Notes from Real and Functional Analysis

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October 3, 2022

1 Lesson 12/09/2022

Element of set theory

Let X be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \tag{\textbf{Power Set}}$$

Let $I \subseteq \mathbb{R}$ be a set of indexes. A family of sets induced by I is

$$\{E_i\}_{i \in I}, \quad E_i \subseteq X \tag{\textbf{Family/Collection}}$$

If $I = \mathbb{N}$ is called a

$$\{E_n\}_{n \in \mathbb{N}} \tag{\textbf{Sequence}}$$

Definition 1.1
$\{E_n\} \subseteq \mathcal{P}(X)$ is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1} \, \forall n \qquad (\text{resp. } E_n \supseteq E_{n+1} \, \forall n)$$

and is written as

$$\{E_n\} \nearrow \qquad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets $\{E_i\}_{i \in I} \subseteq \mathcal{P}(X)$, will be often considered

$$\bigcup_{i \in I} E_i = \{x \in X : \exists\, i \in I \; s.t. \; x \in E_i\}$$

$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i, \, \forall i \in I\}$$

$\{E_i\}$ is said to be **disjoint** if $E_i \cap E_j = \varnothing \; \forall i \neq j$.

Examples:

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

Definition 1.2
$\{E_n\} \subseteq \mathcal{P}(X)$. We define

$$\limsup_n E_n := \bigcap_{k=1}^{\infty} (\bigcup_{n=k}^{\infty} E_n)$$

$$\liminf_n E_n := \bigcup_{k=1}^{\infty} (\bigcap_{n=k}^{\infty} E_n)$$

If these two sets are equal, then

$$\lim_n E_n = \limsup_n E_n = \liminf_n E_n$$

Proposition 1.1
Some limits are:

- $\limsup_n E_n = \{x \in X : \; x \in E_n \text{ for } \infty - \text{many indexes } n\}$
- $\liminf_n E_n = \{x \in X : \; x \in E_n \text{ for all but finitely many indexes } n\}$

- $\liminf_n E_n \subseteq \limsup_n E_n$
- $(\liminf_n E_n)^C = \limsup_n E_n^C$

Definition 1.3

$$x \in \limsup_n E_n \iff x \in \bigcap_{k=1}^{\infty}\left(\bigcup_{n=k}^{\infty} E_n\right)$$
$$\iff \forall k \in \mathbb{N} : \bigcup_{n=k}^{\infty} E_n$$
$$\iff \forall k \in \mathbb{N}\ \exists n_k \geq k\, s.t.\, x \in E_{n_k}$$

So $x \in \limsup_n E_n \implies \exists m_1 = n_1\, s.t.\, x \in E_{n_1}$
$\exists m_2 := n_{m_1+1} \geq m_1 + 1\, s.t.\, x \in E_{n_2}$
$\vdots$
$\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1\, s.t.\, x \in E_{n_k}$
$\vdots$
$x \in E_{m_1}, \ldots, E_{m_k}, \ldots$

On the other hand, assume that $x \in E_n$ for ∞-many indexes. We claim that $\forall k \in \mathbb{N}\, \exists n_k \geq k\, s.t.\, x \in E_{n_k} \iff x \in \limsup_n E_n$. If that claim is not true, then $\exists \bar{k}\, s.t.\, x \notin E_n\, \forall n > \bar{k} \implies x$ belongs at most to $E_1, \ldots, E_{\bar{k}}$, a contradiction. ★

Definition 1.4
$\{E_i\}_{i\in I}$ is a **covering** of X if
$$X \subseteq \bigcup_{i\in I} E_i$$
A subfamily of E_i that is still a covering is called a **subcovering**

Definition 1.5
Let $E \subseteq X$. The function $\chi_E : X \to \mathbb{R}$
$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \backslash E \end{cases}$$
is called **characteristic function** of E

Let E_1, E_2 be sets:
$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$
$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$
$$\{E_n\} \subseteq \mathcal{P}(X),\ \text{disjoint}, E = \bigcup_{n=1}^{\infty} E_n \implies \mathcal{X}_{\mathcal{E}} = \sum_{n=1}^{\infty} \chi_{E_n}$$
$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \implies \chi_P = \liminf \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that $\limsup_n a_n = \lim_{k\to\infty} \sup_{n\geq k} a_n$ and $\liminf_n a_n = \lim_{k\to\infty} \inf_{n\geq k} a_n$

Let's also check that $\chi_Q = \limsup_n \chi_{E_n}$

$$x \in \limsup_n E_n \iff \chi_Q(x) = 1$$
$$\iff \forall\, k \in \mathbb{N}\, \exists\, n_k \geq k\ s.t.\ x \in E_{n_k}$$

If we fix k then

$$\sup_{n\geq k} \chi_{E_n}(x) = \chi_{E_{n_k}}(x) = 1$$
$$\lim_{k\to\infty} \sup_{n\geq k} \chi_{E_n}(x) = \limsup_{n} \chi_{E_n}(x) = 1$$

Let now $x \notin \limsup E_n \iff \chi_Q(x) = 0$. Then x belongs at most to finitely many $E_n \implies \exists\, \bar{k}$ *s.t.* $x \notin E_n, \forall n \geq \bar{k}$

If $k \geq \bar{k}$, then $\sup_{n\geq k} \chi_{E_n}(x) = 0 \implies \lim_{k\to\infty} \sup_{n\geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 0$

Relations

Given X, Y sets, is called a **relation** of X and Y a subset of $X \times Y$

$$R \subseteq X + Y \quad R = \{(x,y) \,:\, x \in X, y \in Y\}$$

$$(x,y) \in R \iff xRy$$

$$X = \{0,1,2,3\} \quad R = \{(0,1),(1,2),(2,1)\} \text{ is a relation in } X$$

Definition 1.6
A **function** from X to Y is a relation R s.t. for any element x of X $\exists!$ element y of Y s.t. xRy

Definition 1.7
R on X is an **equivalence relation** if

(1) $xRx \;\forall\, x \in X$ (R is **reflexive**)

(2) $xRy \implies yRx$ (R is **symmetric**)

(3) $xRy, yRz \implies xRz$ (R is **transitive**)

If R is an equivalence relation, the set $E_X := \{y \in X \,:\, yRx\}$, $x \in X$ is called the **equivalence class** of X

Definition 1.8
$\frac{X}{R} := \{E_X \,:\, x \in X\}$ is the **quotient set**

Ex: $X = \mathbb{Z}$, let's say that nRm if $n - m$ is even. This is an equivalence relation.

$$E_n = \{\dots, n-4, n-2, n, n+2, n+4, \dots\}$$

in this case if n is even, $E_n = \{\text{even numbers}\}$ and if n is odd, $E_n = \{\text{odd numbers}\}$

Measure theory

Definition 1.9
A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is called a **σ-algebra** if

(1) $X \in \mathcal{M}$

(2) $E \in \mathcal{M} \implies E^C = X \backslash E \in \mathcal{M}$

(3) If $E = \bigcup_{n\in\mathbb{N}}$ and $E_n \in \mathcal{M} \;\forall\, n$, then $E \in \mathcal{M}$

If $\mathcal{M}$ is a σ-algebra, $(X, \mathcal{M})$ is called **measurable space** and the sets in $\mathcal{M}$ are called **measurable**. Ex:

- $(X, \mathcal{P}(X))$ is a measurable space

- Let X be a sets, then $\{\varnothing, X\}$ is a σ-algebra

Remark 2
σ is often used to denote the closure w.r.t. countably many operators. If we replace the countable unions with finite unions in the definition of σ-algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space $(X, \mathcal{M})$:

(1) $\varnothing \in \mathcal{M}$: $\varnothing = X^C$ and $X \in \mathcal{M}$

(2) $\mathcal{M}$ is an algebra, and $E_1, \ldots, E_n \in \mathcal{M}$

$$E_1 \cup \ldots \cup E_n = E_1 \cup \ldots \cup E_n \cup \underbrace{\varnothing}_{\in \mathcal{M}} \cup \varnothing \ldots \in \mathcal{M}$$

(3) $E_n \in \mathcal{M}$, $\bigcap_{n \in \mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n \in \mathbb{N}} E_n = \underbrace{\left(\bigcup_{n \in \mathbb{N}} \underbrace{E_n^C}_{\in \mathcal{M}} \right)}_{\in \mathcal{M}}^C \qquad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \Longrightarrow E \backslash F \in \mathcal{M} = E \backslash F = E \cap F^C \in \mathcal{M}$
- If $\Omega \subset X$, then the **restriction** of $\mathcal{M}$ to Ω, written as

$$\mathcal{M}|_\Omega := \{F \subseteq \Omega : F = E \cap \Omega, \text{ with } E \in \mathcal{M}\}$$

  is a σ-algebra on Ω

Theorem 2.1
$\mathcal{S} \subseteq \mathcal{P}(X)$. Then it is well defined the smallest σ-algebra containing $\mathcal{S}$, the σ-algebra generated by $\mathcal{S} := \sigma_0(\mathcal{S})$:

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$ and thus is a σ-algebra
- $\forall \sigma(\mathcal{M})$ s.t. $\mathcal{M} \supseteq \mathcal{S}$, we have $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

Proof idea.

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \varnothing \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} \; : \; \mathcal{M} \in \mathcal{V}\}$, so it can be proved that this is the desired σ-algebra ★

Borel sets

Given (X, d) metric space, the σ-algebra generated by the open sets is called **Borel** σ-algebra, written as $\mathcal{B}(X)$. The sets in $\mathcal{B}(X)$ are called **Borel sets**. The following sets are Borel sets:

- open sets
- closed sets
- countable intersections of open sets: G_σ sets
- countable unions of closed sets: F_σ sets

Remark 3
$\mathcal{B}(\mathbb{R})$ can be equivalently defined as the σ-algebra generated by

$$\{(a,b):\ a,b\in\mathcal{R}, a<b\}$$

$$\{(-\infty,b):\ b\in\mathcal{R}\}$$
$$\{(a,+\infty):\ a\in\mathcal{R}\}$$
$$\{[a,b):\ a,b\in\mathcal{R}, a<b\}$$
$$\vdots$$

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Question: What is $\mathcal{B}(\mathbb{R})$? Is $\mathcal{B}(\mathbb{R})\neq\mathcal{P}(\mathbb{R})$? No.

Definition 4.1
$(X,\mathcal{M})$ measurable space. A function $\mu:\ \mathcal{M}\to[0,+\infty]$ is called a **positive measure** if $\mu(\varnothing)=0$ and if μ is countably additive, that is

$$\forall\{E_n\}\subseteq\mathcal{M}\quad\text{disjoint}$$

we have that

$$\mu\left(\bigcup_{n=1}^{\infty}\right)=\sum_{n=1}^{\infty}\mu(E_n)\qquad\sigma\text{-additivity}$$

Remark 5
a set A is countable if $\exists\ f$ $1-1$ s.t. $f:A\to\mathbb{N}$ Examples: $\mathbb{Z},\mathbb{Q}$ are countable, while $\mathbb{R}$ is not, also $(0,1)$ is uncountable.

We always assume that $\exists\ E\neq\varnothing, E\in\mathcal{M}$ s.t. $\mu(E)\neq\infty$.

If $(X,\mathcal{M})$ is a measurable space, and μ is a measure on it, then $(X,\mathcal{M},\mu)$ is a measure space.

Then:

(1) μ is **finitely additive**:

$$\forall\ E,F\in\mathcal{M},\ \text{with } E\cap F\neq\varnothing\Longrightarrow\mu(E\cup F)=\mu(E)+\mu(F)$$

(2) the **excision property**

$$\forall\ E,f\in\mathcal{M},\ \text{with } E\subset F \text{ and } \mu(E)<+\infty\Longrightarrow\mu(F\backslash E)=\mu(F)-\mu(E)$$

(3) **monotonicity**

$$\forall\ E,F\in\mathcal{M},\ \text{with } E\subset F\Longrightarrow\mu(E)\leq\mu(F)$$

(4) if $\Omega\in\mathcal{M}$ then $(\Omega,\mathcal{M}|_{\Omega},\mu|_{\mathcal{M}|_{\Omega}})$ is a measure space

Proof. (1) $E_1=E, E_2=F, E_3=\ldots=E_n=\ldots=\varnothing$ This is a disjoint sequence $\Longrightarrow$ by σ-additivity.

$$\mu(E\cup F)=\mu\left(\bigcup_n E_n\right)=\sum_n\mu(E_n)=\mu(E)+\mu(F)+\underbrace{\mu(E_k)}_{=\mu(\varnothing)}$$

(2) $E \subset F$, so $F = E \cup (F \backslash E)$ and this is disjoint $\overset{(i)}{\Longrightarrow} \mu(F) = \mu(E) + \mu(F \backslash E)$, and since $\mu(E) < \infty$, the property follows.

(3) $E \subset F \Longrightarrow \mu(F) = \mu(E) + \underbrace{\mu(F \backslash E)}_{\geq 0} \geq \mu(E)$

(4)

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Definition 5.1
$(X, \mathcal{M}, \mu)$ measure space.

- If $\mu(X) < +\infty$, we say that μ is **finite**.
- If $\mu(X) = +\infty$, and $\exists \ \{E_n\} \subset \mathcal{M}$ s.t. $X = \bigcup_n E_n$ and each E_n has finite measure, then we say that μ is σ-finite.
- If $\mu(X) = 1$ we say that μ is a **probability measure**.

Some examples:

- Trivial Measure: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(E) = 0 \quad \forall \ E \in \mathcal{M}$
- Counting Measure: $(X, \mathcal{P}(X))$ measurable space. We define

$$\mu_C : \mathcal{P}(X) \to [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure: $(X, \mathcal{P}(X))$ measurable space, $t \in X$. We define

$$\delta_t : \mathcal{P}(X) \to [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

<u>Continuity of the measure along monotone sequences</u>
$(X, \mathcal{M}, \mu)$ measure space

(1) $\{E_i\} \subset \mathcal{M}, \ E_i \subseteq E_{i+1} \ \forall i$ and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:

$$\mu(E) = \lim_i \mu(E_i)$$

(2) $\{E_i\} \subset \mathcal{M}, \ E_{i+1} \subseteq E_i \ \forall i$ and let $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$.

Proof. (1) if $\exists \ i$ s.t. $\mu(E_i) = +\infty$, then is trivial. Assume then that every E_i has a finite measure, so that $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty} (E_{i+1} \backslash E_i)$ with $E_0 = \varnothing$.

So, by σ-additivity

$$\mu(E) = \mu\left(\bigcup_{i=0}^{\infty} (E_{i+1} \backslash E_i)\right) =$$

$$= \sum_{i=0}^{\infty} \mu(E_{i+1} \backslash E_i) \overset{(excision)}{=} \sum_{i=0}^{\infty} (\mu(E_{i+1} - \mu(E_i))) =$$

$$\overset{(telescopic\ series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n)$$

(2) For simplicity, suppose $\tau = 1$, and define $F_k = E_i \backslash E_k$

$$\{E_k\} \searrow \Longrightarrow \{F_k\} \nearrow$$

$$\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \backslash (\bigcap_k E_k)$$

$$\mu(E_i) = \mu(\bigcup_k F_k) + \underbrace{\mu(\bigcap_k E_k)}_{\mu(E)} =$$

$$\overset{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k (\mu(E_i) - \mu(E_k)) + \mu(E)$$

Since $\mu(E_i) < \infty$ we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$

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Counterexample: given $(\mathcal{N}, \mathcal{P}(\mathbb{N}), \mu_C)$ measure space. Let $E_n = \{n, n+1, n+2, \dots\}$. In this case $\mu_C(E_n) = +\infty, E_{n+1} \subseteq E_n \forall\, n$, but $\bigcap_n E_n = \varnothing \Longrightarrow \mu\left(\bigcap_n E_n\right) = 0$

Theorem 5.1 (σ-subadditivity of the measure)
$(X, \mathcal{M}, \mu)$ is a measure space. $\forall \{E_n\} \subseteq \mathcal{M}$ (not necessarily disjoint): $\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n)$

***Proof*.** $E_1, E_2 \in \mathcal{M}$ and also $E_1 \cup E_2 = E_1 \cup (E_2 \backslash E_1)$ disjoint sets.

$$\mu(E_1 \cup E_2) = \mu(\underbrace{E_2 \backslash E_1}_{\subseteq E_2}) \overset{(monotonicity)}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.

$$A = \bigcup_{n=1}^{\infty} E_n, \quad A_k = \bigcup_{n=1}^{k} E_n$$

$$\{A_k\} \nearrow,\ A_{k+1} \supseteq A_k,\ \lim_k A_k = A$$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{(continuity)}{=} \lim_k \mu(A_k) = \lim_k \mu\left(\bigcup_{n=1}^{k} E_n\right) \leq$$

$$\leq \lim_k \sum_{n=1}^{k} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

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Exercise: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, +\infty]$ s.t. μ is finitely additive, σ-subadditive and $\mu(\varnothing) = 0 \Longrightarrow \mu$ is σ-additive, and hence is a measure.

Exercise: the Borel-Cantelli lemma states that, given $(X, \mathcal{M}, \mu)$ and $\{E_n\} \subseteq \mathcal{M}$. Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \Longrightarrow \mu(\limsup_n E_n) = 0$$

It can be phrased as:

> If the series of the probability of the events E_n is convergent, then the probability that ∞-many events occur is 0

Proof. The thesis is:

$$\mu(\limsup_n E_n) = \mu\left(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k\geq n} E_k}_{A_n := \bigcup_{k\geq n} E_k}\right)$$

Is it true that $\{A_n\} \searrow$? Yes.

$$A_{n+1} = \bigcup_{k\geq n+1} E_k \subseteq \bigcup_{k\geq n} E_k = A_n$$

Does some A_n have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k\geq n} E_k\right) \leq \sum_{k\geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k\geq n} E_k\right) \overset{\sigma-sub.}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

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Sets of 0 measure

$(X, \mathcal{M}, \mu)$ measure space.

- $N \subseteq X$ is a set of 0 measure if $N \in \mathcal{M}$ and $\mu(N) = 0$
- $E \subseteq X$ is called **negligible set** if $\exists\ N \in \mathcal{M}$ with 0 measure s.t. $E \subseteq N$ (E does not necessarily stays in $\mathcal{M}$)

Definition 5.2
$(X, \mathcal{M}, \mu)$ measure space s.t. every negligible set is measurable (and hence of 0 measure), then $(X, \mathcal{M}, \mu)$ is said to be a **complete measure space**.

A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists\ F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \backslash F) = 0\}$$

Clearly $\mathcal{M} \subseteq \overline{\mathcal{M}}$. For $E \in \overline{\mathcal{M}}$, take F and G as above and let $\bar{\mu}(E) = \bar{\mu}(F)$ then $\bar{\mu}|_{\mathcal{M}} = \mu$, and moreover:

Theorem 5.2
$(X, \mathcal{M}, \mu)$ is a complete measure space. Let's observe that $\bar{\mu}$ is well defined: let $E \subseteq X$ and F_1, F_2, G_1, G_2 s.t. $F_i \subset E \subset G_i \quad i = 1, 2$. Then $\mu(G_i \backslash F_i) = 0$. Now we have to check that $\mu(F_1) = \mu(F_2)$.

Since

$$F_1 \backslash F_2 \subseteq E \backslash F_2 \subseteq G_2 \backslash F_2$$

and $G_2 \backslash F_2$ has 0 measure $\Longrightarrow \mu(F_1 \backslash F_2) = 0$. Then $F_1 = (F_1 \backslash F_2) \cup (F_1 \cap F_2) \Longrightarrow \mu(F_1) = \mu(F_1 \cap F_2)$. In the same way, $\mu(F_2) = \mu(F_1 \cap F_2)$

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The elements of $\overline{\mathcal{M}}$ are sets of the type $E \cup N$, with $E \in \mathcal{M}$ and $\bar{\mu}(N) = 0$.

Outer measure

We wish to defin

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(1) $((X, \mathcal{M}))$ is a measurable space obtained by means of an outer measure. Ex: $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$, (Y, d_y) metric space If $X \to Y$ is (Lebesgue) measurable $\iff (\mathcal{M}, \mathcal{B}(Y))$ is measurable

(2) $(X, d_X), (Y, d_Y)$ are metric spaces $\longrightarrow (X, \mathcal{B}(X))$ If $X \to Y$ are borel measurable $\iff$ $(\mathcal{B}(X), \mathcal{B}(Y))$measurable

Remark 8
f is Lebesgue measurable if the continuity of the borel set is a Lebesgue-measurable set.

Proposition 8.1 (1) $(X, d_X), (Y, d_Y)$ metric spaces. If $X \to Y$ is continuous, then is Borel measurable

(2) (Y, d_Y) metric space. If $\mathbb{R}^n \to Y$ is continuous, then it is a Lebesgue measure.

Proof. (1) f is continuous $\iff f^{-1}(A)$ is open $\forall\ A \in Y$ open $\implies f^{-1}(A) \in \mathcal{B}(Y)\ \forall\ A \in Y$ open Since $\mathcal{B}(Y) = \sigma_0$(open sets) by proposition 1 thus implies that f is Borel measurable

(2) f is continuous $\implies f$ is Borel measurable mancano pezzi namely f is Lebesgue measurable ★

Proposition 8.2
$(X, \mathcal{M})$ measurable space, $(X, d_Y), (Y, d_Y)$ metric spaces. if $f : X \to Y$ is $\mathcal{M}, \mathcal{B}(Y)$-measurable and $g : Y \to Z$ is continuous $\implies g \circ f : x \to Z$ is $\mathcal{M}, \mathcal{B}(Y)$-measurable

Proposition 8.3
$(X, \mathcal{M})$ measurable space Let $\Phi : \mathbb{R}^n \to Y$ be continuous where (Y, d_Y) is a metric space. Then $h : X \to Y$ defined by $h(x) = \Phi(u(x), boh)$ is $\mathcal{M}, \mathcal{B}(Y)$-measurable.

Proof. Define $f : X \to \mathbb{R}^n$, $f(x) = u(x), v(x)$. By def $h = \Phi \circ f$ by prop 3 if we show that f is measurable, then h is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0\left(\{(a_1, b_1) \times (a_2, b_2) : a, b \in \mathbb{R}\}\right)$$

pezzi $f^{-1}(\mathcal{R} \in \mathcal{M}) \quad \forall$open rectangle in $\mathcal{R}^2$ $R = I \times J$ $F^{-1} = \{x \in X\}$ ★

Remark 9
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$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

cosine $(X, \mathcal{M})$ measurable space, then such a function f is measurable iff

$$f^{-1}(a, +\infty)] \in \mathcal{M} \quad \forall a \in \mathcal{R}$$

LEt now $\{f_n\}$ be a Sequence of measurable functions from X to $\bar{\mathcal{R}}$. Then we define

$$(\inf_n f_n)(x) = \inf_n f_n(x)$$

$$(\sup_n f_n)(x) = \sup_n f_n(x)$$

$$(\liminf_n f_n)(x) = \liminf_n f_n(x)$$

$$(\limsup_n f_n)(x) = \limsup_n f_n(x)$$

$$(\lim_n f_n)(x) = \lim_n f_n(x) \quad \text{if the limit exists}$$

Proposition 9.1
$(X, \mathcal{M})$ measurable space, $f_n : X \to \bar{\mathcal{R}}$ measurable, then sup inf lim inf lim sup of f_n are measurable, in particular if $\lim f_n$ exists, then f is measurable

Proof. $(\sup f_n)^{-1}((a, \infty]) = \{x \in X : \sup f_n(x) > a\}$ (manca pezzi)

$$\bigcup \{x \in X : f_n(x) > a\}$$

Then $(\sup f_n)^{-1}((a, \infty])$ is measurable, cose da aggiungere Noe the limsup

$$\limsup_n f_n = \lim_n (\sup_{k>n} f_n(x))$$

cose cose ★

Simple functions

Definition 9.1
$(X, \mathcal{M})$ measurable space. A measurable function s: $X \to \bar{\mathcal{R}}$ is said to be simple if $s(X)$ is a finite set altre cose Then $s(x) = \sum_{n=1} a_n \chi_{E_n}(x)$ where E_n is a measurable set sistemare.

Particular case: if s:$\mathbb{R} \to \bar{\mathbb{R}}$ and each E_n is a finite union of intervals, then s is said to be a STEP FUNCTION.

The idea is to approximate functions with simple functions.

Theorem 9.1
$(X, \mathcal{M})$ measurable space, $f : X \to [0, \infty]$ measurable. Then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq f \quad \text{pointwise}$$

and $s_n(x) \to f(x)$ Moreover if f is bounded then $s_n \to f$ uniformly on X as $n \to \infty$

f is bounded. Fix $n \in \mathbb{N}$ and divide $[0, n)$ in $n \cdot 2^n$ intervals called $I_j = [a_j, b_j)$ with lenght $\frac{1}{2^n}$

Let $E_0 = f^{-1}([n, \infty)), E_j = f^{-1}([a_j, b_j))$ for $j = 1, \ldots, n \cdot 2^n$

We let Array

Namely we define

$$s_n(x) = n\chi_{E_0}(X) + \sum_{j=1}^{n \cdot 2^n} a_j \chi_{E_j}(x)$$

Then $s_n \leq s_{n+1}$ by contradiction

Then any $x \in X$ stays in $f^{-1}([a_j, b_j))$ for some j $\Longrightarrow$ ★