

Stochastic Finance (FIN 519)

Homework Solutions

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HW 1-1 Using martingale property, re-drive that

$$E(\tau) = AB \quad \text{for} \quad \tau = \min\{n : S_n = A \text{ or } S_n = -B\}.$$

Answer You can find the answer in the textbook section 2.3.

HW 1-2. SCFA Exercise 2.4

Answer From the hint, let us define

$$A_{n+1} = A_n + E[(M_{n+1} - M_n)^2 | \mathcal{F}_n]$$

and prove the three required properties:

(i) N_n is a martingale.

$$\begin{aligned} E[N_{n+1} | \mathcal{F}_n] &= E[M_{n+1}^2 - A_{n+1} | \mathcal{F}_n] \\ &= E[2M_{n+1}M_n - M_n^2 - A_n | \mathcal{F}_n] \\ &= 2E[M_{n+1} | \mathcal{F}_n] M_n - M_n^2 - A_n \\ &= M_n^2 - A_n = N_n. \end{aligned}$$

(ii) $A_{n+1} \geq A_n$ is trivial.

(iii) A_n is non-anticipating because it is defined via the expectation under \mathcal{F}_n .

HW 2-1 Compute the self-covariance of Brownian bridge, $\text{Cov}(U_s, U_t)$ where $U_t = B_t - tB_1$.

Answer

$$\begin{aligned} \text{Cov}(U_s, U_t) &= E\left((B_s - sB_1)(B_t - tB_1)\right) = E\left(B_s B_t - sB_1 B_t - tB_s B_1 + stB_1^2\right) \\ &= \min(s, t) - s \min(1, t) - t \min(s, 1) + st = s(1 - t) \end{aligned}$$

HW 2-2 Calculate $\text{Var}(aB_t + bB_s)$ for constants a and b .

Answer

$$\begin{aligned} \text{Var}(aB_t + bB_s) &= E[a^2 B_t^2 + 2ab B_t B_s + b^2 B_s^2] \\ &= a^2 t + 2ab \min(s, t) + b^2 s \end{aligned}$$

HW 2-3 Derive the price of down-and-out call option with knock-out strike K_D and option strike K . (Obviously, $K_D < F$ and $K_D < K$) See the derivation for up-and-out call option and down-and-out digital option from the previous HW and exams.

Answer Assume $B_T^M = \max_{0 \leq t \leq T} B_t$ and $B_T^m = \min_{0 \leq t \leq T} B_t$. From textbook and class, we know

$$P(\sigma B_T^M < v, \sigma B_T < u) = P(B_T^M < v/\sigma, B_T < u/\sigma) = \Phi\left(\frac{u}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{u-2v}{\sigma\sqrt{t}}\right).$$

Using the reflection, $B_t \rightarrow -B_t$, we get

$$(-B)_T^m = \min_{0 \leq t \leq T} (-B_t) = -\max_{0 \leq t \leq T} B_t = -B_T^M$$

and

$$P(\sigma B_T^m > v, \sigma B_T > u) = \Phi\left(\frac{-u}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{2v-u}{\sigma\sqrt{T}}\right).$$

As the stock price is given as $S_T = F + \sigma B_T$,

$$P(S_T^m > v, S_T > u) = \Phi\left(\frac{F-u}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{2v-u-F}{\sigma\sqrt{T}}\right).$$

The probability density function on u with the joint condition, $\sigma B_T^m > v$ is obtained from the partial derivative w.r.t. u (with negative sign),

$$f(u) = \frac{1}{\sigma\sqrt{T}} \left(n\left(\frac{F-u}{\sigma\sqrt{T}}\right) - n\left(\frac{2v-u-F}{\sigma\sqrt{T}}\right) \right) \quad \text{for } -\infty < v \leq u.$$

Let $z = (u-F)/\sigma\sqrt{T}$, $d = (F-K)/\sigma\sqrt{T}$ and $d^* = (F-K_D)/\sigma\sqrt{T}$. Then, the down-and-out call option price is given as

$$\begin{aligned} C(K, K_D) &= \int_{u=K}^{\infty} (u-K) f(u) du = \int_{z=-d}^{\infty} (F-K + \sigma\sqrt{T}z)(n(z) - n(z+2d^*)) dz \\ &= (F-K)N(d) + \sigma\sqrt{T}n(d) - (F-K-2d^*\sigma\sqrt{T})N(d-2d^*) - \sigma\sqrt{T}n(d-2d^*). \end{aligned}$$

The first two terms are exactly the regular call option price, $C(K) = (F-K)N(d) + \sigma\sqrt{T}n(d)$. Therefore, the down-and-out option is cheaper than the regular option by $(F-K-2d^*\sigma\sqrt{T})N(d-2d^*) + \sigma\sqrt{T}n(d-2d^*)$.

We can verify two cases:

1. If $K_D \rightarrow -\infty$ ($d^* \rightarrow \infty$), $C(K, K_D) = C(K)$ because the probability of being knocked out is zero. It is indeed the case because $N(2d^*-d) = n(2d^*-d) = 0$.
2. If $K_D \rightarrow F$ from below ($d^* \rightarrow 0$) on the other hand, the knock-out probability approaches to 100%, so the price should be zero. This is also the case from the formula.

HW 3-1 SCFA Exercise 8.2 The notation $h \in C^1(\mathbb{R}^+)$ means that the function $h(s)$ is differentiable for $s > 0$.

HW 3-2 For a standard BM B_t , let

$$N_t = B_t^3 - 3t B_t.$$

(i) Prove that N_t is a martingale. (Hint: use Proposition 8.1) (ii) By applying Itô's lemma, express N_t as a stochastic integration. (iii) Calculate the variance of N_t .

HW 3-3 SCFA Exercise 8.4 The sub-problem (b) is understood better after **HW 3-2** is solved.

HW 3-4 SCFA Exercise 9.6

HW 3-5 (Martingale representation theory) For a standard BM B_t ($0 \leq t \leq T$), find the martingale representation of $X_t = E(B_T^3 | \mathcal{F}_t)$. (In class, we did the same for $X_t = E(B_T^2 | \mathcal{F}_t)$)