# Lesson 12/10/2022

... typewriter sequence But we do have convergence in measure

**Remark 1**

$f_p \nrightarrow 0$ a.e. on [0, 1] But consider $\{f_{p(n,1)} : n \in \mathbb{N}\}$. This is a subsequence and, by def

For this subsequence, we have $f_{p(n,1)}(x) \to 0$ as $n \to \infty$ $\forall x \in (0, 1]$ ..

This is not random

**Proposition 1.1**

If $\mu(X) < \infty$ and $f_n \to f$ in measure, then $\exists$

Now we analize ..

**Theorem 1.1**

$\{f_n \subset L^1(X), f \in L^1(X)\}$ If $f_n \to f$ in $L^1(X)$ then $f_n \to f$ in measure on $X$

***Proof.*** By contradiction. suppose that $f_n \nrightarrow f$ in measure on X: $\exists \bar{\alpha} > 0$ s.t.

$$\limsup_{n\to\infty} \mu(\{f_n - f \geq \bar{\alpha}\}) > 0$$

$\Rightarrow \exists \bar{\epsilon}$ and a subsequence $\{f_{n_k}\}$ s.t.

$$\mu(\{f_{n_k} - f \geq \bar{\alpha}\}) > \bar{\epsilon}$$

....

But, by assumption, $d_1(f_n, f) \to 0$

$$\Rightarrow d_1(f_{n_k}, f) \to 0$$

contradiction. ★

**Remark 2**

the convergence in measure doesn't imply the convergence in $L^1$

for example, consider on the other hand

convergence a.e. $\nRightarrow$ convergence in $L^1$ use the same example above, $f_n \to 0$ a.e. on $[0, 1] \nRightarrow f_n \to 0$ in $L^1$ Consider the typewriter sequence: ..... But we don't have a.e. convergence However, recall the dominated convergence theorem: (DOM)

$$f_n \to f a.e. + \exists \text{ of a dom function} \Rightarrow d(f_n, f) \to 0$$

It is also possible to show a reverse dom: .... s.t.

(1) $f_{n_k} \to f$ a.e. on X

(2) $\|f_{n_k}\| \leq w(x)$ for a.e. x $\in X$

## Derivatives of measures

.....

necessary condition

$$\exists \frac{d\nu}{d\mu} \Rightarrow \nu << \mu$$

***Proof.*** $\nu(E) = \int_E ()..$ ★

.....

**Theorem 2.1**
Radon Nykodim Theorem ....

**Remark 3**
if $\mu$ is not sigma finite the theorem may fail. .....

mi sto addormentando io ci sto provando anche altre cose su radon

## Product Measure

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. the goal is to define a measure space on $X \times Y$

**Definition 3.1**
we call measurable rectangle in $X \times Y$ a set of type $A \times B$ where $A \in \mathcal{M}, B \in \mathcal{N}$

$$R = \{A \times B \subset X \times Y....\}$$

We define the product $\sigma$ algebra ...

**Definition 3.2**
let $E \subset X \times Y$ For $\bar{x} \in X$ and $\bar{y} \in Y$ we define

(1) $E_{\bar{x}} = \{y \in Y : (\bar{x}, y) \in E\} \subset Y$

(2) $E_{\bar{y}} = \{x \in X : (x, \bar{y}) \in Y\} \subset E$

**Proposition 3.1**
$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. ....

...

**Theorem 3.1**
If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are $\sigma$ finite spaces, then:

(1) if $\phi$ is $\mathcal{M}$ measurable and $\psi$ is $\mathcal{N}$ meas

(2) we have that

using ....

**Theorem 3.2**
iterated integrals for characteristic functions
$\mu \times \nu : \mathcal{M} \otimes \mathcal{N} \to \mathbb{R}$ defined by

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x)\, d\mu = \int_Y \mu(E_y)\, d\nu$$

is a measure, the product measure

...

**Theorem 3.3**
Let $\lambda_n$ be the lebesgue measure in $\mathbb{R}^n$. If $n = K + m$, then $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$ is the complection of $\mathbb{R}^k \times \mathbb{R}^m.., \lambda_k \otimes \lambda_m$