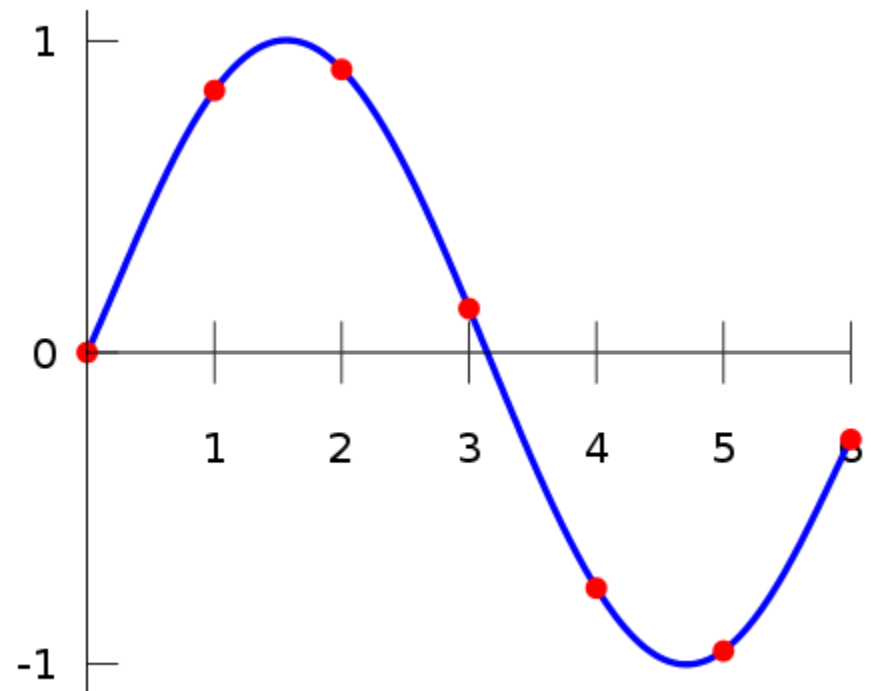
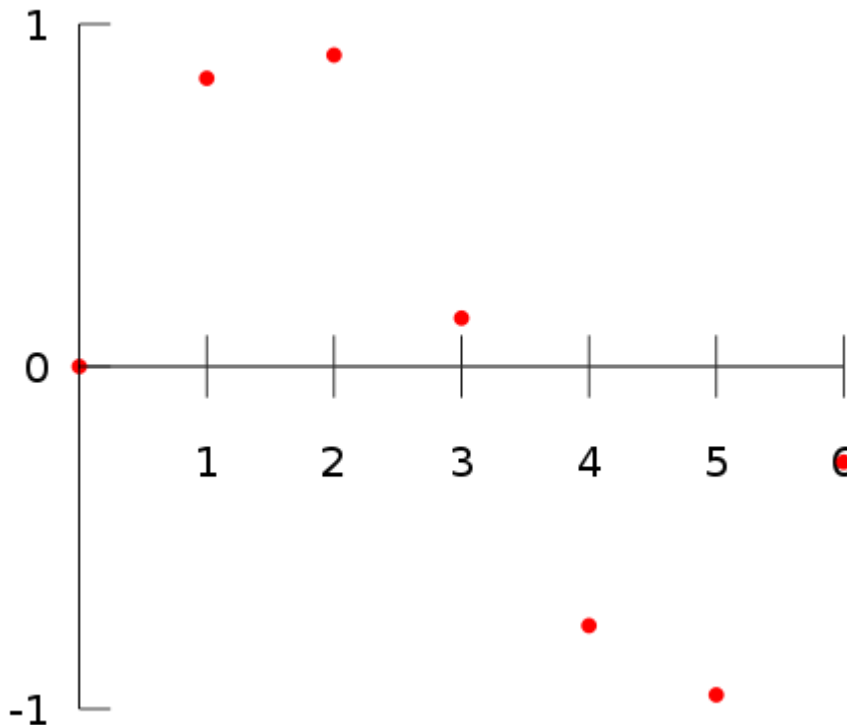


Interpolation – Part 2

Mei-Chen Yeh

Review: Interpolating data

- Given a collection of **data samples** $\{(x_i, y_i)\}_{i=1}^n$, we want to find a function $P(x)$ which can be used to estimate samples for any $x \neq x_i$.



Review: Lagrange interpolation

- $P(x)$: a polynomial in this form:

$$P(x) = y_1 L_1(x) + y_2 L_2(x) + \cdots + y_n L_n(x)$$

$$L_k(x) = \frac{(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$

- For example, given 3 points:

$$P_2(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Review: Lagrange interpolation

- $P(x)$: a polynomial in this form:

$$P(x) = y_1 L_1(x) + y_2 L_2(x) + \cdots + y_n L_n(x)$$

$$L_k(x) = \frac{(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$

- The degree of $P(x)$ is ? **at most $n - 1$**

$$\begin{aligned} P(x) &= y_1 L_1(x) + y_2 L_2(x) + \cdots + y_n L_n(x) \\ &= a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \end{aligned}$$

Theorem. Let $(x_1, y_1), \dots, (x_n, y_n)$ be n points on a plane with distinct x_i . Then there exists **one and only one** polynomial P of degree $n-1$ or less that satisfies $P(x_i) = y_i$ for $i = 1, \dots, n$.

Today

- Interpolation error
- Runge phenomenon
- Newton's divided differences

Interpolation error

- Assume we have $y_i = f(x_i)$, $i = 0, 1, \dots, n$ and an interpolating polynomial $P(x)$. The **interpolation error** at x is

$$f(x) - P(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{n!} f^{(n)}(c)$$

c lies between the smallest and largest of x_i .

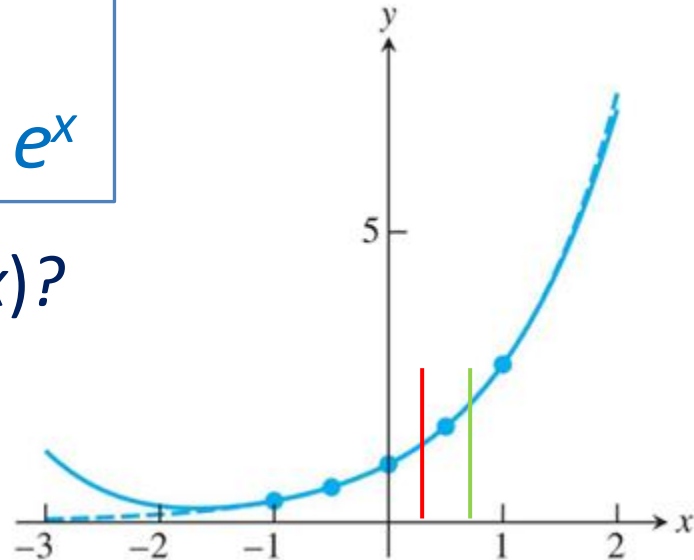
Example

- Find an upper bound for the difference at $x = 0.25$ and $x = 0.75$ between $f(x) = e^x$ and the polynomial that interpolates it at the points $-1, -0.5, 0, 0.5, 1$.

solid: $P(x)$

dash: $f(x) = e^x$

degree of $P(x)$?



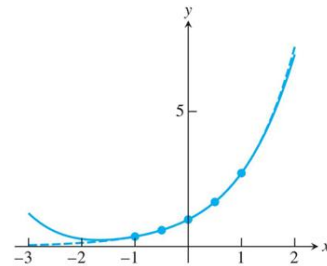
$$x_i = -1, -0.5, 0, 0.5, 1,$$

$$f(x) = e^x \rightarrow f'(x) = e^x$$

$$f(x) - P(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{n!} f^{(n)}(c)$$

$$= \frac{(x + 1) \left(x + \frac{1}{2}\right) x \left(x - \frac{1}{2}\right) (x - 1)}{5!} f^{(5)}(c)$$

- Range of c ? **-1 to 1**
- Maximal value of $|f^{(5)}(c)|$ on $[-1, 1]$? **e^1**



$$|f(x) - P(x)| \leq \frac{(x + 1) \left(x + \frac{1}{2}\right) x \left(x - \frac{1}{2}\right) (x - 1)}{5!} e$$

$$|f(x) - P(x)| \leq \frac{(x+1)\left(x+\frac{1}{2}\right)x\left(x-\frac{1}{2}\right)(x-1)}{5!}e$$

- At $x = 0.25$

$$|f(x) - P(x)| \leq \frac{(1.25)(0.75)0.25(-0.25)(-0.75)}{5!}e$$

$$\approx 0.000995$$

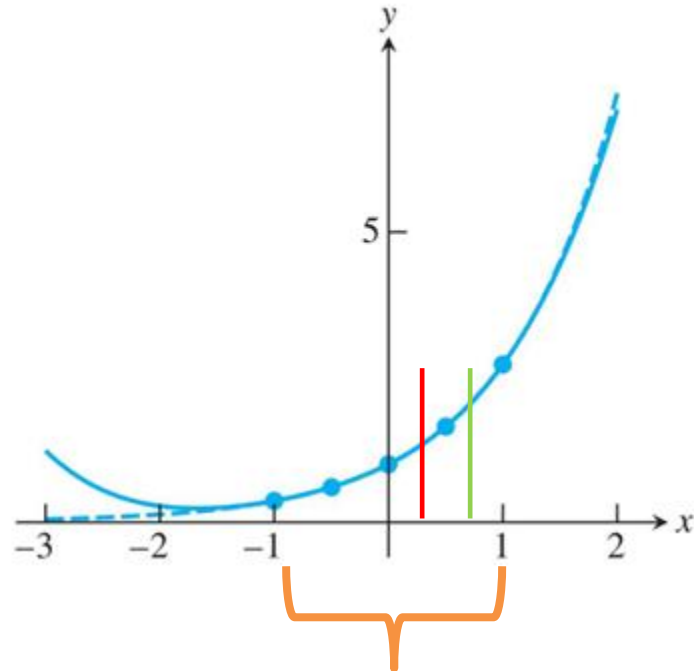
- At $x = 0.75$

$$|f(x) - P(x)| \leq \frac{(1.75)(1.25)0.75(0.25)(-0.25)}{5!}e$$

$$\approx 0.002323$$

- error = 0.000995 at $x = 0.25$
- error = 0.002323 at $x = 0.75$

The interpolation error will be smaller close to the center of the interpolation interval.



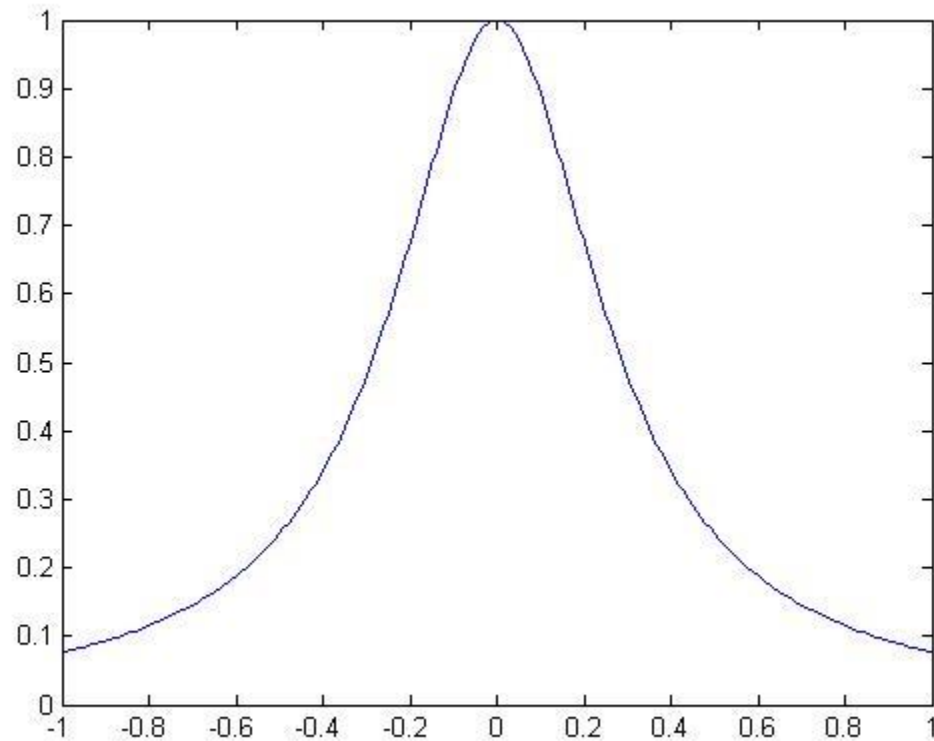
Today

- Interpolation error
- Runge phenomenon
- Newton's divided differences

Runge phenomenon

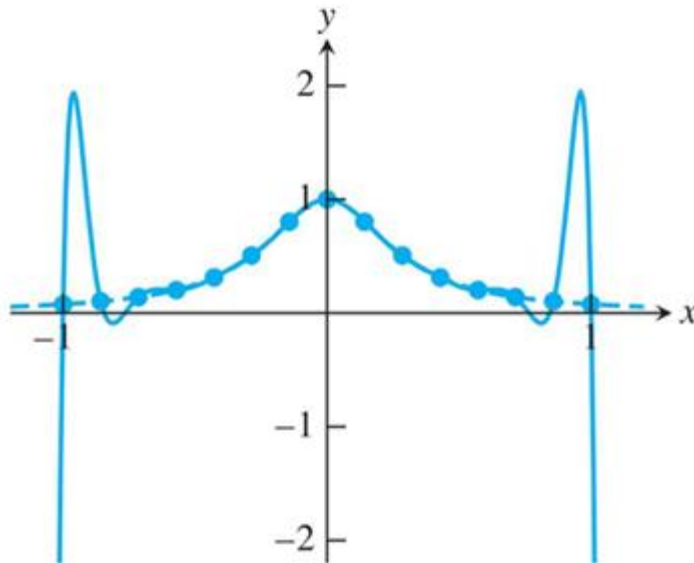
- Polynomials can fit any set of data points.
- But, there are some shapes that polynomials prefer over others.
- Example
 - Interpolate $f(x) = 1/(1+12x^2)$ at evenly spaced points in $[-1, 1]$.
 - The shape $f(x)$?

$$f(x) = 1/(1+12x^2)$$

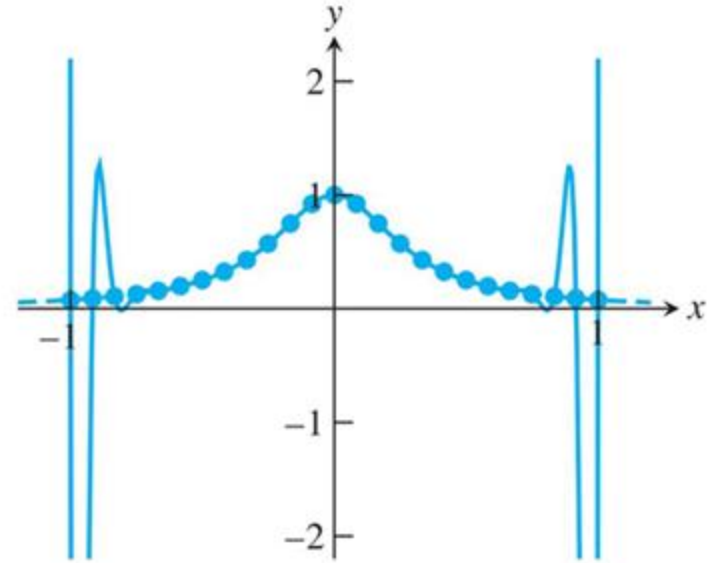


Polynomial interpolation of $f(x)$

15 base points



25 base points



Runge phenomenon: polynomial wiggle near the ends of the interpolation interval.

Today

- Interpolation error
- Runge phenomenon
- Newton's divided differences

Newton's divided differences

- $P(x)$: a polynomial in this form: (given n points)

$$P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) + \cdots + c_{n-1}(x - x_1) \cdots (x - x_{n-1})$$

- Example, given 3 points:

$$P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2)$$

- Degree of P ? **at most $n - 1$**
- How to compute c_i ?

Computing c_i

- $P(x)$: a polynomial in this form:

$$P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) + \cdots + c_{n-1}(x - x_1) \cdots (x - x_{n-1})$$

- $P(x_1) = c_0 = y_1$
- $P(x_2) = c_0 + c_1(x_2 - x_1) = y_2 \Rightarrow c_1 = \frac{y_2 - y_1}{x_2 - x_1}$
- $P(x_3) = c_0 + c_1(x_3 - x_1) + c_2(x_3 - x_1)(x_3 - x_2) = y_3$
 $\Rightarrow c_2 = ?$

“Divided Difference”

$$P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) + \cdots + c_{n-1}(x - x_1) \cdots (x - x_{n-1})$$

||

$$P(x) = \overset{c_0}{f[x_1]} + \overset{c_1}{f[x_1 \ x_2]}(x - x_1) + \overset{c_2}{f[x_1 \ x_2 \ x_3]}(x - x_1)(x - x_2) + \overset{c_3}{f[x_1 \ x_2 \ x_3 \ x_4]}(x - x_1)(x - x_2)(x - x_3) + \cdots + \overset{c_{n-1}}{f[x_1 \cdots x_n]}(x - x_1) \cdots (x - x_{n-1}).$$

Denoted by $f[x_1 \dots x_n]$ the coefficient of the x^{n-1} term

$$f[x_1 \dots x_n] \equiv c_{n-1}$$

Newton's divided differences

- 0-th divided difference

$$f[x_i] = P(x_i) = y_i$$

$$f[x_1] = y_1$$

- 1-st divided difference

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

$$f[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}$$

- k -th divided difference

$$f[x_i \cdots x_{i+k}] = \frac{f[x_{i+1} \cdots x_{i+k}] - f[x_i \cdots x_{i+k-1}]}{x_{i+k} - x_i}$$

For example,

- The 0-th divided difference is $f[x_1] = P(x_1) = y_1$
- The 1-st divided difference is

$$f[x_1 \ x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

- The 2-nd divided difference is

$$f[x_1 \ x_2 \ x_3] = \frac{f[x_2 \ x_3] - f[x_1 \ x_2]}{x_3 - x_1}$$

- The 3-rd divided difference is

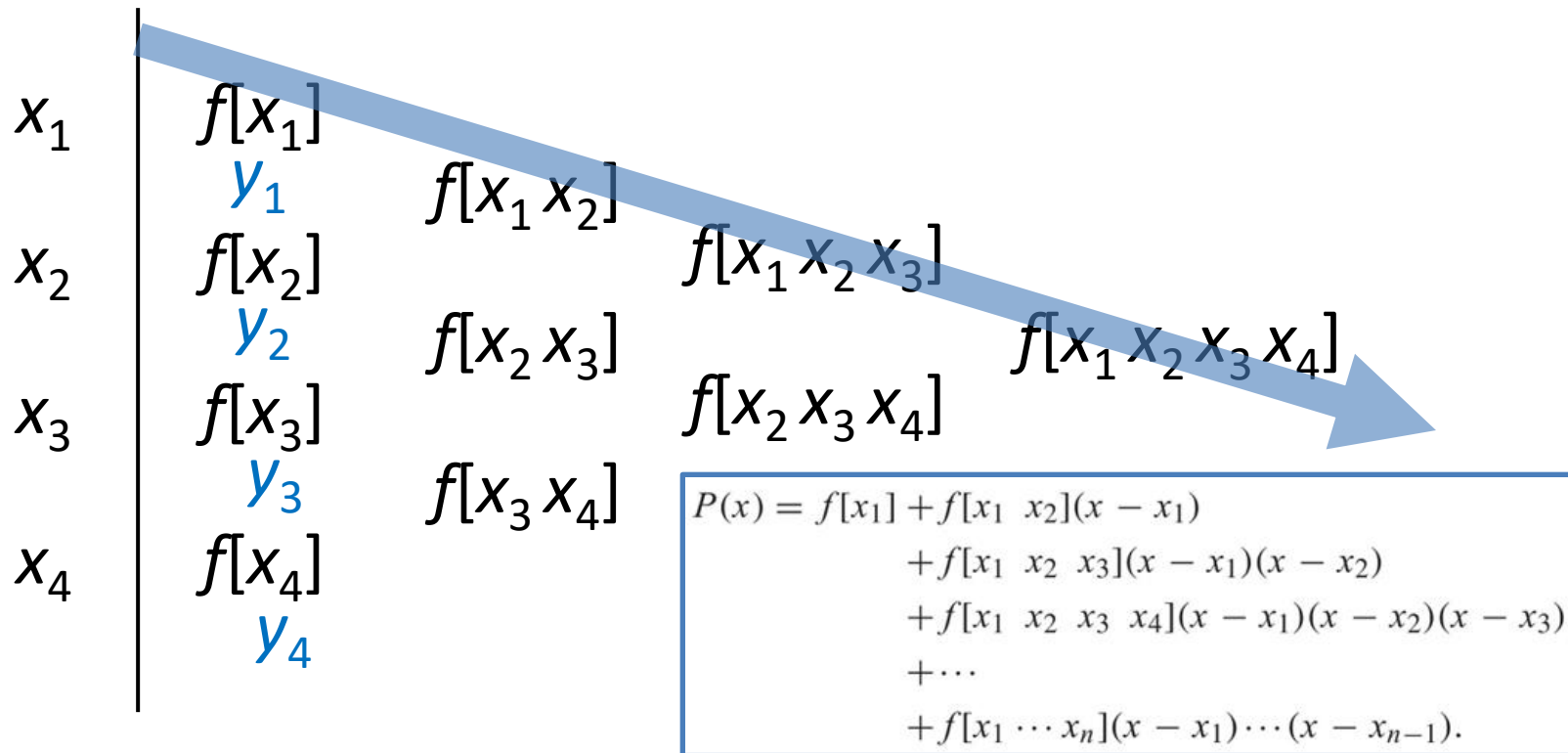
$$f[x_1 \ x_2 \ x_3 \ x_4] = \frac{f[x_2 \ x_3 \ x_4] - f[x_1 \ x_2 \ x_3]}{x_4 - x_1}$$

Computing the divided differences

- The divided differences are defined *recursively*; they are ratios of differences of previously computed ratios.

Newton's divided differences

- The divided differences are best arranged in a triangular array:



Newton's divided differences

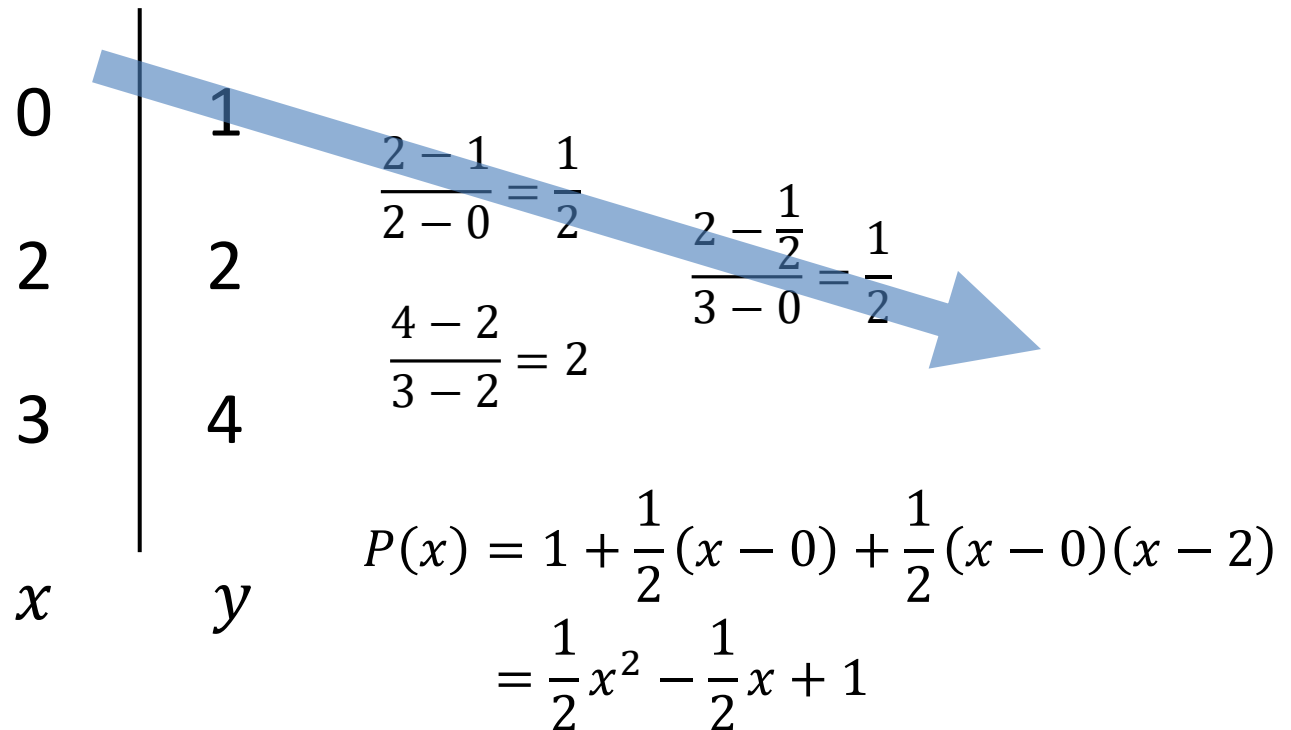
- The divided differences are best arranged in a triangular array:

x_1	$f[x_1]$ y_1	$\frac{f[x_2] - f[x_1]}{x_2 - x_1}$	$\frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	$\frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
x_2	$f[x_2]$ y_2	$\frac{f[x_3] - f[x_2]}{x_3 - x_2}$	$\frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
x_3	$f[x_3]$ y_3	$\frac{f[x_4] - f[x_3]}{x_4 - x_3}$		
x_4	$f[x_4]$ y_4			

$$\begin{aligned}
 P(x) = & f[x_1] + f[x_1, x_2](x - x_1) \\
 & + f[x_1, x_2, x_3](x - x_1)(x - x_2) \\
 & + f[x_1, x_2, x_3, x_4](x - x_1)(x - x_2)(x - x_3) \\
 & + \dots \\
 & + f[x_1, \dots, x_n](x - x_1) \cdots (x - x_{n-1}).
 \end{aligned}$$

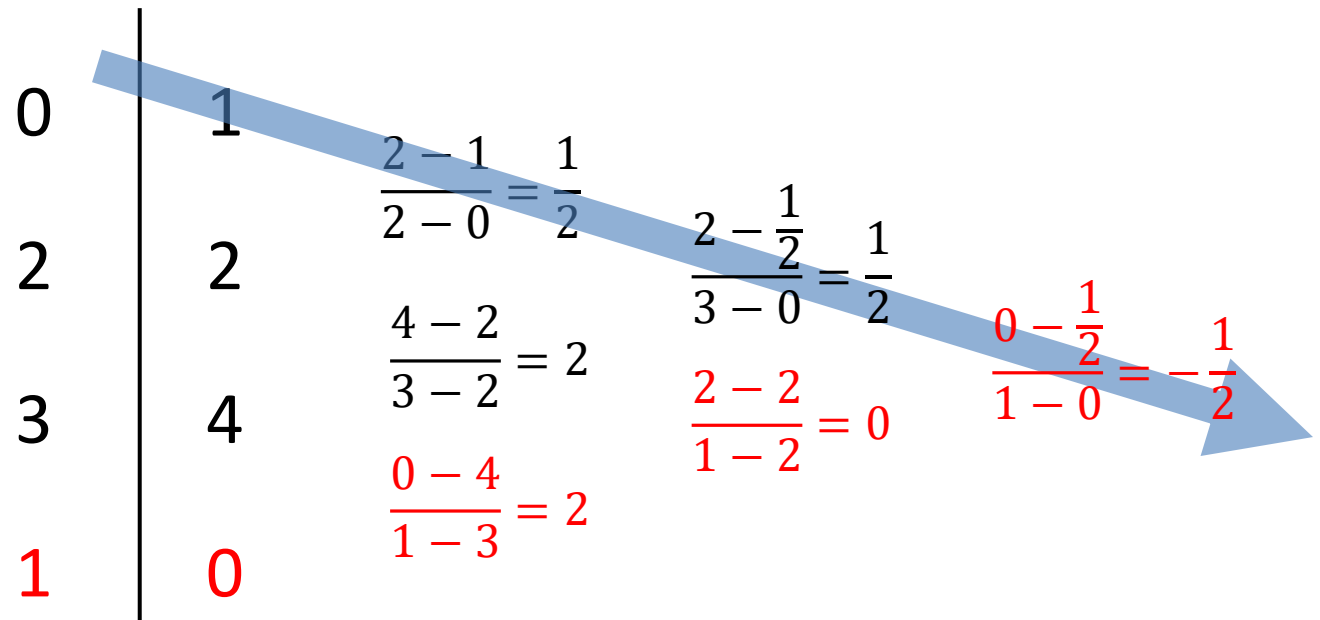
Example 1

- Use Newton's divided differences to find the interpolating polynomial passing through (0,1), (2,2), (3,4).



Example 2

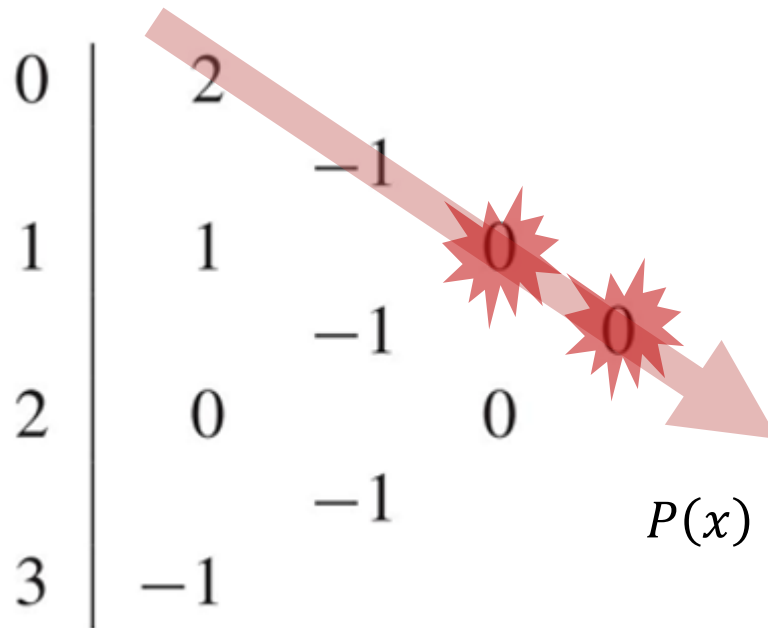
- Use Newton's divided differences to find the interpolating polynomial passing through (0,1), (2,2), (3,4), (1,0).



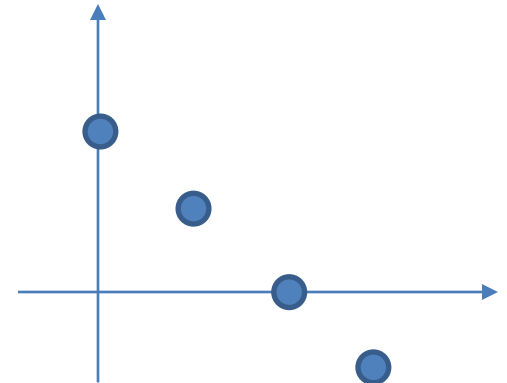
$$P(x) = 1 + \frac{1}{2}(x-0) + \frac{1}{2}(x-0)(x-2) - \frac{1}{2}(x-0)(x-2)(x-3)$$

Example 3

- Use Newton's divided differences to find the interpolating polynomial passing through (0,2), (1,1), (2,0), (3,-1).



$$P(x) = 2 - 1(x - 0) = 2 - x$$



Evaluating Newton's polynomial

- Newton's polynomial

$$P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) \\ + \cdots + c_{n-1}(x - x_1) \cdots (x - x_{n-1})$$

$$P(z) = ?$$

$$P(z) = c_0 + (z - x_1)(c_1 + (z - x_2)(c_2 + \cdots \\ + (z - x_{n-1})(c_{n-1}) \cdots))$$

a procedure similar to Horner's rule

程式練習

And, please upload your program on moodle.

- Use Newton's divided differences to find the interpolating polynomial for the data.

year	CO ₂ (ppm)
1800	280
1850	283
1900	291
2000	370

- Please calculate the CO₂ concentration in 1950.
Bonus! Using Horner's method to evaluate the z values!