Solving nonlinear equations in one variable (II)

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Last week

- Honer's method: Evaluating a polynomial
- Bisection: Solving $f(x) = 0$

Bisection

- Given a function *f*(.)
- Given a range [*a*, *b*]
- Repeat computing the middle point until convergence, e.g., (b-a)/2 < TOL
- 猜數字

Bisection: How accurate and how fast?

- The interval length after n bisection steps is: $\dfrac{b-a}{2^n}$

$$\text{Solution error} = |x_n - x| < \frac{b-a}{2^{n+1}}$$

x_n: the midpoint of the n-th interval

- If we want the error to satisfy $|x_n - x| \le \varepsilon$, it suffices to have $(b-a)/2^n \le \varepsilon$, so that

$$\boxed{n > \log_2\left(\frac{b-a}{\varepsilon}\right)}$$

Bisection: Properties

- Simple ☺
- Safe, robust ☺
- Requires only that *f* be continuous ☺
- Slow ☹
- Hard to generalize to systems ☹

Today

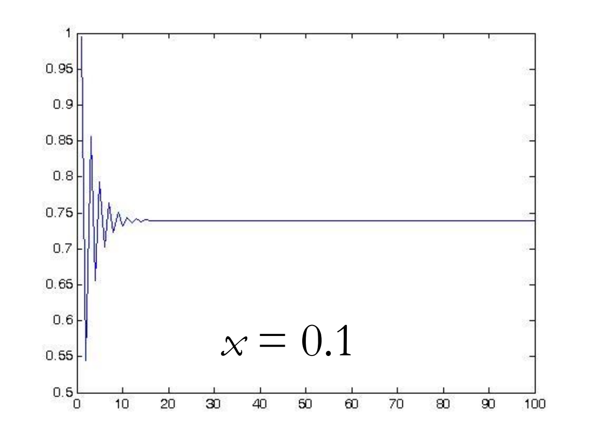
- Fixed-point iteration for solving $f(x) = 0$

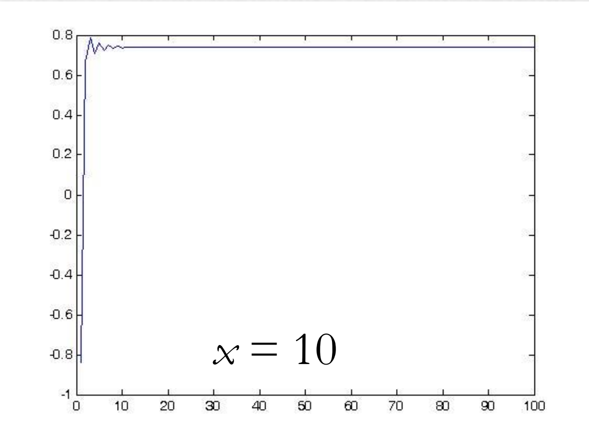
Fixed point iteration

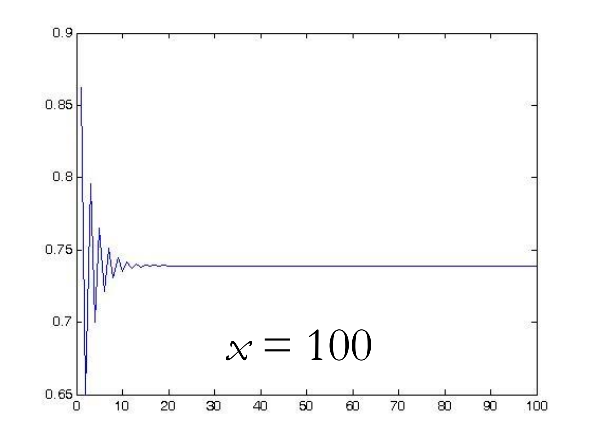
What happens in the following example?

- Let x be an arbitrary number.
- Repeat computing $x = \cos(x)$

The number converges to 0.7390851332.







Fixed point iteration

- Problem $f(x) = 0$ can be rewritten as

$$x = g(x).$$

(There are many ways to do this.)

We are looking for a **fixed point** of $g(x)$.

Fixed point iteration

- Definition: The real number x is a **fixed point** of a function g if $g(x) = x$.
- Example
  - The fixed point of $\cos(x)$ is 0.7390851332.
- What is (or are) the fixed point(s) of $g(x) = x^3$?

Fixed point iteration: Approach

- Given a function $f(x)$, select a function $g(x)$ such that

$$f(x) = 0 \longrightarrow g(x) = x.$$

- Then
  - x_0 = initial guess
  - $x_{i+1} = g(x_i)$ for $i = 0, 1, 2, \ldots$
- Until x_{i+1} satisfies some termination criterion

注!意

- There are many ways to transform $f(x) = 0$ into fixed point form! Not all of them are "good" in terms of convergence.
- Example: $x^3 + x - 1 = 0$

$$f(x) = 0 \rightarrow g(x) = x$$

1

$$x = 1 - x^3$$

2

$$x = \sqrt[3]{1 - x}$$

3

$$x = \frac{1 + 2x^3}{1 + 3x^2}$$

(+$2x^3$ on both sides)

$$3x^3 + x - 1 = 2x^3$$

$$(3x^2 + 1)x = 2x^3 + 1$$

注!意

- There are many ways to transform $f(x) = 0$ into fixed point form! Not all of them are "good" in terms of convergence.
- Example: $x^3 + x - 1 = 0$

$$f(x) = 0 \longrightarrow g(x) = x$$

1. $x = 1 - x^3 \quad \Rightarrow g(x) = 1 - x^3$
2. $x = \sqrt[3]{1-x} \quad \Rightarrow g(x) = \sqrt[3]{1-x}$
3. $x = \dfrac{1+2x^3}{1+3x^2} \quad \Rightarrow g(x) = \dfrac{1+2x^3}{1+3x^2}$

Which one will work?

- There are many ways to transform $f(x) = 0$ into fixed point form! Not all of them are "good" in terms of convergence.
- Example: $x^3 + x - 1 = 0$

$$f(x) = 0 \longrightarrow g(x) = x$$

1. $$x = 1 - x^3 \quad \Rightarrow g(x) = 1 - x^3$$
2. $$x = \sqrt[3]{1 - x} \quad \Rightarrow g(x) = \sqrt[3]{1 - x}$$
3. $$x = \frac{1 + 2x^3}{1 + 3x^2} \quad \Rightarrow g(x) = \frac{1 + 2x^3}{1 + 3x^2}$$

$$x_0 = 0.5$$

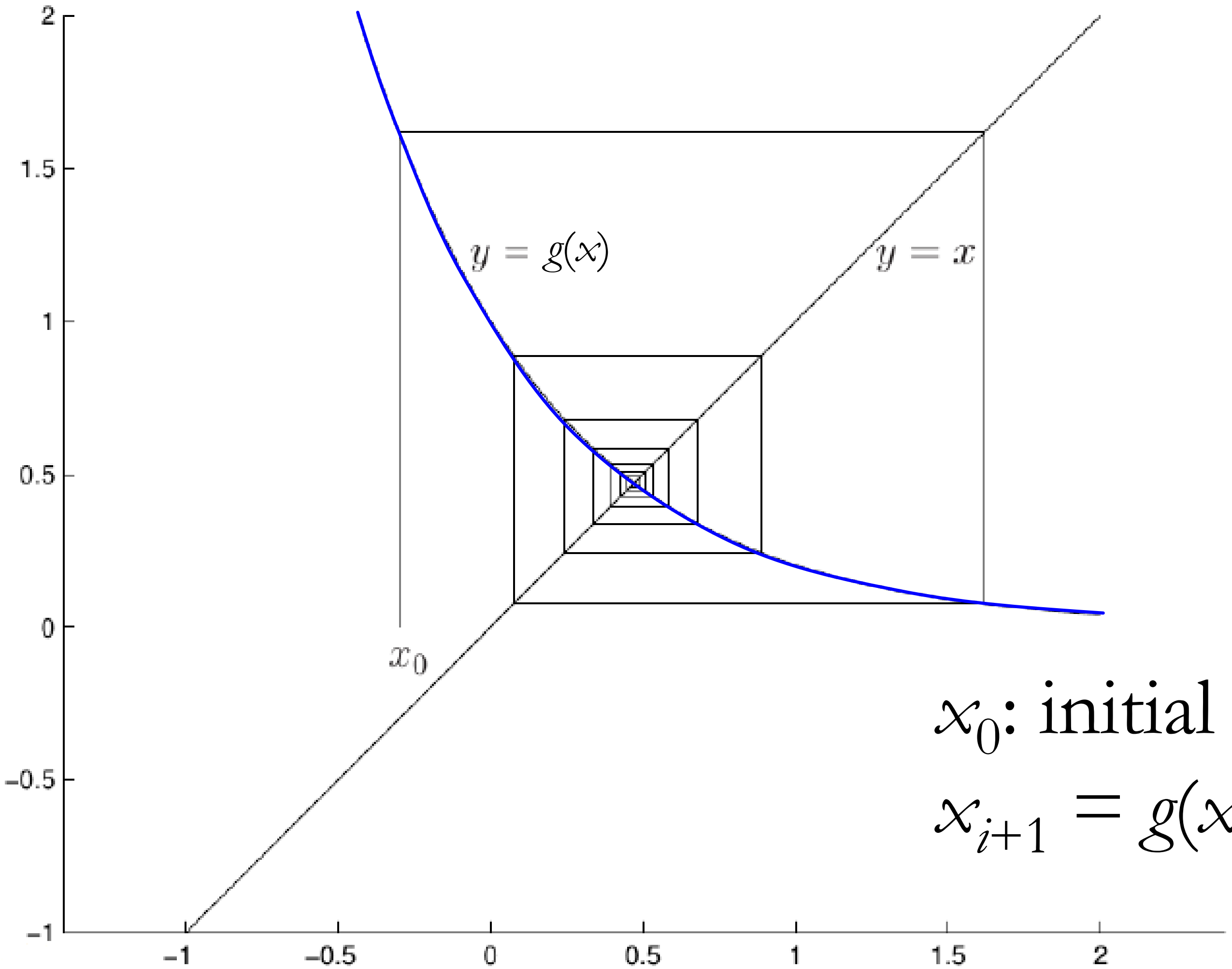
$g(x)=1-x^3$	$g(x)=\sqrt[3]{1-x}$		$g(x)=\dfrac{1+2x^3}{1+3x^2}$
0.5000	0.5000		0.5000
0.8750	0.7937		0.7143
0.3301	0.5909		0.6832
0.9640	0.7424		0.6823
0.1041	0.6363	0.6807	0.6823
0.9989	0.7138	0.6835	0.6823
0.0034	0.6590	0.6815	0.6823
1.0000	0.6986	0.6829	0.6823
0.0000	0.6704	0.6819	0.6823
1.0000	0.6907	0.6826	0.6823
0.0000	0.6763	0.6821	0.6823
1.0000	0.6866	0.6825	0.6823
0.0000	0.6792	0.6822	0.6823
	0.6845	0.6824	
		0.6823	
		0.6824	

start with x_0 on the x-axis

go parallel to the y-axis to the graph of $F \equiv g$

move parallel to the x-axis to the graph $y = x$

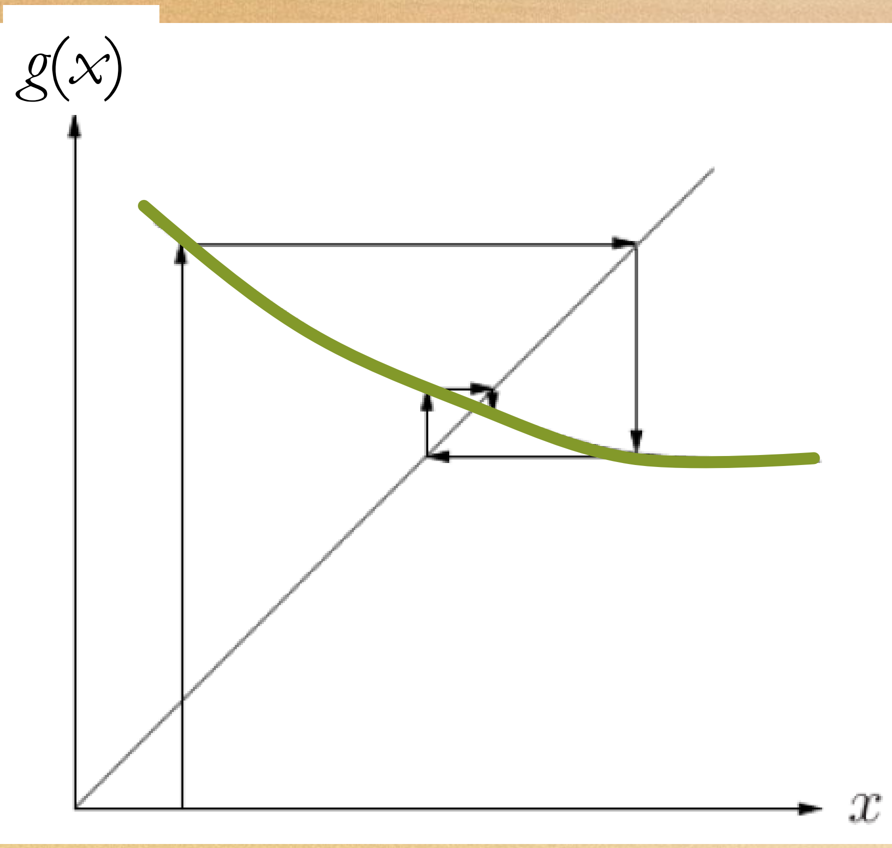
go parallel to the y-axis to the graph of F

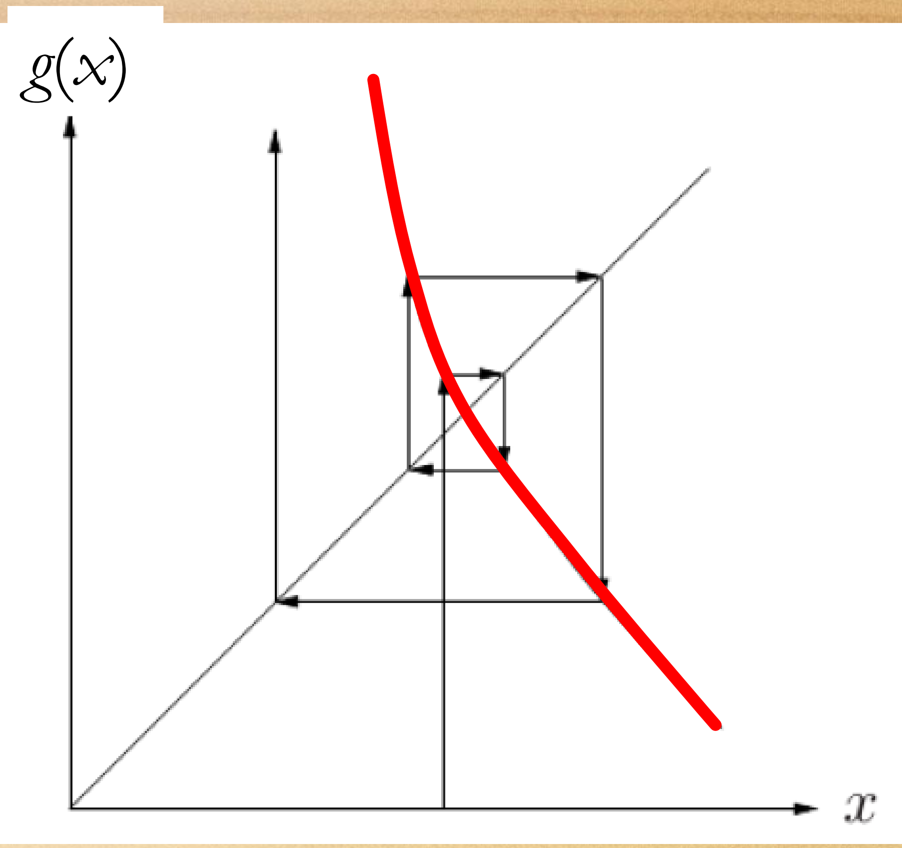


x_0: initial guess

$x_{i+1} = g(x_i)$

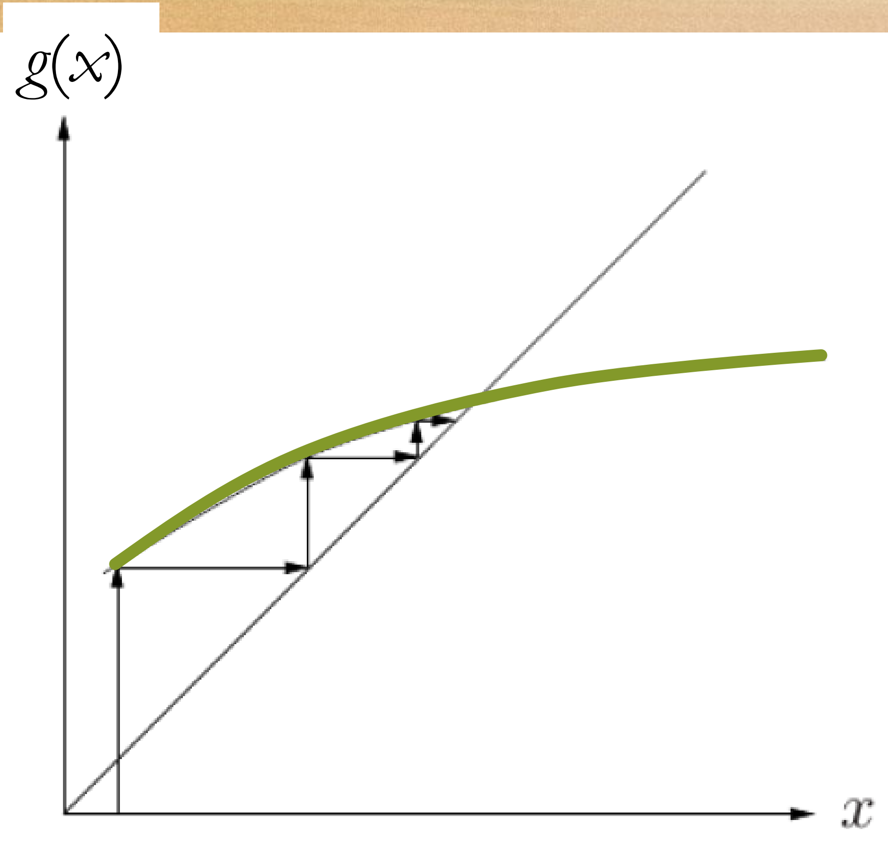
Geometric interpretation

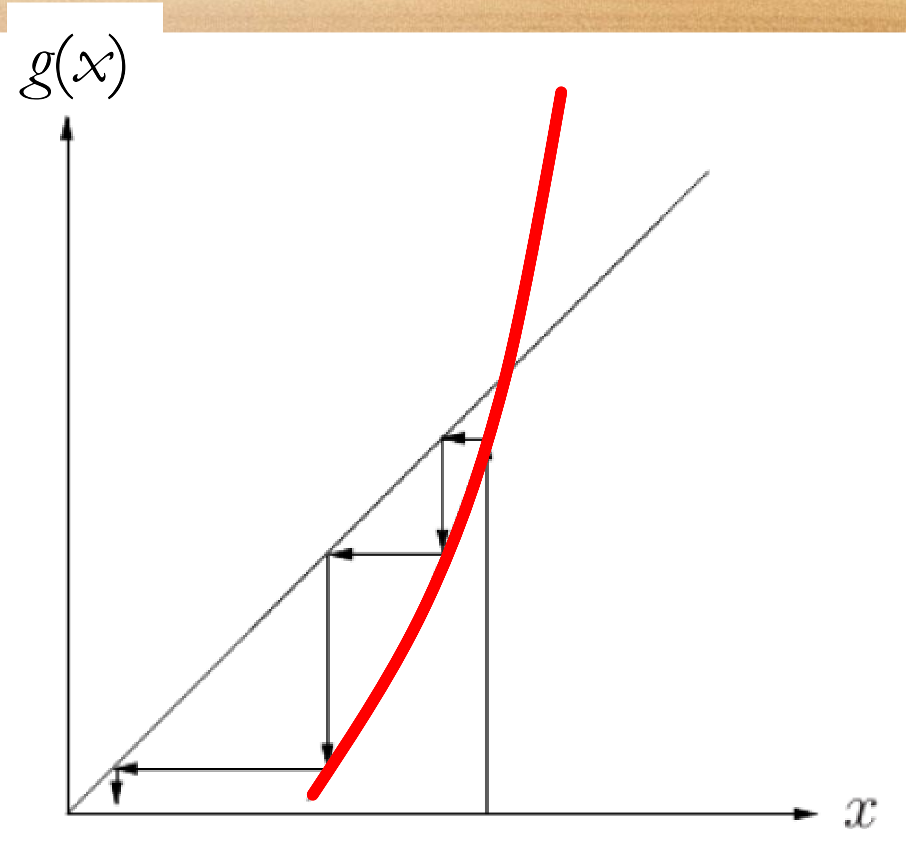




x_0 is close to the root

Geometric interpretation





$$g(x) = 1 - x^3$$

$$g(x) = \sqrt[3]{1 - x}$$

$$g(x) = \frac{1 + 2x^3}{1 + 3x^2}$$

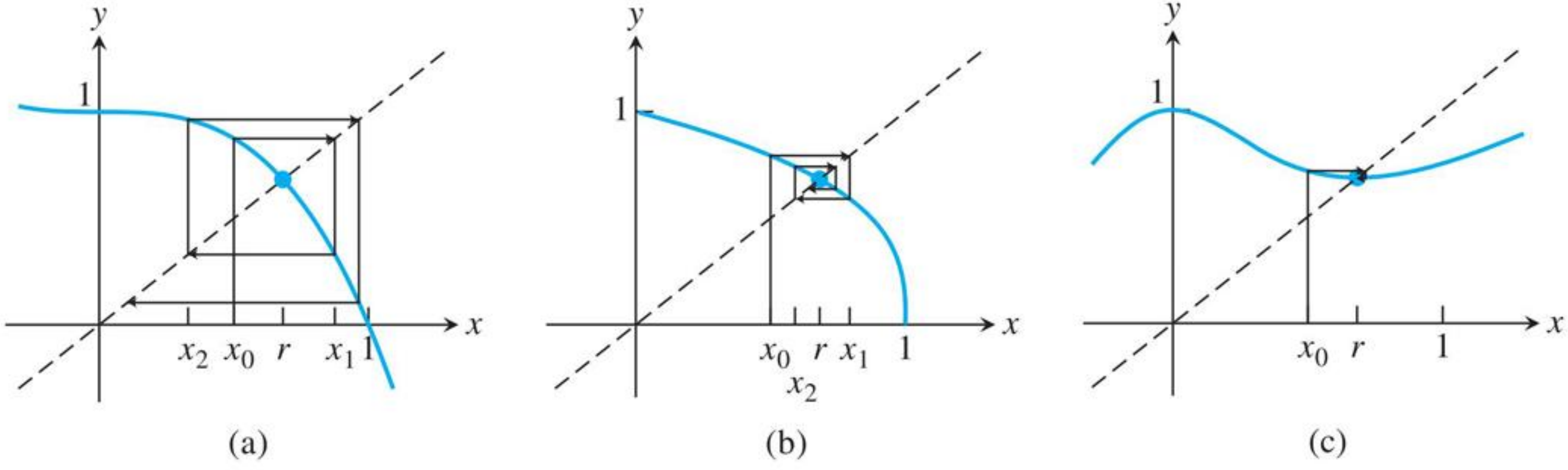


Figure 1.3 Geometric view of FPI. The fixed point is the intersection of $g(x)$ and the diagonal line. Three examples of $g(x)$ are shown together with the first few steps of FPI. (a) $g(x) = 1 - x^3$ (b) $g(x) = (1 - x)^{1/3}$ (c) $g(x) = (1 + 2x^3)/(1 + 3x^2)$

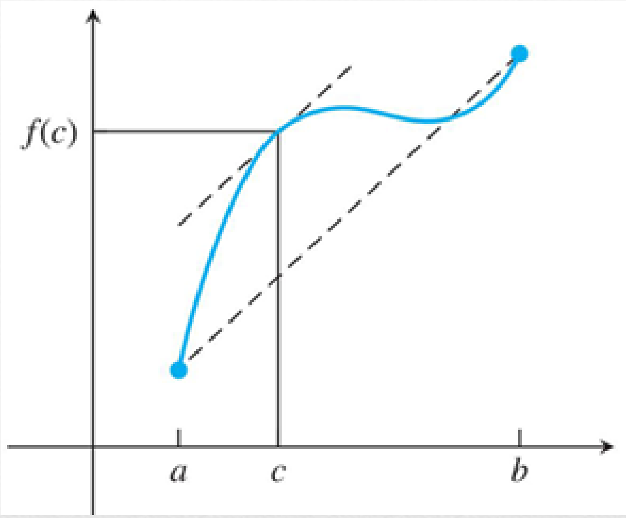
Convergence

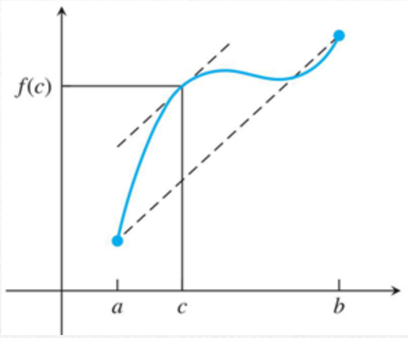
- $S = |g'(r)| < 1$

Will explain this now. Stay tuned. ☺

Mean Value Theorem

- Let *f* be a continuous function on the interval [*a*, *b*]. Then there exists a number *c* between a and b such that $f'(c) = (f(b) - f(a)) / (b-a)$





Convergence

- Let x_i denote the iterate at step i. There exists a number c_i between x_i and r such that

$$g'(c) = (g(x_i) - g(r)) \;/\; (x_i - r)$$

$$= (x_{i+1} - r) \;/\; (x_i - r)$$

$$(x_{i+1} - r) = g'(c)(x_i - r)$$

Define $e_i = |x_i - r|$

$$e_{i+1} = |g'(c)|\, e_i$$

Linear convergence

- Let e_i denote the error at step i of an iterative method. If

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = S < 1,$$

the method is said to obey **linear convergence** with rate S.

$$x^3 + x - 1 = 0,\ r \approx 0.6823$$

$$\boxed{g(x) = 1 - x^3}$$

$$g'(x) = -3x^2$$

$$|g'(r)| = 1.3966$$

$$\boxed{g(x) = \sqrt[3]{1-x}}$$

$$g'(x) = \frac{1}{3}(1-x)^{-2/3}(-1)$$

$$|g'(r)| = 0.0716$$

$$\boxed{g(x) = \frac{1+2x^3}{1+3x^2}}$$

$$g'(x) = \frac{6x^2(1+3x^2) - (1+2x^3)6x}{(1+3x^2)^2}$$

$$|g'(r)| = \text{-4.7495e-005}$$

$\lvert g'(r)\rvert = 1.3966$	$\lvert g'(r)\rvert = 0.0716$		$\lvert g'(r)\rvert = \text{-4.7495e-005}$
$\boxed{g(x) = 1 - x^3}$	$\boxed{g(x) = \sqrt[3]{1-x}}$		$\boxed{g(x) = \dfrac{1+2x^3}{1+3x^2}}$
0.5000	0.5000		0.5000
0.8750	0.7937		0.7143
0.3301	0.5909		0.6832
0.9640	0.7424		0.6823
0.1041	0.6363	0.6807	0.6823
0.9989	0.7138	0.6835	0.6823
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	0.6845	0.6824	
		0.6823	
		0.6824	

Practice

- Explain why the fixed-point iteration $g(x) = \cos(x)$ converges.

$r \approx 0.74$

$g'(x) = -\sin x$

$g'(r) = -\sin 0.74 \approx -0.67$

$|g'(r)| < 1$

Bisection vs. Fixed-point iteration

- Which one is faster?
- Depending on $S = |g'(r)|$ is smaller or larger than ½.

程式練習

- 請寫一個程式(使用FPI)計算方程式的根
- 請用你的程式計算

$$x^3 = 2x + 2$$

Two specific examples of FPI

- Newton's algorithm
  - A refined version of FPI where S is designed to be zero
- Secant method

Algorithm: Newton's iteration

Given a scalar differentiable function $f(x)$,

1. Start from an initial guess x_0.
2. For $i = 0, 1, 2, \ldots,$ compute

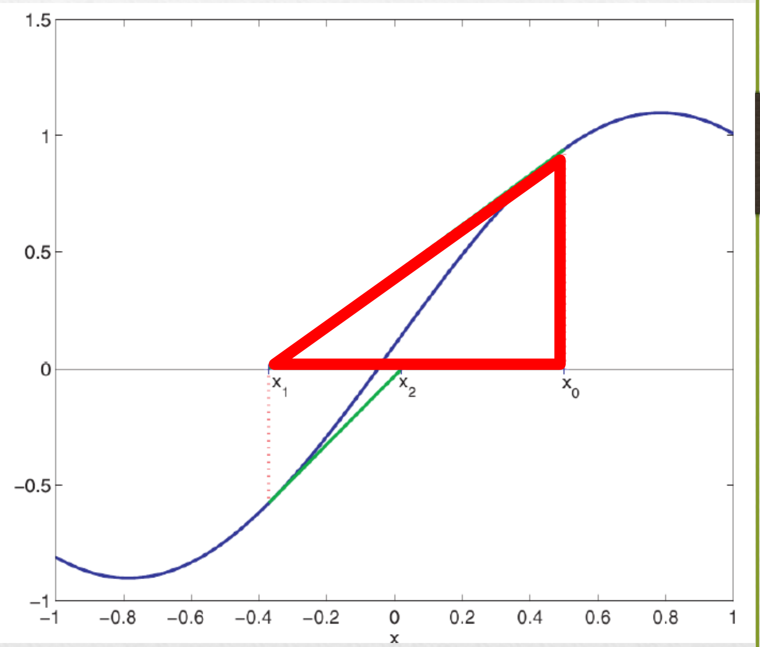
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

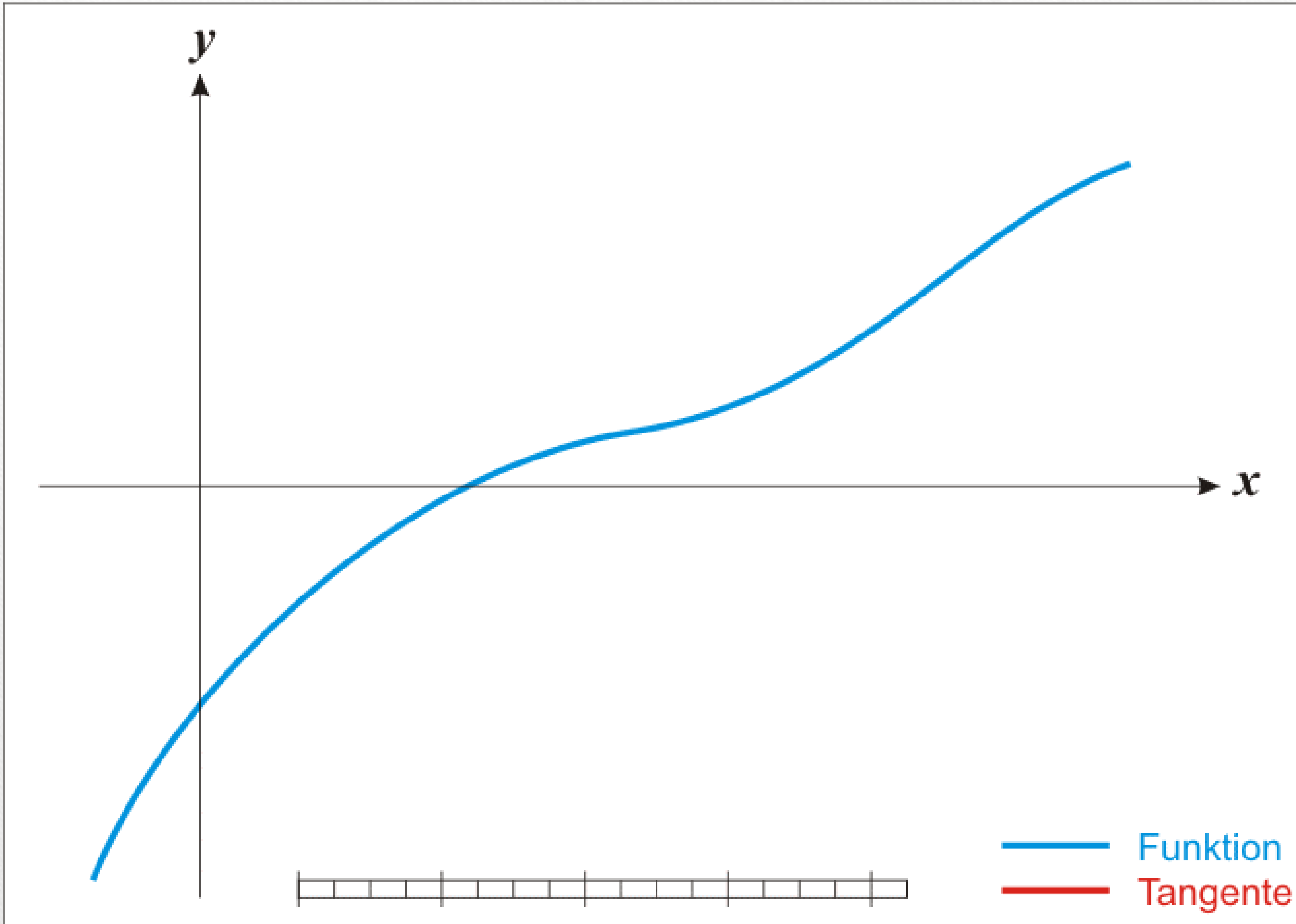
until x_{i+1} satisfies some termination criterion

Newton’s iteration

- $f'(x_0) = f(x_0) / (x_0 - x_1)$
- $x_1 = x_0 - f(x_0) / f'(x_0)$
- $x_{i+1} = x_i - f(x_i) / f'(x_i),\ i = 0, 1, 2, \ldots$
- Newton's method is a fixed point iteration with iteration function

  $g(x) = x - f(x) / f'(x)$

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x
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Example

- Find the Newton's Method formula for $x^3 + x - 1 = 0$.
- $f(x) = x^3 + x - 1 = 0$
- $f'(x) = 3x^2 + 1$
- The iteration formula is

$$x_{i+1} = x_i - \frac{x_i^3 + x_i - 1}{3{x_i}^2 + 1}$$

i	x_i	$e_i = \lvert x_i - r \rvert$	e_i / e_{i-1}^2
0	−0.70000000	1.38232780	
1	0.12712551	0.55520230	0.2906
2	0.95767812	0.27535032	0.8933
3	0.73482779	0.05249999	0.6924
4	0.68459177	0.00226397	0.8214
5	0.68233217	0.00000437	0.8527
6	0.68232780	0.00000000	0.8541
7	0.68232780	0.00000000	

Order of Convergence (e_i vs. e_{i+1})

The method is

- **linearly convergent** if

$$\lim_{i\to\infty}\frac{e_{i+1}}{e_i} = S < 1,$$

- **quadratically convergent** if

$$\lim_{i\to\infty}\frac{e_{i+1}}{e_i^2} = M \text{ (a constant).}$$

Convergence of Newton's method

- Recall that the convergence rate in fixed-point iteration is $|g'(r)|$, will show $|g'(r)|<1$.

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

$g'(r) = 0$ **(Why?)**

Theorem: Let f be twice continuously differentiable and $f(r) = 0$. If $f\,'(r) \neq 0$, then Newton's method is **quadratically convergent** to r, starting with x_0 close to r.

Proof

$$f(r) = f(x_i) + (r - x_i)f'(x_i) + \frac{(r - x_i)^2}{2} f''(x_i)$$

$$0 = f(x_i) + (r - x_i)f'(x_i) + \frac{(r - x_i)^2}{2} f''(x_i)$$

$$-\frac{f(x_i)}{f'(x_i)} = r - x_i + \frac{(r - x_i)^2}{2} \frac{f''(x_i)}{f'(x_i)}$$

$$\underbrace{\left(x_i - \frac{f(x_i)}{f'(x_i)}\right)}_{x_{i+1}} - r = \frac{(r - x_i)^2}{2} \frac{f''(x_i)}{f'(x_i)}$$

$$e_{i+1} = e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

$$\lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} = \left| \frac{f''(r)}{2\,f'(r)} \right|$$

$$\boxed{\lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} = M \text{ (a constant).}}$$

Secant method

- Replaces the tangent line (the function's derivative) with the secant line.

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

- Two starting guesses are needed to begin the Secant method.

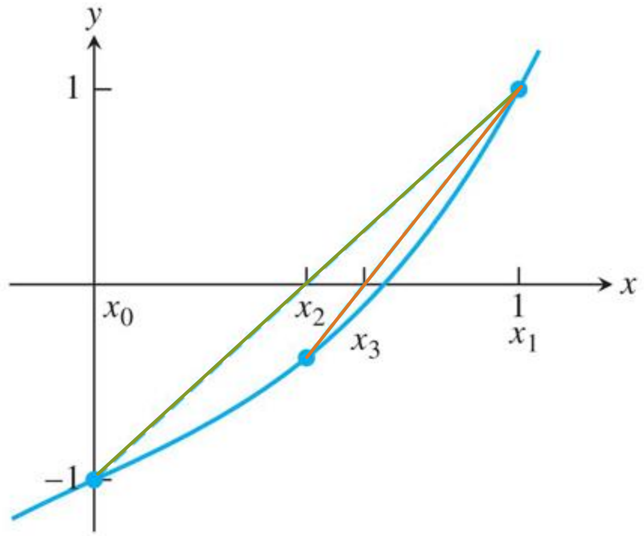

Figure 1.11 Two steps of the Secant Method. Illustration of Example 1.16. Starting with $x_0 = 0$ and $x_1 = 1$, the Secant Method iterates are plotted along with the secant lines.

Convergence of Secant method

- The secant method is **superlinearly convergent**, meaning that it lies between linearly and quadratically convergent methods.

Without proof.

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Example

- Apply the Secant method with $x_0 = 0$, $x_1 = 1$ to find the root of $f(x) = x^3 + x - 1$.
- The iteration formula is

$$x_{i+1} = x_i - \frac{(x_i^3 + x_i - 1)(x_i - x_{i-1})}{x_i^3 + x_i - (x_{i-1}^3 + x_{i-1})}$$

i	x_i
0	0.00000000000000
1	1.00000000000000
2	0.50000000000000
3	0.63636363636364
4	0.69005235602094
5	0.68202041964819
6	0.68232578140989
7	0.68232780435903
8	0.68232780382802
9	0.68232780382802

程式練習

- Each equation has one root. Please use Newton (or Secant) method to find the root of

$$x^5 + x = 1$$ ①

$$\ln x + x^2 = 3$$ ②

$$\sin x = 6x + 5$$ ③

輸出至小數點以下四位