Chapter 2 Derivatives

CHAPTER EXPECTATIONS

In this chapter, you will

- **understand and determine derivatives of polynomial and simple rational functions from first principles, Section 2.1**
- **identify examples of functions that are not differentiable, Section 2.1**
- **justify and use the rules for determining derivatives, Sections 2.2, 2.3, 2.4, 2.5**
- **identify composition as two functions applied in succession, Section 2.5**
- **determine the composition of two functions expressed in notation, and decompose a given composite function into its parts, Section 2.5**
- **use the derivative to solve problems involving instantaneous rates of change,**
Sections 2.2, 2.3, 2.4, 2.5

Recall

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This limit has two interpretations:

1)

2)

This limit has two interpretations:
1) the slope of the tangent to the graph y=f(x)at the point (a,f(a)).

2) The instantaneous rate of change of y=f(x) with respect to x, at x=a.

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

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- **Since this limit plays a central role in calculus, it is given a name and a concise notation. It is called the derivative of $f(x)$ at x=a**
- **It is denoted by $f'(a)$ and is read as "*f* prime of *a*."**

The derivative of f at the number a is given by $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided that this limit exists.

First Principles Definition of the Derivative

The derivative of a function $f(x)$ is a new function $f'(x)$ defined by

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, if the limit exists.

Example 1 Determine a Derivative Using the First Principles Definition

a) State the domain of the function $f(x) = x^2$.

b) Use the first principles definition to determine the derivative of $f(x) = x^2$. What is the derivative's domain?

c) What do you notice about the nature of the derivative? Describe the relationship between the function and its derivative.

Determine the derivative of $f(x) = x^2$ at $x = -3$.

Solution

Using the definition, the derivative at $x = -3$ is given by

$$\begin{aligned}
f'(-3) &= \lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h} \\
&= \lim_{h\to 0} \frac{(-3+h)^2 - (-3)^2}{h} \\
&= \lim_{h\to 0} \frac{9 - 6h + h^2 - 9}{h} \\
&= \lim_{h\to 0} \frac{h(-6+h)}{h} \\
&= \lim_{h\to 0} (-6+h) \\
&= -6
\end{aligned}$$

Determine the derivative $f'(t)$ of the function $f(t) = \sqrt{t}, t \geq 0$.

Using the definition, $f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} \left(\frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + \sqrt{t}} \right)$$ (Rationalize th numerato

$$= \lim_{h \to 0} \frac{(t+h) - t}{h(\sqrt{t+h} + \sqrt{t})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{t+h} + \sqrt{t})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{t+h} + \sqrt{t}}$$

$$= \frac{1}{2\sqrt{t}}, \text{ for } t > 0$$

Determine an equation of the tangent to the graph of $f(x) = \frac{1}{x}$ at the point where $x = 2$.

Determine an equation of the line that is perpendicular to the tangent to the graph of $f(x) = \frac{1}{x}$ at $x = 2$ and that intersects it at the point of tangency.

Differentiability

f(x) is differentiable at x=a if the following limit exists:

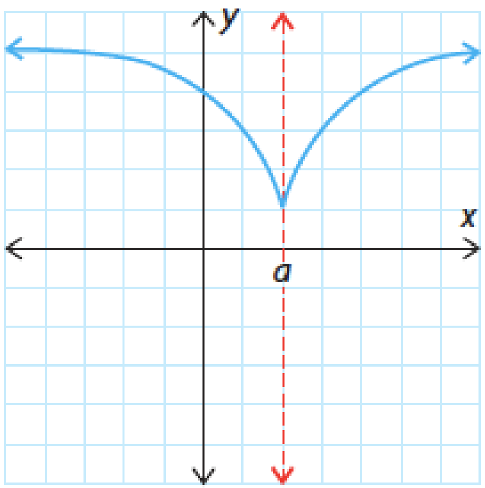
$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

OR

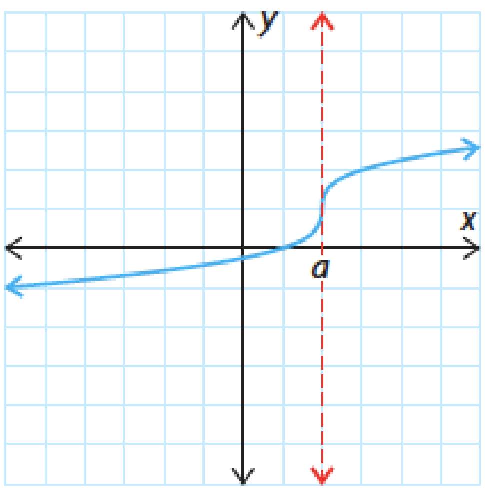
$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h},$$

The Existence of Derivatives

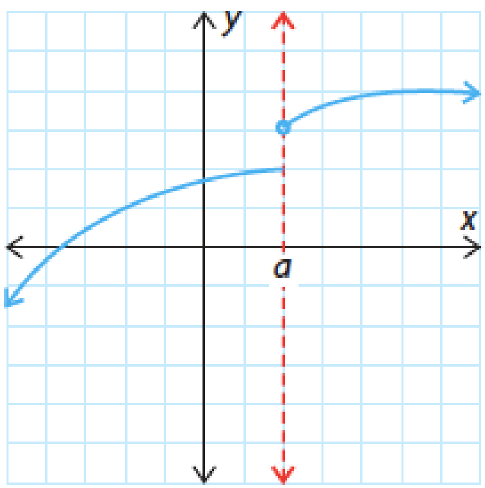
A function f is said to be **differentiable** at a if $f'(a)$ exists. At points where f is not differentiable, we say that the *derivative does not exist*. Three common ways for a derivative to fail to exist are shown.



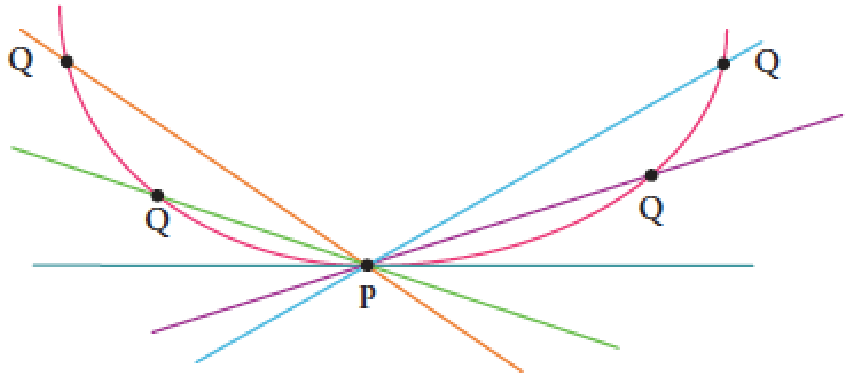
Cusp



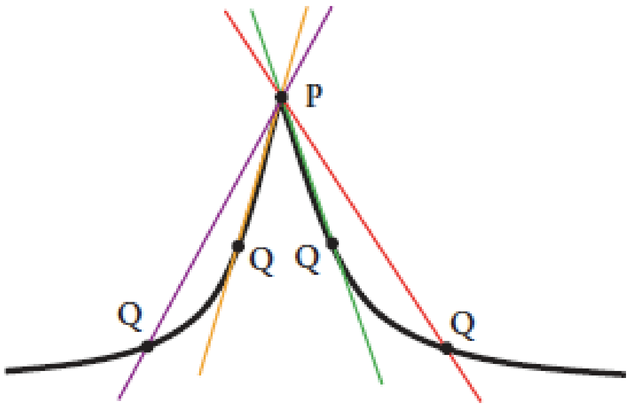
Vertical Tangent



Discontinuity



Curve A



Curve B

Show that the absolute value function $f(x) = |x|$ is not differentiable at $x = 0$.

Existence of Limit, Continuity, Differentiability

- **p.73 #1, 6c, 7b, 8 (no sketch- use formula) , 14, 15, 18, 19 (14-19 use formulas if needed)**

- *Section 2.2*—The Derivatives of Polynomial
- Functions

Leibniz Notation expresses the derivative of the function $y = f(x)$ as $\frac{dy}{dx}$, read as "dee y by dee x." Leibniz's form can also be written $\frac{d}{dx}f(x)$.

The expression $\left.\frac{dy}{dx}\right|_{x=a}$ in Leibniz notation means "determine the value of the derivative when $x = a$."

The Constant Function Rule

If $f(x) = k$, where k is a constant, then $f'(x) = 0$.

In Leibniz notation, $\frac{d}{dx}(k) = 0$.

The Power Rule

If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.
In Leibniz notation, $\frac{d}{dx}(x^n) = nx^{n-1}$.

The Constant Multiple Rule

If $f(x) = kg(x)$, where k is a constant, then $f'(x) = kg'(x)$.
In Leibniz notation, $\frac{d}{dx}(ky) = k\frac{dy}{dx}$.

Differentiate the following functions:

a. $f(x) = 7x^3$

b. $y = 12x^{\frac{4}{3}}$

The Sum Rule

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$.

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$.

The Difference Rule

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$.

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) - \frac{d}{dx}(q(x))$.

Differentiate the following functions:

a. $f(x) = 3x^2 - 5\sqrt{x}$

b. $y = (3x + 2)^2$

Example 2 Rational Exponents and the Power Rule

Determine $\frac{dy}{dx}$ for each function. Express your answers using positive exponents.

a) $y = \sqrt[3]{x}$

b) $y = \frac{1}{x}$

c) $y = -\frac{1}{x^5}$

Determine the equation of the tangent to the graph of $f(x) = -x^3 + 3x^2 - 2$ at $x = 1$.

Determine points on the graph in Example 5 where the tangents are horizontal.

p.82
2(c,d,f) 3(c,e), 4(a,e,f) 5b, 6b, 7a, 8b, 9b, 11, 14, 20(b,c), 25 (I, ii)

Do more similar problems if you feel you need extra practice

- **2.3 The Product Rule**

The Product Rule

If $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + f(x)g'(x)$.

If u and v are functions of x, $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$.

The Product Rule

If $P(x) = f(x)g(x)$, where $f(x)$ and $g(x)$ are differentiable functions, then $P'(x) = f'(x)g(x) + f(x)g'(x)$.

In Leibniz notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Applying the product rule

Differentiate $h(x) = (x^2 - 3x)(x^5 + 2)$ using the product rule.

Selecting an efficient strategy to determine the value of the derivative

Find the value $h'(-1)$ for the function $h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$.

Example 2 Find Equations of Tangents Using the Product Rule

Determine the equation of the tangent to the curve $y = (x^2 - 1)(x^2 - 2x + 1)$ at $x = 2$. Use technology to confirm your solution.

- Homework
- p.90 #1(d,e), 5(c,f), 6, 7b, 14

2-4 The Quotient Rule

The Quotient Rule

If $h(x) = \frac{f(x)}{g(x)}$, then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

In Leibniz notation, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$.

Determine the derivative of $h(x) = \frac{3x - 4}{x^2 + 5}$.

Determine the equation of the tangent to $y = \dfrac{2x}{x^2 + 1}$ at $x = 0$.

Quotient Rule

p.97 4cf, 5bd, 6,7,8,

- 2.5 The Chain Rule

Definition of a composite function

Given two functions f and g, the **composite function** $(f \circ g)$ is defined by $(f \circ g)(x) = f(g(x))$.

Reflecting on the process of composition

If $f(x) = \sqrt{x}$ and $g(x) = x + 5$, find each of the following values:

a. $f(g(4))$ b. $g(f(4))$ c. $f(g(x))$ d. $g(f(x))$

The Chain Rule

If f and g are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by $h'(x) = f'(g(x))g'(x)$.

The Chain Rule

Given two differentiable functions $g(x)$ and $h(x)$, the derivative of the composite function $f(x) = g[h(x)]$ is $f'(x) = g'[h(x)] \times h'(x)$.

The chain rule states how to compute the derivative of the composite function $h(x) = f(g(x))$ in terms of the derivatives of f and g.

If we interpret derivatives as rates of change, the chain rule states that if y is a function of x through the intermediate variable u, then the rate of change of y with respect to x is equal to the product of the rate of change of y with respect to u and the rate of change of u with respect to x.

Example 1 Apply the Chain Rule

Differentiate each function using the chain rule.

a) $f(x) = (3x - 5)^4$

b) $f(x) = \sqrt{4 - x^2}$

Alternate Form of the Chain Rule

Consider the function in Example 1 part b): $f(x) = \sqrt{4 - x^2}$.

Let $y = \sqrt{u}$. Let $u = 4 - x^2$.

Example 3 Combine the Chain Rule and the Product Rule

Determine the equation of the tangent to $f(x) = 3x(1 - x)^2$ at $x = 0.5$.

Differentiate $h(x) = (x^2 + 3)^4(4x - 5)^3$. Express your answer in a simplified factored form.

Using the derivative to solve a problem

The position s, in centimetres, of an object moving in a straight line is given by $s = t(6 - 3t)^4$, $t \geq 0$, where t is the time in seconds. Determine the object's velocity at $t = 2$.

- Chain rule

p.105 4(a,f), 8, 9a, 12, 15,16