbb{R}, \mathcal{E} = \{(a, b) : a \leq b, a, b \in \mathbb{R}\}$  family of open intervals:

$$\rho((a, b)) = b - a$$

- (2)  $X = \mathbb{R}^n, \mathcal{E} = \{(a_1, b_1) \times \dots \times (a_n, b_n) : a_i \leq b_i, a_i, b_i \in \mathbb{R}\}$ :

$$\rho((a_1, b_1) \times \dots \times (a_n, b_n)) = (b_1 - a_1) \cdot \dots \cdot (b_n - a_n)$$

**Remark 7**

$E \in \mathcal{E} \implies \mu^*(E) = \rho(E)$ .

In examples 1 and 2, we have in fact  $\mu^*((a, b)) = b - a, \mu^*((a_1, b_1) \times \dots \times (a_n, b_n)) = \prod_{i=1}^n (b_i - a_i)$

To pass from the outer measure to a measure there is a condition

**Definition 7.1** (Caratheodory condition)

If  $\mu^*$  is an outer measure on  $X$ , a set  $A \subset X$  is called  $\mu^*$ -measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E \subset X$$

**Remark 8**

If  $E$  is a “nice” set containing  $A$ , then the above equality says that the outer measure of  $A$ ,  $\mu^*(E \cap A)$ , is equal to  $\mu^*(E) - \mu^*(E \cap A^C)$ , which can be thought as an “inner measure”. So basically we are saying that  $A$  is measurable if the outer and inner measure coincide. (Like the definition of Riemann integration with lower and upper sum)

**Remark 9**

$\mu^*$  is subadditive by def  $\implies \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E, A \subset X$ . So, to prove that a set is  $\mu^*$ -measurable it is enough to prove the reverse inequality,  $\forall E \subset X$ . In fact, if  $\mu^*(E) = +\infty$ , then  $+\infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$ , and hence  $A$  is  $\mu^*$ -measurable iff

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E \subset X \text{ with } \mu^*(E) < +\infty$$

Their relevance to the notion of  $\mu^*$ -measurability is clarified by the following

**Theorem 9.1** (Caratheodory)

If  $\mu^*$  is an outer measure on  $X$ , the family

$$\mathcal{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$$

is a  $\sigma$ -algebra and  $\mu^*|_{\mathcal{M}}$  is a complete measure.

**Lemma 9.1**

If  $A \subset X$  and  $\mu^*(A) = 0$ , then  $A$  is  $\mu^*$ -measurable.

**Proof.** Let  $E \subset X$  with  $\mu^*(E) < +\infty$ . Then

$$\mu^*(E) \geq \mu^*(E) + \overset{\dagger}{\mu^*(A)} \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

This implies that  $A$  is  $\mu^*$ -measurable. ★

To sum up:  $X$  set,  $(\mathcal{E}, \rho)$  elementary and measurable sets, so  $\mu^*$  is an outer measure. Then given  $\mu^*$  and the Caratheodory condition, we have  $(X, \mathcal{M}, \mu)$  that is a complete measure space.

**Remark 10**

So far we did not prove that  $\mathcal{E} \subseteq \mathcal{M}$ . We will do it in a particular case.

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$\dagger E \cap A^C \subseteq E$  and  $E \cap A \subseteq A$  + monotonicity

## Lebesgue measure

- $X = \mathbb{R}$ ,  $\mathcal{E}$  family of open intervals,  $\rho((a, b)) = b - a = \lambda((a, b))$ , the complete measure space is  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  with  $\mathcal{L}(\mathbb{R})$  the Lebesgue-measurable sets on  $\mathbb{R}$  and  $\lambda$  the Lebesgue measure on  $\mathbb{R}$ .
- $X = \mathbb{R}^n$ ,  $\mathcal{E} = \{\prod_{k=1}^n (a_k, b_k) : a_k \leq b_k \quad \forall k = 1, \dots, n\}$ ,  $\rho(\prod_{k=1}^n (a_k, b_k)) = \prod_{k=1}^n (b_k - a_k)$  and this is a complete measure space  $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$

## 11 Lesson 21/09/2022

### Lebesgue measure

## 12 Lesson 22/09/2022

- (1)  $((X, \mathcal{M}))$  is a measurable space obtained by means of an outer measure. Ex:  $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$ ,  $(Y, d_Y)$  metric space If  $X \rightarrow Y$  is (Lebesgue) measurable  $\iff (\mathcal{M}, \mathcal{B}(Y))$  is measurable
- (2)  $(X, d_X), (Y, d_Y)$  are metric spaces  $\rightarrow (X, \mathcal{B}(X))$  If  $X \rightarrow Y$  are borel measurable  $\iff (\mathcal{B}(X), \mathcal{B}(Y))$  measurable

### Remark 13

$f$  is Lebesgue measurable if the continuity of the borel set is a Lebesgue-measurable set.

**Proposition 13.1** (1)  $(X, d_X), (Y, d_Y)$  metric spaces. If  $X \rightarrow Y$  is continuous, then is Borel measurable

- (2)  $(Y, d_Y)$  metric space. If  $\mathbb{R}^n \rightarrow Y$  is continuous, then it is a Lebesgue measure.

**Proof.** (1)  $f$  is continuous  $\iff f^{-1}(A)$  is open  $\forall A \in Y$  open  $\implies f^{-1}(A) \in \mathcal{B}(Y) \forall A \in Y$  open Since  $\mathcal{B}(Y) = \sigma_0(\text{open sets})$  by proposition 1 thus implies that  $f$  is Borel measurable

- (2)  $f$  is continuous  $\implies f$  is Borel measurable mancano pezzi namely  $f$  is Lebesgue measurable

★

### Proposition 13.2

$(X, \mathcal{M})$  measurable space,  $(X, d_X), (Y, d_Y)$  metric spaces. if  $f : X \rightarrow Y$  is  $\mathcal{M}, \mathcal{B}(Y)$ -measurable and  $g : Y \rightarrow Z$  is continuous  $\implies g \circ f : x \rightarrow Z$  is  $\mathcal{M}, \mathcal{B}(Y)$ -measurable

### Proposition 13.3

$(X, \mathcal{M})$  measurable space Let  $\Phi : \mathbb{R}^n \rightarrow Y$  be continuous where  $(Y, d_Y)$  is a metric space. Then  $h : X \rightarrow Y$  defined by  $h(x) = \Phi(u(x), v(x))$  is  $\mathcal{M}, \mathcal{B}(Y)$ -measurable.

**Proof.** Define  $f : X \rightarrow \mathbb{R}^n$ ,  $f(x) = (u(x), v(x))$ . By def  $h = \Phi \circ f$  by prop 3 if we show that  $f$  is measurable, then  $h$  is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0(\{(a_1, b_1) \times (a_2, b_2) : a, b \in \mathbb{R}\})$$

pezzi  $f^{-1}(\mathcal{R} \in \mathcal{M}) \quad \forall \text{open rectangle in } \mathbb{R}^2 \quad R = I \times J \quad f^{-1} = \{x \in X\}$

★

### Remark 14

roba

$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

cosine  $(X, \mathcal{M})$  measurable space, then such a function  $f$  is measurable iff

$$f^{-1}(a, +\infty] \in \mathcal{M} \quad \forall a \in \mathcal{R}$$

Let now  $\{f_n\}$  be a Sequence of measurable functions from  $X$  to  $\bar{\mathcal{R}}$ . Then we define

$$\begin{aligned} (\inf_n f_n)(x) &= \inf_n f_n(x) \\ (\sup_n f_n)(x) &= \sup_n f_n(x) \\ (\liminf_n f_n)(x) &= \liminf_n f_n(x) \\ (\limsup_n f_n)(x) &= \limsup_n f_n(x) \\ (\lim_n f_n)(x) &= \lim_n f_n(x) \quad \text{if the limit exists} \end{aligned}$$

### Proposition 14.1

$(X, \mathcal{M})$  measurable space,  $f_n : X \rightarrow \bar{\mathcal{R}}$  measurable, then  $\sup \inf \liminf \limsup$  of  $f_n$  are measurable, in particular if  $\lim f_n$  exists, then  $f$  is measurable

**Proof.**  $(\sup f_n)^{-1}((a, \infty]) = \{x \in X : \sup f_n(x) > a\}$  (manca pezzi)

$$\bigcup \{x \in X : f_n(x) > a\}$$

Then  $(\sup f_n)^{-1}((a, \infty])$  is measurable, cose da aggiungere Noe the limsup

$$\limsup_n f_n = \lim_n (\sup_{k>n} f_k(x))$$

cose cose

★

## Simple functions

### Definition 14.1

$(X, \mathcal{M})$  measurable space. A measurable function  $s : X \rightarrow \bar{\mathcal{R}}$  is said to be simple if  $s(X)$  is a finite set altre cose Then  $s(x) = \sum_{n=1} a_n \chi_{E_n}(x)$  where  $E_n$  is a measurable set sistemare.

Particular case: if  $s : \mathbb{R} \rightarrow \bar{\mathbb{R}}$  and each  $E_n$  is a finite union of intervals, then  $s$  is said to be a STEP FUNCTION.

The idea is to approximate functions with simple functions.

### Theorem 14.1

$(X, \mathcal{M})$  measurable space,  $f : X \rightarrow [0, \infty]$  measurable. Then  $\exists$  a sequence  $\{s_n\}$  of simple functions s.t.

$$0 \leq s_1 \leq \dots \leq s_n \leq \dots \leq f \quad \text{pointwise}$$

and  $s_n(x) \rightarrow f(x)$  Moreover if  $f$  is bounded then  $s_n \rightarrow f$  uniformly on  $X$  as  $n \rightarrow \infty$

*f is bounded.* Fix  $n \in \mathbb{N}$  and divide  $[0, n]$  in  $n \cdot 2^n$  intervals called  $I_j = [a_j, b_j)$  with lenght  $\frac{1}{2^n}$

Let  $E_0 = f^{-1}([n, \infty))$ ,  $E_j = f^{-1}([a_j, b_j))$  for  $j = 1, \dots, n \cdot 2^n$

We let Array

Namely we define

$$s_n(x) = n \chi_{E_0}(x) + \sum_{j=1}^{n \cdot 2^n} a_j \chi_{E_j}(x)$$

Then  $s_n \leq s_{n+1}$  by contradiction

Then any  $x \in X$  stays in  $f^{-1}([a_j, b_j))$  for some  $j \implies$

★

## 15 06/10/2022

$f \notin R(I)$ . Is it true that  $\exists g \in R(I)$  s.t.  $g = f$  almost everywhere (a.e.) on  $I$ ? No.

For instance, consider  $T_\varepsilon$ , the generalized Cantor set ( $\lambda(T_\varepsilon)$ ). Consider  $\chi_\varepsilon$ . In general,  $\chi_A$  is discontinuous on  $\delta A$ . But  $T_\varepsilon$  has no interior parts  $\implies T_\varepsilon = \delta T_\varepsilon \implies \chi_{T_\varepsilon}$  is discontinuous on  $T_\varepsilon$ . cosine

Clearly

$$\int_{[0,1]} \chi_{T_\varepsilon} d\lambda = \lambda(T_\varepsilon)$$

so  $\chi_{T_\varepsilon} \in \mathcal{L}^1([0,1])$ . If  $g = \chi_{T_\varepsilon}$  a.e., then  $g$  is discontinuous at almost every part of  $T_\varepsilon \implies g$  is discontinuous on a set of positive measure  $\implies g \notin R(I)$ . So, the Lebesgue integral is a true extension of the Riemann one.

Regarding generalized integrals we have

### Theorem 15.1

$-\infty \leq a < b \leq +\infty$ ,  $f \in R^g([a,b])$  where

$$R^g([a,b]) = \{\text{Riemann-int functions on } [a,b] \text{ in the generalized sense}\}$$

Then,  $f$  is  $([a,b], \mathcal{L}([a,b]))$ -measurable. Moreover

$$(1) \quad f \geq 0 \text{ on } [a,b] \implies f \in \mathcal{L}^1([a,b])$$

$$(2) \quad |f| \in R^g([a,b]) \implies f \in \mathcal{L}^1([a,b])$$

and in both cases

$$\int_{[a,b]} f d\lambda = \int_a^b f(x) dx$$

If  $f$  is in  $R^g([a,b])$ , but  $|f| \notin R^g([a,b])$ , then the two notions of  $\int$  are not really related

$$\text{Ex: } f(x) = \frac{\sin x}{x}, \quad x \in [1, \infty]$$

$$\int_1^\infty |f(x)| dx = +\infty \implies f \notin \mathcal{L}^1([1, +\infty])$$

. But on the other hand

$$\int_1^\infty \frac{\sin x}{x} dx = \lim_{\omega \rightarrow \infty} \int_1^\omega \frac{\sin x}{x} dx = \frac{\pi}{2}$$

### Spaces of integrable functions

$(X, \mathcal{M}, \mu)$  complete measure space.

$$\mathcal{L}^1 = \{f : X \rightarrow \mathbb{R} : f \text{ is integrable}\}$$

$\mathcal{L}^1$  is a vector space. On  $\mathcal{L}^1$  we can introduce  $d : \mathcal{L}^1 \times \mathcal{L}^1 \rightarrow [0, +\infty)$  defined by

$$d_1(f, g) = \int_X |f - g|$$

cose

However,  $d_1$  is not a distance on  $\mathcal{L}^1(X)$ , since

$$d_1(f, g) = 0 \implies f = g \quad \text{a.e on } X \quad (\text{Pseudo-distance})$$

To overcome this problem, we introduce an equivalent relation in  $\mathcal{L}^1(X)$ : we say that

$$f \sim g \iff f = g \quad \text{a.e. on } X$$

If  $f \in \mathcal{L}^1(X)$ , we can consider the equivalence class

$$[f] = \{g \in \mathcal{L}^1(X) : g = f \text{ a.e. on } X\}$$

We define

$$L^1(X) = \underline{\mathcal{L}^1(X)}$$

$L^1(X)$  is a vector space, and on  $L^1(X)$  the function  $d_1$  is a distance:

$$d_1([f], [g]) = \int |f - g|$$

To simplify the notations, the elements of  $L^1(X)$  are called functions, and one writes  $f \in L^1(X)$ . With this, we mean that we choose a representative in  $[f]$ , and  $f$  denotes both the representative and the equivalence class. The representative can be arbitrarily modified on any set with 0 measure.

Another relevant space of measurable functions is the space of **essentially bounded** functions

**Definition 15.1**

$f : X \rightarrow \bar{\mathbb{R}}$  measurable is called essentially bounded if  $\exists M > 0$  s.t.

$$\mu(\{x \in X : |f(x)| \geq M\}) = 0$$

Ex:

$$f(x) = \begin{cases} 1 & x > 0 \\ +\infty & x = 0 \\ 0 & x < 0 \end{cases}$$

For  $M > 1$ ,  $\lambda(\{x \in \mathbb{R} : |f(x)| > M\}) = \lambda(\{0\}) = 0 \implies f$  is essentially bounded