Solving Linear Systems

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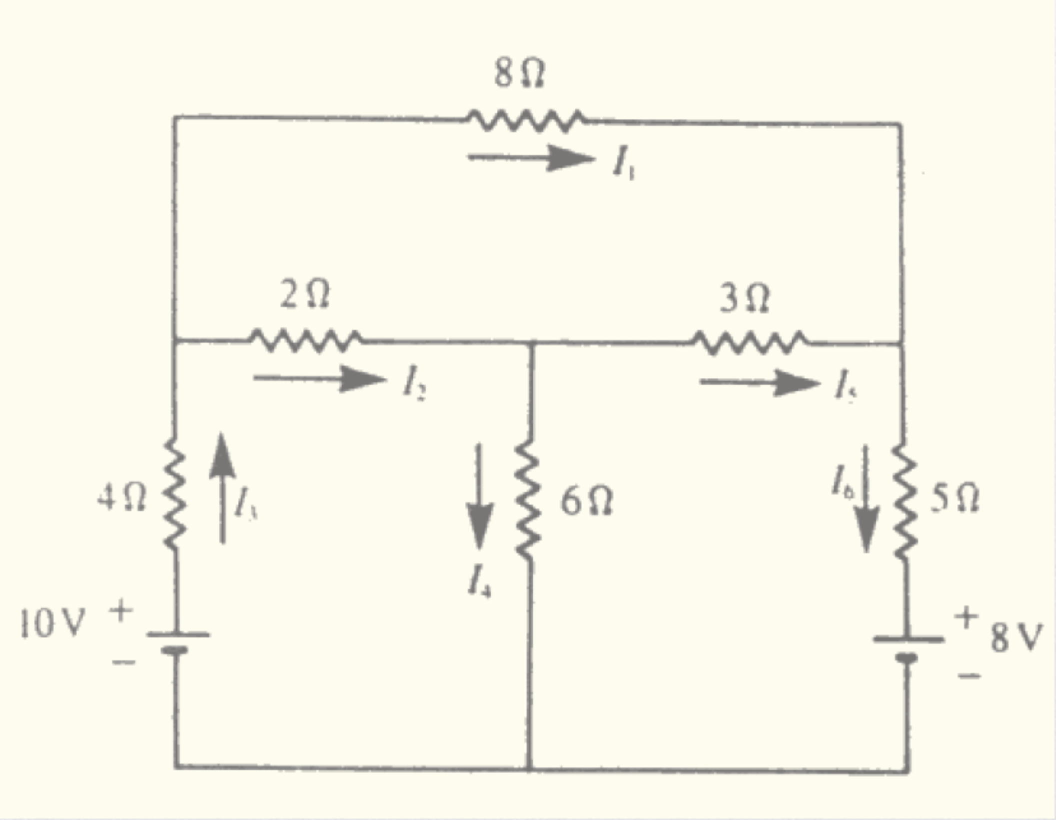
What do we have so far?

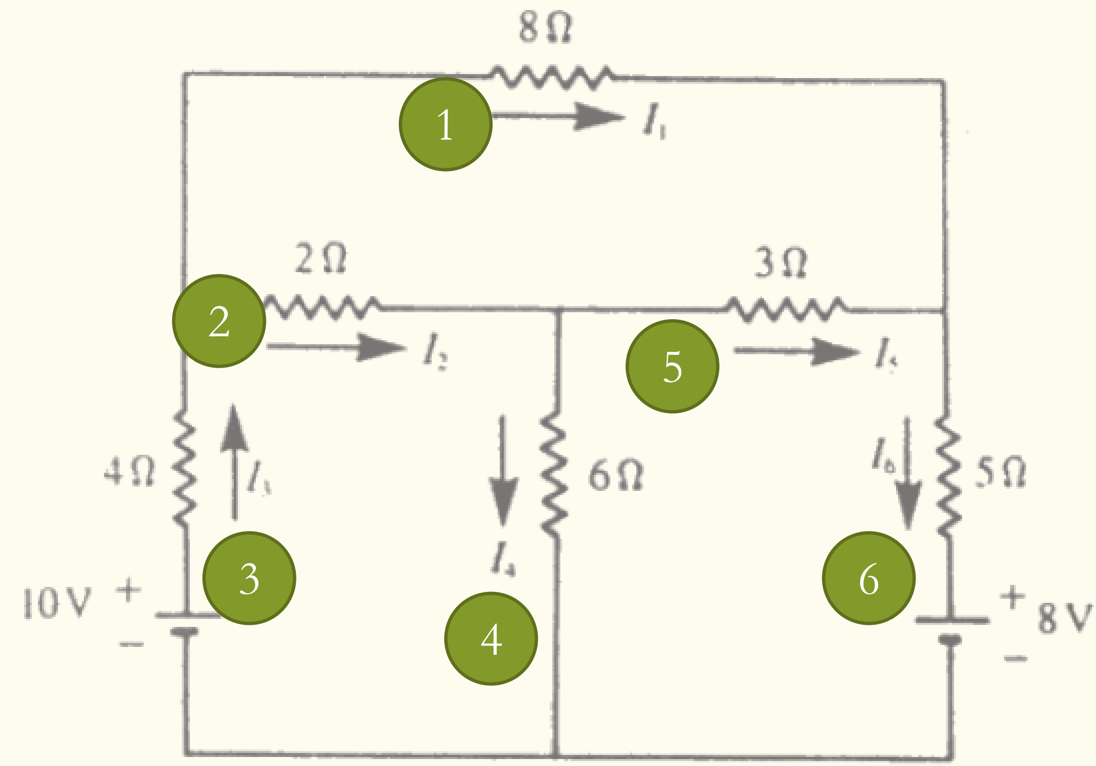
- Homer's method for evaluating a polynomial
- Solving equations 電腦解方程式
    - Bisection
    - Fixed point iteration
    - Newton
    - Secant

Today

- Solving linear systems of equations
  - Gaussian elimination
  - LU factorization

請問通過每個電阻的電流為?





$$x_1 + x_2 - x_3 = 0$$

$$x_2 - x_4 - x_5 = 0$$

$$x_1 + x_5 - x_6 = 0$$

$$2x_2 + 4x_3 + 6x_4 - 10 = 0$$

$$-6x_4 + 3x_5 + 5x_6 + 8 = 0$$

$$8x_1 - 2x_2 - 3x_5 = 0$$

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 4 & 6 & 0 & 0 \\ 0 & 0 & 0 & -6 & 3 & 5 \\ 8 & -2 & 0 & 0 & -3 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \\ -8 \\ 0 \end{bmatrix}}_{b}$$

- Consider the system

$$x + y = 3$$

$$3x - 4y = 2$$

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

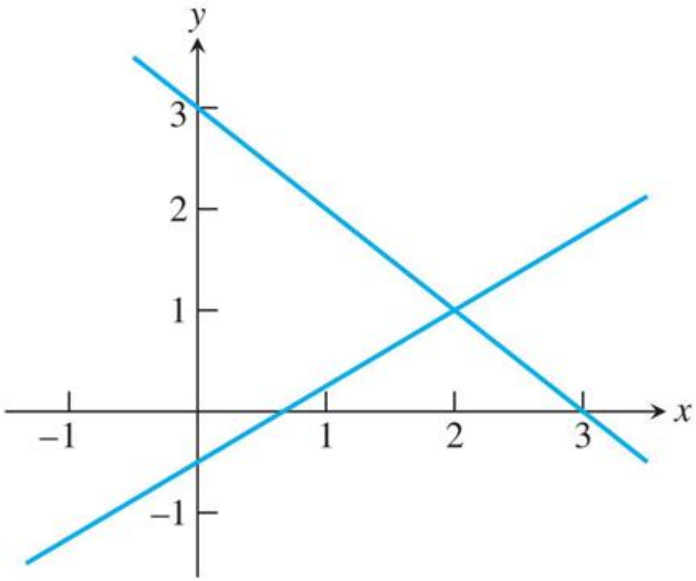


Figure 2.1 Geometric solution of a system of equations. Each equation of (2.1) corresponds to a line in the plane. The intersection point is the solution.

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Linear system: Problem statement

We consider linear systems of the form

$$\sum_{k=1}^{n} a_{ik} x_k = b_i, \qquad i = 1, \dots, n$$

or

$$A\underline{x} = \underline{b}.$$

The matrix element a_{ik} and the right-hand side elements b_i are given. We are looking for the unknowns x_k.

Back substitution

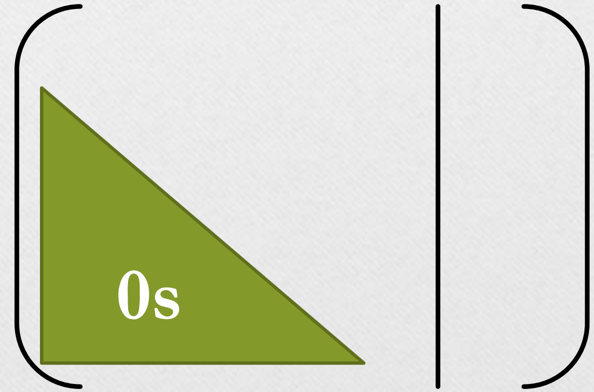
$x + y = 3$

$3x - 4y = 2$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -4 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -7 & -7 \end{array}\right]$$

$-7y = -7 \rightarrow y = 1$

$x + 1 = 3 \rightarrow x = 2$



Naïve Gaussian elimination

$x + y = 3$

$3x - 4y = 2$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -4 & 2 \end{array}\right] \xrightarrow{} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -7 & -7 \end{array}\right]$$

(2) – (1)×3

Two useful operations

1. Multiply an equation by a nonzero constant
2. Add or subtract a multiple of one equation from another

Example

- Apply Gaussian elimination for solving the system:

$$x + 2y - z = 3$$

$$2x + y - 2z = 3$$

$$-3x + y + z = -6$$

$$x + 2y - z = 3$$
$$2x + y - 2z = 3$$
$$-3x + y + z = -6$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ \boxed{2} & 1 & -2 & 3 \\ -3 & 1 & 1 & -6 \end{array}\right] \xrightarrow{(2)-(1)\times 2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ \boxed{-3} & 1 & 1 & -6 \end{array}\right]$$

$$\xrightarrow{(3)+(1)\times 3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & \boxed{7} & -2 & 3 \end{array}\right]$$

$$\xrightarrow{(3)+(2)\times(7/3)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$$z = 2$$
$$y = 1$$
$$x = 3$$

#Operations vs. input size

- *n* equation, *n* unknowns

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right]$$

Elimination step

```
for j = 1 : n-1
    eliminate column j
end
```

```
for j = 1 : n-1
    for i = j+1 : n
        eliminate entry a(i, j)
    end
end
```

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

Eliminate entry *a*(i, j)

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \end{matrix}$$

$$\Downarrow$$

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & \dots & a_{2n} - \frac{a_{21}}{a_{11}} a_{1n} & | & b_2 - \frac{a_{21}}{a_{11}} b_1 \end{matrix}$$

$2n + 1$ operations (1 division, n multiplications, n addition/subtractions)

$$
\begin{bmatrix}
0 & & & & & & \\
2n+1 & 0 & & & & & \\
2n+1 & 2(n-1)+1 & 0 & & & & \\
\vdots & \vdots & \vdots & & & & \\
2n+1 & 2(n-1)+1 & 2(n-2)+1 & \dots & 2(3)+1 & 0 & \\
2n+1 & 2(n-1)+1 & 2(n-2)+1 & \dots & 2(3)+1 & 2(2)+1 & 0
\end{bmatrix}
$$

```
for j = 1 : n-1
   eliminate column j
end
```

$$
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
\vdots & \vdots & \dots & \vdots & \vdots \\
a_{n1} & a_{n2} & \dots & a_{nn} & b_n
\end{array}\right]
$$

```
for j = 1 : n-1
   for i = j+1 : n
      eliminate entry a(i, j)
   end
end
```

$$
\sum_{j=1}^{n-1} \sum_{i=j+1}^{n} 2(n+1-j)+1
$$

$$
= \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n
$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{nn}x_n &= b_n
\end{aligned}
$$

Substitution step

$$
\begin{aligned}
x_1 &= \frac{b_1 - a_{12}x_2 - \cdots - a_{1n}x_n}{a_{11}} \\
x_2 &= \frac{b_2 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}} \\
&\vdots \\
x_n &= \frac{b_n}{a_{nn}}.
\end{aligned}
$$

$$1 + 3 + \cdots + (2n - 1)$$

$$= \sum_{i=1}^{n} (2i - 1)$$

$$= n^2$$

Operation count of Gaussian elimination

$$O(n^3) + O(n^2) = O(n^3)$$

The computational cost is dominated by the elimination step!

Today

- Solving systems of equations
  - Gaussian elimination
  - LU factorization

Upper triangular matrix

LU Factorization

Lower triangular matrix

- A matrix representation of Gaussian elimination
- Example

$x + y = 3$

$3x - 4y = 2$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -4 & 2 \end{array}\right] \rightarrow \overset{U}{\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -7 & -7 \end{array}\right]}$$

$(2) - (1) \times 3$

$$\underset{L}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}$$

Upper triangular matrix

LU Factorization

Lower triangular matrix

$$x + y = 3$$
$$3x - 4y = 2$$

$$LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = A.$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} \xrightarrow[(2)-(1)\times 2]{} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[(3)-(1)\times -3]{} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 7 & -2 \end{bmatrix}$$

$$\xrightarrow[(3)-(2)\times -(7/3)]{} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix}$$

L U

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} = A.$$

- Next, how do we get the solution $\underline{x}$?
- $A\underline{x} = \underline{b} \rightarrow LU\underline{x} = \underline{b}$
- Define $\underline{c} = U\underline{x}$

  Solve $L\underline{c} = \underline{b}$ for $\underline{c}$ (forward substitution)

  Solve $U\underline{x} = \underline{c}$ for $\underline{x}$ (backward substitution)

What is the benefit using the LU factorization

LU factorization: complexity

- Lessons from Gaussian elimination analysis
  - Elimination: more expensive $O(n^3)$
  - Substitution: less expensive $O(n^2)$
- Need to solve a number of different problems with the **same A** and **different $\underline{\boldsymbol{b}}$**

$$
\begin{gathered}
A\underline{x}_1 = \underline{b}_1 \\
A\underline{x}_2 = \underline{b}_2 \\
\vdots \\
A\underline{x}_k = \underline{b}_k
\end{gathered}
$$

Complexity comparison

- Naïve Gaussian elimination

$$(2/3)n^3 \times k$$

- LU approach

$$(2/3)n^3 + 2kn^2$$

Does a matric always have an LU factorization

- **No**
- Consider the following example

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} b & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} b & c \\ ab & ac + d \end{bmatrix}$$

$b = 0$, $ab = 1$, a contradiction!

程式練習

- Please use Gaussian elimination to solve

$$\begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 5 \\ -9 & 7 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22 \\ -7 \end{bmatrix}$$

- and do LU factorization of the above matrix