Lesson 13/10/2022

Integration on product spaces

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. $f : X \times Y \to \bar{\mathbb{R}}$ measurable.

If $f \geq 0$, then

$$\iint_{X\times Y} f d\mu \otimes d\nu$$

Goal: obtain a formula of iterated integral like the one in Analysis 2.

$\forall \bar{x} \in X$ and $\bar{y} \in Y$

$$cose$$

Proposition 0.1
If f is measurable $\Rightarrow$ $f_{\bar{x}}$ is $(\mathcal{N}, \mathcal{B}(\mathbb{R}))$-measurable and $f_{\bar{y}}$ is $(\mathcal{M}, \mathcal{B}(\bar{\mathbb{R}}))$-measurable. Then we can conclude $\phi : X \to \bar{\mathbb{R}}$:

$$\phi(x) = \int_Y f_x d\nu = \int_Y f(x, y) d\nu(y)$$

and $\psi : Y \to \bar{\mathbb{R}}$

$$\psi(y) = \int_X f_y d\mu = \int_X f(x, y) d\mu(x)$$

Questions: what is the solution of $\iint_{X\times Y}$ cose cose

Theorem 0.1 (Tonelli's theorem)
$(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ complete measure spaces and σ-finite. Suppose that f is $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}(\bar{\mathbb{R}}))$-measurable and that $f > 0$ a.e. on $X \times Y$. Then ψ and ϕ are measurable and

$$\iint_{X\times Y} f d\mu \otimes d\nu = cose$$

Equally holds also if one of the integrals is ∞.

Remark 1
The double integral can be reduced to single integrals, iterated. Moreover we can always change the order of the integrals For sign changing functions the situation is more involved.

Theorem 1.1 (Fubini's theorem)
$(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ complete measure spaces and σ-finite. If $f \in L^1(X \times Y)$, then ψ and ϕ defined above are measurable, and cose holds, and all the integrals are finite.

Question: how to check if $f \in L^1(X \times Y)$? Typically, to check cosette

If $\iiiint_{X\times Y} |f| d\mu \otimes d\nu < \infty$ then we can apply Fubini for $\iint_{X\times Y} f d\mu \otimes d\nu$

Remark 2
the proof of Fubini's and Tonelli's theorems is based for the iterated integrals for characteristic functions. (Note that $(\mu \otimes \nu)(E) = \int_X ()$ e altre cosette)