

# Notes from Real and Functional Analysis

Andrea Bonifacio & Ilaria Bonfanti

October 19, 2022

# 1 Lesson 12/09/2022

## Element of set theory

Let  $X$  be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \quad (\text{Power Set})$$

Let  $I \subseteq \mathbb{R}$  be a set of indexes. A family of sets induced by  $I$  is

$$\{E_i\}_{i \in I}, \quad E_i \subseteq X \quad (\text{Family/Collection})$$

If  $I = \mathbb{N}$  is called a

$$\{E_n\}_{n \in \mathbb{N}} \quad (\text{Sequence})$$

### Definition 1.1

$\{E_n\} \subseteq \mathcal{P}(X)$  is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1} \forall n \quad (\text{resp. } E_n \supseteq E_{n+1} \forall n)$$

and is written as

$$\{E_n\} \nearrow \quad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets  $\{E_i\}_{i \in I} \subseteq \mathcal{P}(X)$ , will be often considered

$$\bigcup_{i \in I} E_i = \{x \in X : \exists i \in I \text{ s.t. } x \in E_i\}$$

$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i, \forall i \in I\}$$

$\{E_i\}$  is said to be **disjoint** if  $E_i \cap E_j = \emptyset \forall i \neq j$ .

Examples:

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

### Definition 1.2

$\{E_n\} \subseteq \mathcal{P}(X)$ . We define

$$\limsup_n E_n := \bigcap_{k=1}^{\infty} \left( \bigcup_{n=k}^{\infty} E_n \right)$$

$$\liminf_n E_n := \bigcup_{k=1}^{\infty} \left( \bigcap_{n=k}^{\infty} E_n \right)$$

If these two sets are equal, then

$$\lim_n E_n = \limsup_n E_n = \liminf_n E_n$$

### Proposition 1.1

Some limits are:

- $\limsup_n E_n = \{x \in X : x \in E_n \text{ for } \infty - \text{many indexes } n\}$
- $\liminf_n E_n = \{x \in X : x \in E_n \text{ for all but finitely many indexes } n\}$

- $\liminf_n E_n \subseteq \limsup_n E_n$
- $(\liminf_n E_n)^C = \limsup_n E_n^C$

**Definition 1.3**

We can define:

$$\begin{aligned}
 x \in \limsup_n E_n &\iff x \in \bigcap_{k=1}^{\infty} \left( \bigcup_{n=k}^{\infty} E_n \right) \\
 &\iff \forall k \in \mathbb{N} : \bigcup_{n=k}^{\infty} E_n \\
 &\iff \forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k}
 \end{aligned}$$

$$\begin{aligned}
 \text{So } x \in \limsup_n E_n &\implies \exists m_1 = n_1 \text{ s.t. } x \in E_{n_1} \\
 &\exists m_2 := n_{m_1+1} \geq m_1 + 1 \text{ s.t. } x \in E_{n_2} \\
 &\vdots \\
 &\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1 \text{ s.t. } x \in E_{n_k} \\
 &\vdots \\
 &x \in E_{m_1}, \dots, E_{m_k}, \dots
 \end{aligned}$$

On the other hand, assume that  $x \in E_n$  for  $\infty$ -many indexes. We claim that  $\forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k} \iff x \in \limsup_n E_n$ . If that claim is not true, then  $\exists \bar{k} \text{ s.t. } x \notin E_n \forall n > \bar{k} \implies x$  belongs at most to  $E_1, \dots, E_{\bar{k}}$ , a contradiction. ★

**Definition 1.4**

$\{E_i\}_{i \in I}$  is a **covering** of  $X$  if

$$X \subseteq \bigcup_{i \in I} E_i$$

A subfamily of  $E_i$  that is still a covering is called a **subcovering**

**Definition 1.5**

Let  $E \subseteq X$ . The function  $\chi_E : X \rightarrow \mathbb{R}$

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \setminus E \end{cases}$$

is called **characteristic function** of  $E$

Let  $E_1, E_2$  be sets:

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

$$\{E_n\} \subseteq \mathcal{P}(X), \text{ disjoint, } E = \bigcup_{n=1}^{\infty} E_n \implies \chi_E = \sum_{n=1}^{\infty} \chi_{E_n}$$

$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \implies \chi_P = \liminf_n \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that  $\limsup_n a_n = \lim_{k \rightarrow \infty} \sup_{n \geq k} a_n$  and  $\liminf_n a_n = \lim_{k \rightarrow \infty} \inf_{n \geq k} a_n$

Let's also check that  $\chi_Q = \limsup_n \chi_{E_n}$

$$\begin{aligned}
 x \in \limsup_n E_n &\iff \chi_Q(x) = 1 \\
 &\iff \forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k}
 \end{aligned}$$

If we fix  $k$  then

$$\sup_{n \geq k} \chi_{E_n}(x) = \chi_{E_{n_k}}(x) = 1$$

$$\lim_{k \rightarrow \infty} \sup_{n \geq k} \chi_{E_n}(x) = \lim_n \sup \chi_{E_n}(x) = 1$$

Let now  $x \notin \limsup E_n \iff \chi_Q(x) = 0$ . Then  $x$  belongs at most to finitely many  $E_n \implies \exists \bar{k}$  s.t.  $x \notin E_n, \forall n \geq \bar{k}$

If  $k \geq \bar{k}$ , then  $\sup_{n \geq k} \chi_{E_n}(x) = 0 \implies \lim_{k \rightarrow \infty} \sup_{n \geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 0$

## Relations

Given  $X, Y$  sets, is called a **relation** of  $X$  and  $Y$  a subset of  $X \times Y$

$$R \subseteq X \times Y \quad R = \{(x, y) : x \in X, y \in Y\}$$

$$(x, y) \in R \iff xRy$$

$$X = \{0, 1, 2, 3\} \quad R = \{(0, 1), (1, 2), (2, 1)\} \text{ is a relation in } X$$

### Definition 1.6

A **function** from  $X$  to  $Y$  is a relation  $R$  s.t. for any element  $x$  of  $X$   $\exists!$  element  $y$  of  $Y$  s.t.  $xRy$

### Definition 1.7

$R$  on  $X$  is an **equivalence relation** if

- (1)  $xRx \forall x \in X$  ( $R$  is **reflexive**)
- (2)  $xRy \implies yRx$  ( $R$  is **symmetric**)
- (3)  $xRy, yRz \implies xRz$  ( $R$  is **transitive**)

If  $R$  is an equivalence relation, the set  $E_x := \{y \in X : yRx\}$ ,  $x \in X$  is called the **equivalence class** of  $X$

### Definition 1.8

$\frac{X}{R} := \{E_x : x \in X\}$  is the **quotient set**

Ex:  $X = \mathbb{Z}$ , let's say that  $nRm$  if  $n - m$  is even. This is an equivalence relation.

$$E_n = \{\dots, n-4, n-2, n, n+2, n+4, \dots\}$$

in this case if  $n$  is even,  $E_n = \{\text{even numbers}\}$  and if  $n$  is odd,  $E_n = \{\text{odd numbers}\}$

## Measure theory

### Definition 1.9

A family  $\mathcal{M} \subseteq \mathcal{P}(X)$  is called a  **$\sigma$ -algebra** if

- (1)  $X \in \mathcal{M}$
- (2)  $E \in \mathcal{M} \implies E^C = X \setminus E \in \mathcal{M}$
- (3) If  $E = \bigcup_{n \in \mathbb{N}} E_n$  and  $E_n \in \mathcal{M} \forall n$ , then  $E \in \mathcal{M}$

If  $\mathcal{M}$  is a  $\sigma$ -algebra,  $(X, \mathcal{M})$  is called **measurable space** and the sets in  $\mathcal{M}$  are called **measurable**. Ex:

- $(X, \mathcal{P}(X))$  is a measurable space

- Let  $X$  be a set, then  $\{\emptyset, X\}$  is a  $\sigma$ -algebra

**Remark 1.1**

$\sigma$  is often used to denote the closure w.r.t. countably many operators. If we replace the countable unions with finite unions in the definition of  $\sigma$ -algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space  $(X, \mathcal{M})$ :

- (1)  $\emptyset \in \mathcal{M}$ :  $\emptyset = X^C$  and  $X \in \mathcal{M}$
- (2)  $\mathcal{M}$  is an algebra, and  $E_1, \dots, E_n \in \mathcal{M}$

$$E_1 \cup \dots \cup E_n = E_1 \cup \dots \cup E_n \cup \underbrace{\emptyset}_{\in \mathcal{M}} \cup \emptyset \dots \in \mathcal{M}$$

- (3)  $E_n \in \mathcal{M}$ ,  $\bigcap_{n \in \mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n \in \mathbb{N}} E_n = \left( \bigcup_{n \in \mathbb{N}} \underbrace{E_n^C}_{\in \mathcal{M}} \right)^C \quad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \implies E \setminus F \in \mathcal{M} = E \setminus F = E \cap F^C \in \mathcal{M}$
- If  $\Omega \subset X$ , then the **restriction** of  $\mathcal{M}$  to  $\Omega$ , written as

$$\mathcal{M}|_{\Omega} := \{F \subseteq \Omega : F = E \cap \Omega, \text{ with } E \in \mathcal{M}\}$$

is a  $\sigma$ -algebra on  $\Omega$

**Theorem 1.1**

$\mathcal{S} \subseteq \mathcal{P}(X)$ . Then it is well defined the smallest  $\sigma$ -algebra containing  $\mathcal{S}$ , the  $\sigma$ -algebra generated by  $\mathcal{S} := \sigma_0(\mathcal{S})$ :

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$  and thus is a  $\sigma$ -algebra
- $\forall \sigma(\mathcal{M})$  s.t.  $\mathcal{M} \supseteq \mathcal{S}$ , we have  $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

*Proof idea.*

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \emptyset \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define  $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} : \mathcal{M} \in \mathcal{V}\}$ , so it can be proved that this is the desired  $\sigma$ -algebra ★

**Borel sets**

Given  $(X, d)$  metric space, the  $\sigma$ -algebra generated by the open sets is called **Borel**  $\sigma$ -algebra, written as  $\mathcal{B}(X)$ . The sets in  $\mathcal{B}(X)$  are called **Borel sets**. The following sets are Borel sets:

- open sets
- closed sets
- countable intersections of open sets:  $G_{\sigma}$  sets
- countable unions of closed sets:  $F_{\sigma}$  sets

**Remark 1.2**

$\mathcal{B}(\mathbb{R})$  can be equivalently defined as the  $\sigma$ -algebra generated by

$$\begin{aligned} &\{(a, b) : a, b \in \mathcal{R}, a < b\} \\ &\{(-\infty, b) : b \in \mathcal{R}\} \\ &\{(a, +\infty) : a \in \mathcal{R}\} \\ &\{[a, b) : a, b \in \mathcal{R}, a < b\} \\ &\vdots \end{aligned}$$

**2 Lesson 14/09/2022**

Question: What is  $\mathcal{B}(\mathbb{R})$ ? Is  $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$ ? No.

**Definition 2.1**

$(X, \mathcal{M})$  measurable space. A function  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  is called a **positive measure** if  $\mu(\emptyset) = 0$  and if  $\mu$  is countably additive, that is

$$\forall \{E_n\} \subseteq \mathcal{M} \text{ disjoint}$$

we have that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) \quad \sigma\text{-additivity}$$

**Remark 2.1**

a set  $A$  is countable if  $\exists f : \mathbb{N} \rightarrow A$  s.t.  $f$  is 1-1. Examples:  $\mathbb{Z}, \mathbb{Q}$  are countable, while  $\mathbb{R}$  is not, also  $(0, 1)$  is uncountable.

We always assume that  $\exists E \neq \emptyset, E \in \mathcal{M}$  s.t.  $\mu(E) < \infty$ .

If  $(X, \mathcal{M})$  is a measurable space, and  $\mu$  is a measure on it, then  $(X, \mathcal{M}, \mu)$  is a measure space.

Then:

(1)  $\mu$  is **finitely additive**:

$$\forall E, F \in \mathcal{M}, \text{ with } E \cap F = \emptyset \implies \mu(E \cup F) = \mu(E) + \mu(F)$$

(2) the **excision property**

$$\forall E, F \in \mathcal{M}, \text{ with } E \subset F \text{ and } \mu(E) < +\infty \implies \mu(F \setminus E) = \mu(F) - \mu(E)$$

(3) **monotonicity**

$$\forall E, F \in \mathcal{M}, \text{ with } E \subset F \implies \mu(E) \leq \mu(F)$$

(4) if  $\Omega \in \mathcal{M}$  then  $(\Omega, \mathcal{M}|_{\Omega}, \mu|_{\mathcal{M}|_{\Omega}})$  is a measure space

**Proof.** (1)  $E_1 = E, E_2 = F, E_3 = \dots = E_n = \dots = \emptyset$  This is a disjoint sequence  $\implies$  by  $\sigma$ -additivity.

$$\mu(E \cup F) = \mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n) = \mu(E) + \mu(F) + \underbrace{\mu(E_k)}_{=\mu(\emptyset)}$$

(2)  $E \subset F$ , so  $F = E \cup (F \setminus E)$  and this is disjoint  $\xRightarrow{(i)} \mu(F) = \mu(E) + \mu(F \setminus E)$ , and since  $\mu(E) < \infty$ , the property follows.

(3)  $E \subset F \implies \mu(F) = \mu(E) + \underbrace{\mu(F \setminus E)}_{\geq 0} \geq \mu(E)$

(4)

★

### Definition 2.2

$(X, \mathcal{M}, \mu)$  measure space.

- If  $\mu(X) < +\infty$ , we say that  $\mu$  is **finite**.
- If  $\mu(X) = +\infty$ , and  $\exists \{E_n\} \subset \mathcal{M}$  s.t.  $X = \bigcup_n E_n$  and each  $E_n$  has finite measure, then we say that  $\mu$  is  $\sigma$ -finite.
- If  $\mu(X) = 1$  we say that  $\mu$  is a **probability measure**.

Some examples:

- Trivial Measure:  $(X, \mathcal{M})$  measurable space.  $\mu : \mathcal{M} \rightarrow [0, \infty]$  defined by  $\mu(E) = 0 \quad \forall E \in \mathcal{M}$
- Counting Measure:  $(X, \mathcal{P}(X))$  measurable space. We define

$$\mu_C : \mathcal{P}(X) \rightarrow [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure:  $(X, \mathcal{P}(X))$  measurable space,  $t \in X$ . We define

$$\delta_t : \mathcal{P}(X) \rightarrow [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

### Continuity of the measure along monotone sequences

$(X, \mathcal{M}, \mu)$  measure space

(1)  $\{E_i\} \subset \mathcal{M}$ ,  $E_i \subseteq E_{i+1} \forall i$  and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:

$$\mu(E) = \lim_i \mu(E_i)$$

(2)  $\{E_i\} \subset \mathcal{M}$ ,  $E_{i+1} \subseteq E_i \forall i$  and let  $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$ .

**Proof.** (1) if  $\exists i$  s.t.  $\mu(E_i) = +\infty$ , then is trivial. Assume then that every  $E_i$  has a finite measure, so that  $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i)$  with  $E_0 = \emptyset$ .

So, by  $\sigma$ -additivity

$$\mu(E) = \mu \left( \bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i) \right) =$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \mu(E_{i+1} \setminus E_i) \stackrel{(excision)}{=} \sum_{i=0}^{\infty} (\mu(E_{i+1}) - \mu(E_i)) = \\
&\stackrel{(telescopic \ series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n)
\end{aligned}$$

(2) For simplicity, suppose  $\tau = 1$ , and define  $F_k = E_i \setminus E_k$

$$\begin{aligned}
&\{E_k\} \searrow \implies \{F_k\} \nearrow \\
&\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \setminus \left(\bigcap_k E_k\right) \\
&\mu(E_i) = \mu\left(\bigcup_k F_k\right) + \underbrace{\mu\left(\bigcap_k E_k\right)}_{\mu(E)} = \\
&\stackrel{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k (\mu(E_i) - \mu(E_k)) + \mu(E)
\end{aligned}$$

Since  $\mu(E_i) < \infty$  we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$

★

Counterexample: given  $(\mathcal{N}, \mathcal{P}(\mathbb{N}), \mu_C)$  measure space. Let  $E_n = \{n, n+1, n+2, \dots\}$ . In this case  $\mu_C(E_n) = +\infty$ ,  $E_{n+1} \subseteq E_n \forall n$ , but  $\bigcap_n E_n = \emptyset \implies \mu(\bigcap_n E_n) = 0$

**Theorem 2.1** ( $\sigma$ -subadditivity of the measure)

$(X, \mathcal{M}, \mu)$  is a measure space.  $\forall \{E_n\} \subseteq \mathcal{M}$  (not necessarily disjoint):  $\mu(\bigcup_n E_n) \leq \sum_n \mu(E_n)$

**Proof.**  $E_1, E_2 \in \mathcal{M}$  and also  $E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1)$  disjoint sets.

$$\mu(E_1 \cup E_2) = \mu\left(\underbrace{E_2 \setminus E_1}_{\subseteq E_2}\right) \stackrel{(monotonicity)}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.

$$\begin{aligned}
&A = \bigcup_{n=1}^{\infty} E_n, \quad A_k = \bigcup_{n=1}^k E_n \\
&\{A_k\} \nearrow, \quad A_{k+1} \supseteq A_k, \quad \lim_k A_k = A \\
&\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \stackrel{(continuity)}{=} \lim_k \mu(A_k) = \lim_k \mu\left(\bigcup_{n=1}^k E_n\right) \leq \\
&\leq \lim_k \sum_{n=1}^k \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)
\end{aligned}$$

★



Exercise:  $(X, \mathcal{M})$  measurable space.  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  s.t.  $\mu$  is finitely additive,  $\sigma$ -subadditive and  $\mu(\emptyset) = 0 \implies \mu$  is  $\sigma$ -additive, and hence is a measure.

Exercise: the Borel-Cantelli lemma states that, given  $(X, \mathcal{M}, \mu)$  and  $\{E_n\} \subseteq \mathcal{M}$ . Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \implies \mu(\limsup_n E_n) = 0$$

It can be phrased as:

If the series of the probability of the events  $E_n$  is convergent, then the probability that  $\infty$ -many events occur is 0

**Proof.** The thesis is:

$$\mu(\limsup_n E_n) = \mu\left(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k \geq n} E_k}_{A_n := \bigcup_{k \geq n} E_k}\right)$$

Is it true that  $\{A_n\} \searrow$ ? Yes.

$$A_{n+1} = \bigcup_{k \geq n+1} E_k \subseteq \bigcup_{k \geq n} E_k = A_n$$

Does some  $A_n$  have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k \geq n} E_k\right) \stackrel{\sigma\text{-sub.}}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

★

## Sets of 0 measure

$(X, \mathcal{M}, \mu)$  measure space.

- $N \subseteq X$  is a set of 0 measure if  $N \in \mathcal{M}$  and  $\mu(N) = 0$
- $E \subseteq X$  is called **negligible set** if  $\exists N \in \mathcal{M}$  with 0 measure s.t.  $E \subseteq N$  ( $E$  does not necessarily stay in  $\mathcal{M}$ )

### Definition 2.3

$(X, \mathcal{M}, \mu)$  measure space s.t. every negligible set is measurable (and hence of 0 measure), then  $(X, \mathcal{M}, \mu)$  is said to be a **complete measure space**.

A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \setminus F) = 0\}$$

Clearly  $\mathcal{M} \subseteq \overline{\mathcal{M}}$ . For  $E \in \overline{\mathcal{M}}$ , take  $F$  and  $G$  as above and let  $\bar{\mu}(E) = \bar{\mu}(F)$  then  $\bar{\mu}|_{\mathcal{M}} = \mu$ , and moreover:

**Theorem 2.2**

$(X, \mathcal{M}, \mu)$  is a complete measure space. Let's observe that  $\bar{\mu}$  is well defined: let  $E \subseteq X$  and  $F_1, F_2, G_1, G_2$  s.t.  $F_i \subset E \subset G_i$   $i = 1, 2$ . Then  $\mu(G_i \setminus F_i) = 0$ . Now we have to check that  $\mu(F_1) = \mu(F_2)$ .

Since

$$F_1 \setminus F_2 \subseteq E \setminus F_2 \subseteq G_2 \setminus F_2$$

and  $G_2 \setminus F_2$  has 0 measure  $\implies \mu(F_1 \setminus F_2) = 0$ . Then  $F_1 = (F_1 \setminus F_2) \cup (F_1 \cap F_2) \implies \mu(F_1) = \mu(F_1 \cap F_2)$ . In the same way,  $\mu(F_2) = \mu(F_1 \cap F_2)$

### 3 Lesson 15/09/2022

The elements of  $\overline{\mathcal{M}}$  are sets of the type  $E \cup N$ , with  $E \in \mathcal{M}$  and  $\bar{\mu}(N) = 0$ .

**Outer measure**

We wish to define a measure  $\lambda$  “on  $\mathcal{R}$ ” with the following properties:

- (1)  $\lambda((a, b)) = b - a$
- (2)  $\lambda(E + t)^\dagger = \lambda(E)$  for every measurable set  $E \subset \mathbb{R}$  and  $t \in \mathbb{R}$

It would be nice to define such a measure on  $\mathcal{P}(\mathbb{R})$ . In such case, note that  $\lambda(\{x\}) = 0, \forall x \in \mathbb{R}$   
But then

**Theorem 3.1** (Ulam)

The only measure on  $\mathcal{P}(\mathbb{R})$  s.t.  $\lambda(\{x\}) = 0 \quad \forall x$  is the trivial measure. Thus, a measure satisfying the two properties of the outer measure cannot be defined on  $\mathcal{P}(\mathcal{R})$

We'll learn in what follows how to create a measure space on  $\mathcal{R}$ , with a  $\sigma$ -algebra including all the Borel sets, and a measure satisfying properties of the outer measure. This is the so called **Lebesgue measure**.

**Definition 3.1**

Given a set  $X$ . An **outer measure** is a function  $\mu^* : \mathcal{P}(\mathbb{R}) \rightarrow [0, +\infty]$  s.t.

- $\mu^*(\emptyset) = 0$
- $\mu^*(A) \leq \mu^*(B)$  if  $A \subseteq B$  (Monotonicity)
- $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$  ( $\sigma$ -subadditivity)

The common way to define an outer measure is to start with a family of elementary sets  $\mathcal{E}$  on which a notion of measure is defined (e.g. intervals on  $\mathcal{R}$ , rectangles on  $\mathcal{R}^2, \dots$ ) and then to approximate arbitrary sets from outside by **countable** unions of members of  $\mathcal{E}$ .

**Proposition 3.1**

Let  $\mathcal{E} \subset \mathcal{P}(\mathbb{R})$  and  $\rho : \mathcal{E} \rightarrow [0, +\infty]$  be such that  $\emptyset \in \mathcal{E}, X \in \mathcal{E}$  and  $\rho(\emptyset) = 0$ . For any  $A \in \mathcal{P}(X)$ , let

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) : E_n \in \mathcal{E} \text{ and } A \subset \bigcup_{n=1}^{\infty} E_n \right\}$$

Then  $\mu^*$  is an outer measure, the outer measure generated by  $(\mathcal{E}, \rho)$ .

---

$^\dagger \{x \in \mathbb{R} : x = y + t, \text{ with } y \in E\}$

**Proof.**  $\forall A \subset X \exists \{E_n\} \subset \mathcal{E}$  s.t.  $A \subset \bigcup_n E_n$  : take  $E_n = X \forall n$  then  $\mu^*$  is well defined. Obviously,  $\mu^*(\emptyset) = 0$  (with  $E_n = \emptyset \forall n$ ), and  $\mu^*(A) \leq \mu^*(B)$  for  $A \subset B$  (any covering of  $B$  with elements of  $\mathcal{E}$  is also a covering of  $A$ .)

We have to prove the  $\sigma$ -subadditivity. Let  $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X)$  and  $\varepsilon > 0$ . For each  $n, \exists \{E_{n_j}\}_{j \in \mathbb{N}} \in \mathcal{E}$  s.t.  $A_n \subset \bigcup_{j=1}^{\infty} E_{n_j}$  and  $\sum_{j=1}^{\infty} \rho(E_{n_j}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$ . But then, if  $A = \bigcup_{n=1}^{\infty} A_n$ , we have that  $A \subset \bigcup_{n,j \in \mathbb{N}^2} E_{n_j}$  and

$$\mu^*(A) \leq \sum_{n,j} \rho(E_{n_j}) \leq \sum_n \left( \mu^*(A_n) + \frac{\varepsilon}{2^n} \right) = \sum_n \mu^*(A_n) + \varepsilon$$

Since  $\varepsilon$  is arbitrary, we are done. ★

Ex:

(1)  $X \in \mathbb{R}, \mathcal{E} = \{(a, b) : a \leq b, a, b \in \mathbb{R}\}$  family of open intervals:

$$\rho((a, b)) = b - a$$

(2)  $X = \mathbb{R}^n, \mathcal{E} = \{(a_1, b_1) \times \dots \times (a_n, b_n) : a_i \leq b_i, a_i, b_i \in \mathbb{R}\}$ :

$$\rho((a_1, b_1) \times \dots \times (a_n, b_n)) = (b_1 - a_1) \cdot \dots \cdot (b_n - a_n)$$

### Remark 3.1

$E \in \mathcal{E} \implies \mu^*(E) = \rho(E)$ .

In examples 1 and 2, we have in fact  $\mu^*((a, b)) = b - a, \mu^*((a_1, b_1) \times \dots \times (a_n, b_n)) = \prod_{i=1}^n (b_i - a_i)$

To pass from the outer measure to a measure there is a condition

### Definition 3.2 (Caratheodory condition)

If  $\mu^*$  is an outer measure on  $X$ , a set  $A \subset X$  is called  $\mu^*$ -**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E \subset X$$

### Remark 3.2

If  $E$  is a “nice” set containing  $A$ , then the above equality says that the outer measure of  $A$ ,  $\mu^*(E \cap A)$ , is equal to  $\mu^*(E) - \mu^*(E \cap A^C)$ , which can be thought as an “inner measure”. So basically we are saying that  $A$  is measurable if the outer and inner measure coincide. (Like the definition of Riemann integration with lower and upper sum)

### Remark 3.3

$\mu^*$  is subadditive by def  $\implies \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E, A \subset X$ . So, to prove that a set is  $\mu^*$ -measurable it is enough to prove the reverse inequality,  $\forall E \subset X$ . In fact, if  $\mu^*(E) = +\infty$ , then  $+\infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$ , and hence  $A$  is  $\mu^*$ -measurable iff

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E \subset X \text{ with } \mu^*(E) < +\infty$$

Their relevance to the notion of  $\mu^*$ -measurability is clarified by the following

### Theorem 3.2 (Caratheodory)

If  $\mu^*$  is an outer measure on  $X$ , the family

$$\mathcal{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$$

is a  $\sigma$ -algebra and  $\mu^*|_{\mathcal{M}}$  is a complete measure.

**Lemma 3.1**

If  $A \subset X$  and  $\mu^*(A) = 0$ , then  $A$  is  $\mu^*$ -measurable.

**Proof.** Let  $E \subset X$  with  $\mu^*(E) < +\infty$ . Then

$$\mu^*(E) \geq \mu^*(E) + \mu^*(A) \stackrel{\dagger}{=} \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

This implies that  $A$  is  $\mu^*$ -measurable. ★

To sum up:  $X$  set,  $(\mathcal{E}, \rho)$  elementary and measurable sets, so  $\mu^*$  is an outer measure. Then given  $\mu^*$  and the Caratheodory condition, we have  $(X, \mathcal{M}, \mu)$  that is a complete measure space.

**Remark 3.4**

So far we did not prove that  $\mathcal{E} \subseteq \mathcal{M}$ . We will do it in a particular case.

**Lebesgue measure**

- $X = \mathbb{R}$ ,  $\mathcal{E}$  family of open intervals,  $\rho((a, b)) = b - a = \lambda((a, b))$ , the complete measure space is  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  with  $\mathcal{L}(\mathbb{R})$  the Lebesgue-measurable sets on  $\mathbb{R}$  and  $\lambda$  the Lebesgue measure on  $\mathbb{R}$ .
- $X = \mathbb{R}^n$ ,  $\mathcal{E} = \{\prod_{k=1}^n (a_k, b_k) : a_k \leq b_k \quad \forall k = 1, \dots, n\}$ ,  $\rho(\prod_{k=1}^n (a_k, b_k)) = \prod_{k=1}^n (b_k - a_k)$  and this is a complete measure space  $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$

## 4 Lesson 21/09/2022

**Lebesgue measure**

$\mathcal{E}$  = family of open intervals  $(a, b)$ ,  $a, b \in \mathbb{R}^*$ ,  $a < b$ .  $\rho = \text{lenght } l$ .  $\rho((a, b)) = b - a$ .

Notations: open interval  $I$  with lenght  $l(I)$

**Outer measure**

$E \subset \mathbb{R}$ . The outer measure of  $E$  is

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{+\infty} l(I_n) \mid I_n \text{ is an open interval, } E \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

**Caratheodory condition (CC)**

$A \subset \mathbb{R}$  is  $\lambda^*$ -measurable if

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^C) \quad \forall E \subset \mathbb{R}$$

$$\{A \subset \mathbb{R} : A \text{ is } \lambda^*\text{-measurable}\} =: \mathcal{L}(\mathbb{R}) \quad (\text{Lebesgue } \sigma\text{-algebra})$$

$$\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})} \quad (\text{Lebesgue measure on } \mathbb{R})$$

Then,  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  is a complete measure space. In particular,  $\lambda^*(A) = 0 \implies A \in \mathcal{L}(\mathbb{R})$  and  $\lambda(A) = 0$ .

---

$\dagger E \cap A^C \subseteq E$  and  $E \cap A \subseteq A$  + monotonicity

**Remark 4.1** (CC-Criterion for measurability)

To check that  $A$  is  $\lambda^*$ -measurable, it is sufficient to check that

$$\lambda^* \geq \lambda^*(E \cap A) + \lambda^*(E \cap A^C)$$

for every  $E \subset \mathbb{R}$  with  $\lambda^*(E) < +\infty$

**Proposition 4.1**

Any countable set is measurable, with 0 Lebesgue measure.

**Proof.** Let  $a \in \mathbb{R}$ ,

$$\{a\} \subseteq (a - \varepsilon, a + \varepsilon), \forall \varepsilon > 0 \xrightarrow{\text{by def.}} \lambda^*(\{a\}) \leq 2\varepsilon \xrightarrow{\lim} \lambda^*(\{a\}) = 0$$

$\{a\}$  is measurable with  $\lambda(\{a\}) = 0, \forall a \in \mathbb{R}$ . Now if a set  $A$  is countable,  $A = \{a_n\}_{n \in \mathbb{N}} = \bigcup_n \{a_n\}$  (disjoint)  $\implies \lambda(A) \stackrel{\sigma\text{-add}}{=} \sum_n \lambda(\{a_n\}) = 0$  ★

**Remark 4.2**

$\lambda(\mathbb{Q}) = 0$ .  $\mathbb{Q}$  is dense on  $\mathbb{R}$ ,  $\bar{\mathbb{Q}} = \mathbb{R}$ . In general, measure theoretical info and topological info cannot be compared.

**Proposition 4.2**

$$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$$

**Remark 4.3**

So far we didn't prove the fact that open intervals are  $\mathcal{L}$ -measurable.

**Proof.** We know that  $\mathcal{B}(\mathbb{R})$  is generated by  $\{(a, +\infty) : a \in \mathbb{R}\}$ . Then, we can directly show that  $(a, +\infty) \in \mathcal{L}(\mathbb{R}) \quad \forall a \in \mathbb{R}$ . Let  $a \in \mathbb{R}$  be fixed. We use the criterion for measurability and we check that

$$\lambda^*(E) \geq \lambda^* \underbrace{(E \cap (a, +\infty))}_{=: E_1} + \lambda^* \underbrace{(E \cap (-\infty, a])}_{=: E_2} \quad \forall E \subset \mathbb{R}, \lambda^* < +\infty$$

Since  $\lambda^*(E) < +\infty$ ,  $\exists$  a countable union  $\bigcup_n I_n \supset E$ , where  $I_n$  is an open interval  $\forall n$  and

$$\sum_n l(I_n) \leq \lambda^*(E) + \varepsilon$$

Let  $I_n^1 := I_n \cap E_1, I_n^2 := I_n \cap (-\infty, a + \frac{\varepsilon}{2^n})$ . These are open intervals:

$$E_1 \subset \bigcup_n I_n^1 \quad E_2 \subset \bigcup_n I_n^2 \quad \text{countable unions}$$

and moreover

$$l(I_n) \geq l(I_n^1) + l(I_n^2) - \frac{\varepsilon}{2^n}$$

By definition of  $\lambda^*$ ,  $\lambda^*(E_1) \leq \sum_n l(I_n^1)$  and  $\lambda^*(E_2) \leq \sum_n l(I_n^2)$ , therefore

$$\lambda^*(E_1) + \lambda^*(E_2) \leq \sum_n l(I_n^1) + \sum_n l(I_n^2) \leq \sum_n \left( l(I_n) + \frac{\varepsilon}{2^n} \right) = \left( \sum_n l(I_n) \right) + \varepsilon \leq \lambda^*(E) + 2\varepsilon$$

Since  $\varepsilon$  was arbitrarily chosen, we have

$$\lambda^*(E) \geq \lambda^*(E_1) + \lambda^*(E_2)$$

which is the thesis. ★

So, the Lebesgue measure measures all the open, closed  $G_\delta$ ,  $F_\delta$  sets. Clearly

$$\lambda((a, b)) = b - a$$

One can also show that  $\lambda$  is invariant under translation.

Questions:  $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$ , is it a strict inclusion or not?

- By Ulam's theorem, if a measure is such that  $\lambda(\{a\}) = 0, \forall a$  and all the sets in  $\mathcal{P}(\mathbb{R})$  are measurable, then  $\lambda \equiv 0$ . This and the fact that  $\lambda((a, b)) \neq 0$  simply that  $\mathcal{L}(\mathbb{R}) \subsetneq \mathcal{P}(\mathbb{R})$  :  $\exists$  non-measurable sets called Vitali sets. Every measurable set with positive measure contains a Vitali set. (Explanation)
- $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$ . The construction of a  $\mathcal{L}$ -measurable set which is not a Borel set will be done during exercise classes.

The relation between  $\mathcal{B}(\mathbb{R})$  and  $\mathcal{L}(\mathbb{R})$  is clarified by

**Theorem 4.1** (Regularity of  $\lambda$ )

The following sentences are equivalent:

- (1)  $E \in \mathcal{L}(\mathbb{R})$
- (2)  $\forall \varepsilon > 0 \exists A \supset E, A$  open s.t.  
 $\lambda(A \setminus E) < \varepsilon$
- (3)  $\exists G \supset E, G$  of class  $G_\delta$ , s.t.  
 $\lambda(G \setminus E) = 0$
- (4)  $\exists C \subset E, C$  closed, s.t.  
 $\lambda(E \setminus C) = 0$
- (5)  $\exists F \subset E, F$  of class  $F_\delta$ , s.t.  
 $\lambda(E \setminus F) = 0$

**Consequence:**  $E \in \mathcal{L}(\mathbb{R}) \implies E = F \cup N$ , where  $F$  is of class  $F_\delta$ , and  $\lambda(N) = 0$ .

*Partial proof.* For simplicity, we will consider only sets with finite measure.

- (1)  $\implies$  (2)  $E \in \mathcal{L}(\mathbb{R})$ . By definition of  $\lambda^*$ ,  $\forall \varepsilon > 0 \exists \bigcup_n I_n \supset E$  s.t. each  $I_n$  is an open interval, and

$$\lambda(E) = \lambda^*(E) \geq \sum_n l(I_n) - \varepsilon$$

We define  $A = \bigcup_n I_n$ , which is open. Also  $A \supset E$  and

$$\lambda(A) = \lambda\left(\bigcup_n I_n\right) \stackrel{\sigma\text{-sub.}}{\leq} \sum_n l(I_n) \leq \lambda(E) + \varepsilon$$

Then, by excision

$$\lambda(A \setminus E) = \lambda(A) - \lambda(E) \leq \varepsilon$$

- (2)  $\implies$  (3) Define, for every  $K \in \mathbb{N}$ , an open set  $A_k$  s.t.  $A_k \supset E$  and  $\lambda(A_k \setminus E) < \frac{1}{k}$ . Let  $A = \bigcap_k A_k$ . This is a  $G_\delta$  set, it contains  $E$  (since each  $A_k$  contains  $E$ ) and

$$\lambda(A \setminus E) \stackrel{(A \subset \bigcap_k A_k \forall k)}{\leq} \lambda(A_k \setminus E) < \frac{1}{k} \implies \lambda(A \setminus E) = 0 \quad \forall k$$

---

<sup>†</sup>I had no choice

(3)  $\Rightarrow$  (1)] If  $E \subset \mathbb{R}$  and  $\exists G \supset E$ , with  $G$  of class  $G_\delta$ , s.t.  $\lambda(G \setminus E) = 0$ , then

$$E = G \setminus (G \setminus E) \text{ is measurable}$$

since  $G$  is a Borel set and  $(G \setminus E)$  has 0 measure, then both are in  $\mathcal{L}$

★

#### Remark 4.4

Any countable set has 0 measure. the inverse is false. An example is given by the **Cantor set**.

Let  $T_0 = [0, 1]$ . Then we define  $T_{n+1}$  starting from  $T_n$  in the following way: given  $T_n$ , finite union of closed disjoint intervals of length  $l_n(\frac{1}{3})^n$ ,  $T_{n+1}$  is obtained by removing from each interval of  $T_n$ , the open central subinterval of length  $\frac{l_n}{3}$ .

The Cantor set is  $T := \bigcap_{k=0}^{+\infty} T_k$ . It can be proved that  $T$  is compact,  $\lambda(T) = 0$  and  $T$  is uncountable.

If, instead of removing intervals of size  $\frac{1}{3}, \frac{1}{9}, \dots, \frac{1}{3^k}$ , we remove sets of size  $(\frac{\varepsilon}{3})^k$ , with  $\varepsilon \in (0, 1)$ , we obtain the **generalized Cantor set** (or **fat Cantor set**)  $T_\varepsilon$ .  $T_\varepsilon$  is uncountable, compact and has no interior points (it contains no intervals). However,  $\lambda(T_\varepsilon) = \frac{3(1-\varepsilon)}{3-2\varepsilon} > 0$

#### Remark 4.5

We worked on  $\mathbb{R}$ , but everything can be adapted to  $\mathbb{R}^n$

### Measurable functions and integration

#### Definition 4.1

$f : X \rightarrow Y$ , then it is well defined the counterimage

$$f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$$

$$E \mapsto f^{-1}(E) = \{x \in X : f(x) \in E\}$$

#### Definition 4.2

$(X, \mathcal{M}), (Y, \mathcal{N})$  measurable spaces.  $f : X \rightarrow Y$  is called **measurable** or  $(\mathcal{M}, \mathcal{N})$ -measurable if

$$f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{N}$$

so, the counterimage of measurable sets in  $Y$  is a measurable set on  $X$ .

## 5 Lesson 22/09/2022

To check if a function is measurable or not, it is often used the following proposition

#### Proposition 5.1

$(X, \mathcal{M}), (Y, \mathcal{N})$  measurable spaces. Let  $\mathcal{F} \subseteq \mathcal{P}(Y)$  be s.t.  $\mathcal{N} = \sigma_0(\mathcal{F})$ . Then

$$f : X \rightarrow Y \text{ is } (\mathcal{M}, \mathcal{N})\text{-measurable} \iff f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{F}$$

We will mainly focus on 2 situations:

- (1)  $((X, \mathcal{M}))$  is a measurable space obtained by means of an outer measure. Ex:  $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n)), (Y, d_Y)$  metric space  $\rightarrow (Y, \mathcal{B}(Y))$ .

If  $X \rightarrow Y$  is (Lebesgue) measurable  $\iff (\mathcal{M}, \mathcal{B}(Y))$  is measurable

- (2)  $(X, d_X), (Y, d_Y)$  are metric spaces  $\rightarrow (X, \mathcal{B}(X)), (Y, \mathcal{B}(Y))$   $f : X \rightarrow Y$  is Borel measurable  $\iff (\mathcal{M}(X), \mathcal{B}(Y))$ -measurable.

**Remark 5.1**

$f$  is Lebesgue measurable if the continuity of the borel set is a Lebesgue-measurable set.

**Proposition 5.2**

There are two parts:

- (1)  $(X, d_X), (Y, d_Y)$  metric spaces. If  $f : X \rightarrow Y$  is continuous, then is Borel measurable
- (2)  $(Y, d_Y)$  metric space. If  $f : \mathbb{R}^n \rightarrow Y$  is continuous, then it is a Lebesgue measure.

**Proof.** The proof is divided in:

- (1)  $f$  is continuous  $\iff f^{-1}(A)$  is open  $\forall A \subset Y$  open  $\implies f^{-1}(A) \in \mathcal{B}(Y) \forall A \subset Y$  open  
Since  $\mathcal{B}(Y) = \sigma_0(\text{open sets})$  by proposition (1) this implies that  $f$  is Borel measurable
- (2)  $f$  is continuous  $\xRightarrow{(1)}$   $f$  is Borel measurable.  $f^{-1}(A) \in \mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{L}(\mathbb{R}^n) \forall A \in \mathcal{B}(Y)$ .  
Namely  $f$  is Lebesgue measurable

★

**Proposition 5.3**

$(X, \mathcal{M})$  measurable space,  $(X, d_X), (Y, d_Y)$  metric spaces. If  $f : X \rightarrow Y$  is  $(\mathcal{M}, \mathcal{B}(Y))$ -measurable and  $g : Y \rightarrow Z$  is continuous  $\implies g \circ f : x \rightarrow Z$  is  $(\mathcal{M}, \mathcal{B}(Z))$ -measurable

**Proposition 5.4**

$(X, \mathcal{M})$  measurable space,  $u, v : X \rightarrow \mathbb{R}$  measurable functions. Let  $\Phi : \mathbb{R}^2 \rightarrow Y$  be continuous where  $(Y, d_Y)$  is a metric space. Then  $h : X \rightarrow Y$  defined by  $h(x) = \Phi(u(x), v(x))$  is  $\mathcal{M}, \mathcal{B}(Y)$ -measurable.

Consequence:  $u, v$  measurable  $\implies u + v$  is measurable.

**Proof.** Define  $f : X \rightarrow \mathbb{R}^2$ ,  $f(x) = (u(x), v(x))$ . By definition  $h = \Phi \circ f$  by proposition (3) if we show that  $f$  is  $(\mathcal{M}, \mathcal{B}(\mathbb{R}^2))$ -measurable, then  $h$  is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0(\underbrace{\{(a_1, b_1) \times (a_2, b_2) : a, b \in \mathbb{R}\}}_{\text{open rectangle}})$$

Thanks to proposition (1), to check that  $f$  is measurable. We can simply check that  $f^{-1}(\mathcal{R} \in \mathcal{M}) \forall$  open rectangle in  $\mathbb{R}^2$  and  $R = I \times J$ , with  $I$  and  $J$  open intervals:

$$\begin{aligned} F^{-1}(\mathbb{R}) &= \{x \in X : (u(x), v(x)) \in \mathbb{R}\} \\ &\quad \updownarrow \\ &= \{x \in X : u(x) \in I \text{ and } v(x) \in J\} \\ &= \underbrace{u^{-1}(I)}_{\in \mathcal{M}} \cap \underbrace{v^{-1}(J)}_{\in \mathcal{M}} \in \mathcal{M} \\ &\quad \text{since both } u, v \text{ are measurable} \end{aligned}$$

This completes the proof

★

Consequences: by proposition 3 and 4, if  $u$  and  $v$  are measurable, then also  $u + v$ ,  $u \cdot v$ . Other measurable functions include  $u^+ = \max\{u, 0\}$ ,  $u^- = -\min\{u, 0\}$ ,  $|u| = u^+ + u^-$ ,  $u^2, \dots$   
Recall that  $u = u^+ - u^-$



**Remark 5.2**

$u^+$  is measurable since  $u^+ = g \circ u$ , where:

$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

Most of the times we will work with functions  $f : X \rightarrow \mathbb{R}$  or  $f : X \rightarrow \underbrace{\overline{\mathbb{R}}}_{\mathbb{R} \cup \{\pm\infty\}} (X, \mathcal{M})$  measurable space, then such a function  $f$  is measurable iff

$$f^{-1}((a, +\infty)]^\dagger \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

or equivalently

$$f^{-1}([a, +\infty)) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

Let now  $\{f_n\}$  be a sequence of measurable functions from  $X$  to  $\overline{\mathbb{R}}$ . Then we define

$$\begin{aligned} (\inf_n f_n)(x) &= \inf_n f_n(x) \\ (\sup_n f_n)(x) &= \sup_n f_n(x) \\ (\liminf_n f_n)(x) &= \liminf_n f_n(x) \\ (\limsup_n f_n)(x) &= \limsup_n f_n(x) \\ (\lim_n f_n)(x) &= \lim_n f_n(x) \quad \text{if the limit exists} \end{aligned}$$

**Proposition 5.5**

$(X, \mathcal{M})$  measurable space,  $f_n : X \rightarrow \overline{\mathbb{R}}$  measurable, then

$$\sup_n f_n \quad \inf_n f_n \quad \liminf_n f_n \quad \limsup_n f_n$$

are measurable, in particular if  $\lim_n f_n$  is well defined, then  $f$  is measurable

**Proof.**  $(\sup_n f_n)^{-1}((a, \infty]) = \{x \in X : \sup_n f_n(x) > a\}$   
 $\Downarrow$   
 $\exists \text{ some indexes } n \text{ s.t. } f_n(x) > a$

$$\bigcup_n \{x \in X : f_n(x) > a\} = \bigcup_n \underbrace{f_n^{-1}((a, +\infty))}_{\in \mathcal{M}}$$

Then  $(\sup_n f_n)^{-1}((a, \infty])$  is measurable, since it is the countable union of measurable sets.

Now we check that the  $\limsup_n f_n$  is measurable

$$\limsup_n f_n(x) = \lim_n \underbrace{(\sup_{k \geq n} f_k(x))}_{\text{is decreasing on } n} = \inf_n (\sup_{k \geq n} f_k(x))$$

If we write  $g_n(x) = \sup_{k \geq n} f_k(x)$ , then

- $g_n$  is measurable, by what we proved previously
- $\limsup_n f_n = \inf_n g_n$  is measurable




---

<sup>†</sup>We use  $)$  if  $f$  takes values in  $\mathbb{R}$  and  $]$  if  $f$  takes values in  $\overline{\mathbb{R}}$

## Simple functions

### Definition 5.1

$(X, \mathcal{M})$  measurable space. A measurable function  $s : X \rightarrow \overline{\mathbb{R}}$  is said to be simple if  $s(X)$  is a finite set.

$$s(X) = \{a_1, \dots, a_n\} \text{ for some } n \in \mathbb{N}, a_i \neq a_j$$

Then  $s(x) = \sum_{n=1}^N a_n \chi_{E_n}(x)$ , where  $E_n$  is a measurable set,  $E_n = \{x \in X : s(x) = a_n\}$ , and  $E_i \cap E_j = \emptyset$  for  $i \neq j$ , and  $\bigcup_{n=1}^N E_n = X$ .

Particular case: if  $s : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ , and each  $E_n$  is a finite union of intervals, then  $s$  is said to be a STEP FUNCTION.

Goal: to approximate arbitrary measurable functions with simple functions.

### Theorem 5.1

$(X, \mathcal{M})$  measurable space,  $f : X \rightarrow [0, \infty]$  measurable. Then  $\exists$  a sequence  $\{s_n\}$  of simple functions s.t.

$$0 \leq s_1 \leq \dots \leq s_n \leq \dots \leq f \quad (\text{pointwise})$$

$\forall x \in X$

and  $s_n(x) \rightarrow f(x) \forall x \in X$  as  $n \rightarrow \infty$  Moreover if  $f$  is bounded then  $s_n \rightarrow f$  uniformly on  $X$  as  $n \rightarrow \infty$

for  $f$  bounded. Fix  $n \in \mathbb{N}$  and divide  $[0, n]$  in  $n \cdot 2^n$  intervals called  $I_j = [a_j, b_j)$  with length  $\frac{1}{2^n}$

Let  $E_0 = f^{-1}([n, +\infty))$ ,  $E_j = f^{-1}([a_j, b_j))$  for  $j = 1, \dots, n \cdot 2^n$

We let  $s_n(x) = a_j$  for  $x \in E_j$   
 $s_n(x) = n$  for  $x \in E_0$

Namely we define the simple function  $s_n$  as

$$s_n(x) = n \chi_{E_0}(x) + \sum_{j=1}^{n \cdot 2^n} a_j \chi_{E_j}(x)$$

Then  $s_n \leq s_{n+1}$  by contradiction, and, since  $f$  is bounded,  $E_0 = \emptyset$  for  $n$  sufficiently large ( $n > \sup f$ ).

Then any  $x \in X$  stays in  $f^{-1}([a_j, b_j))$  for some  $j$

$$\begin{aligned} \implies a_j &\leq f(x) < b_j \\ &\parallel \\ &s_n(x) \\ \implies 0 &\leq f(x) - s_n(x) < b_j - a_j = \frac{1}{2^n} \\ \implies \sup_{x \in X} |f(x) - s_n(x)| &< \frac{1}{2^n} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Namely,  $s_n \rightarrow f$  uniformly on  $X$ .

★

## 6 Lesson 29/09/2022

### Remark 6.1

On the relation between  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$  and  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  ( $\lambda =$  Lebesgue measure)

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$  is not complete. In fact,  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  is the completion of  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ .

Note that,  $\forall E \in \mathcal{L}(\mathbb{R}) \exists$  a  $G_\delta$ -set  $A$  and an  $F_\delta$ -set  $B$  s.t.

$$\begin{aligned} A &\supset E \text{ and } \lambda(A \setminus E) = 0 \\ B &\subset E \text{ and } \lambda(E \setminus B) = 0 \end{aligned}$$

$(X, \mathcal{M}, \mu)$  a complete measure space. Let  $P(x)$  be a proposition depending on  $x \in X$ . We say that  $P(x)$  is true  $(\mu-)$ almost everywhere if

$$\mu(\{x \in X : P(x) \text{ is false}\}) = 0$$

$P(x)$  is true  $\underset{(\mu-\text{a.e.})}{\text{a.e.}}$  on  $X$ .

Ex:  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ ,  $f(x) = x^2$ . Then  $f(x) > 0$  a.e. on  $\mathbb{R}$  (for a.e.  $x$ ):

$$\{f(x) \leq 0\} = \{0\}, \text{ and } \lambda(\{0\}) = 0$$

$(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu_C)$  with  $\mu_C$  counting measure. Then it is not true that  $f(x) > 0$   $\mu_C$ -a.e.

$$\mu_C(\{0\}) = 1$$

It will be useful to consider sequences converging a.e.:

$$f_n \rightarrow f \quad \text{a.e. on } X_{\text{mboxtext}}$$

if  $\mu(\{x \in X : \lim_n f_n(x) \neq f(x), \text{ or does not exist}\}) = 0$

### Proposition 6.1

$(X, \mathcal{M}, \mu)$  complete measure space.

(1)  $f : X \rightarrow \mathbb{R}$  is measurable, and  $g = f$  a.e. on  $X$ , then  $g$  is measurable

(2)  $f_n \rightarrow f$  a.e. on  $X$ ,  $f_n : X \rightarrow \mathbb{R}$  measurable for all  $n$ , then  $f$  is measurable

## Integration of non-negative functions

Notation:

$$\begin{aligned} \{x \in X : f(x) \geq 0\} &= \{f \geq 0\} \\ \{x \in X : f(x) > 0\} &= \{f > 0\} \\ &\vdots \end{aligned}$$

$(X, \mathcal{M}, \mu)$  complete measure space. We consider measurable functions  $f : X \rightarrow [0, +\infty]$

Convention: we define

$$\begin{aligned} a + \infty &= +\infty \quad \forall a \in \mathbb{R} \\ a \cdot (+\infty) &= \begin{cases} +\infty & \text{if } a \neq 0, a > 0 \\ 0 & \text{if } a = 0 \end{cases} \end{aligned}$$

With this convention,  $+$  and  $\cdot$  of measurable functions are measurable functions.

### Definition 6.1

Let  $s : X \rightarrow [0, +\infty]$  be a measurable simple function,

$$s(x) = \sum_{n=1}^m a_n \chi_{D_n}(\bar{x})$$

where  $D_1, \dots, D_m$  are measurable, disjoint, and  $\bigcup_{n=1}^m D_n = X$ . Let also  $E \in \mathcal{M}$ . Then we define

$$\int_E s \, d\mu := \sum_{n=1}^m a_n \mu(D_n \cap E)$$

**Remark 6.2**

Given a simple function  $s$ :

$$s : [a, b] \rightarrow \mathbb{R}, \lambda = \mu \implies \int_E s d\mu \text{ is the area under the curve}$$

**Remark 6.3**

There are several points:

- In the definition we have already used the convention  $\mu(D_n \cap E = +\infty)$  for some  $n$
- $E \in \mathcal{M} \implies \chi_E$  is a simple function

$$\chi_E(x) = 1 \cdot \chi_E + 0 \cdot \chi_{X \setminus E}(x)$$

In this case

$$\int_X \chi_E d\mu = 1 \cdot \mu(E) + 0 \cdot \mu(X \setminus E) = \mu(E)$$

- $s\chi_E = \sum_{n=1}^m a_n \chi_{D_n \cap E} \implies \int_E s d\mu = \int_X s\chi_E d\mu$

**Definition 6.2**

$f : X \rightarrow [0, +\infty]$  measurable,  $E \in \mathcal{M}$ . The **Lebesgue integral** of  $f$  on  $E$ , with respect to (w.r.t.)  $\mu$ , is

$$\int_E f d\mu = \sup \left\{ \int_E s d\mu \mid \begin{array}{l} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

- (1) If  $f$  is simple, the definitions are consistent
- (2) Also for  $f$  measurable:  $\int_E f d\mu = \int_X f\chi_E d\mu$
- (3)  $(\mathbb{N}, \mathbb{N}, \mu_c)$ .  $f : \mathbb{N} \rightarrow \mathbb{R}$  is a sequence  $\{a_n\}_{n \in \mathbb{N}}$

$$\int_{\mathbb{N}} \{a_n\} d\mu_c = \sum_{n=0}^{\infty} a_n$$

Basic Properties.

Let  $f, g : X \rightarrow [0, \infty]$  measurable.  $E, F \in \mathcal{M}$ ,  $\alpha \geq 0$ . Then:

- (1)  $\mu(E) = 0 \implies \int_E f d\mu = 0$
- (2)  $f \leq g$  on  $E \implies \int_E f d\mu \leq \int_E g d\mu$
- (3)  $E \subset F \implies \int_E f d\mu \leq \int_F f d\mu$
- (4)  $\alpha \geq 0 \implies \int_E \alpha f d\mu = \alpha \int_E f d\mu$

**Remark 6.4**

$([0, 1], \mathcal{L}([0, 1]), \lambda)$

Consider  $\chi_{\mathbb{Q}}$ , it is the Dirichlet function on  $[0, 1]$ . This is not Riemann integrable. However,  $\int_{[0, 1]} \chi_{\mathbb{Q}} d\lambda = \lambda(\mathbb{Q} \cap [0, 1]) = 0$

**Theorem 6.1** (Chebychev's inequality)

$f : X \rightarrow [0, \infty]$  measurable,  $c > 0$ . Then

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int \{f \geq c\} f d\mu \leq \frac{1}{c} \int_X f d\mu$$

**Proof.**  $\int_X f d\mu \stackrel{X \supset \{f \geq c\}}{\geq} \int_{\{f \geq c\}} f d\mu \geq \int_{\{f \geq c\}} c d\mu = c \int_{\{f \geq c\}} d\mu = c\mu(\{f \geq c\})$  ★

**Theorem 6.2**

$s : X \rightarrow [0, \infty]$  simple. Define  $\varphi : \mathbb{M} \rightarrow [0, \infty]$

$$\varphi(E) = \int_E s d\mu$$

$\Rightarrow \varphi$  is a measure.

**Proof.**  $\mu(\emptyset) = 0 \Rightarrow \varphi(\emptyset) = 0$  by definition.

**Definition 6.3** (sigma additivity)

$\{E_n \subset \mathbb{M}\}$  disjoint, and let  $E = \bigcup_{n=1}^{\infty} E_n \Rightarrow s = \sum_{k=1}^m a_k \chi_{D_k} \quad D_k \in \mathbb{M}$

Then, by definition and since  $\mu$  is a measure and  $E \cap D_k = \bigcup_n (E_n \cap D_k)$   
 $\varphi(E) = \sum_{k=1}^m a_k \mu(D_k \cap E) = \sum_{k=1}^m a_k \sum_{n=1}^{\infty} \mu(E_n \cap D_k) = \sum_{n=1}^{\infty} (\sum_{k=1}^m a_k \mu(E_n \cap D_k)) = \sum_{n=1}^{\infty} \int_{E_n} s d\mu = \sum_{n=1}^{\infty} \varphi(E_n)$  ★

**Theorem 6.3** (Vanishing Lemma)

$f : X \rightarrow [0, \infty]$  measurable.  $E \subset X$  measurable

$$\int_E f d\mu = 0 \iff f = 0 \text{ a.e. on } E$$

**Proof.**  $\Leftarrow$  easy //  $\Rightarrow$  Consider  $E \cap \{f > 0\} = \bigcup_{n=1}^{\infty} \underbrace{E_n \{f \geq \frac{1}{n}\}}_{=: E_n}$  Then  $\{E_n\}$  is an increasing sequence. By Chebyshev

$$\mu(E_n) \leq \frac{1}{\frac{1}{n}} \int_E f d\mu = 0 \quad \forall n \Rightarrow \mu(E_n) = 0 \quad \forall n$$

$\mu(E) \cup \{f > 0\} \stackrel{\text{continuity}}{=} \lim_n \mu(E_n) = 0$ , namely  $f = 0$  a.e. on  $E$  ★

The  $\int$  does not see sets with 0 measure.

**Definition 6.4**

If  $f : X \rightarrow [0, \infty]$  is measurable, and  $\int_X f d\mu < \infty$  then we say that  $f$  is integrable

**Theorem 6.4** (Monotone Convergence - Beppo Levi)

$f_n : X \rightarrow [0, \infty]$  measurable. Suppose that

- (1)  $f_n(x) \leq f_{n+1}(x)$  for a.e.  $x \in X$  for every  $n$
- (2)  $f_n \rightarrow f$  a.e. on  $X$

Then

$$\int_X f d\mu = \lim_n \int_X f_n d\mu$$

**Proof.** Part 1. Assume that 1) and 2) hold everywhere. First, if  $f$  is measurable  $\int_X f_n d\mu \nearrow \Rightarrow \exists \alpha = \lim_n \int_X f_n d\mu$  Also,  $f_n \leq f$  everywhere  $\Rightarrow \int_X f_n d\mu \leq \int_X f d\mu \quad \forall n$   
 $\Rightarrow \alpha \leq \int_X f d\mu$  We want to show that also  $\geq$  is true. Let  $s$  be a simple function s.t.  $0 \leq s \leq f$  and  $c \in (0, 1)$  Let  $E_n = \{f_n \geq cs\} \in \mathbb{M}$

- (1)  $E_n \in E_{n+1} \quad \forall n$  :  
if  $x \in E_n$ , then  $f_n(x) \geq cs(x) \Rightarrow f_{n+1}(x) \geq cs(x)$   
 $\Rightarrow f_{n+1}(x) \geq f_n(x) \geq cs(x) \Rightarrow x \in E_{n+1}$

(2) Moreover,  $X = \bigcup_{n=1}^{\infty} E_n$ . Indeed:

- if  $f(x) = 0$ , then  $s(x) = 0 \Rightarrow f_1(x) = 0 = cs(x)$ ,  $x \in E_1$
- if  $f(x) > 0$ , then  $cs(x) < f(x) = \lim_n f_n(x)$  since  $s \leq f$  and  $c < 1$   
 $\Rightarrow cs(x) < f_n(x)$  for  $n$  sufficiently large, namely  $x \in E_n$  for  $n$  sufficiently large.

Therefore,  $\alpha \geq \int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq c \int_{E_n} s d\mu = c\varphi(E_n) \forall n, \forall 0 \leq s \leq f, \forall c \in [0, 1]$   $\varphi(E_n) = \int_{E_n} s d\mu$ .  $\varphi$  is a measure, and  $\{E_n\} \nearrow$  Therefore, taking the lim when  $n \rightarrow \infty$  by continuity  $\alpha \geq \lim_n c \int_{E_n} s d\mu = c \int_X s d\mu \forall c \in [0, 1]$  Take the limit when  $c \rightarrow 1^-$  :  $\alpha \geq \int_X s d\mu \forall 0 \leq s \leq f$  Take the sup over  $s$ :  $\alpha \geq \int_X f d\mu$  We proved both inequalities, so the thesis holds.

Part 2. Note that  $\{x \in X : \text{assumptions (1) and (2) of the theorem do not hold}\}$  is a set of zero measure. Take  $F$ .  $X = E \cup F$  since we have the assumption on  $E$  and  $\mu(F) = 0$  Then, by the Vanishing Lemma, since  $(f - f\chi_E) = 0$  a.e. and  $(f_n - f_n\chi_E) = 0$  we have that

$$\int_X f d\mu = \int_E f d\mu = \lim_n \int_E f_n d\mu = \lim_n \int_X f_n d\mu$$

★

## 7 Lesson 05/10/2022

**Theorem 7.1** (Monotone Convergence (or Beppo Levi's theorem))

$f_n : X \rightarrow [0, +\infty]$  measurable. Suppose that

- (1)  $f_n(x) \leq f_{n+1}(x)$  for a.e.  $x \in X$ , for every  $n$
- (2)  $f_n \rightarrow f$  a.e. on  $X$

Then

$$\int_X f d\mu = \lim_n \int_X f_n d\mu$$

**Corollary 7.1**

$f_n : X \rightarrow [0, +\infty]$  measurable, then

$$\int_X \left( \sum_{n=0}^{\infty} f_n \right) d\mu = \sum_{n=0}^{\infty} \int_X f_n d\mu$$

**Theorem 7.2** (Approximation with simple functions)

Given  $(X, \mathcal{M})$  measure space,  $f : X \rightarrow [0, +\infty]$  measurable, then  $\exists$  a sequence  $\{s_n\}$  of simple functions s.t.

$$0 \leq s_1 \leq \dots \leq s_n \leq \dots \leq f \quad \text{pointwise } \forall x \in X$$

and

$$s_n(x) \rightarrow f(x) \quad \forall x \in X \text{ as } n \rightarrow \infty$$

Moreover, if  $f$  is bounded, then  $s_n \rightarrow f$  uniformly on  $X$  as  $n \rightarrow \infty$ .

**Remark 7.1**

$$\int_X f d\mu = \sup \left\{ \int_X s d\mu \mid \begin{array}{l} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

## 8 Lesson 06/10/2022

$f \notin R(I)$ . Is it true that  $\exists g \in R(I)$  s.t.  $g = f$  almost everywhere (a.e.) on  $I$ ? No.

For instance, consider  $T_{\mathcal{E}}$ , the generalized Cantor set ( $\lambda(T_{\mathcal{E}})$ ). Consider  $\chi_{\mathcal{E}}$ . In general,  $\chi_A$  is discontinuous on  $\delta A$ . But  $T_{\mathcal{E}}$  has no interior parts  $\implies T_{\mathcal{E}} = \delta T_{\mathcal{E}} \implies \chi_{T_{\mathcal{E}}}$  is discontinuous on  $T_{\mathcal{E}}$ . cosine

Clearly

$$\int_{[0,1]} \chi_{T_{\mathcal{E}}} d\lambda = \lambda(T_{\mathcal{E}})$$

so  $\chi_{T_{\mathcal{E}}} \in \mathcal{L}^1([0,1])$ . If  $g = \chi_{T_{\mathcal{E}}}$  a.e., then  $g$  is discontinuous at almost every part of  $T_{\mathcal{E}} \implies g$  is discontinuous on a set of positive measure  $\implies g \notin R(I)$ . So, the Lebesgue integral is a true extension of the Riemann one.

Regarding generalized integrals we have

### Theorem 8.1

$-\infty \leq a < b \leq +\infty$ ,  $f \in R^g([a,b])$  where

$$R^g([a,b]) = \{\text{Riemann-int functions on } [a,b] \text{ in the generalized sense}\}$$

Then,  $f$  is  $([a,b], \mathcal{L}([a,b]))$ -measurable. Moreover

$$(1) \quad f \geq 0 \text{ on } [a,b] \implies f \in \mathcal{L}^1([a,b])$$

$$(2) \quad |f| \in R^g([a,b]) \implies f \in \mathcal{L}^1([a,b])$$

and in both cases

$$\int_{[a,b]} f d\lambda = \int_a^b f(x) dx$$

If  $f$  is in  $R^g([a,b])$ , but  $|f| \notin R^g([a,b])$ , then the two notions of  $\int$  are not really related

$$\text{Ex: } f(x) = \frac{\sin x}{x}, \quad x \in [1, \infty]$$

$$\int_1^{\infty} |f(x)| dx = +\infty \implies f \notin \mathcal{L}^1([1, +\infty])$$

. But on the other hand

$$\int_1^{\infty} \frac{\sin x}{x} dx = \lim_{\omega \rightarrow \infty} \int_1^{\omega} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

### Spaces of integrable functions

$(X, \mathcal{M}, \mu)$  complete measure space.

$$\mathcal{L}^1 = \{f : X \rightarrow \overline{\mathbb{R}} : f \text{ is integrable}\}$$

$\mathcal{L}^1$  is a vector space. On  $\mathcal{L}^1$  we can introduce  $d : \mathcal{L}^1 \times \mathcal{L}^1 \rightarrow [0, +\infty)$  defined by

$$d_1(f, g) = \int_X |f - g|$$

cose

However,  $d_1$  is not a distance on  $\mathcal{L}^1(X)$ , since

$$d_1(f, g) = 0 \implies f = g \quad \text{a.e on } X \quad (\text{Pseudo-distance})$$

To overcome this problem, we introduce an equivalent relation in  $\mathcal{L}^1(X)$ : we say that

$$f \sim g \iff f = g \quad \text{a.e. on } X$$

If  $f \in \mathcal{L}^1(X)$ , we can consider the equivalence class

$$[f] = \{g \in \mathcal{L}^1(X) : g = f \text{ a.e. on } X\}$$

We define

$$L^1(X) = \underline{\mathcal{L}^1(X)}$$

$L^1(X)$  is a vector space, and on  $L^1(X)$  the function  $d_1$  is a distance:

$$d_1([f], [g]) = \int_X |f - g| \, d\mu$$

To simplify the notations, the elements of  $L^1(X)$  are called functions, and one writes  $f \in L^1(X)$ . With this, we mean that we choose a representative in  $[f]$ , and  $f$  denotes both the representative and the equivalence class. The representative can be arbitrarily modified on any set with 0 measure.

Another relevant space of measurable functions is the space of **essentially bounded** functions

**Definition 8.1**

$f : X \rightarrow \overline{\mathbb{R}}$  measurable is called essentially bounded if  $\exists M > 0$  s.t.

$$\mu(\{x \in X : |f(x)| \geq M\}) = 0$$

Ex:

$$f(x) = \begin{cases} 1 & x > 0 \\ +\infty & x = 0 \\ 0 & x < 0 \end{cases}$$

For  $M > 1$ ,  $\lambda(\{x \in \mathbb{R} : |f(x)| > M\}) = \lambda(\{0\}) = 0 \implies f$  is essentially bounded. If  $f$  is essentially bounded, it is well defined the **essential supremum** of  $f$ .

$$esssup_X f := \inf \{M > 0 \text{ s.t. } f \leq M \text{ a.e. on } X\}$$

It can also be defined on essential inf.

**Remark 8.1**

Note that, by def of inf,  $\forall \varepsilon > 0$  we have

$$f \leq (esssup_X f) + \varepsilon$$

We define

$$L^\infty(X, \mathcal{M}, \mu) = \underline{\mathcal{L}^\infty(X, \mathcal{M}, \mu)}$$

$L^\infty(X)$  is a vector space, and it is also a metric space for  $d_\infty(f, g) = esssup_X |f - g|$



## Relation between different types of convergence

$\{f_n\}$  sequence of measurable functions  $X \rightarrow \overline{\mathbb{R}}$

- recupera
- $f_n \rightarrow f$  pointwise
- $f_n \rightarrow f$  uniformly
- $f_n \rightarrow f$
- Convergence in  $L^1(X)$
- Convergence in measure/probability

cose cose parlavo con ila

**Theorem 8.2** (Egorov)

Let  $\mu(X) < +\infty$ , and suppose that  $f_n \rightarrow f$  a.e. on  $X$ . Then,  $\forall \varepsilon > 0, \exists X_\varepsilon \subset X$ , measurable, s.t.

$$\mu(X \setminus X_\varepsilon) < \varepsilon$$

and  $f_n \rightarrow f$  uniformly on  $X_\varepsilon$

**Theorem 8.3**

If  $\mu(X) < +\infty$  and  $f_n \rightarrow f$  a.e. on  $X \implies f_n \rightarrow f$  is measure on  $X$

**Proof.** Let  $\alpha > 0$ . We want to show that  $\forall \varepsilon > 0 \exists \bar{n} \in \mathbb{N}$  s.t.

$$n > \bar{n} \implies \mu(\{ \})$$

altre cosette

★

**Remark 8.2**

$\mu(X) < +\infty$  is essential

For example, in  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  consider

$$f_n(x) = \chi_{[n, n+1)}(x)$$

$f_n(x) \rightarrow 0$  for every  $x \in \mathbb{R}$ . However,  $\lambda(\{|f_n| \geq \frac{1}{2}\}) = \lambda([n, n+1)) = 1$  not 0

## 9 Lesson 12/10/2022

Typewriter sequence che però aveva iniziato la lezione scorsa

**Remark 9.1**

$f_p \not\rightarrow 0$  a.e. on  $[0, 1]$ . But consider  $\{f_{p(n,1)} : n \in \mathbb{N}\}$ . This is a subsequence and, by definition  $f_{p(n,1)}(x) = \chi_{n,1}(x) = \chi_{[0, \frac{1}{n}]}(x)$ . For this subsequence, we have  $f_{p(n,1)}(x) \rightarrow 0$  as  $n \rightarrow \infty \forall x \in (0, 1]$ , then a.e. on  $[0, 1]$

This is not random!

**Proposition 9.1**

If  $\mu(X) < \infty$  and  $f_n \rightarrow f$  in measure, then  $\exists$  a subsequence  $\{f_{n_k}\}$  s.t.  $f_{n_k} \rightarrow f$  a.e. on  $X$ .

Now we analyze the relation between convergence in  $L^1(X)$  and the other convergences.

**Theorem 9.1**

$\{f_n\} \subset L^1(X), f \in L^1(X)$ . If  $f_n \rightarrow f$  in  $L^1(X)$  then  $f_n \rightarrow f$  in measure on  $X$

**Proof.** By contradiction. Suppose that  $f_n \not\rightarrow f$  in measure on  $X$ :  $\exists \bar{\alpha} > 0$  s.t.

$$\limsup_{n \rightarrow \infty} \mu(\{|f_n - f| \geq \bar{\alpha}\}) > 0$$

$\Rightarrow \exists \bar{\varepsilon}$  and a subsequence  $\{f_{n_k}\}$  s.t.

$$\mu(\{|f_{n_k} - f| \geq \bar{\alpha}\}) > \bar{\varepsilon}$$

Consider then  $d_1(f_{n_k}, f) = \int_X |f_{n_k} - f| d\mu \geq \int_{\{|f_{n_k} - f| \geq \bar{\alpha}\}} 1 d\mu = \bar{\alpha} \mu(\{|f_{n_k} - f| \geq \bar{\alpha}\}) > \bar{\alpha} \bar{\varepsilon}$  But, by assumption,  $d_1(f_n, f) \rightarrow 0$

$$\Rightarrow d_1(f_{n_k}, f) \rightarrow 0$$

contradiction. ★

**Remark 9.2**

the convergence in measure doesn't imply the convergence in  $L^1$ . For example, consider  $f_n(x) = n\chi_{[0, \frac{1}{n}]}(x)$   $\mu(\{|f_n| \geq \alpha\}) \rightarrow 0$  for every  $\alpha$

On the other hand  $\int_{[0,1]} n\chi_{[0, \frac{1}{n}]} d\lambda = \int_{[0, \frac{1}{n}]} n d\lambda = n \frac{1}{n} = 1$   $f_n \not\rightarrow 0$  in  $L^1$

Convergence a.e.  $\not\Rightarrow$  convergence in  $L^1$ :

use the same example above,  $f_n \rightarrow 0$  a.e. on  $[0, 1] \not\Rightarrow f_n \rightarrow 0$  in  $L^1$

Convergence in  $L^1 \not\Rightarrow$  convergence a.e. Consider the typewriter sequence:  $d_1(f_{p(n,k)}, 0) \rightarrow 0$  when  $p \rightarrow \infty$

But we don't have a.e. convergence. However, recall the dominated convergence theorem: (DOM)

$$f_n \rightarrow f \text{ a.e.} + \exists \text{ of a dom function} \Rightarrow d(f_n, f) \rightarrow 0$$

It is also possible to show a reverse DOM: if  $f_n \rightarrow f$  in  $L^1(X)$ , then  $\exists$  a subsequence  $\{f_{n_k}\}$  and  $w \in L^1(X)$  s.t.

$$(1) f_{n_k} \rightarrow f \text{ a.e. on } X$$

$$(2) \|f_{n_k}\| \leq w(x) \text{ for a.e. } x \in X$$

**Derivatives of measures**

$(X, \mathcal{M}, \mu)$  measure space.  $\phi : X \rightarrow [0, \infty]$  measurable. We learned that  $\nu : \mathcal{M} \rightarrow [0, \infty]$  by

$$\nu(E) = \int_E \phi d\mu$$

is a measure on  $X, \mathcal{M}$ .

If the equation above holds, then we say that  $\phi$  is the Radon Nykodym derivative of  $\nu$  with respect to  $\mu$  and we write

$$\phi = \frac{d\nu}{d\mu}$$

**Definition 9.1**

$\mu, \nu$  measures on  $(X, \mathcal{M})$ . We say that  $\nu$  is absolutely continuous with respect to  $\mu$ ,  $\nu \ll \mu$  if

$$\mu(E) = 0 \Rightarrow \nu(E) = 0$$

**Lemma 9.1**

There is a necessary condition:

$$\exists \frac{d\nu}{d\mu} \Rightarrow \nu \ll \mu$$

**Proof.**  $\nu(E) = \int_E (\frac{d\nu}{d\mu}) d\mu = 0$  if  $\mu(E) = 0$  by basic properties of  $\int$  ★

**Theorem 9.2** (Radon Nykodim Theorem)

$(X, \mathcal{M})$  measurable space,  $\mu, \nu$  measures. If  $\nu \ll \mu$  and moreover  $\mu$  is  $\sigma$  finite, then  $\phi : \rightarrow [0, \infty]$  measurable s.t.  $\phi = \frac{d\nu}{d\mu}$  namely  $\nu(E) = \int_E \phi d\mu \forall E \in \mathcal{M}$

**Remark 9.3**

if  $\mu$  is not sigma finite the theorem may fail. In  $([0, 1], \mathcal{L}([0, 1]))$  consider the counting measure  $\mu = \mu_c$  and the lebesgue measure  $\nu = \lambda$   $\nu \ll \mu$  since  $\mu(E) = 0 \iff E = \emptyset \Rightarrow \lambda(E) = \nu(E) = 0$

But we can check that  $\nexists \phi : [0, 1] \rightarrow [0, \infty]$  measurable s.t.  $\lambda(E) = \int_E \phi d\mu_c$

Check by contradiction: assume that  $\phi$  does exist, and take  $x_0 \in [0, 1]$

$$0 = \lambda(\{x_0\}) = \int_{\{x_0\}} \phi d\mu_c = \phi(x_0)\mu_c(\{x_0\}) = \phi(x_0)$$

$\Rightarrow \phi(x_0) = 0 \forall x_0 \in [0, 1]$ . But then  $1 = \lambda([0, 1]) = \int_{[0, 1]} 0 d\mu_c = 0$ . Contradiction Note that  $\mu_c([0, 1]) = \infty$  and  $([0, 1], \mathcal{L}([0, 1]), \mu_c)$  is not  $\sigma$  - finite ( $[0, 1]$  is uncountable)

**Product Measure**

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  measure spaces. The goal is to define a measure space on  $X \times Y$

**Definition 9.2**

we call measurable rectangle in  $X \times Y$  a set of type  $A \times B$  where  $A \in \mathcal{M}, B \in \mathcal{N}$

$$R = \{A \times B \subset X \times Y \text{ s.t. } A \in \mathcal{M}, B \in \mathcal{N}\}$$

We define the product  $\sigma$  algebra  $\mathcal{M} \otimes \mathcal{N}$  as  $\sigma_0(R)$ .

This is a  $\sigma$  algebra in  $X \times Y$

**Definition 9.3**

let  $E \subset X \times Y$  For  $\bar{x} \in X$  and  $\bar{y} \in Y$  we define

$$\begin{aligned} E_{\bar{x}} &= \{y \in Y : (\bar{x}, y) \in E\} \subseteq Y & \bar{x}\text{-section of } E \\ E_{\bar{y}} &= \{x \in X : (x, \bar{y}) \in E\} \subseteq X & \bar{y}\text{-section of } E \end{aligned}$$

**Proposition 9.2**

$(X, \mathcal{M}), (Y, \mathcal{N})$  measurable spaces.  $E \in \mathcal{M} \otimes \mathcal{N}$

Then  $E_x \in \mathcal{M}$  and  $E_y \in \mathcal{N} \Rightarrow$  we can define

$$\begin{aligned} \varphi : X &\rightarrow [0, \infty] & \psi : Y &\rightarrow [0, \infty] \\ x &\mapsto \nu(E_x) & y &\mapsto \mu(E_y) \end{aligned}$$

**Theorem 9.3**

If  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  are  $\sigma$  finite spaces, then:

(1)  $\varphi$  is  $\mathcal{M}$  measurable and  $\psi$  is  $\mathcal{N}$  meas

(2) we have that  $\int_X \nu(E_x) d\mu = \int_Y \mu(E_y) d\nu$

Using the fact that  $\mu$  and  $\nu$  are measures, and that  $\int$  of non negative function is a measure, we deduce the following

**Theorem 9.4** (Iterated integrals for characteristic functions)

$\mu \otimes \nu : \mathcal{M} \otimes \mathcal{N} \rightarrow \mathbb{R}$  defined by

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x) d\mu = \int_Y \mu(E_y) d\nu$$

is a measure, the product measure.

**Remark 9.4**

On the completion of product measure spaces:

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  complete measures spaces. In general it is not true that  $(X \times Y, (M) \otimes)$  ...

**Theorem 9.5**

Let  $\lambda_n$  be the Lebesgue measure in  $\mathbb{R}^n$ . If  $n = K + m$ , then  $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$  is the completion of  $(\mathbb{R}^k \times \mathbb{R}^m, \mathcal{L}(\mathbb{R}^k) \otimes \mathcal{L}(\mathbb{R}^m), \lambda_k \otimes \lambda_m)$

## 10 Lesson 13/10/2022

**Integration on product spaces**

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  measure spaces.  $f : X \times Y \rightarrow \overline{\mathbb{R}}$  measurable.

If  $f \geq 0$ , then

$$\iint_{X \times Y} f d\mu \otimes d\nu$$

Goal: obtain a formula of iterated integral like the one in Analysis 2.

$\forall \bar{x} \in X$  and  $\bar{y} \in Y$

*cose*

**Proposition 10.1**

If  $f$  is measurable  $\Rightarrow f_{\bar{x}}$  is  $(\mathcal{N}, \mathcal{B}(\mathbb{R}))$ -measurable and  $f_{\bar{y}}$  is  $(\mathcal{M}, \mathcal{B}(\overline{\mathbb{R}}))$ -measurable. Then we can conclude  $\varphi : X \rightarrow \overline{\mathbb{R}}$ :

$$\varphi(x) = \int_Y f_x d\nu = \int_Y f(x, y) d\nu(y)$$

and  $\psi : Y \rightarrow \overline{\mathbb{R}}$

$$\psi(y) = \int_X f_y d\mu = \int_X f(x, y) d\mu(x)$$

Questions: what is the solution of  $\iint_{X \times Y} \text{cose cose}$

**Theorem 10.1** (Tonelli's theorem)

$(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  complete measure spaces and  $\sigma$ -finite. Suppose that  $f$  is  $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}(\overline{\mathbb{R}}))$ -measurable and that  $f \geq 0$  a.e. on  $X \times Y$ . Then  $\psi$  and  $\varphi$  are measurable and

$$\iint_{X \times Y} f d\mu \otimes d\nu = \text{cose}$$

Equally holds also if one of the integrals is  $\infty$ .

**Remark 10.1**

The double integral can be reduced to single integrals, iterated. Moreover we can always change the order of the integrals For sign changing functions the situation is more involved.

**Theorem 10.2** (Fubini's theorem)

$(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  complete measure spaces and  $\sigma$ -finite. If  $f \in L^1(X \times Y)$ , then  $\psi$  and  $\varphi$  defined above are measurable, and Fubini's theorem holds, and all the integrals are finite.

Question: how to check if  $f \in L^1(X \times Y)$ ? Typically, to check Fubini's theorem

If  $\iint_{X \times Y} |f| d\mu \otimes d\nu < \infty$  then we can apply Fubini for  $\iint_{X \times Y} f d\mu \otimes d\nu$

**Remark 10.2**

the proof of Fubini's and Tonelli's theorems is based for the iterated integrals for characteristic functions. (Note that  $(\mu \otimes \nu)(E) = \int_X (\int_Y \chi_E(x, y) d\nu) d\mu$ )

**Remark 10.3**

Sometimes double integrals are very useful to compute single integrals.

$$\text{Ex: } \int_{-\infty}^{+\infty} \exp -x^2 = \sqrt{\pi}$$

**The first fundamental theorem of calculus**

Consider  $f \in L^1([a, b])$ . We can define the **integral function**

$$F(x) = \int_{[a, x]} f d\lambda = \int_a^x f(t) dt,$$

If the function is continuous

What happens if ?