

Stochastic Finance (FIN 519)

Final Exam

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BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively. Assume that B_t is a standard **BM**.

1. ($1 \times 4 = 4$ points) Give **True or False**.

- (a) $(dB_t)^2 = dt$
- (b) $(dB_t)^4 = (dt)^2$
- (c) $d(B_t^2) = 2(B_t dB_t + dt)$
- (d) $d \sinh(B_t) = \cosh(B_t)dB_t + \frac{1}{2} \sinh(B_t)dt$ where $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$

Solution:

- (a) True. $(dB_t)^2 = dt$.
- (b) False. $(dB_t)^4 = 3(dt)^2$ from $E(B_t^4) = 3t^2$.
- (c) False. $d(B_t^2) = 2B_t dB_t + dt$
- (d) True. $d \sinh(B_t) = \cosh(B_t)dB_t + \frac{1}{2} \sinh(B_t)dt$ since $\sinh'(x) = \cosh(x)$ and $\sinh''(x) = \sinh(x)$.

2. ($2 \times 3 = 6$ points) **Black-Scholes and Importance sampling** Assume that $r = q = 0$ in this problem.

- (a) Suppose that a stock price follows $dS_t/S_t = \sigma dB_t$ and you want to price a call option with strike price K and maturity T . How do you get the option price by Monte-Carlo simulation? In other words, what should be the expression for $f(Z)$ in the following Monte-Carlo formula:

$$C = \frac{1}{N} \sum_{k=1}^N f(Z_k),$$

where Z_k is a sequence of i.i.d. standard normal RVs.

- (b) Using Black-Scholes formula, find the call option price for $S_0 = 100, K = 2000, T = 1, \sigma = 0.5$. For computation, use approximation $20 \approx e^3$. You may also use the following CDF values for the standard normal distribution $N(z)$.

z	-6.5	-6.25	-6.0	-5.75	-5.5
$N(z)$	4.0×10^{-11}	2.1×10^{-10}	9.9×10^{-10}	4.5×10^{-9}	1.9×10^{-8}

- (c) As observed in (b), the call option value can be very small if K is extremely high compared to S_0 ($K \gg S_0$). Therefore, the Monte-Carlo price from (a) would be zero because no Z_k will make the stock price in-the-money. Apply important sampling and express the option price. (Hint: Select the shift amount to make the stock price at-the-money.)

Solution:

- (a) The security price can be simulated as

$$S_T = S_0 \exp(\sigma\sqrt{T}Z - \sigma^2 T/2) \quad \text{for } Z \sim N(0, 1).$$

and the call option price is obtained as

$$C = \frac{1}{N} \sum_{k=1}^N \left(S_0 \exp(\sigma\sqrt{T}Z_k - \sigma^2 T/2) - K \right)^+$$

for i.i.d. samples $\{Z_k\}$ (assume $r = 0$) and $(x)^+ = \max(x, 0)$.

- (b) From $d_{1,2} = \log(S_0/K)/\sigma\sqrt{T} \pm \sigma\sqrt{T}/2 = -6 \pm 0.25 = -5.75, -6.25$,

$$C = 100N(-5.75) - 2000N(-6.25) = 100 \times 4.5 \times 10^{-9} - 2000 \times 2.1 \times 10^{-10} = 3 \times 10^{-8}$$

- (c) Remind that

$$E[f(Z)] = E[f(Z + \mu) e^{-\mu Z - \mu^2/2}].$$

Therefore,

$$C = \frac{1}{N} \sum_{k=1}^N \left(S_0 \exp(\sigma\sqrt{T}(Z_k + \mu) - \sigma^2 T/2) - K \right)^+ e^{-\mu Z_k - \mu^2/2}$$

where μ is determined as

$$S_0 \exp(\sigma\sqrt{T}\mu - \sigma^2 T/2) = K \quad \Rightarrow \quad \mu = \frac{\log(K/S_0)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}.$$

Notice that $\mu = -d_2$ in the Black-Scholes formula.

3. (4 points) **Solving SDE** Solve the modified “3/2” volatility process defined by

$$dV_t = \frac{3\sigma^2}{4} V_t^2 dt + \sigma V_t^{3/2} dB_t.$$

The original “3/2” model has the drift term $a(b - V_t)V_t dt$ instead of $(3\sigma^2/4)V_t^2 dt$ and was originally proposed for a stochastic model for volatility.

Solution: First we remove $\sqrt{V_t}$ from the coefficient of dB_t

$$\frac{dV_t}{V_t^{3/2}} = \frac{3\sigma^2}{4} \sqrt{V_t} dt + \sigma dB_t.$$

It turns out that the first candidate $\frac{dx}{x^{3/2}}$ works. Since $\int \frac{dx}{x^{3/2}} = -\frac{2}{\sqrt{x}}$,

$$-d\left(\frac{2}{\sqrt{V_t}}\right) = \frac{dV_t}{V_t^{3/2}} - \frac{3(dV_t)^2}{4V_t^{5/2}} = \frac{3\sigma^2}{4} \sqrt{V_t} dt + \sigma dB_t - \frac{3\sigma^2}{4} \sqrt{V_t} dt = \sigma dB_t$$

$$\frac{2}{\sqrt{V_0}} - \frac{2}{\sqrt{V_t}} = \sigma B_t \quad \Rightarrow \quad V_t = \left(\frac{1}{\sqrt{V_0}} - \frac{\sigma B_t}{2} \right)^{-2}$$

4. ($2 \times 4 = 8$ points) **Interest rate and bond price SDE**

- (a) Find the mean and variance of the following expression:

$$I = \int_0^T B_t dt.$$

You may find that $TB_T = \int_0^T TdB_t$ is a helpful trick. Is I normally distribution?

- (b) Suppose now you live in a real world where the risk-free rate, r_t , is stochastic rather than constant. Therefore the saving account β_T exponentially grows as $\beta_T = \exp\left(\int_0^T r_t dt\right)$. Show that, in risk-neutral measure Q , the zero bond price $P(t, T)$ is given as

$$P(0, T) = E^Q \left[\exp \left(- \int_0^T r_t dt \right) \right]$$

- (c) Suppose that the SDE for r_t is given as

$$r_t = r_0 + \alpha t + \beta B_t,$$

where B_t is a standard BM under Q ($B_t = B_t^Q$). Calculate $P(0, T)$ using the results from (a) and (b).

- (d) Suppose that the SDE for the bond price $P(t, T)$ in Q is given as

$$\frac{dP(t, T)}{P(t, T)} = \mu_P dt + \sigma_P dB_t.$$

From the result of (c), find $P(t, T)$ by replacing r_0 with r_t and T with $T - t$. Then, apply Itô's lemma to $\log P(t, T)$. By matching drift and volatility, find the expressions for μ_P and σ_P . (Hint: The answers are quite simple, so check your calculation again if the expressions are complicated. Especially μ_P is almost obvious in the risk-neutral measure. Even if you cannot get the derivation right, you still earn partial credit by correctly guessing μ_P . When applying Itô's lemma to $\log P(t, T)$, t is the time variable and T should be considered as a constant.)

Solution:

- (a) (See the Exercise 6.2 of **SCFA**) Using Itô's lemma applied to tB_t , $d(tB_t) = tdB_t + B_t dt$, we can express I as

$$I = TB_T - \int_0^T t dB_t = T \int_0^T dB_t - \int_0^T t dB_t = \int_0^T (T - t) dB_t.$$

We get

$$E(I) = 0 \quad \text{and} \quad \text{Var}(I) = \int_0^T (T - t)^2 dt = \frac{1}{3}T^3.$$

and I is normally distributed.

- (b) The bond price $P(0, T)$ is the $t = 0$ value of \$ 1 at $t = T$. Under the risk-neutral measure Q , any security value measured in the unit of β_t is a martingale,

$$\frac{P(0, T)}{\beta_0 = 1} = E_Q \left[\frac{P(T, T) = 1}{\beta_T} \right].$$

Therefore,

$$P(0, T) = E_Q \left[\exp \left(- \int_0^T r_t dt \right) \right]$$

or, more generally,

$$P(t, T) = E_Q \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right]$$

- (c) From

$$- \int_0^T r_t dt = -Tr_0 - \frac{1}{2}\alpha T^2 - \beta I,$$

we get

$$\begin{aligned} P(0, T) &= \exp \left(-Tr_0 - \frac{1}{2}\alpha T^2 \right) E_Q(e^{-\beta I}) \\ &= \exp \left(-Tr_0 - \frac{1}{2}\alpha T^2 + \frac{1}{6}\beta^2 T^3 \right) \end{aligned}$$

Note that this is a simplified version of the Ho-Lee model ([WIKIPEDIA](#)). In the original model, the drift α is time-dependent:

$$dr_t = \alpha_t dt + \sigma dB_t.$$

(d) From

$$\log P(t, T) = -(T-t)r_t - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\beta^2(T-t)^3$$

we get the SDE for $\log P(t, T)$:

$$\begin{aligned} d\log P(t, T) &= -(T-t)dr_t + \left(r_t + \alpha(T-t) - \frac{1}{2}\beta^2(T-t)^2 \right) dt \\ &= \left(r_t - \frac{1}{2}\beta^2(T-t)^2 \right) dt - \beta(T-t)dB_t. \end{aligned}$$

and this should be equal to

$$d\log P(t, T) = (\mu_P - \sigma_P^2/2)dt + \sigma_P dB_t.$$

Therefore we get

$$\mu_P = r_t \quad \text{and} \quad \sigma_P = -\beta(T-t).$$

Note that (i) $\mu_P = r_t$ is obvious in the risk-neutral measure, (ii) the bond volatility σ_P is negative because the bond price moves in an opposite way to the interest rate, and (iii) $\sigma_P = 0$ at maturity $t = T$ because the bond price $P(t, T)$ converges to 1.

5. ($2 \times 4 = 8$ points) **Siegel's paradox** In foreign exchange market, each currency has a symbol of three characters. For example, USD is for US Dollar, CNY is for Chinese Yuan, and EUR is for Euro. Also, foreign exchange rate is quoted in one fixed direction such that the exchange rate is bigger than one. For example, the exchange between USD and CNY is always quoted in USD/CNY, not in CNY/USD. USD/CNY means the price of 1 USD in terms of CNY, which is around 6.3 as of today.

- (a) Due to the recent trade war between US and China, suppose that the FX rate USD/CNY is going to be very volatile. Assuming USD/CNY = 6.0 today and it can change to either 4.0 or 8.0 after one year. Under the risk-neutral measure Q , what are the probabilities for the two scenarios? Assume that risk-free rate is zero in both US and China. (Hint: You can think of USD/CNY as the price of an asset (1 USD) in CNY unit.)
- (b) Now consider the scenario from the perspective of an investor in US. Suppose a US investor invests in CNY by converting 1 USD to 6 CNY today and holding 6 CNY for one year. What is her expected return in terms of USD after one year? If the expected return is not zero, does it conflict with that the expected return under the risk-neutral measure Q should be same as the risk-less rate (which is zero in this problem)? (Hint: Consider the numeraire.)
- (c) We can demonstrate this using stochastic calculus. If $X_t = \text{USD/CNY}$, X_t is a martingale (no drift), so the SDE is given by

$$\frac{dX_t}{X_t} = \sigma dB_t^Q.$$

By applying Itô's lemma, find the SDE for X_0/X_t , which is the investment value in USD. Is the drift positive, negative or zero?

- (d) In class, we show that, for a numeraire asset with volatility σ_N , the relation between the two standard BMs between the probability measures, Q^N and Q (risk-neutral measure), is given as

$$dB_t^{Q^N} + \sigma_N dt = dB_t^Q.$$

What is the numeraire to make X_0/X_t a martingale? From the relation above, show that X_0/X_t is indeed a martingale under the measure associated with the numeraire.

Solution: See ([WIKIPEDIA](#)) for **Siegel's paradox**.

(a)

$$4p + 8(1 - p) = 6 \text{ CNY} \Rightarrow p = 0.5$$

(b) After one year, the investor receives in

$$\frac{6}{4} \cdot \frac{1}{2} + \frac{6}{8} \cdot \frac{1}{2} = \frac{9}{8} \text{ USD}$$

Therefore the return is 12.5%. This does not contract with the zero expected return of the risk-neutral measure because the numeraire of the risk-neutral measure is the CNY cash while the numeraire for the US investor is USD cash.

(c) In general, the SDE for the exchange rate is given as

$$\frac{dX_t}{X_t} = (r_d - r_f) + \sigma dB_t^Q,$$

where r_d is the risk-free rate of domestic currency (CNY in this case) and r_f is the foreign currency (USD in this case). We assume $r_d = r_f = 0$ for this problem, but we derive general result with r_d and r_f .

$$\begin{aligned} d \log X_t &= (r_f - r_d - \sigma^2/2)dt + \sigma dB_t^Q \\ d \log(X_0/X_t) &= (r_d - r_f + \sigma^2/2)dt - \sigma dB_t^Q \\ \frac{d(X_0/X_t)}{(X_0/X_t)} &= (r_d - r_f + \sigma^2)dt - \sigma dB_t^Q \end{aligned}$$

Assuming $r_d = r_f = 0$,

$$\frac{d(X_0/X_t)}{(X_0/X_t)} = \sigma^2 dt - \sigma dB_t^Q.$$

(d) The USD cash should be the numeraire and the price of USD in terms of the risk-neutral numeraire CNY is nothing but X_t ! Therefore the volatility is σ and we get

$$dB_t^{\text{USD}} + \sigma dt = dB_t^{\text{CNY}} (= dB_t^Q).$$

The SDE from (c) becomes

$$\frac{d(X_0/X_t)}{(X_0/X_t)} = \sigma (\sigma dt - dB_t^{\text{CNY}}) = \sigma dB_t^{\text{USD}}$$

and X_0/X_t is indeed a martingale under the USD numeraire.