1 Lesson 14/09/2022

Question: What is $\mathcal{B}(\mathbb{R})$? Is $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$? No.

Definition 1.1
$(X, \mathcal{M})$ measurable space. A function $\mu : \mathcal{M} \to [0, +\infty]$ is called a **positive measure** if $\mu(\varnothing) = 0$ and if μ is countably additive, that is

$$\forall \{E_n\} \subseteq \mathcal{M} \quad \text{disjoint}$$

we have that

$$\mu\left(\bigcup_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \mu(E_n) \qquad \sigma\text{-additivity}$$

Remark 2
a set A is countable if $\exists\, f\ 1-1$ s.t. $f : A \to \mathbb{N}$ Examples: $\mathbb{Z}, \mathbb{Q}$ are countable, while $\mathbb{R}$ is not, also $(0,1)$ is uncountable.

We always assume that $\exists\, E \neq \varnothing, E \in \mathcal{M}$ s.t. $\mu(E) \neq \infty$.

If $(X, \mathcal{M})$ is a measurable space, and μ is a measure on it, then $(X, \mathcal{M}, \mu)$ is a measure space.

Then:

(1) μ is **finitely additive**:

$$\forall\, E, F \in \mathcal{M}, \text{ with } E \cap F \neq \varnothing \Longrightarrow \mu(E \cup F) = \mu(E) + \mu(F)$$

(2) the **excision property**

$$\forall\, E, f \in \mathcal{M}, \text{ with } E \subset F \text{ and } \mu(E) < +\infty \Longrightarrow \mu(F \backslash E) = \mu(F) - \mu(E)$$

(3) **monotonicity**

$$\forall\, E, F \in \mathcal{M}, \text{ with } E \subset F \Longrightarrow \mu(E) \leq \mu(F)$$

(4) if $\Omega \in \mathcal{M}$ then $(\Omega, \mathcal{M}|_\Omega, \mu|_{\mathcal{M}|_\Omega})$ is a measure space

Proof. (1) $E_1 = E, E_2 = F, E_3 = \ldots = E_n = \ldots = \varnothing$ This is a disjoint sequence $\Longrightarrow$ by σ-additivity.

$$\mu(E \cup F) = \mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n) = \mu(E) + \mu(F) + \underbrace{\mu(E_k)}_{=\mu(\varnothing)}$$

(2) $E \subset F$, so $F = E \cup (F \backslash E)$ and this is disjoint $\overset{(i)}{\Longrightarrow}$ $\mu(F) = \mu(E) + \mu(F \backslash E)$, and since $\mu(E) < \infty$, the property follows.

(3) $E \subset F \Longrightarrow \mu(F) = \mu(E) + \underbrace{\mu(F \backslash E)}_{\geq 0} \geq \mu(E)$

(4)

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Definition 2.1
$(X, \mathcal{M}, \mu)$ measure space.

- If $\mu(X) < +\infty$, we say that μ is **finite**.
- If $\mu(X) = +\infty$, and $\exists \ \{E_n\} \subset \mathcal{M}$ s.t. $X = \bigcup_n E_n$ and each E_n has finite measure, then we say that μ is σ-finite.
- If $\mu(X) = 1$ we say that μ is a **probability measure**.

Some examples:

- Trivial Measure: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(E) = 0 \quad \forall \ E \in \mathcal{M}$
- Counting Measure: $(X, \mathcal{P}(X))$ measurable space. We define

$$\mu_C : \mathcal{P}(X) \to [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure: $(X, \mathcal{P}(X))$ measurable space, $t \in X$. We define

$$\delta_t : \mathcal{P}(X) \to [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

<u>Continuity of the measure along monotone sequences</u>

$(X, \mathcal{M}, \mu)$ measure space

(1) $\{E_i\} \subset \mathcal{M}, \ E_i \subseteq E_{i+1} \ \forall i$ and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:

$$\mu(E) = \lim_i \mu(E_i)$$

(2) $\{E_i\} \subset \mathcal{M}, \ E_{i+1} \subseteq E_i \ \forall i$ and let $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$.

Proof. (1) if $\exists \ i$ s.t. $\mu(E_i) = +\infty$, then is trivial. Assume then that every E_i has a finite measure, so that $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty} (E_{i+1} \backslash E_i)$ with $E_0 = \varnothing$.

So, by σ-additivity

$$\mu(E) = \mu\left(\bigcup_{i=0}^{\infty} (E_{i+1} \backslash E_i)\right) =$$

$$= \sum_{i=0}^{\infty} \mu(E_{i+1} \backslash E_i) \overset{(excision)}{=} \sum_{i=0}^{\infty} (\mu(E_{i+1} - \mu(E_i))) =$$

$$\overset{(telescopic\ series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n)$$

(2) For simplicity, suppose $\tau = 1$, and define $F_k = E_i \backslash E_k$

$$\{E_k\} \searrow \Longrightarrow \{F_k\} \nearrow$$

$$\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \backslash (\bigcap_k E_k)$$

$$\mu(E_i) = \mu(\bigcup_k F_k) + \underbrace{\mu(\bigcap_k E_k)}_{\mu(E)} =$$

$$\overset{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k \left(\mu(E_i) - \mu(E_k)\right) + \mu(E)$$

Since $\mu(E_i) < \infty$ we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$

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Counterexample: given $(\mathcal{N}, \mathcal{P}(\mathbb{N}), \mu_C)$ measure space. Let $E_n = \{n, n+1, n+2, \ldots\}$. In this case $\mu_C(E_n) = +\infty, E_{n+1} \subseteq E_n \forall\, n$, but $\bigcap_n E_n = \varnothing \implies \mu\left(\bigcap_n E_n\right) = 0$

Theorem 2.1 (σ-subadditivity of the measure)
$(X, \mathcal{M}, \mu)$ is a measure space. $\forall \{E_n\} \subseteq \mathcal{M}$ (not necessarily disjoint): $\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n)$

Proof. $E_1, E_2 \in \mathcal{M}$ and also $E_1 \cup E_2 = E_1 \cup (E_2 \backslash E_1)$ disjoint sets.

$$\mu(E_1 \cup E_2) = \mu(\underbrace{E_2 \backslash E_1}_{\subseteq E_2}) \overset{(monotonicity)}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.

$$A = \bigcup_{n=1}^{\infty} E_n, \quad A_k = \bigcup_{n=1}^{k} E_n$$

$$\{A_k\} \nearrow,\ A_{k+1} \supseteq A_k,\ \lim_k A_k = A$$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{(continuity)}{=} \lim_k \mu(A_k) = \lim_k \mu\left(\bigcup_{n=1}^{k} E_n\right) \leq$$

$$\leq \lim_k \sum_{n=1}^{k} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

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Exercise: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, +\infty]$ s.t. μ is finitely additive, σ-subadditive and $\mu(\varnothing) = 0 \implies \mu$ is σ-additive, and hence is a measure.

Exercise: the Borel-Cantelli lemma states that, given $(X, \mathcal{M}, \mu)$ and $\{E_n\} \subseteq \mathcal{M}$. Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \implies \mu(\limsup_n E_n) = 0$$

It can be phrased as:

> If the series of the probability of the events E_n is convergent, then the probability that ∞-many events occur is 0

Proof. The thesis is:

$$\mu(\limsup_n E_n) = \mu\Bigg(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k\geq n} E_k}_{A_n := \bigcup_{k\geq n} E_k}\Bigg)$$

Is it true that $\{A_n\} \searrow$? Yes.

$$A_{n+1} = \bigcup_{k\geq n+1} E_k \subseteq \bigcup_{k\geq n} E_k = A_n$$

Does some A_n have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k\geq n} E_k\right) \leq \sum_{k\geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k\geq n} E_k\right) \overset{\sigma-sub.}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

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Sets of 0 measure

$(X, \mathcal{M}, \mu)$ measure space.

- $N \subseteq X$ is a set of 0 measure if $N \in \mathcal{M}$ and $\mu(N) = 0$
- $E \subseteq X$ is called **negligible set** if $\exists\ N \in \mathcal{M}$ with 0 measure s.t. $E \subseteq N$ (E does not necessarily stays in $\mathcal{M}$)

Definition 2.2
$(X, \mathcal{M}, \mu)$ measure space s.t. every negligible set is measurable (and hence of 0 measure), then $(X, \mathcal{M}, \mu)$ is said to be a **complete measure space**.

A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists\ F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \backslash F) = 0\}$$

Clearly $\mathcal{M} \subseteq \overline{\mathcal{M}}$. For $E \in \overline{\mathcal{M}}$, take F and G as above and let $\bar{\mu}(E) = \bar{\mu}(F)$ then $\bar{\mu}|_{\mathcal{M}} = \mu$, and moreover:

Theorem 2.2
$(X, \mathcal{M}, \mu)$ is a complete measure space. Let's observe that $\bar{\mu}$ is well defined: let $E \subseteq X$ and F_1, F_2, G_1, G_2 s.t. $F_i \subset E \subset G_i \quad i = 1, 2$. Then $\mu(G_i \backslash F_i) = 0$. Now we have to check that $\mu(F_1) = \mu(F_2)$.

Since

$$F_1 \backslash F_2 \subseteq E \backslash F_2 \subseteq G_2 \backslash F_2$$

and $G_2 \backslash F_2$ has 0 measure $\implies \mu(F_1 \backslash F_2) = 0$. Then $F_1 = (F_1 \backslash F_2) \cup (F_1 \cap F_2) \implies \mu(F_1) = \mu(F_1 \cap F_2)$. In the same way, $\mu(F_2) = \mu(F_1 \cap F_2)$