1 Lesson 22/09/2022

We will mainly focus on 2 situations:

(1) $((X, \mathcal{M}))$ is a measurable space obtained by means of an outer measure. Ex: $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$, (Y, d_y) metric space $\to (Y, \mathcal{B}(Y))$.

If $X \to Y$ is (Lebesgue) measurable $\iff (\mathcal{M}, \mathcal{B}(Y))$ is measurable

(2) $(X, d_X), (Y, d_Y)$ are metric spaces $\longrightarrow (X, \mathcal{B}(X)), (Y, \mathcal{B}(Y))$ $f : X \to Y$ is Borel measurable $\iff (\mathcal{M}(X), \mathcal{B}(Y))$-measurable.

Remark 2
f is Lebesgue measurable if the continuity of the borel set is a Lebesgue-measurable set.

Proposition 2.1
There are two parts:

(1) $(X, d_X), (Y, d_Y)$ metric spaces. If $f : X \to Y$ is continuous, then is Borel measurable

(2) (Y, d_Y) metric space. If $f : \mathbb{R}^n \to Y$ is continuous, then it is a Lebesgue measure.

Proof. The proof is divided in:

(1) f is continuous $\iff f^{-1}(A)$ is open $\forall\ A \subset Y$ open $\implies f^{-1}(A) \in \mathcal{B}(Y)\ \forall\ A \subset Y$ open Since $\mathcal{B}(Y) = \sigma_0$(open sets) by proposition (1) this implies that f is Borel measurable

(2) f is continuous $\overset{(1)}{\implies}$ f is Borel measurable. $f^{-1}(A) \in \mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{L}(\mathbb{R}^n) \forall\ A \in \mathcal{B}(Y)$. Namely f is Lebesgue measurable

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Proposition 2.2
$(X, \mathcal{M})$ measurable space, $(X, d_Y), (Y, d_Y)$ metric spaces. if $f : X \to Y$ is $\mathcal{M}, \mathcal{B}(Y)$-measurable and $g : Y \to Z$ is continuous $\implies g \circ f : x \to Z$ is $\mathcal{M}, \mathcal{B}(Y)$-measurable

Proposition 2.3
$(X, \mathcal{M})$ measurable space Let $\Phi : \mathbb{R}^n \to Y$ be continuous where (Y, d_Y) is a metric space. Then $h : X \to Y$ defined by $h(x) = \Phi(u(x), boh)$ is $\mathcal{M}, \mathcal{B}(Y)$-measurable.

Proof. Define $f : X \to \mathbb{R}^n$, $f(x) = u(x), v(x)$. By def $h = \Phi \circ f$ by prop 3 if we show that f is measurable, then h is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0 \left(\{ (a_1, b_1) \times (a_2, b_2) : a, b \in \mathbb{R} \} \right)$$

pezzi $f^{-1}(\mathcal{R} \in \mathcal{M})$ $\quad\forall$open rectangle in $\mathcal{R}^2$ $R = I \times J$ $F^{-1} = \{x \in X\}$ ★

Remark 3
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$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

cosine $(X, \mathcal{M})$ measurable space, then such a function f is measurable iff

$$f^{-1}(a, +\infty)] \in \mathcal{M} \quad \forall a \in \mathcal{R}$$

LEt now $\{f_n\}$ be a Sequence of measurable functions from X to $\bar{\mathcal{R}}$. Then we define

$$(\inf_n f_n)(x) = \inf_n f_n(x)$$

$$(\sup_n f_n)(x) = \sup_n f_n(x)$$

$$(\liminf_n f_n)(x) = \liminf_n f_n(x)$$

$$(\limsup_n f_n)(x) = \limsup_n f_n(x)$$

$$(\lim_n f_n)(x) = \lim_n f_n(x) \quad \text{if the limit exists}$$

Proposition 3.1
$(X, \mathcal{M})$ measurable space, $f_n : X \to \bar{\mathcal{R}}$ measurable, then $\sup \inf \liminf \limsup$ of f_n are measurable, in particular if $\lim f_n$ exists, then f is measurable

Proof. $(\sup f_n)^{-1}((a, \infty]) = \{x \in X : \sup f_n(x) > a\}$ (manca pezzi)

$$\bigcup \{x \in X : f_n(x) > a\}$$

Then $(\sup f_n)^{-1}((a, \infty])$ is measurable, cose da aggiungere Noe the limsup

$$\limsup_n f_n = \lim_n (\sup_{k>n} f_n(x))$$

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Simple functions

Definition 3.1
$(X, \mathcal{M})$ measurable space. A measurable function s: $X \to \bar{\mathcal{R}}$ is said to be simple if $s(X)$ is a finite set altre cose Then $s(x) = \sum_{n=1} a_n \chi_{E_n}(x)$ where E_n is a measurable set sistemare.

Particular case: if s:$\mathbb{R} \to \bar{\mathbb{R}}$ and each E_n is a finite union of intervals, then s is said to be a STEP FUNCTION.

The idea is to approximate functions with simple functions.

Theorem 3.1
$(X, \mathcal{M})$ measurable space, $f : X \to [0, \infty]$ measurable. Then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq f \quad \text{pointwise}$$

and $s_n(x) \to f(x)$ Moreover if f is bounded then $s_n \to f$ uniformly on X as $n \to \infty$

f is bounded. Fix $n \in \mathbb{N}$ and divide $[0, n)$ in $n \cdot 2^n$ intervals called $I_j = [a_j, b_j)$ with lenght $\frac{1}{2^n}$

Let $E_0 = f^{-1}([n, \infty)), E_j = f^{-1}([a_j, b_j))$ for $j = 1, \ldots, n \cdot 2^n$

We let Array

Namely we define

$$s_n(x) = n\chi_{E_0}(X) + \sum_{j=1}^{n \cdot 2^n} a_j \chi_{E_j}(x)$$

Then $s_n \leq s_{n+1}$ by contradiction

Then any $x \in X$ stays in $f^{-1}([a_j, b_j))$ for some j $\Longrightarrow$ ★