

Notes from Real and Functional Analysis

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1 Lesson 12/09/2022

Element of set theory

Let X be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \quad (\text{Power Set})$$

Let $I \subseteq \mathbb{R}$ be a set of indexes. A family of sets induced by I is

$$\{E_i\}_{i \in I}, \quad E_i \subseteq X \quad (\text{Family/Collection})$$

If $I = \mathbb{N}$ is called a

$$\{E_n\}_{n \in \mathbb{N}} \quad (\text{Sequence})$$

Definition 1.1

$\{E_n\} \subseteq \mathcal{P}(X)$ is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1} \forall n \quad (\text{resp. } E_n \supseteq E_{n+1} \forall n)$$

and is written as

$$\{E_n\} \nearrow \quad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets $\{E_i\}_{i \in I} \subseteq \mathcal{P}(X)$, will be often considered

$$\bigcup_{i \in I} E_i = \{x \in X : \exists i \in I \text{ s.t. } x \in E_i\}$$

$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i, \forall i \in I\}$$

$\{E_i\}$ is said to be **disjoint** if $E_i \cap E_j = \emptyset \forall i \neq j$.

Examples:

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

Definition 1.2

$\{E_n\} \subseteq \mathcal{P}(X)$. We define

$$\limsup_n E_n := \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} E_n \right)$$

$$\liminf_n E_n := \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right)$$

If these two sets are equal, then

$$\lim_n E_n = \limsup_n E_n = \liminf_n E_n$$

Proposition 1.1

Some limits are:

- $\limsup_n E_n = \{x \in X : x \in E_n \text{ for } \infty - \text{many indexes } n\}$
- $\liminf_n E_n = \{x \in X : x \in E_n \text{ for all but finitely many indexes } n\}$

- $\liminf_n E_n \subseteq \limsup_n E_n$
- $(\liminf_n E_n)^C = \limsup_n E_n^C$

Definition 1.3

$$\begin{aligned}
x \in \limsup_n E_n &\iff x \in \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} E_n \right) \\
&\iff \forall k \in \mathbb{N} : \bigcup_{n=k}^{\infty} E_n \\
&\iff \forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k}
\end{aligned}$$

So $x \in \limsup_n E_n \implies \begin{aligned} &\exists m_1 = n_1 \text{ s.t. } x \in E_{n_1} \\ &\exists m_2 := n_{m_1+1} \geq m_1 + 1 \text{ s.t. } x \in E_{n_2} \\ &\vdots \\ &\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1 \text{ s.t. } x \in E_{n_k} \\ &\vdots \\ &x \in E_{m_1}, \dots, E_{m_k}, \dots \end{aligned}$

On the other hand, assume that $x \in E_n$ for ∞ -many indexes. We claim that $\forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k} \iff x \in \limsup_n E_n$. If that claim is not true, then $\exists \bar{k} \text{ s.t. } x \notin E_n \forall n > \bar{k} \implies x$ belongs at most to $E_1, \dots, E_{\bar{k}}$, a contradiction. ★

Definition 1.4

$\{E_i\}_{i \in I}$ is a **covering** of X if

$$X \subseteq \bigcup_{i \in I} E_i$$

A subfamily of E_i that is still a covering is called a **subcovering**

Definition 1.5

Let $E \subseteq X$. The function $\chi_E : X \rightarrow \mathbb{R}$

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \setminus E \end{cases}$$

is called **characteristic function** of E

Let E_1, E_2 be sets:

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

$$\{E_n\} \subseteq \mathcal{P}(X), \text{ disjoint, } E = \bigcup_{n=1}^{\infty} E_n \implies \chi_E = \sum_{n=1}^{\infty} \chi_{E_n}$$

$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \implies \chi_P = \liminf_n \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that $\limsup_n a_n = \lim_{k \rightarrow \infty} \sup_{n \geq k} a_n$ and $\liminf_n a_n = \lim_{k \rightarrow \infty} \inf_{n \geq k} a_n$

Let's also check that $\chi_Q = \limsup_n \chi_{E_n}$

$$\begin{aligned}
x \in \limsup_n E_n &\iff \chi_Q(x) = 1 \\
&\iff \forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k}
\end{aligned}$$

If we fix k then

$$\sup_{n \geq k} \chi_{E_n}(x) = \chi_{E_{n_k}}(x) = 1$$

$$\lim_{k \rightarrow \infty} \sup_{n \geq k} \chi_{E_n}(x) = \lim_n \sup \chi_{E_n}(x) = 1$$

Let now $x \notin \limsup E_n \iff \chi_Q(x) = 0$. Then x belongs at most to finitely many $E_n \implies \exists \bar{k}$ s.t. $x \notin E_n, \forall n \geq \bar{k}$

If $k \geq \bar{k}$, then $\sup_{n \geq k} \chi_{E_n}(x) = 0 \implies \lim_{k \rightarrow \infty} \sup_{n \geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 0$

Relations

Given X, Y sets, is called a **relation** of X and Y a subset of $X \times Y$

$$R \subseteq X \times Y \quad R = \{(x, y) : x \in X, y \in Y\}$$

$$(x, y) \in R \iff xRy$$

$$X = \{0, 1, 2, 3\} \quad R = \{(0, 1), (1, 2), (2, 1)\} \text{ is a relation in } X$$

Definition 1.6

A **function** from X to Y is a relation R s.t. for any element x of X $\exists!$ element y of Y s.t. xRy

Definition 1.7

R on X is an **equivalence relation** if

- (1) $xRx \forall x \in X$ (R is **reflexive**)
- (2) $xRy \implies yRx$ (R is **symmetric**)
- (3) $xRy, yRz \implies xRz$ (R is **transitive**)

If R is an equivalence relation, the set $E_x := \{y \in X : yRx\}$, $x \in X$ is called the **equivalence class** of X

Definition 1.8

$\frac{X}{R} := \{E_x : x \in X\}$ is the **quotient set**

Ex: $X = \mathbb{Z}$, let's say that nRm if $n - m$ is even. This is an equivalence relation.

$$E_n = \{\dots, n-4, n-2, n, n+2, n+4, \dots\}$$

in this case if n is even, $E_n = \{\text{even numbers}\}$ and if n is odd, $E_n = \{\text{odd numbers}\}$

Measure theory

Definition 1.9

A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is called a **σ -algebra** if

- (1) $X \in \mathcal{M}$
- (2) $E \in \mathcal{M} \implies E^C = X \setminus E \in \mathcal{M}$
- (3) If $E = \bigcup_{n \in \mathbb{N}} E_n$ and $E_n \in \mathcal{M} \forall n$, then $E \in \mathcal{M}$

If \mathcal{M} is a σ -algebra, (X, \mathcal{M}) is called **measurable space** and the sets in \mathcal{M} are called **measurable**. Ex:

- $(X, \mathcal{P}(X))$ is a measurable space

- Let X be a set, then $\{\emptyset, X\}$ is a σ -algebra

Remark 2

σ is often used to denote the closure w.r.t. countably many operators. If we replace the countable unions with finite unions in the definition of σ -algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space (X, \mathcal{M}) :

- (1) $\emptyset \in \mathcal{M}$: $\emptyset = X^C$ and $X \in \mathcal{M}$
- (2) \mathcal{M} is an algebra, and $E_1, \dots, E_n \in \mathcal{M}$

$$E_1 \cup \dots \cup E_n = E_1 \cup \dots \cup E_n \cup \underbrace{\emptyset}_{\in \mathcal{M}} \cup \emptyset \dots \in \mathcal{M}$$

- (3) $E_n \in \mathcal{M}$, $\bigcap_{n \in \mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n \in \mathbb{N}} E_n = \left(\underbrace{\bigcup_{n \in \mathbb{N}} E_n^C}_{\in \mathcal{M}} \right)^C \quad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \implies E \setminus F \in \mathcal{M} = E \setminus F = E \cap F^C \in \mathcal{M}$
- If $\Omega \subset X$, then the **restriction** of \mathcal{M} to Ω , written as

$$\mathcal{M}|_{\Omega} := \{F \subseteq \Omega : F = E \cap \Omega, \text{ with } E \in \mathcal{M}\}$$

is a σ -algebra on Ω

Theorem 2.1

$\mathcal{S} \subseteq \mathcal{P}(X)$. Then it is well defined the smallest σ -algebra containing \mathcal{S} , the σ -algebra generated by $\mathcal{S} := \sigma_0(\mathcal{S})$:

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$ and thus is a σ -algebra
- $\forall \sigma(\mathcal{M})$ s.t. $\mathcal{M} \supseteq \mathcal{S}$, we have $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

Proof idea.

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \emptyset \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} : \mathcal{M} \in \mathcal{V}\}$, so it can be proved that this is the desired σ -algebra ★

Borel sets

Given (X, d) metric space, the σ -algebra generated by the open sets is called **Borel** σ -algebra, written as $\mathcal{B}(X)$. The sets in $\mathcal{B}(X)$ are called **Borel sets**. The following sets are Borel sets:

- open sets
- closed sets
- countable intersections of open sets: G_{σ} sets
- countable unions of closed sets: F_{σ} sets

Remark 3

$\mathcal{B}(\mathbb{R})$ can be equivalently defined as the σ -algebra generated by

$$\begin{aligned} &\{(a, b) : a, b \in \mathcal{R}, a < b\} \\ &\{(-\infty, b) : b \in \mathcal{R}\} \\ &\{(a, +\infty) : a \in \mathcal{R}\} \\ &\{[a, b) : a, b \in \mathcal{R}, a < b\} \\ &\vdots \end{aligned}$$

4 Lesson 14/09/2022

Question: What is $\mathcal{B}(\mathbb{R})$? Is $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$? No.

Definition 4.1

(X, \mathcal{M}) measurable space. A function $\mu : \mathcal{M} \rightarrow [0, +\infty]$ is called a **positive measure** if $\mu(\emptyset) = 0$ and if μ is countably additive, that is

$$\forall \{E_n\} \subseteq \mathcal{M} \text{ disjoint}$$

we have that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) \quad \sigma\text{-additivity}$$

Remark 5

a set A is countable if $\exists f : \mathbb{N} \rightarrow A$ s.t. f is 1-1. Examples: \mathbb{Z}, \mathbb{Q} are countable, while \mathbb{R} is not, also $(0, 1)$ is uncountable.

We always assume that $\exists E \neq \emptyset, E \in \mathcal{M}$ s.t. $\mu(E) < \infty$.

If (X, \mathcal{M}) is a measurable space, and μ is a measure on it, then (X, \mathcal{M}, μ) is a measure space.

Then:

(1) μ is **finitely additive**:

$$\forall E, F \in \mathcal{M}, \text{ with } E \cap F = \emptyset \implies \mu(E \cup F) = \mu(E) + \mu(F)$$

(2) the **excision property**

$$\forall E, F \in \mathcal{M}, \text{ with } E \subset F \text{ and } \mu(E) < +\infty \implies \mu(F \setminus E) = \mu(F) - \mu(E)$$

(3) **monotonicity**

$$\forall E, F \in \mathcal{M}, \text{ with } E \subset F \implies \mu(E) \leq \mu(F)$$

(4) if $\Omega \in \mathcal{M}$ then $(\Omega, \mathcal{M}|_{\Omega}, \mu|_{\mathcal{M}|_{\Omega}})$ is a measure space

Proof. (1) $E_1 = E, E_2 = F, E_3 = \dots = E_n = \dots = \emptyset$ This is a disjoint sequence \implies by σ -additivity.

$$\mu(E \cup F) = \mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n) = \mu(E) + \mu(F) + \underbrace{\mu(E_k)}_{=\mu(\emptyset)}$$

(2) $E \subset F$, so $F = E \cup (F \setminus E)$ and this is disjoint $\xRightarrow{(i)} \mu(F) = \mu(E) + \mu(F \setminus E)$, and since $\mu(E) < \infty$, the property follows.

(3) $E \subset F \implies \mu(F) = \mu(E) + \underbrace{\mu(F \setminus E)}_{\geq 0} \geq \mu(E)$

(4)

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Definition 5.1

(X, \mathcal{M}, μ) measure space.

- If $\mu(X) < +\infty$, we say that μ is **finite**.
- If $\mu(X) = +\infty$, and $\exists \{E_n\} \subset \mathcal{M}$ s.t. $X = \bigcup_n E_n$ and each E_n has finite measure, then we say that μ is σ -finite.
- If $\mu(X) = 1$ we say that μ is a **probability measure**.

Some examples:

- Trivial Measure: (X, \mathcal{M}) measurable space. $\mu : \mathcal{M} \rightarrow [0, \infty]$ defined by $\mu(E) = 0 \quad \forall E \in \mathcal{M}$
- Counting Measure: $(X, \mathcal{P}(X))$ measurable space. We define

$$\mu_C : \mathcal{P}(X) \rightarrow [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure: $(X, \mathcal{P}(X))$ measurable space, $t \in X$. We define

$$\delta_t : \mathcal{P}(X) \rightarrow [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

Continuity of the measure along monotone sequences

(X, \mathcal{M}, μ) measure space

(1) $\{E_i\} \subset \mathcal{M}$, $E_i \subseteq E_{i+1} \forall i$ and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:

$$\mu(E) = \lim_i \mu(E_i)$$

(2) $\{E_i\} \subset \mathcal{M}$, $E_{i+1} \subseteq E_i \forall i$ and let $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$.

Proof. (1) if $\exists i$ s.t. $\mu(E_i) = +\infty$, then is trivial. Assume then that every E_i has a finite measure, so that $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i)$ with $E_0 = \emptyset$.

So, by σ -additivity

$$\mu(E) = \mu \left(\bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i) \right) =$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \mu(E_{i+1} \setminus E_i) \stackrel{(excision)}{=} \sum_{i=0}^{\infty} (\mu(E_{i+1}) - \mu(E_i)) = \\
&\stackrel{(telescopic series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n)
\end{aligned}$$

(2) For simplicity, suppose $\tau = 1$, and define $F_k = E_i \setminus E_k$

$$\begin{aligned}
&\{E_k\} \searrow \implies \{F_k\} \nearrow \\
&\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \setminus \left(\bigcap_k E_k\right) \\
&\mu(E_i) = \mu\left(\bigcup_k F_k\right) + \underbrace{\mu\left(\bigcap_k E_k\right)}_{\mu(E)} = \\
&\stackrel{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k (\mu(E_i) - \mu(E_k)) + \mu(E)
\end{aligned}$$

Since $\mu(E_i) < \infty$ we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$

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Counterexample: given $(\mathcal{N}, \mathcal{P}(\mathbb{N}), \mu_C)$ measure space. Let $E_n = \{n, n+1, n+2, \dots\}$. In this case $\mu_C(E_n) = +\infty, E_{n+1} \subseteq E_n \forall n$, but $\bigcap_n E_n = \emptyset \implies \mu(\bigcap_n E_n) = 0$

Theorem 5.1 (σ -subadditivity of the measure)

(X, \mathcal{M}, μ) is a measure space. $\forall \{E_n\} \subseteq \mathcal{M}$ (not necessarily disjoint): $\mu(\bigcup_n E_n) \leq \sum_n \mu(E_n)$

Proof. $E_1, E_2 \in \mathcal{M}$ and also $E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1)$ disjoint sets.

$$\mu(E_1 \cup E_2) = \mu\left(\underbrace{E_2 \setminus E_1}_{\subseteq E_2}\right) \stackrel{(monotonicity)}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.

$$\begin{aligned}
&A = \bigcup_{n=1}^{\infty} E_n, \quad A_k = \bigcup_{n=1}^k E_n \\
&\{A_k\} \nearrow, \quad A_{k+1} \supseteq A_k, \quad \lim_k A_k = A \\
&\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \stackrel{(continuity)}{=} \lim_k \mu(A_k) = \lim_k \mu\left(\bigcup_{n=1}^k E_n\right) \leq \\
&\leq \lim_k \sum_{n=1}^k \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)
\end{aligned}$$

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Exercise: (X, \mathcal{M}) measurable space. $\mu : \mathcal{M} \rightarrow [0, +\infty]$ s.t. μ is finitely additive, σ -subadditive and $\mu(\emptyset) = 0 \implies \mu$ is σ -additive, and hence is a measure.

Exercise: the Borel-Cantelli lemma states that, given (X, \mathcal{M}, μ) and $\{E_n\} \subseteq \mathcal{M}$. Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \implies \mu(\limsup_n E_n) = 0$$

It can be phrased as:

If the series of the probability of the events E_n is convergent, then the probability that ∞ -many events occur is 0

Proof. The thesis is:

$$\mu(\limsup_n E_n) = \mu\left(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k \geq n} E_k}_{A_n := \bigcup_{k \geq n} E_k}\right)$$

Is it true that $\{A_n\} \searrow$? Yes.

$$A_{n+1} = \bigcup_{k \geq n+1} E_k \subseteq \bigcup_{k \geq n} E_k = A_n$$

Does some A_n have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k \geq n} E_k\right) \stackrel{\sigma\text{-sub.}}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

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Sets of 0 measure

(X, \mathcal{M}, μ) measure space.

- $N \subseteq X$ is a set of 0 measure if $N \in \mathcal{M}$ and $\mu(N) = 0$
- $E \subseteq X$ is called **negligible set** if $\exists N \in \mathcal{M}$ with 0 measure s.t. $E \subseteq N$ (E does not necessarily stay in \mathcal{M})

Definition 5.2

(X, \mathcal{M}, μ) measure space s.t. every negligible set is measurable (and hence of 0 measure), then (X, \mathcal{M}, μ) is said to be a **complete measure space**.

A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \setminus F) = 0\}$$

Clearly $\mathcal{M} \subseteq \overline{\mathcal{M}}$. For $E \in \overline{\mathcal{M}}$, take F and G as above and let $\bar{\mu}(E) = \mu(F)$ then $\bar{\mu}|_{\mathcal{M}} = \mu$, and moreover:

Theorem 5.2

(X, \mathcal{M}, μ) is a complete measure space. Let's observe that $\bar{\mu}$ is well defined: let $E \subseteq X$ and F_1, F_2, G_1, G_2 s.t. $F_i \subseteq E \subseteq G_i$ $i = 1, 2$. Then $\mu(G_i \setminus F_i) = 0$. Now we have to check that $\mu(F_1) = \mu(F_2)$.

Since

$$F_1 \setminus F_2 \subseteq E \setminus F_2 \subseteq G_2 \setminus F_2$$

and $G_2 \setminus F_2$ has 0 measure $\implies \mu(F_1 \setminus F_2) = 0$. Then $F_1 = (F_1 \setminus F_2) \cup (F_1 \cap F_2) \implies \mu(F_1) = \mu(F_1 \cap F_2)$. In the same way, $\mu(F_2) = \mu(F_1 \cap F_2)$

6 Lesson 15/09/2022

The elements of $\overline{\mathcal{M}}$ are sets of the type $E \cup N$, with $E \in \mathcal{M}$ and $\bar{\mu}(N) = 0$.

Outer measure

We wish to define a measure λ “on \mathcal{R} ” with the following properties:

- (1) $\lambda((a, b)) = b - a$
- (2) $\lambda(E + t)^\dagger = \lambda(E)$ for every measurable set $E \subset \mathbb{R}$ and $t \in \mathbb{R}$

It would be nice to define such a measure on $\mathcal{P}(\mathbb{R})$. In such case, note that $\lambda(\{x\}) = 0, \forall x \in \mathbb{R}$. But then

Theorem 6.1 (Ulam)

The only measure on $\mathcal{P}(\mathbb{R})$ s.t. $\lambda(\{x\}) = 0 \quad \forall x$ is the trivial measure. Thus, a measure satisfying the two properties of the outer measure cannot be defined on $\mathcal{P}(\mathbb{R})$

We’ll learn in what follows how to create a measure space on \mathcal{R} , with a σ -algebra including all the Borel sets, and a measure satisfying properties of the outer measure. This is the so called **Lebesgue measure**.

Definition 6.1

Given a set X . An **outer measure** is a function $\mu^* : \mathcal{P}(\mathbb{R}) \rightarrow [0, +\infty]$ s.t.

- $\mu^*(\emptyset) = 0$
- $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$ (Monotonicity)
- $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$ (σ -subadditivity)

The common way to define an outer measure is to start with a family of elementary sets \mathcal{E} on which a notion of measure is defined (e.g. intervals on \mathcal{R} , rectangles on \mathcal{R}^2, \dots) and then to approximate arbitrary sets from outside by **countable** unions of members of \mathcal{E} .

Proposition 6.1

Let $\mathcal{E} \subset \mathcal{P}(\mathbb{R})$ and $\rho : \mathcal{E} \rightarrow [0, +\infty]$ be such that $\emptyset \in \mathcal{E}, X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. For any $A \in \mathcal{P}(X)$, let

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) : E_n \in \mathcal{E} \text{ and } A \subset \bigcup_{n=1}^{\infty} E_n \right\}$$

Then μ^* is an outer measure, the outer measure generated by (\mathcal{E}, ρ) .

Proof. $\forall A \subset X \exists \{E_n\} \subset \mathcal{E}$ s.t. $A \subset \bigcup_n E_n$: take $E_n = X \forall n$ then μ^* is well defined. Obviously, $\mu^*(\emptyset) = 0$ (with $E_n = \emptyset \quad \forall n$), and $\mu^*(A) \leq \mu^*(B)$ for $A \subset B$ (any covering of B with elements of \mathcal{E} is also a covering of A .)

We have to prove the σ -subadditivity. Let $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X)$ and $\epsilon > 0$. For each $n, \exists \{E_{n_j}\}_{j \in \mathbb{N}} \in \mathcal{E}$ s.t. $A_n \subset \bigcup_{j=1}^{\infty} E_{n_j}$ and $\sum_{j=1}^{\infty} \rho(E_{n_j}) \leq \mu^*(A_n) + \frac{\epsilon}{2^n}$. But then, if $A = \bigcup_{n=1}^{\infty} A_n$, we have that $A \subset \bigcup_{n,j \in \mathbb{N}^2} E_{n_j}$ and

$$\mu^*(A) \leq \sum_{n,j} \rho(E_{n_j}) \leq \sum_n \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) = \sum_n \mu^*(A_n) + \epsilon$$

Since ϵ is arbitrary, we are done. ★

[†] $\{x \in \mathbb{R} : x = y + t, \text{ with } y \in E\}$

Ex:

(1) $X \in \mathbb{R}, \mathcal{E} = \{(a, b) : a \leq b, a, b \in \mathbb{R}\}$ family of open intervals:

$$\rho((a, b)) = b - a$$

(2) $X = \mathbb{R}^n, \mathcal{E} = \{(a_1, b_1) \times \dots \times (a_n, b_n) : a_i \leq b_i, a_i, b_i \in \mathbb{R}\}$:

$$\rho((a_1, b_1) \times \dots \times (a_n, b_n)) = (b_1 - a_1) \cdot \dots \cdot (b_n - a_n)$$

Remark 7

$E \in \mathcal{E} \implies \mu^*(E) = \rho(E)$.

In examples 1 and 2, we have in fact $\mu^*((a, b)) = b - a, \mu^*((a_1, b_1) \times \dots \times (a_n, b_n)) = \prod_{i=1}^n (b_i - a_i)$

To pass from the outer measure to a measure there is a condition

Definition 7.1 (Caratheodory condition)

If μ^* is an outer measure on X , a set $A \subset X$ is called μ^* -measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E \subset X$$

Remark 8

If E is a “nice” set containing A , then the above equality says that the outer measure of A , $\mu^*(E \cap A)$, is equal to $\mu^*(E) - \mu^*(E \cap A^C)$, which can be thought as an “inner measure”. So basically we are saying that A is measurable if the outer and inner measure coincide. (Like the definition of Riemann integration with lower and upper sum)

Remark 9

μ^* is subadditive by def $\implies \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E, A \subset X$. So, to prove that a set is μ^* -measurable it is enough to prove the reverse inequality, $\forall E \subset X$. In fact, if $\mu^*(E) = +\infty$, then $+\infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$, and hence A is μ^* -measurable iff

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E \subset X \text{ with } \mu^*(E) < +\infty$$

Their relevance to the notion of μ^* -measurability is clarified by the following

Theorem 9.1 (Caratheodory)

If μ^* is an outer measure on X , the family

$$\mathcal{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$$

is a σ -algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

Lemma 9.1

If $A \subset X$ and $\mu^*(A) = 0$, then A is μ^* -measurable.

Proof. Let $E \subset X$ with $\mu^*(E) < +\infty$. Then

$$\mu^*(E) \geq \mu^*(E) + \mu^*(A) \stackrel{\dagger}{\geq} \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

This implies that A is μ^* -measurable. ★

To sum up: X set, (\mathcal{E}, ρ) elementary and measurable sets, so μ^* is an outer measure. Then given μ^* and the Caratheodory condition, we have (X, \mathcal{M}, μ) that is a complete measure space.

Remark 10

So far we did not prove that $\mathcal{E} \subseteq \mathcal{M}$. We will do it in a particular case.

$\dagger E \cap A^C \subseteq E$ and $E \cap A \subseteq A$ + monotonicity

Lebesgue measure

- $X = \mathbb{R}$, \mathcal{E} family of open intervals, $\rho((a, b)) = b - a = \lambda((a, b))$, the complete measure space is $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ with $\mathcal{L}(\mathbb{R})$ the Lebesgue-measurable sets on \mathbb{R} and λ the Lebesgue measure on \mathbb{R} .
- $X = \mathbb{R}^n$, $\mathcal{E} = \{\prod_{k=1}^n (a_k, b_k) : a_k \leq b_k \quad \forall k = 1, \dots, n\}$, $\rho(\prod_{k=1}^n (a_k, b_k)) = \prod_{k=1}^n (b_k - a_k)$ and this is a complete measure space $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$

11 Lesson 21/09/2022

Lebesgue measure

12 Lesson 22/09/2022

- (1) $((X, \mathcal{M}))$ is a measurable space obtained by means of an outer measure. Ex: $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$, (Y, d_Y) metric space If $X \rightarrow Y$ is (Lebesgue) measurable $\iff (\mathcal{M}, \mathcal{B}(Y))$ is measurable
- (2) $(X, d_X), (Y, d_Y)$ are metric spaces $\rightarrow (X, \mathcal{B}(X))$ If $X \rightarrow Y$ are borel measurable $\iff (\mathcal{B}(X), \mathcal{B}(Y))$ measurable

Remark 13

f is Lebesgue measurable if the continuity of the borel set is a Lebesgue-measurable set.

Proposition 13.1 (1) $(X, d_X), (Y, d_Y)$ metric spaces. If $X \rightarrow Y$ is continuous, then is Borel measurable

- (2) (Y, d_Y) metric space. If $\mathbb{R}^n \rightarrow Y$ is continuous, then it is a Lebesgue measure.

Proof. (1) f is continuous $\iff f^{-1}(A)$ is open $\forall A \in Y$ open $\implies f^{-1}(A) \in \mathcal{B}(Y) \forall A \in Y$ open Since $\mathcal{B}(Y) = \sigma_0(\text{open sets})$ by proposition 1 thus implies that f is Borel measurable

- (2) f is continuous $\implies f$ is Borel measurable mancano pezzi namely f is Lebesgue measurable

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Proposition 13.2

(X, \mathcal{M}) measurable space, $(X, d_X), (Y, d_Y)$ metric spaces. if $f : X \rightarrow Y$ is $\mathcal{M}, \mathcal{B}(Y)$ -measurable and $g : Y \rightarrow Z$ is continuous $\implies g \circ f : x \rightarrow Z$ is $\mathcal{M}, \mathcal{B}(Y)$ -measurable

Proposition 13.3

(X, \mathcal{M}) measurable space Let $\Phi : \mathbb{R}^n \rightarrow Y$ be continuous where (Y, d_Y) is a metric space. Then $h : X \rightarrow Y$ defined by $h(x) = \Phi(u(x), v(x))$ is $\mathcal{M}, \mathcal{B}(Y)$ -measurable.

Proof. Define $f : X \rightarrow \mathbb{R}^n$, $f(x) = (u(x), v(x))$. By def $h = \Phi \circ f$ by prop 3 if we show that f is measurable, then h is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0(\{(a_1, b_1) \times (a_2, b_2) : a, b \in \mathbb{R}\})$$

pezzi $f^{-1}(\mathcal{R} \in \mathcal{M}) \quad \forall \text{open rectangle in } \mathbb{R}^2 \quad R = I \times J \quad f^{-1} = \{x \in X\}$

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Remark 14

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$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

cosine (X, \mathcal{M}) measurable space, then such a function f is measurable iff

$$f^{-1}(a, +\infty] \in \mathcal{M} \quad \forall a \in \mathcal{R}$$

Let now $\{f_n\}$ be a Sequence of measurable functions from X to $\bar{\mathcal{R}}$. Then we define

$$\begin{aligned} (\inf_n f_n)(x) &= \inf_n f_n(x) \\ (\sup_n f_n)(x) &= \sup_n f_n(x) \\ (\liminf_n f_n)(x) &= \liminf_n f_n(x) \\ (\limsup_n f_n)(x) &= \limsup_n f_n(x) \\ (\lim_n f_n)(x) &= \lim_n f_n(x) \quad \text{if the limit exists} \end{aligned}$$

Proposition 14.1

(X, \mathcal{M}) measurable space, $f_n : X \rightarrow \bar{\mathcal{R}}$ measurable, then $\sup \inf \liminf \limsup$ of f_n are measurable, in particular if $\lim f_n$ exists, then f is measurable

Proof. $(\sup f_n)^{-1}((a, \infty]) = \{x \in X : \sup f_n(x) > a\}$ (manca pezzi)

$$\bigcup \{x \in X : f_n(x) > a\}$$

Then $(\sup f_n)^{-1}((a, \infty])$ is measurable, cose da aggiungere Noe the limsup

$$\limsup_n f_n = \lim_n (\sup_{k>n} f_k(x))$$

cose cose

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Simple functions

Definition 14.1

(X, \mathcal{M}) measurable space. A measurable function $s : X \rightarrow \bar{\mathcal{R}}$ is said to be simple if $s(X)$ is a finite set altre cose Then $s(x) = \sum_{n=1} a_n \chi_{E_n}(x)$ where E_n is a measurable set sistemare.

Particular case: if $s : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ and each E_n is a finite union of intervals, then s is said to be a STEP FUNCTION.

The idea is to approximate functions with simple functions.

Theorem 14.1

(X, \mathcal{M}) measurable space, $f : X \rightarrow [0, \infty]$ measurable. Then \exists a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \dots \leq s_n \leq \dots \leq f \quad \text{pointwise}$$

and $s_n(x) \rightarrow f(x)$ Moreover if f is bounded then $s_n \rightarrow f$ uniformly on X as $n \rightarrow \infty$

f is bounded. Fix $n \in \mathbb{N}$ and divide $[0, n]$ in $n \cdot 2^n$ intervals called $I_j = [a_j, b_j)$ with lenght $\frac{1}{2^n}$

Let $E_0 = f^{-1}([n, \infty))$, $E_j = f^{-1}([a_j, b_j))$ for $j = 1, \dots, n \cdot 2^n$

We let Array

Namely we define

$$s_n(x) = n \chi_{E_0}(x) + \sum_{j=1}^{n \cdot 2^n} a_j \chi_{E_j}(x)$$

Then $s_n \leq s_{n+1}$ by contradiction

Then any $x \in X$ stays in $f^{-1}([a_j, b_j))$ for some $j \implies$

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