

Announcement

- No class next week (University Sports Day)

Interpolation – Part 1

Mei-Chen Yeh

Today

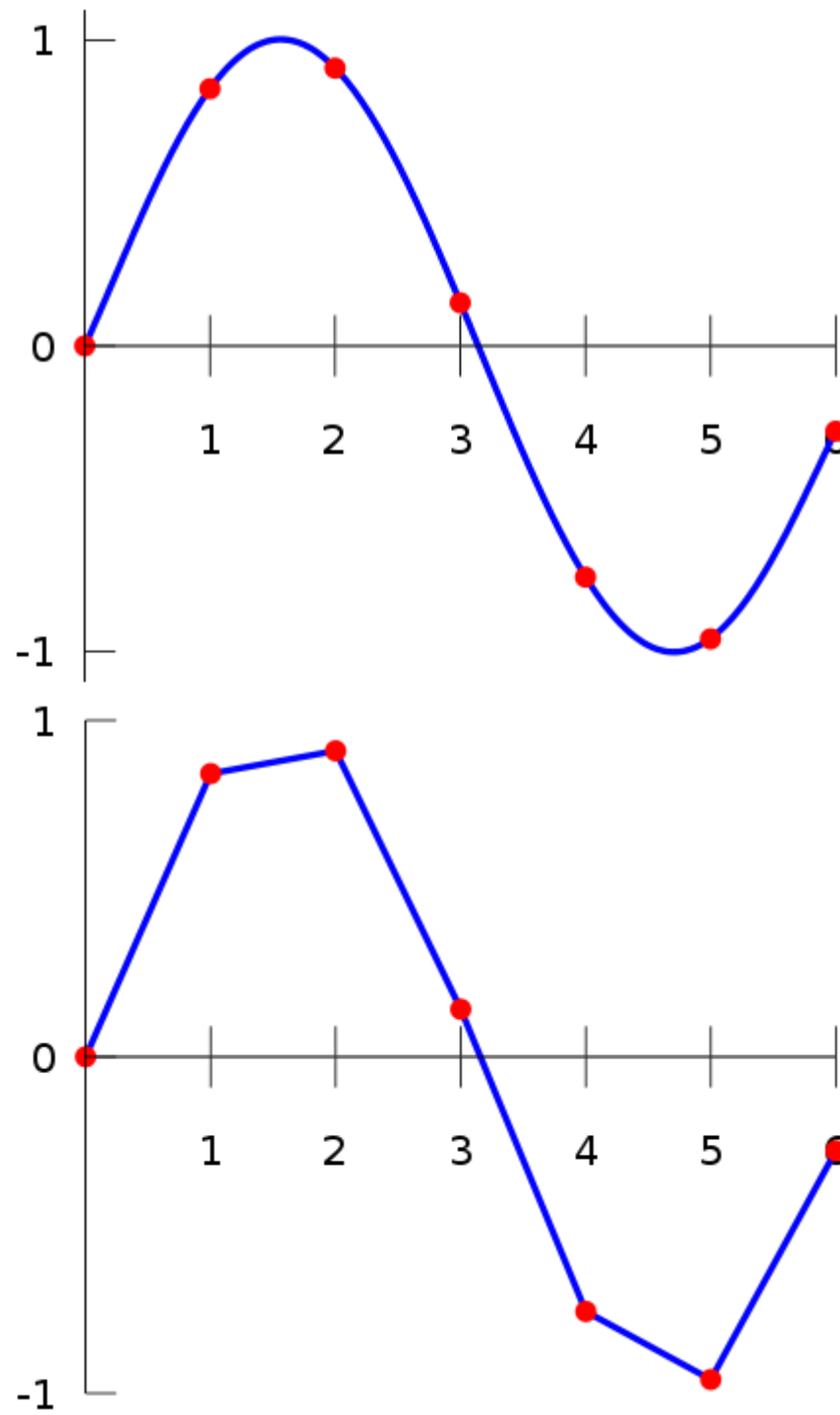
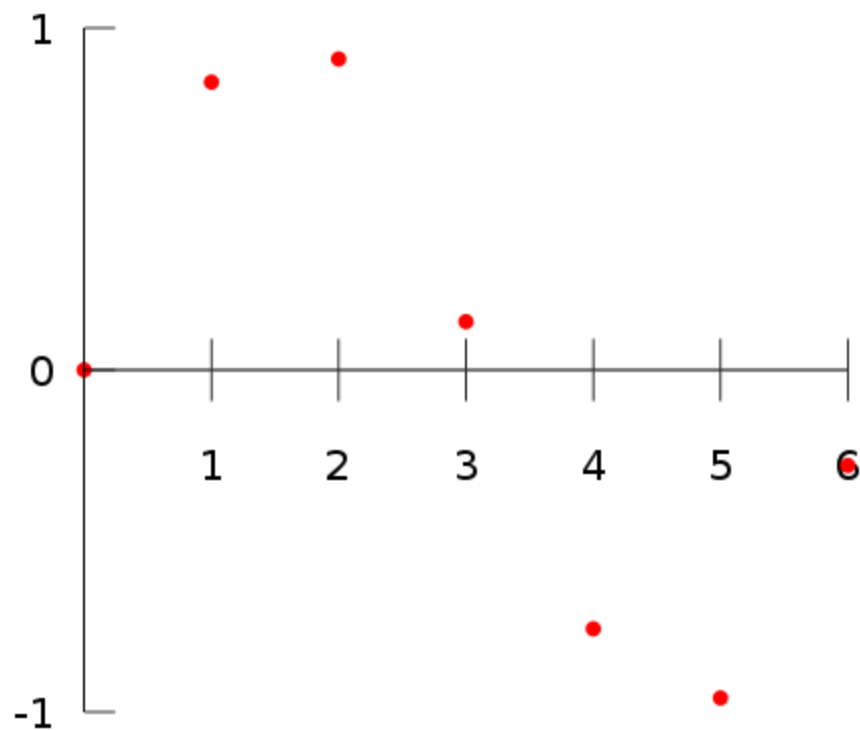
- What is interpolation?
- Lagrange interpolation

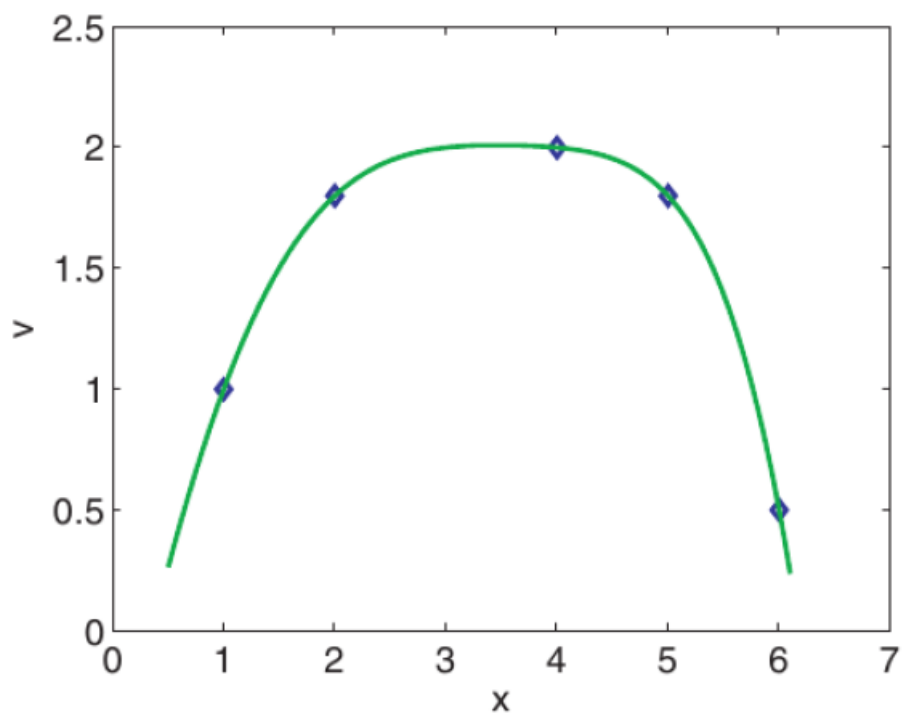
Interpolating data

- We are given a collection of **data samples** $\{(x_i, y_i)\}_{i=1}^n$.
 - x_i : the abscissas
 - y_i : the data values
- Want to find a function $v(x)$ which can be used to estimate sampled function for $x \neq x_i$.

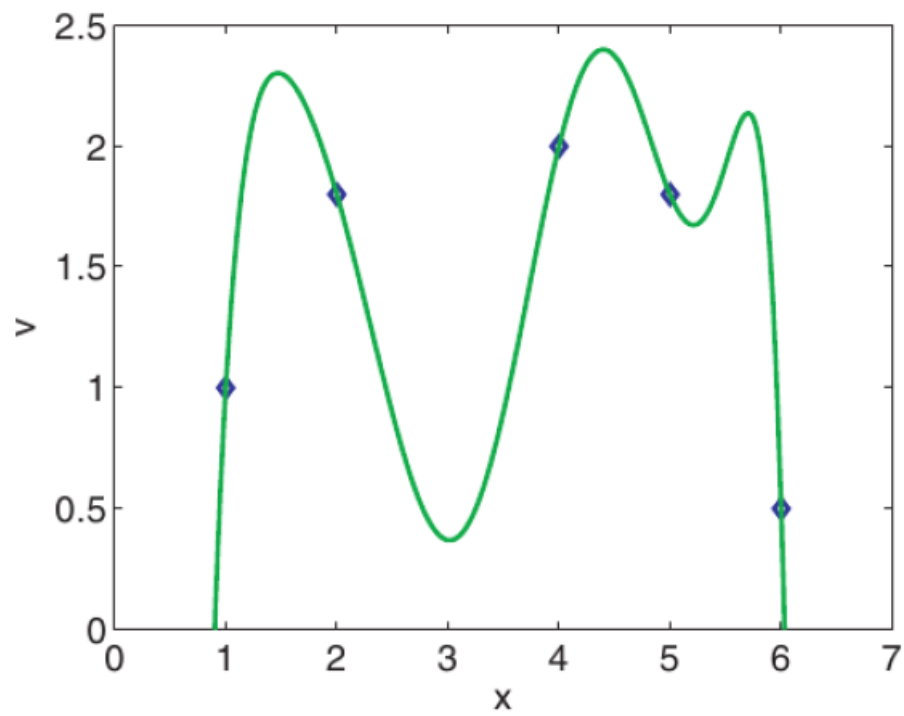
Interpolation:	$v(x_i) = y_i, \quad i = 0, 1, \dots, n.$
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Examples





Reasonable



Unreasonable

Interpolating data

- **Interpolation** is the reverse of **evaluation**.
 - **Evaluation** (e.g., nested multiplication): given a polynomial, evaluate a y -value for a given x -value.
 - **Interpolation**: given these points, compute a polynomial that can generate them.

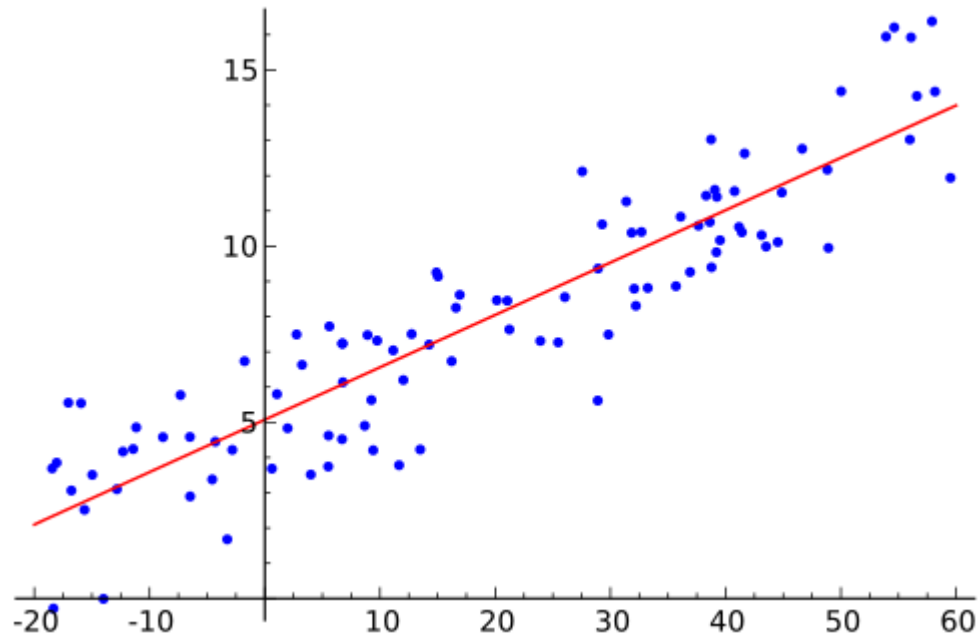
Interpolating data

- Why?
 - We often get discrete data from sensors or computations, but we want information as if the data were not discretely sampled.
- Want a reasonable looking interpolation. If possible, $v(x)$ should be inexpensive to evaluate for a given x .

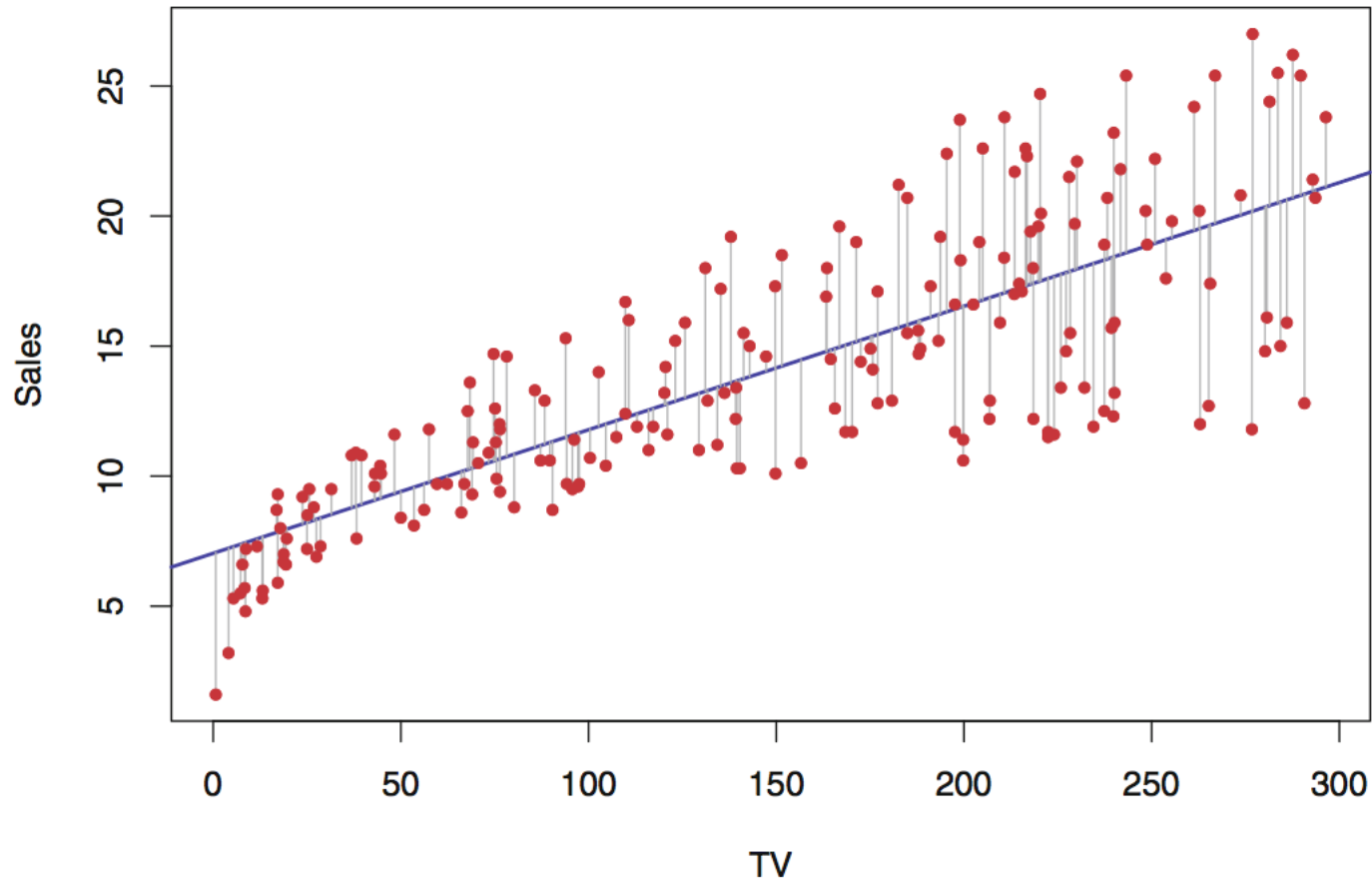
Interpolation, extrapolation, approximation

- Interpolation
 - The new value $x \neq x_i$ is **inside** the range of the interpolation points x_0, x_1, \dots, x_n .
- Extrapolation
 - The new value z is outside this range.
- Approximation
 - Some norm $\|v - y\|$ of the difference of the vectors $\underline{v} = [v(x_0), \dots, v(x_n)]$ and $\underline{y} = [y_0, \dots, y_n]$ is minimized.

Approximation: Example



Approximation: Example



Today

- What is interpolation?
- Lagrange interpolation

Lagrange interpolation

- Given n data points $(x_1, y_1), \dots, (x_n, y_n)$, the polynomial that interpolates the points is:

$$P(x) = y_1 L_1(x) + y_2 L_2(x) + \dots + y_n L_n(x)$$

$$L_k(x) = \frac{(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$

- Degree of $P(x)$?

Lagrange interpolation ($n = 3$)

- Given n data points $(x_1, y_1), \dots, (x_n, y_n)$, the polynomial of degree $d = n-1$ that interpolates the points is:

$$P_2(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

$$P(x_1) = ?$$

$$P(x_2) = ?$$

$$P(x_3) = ?$$

Lagrange interpolation: Example 1

- Find an interpolating polynomial for the data points $(0, 1)$, $(2, 2)$ and $(3, 4)$.

$$\begin{aligned}P(x) &= 1 \frac{(x-2)(x-3)}{(0-2)(0-3)} + 2 \frac{(x-0)(x-3)}{(2-0)(2-3)} + 4 \frac{(x-0)(x-2)}{(3-0)(3-2)} \\&= \frac{1}{6}(x^2 - 5x + 6) + 2 \left(-\frac{1}{2}\right)(x^2 - 3x) + 4 \left(\frac{1}{3}\right)(x^2 - 2x) \\&= \frac{1}{2}x^2 - \frac{1}{2}x + 1.\end{aligned}$$

Check $P(0) = ?$, $P(2) = ?$, $P(3) = ?$

Lagrange interpolation: Example 1

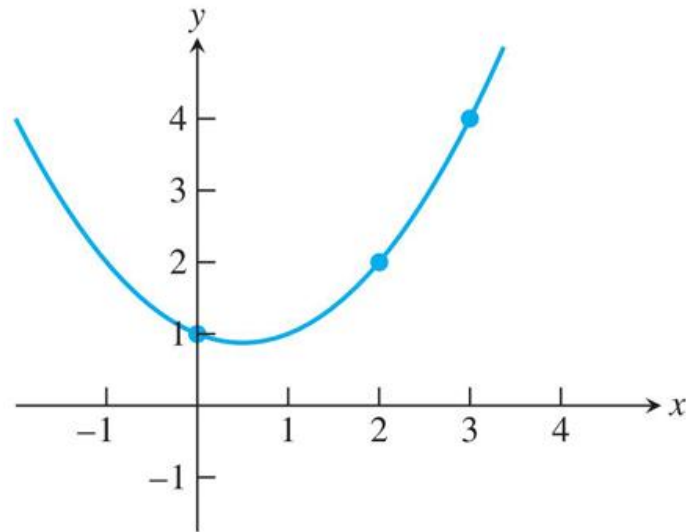
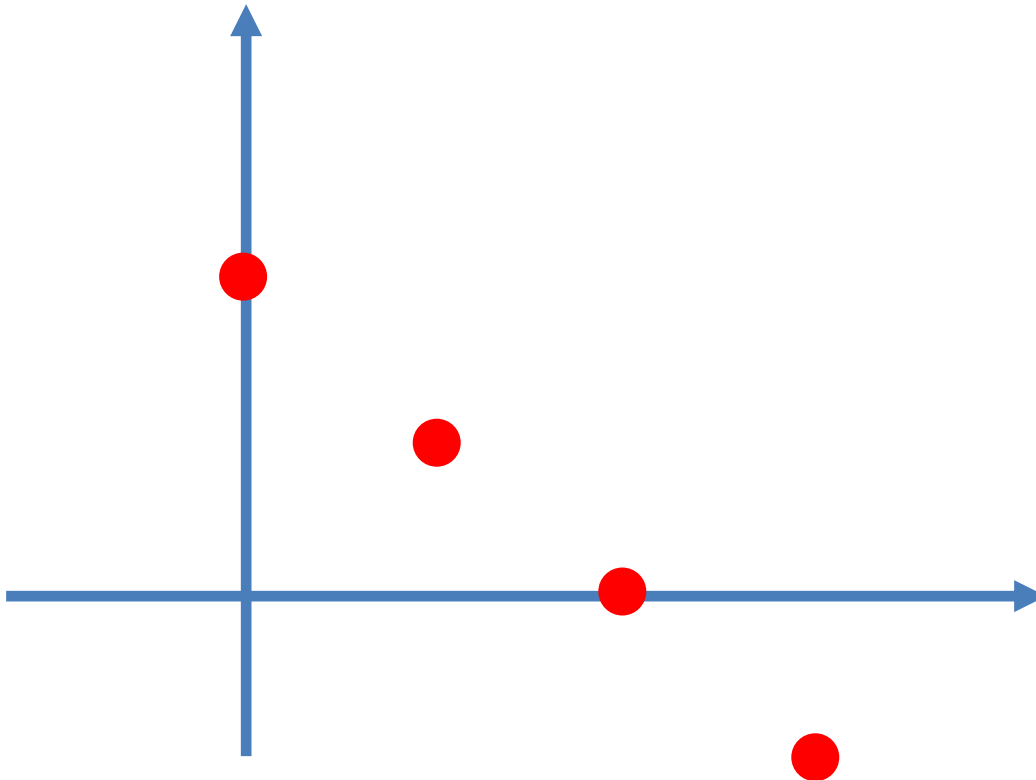


Figure 3.1 Interpolation by parabola. The points $(0,1)$, $(2,2)$, and $(3,4)$ are interpolated by the function $P(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 1$.

Lagrange interpolation: Example 2

- Find an interpolating polynomial for the data points $(0, 2)$, $(1, 1)$, $(2, 0)$ and $(3, -1)$.



Lagrange interpolation: Example 2

- Find an interpolating polynomial for the data points $(0, 2)$, $(1, 1)$, $(2, 0)$ and $(3, -1)$.

$$\begin{aligned} P(x) &= 2 \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} + 1 \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} \\ &\quad + 0 \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} - 1 \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} \\ &= -\frac{1}{3}(x^3 - 6x^2 + 11x - 6) + \frac{1}{2}(x^3 - 5x^2 + 6x) - \frac{1}{6}(x^3 - 3x^2 + 2x) \\ &= -x + 2. \end{aligned}$$

A linear function!

Question



- Given n data points
 - How many polynomials that interpolate the points?
 - How many polynomials of **degree $n - 1$ or less** that interpolate the points?
- Lagrange interpolation: Is $P(x)$ ***unique***?

$$\begin{aligned} P(x) &= y_1 L_1(x) + y_2 L_2(x) + \cdots + y_n L_n(x) \\ &= a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \end{aligned}$$

Yes

Theorem. Let $(x_1, y_1), \dots, (x_n, y_n)$ be n points in a plane with distinct x_i . Then there exists **one and only one** polynomial P of degree $n-1$ or less that satisfies $P(x_i) = y_i$ for $i = 1, \dots, n$.

Proof sketch (by contradiction)

- Suppose $P(x)$ and $Q(x)$ have degree at most $n-1$ and both interpolate all n points.
- Now, define $H(x) = P(x) - Q(x)$.
 - Degree of $H(x)$?
 - $H(x_i) = P(x_i) - Q(x_i) = y_i - y_i = 0$
 - $0 = H(x_1) = H(x_2) = \dots = H(x_n)$
 - $H(x)$ has a degree of at most $n-1$ and it has n roots.

Proof sketch (by contradiction)

- Suppose $P(x)$ and $Q(x)$ have degree at most $n-1$ and both interpolate all n points.
- Now, define $H(x) = P(x) - Q(x)$.
 - Degree of $H(x)$?
 - $H(x_i) = P(x_i) - Q(x_i) = y_i - y_i = 0$
 - $0 = H(x_1) = H(x_2) = \dots = H(x_n)$
- H is the identically zero polynomial, and $P(x) \equiv Q(x)$.

Side information

A degree d polynomial can have at most d roots, unless it is the identically zero polynomial.

程式練習

And, please upload your program on moodle.

- Please estimate $f(1.09)$.

x	$f(x)$
1.00	0.1924
1.05	0.2414
1.09	?
1.10	0.2933
1.15	0.3492