COMELEC department UE COM105

Exercise 7

Exam of Year 2015/2016
Duration : 1h30 - 3 problems - Open book exam, no electronic devices allowed

Problem 1 : Error correcting codes (5 points)

Consider the linear code defined by the following generator matrix

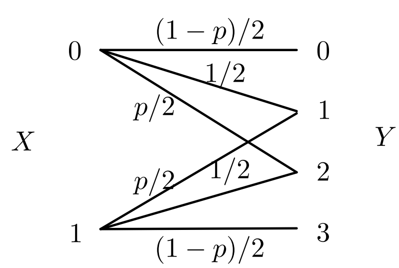
$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Questions :

1. *(1 pt)* Find a parity check matrix for this code. Find the values of n and k.
2. *(1 pt)* The minimum distance $d_{\min}$ of this code is 3. How many errors can it correct ? And how many errors can it detect ?
3. *(1 pt)* The code is used on a BSC. If the channel changes only the bit at position 4, what is the syndrome of the received word ?
4. *(1 pt)* We build a new code by removing one of the columns of H. Find n and k of the new code. Is the minimum distance of the new code « greater or equal » or « smaller or equal » to the minimum distance of the original code ?
5. *(1 pt)* We build a new code by removing one of the rows of H. Is the minimum distance of the new code « greater or equal » or « smaller or equal » to the minimum distance of the original code ?

Problem 2 : Information theory (5 points)

We consider a DMC with the following channel diagram : where p is a real constant taking value in the interval $[0, 1]$.



Questions :

1. *(1 pt)* Let $p = 0$. For this special case, redraw the diagram of the channel and find its capacity.
2. *(1 pt)* Let $p = 1$. For this special case, redraw the diagram of the channel and find its capacity.
3. *(1 pt)* For an arbitrary $p \in (0, 1)$ and for X following a Bernoulli-$1/2$ distribution, find $H(Y|X = 0)$, $H(Y|X = 1)$ and $H(Y|X)$.
4. *(1 pt)* Note that for an arbitrary $p \in (0, 1)$ and for X following a Bernoulli-$1/2$ distribution, $H(Y) = 1 + H_b(\frac{1-p}{2})$. Show that, for any $p \in (0, 1)$, the channel capacity satisfies the inequality
$$C \geq H_{\mathrm{b}}\left(\frac{1-p}{2}\right) - \frac{1}{2}H_{\mathrm{b}}\left(1-p\right).$$
5. *(1 pt)* Find a finite upper bound on the channel capacity.

Problem 3 : Modulations (6 points)

We consider a system with **two transmit antennas** and **one receive antenna**, where we study two different ways of assigning the transmitted symbols $s(1), s(2), \ldots, s(N)$ (supposed i.i.d., uniform over $\{-A, A\}$ and N even) to the transmit antennas.

1) Naive communication system : *The two transmit antennas send the same symbol.* Consequently, the sequence $z(1), \ldots, z(N)$ observed at the receiver is given by

$$z(n) = h_1 s(n) + h_2 s(n) + w(n), \qquad n = 1, \ldots, N, \tag{1}$$

where

— h_1 and h_2 are constant non-zero coefficients [1],

— $w(n)$ is a sequence of i.i.d. Gaussian noise of mean zero and variance $N_0/2$.

Questions :

1. *(1 pt)* Find the optimal detector, that is, the one that minimizes the symbol error probability.
2. *(1 pt)* Find the minimum error probability expressed in terms of h_1, h_2, E_b and N_0.

2) Alamouti communication system : *The two transmit antennas send at odd time slots the two different symbols s (from antenna 1) and $\tilde{s}$ (from antenna 2) and at even time slots the two symbols $\tilde{s}$ (from antenna 1) and $-s$ (from antenna 2).* Consequently, the receiver observes $z(1), \ldots, z(N)$ where for $k = 1, \ldots, N/2$:

$$z(2k-1) = h_1 s(2k-1) + h_2 s(2k) + w(2k-1),$$

and

$$z(2k) = h_1 s(2k) - h_2 s(2k-1) + w(2k).$$

Each symbol $s(n)$ contributes thus to two consecutive symbols of the sequence $z(1), \ldots, z(N)$.

Questions :

3. *(1 pt)* Show that for each k, from $z(2k-1)$ and $z(2k)$ it is possible to compute
$$\tilde{z}(2k-1) = (h_1^2 + h_2^2)s(2k-1) + \tilde{w}(2k-1)$$
and
$$\tilde{z}(2k) = (h_1^2 + h_2^2)s(2k) + \tilde{w}(2k),$$
where $\tilde{w}(n)$ is a noise sequence that needs to be described.

1. In practice, h_1 represents the channel between the receiver and the first transmit antenna, while h_2 represents the channel between the receiver and the second transmit antenna

4. *(1 pt)* From the sequence $\tilde{z}(1), \ldots, \tilde{z}(N)$, find the optimal detector that minimizes the symbol error probability. To do that, start by showing that the sequence $\tilde{w}(n)$ is an i.i.d. Gaussian sequence of mean zero and variance equal to $(h_1^2 + h_2^2)N_0/2$.
5. *(1 pt)* Find E_b. Find the minimum error probability that should be expressed in terms of h_1, h_2, E_b and N_0.
6. *(1 pt)* Which one of the two systems would you use ? What is the difference in RSB between the Alamouti system and the Naive system ?