This document covers nearly all problems on the UCLA Basic Exam from Fall 2001 to Spring 2013. Problems are listed by category and by exam and linked below. Relevant definitions are listed at the start of most categories below. The linear algebra section in particular starts with many standard theorems. I cannot guarantee this material is completely accurate, but it should at least help you along the way. Good luck!

Problems listed by category:

Analysis
- Countability
- Metric space topology
- Topology on reals
- Fixed point
- Inverse and Implicit Function Theorems
- Infinite sequences and series
- Partial derivatives
- Differentiation
- Riemann integration
- Taylor Series
- Jacobian
- Lagrange Multipliers
- Miscellaneous

Linear Algebra
- Recurring Problems
- Other Problems

Problems listed by exam:

Fall 2001: 1 2 3 4 5 6 7 8 9 10
Winter 2002: 1 2 3 4 5 6 7 8 9 10 11
Spring 2002: 1 2 3 4 5 6 7 8 9 10 11
Fall 2002: 1 2 3 4 5 6 7 8 9 10
Spring 2003: 1 2 3 4 5 6 7 8 9 10
Fall 2003: 1 2 3 4 5 6 7 8 9 10
Spring 2004: 1 2 3 4 5 6 7 8 9 10
Fall 2004: 1 2 3 4 5 6 7 8 9 10
Spring 2005: 1 2 3 4 5 6 7 8 9 10 11 12 13
Fall 2005: 1 2 3 4 5 6 7 8 9 10
Winter 2006: 1 2 3 4 5 6 7 8 9 10
Spring 2006: 1 2 3 4 5 6 7 8 9 10
Spring 2007: 1 2 3 4 5 6 7 8 9 10 11 12
Fall 2007: 1 2 3 4 5 6 7 8 9 10 11 12
Spring 2008: 1 2 3 4 5 6 7 8 9 10 11 12
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Spring 2009: 1 2 3 4 5 6 7 8 9 10 11 12
Fall 2009: 1 2 3 4 5 6 7 8 9 10 11 12
Spring 2010: 1 2 3 4 5 6 7 8 9 10 11 12
Fall 2010: 1 2 3 4 5 6 7 8 9 10 11 12
Spring 2011: 1 2 3 4 5 6 7 8 9 10 11 12
Fall 2011: 1 2 3 4 5 6 7 8 9 10 11 12
Spring 2012: 1 2 3 4 5 6 7 8 9 10 11 12
Fall 2012: 1 2 3 4 5 6 7 8 9 10 11 12
Spring 2013: 1 2 3 4 5 6 7 8 9 10 11 12

Analysis

Countability

A set S is countable if there exists a one-to-one map $f : S \to \mathbb{N}$.

Fall 2001 # 4. Let S be the set of all sequences $(x_1, x_2, \ldots)$ such that for all n,

$$x_n \in \{0, 1\}.$$

Prove that there does not exist a one-to-one mapping from the set $N = \{1, 2, \ldots\}$ *onto* the set S.

Suppose for the sake of contradiction that there exists $f : \mathbb{N} \to S$ such that f is a bijection (one-to-one and onto). Define the sequence $(x_n)_{n=0}^{\infty}$ so that for all natural numbers n, $x_n = 0$ if $(f(n))_n = 1$ and $x_n = 1$ if $(f(n))_n = 0$. Then $(x_n)_{n=1}^{\infty} \in S$. Hence there exists some natural number M such that

$$(x_n)_{n=1}^{\infty} = f(M).$$

But this means

$$x_M = (f(M))_M,$$

which contradicts the construction of x_M. Thus no such f exists.

Fall 2003 # 1. Prove that $\mathbb{R}$ is uncountable. If you like to use the Baire category theorem, you have to prove it.

Suppose for the sake of contradiction that the real numbers are countable, so there exists a sequence $(r_n)_{n=1}^{\infty}$ such that $\{r_n : n \geq 1\} = \mathbb{R}$. Then we can choose a closed interval $[a_1, b_1]$ such that $r_1 \notin [a_1, b_1]$. Next, choose a subinterval $[a_2, b_2] \subset [a_1, b_1]$ such that $r_2 \notin [a_2, b_2]$. Repeating this procedure inductively, we select a decreasing sequence of intervals $([a_n, b_n])_{n=1}^{\infty}$ such that for all n, $r_n \notin [a_n, b_n]$.

Now the a_n form an increasing sequence, the b_n form a decreasing sequence, and for all n we have $a_n \leq b_n$. In fact, for any natural numbers n, m with $n \leq m$, $a_n \leq a_m \leq b_m$. Letting $m \to \infty$,

$$a_n \leq \lim_{m\to\infty} b_m = \inf_{m\in\mathbb{N}} b_m.$$

Then taking the limit as $n \to \infty$,

$$\sup_{n\in\mathbb{N}} a_n = \lim_{n\to\infty} a_n \leq \inf_{m\in\mathbb{N}} b_m.$$

In particular, there exists some $x \in [\sup_{n\in\mathbb{N}} a_n, \inf_{n\in\mathbb{N}} b_n]$. Then for all n we have $x \in [a_n, b_n]$, so $x \neq r_n$ by construction of a_n, b_n. But since $x \in \mathbb{R} = \{r_n : n \geq 1\}$, $x = r_n$ for some n, a contradiction. Hence $\mathbb{R}$ is uncountable.

Fall 2005 #1. A real number α is said to be *algebraic* if for some finite set of integers $a_0, \ldots, a_n$, not all 0,

$$a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0.$$

Prove that the set of algebraic real numbers is countable.

We assume that a countable union of countable sets is countable.

Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of countable sets. Define

$$S = \bigcup_{n\in\mathbb{N}} S_n.$$

For all $n \in \mathbb{N}$, let F_n denote the set of all 1-1 maps from S_n to $\mathbb{N}$. Since S_n is countable, F_n is non-empty.

Using the axiom of countable choice, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \in F_n$ for all $n \in \mathbb{N}$. Let $\phi : S \to \mathbb{N} \to \mathbb{N}$ be the mapping defined by

$$\phi(x) = (n, f_n(x)),$$

where n is the smallest natural number such that $x \in S_n$. (Clearly $\{S_n : x \in S_n\}$ is non-empty, so the Well-Ordering Principle ensures that such an n exists.) Since each f_n is 1-1, ϕ must be 1-1. By the Fundamental Theorem of Arithmetic, $g(n,m) = 2^n 3^m$ is an injection from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Hence $g \circ \phi$ is an injection from S into $\mathbb{N}$, so S is countable.

Clearly the set of integers is countable. Since each corresponds to a finite selection of integers, the set of polynomials of degree n with integer coefficients is countable for each n. The set $\mathbb{Z}[x]$ of all polynomials with integer coefficients is the union over all n of sets of polynomials of degree n. As a countable union of countable sets, $\mathbb{Z}[x]$ is thus countable.

Note that the set A of algebraic real numbers is

$$A = \bigcup_{p\in\mathbb{Z}[x]} \{x \in \mathbb{R} : p(x) = 0\}.$$

Now each $p \in \mathbb{Z}[x]$ has some finite degree n, and then can only have at most n real roots. Thus the set $\{x \in \mathbb{R} : p(x) = 0\}$ is finite for each $p \in \mathbb{Z}[x]$. Hence as a countable union of finite sets, A is countable, as desired.

Fall 2005 #5. Prove carefully that $\mathbb{R}^2$ is not a (countable) union of sets S_i, $i = 1, 2, \ldots$ with each S_i being a subset of some straight line L_i in $\mathbb{R}^2$.

Suppose for the sake of contradiction that $\mathbb{R}^2$ is a countable union of sets S_i, $i = 1, 2, \ldots$ with each S_i being a subset of some straight line L_i in $\mathbb{R}^2$. Since $\mathbb{R}$ is uncountable, there exists some $x \in \mathbb{R}$ such that the line $L_x = \{(x, y) : y \in \mathbb{R}\}$ is not equal to any S_i. (Otherwise $\{(x, 0) : x \in \mathbb{R}\}$ would be countable, implying that $\mathbb{R}$ was countable.) Now each line S_i intersects S_i at either zero or one point. Thus the set

$$\bigcup_{i\in\mathbb{N}} L_x \cap S_i$$

is countable. Since $\mathbb{R}^2$ is a countable union of the S_i however,

$$\bigcup_{i\in\mathbb{N}} L_x \cap S_i = L_x,$$

so L_x is countable. But this implies that $\mathbb{R}$ is countable, a contradiction.

Spring 2008 #7. Let $a(x)$ be a function on $\mathbb{R}$ such that

(i) $a(x) \geq 0$ for all x, and

(ii) There exists $M < \infty$ such that for all *finite* $F \subset \mathbb{R}$,

$$\sum_F a(x) \leq M.$$

Prove $\{x : a(x) > 0\}$ is countable.

Define

$$S_n = \left\{x : f(x) > \frac{1}{n}\right\}.$$

Fix n and suppose for the sake of contradiction that $\#(S_n) > Mn$. Then there exists a subset F of S_n of cardinality $Mn+1$. By property (ii),

$$\sum_F a(x) \leq M.$$

However,

$$\sum_F a(x) \geq \sum_F \frac{1}{n} = \frac{Mn+1}{n} > M,$$

a contradiction. Thus S_n is finite for each n.

We have

$$\{x : a(x) > 0\} = \bigcup_{n\in\mathbb{N}} \left\{x : a(x) > \frac{1}{n}\right\} = \bigcup_{n\in\mathbb{N}} S_n,$$

which is a countable union of finite sets, and thus countable. (Apparently assumption (i) is not needed.)

Fall 2011 #3. Prove that the set of real numbers can be written as the union of uncountably many pairwise disjoint subsets, each of which is uncountable.

Define the map $f : (0,1)\times(0,1) \to (0,1)$ so that

$$f(x,y) = 0.x_0y_0x_1y_1\ldots$$

where $x = 0.x_0x_1\ldots$ and $y = 0.y_0y_1\ldots$. Here we replace any infinite chain of 9's in x or y by incrementing the digit preceding the chain and replacing the chain of 9's by a chain of 0's, then evaluate f. Then this map is well-defined and actually an injection. Consider the set S of all vertical lines in $(0,1)\times(0,1)$. There are uncountably many, and their union is $(0,1)\times(0,1)$. Also, each vertical line is an uncountable set of points. Now consider $f(S)$. Note the only decimals that f misses form a countable set. Since f is an injection, for each line $L \in S$, $f(S)$ is uncountable. Also, the images of distinct lines of S under f are disjoint. Thus

$$(0,1) = f(S) = \bigcup_{L\in S} f(L) \cup \bigcup(\text{countable collection of points})$$

can be written as a union of uncountably many pairwise disjoint subsets, each of which is uncountable.

Let g be a bijection between $(0,1)$ and $\mathbb{R}$ ($\tan(\pi(x-1/2))$, for instance). Composing g with f above, we can write the set of real numbers as the desired union.

Metric space topology

X is compact if every open cover of X has a finite subcover.

X is complete if every Cauchy sequence of elements in X converges to some element of X.

X is connected if for every pair of non-empty open sets A and B with $A \cup B = X$, $A \cap B \neq \emptyset$.

X is sequentially compact if every sequence of elements in X has a convergent subsequence.

X is totally bounded if for all $\varepsilon > 0$, there exists a finite open cover of X using balls of radius ε.

X is separable if there is a dense subset of X that is countable. (A dense subset $S \subset X$ is one which satisfies $\overline{S} = X$.)

A base of open sets for X is a family B of open subsets of X such that every open subset of X is the union of sets in B.

X is second-countable if there is a base of open sets of X that is at most countable.

An accumulation point of a sequence $(x_n)_{n=1}^{\infty}$ is a point x such that for each neighborhood B of x there are infinitely many natural numbers i such that $x_i \in B$.

A homeomorphism is a bijection f such that both f and f^{-1} are continuous.

Spring 2009 #4; Spring 2005 #13. Spring 2013 #3. Let (X, d) be an arbitrary metric space.

(a) Give a definition of *compactness* of X involving open covers.

(b) Define *completeness* of X.

(c) Define *connectedness* of X.

(d) Is the set of rational numbers $\mathbb{Q}$ (with the usual metric) connected? Justify your answer.

(e) Suppose X is complete. Show that X is compact in the sense of part (a) if and only if for every $r > 0$, X can be covered by finitely many balls of radius r. (X is totally bounded.)

(a) X is compact if every open cover of X has a finite subcover.

(b) X is complete if every Cauchy sequence of elements in X converges to some element of X.

(c) X is connected if for every pair of non-empty open sets A and B with $A \cup B = X$, $A \cap B \neq \emptyset$.

(d) No, the set of rational numbers (with the usual metric) is not connected. Let α be an irrational number. Consider the open sets $S = \mathbb{Q} \cap (-\infty, \alpha)$ and $T = \mathbb{Q} \cap (\alpha, \infty)$. Clearly S and T are non-empty, have union $\mathbb{Q}$, and $S \cap T = \emptyset$. Let $\varepsilon > 0$. For any $s \in S$, $B(s, \alpha - s) \subset S$, hence S is open. Likewise, for any $t \in T$, $B(t, t - \alpha) \subset T$, hence T is open. (Here the balls are with respect to the usual metric restricted to $\mathbb{Q} \times \mathbb{Q}$.) Thus we have exhibited two non-empty disjoint open subsets of $\mathbb{Q}$ with union $\mathbb{Q}$, so $\mathbb{Q}$ is not connected.

(e) Suppose X is complete. We show that X is compact if and only if X is totally bounded in multiple steps:

Step 1: If X is compact, then X is sequentially compact.

Find a point to converge to: Let $\{y_j\}_{j=1}^{\infty}$ be a sequence in X. Suppose for the sake of contradiction that for each $x \in X$, there exists $\varepsilon = \varepsilon(x) > 0$ such that only finitely many terms of the sequence $\{y_j\}$ lie in $B(x, \varepsilon(x))$. Note that the set of open balls $\{B(x, \varepsilon(x)) : x \in X\}$ forms an open cover of X. Since X is compact, there is a finite subcover

$$X = B(x_1, \varepsilon(x_1)) \cup \cdots \cup B(x_m, \varepsilon(x_m)).$$

Since y_j belongs to $B(x_i, \varepsilon(x_i))$ for only finitely many indices j, we conclude that y_j belongs to X for only finitely many indices j, contradicting that $y_i \in X$ for all i. Hence there exists $x \in X$ such that for each $\varepsilon > 0$, $B(x, \varepsilon)$ contains infinitely many terms of the sequence $\{y_j\}$.

Construct a convergent subsequence: Choose j_1 so that $y_{j_1} \in B(x, 1)$. Inductively choose j_{n+1} so that $j_{n+1} > j_n$ and $y_{j_{n+1}} \in B(x; 1/(n+1))$. Then $\{y_{j_n}\}_{n=1}^{\infty}$ is a subsequence of $\{y_j\}$ that converges to x. Thus X is sequentially compact.

Step 2: If X is sequentially compact, then X is totally bounded.

Select points which are spread out to form a sequence; at some point you must stop. Let $\varepsilon > 0$. Let $y_1 \in X$. If $X = B(y_1, \varepsilon)$, we are finished. Otherwise, let y_2 be any point in $X \setminus B(y_1, \varepsilon)$. As long as $\bigcup_{j=1}^{n} B(y_j, \varepsilon) \neq X$, select

$$y_{n+1} \in X \setminus (\bigcup_{j=1}^{n} B(y_j, \varepsilon)).$$

Suppose for the sake of contradiction that this procedure does not terminate. Then the points $y_1, y_2, \ldots$ satisfy

$$d(y_k, y_j) \geq \varepsilon$$

for all $1 \leq j < k$. It follows that $\{y_j\}_{j=1}^{\infty}$ has no convergent subsequence, contradicting the sequential compactness of X. Thus the procedure does terminate, so there exists N with

$$X = \bigcup_{j=1}^{N} B(y_j, \varepsilon).$$

Hence X is totally bounded.

Step 3: If X is totally bounded, then X is sequentially compact.

Construct a sequence of subsequences using the pigeonhole principle, then diagonalize. Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in X. Rewrite the sequence in the form $\{x_{1j}\}_{j=1}^{\infty}$. By induction, we construct sequences $\{x_{kj}\}_{j=1}^{\infty}$, $k \geq 2$, with the properties

(i) $\{x_{kj}\}_{j=1}^{\infty}$ is a subsequence of $\{x_{k-1,j}\}_{j=1}^{\infty}$, $k \geq 2$.

(ii) $\{x_{kj}\}_{j=1}^{\infty}$ is contained in a ball of radius $1/k$, $k \geq 2$.

Suppose $k \geq 2$ and we already have the sequences $\{x_{ij}\}_{j=1}^{\infty}$ for $i < k$. Let $B_1, \ldots B_n$ be a finite number of open balls of radius $1/k$ that cover X. Since there are infinitely many indices j and only finitely many balls, there must exist at least one ball, say B_m such that $x_{k-1,j} \in B_m$ for infinitely many $j \geq 1$. Now let x_{k1} be the first of the $x_{k-1,j}$'s that belong to B_m, let x_{k2} be the second, etc. Then $\{x_{kj}\}_{j=1}^{\infty}$ has properties (i) and (ii).

Now set $y_n = x_{nn}$, so that $\{y_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_j\}_{j=1}^{\infty}$. Also note that $\{y_n\}_{n=k}^{\infty}$ is a subsequence of $\{x_{kj}\}_{j=1}^{\infty}$ for each k. Thus by construction of the x_{kj}, for any $n, m \geq k$,

$$d(y_n, y_m) < 2/k.$$

Thus $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence, so $\{y_n\}_{n=1}^{\infty}$ converges, and X is sequentially compact.

Step 4: If X is totally bounded, then X is separable.

Direct approach. Let n be a positive integer. Then there exist $x_{n1}, \ldots, x_{nm}$ such that the open balls with centers at the x_{nj} and radii $1/n$ cover X. The family $\{x_{nj} : 1 \leq j \leq m_n, 1 \leq n < \infty\}$ is then a countable subset of X. For each $x \in X$ and each integer n, there is an x_{nj} such that $d(x_{nj}, x) < 1/n$. Consequently the x_{nj} are dense in X.

Step 5: If X is separable, then X is second-countable.

Direct approach, similar to Step 4. Let $\{x_j\}_{j=1}^{\infty}$ be a dense sequence in X. Consider the family of open sets

$$B = \{B(x_j, 1/n) : j \geq 1, n \geq 1\}.$$

Let U be an open set of X and let $x \in U$. For some $n \geq 1$, we have $B(x, 2/n) \subset U$. Choose j so that $d(x_j, x) < 1/n$. Then $x \in B(x_j, 1/n)$, and the triangle inequality shows that $B(x_j, 1/n) \subset B(x, 2/n) \subset U$. Thus each $x \in U$ has some associated $V_x \in B$ such that $x \in V_x$ and $V_x \subset U$. It follows that

$$U = \bigcup_{x \in U} V_x$$

represents U as a union of sets in B. Thus B is a base of open sets. Since B is countable, X is second-countable.

Step 6: If X is second-countable, then every open cover of X has a countable subcover. (Lindelof's Theorem).

Pick elements of the base which are inside sets from the open cover. They cover X. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X, where A is some index set. Let B be a countable base of open sets. Let C be the subset of B consisting of those sets $V \in B$ such that $V \subset U_\alpha$ for some α. We claim that C is a cover of X. Indeed, if $x \in X$, then there is some index α such that $x \in U_\alpha$. Since B is a base and U_α is open, there exists $V \in B$ such that $x \in V$ and $V \subset U_\alpha$. In particular, $V \in C$, so C covers X.

For each $V \in C$, select one index $\alpha = \alpha(V)$ such that $V \subset U_{\alpha(V)}$. Then the sets $\{U_{\alpha(V)} : V \in C\}$ cover X. Since B is countable, so is C, so that the $U_{\alpha(V)}$'s form a countable subcover of X.

Step 7: If X is sequentially compact and every open cover of X has a countable subcover, then every open cover of X has a finite subcover.

Argue by contradiction. Make a sequence, get subsequence, use completeness of X, and that a set is closed. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open subsets of X that cover X. Suppose for the sake of contradiction that for all positive integers m,

$$X \neq U_1 \cup \cdots \cup U_m.$$

For each m, let x_m be any point in $X \setminus (\bigcup_{j=1}^m U_j)$. Since X is sequentially compact, the sequence $(x_m)_{m=1}^{\infty}$ has a subsequence which converges. Since X is complete, this subsequence converges to some $x \in X$. Now $x_j \in X \setminus (\bigcup_{j=1}^m U_j)$ for all $j \geq m$. Thus since $X \setminus (\bigcup_{j=1}^m U_j)$ is closed, $x \in X \setminus (\bigcup_{j=1}^m U_j)$. But this is true for all m, hence

$$x \in X \setminus (\bigcup_{j=1}^{\infty} U_j) = \emptyset,$$

a contradiction. Thus $\{U_n\}_{n=1}^{\infty}$ has a finite subcover.

Note: Tao's Analysis II, Ch. 12 also has an argument that sequential compactness implies compactness.

Fall 2004 #4. Suppose that (M, ρ) is a metric space, $x, y \in M$, and that $\{x_n\}$ is a sequence in this metric space such that $x_n \to x$. Prove that $\rho(x_n, y) \to \rho(x, y)$.

Let $\varepsilon > 0$. Since $x_n \to x$, there exists N such that for all $n \geq N$, $\rho(x_n, x) \leq \varepsilon$. By the reverse triangle inequality, for any $n \geq N$,

$$|\rho(x_n, y) - \rho(y, x)| \leq \rho(x_n, x) \leq \varepsilon.$$

Thus $\rho(x_n, y) \to \rho(x, y)$.

Fall 2002 #1. Let K be a compact subset and F be a closed subset in the metric space X. Suppose $K \cap F = \emptyset$. Prove that

$$0 < \inf\{d(x, y) : x \in K, y \in F\}.$$

Suppose for the sake of contradiction that $\inf\{d(x,y) : x \in K, y \in F\} = 0$. Then for each n, there exist $x_n \in K$ and $y_n \in F$ such that $d(x_n, y_n) < 1/n$. Since K is compact, K is sequentially compact and complete, so there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ that approaches some $x \in K$. It follows that $y_{n_j} \to x$ as $j \to \infty$. Since F is closed, $x \in F$. Hence $x \in K \cap F$, a contradiction. Thus

$$0 < \inf\{d(x,y) : x \in K, y \in F\}.$$

Fall 2008 #4. (a) Suppose that K and F are subsets of $\mathbb{R}^2$ with K closed and bounded and F closed. Prove that if $K \cap F = \emptyset$, then $d(K,F) > 0$. Recall that

$$d(K,F) = \inf\{d(x,y) : x \in K, y \in F\}.$$

Fall 2010 #1. Also show the converse, that if $K \subset X$ is compact and

$$\inf_{x \in K, y \in F} d(x,y) > 0,$$

then $K \cap F = \emptyset$.

(b) Is (a) true if K is just closed? Prove your assertion.

(a) This is the above exercise with $X = \mathbb{R}^2$, since $\mathbb{R}^2$ is complete with respect to the standard metric.

For the converse, suppose for the sake of contradiction that $K \cap F \neq \emptyset$. Then there exists $x \in K \cap F$, and $0 = d(x,x) \in \{d(x,y) : x \in K, y \in F\}$, so $d(K,F) = 0$, a contradiction. Thus $K \cap F = \emptyset$.

No, part (a) is no longer true if K is just closed. Let

$$K = \{(n,0) : n \in \mathbb{N}\} \qquad \text{and} \qquad F = \{(n + \frac{1}{2n}, 0) : n \in \mathbb{N} \setminus \{0\}\}.$$

Note that K and F contain all their limit points, so they are closed. However, there are points in F and K with distance $\frac{1}{2n}$ for each $n \in \mathbb{N}$, so $d(K,F) = 0$.

Spring 2002 #3. Suppose that X is a compact metric space (in the covering sense of the word compact). Prove that every sequence $\{x_n : x_n \in X,\ n = 1,2,3\ldots\}$ has a convergent subsequence. [Prove this directly. Do not just quote a theorem.]

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Let $\varepsilon > 0$. Suppose for the sake of contradiction that there does not exist $x \in X$ such that $\{i \in \mathbb{N} \setminus \{0\} : x_i \in B(x,\varepsilon)\}$ is infinite. Clearly $\bigcup_{x \in X} B(x,\varepsilon)$ covers X, so since X is compact, there exists a finite cover

$$X = B(x_1,\varepsilon) \cup \cdots \cup B(x_N,\varepsilon).$$

Now

$$\{i \in \mathbb{N} \setminus \{0\} : x_i \in X\} = \bigcup_{j=1}^{N} \{i \in \mathbb{N} \setminus \{0\} : x_i \in B(x_j,\varepsilon)\}.$$

Clearly the left hand side is an infinite set, but by assumption, each set in the union on the right hand side is finite, so the union on the right hand side is finite, a contradiction. Hence there exists some $x \in X$ such that $\{i \in \mathbb{N} \setminus \{0\} : x_i \in B(x,\varepsilon)\}$ is infinite.

Select $x_{n_1} \in B(x,1)$. Then for each $k \geq 1$, inductively select $x_{n_{k+1}} > x_{n_k}$ with $x_{n_{k+1}} \in B(x, 1/(k+1))$. It follows that $(x_{n_k})_{k=1}^{\infty}$ converges to x, hence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Spring 2005 #12. Let (X,d) be a metric space. Prove that the following are equivalent:

(a) There is a countable dense set.

(b) There is a countable basis for the topology.

Recall that a collection of open sets U is called a basis if every open set can be written as a union of elements of U.

Let $\{x_n\}_{n=1}^{\infty}$ be a countable dense set in X. Let B be a basis for the topology. It follows that

$$\{B(x_n, 1/m), n \geq 1, m \geq 1\}$$

is a countable basis for the topology. To see this, let U be an open set in X and let $x \in U$. For some $m \geq 1$, we have $B(x, 2/m) \subset U$. Choose n so that $d(x_n, x) < 1/m$. Then $x \in B(x_n, 1/m)$, and the triangle inequality implies $B(x_n, 1/m) \subset B(x, 2/m) \subset U$. Thus each $x \in U$ has some associated $V_x \in B$ such that $x \in V_x$ and $V_x \subset U$. It follows that

$$U = \bigcup_{x \in U} V_x$$

represents U as a union of sets in B. Thus B is a countable basis for the topology.

Conversely, suppose $\{B_n\}_{n=1}^{\infty}$ is a countable basis for the topology. Choose $x_n \in B_n$ for each $n \geq 1$. It follows that $\{x_n\}$ is dense in X, so there is a countable dense set in X.

Spring 2005 #6. Let X be the set of all infinite sequences $\{\sigma_n\}_{n=1}^{\infty}$ of 1's and 0's endowed with the metric

$$\text{dist}(\{\sigma_n\}_{n=1}^{\infty}, \{\sigma'_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\sigma_n - \sigma'_n|.$$

Give a direct proof that every infinite subset of X has an accumulation point.

An accumulation point of a sequence $(x_n)_{n=1}^{\infty}$ is a point x such that for each neighborhood B of x there are infinitely many natural numbers i such that $x_i \in B$. Let S_0 be an infinite subset of S. Now either infinitely many sequences in S_0 start with 0 or infinitely many sequences in S_0 start with 1. Let x_1 be 0 if infinitely many sequences in S_0 start with 0 and otherwise, let $x_1 = 1$. Note that if $x_1 = 1$, then infinitely many sequences in S_0 start with 1. Now for each $n \geq 0$, inductively let S_{n+1} be the set of sequences in S_n whose n-th digit is x_n. By construction S_{n+1} is infinite. Define x_{n+1} to be 0 if infinitely many sequences in S_{n+1} have $n+1$-st digit 0, and $x_{n+1} = 1$ otherwise.

Thus we form a sequence $(x_n)_{n=1}^{\infty}$ such that for each $N \geq 1$, there exists $\{\sigma_n\}_{n=1}^{\infty} \in S$ such that $\sigma_n = s_n$ for all $n \leq N$. Hence

$$\text{dist}(\{\sigma_n\}_{n=1}^{\infty}, \{x_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\sigma_n - x_n| = \sum_{n=N+1} \frac{1}{2^n} |\sigma_n - x_n| \leq \sum_{n=N+1} \frac{1}{2^n} \leq \frac{1}{2^N}.$$

Since N is arbitrary, this shows that $\{x_n\}_{n=1}^{\infty}$ is an accumulation point of S.

Spring 2005 #7. Let X, Y be two topological spaces. We say that a continuous function $f : X \to Y$ is *proper* if $f^{-1}(K)$ is compact for any compact set $K \subset Y$.

(a) Give an example of a function that is proper but not a homeomorphism.

(b) Give an example of a function that is continuous but not proper.

(c) Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 (that is, has a continuous derivative) and for all $x \in \mathbb{R}$,

$$|f'(x)| \geq 1.$$

Show that f is proper.

(a) Pick any continuous function which is not a bijection. For example, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$ is continuous and proper, but f is clearly not a bijection, so f is not a homeomorphism.

(b) Let X be a non-compact metric space ($\mathbb{R}$ for example) and let $Y = \{0\}$. Then the constant function $f : X \to Y$ is continuous, but $f^{-1}(\{0\}) = X$ is not compact, and $\{0\}$ is compact. Hence f is not proper.

(c) Let $K \subset Y$ be a compact set. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f^{-1}(K)$. It follows that $\{f(U_\alpha)\}_{\alpha \in I}$ is an open cover of $f(f^{-1}(K)) = K$. Since K is compact, there exists a finite subcover $\{f(U_n)\}_{n=1}^N$ of K.

Let $x \in f^{-1}(K)$. Then since $\{f(U_n)\}_{n=1}^N$ covers K, $f(x) \in f(U_j)$ for some $1 \leq j \leq n$. Hence there exists $y \in U_j$ such that $f(x) = f(y)$.

By the mean value theorem (valid since f is C^1), there exists $c \in (x, y)$ such that

$$f(x) - f(y) = f'(c)(x - y).$$

Thus by the given property of f,

$$0 = |f(x) - f(y)| = |f'(c)||x - y| \geq |x - y|.$$

Thus $x = y$, so $x \in U_j$. Thus $\{U_n\}_{n=1}^N$ is a finite cover of $f^{-1}(K)$, so $f^{-1}(K)$ is compact, and f is proper.

Spring 2008 #6. Let Y be a complete *countable* metric space. Prove there is $y \in Y$ such that $\{y\}$ is open.

Suppose for the sake of contradiction that $\{y\}$ has non-empty interior for each $y \in Y$. Then $\{y\}$ is nowhere dense, and $Y = \bigcup_{y \in Y} \{y\}$ is a countable union of closed nowhere-dense sets since Y is countable. But this contradicts the Baire Category Theorem, since Y is a complete metric space. Hence there exists some $y \in Y$ such that $\{y\}$ has non-empty interior. It follows that y is an interior point of $\{y\}$, hence $\{y\}$ is open.

Spring 2010 #8. Let (X, d) be a complete metric space and let K be a closed subset of X such that for any $\varepsilon > 0$, K can be covered by a finite number of sets $B_\varepsilon(x)$, where

$$B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}.$$

Prove that K is compact.

Follow the proof in Spring 2009 #4.

Fall 2012 #3; Fall 2011 #6; Spring 2008 #4; Winter 2006 #4. Let $\{f_n(x)\}$ be a sequence of non-negative continuous functions on a compact metric space X. Assume $f_n(x) \geq f_{n+1}(x)$ for all n and x, so that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for every $x \in X$. Prove f is continuous if and only if f_n converges to f uniformly on X.

The forward direction is called Dini's Theorem. Let $\varepsilon > 0$. For each n, let $g_n = f_n - f$, and define $E_n := \{x \in X : g_n(x) < \varepsilon\}$. Each g_n is continuous, so each E_n is open. Since $\{f_n\}$ is monotonically decreasing, $\{g_n\}$ is monotonically decreasing, so $E_n \subset E_{n+1}$ for all $n \geq 1$. Since f_n converges pointwise to f, it follows that the collection $\{E_n\}$ is an open cover of X. Since X is compact, there exist $n_1 < n_2 < \cdots < n_K$ with

$$X = E_{n_1} \cup E_{n_2} \cup \cdots \cup E_{n_K} = E_{n_K}.$$

Thus for any $n \geq n_K$ and $x \in X$, $x \in E_{n_K}$, so

$$|f_n(x) - f(x)| = f_n(x) - f(x) \leq f_{n_K}(x) - f(x) = g_{n_K}(x) \leq \varepsilon.$$

Thus f_n converges to f uniformly on X.

Suppose f_n converges to f uniformly on x. Let $\varepsilon > 0$. Select N such that for all $n \geq N$ and all $x \in X$,

$$|f_n(x) - f(x)| \leq \varepsilon/3.$$

Since f_N is continuous, there exists $\delta > 0$ such that if $x, y \in X$ with $d(x, y) \leq \varepsilon$, then

$$|f_N(x) - f_N(y)| \leq \varepsilon/3.$$

It follows that for any $x, y \in X$ with $d(x, y) \leq \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus f is continuous.

Fall 2012 #4. A subset K of a metric space (X, d) is called *nowhere dense* if K has empty interior, (i.e., if $U \subset K$, U open in X imply $U = \emptyset$.) Prove the Baire theorem that if (X, d) is a complete metric space, then X is not a countable union of closed nowhere-dense sets. Hint: Assume $X = \cup_n K_n$ where each K_n is closed and nowhere dense. Show there is $x_1 \in X$ and $0 < \delta_1 < 1/2$ such that $B_1 = B(x_1, \delta_1) = \{y \in X : d(y, x) < \delta_1\}$ satisfies $B_1 \cap K_1 = \emptyset$ and there is $x_2 \in X$ and $0 < \delta_2 < \frac{\delta_1}{2}$ such that $B_2 = B(x_2, \delta_2)$ satisfies $B_2 \subset B_1$ and $B_2 \cap K_2 = \emptyset$. Then continue by induction to find a sequence $\{x_n\}$ in X that converges to $x \in X \setminus \bigcup_{n=1}^\infty K_n$.

Let K_n, $n \geq 1$ be closed nowhere-dense sets such that $X = \bigcup_{n \geq 1} K_n$. Choose some $x_1 \in X \setminus K_1$. Since K_1 is closed, $X \setminus K_1$ is open, so there exists $0 < \delta_1 < 1/2$ such that $B_1 := B(x_1, \delta_1) \subset X \setminus K_1$. Then $B_1 \cap K_1 = \emptyset$. Suppose inductively that $n \geq 2$ and we have selected $B_n = B(x_n, \delta_n)$ such that $B_n \subset B_{n-1}$ and $B_n \cap K_n = \emptyset$. Select $x_{n+1} \in B_n \setminus K_{n+1}$. (Since K_{n+1} is nowhere dense, it cannot contain B_n, so some such x_{n+1} exists.) Now K_{n+1} is closed, so $B_n \setminus K_{n+1}$ is open, hence there exists $0 < \delta_{n+1} < \delta_n/2$ such that $B(x_{n+1}, \delta_{n+1}) \subset (B_n \setminus K_{n+1})$. Then $B_{n+1} \subset B_n$ and $B_{n+1} \cap K_{n+1} = \emptyset$, completing the induction.

Choosing countably many x_n and B_n in this way requires the axiom of countable choice. It follows that $\delta_n \leq \delta_1/2^{n-1}$, so the sequence $(x_n)_{n=1}^\infty$ is a Cauchy sequence. Since (X, d) is complete, (X, d) converges to some $x \in X$. Because $B_{n+1} \subset B_n$ for each n, it follows that $x \in B_n$ for each $n \geq 1$. Since $B_n \cap K_n = \emptyset$, we must have $x \notin K_n$ for each $n \geq 1$, hence

$$x \in X \setminus \bigcup_{n=1}^{\infty} K_n = \emptyset,$$

a contradiction. Hence X is not a countable union of closed nowhere-dense sets.

Winter 2002 #1. Spring 2012 #1. Let Ω denote the set of all closed subsets of $[0, 1]$ and let $\rho : \Omega \times \Omega \to [0, 1]$ be defined by

$$\rho(A, B) := \max\{\sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y|\}.$$

Show that (Ω, ρ) is a metric space.

Clearly ρ is non-negative and symmetric.

Suppose A and B are closed subsets of $[0, 1]$ with $\rho(A, B) = 0$. Then

$$\sup_{x \in A} \inf_{y \in B} |x - y| = 0,$$

which implies that for any $x \in A$,

$$\inf_{y \in B} |x - y| = 0.$$

Thus x is a limit point of B, and since B is closed, $x \in B$. Hence $A \subseteq B$.

Likewise, $\rho(A, B) = 0$ implies $\sup_{y \in B} \inf_{x \in A} |x - y| = 0$; reasoning as above implies $B \subseteq A$. Thus $A = B$.

Finally, we verify that ρ satisfies the triangle inequality on Ω. Let A, B, C be closed subsets of $[0,1]$. For any $a \in A$, $b \in B$, $c \in C$,

$$|a-b| \leq |a-c| + |c-b|.$$

Taking the infimum of both sides over all $b \in B$,

$$\inf_{b \in B} |a-b| \leq |a-c| + \inf_{b \in B} |c-b|.$$

It follows that

$$\inf_{b \in B} |a-b| \leq |a-c| + \sup_{c \in C} \inf_{b \in B} |c-b|,$$

Then taking the infimum over all $c \in C$,

$$\inf_{b \in B} |a-b| \leq \inf_{c \in C} |a-c| + \sup_{c \in C} \inf_{b \in B} |c-b|.$$

Finally, taking the supremum over all $a \in A$,

$$\sup_{a \in A} \inf_{b \in B} |a-b| \leq \sup_{a \in A} \inf_{c \in C} |a-c| + \sup_{c \in C} \inf_{b \in B} |c-b|.$$

Thus

$$\sup_{a \in A} \inf_{b \in B} |a-b| \leq \rho(A,C) + \rho(C,B).$$

By symmetry of A and B,

$$\sup_{b \in B} \inf_{a \in A} |a-b| \leq \rho(A,C) + \rho(C,B).$$

Thus

$$\rho(A,B) \leq \rho(A,C) + \rho(C,B).$$

Topology on reals

A family of functions is equicontinuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, then for any $f \in F$, $d(f(x_1), f(x_2)) < \varepsilon$.

Winter 2006 #6. Let $-\infty < a < b < \infty$. Prove that a continuous function $f : [a, b] \to \mathbb{R}$ attains all values in $[f(a), f(b)]$.

Let $y \in [f(a), f(b)]$. If $y = f(a)$ or $y = f(b)$, we are done. Otherwise, $f(a) < y < f(b)$. Define

$$E := \{x \in [a, b] : f(x) < y\}.$$

Clearly E is a subset of $[a, b]$ and is hence bounded. Also, $a \in E$, so E is non-empty. By the least upper bound principle,

$$c := \sup(E)$$

is finite. Clearly $c \in [a, b]$.

Select N such that $c - \frac{1}{N} \geq a$. By definition of supremum, there must exist $x_n \in E$ with

$$c - \frac{1}{n} \leq x_n \leq c$$

for all $n \geq N$. Letting $N \to \infty$, it follows that $\lim_{n\to\infty} x_n = c$. Since f is continuous at c, this implies

$$\lim_{n\to\infty} f(x_n) = f(c).$$

But since $x_n \in E$, $f(x_n) < y$ for every n. This implies $f(c) \leq y$.

Since $f(c) \leq y < f(b)$, $c < b$. Choose N such that $c + \frac{1}{n} < b$ for all $n \geq N$. Then $c + \frac{1}{n} \notin E$ for all $n \geq N$, so

$$f(c + \frac{1}{n}) \geq y$$

for all $n \geq N$. Taking the limit as $n \to \infty$ and using the continuity of f,

$$f(c) \geq y.$$

Thus $f(c) = y$, hence f attains all values in $[f(a), f(b)]$.

Fall 2004 #2. State and prove Rolle's Theorem. (You can use without proof theorems about the maxima and minima of continuous or differentiable functions.)

Rolle's Theorem: Let $a < b$, f be a continuous function on $[a, b]$ which is differentiable on (a, b), and suppose $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof: Since f is continuous on the compact set $[a, b]$, it attains its maximum and minimum on $[a, b]$. If both the maximum and minimum occur at the endpoints a and b, then $f(a) = f(b)$ implies that f is constant, so taking $c = (a+b)/2 \in (a, b)$, $f'(c) = 0$. Otherwise, there exists $c \in (a, b)$ such that $f(c)$ is either the maximum or minimum of f on $[a, b]$.

Suppose that $f(c)$ is the maximum of f on $[a, b]$ (the other case is similar). For every $h > 0$,

$$\frac{f(c + h) - f(c)}{h} \leq 0,$$

thus letting $h \to 0$ from the right,

$$f'(c+) \leq 0.$$

Likewise, for every $h < 0$,

$$\frac{f(c + h) - f(c)}{h} \geq 0,$$

thus letting $h \to 0$ from the left,

$$f'(c-) \geq 0.$$

Since f is differentiable at c, $f'(c) = f'(c+) = f'(c-)$, so $f'(c) = 0$.

Spring 2011 #7. Prove that there is a real number x such that

$$x^5 - 3x + 1 = 0.$$

Let $f(x) = x^5 - 3x + 1$. Clearly f is continuous. Also, $f(-2) = -25$ and $f(0) = 1$. Thus by the intermediate value theorem, there exists $x \in [-2, 0]$ such that $f(x) = 0$.

Fall 2001 #1. Let K be a compact set of real numbers and let $f(x)$ be a continuous real-valued function on K. Prove there exists $x_0 \in K$ such that $f(x) \leq f(x_0)$ for all $x \in K$.

Steps: Show f is bounded, so supremum of f on K exists. Find a sequence that converges to the supremum. Use sequential compactness to get a subsequence converging to some $d \in [a, b]$. Use continuity.

Because K is compact, the Heine-Borel Theorem implies it is closed and bounded. By the Bolzano-Weierstrauss Theorem, K is also sequentially compact.

Suppose for the sake of contradiction that f is unbounded. Then for each natural number n, there exists $x_n \in K$ such that $f(x_n) > n$. Since K is sequentially compact, there exists a subsequence of $\{x_n\}_{n=0}^{\infty}$ converging to some $x \in \mathbb{R}$. Since K is closed, $x \in K$. Since f is continuous, $f(x_n) \to f(x)$. But $f(x_n)$ is unbounded as $n \to \infty$, a contradiction. Hence f is bounded.

Thus by the least upper bound principle, $M := \sup(f(K))$ exists. Select a sequence $(x_n)_{n=1}^{\infty}$ such that

$$M - \frac{1}{n} \leq f(x_n) \leq M$$

for each n. Thus $(f(x_n))_{n=1}^{\infty}$ converges to M. Since K is sequentially compact, there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ which converges to some $x_0 \in \mathbb{R}$. Since K is closed, $x_0 \in K$. Now $(f(x_{n_k}))_{k=1}^{\infty}$ must converge to M. By continuity of f, this implies $f(x_0) = M$. Thus f attains its maximum on K.

Fall 2002 #2. Show why the Least Upper Bound Property (every set bounded above has a least upper bound) implies the Cauchy Completeness Property (every Cauchy sequence has a limit) of the real numbers.

Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence of real numbers. We first show that $(x_n)_{n=1}^{\infty}$ is bounded. Fix $\varepsilon > 0$ and let N be such that $|x_n - x_m| < \varepsilon$ for $n, m \geq N$. Let $R = \max(d(x_N, x_1), \ldots, d(x_{N-1}, x_N), \varepsilon)$. Then the entire sequence $(x_n)_{n=1}^{\infty}$ is contained in $B(x_n, 2R)$. Thus $(x_n)_{n=1}^{\infty}$ is bounded.

By the least upper bound property, we can define

$$z_n := \sup\{x_k : k \geq n\}$$

for each $n \geq 1$. Clearly $(z_n)_{n=1}^{\infty}$ is decreasing and bounded since $(x_n)_{n=1}^{\infty}$ is bounded. The least upper bound property implies the greatest lower bound property, thus we can define

$$x = \inf\{z_n : n \geq 1\} = \lim_{n\to\infty} z_n.$$

We show that $(x_n)_{n=1}^{\infty} \to x$. First, we exhibit a subsequence of $(x_n)_{n=1}^{\infty}$ which converges to x. Let j be a positive integer. Since x is the limit of the z_n, there exists N_j such that for all $n \geq N_j$,

$$|z_n - x| \leq \frac{1}{2j}.$$

Since $z_{N_j} = \sup\{x_k : k \geq n\}$, there exists $n_j \geq N_j$ such that

$$|z_{N_j} - x_{n_j}| \leq \frac{1}{2j}.$$

Thus we obtain infinitely many distinct n_j; re-index the distinct n_j to obtain the subsequence $\{x_{n_\ell}\}_{\ell=1}^\infty$. By construction,

$$|x_{n_\ell} - x| \leq |x_{n_\ell} - z_{N_\ell}| + |z_{N_\ell} - x| \leq \frac{1}{2\ell} + \frac{1}{2\ell} = \frac{1}{\ell}.$$

Thus $(x_{n_\ell})_{\ell=1}^\infty$ converges to x.

Now since $(x_n)_{n=1}^\infty$ is a Cauchy sequence, it follows that the whole sequence converges to x. Let $\varepsilon > 0$. There exists N such that for all $\ell \geq N$,

$$|x_{n_\ell} - x| \leq \varepsilon/2.$$

There also exists N' such that if $j, k \geq N'$,

$$|x_j - x_k| \leq \varepsilon/2.$$

Hence for all $n \geq \max(n_N, N')$,

$$|x_n - x| \leq |x_n - x_{n_N}| + |x_{n_N} - x| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $(x_n)_{n=1}^\infty$ converges to x.

Winter 2002 #4. Prove that the set of irrational numbers in $\mathbb{R}$ is not a countable union of closed sets.

Suppose for the sake of contradiction that the set of irrational numbers I can be represented as

$$I = \bigcup_{n \in \mathbb{N}} F_N$$

where the F_N are closed. Then

$$\mathbb{R} = (\bigcup_{n \in \mathbb{N}} F_n) \cup (\bigcup_{r \in \mathbb{Q}} \{r\}).$$

Since $\mathbb{R}$ has non-empty interior, the Baire Category Theorem implies that one of the sets in the union on the right hand side has non-empty interior. Clearly it is not $\{r\}$ for some rational r, so some F_n must have non-empty interior. Thus there exists $x \in F_n$ such that $B(x, \varepsilon) \subset F_n \subset I$. But the rationals are dense in $\mathbb{R}$, so some rational number is an element of $B(x, \varepsilon)$ and thus an element of I, a contradiction.

Fall 2002 # 3; Spring 2002 #2 Show that the set $\mathbb{Q}$ of rational numbers in $\mathbb{R}$ is not expressible as the intersection of a countable collection of open subsets of $\mathbb{R}$.

Fall 2012 #5. Use the Baire Category Theorem to prove this.

Suppose for the sake of contradiction that $\mathbb{Q} = \bigcap_{n \in \mathbb{N}} U_n$, where U_n is open for each n. Clearly $\mathbb{Q} \subset U_n$ for each n, and since the rational numbers are dense in $\mathbb{R}$, each U_n is dense in $\mathbb{R}$. For each rational number r, $X \setminus \{r\}$ is open and dense in $\mathbb{R}$. Thus

$$\emptyset = I \cap Q = \left(\bigcap_{r \in \mathbb{Q}} X \setminus \{r\} \right) \cap \bigcap_{n \in \mathbb{N}} U_n$$

is a countable intersection of dense open sets. Applying the Baire Category Theorem however, we expect that $\emptyset$ is dense in $\mathbb{R}$, a contradiction.

Spring 2003 #3. Find a subset S of the real numbers $\mathbb{R}$ such that both (i) and (ii) hold for S:

(i) S is not the countable union of closed sets.

(ii) S is not the countable intersection of open sets.

Let A be a subset of $[0,1]$ that is not a countable union of closed sets, and let B be a subset of $[2,3]$ that is not a countable intersection of open sets. (Irrationals and rationals for instance, respectively.) We show that $S := A \cup B$ satisfies (i) and (ii).

Suppose for the sake of contradiction that S is the countable union of closed sets $\{F_n\}_{n=1}^{\infty}$. Then

$$A = S \cap [0,1] = \left(\bigcup_{n=1}^{\infty} F_n\right) \cap [0,1] = \bigcup_{n=1}^{\infty} (F_n \cap [0,1]).$$

Note that $F_n \cap [0,1]$ is closed for each n, so A is a countable union of closed sets, a contradiction.

Likewise, if S is the countable intersection of open sets, it follows that B is the countable intersection of open sets, a contradiction. Hence S satisfies (i) and (ii).

Spring 2002 #1. Prove that the closed interval $[0,1]$ is connected.

Suppose there exist disjoint non-empty open sets A, B such that $A \cup B = [0,1]$. Suppose without loss of generality that $1 \in B$. Clearly $[0,1]$ is bounded, thus A is bounded, so by the least upper bound principle, we can define

$$c = \sup(A).$$

Since $[0,1]$ is closed, $c \in [0,1]$. We show that c cannot be in either A or B. Suppose for the sake of contradiction that $c \in A$. Note that $c < 1$ since $1 \in B$ and A and B are disjoint. Since A is open, there exists some ball of radius $\varepsilon > 0$ at c such that $B(c,\varepsilon) \cap [0,1] \in A$. But then $c + \min(\varepsilon, 1-c) \in A$, contradicting that $c = \sup(A)$.

Suppose for the sake of contradiction that $c \in B$. If $c = 0$, then $A = \{0\}$, which is closed, a contradiction. Hence $c > 0$. Since B is open, there exists some ball of radius ε at c such that $B(c,\varepsilon) \cap [0,1] \in B$. But then $c - \min(\varepsilon, c)$ is an upper bound for A, contradicting that $c = \sup(A)$.

Winter 2002 #3. Prove that the open ball in $\mathbb{R}^2$

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

is connected. [You may assume that intervals in $\mathbb{R}$ are connected. You should not just quote other general results, but give a direct proof.]

Lemma: The image of a connected set under a continuous function is connected.

Proof: Let S be connected and f be continuous. Suppose for the sake of contradiction that $f(S)$ is disconnected, so $f(S) = A \cup B$, with A and B disjoint non-empty open sets. It follows that $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint and non-empty. Since f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are also open. But $S = f^{-1}(A) \cup f^{-1}(B)$, so S is not connected, a contradiction.

Lemma: Let $S_\alpha \in M \subset X$ be connected sets. Suppose $\cap_\alpha S_\alpha \neq \emptyset$. Then $\bigcup S_\alpha$ is connected.

Proof: Let $S = \bigcup S_\alpha = G \cup H$, where G, H are non-empty disjoint open sets. Choose $x_0 \in \bigcap_\alpha S_\alpha$. Fix α. Note $S_\alpha = (S_\alpha \cap G) \cup (S_\alpha \cap H)$ and $x_0 \in S_\alpha$. If $x_0 \in G$, since S_α is connected, we get $S_\alpha \cap H = \emptyset$. Since this holds for all α, $S \cap H = \emptyset$. Since $H \subset \cup S_\alpha$, $H = \emptyset$, a contradiction.

Let $\theta \in [0, 2\pi)$ and define $f : \mathbb{R} \to \mathbb{R}^2$ by $f_\theta(t) = (t\cos(\theta), t\sin(\theta))$. Then f is continuous, so by the first lemma $f_\theta([0,1])$ is open for each θ. We may write the open unit ball as $\bigcup_{\theta \in [0,2\pi)} f_\theta([0,1])$. Note also that $(0,0) \in f_\theta([0,1])$ for each θ. Hence by the second lemma, $\bigcup_{\theta\in[0,2\pi)} f_\theta([0,1])$ is connected, so the unit ball is connected.

Winter 2002 #2. Prove that the unit interval $[0,1]$ is sequentially compact, i.e., that every infinite sequence has a convergent subsequence. [Prove this directly. Do not just quote general theorems like Heine-Borel].

Let $(x_n)_{n=1}^{\infty}$ be an infinite sequence in $[0,1]$. Clearly this sequence is bounded. Let $I_0 = [0,1]$. Let $n_0 = 0$. If the left half of I_0 contains infinitely many terms of $(x_n)_{n=2}^{\infty}$, set $I_1 = [0, 1/2]$. Otherwise, the right half of I_0 must contain infinitely many terms of the sequence; set $I_1 = [1/2, 1]$. Now assume inductively that $k \geq 1$

and we have chosen $n_j \in I_j$ for all $j < k$ and constructed I_k of length 2^{-k} such that infinitely many terms of the sequence $(x_n)_{n=1}^{\infty}$ lie in I_k. Select $n_k > n_{k-1}$ such that $n_k \in I_k$. If the left half of I_k contains infinitely many terms of $(x_n)_0^{\infty}$, set I_{k+1} to be the left half of I_k. Otherwise, the right half of I_k contains infinitely many terms of $(x_n)_{n=0}^{\infty}$; set I_{k+1} to be the right half of I_k. This completes the induction, so we have an infinite subsequence $(x_{n_k})_{k=0}^{\infty}$ such that $x_{n_k} \in I_k$ for each k. Since $I_{k+1} \subset I_k$ for each k, with the length of I_k given by 2^{-k}, $(x_{n_k})_{k=0}^{\infty}$ is a Cauchy sequence. Thus this subsequence converges to some $x \in [0,1]$, since $[0,1]$ is closed.

Fall 2009 #1. (i) For each $n \in \mathbb{N}$ let $f_n : \mathbb{N} \to \mathbb{R}$ be a function with $|f_n(m)| \leq 1$ for all $m, n \in \mathbb{N}$. Prove that there is an infinite subsequence of distinct positive integers n_i, such that for each $m \in \mathbb{N}$, $f_{n_i}(m)$ converges.

(ii) For n_i as in (i), assume that in addition $\lim_{m\to\infty} \lim_{i\to\infty} f_{n_i}(m)$ exists and equals 0. Prove or disprove: The same holds for the reverse double limit $\lim_{i\to\infty} \lim_{m\to\infty} f_{n_i}(m)$.

(i) Consider the set Ω of functions from $\mathbb{N}$ to $\mathbb{R}$ whose images lie in $[-1,1]$. Define the norm

$$d(f,g) = \sup_{m\in\mathbb{N}} |f(m) - g(m)|$$

on Ω, so that (Ω, d) is a metric space. Note that $d(f,g) \leq 2$ for any $f, g \in \Omega$, so clearly (Ω, d) is totally bounded. It also follows easily that (Ω, d) is closed. Thus by a well-known theorem for metric spaces, Ω is sequentially compact, so there exists a subsequence $(f_{n_i})_{i=0}^{\infty}$ of $(f_n)_{n=0}^{\infty}$ such that $(f_{n_i})_{i=0}^{\infty}$ converges with respect to d. This implies that $(f_{n_i}(m))_{i=1}^{\infty}$ converges with respect to the usual norm on $\mathbb{R}$ for all $m \in \mathbb{N}$.

(ii) Consider $f_n(m) = 1$ for $n < m$ and $f_n(m) = 0$ for $n \geq m$. Then $f_n \in \Omega$, and

$$\lim_{m\to\infty} \lim_{i\to\infty} f_{n_i}(m) = \lim_{m\to\infty} 0 = 0.$$

However,

$$\lim_{i\to\infty} \lim_{m\to\infty} f_{n_i}(m) = \lim_{i\to\infty} 1 = 1.$$

This serves as a counterexample to the given statement.

Spring 2002 #4; Spring 2003 #1; Spring 2009 #6. (a) Define *uniform continuity* of a function $F : X \to \mathbb{R}$, X a metric space.

(b) Prove that a function $f : (0,1) \to \mathbb{R}$ is the restriction to $(0,1)$ of a continuous function $F : [0,1] \to \mathbb{R}$ if and only if f is uniformly continuous on $(0,1)$.

(a) $F : X \to \mathbb{R}$ is uniformly continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $d(x,y) < \delta$, $|F(x) - F(y)| \leq \varepsilon$.

(b) Suppose $F : [0,1] \to \mathbb{R}$ is continuous and $f = F|_{(0,1)}$. Show f is uniformly continuous. Let $\varepsilon > 0$. For each $x \in [0,1]$, let $\delta_x > 0$ be such that if $|x - y| \leq \delta_x$, then $|F(x) - F(y)| \leq \varepsilon$. Note that $\bigcup_{n\in\mathbb{N}} \{x : \delta_x < 1/(n+1)\} = [0,1]$. Since $[0,1]$ is compact, there exists a finite subcover of $[0,1]$ consisting of sets $\{x : \delta_x < 1/(n+1)\}$. Since these are decreasing as n increases, there exists some natural number N such that $\{x : \delta_x < 1/N\} = [0,1]$. Thus f is uniformly continuous. (Faster: F is continuous on a compact set, so F is uniformly continuous. Thus f is uniformly continuous.)

Suppose f is uniformly continuous on $(0,1)$. Show there exists a continuous function $F : [0,1] \to \mathbb{R}$ such that $F|_{(0,1)} = f$. Define $F : [0,1] \to \mathbb{R}$ such that $F(x) = f(x)$ for $x \in (0,1)$ and $F(0) = f(0+)$, $F(1) = f(1-)$. By construction, F is continuous at 0 and 1. Since f is uniformly continuous on $(0,1)$, F is continuous on $(0,1)$. Thus F is continuous on $[0,1]$. We also have $F|_{(0,1)} = f$ by construction.

Spring 2004 #2. Is $f(x) = \sqrt{x}$ uniformly continuous on $[0, \infty)$? Prove your assertion.

Spring 2006 #5. Prove that if $0 < \alpha < 1$, then $F(x) = x^\alpha$ is unformly continuous on $[0, \infty)$.

Fall 2008 #1. For which of the values $a = 0, 1, 2$ is the function $f(t) = t^a$ uniformly continuous on $[0, \infty)$? Prove your assertions.

Consider $f(t) = t^\alpha$ for $0 < \alpha < 1$. Let $\varepsilon > 0$ and take $\delta = \varepsilon/\alpha$. If $x, y \in [1, \infty)$ and $|x - y| < \delta$, then by the mean value theorem, there exists $c \in (x, y)$ such that

$$f(x) - f(y) = f'(c)(x - y) = \alpha c^{\alpha-1}(x - y) \leq \alpha(x - y) \leq \varepsilon.$$

Thus f is uniformly continuous on $[1, \infty)$. Now x^α is continuous on $[0, 1]$, so since $[0, 1]$ is compact, x^α is uniformly continuous on $[0, 1]$.

For each $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $x, y \in [0, 1]$ and $|x-y| \leq \delta_1$, then $|f(x)-f(y)| \leq \varepsilon/2$. There also exists $\delta_2 > 0$ such that if $x, y \in [1, \infty)$ and $|x - y| \leq \delta_2$, then $|f(x) - f(y)| \leq \varepsilon/2$. Let $\delta = \min(\delta_1, \delta_2)$. Suppose $x, y \in [0, \infty)$. If $x, y \in [0, 1]$ or $x, y \in [1, \infty)$, clearly $|f(x) - f(y)| \leq \varepsilon$. Otherwise, suppose without loss of generality that $x \in [0, 1]$ and $y \in [1, \infty)$. Then if $|x - y| \leq \delta$, $|x - 1| \leq \delta$ and $|y - 1| \leq \delta$, so

$$|f(x) - f(y)| \leq |f(x) - f(1)| + |f(1) - f(y)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus f is uniformly continuous on $[0, \infty)$.

Clearly $f(t) = t^a$ is uniformly continuous on $[0, \infty)$ for $a = 0, 1$. But $f(t) = t^2$ is not uniformly continuous on $[0, \infty)$. Let $\delta > 0$. Note that by selecting $x = n$, $y = n + \delta$ for $n \geq 1/(2\delta)$, we have $|x - y| \leq \delta$, but

$$|f(y) - f(x)| = 2n\delta + \delta^2 \geq 2n\delta \geq 1.$$

Thus f is not uniformly continuous on $[0, \infty)$.

Fall 2009 #6. Consider the function $f(x, y) = \sin^3(xy) + y^2|x|$ defined on the region $S \subset \mathbb{R}^2$ given by

$$S = \{(x, y) \in \mathbb{R}^2; x^{2010} + y^{2010} \leq 1\}.$$

Define what it means for f to be uniformly continuous on S and prove that f is indeed uniformly continuous. (You can use any theorem you wish in the proof, as long as it is stated correctly and you justify properly why it can be applied, e.g., if you are using a general theorem on continuous functions, show that the function in question is indeed continuous, and if you are using a metric property of a set explain why it has it.)

We say f is uniformly continuous on S if given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $(x, y), (x', y') \in S$ with $|(x, y) - (x', y')|_2 \leq \delta$, we have

$$|f(x, y) - f(x', y')| \leq \varepsilon.$$

Clearly the projections from $\mathbb{R}^2$ to $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ are continuous. Also, $|x|$ and $\sin(x)$ are continuous. Then $f(x, y)$ is the composition, addition, and product of continuous functions, hence it is continuous.

We show that S is a compact set, so since f is continuous on the compact set S, f is uniformly continuous on S. If $|x| > 1$ or $|y| > 1$, then $x^{2010} + y^{2010} > 1$, so $(x, y) \notin S$. Thus S is bounded. Let $((x_n, y_n))_{n=1}^\infty$ be a sequence of points in S which converges to some (x, y). By definition of S,

$$x_n^{2010} + y_n^{2010} \leq 1$$

for all n. Now $g((x, y)) = x^{2010} + y^{2010}$ is a continuous function on $\mathbb{R}^2$ (iterating that a product or sum of continuous functions is continuous). Thus $(g((x_n, y_n)))_{n=1}^\infty$ approaches $g(x, y)$. The terms in $(g((x_n, y_n)))_{n=1}^\infty$ are inside the closed set $[0, 1]$, hence $g(x, y) \in [0, 1]$. Hence $x^{2010} + y^{2010} \leq 1$, so S is closed. As a closed and bounded set in $\mathbb{R}^2$, S is compact.

Spring 2012 #3. Prove the Bolzano-Weierstrass theorem in the following form: Each sequence $(a_n)_{n\in\mathbb{N}}$ of numbers a_n in the closed interval $[0, 1]$ has a convergent subsequence.

Keep bisecting the range and continuing on in the region with infinitely many terms of the sequence.

Fall 2004 #6. The Bolzano-Weierstrauss Theorem in $\mathbb{R}^n$ states that if S is a bounded closed subset of $\mathbb{R}^n$ and (x_n) is a sequence which takes values in S, then (x_n) has a subsequence which converges to a point in S. Assume this statement known in case $n = 1$, and use it to prove the statement in case $n = 2$.

Suppose S is a bounded closed subset of $\mathbb{R}^2$ and $((x_n, y_n))_{n=1}^{\infty}$ is a sequence which takes values in S. Define S_x and S_y to be the projections of S into the first and second coordinates. Then $(x_n)_{n=1}^{\infty}$ is a sequence in S_x. By the Bolzano-Weierstrauss Theorem on $\mathbb{R}$, $(x_n)_{n=1}^{\infty}$ has a subsequence $(x_{n1})_{n=1}^{\infty}$ which converges to some $x \in S_x$.

By the Bolzano-Weierstrauss Theorem in $\mathbb{R}$, the sequence $(y_{n1})_{n=1}^{\infty}$ in S_y has a subsequence $(y_{n2})_{n=1}^{\infty}$ which converges to $y \in S_y$. It follows that $(x_{n2})_{n=1}^{\infty}$ converges to x in S_x and

$$|(x_{n2}, y_{n2}) - (x, y)|_2 = (|x_{n2} - a|^2 + |y_{n2} - y|^2)^{1/2}.$$

Letting $n \to \infty$, the expression on the right approaches 0, hence

$$((x_{n2}, y_{n2})_{n=1}^{\infty}$$

is a subsequence of

$$((x_n, y_n))_{n=1}^{\infty}$$

which converges to (x, y). Since S is closed, $(x, y) \in S$, so we have shown the Bolzano-Weierstrauss Theorem holds in $\mathbb{R}^2$.

Spring 2007 #10. Suppose the functions $f_n(x)$ on $\mathbb{R}$ satisfy:

(i) $0 \le f_n(x) \le 1$ for all $x \in \mathbb{R}$ and $n \ge 1$.

(ii) $f_n(x)$ is increasing in x for every $n \ge 1$.

(iii) $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in \mathbb{R}$, where f is continuous on $\mathbb{R}$.

(iv) $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$.

Show that $f_n(x) \to f(x)$ uniformly on $\mathbb{R}$.

For any $x, y \in \mathbb{R}$ with $x \le y$, by condition (ii),

$$f_n(x) - f_n(y) \le 0$$

for all $n \ge 1$. Taking the limit as $n \to \infty$ and using condition (iii)

$$f(x) - f(y) \le 0.$$

Thus f is increasing.

Taking the limit as $n \to \infty$ in condition (i) and using condition (iii), we find

$$0 \le f(x) \le 1$$

for all $x \in \mathbb{R}^n$.

Since $[a, b]$ is compact and f is continuous, f is uniformly continuous on $[a, b]$. Let $\varepsilon > 0$. Using condition (iv), select a and b such that $f(a) < \varepsilon/2$ and $f(b) > 1 - \varepsilon/2$. Then for any $x, y < a$, since f is increasing,

$$|f(x) - f(y)| \le f(a) - 0 \le \varepsilon/2 < \varepsilon.$$

Likewise, if $x, y > b$, then

$$|f(x) - f(y)| \le 1 - f(b) \le 1 - (1 - \varepsilon/2) = \varepsilon/2 < \varepsilon.$$

Now since $[a, b]$ is compact and f is continuous, f is uniformly continuous on $[a, b]$. Thus there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| \leq \varepsilon/2$. Choose δ small enough to prohibit $x < a$ and $y > b$. If $x < a$ and $a < y < b$ with $|x - y| \leq \delta$, then $|y - a| \leq \delta$, so

$$|f(x) - f(y)| \leq |f(x) - f(a)| + |f(a) - f(y)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus f is uniformly continuous on $\mathbb{R}$.

Let $\varepsilon > 0$. Select δ such that if $x, y \in [a, b]$ and $|x - y| \leq \delta$, then $|f(x) - f(y)| \leq \varepsilon/2$. Partition $[a, b]$ into a finite number of intervals of length less than δ. That is, choose $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that $x_{i+1} - x_i \leq \delta$ for all i. Since f_n converges to f pointwise, there exists N such that for all $n \geq N$, $|f(x_i) - f_n(x_i)| \leq \varepsilon/4$ for all i. For any $y \in [x_i, x_{i+1}]$ and any $n \geq N$,

$$|f_n(y) - f(y)| \leq |f_n(y) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(y)|$$

$$\leq (f(x_{i+1}) - \varepsilon/4) - (f(x_i) - \varepsilon/4) + |f_n(x_i) - f(x_i)| + |f(x_i) - f(y)|$$

$$\leq \varepsilon + \varepsilon/4 + \varepsilon/2 < 2\varepsilon.$$

Thus the f_n converge to f uniformly on $[a, b]$.

By the choice of a, for any $y < a$, and any $n \geq N$,

$$|f_n(y) - f(y)| \leq f_n(a) - 0 \leq f(a) + \varepsilon/4 \leq \varepsilon/2 + \varepsilon/4 < \varepsilon.$$

Likewise, for any $y > b$, and any $n \geq N$,

$$|f_n(y) - f(y)| \leq \varepsilon.$$

Thus $(f_n)_{n=1}^{\infty}$ converges uniformly to f on $\mathbb{R}$.

Spring 2010 #7. Let $\{f_n\}$ be a sequence of real-valued functions on the line, and assume that there is a $B < \infty$ such that $|f_n(x)| \leq B$ for all n and x. Prove that there is a subsequence $\{f_{n_k}\}$ such that $\lim_{k\to\infty} f_{n_k}(r)$ exists for all rational numbers r.

Define (Ω, d) as in Fall 2009 #1, with Ω consisting of function from $\mathbb{Q} \to \mathbb{R}$, and take d to be the supremum over rational inputs. Then $f_n|_{\mathbb{Q}} \in \Omega$ for each n. Then (Ω, d) is a metric space.

We see that Ω is totally bounded from the condition $|f_n(x)| \leq B$. Let $(f_m)_{m=1}^{\infty}$ be a sequence in Ω which converges to f with respect to d. Clearly $|f(x)| \leq B$ for all $x \in \mathbb{Q}$, so Ω is closed. Thus Ω is sequentially compact, so there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that this subsequence converges to some $f \in \Omega$ with respect to d. It follows that we have pointwise convergence on all rationals, as desired.

Spring 2004 #4. Are there infinite compact subsets of $\mathbb{Q}$? Prove your assertion.

Yes, $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ is an infinite compact subset of $\mathbb{Q}$. Clearly this set is bounded by 1, and it contains its only limit point of 0, hence it is closed. Thus it is compact.

Fall 2008 #2. Suppose that A is a non-empty connected subset of $\mathbb{R}^2$.

(a) Prove that if A is open, then it is path connected.

(b) Is part (a) true if A is closed? Prove your assertion.

(a) Let $a \in A$. Define H to be the subset of points in A which can be joined to a by a path in A. Let $K = A \setminus H$.

Let $x \in H$. Since A is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A$. Given any $y \in B_\varepsilon(x)$, there is a straight line path g in $B_\varepsilon(x) \subset A$ connecting x and y. Since $x \in H$, there is a path f in A joining a to x.

Thus traversing f and then g forms a path from a to y. It follows that $y \in H$, hence $B_\varepsilon(x) \subset H$. Thus H is open.

Let $x \in K$. Since A is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A$. If any point in $B_\varepsilon(x)$ could be joined to a by a path in A, then so could x, a contradiction. Hence $B_\varepsilon \subset K$, so K is open.

Clearly $H \cap K = \emptyset$, $H \cup K = A$, and $a \in H$, so H is non-empty. Since A is connected, we must have $K = \emptyset$, so $H = A$. Thus A is path connected.

(b) No. Consider the set

$$A = \{(x, \sin(\frac{1}{x})) : x \in (0,1]\} \cup \{(0,x) : x \in [-1,1]\}.$$

We show that A is closed and connected, but not path connected. For convenience, let $B = \{(x, \sin(\frac{1}{x})) : x \in (0,1]\}$ and $C = \{(0,x) : x \in [-1,1]\}$. Then C is the set of limit points of B which are not already in B. Since C is closed, $A = B \cup C$ is closed. Defining $f : (0,1] \to \mathbb{R}^2$ by $f(x) = (x, \sin(\frac{1}{x}))$, we see that f is continuous, thus since $(0,1]$ is connected $B = f((0,1])$ is connected as well. Clearly C is connected.

Suppose for the sake of contradiction that $A = B \cup C$ is not connected. Then there exist disjoint non-empty open sets U, V with $A \subset U \cup V$. Since B and C are connected, we can assume without loss of generality that $B \subset U$ and $C \subset V$. Now each element in C is a limit point of B, so every ball centered at some $(0,x) \in C$ must contain infinitely many elements of B. Since U is open, U and V are not disjoint, a contradiction. Hence A is connected.

Now any path in A connecting some point of B with some point of C is a continuous map $f : [0,1] \to A$ with $f(0) \in B$. But such a map must have $f([0,1])$ entirely contained within B. Thus there does not exist a path between an element of B and an element of C.

Spring 2011 #11. Show that a connected subset $A \subseteq \mathbb{R}$ is arcwise connected (= path-connected).

Let A be a connected subset of $\mathbb{R}$. It follows that A is an interval. Let $x, y \in A$. Then necessarily $[x,y] \subset A$. Define $f : [0,1] \to [x,y]$ by $f(t) = x + t(y-x)$. Clearly f is continuous, and $f(0) = x$ and $f(1) = y$, so f is a path connecting x and y. Hence A is path-connected.

Spring 2004 #6. Let $|| \; ||$ be any norm on $\mathbb{R}^n$.

(a) Prove that there exists a constant d with $||x|| \leq d||x||_2$ for all $x \in \mathbb{R}^n$, and use this to show that $N(x) = ||x||$ is continuous in the usual topology on $\mathbb{R}^n$.

(b) Prove that there exists a constant c with $||x|| \geq c||x||_2$. (Hint: use the fact that N is continuous on the sphere $\{x : ||x||_2 = 1\}$).

(c) Show that if L is an n-dimensional subspace of an arbitrary normed vector space V, then L is closed.

(a) Define $d = \sqrt{\sum ||e_i||^2}$. Write $x = \sum x_i e_i$. By the Cauchy-Schwarz inequality,

$$||x|| = ||\sum x_i e_i|| \leq \sum ||x_i e_i|| = \sum |x_i| ||e_i||$$

$$= (|x_1|, \ldots, |x_n|) \cdot (||e_1||, \ldots, ||e_n||) \leq \sqrt{\sum x_i^2} \cdot \sqrt{\sum ||e_i||^2} = d||x||_2.$$

Let $\varepsilon > 0$. For any $x, y \in \mathbb{R}^n$ with $||x-y||_2 \leq \varepsilon/(d+1)$, by the reverse triangle inequality,

$$||x|| - ||y|| \leq ||x-y|| \leq d||x-y||_2 \leq d(\varepsilon/(d+1)) \leq \varepsilon.$$

By symmetry, we conclude

$$|||x|| - ||y||| \leq \varepsilon.$$

Thus N is continuous with respect to the usual topology on $\mathbb{R}^n$.

(b) It follows from (a) that N is continuous on the sphere $T := \{x : ||x||_2 = 1\}$. Since T is a compact set, N achieves its minimum c on T. For any $x \in \mathbb{R}^n$ with $x \neq 0$, since $\frac{x}{||x||_2} \in T$,

$$||x|| = ||x||_2 || \frac{x}{||x||_2} || \geq c||x||_2.$$

(c) Since V is n-dimensional, there exists an isomorphism $T : \mathbb{R}^n \to V$. Define $|| \; || : \mathbb{R}^n \to \mathbb{R}$ by $||x|| = ||Tx||_V$.

From parts (a) and (b), there exist constants c and d such that for any $x \in \mathbb{R}^n$,

$$c||x||_2 \leq ||x|| \leq d||x||_2.$$

Let $(y_k)_{k=1}^{\infty}$ be a sequence of points in L that converges to some $v \in V$ with respect to the norm $|| \; ||_V$ of V. In particular, $(y_k)_{k=1}^{\infty}$ is a Cauchy sequence with respect to $|| \; ||_V$. It follows that

$$c||T^{-1}y_i - T^{-1}y_j||_2 \leq ||T^{-1}y_i - T^{-1}y_j|| = ||y_i - y_j||_V \leq d||T^{-1}y_i - T^{-1}y_j||_2.$$

Let $\varepsilon > 0$. There exists N such that for all $i, j \geq N$, $||y_i - y_j||_V \leq c\varepsilon$. It follows that

$$||T^{-1}y_i - T^{-1}y_j||_2 \leq \varepsilon.$$

Hence $(T^{-1}y_k)_{k=1}^{\infty}$ is a Cauchy sequence with respect to $|| \; ||_2$. Since this norm is complete, there exists some $T^{-1}y \in \mathbb{R}^n$ such that $(T^{-1}(y_n))_{n=1}^{\infty}$ converges to $T^{-1}(y)$. It follows that $(y_k)_{k=1}^{\infty}$ converges to y with respect to $|| \; ||_V$ (using the other side of the inequality above). Hence L is closed.

Fall 2005 #8. For a real $n \times n$ matrix A, let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be the associated linear mapping. Set $||A|| = \sup_{x \in R^n}\{||T_A x|| : ||x|| = 1\}$ (here $||x||$ = usual euclidean norm, i.e.,

$$||(x_1, \ldots, x_n)|| = (x_1^2 + \cdots + x_n^2)^{1/2}).$$

(a) Prove that $||A + B|| \leq ||A|| + ||B||$.

(b) Use part (a) to check that the set M of all $n \times n$ matrices is a metric space if the distance function d is defined by

$$d(A, B) = ||B - A||.$$

(c) Prove that M is a complete metric space with this "distance function". (Suggestion: The ij-th element of $A =< T_A e_j, e_i >$, here $e_i = (0, \ldots, 1, \ldots, 0)$, with a 1 in the ith position.)

(a) By definition of the matrix norm, for all $x \in \mathbb{R}^n$, it follows that

$$||T_A x|| \leq ||A|| ||x||.$$

Now let $x \in \mathbb{R}^n$ with $||x|| = 1$. Then

$$||T_{A+B}x|| = ||T_A x + T_B x|| \leq ||T_A x|| + ||T_B x|| \leq ||A|| \cdot ||x|| + ||B|| \cdot ||x||$$

$$= (||A|| + ||B||) \cdot ||x|| \leq ||A|| + ||B||.$$

Thus taking the supremum over all such x,

$$||A + B|| \leq ||A|| + ||B||.$$

(b) Clearly this distance function is non-negative and symmetric. If $d(A, B) = 0$, then $||B - A|| = 0$. In particular, this implies that $||(T_B - T_A)(e_i)|| = 0$ for all i, so $T_B - T_A = 0$. This implies that $A = B$.

Finally, for matrices A, B, C, by part (a),

$$d(A,B) = ||B-A|| \le ||(B-C)+(C-A)|| \le ||B-C|| + ||C-A|| = d(B,C) + d(C,A).$$

Thus d is a metric on real $n \times n$ matrices.

(c) Suppose $(A_n)_{n=1}^{\infty}$ is a Cauchy sequence with respect to the matrix norm. Thus there exists N such that for all $j, k \ge N$,

$$||A_j - A_k|| \le \varepsilon.$$

This implies that for all $j, k \ge N$,

$$\sum_{\ell=1}^{n} (T_{A_j} - T_{A_k})_{\ell i}^2 = ||(T_{A_j} - T_{A_k})e_i|| \le \varepsilon.$$

Taking $\varepsilon < 1$, it follows that each entry of $T_{A_j} - T_{A_k}$ is less than ε. Thus $((T_{A_n})_{ij})_{n=1}^{\infty}$ forms a Cauchy sequence for each i, j. Define a matrix A by

$$A_{ij} = \lim_{n\to\infty} (T_{A_n})_{ij}.$$

It follows that $(A_n)_{n=1}^{\infty}$ converges to A with respect to the matrix norm. Thus the set of n by n matrices M is a complete metric space with this distance function.

Spring 2005 #10. Consider the set of $f : [0,1] \to \mathbb{R}$ that obey

$$|f(x) - f(y)| \le |x-y| \text{ and } \int_0^1 f(x)dx = 1.$$

Show that this is a compact subset of $C([0,1])$.

Call this set of functions S. The first condition clearly implies that $S \subset C([0,1])$. We will show that S is closed, bounded, and equicontinuous, thus by the Arzela-Ascoli theorem, S is compact.

Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in S converging to f. We want to show $f \in S$. Let $\varepsilon > 0$. Select N such that for all $n \ge N$,

$$\sup_{x\in[0,1]} |f(x) - f_n(x)| \le \varepsilon/2.$$

Then for any $x, y \in [0,1]$,

$$|f(x)-f(y)| \le |f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)| \le \varepsilon + |x-y|.$$

Letting $\varepsilon \to 0$, we have

$$|f(x) - f(y)| \le |x-y|.$$

Since $(f_n)_{n=1}^{\infty}$ converges to f in the supremum norm, they converge uniformly to f. Since the f_n are continuous, it follows that f is continuous. Also, because we have uniform convergence,

$$\int_0^1 f(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} 1 = 1.$$

Hence $f \in S$.

Let $f \in S$. Suppose for the sake of contradiction that there exists $y \in [0,1]$ such that $f(y) \ge 3$. Then for any $x \in [0,1]$,

$$|f(x) - f(y)| \le |x-y| \le 1,$$

hence $f(x) \ge 2$. Then

$$\int_0^1 f(x) \ge \int_0^1 2 = 2,$$

a contradiction. Thus $f(y) \leq 3$ for all $y \in [0,1]$. Likewise, $f(y) \geq -3$ for all $y \in [0,1]$, so f is uniformly bounded.

Finally, uniform equicontinuity on S follows immediately from the first condition of functions in S.

Spring 2006 #6. Let W be the subset of the space $C[0,1]$ of real-valued continuous functions on $[0,1]$ satisfying the conditions:

$$|f(x) - f(y)| < |x - y| \text{ and } \int_0^1 f(x)^2 dx = 1.$$

Edit: The first condition must be strict.

(a) Prove that W is uniformly bounded, i.e., there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [0,1]$. Hint: Show first that $|f(0)| \leq 2$ for all $f \in W$.

(b) Prove that W is a compact subset of $C[0,1]$ under the sup norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$.

Essentially the same proof as the previous exercise. Note f_n^2 converges uniformly to f^2.

Fall 2007 #1. Let S be a subset of $\mathbb{R}^n$ with the distance function $d(x,y) = ((x_1-y_1)^2+\cdots+(x_n-y_n)^2)^{1/2}$ so that $(S, d|_{S\times S})$ is a metric space.

(a) Given $y \in S$, is $E = \{x \in S : d(x,y) \geq r\}$ a closed set in S?

(b) Is the set E in part (a) contained in the closure of $\{x \in S : d(x,y) > r\}$ in S?

(a) Yes. Let $x \in S$ be a limit point of E. Then there exists a sequence $(x_n)_{n=1}^\infty$ in E converging to x. We have

$$d|_{S\times S}(x_n, y) = d(x_n, y) \geq r$$

for each $n \geq 1$. Let $\varepsilon > 0$. Select N such that for all $n \geq N$, $d(x_n, x) < \varepsilon$. It follows that

$$d(x_n, y) \leq d(x_n, x) + d(x, y)$$

thus

$$d(x,y) \geq d(x_n, y) - d(x_n, x) \geq r - \varepsilon.$$

Since ε was arbitrary, $d(x,y) \geq r$. Thus because we assumed $x \in S$, it follows that $x \in E$. Thus E is a closed set in S.

(b) No. Let $S = \{0\} \cup \{1\} \cup \{2\} \subset \mathbb{R}$, $y = 0$, and $r = 1$. Then

$$\{x \in S : d(x,y) > r\} = \{x \in S : d(x,0) > 1\} = \{2\}.$$

The closure of this set in S is $\{2\}$, but

$$E := \{x \in S : d(x,y) \geq r\} = \{x \in S : d(x,0) \geq 1\} = \{1\} \cup \{2\},$$

which is not a subset of $\{2\}$.

Spring 2007 #12. Let c_0 be the normed space of real sequences $x = (x_1, x_2, \ldots)$ such that $\lim_{k\to\infty} x_k = 0$ with the supremum norm $||x|| = \sup_k |x_k|$.

(a) Show that c_0 is complete.

(b) Is the unit ball $\{x \in c_0 : ||x|| \leq 1\}$ compact? Prove your answer.

(c) Is the set $E = \{x \in c_0 : \sum_k k|x_k| \leq 1\}$ compact? Prove your answer.

(a) Let $(x^{(m)})_{m=0}^{\infty}$ be a Cauchy sequence of elements of c_0. Then for any $n, m, k \in \mathbb{N}$, we have

$$|x_k^{(m)} - x_k^{(n)}| \leq ||x^{(m)} - x^{(n)}||$$

so that $x_k^{(m)}$ is a Cauchy sequence of real numbers for each k and thus converges. Define x so that

$$x_k = \lim_{m\to\infty} x_k^{(m)}.$$

We show that x is in c_0. Let $\varepsilon > 0$. Since $x^{(m)}$ is a Cauchy sequence, let N be an integer such that $m, n \geq N$ implies

$$||x^{(m)} - x^{(n)}|| < \varepsilon/2.$$

Since $x^{(N)} \in c_0$, choose M such that $k \geq M$ implies

$$|x_k^{(N)}| < \varepsilon/2.$$

Then for $k \geq M$, we have

$$|x_k| \leq |x_k - x_k^{(N)}| + |x_k^{(N)}| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $x \in c_0$.

For $m \geq N$ and $k \in \mathbb{N}$, we also have

$$|x_k - x_k^{(m)}| \leq |x_k^{(n)} - x_k^{(m)}| + |x_k^{(n)} - x_k|.$$

For large enough n, by the definition of x_k, this is small. Since k was arbitrary,

$$||x - x^{(m)}|| \leq \varepsilon$$

for $m \geq N$, so $x^{(m)} \to x$. Hence c_0 is complete.

(b) No. Suppose for the sake of contradiction that $\{x \in c_0 : ||x|| \leq 1\}$ is compact. For each $n \in \mathbb{N}$, define the sequence $x^{(n)}$ by $x_k^{(n)} = 1$ if $k \leq n$ and 0 if $k > n$. Then each $x^{(n)} \in c_0$, and $||x^{(n)}|| = 1$, so $x^{(n)} \in \{x \in c_0 : ||x|| \leq 1\}$. By sequential compactness and completeness, there exists a subsequence $(x^{(n_k)})_{k=1}^{\infty}$ which converges to some $x \in \{x \in c_0 : ||x|| \leq 1\}$. Thus as $k \to \infty$,

$$||x - x^{(n_k)}|| \to 0.$$

In particular, for each j,

$$|x_j - 1| = \lim_{k\to\infty} |x_j - x_j^{(n_k)}| = 0.$$

Hence $x_j = 1$. But then $\lim_{j\to\infty} x_j = 1$, so $x \notin c_0$, a contradiction.

(c) Yes. Let $x^{(n)}$ be a sequence in E. Note that $|x_k^{(n)}| \leq 1$ for all $k \in \mathbb{N}$, so for each k, $\{x_k^{(n)}\}$ has a convergent subsequence. Thus we may choose a subsequence $\{y^{(n,1)}\}_{n=1}^{\infty}$ of $x^{(n)}$ and $y_1 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} y_1^{(n,1)} = y_1.$$

Continuing, as in the proof of the Arzela-Ascoli Theorem, we may construct subsequences $y^{(n)}$ of $x^{(n)}$ and numbers $y_1, y_2, \ldots$ such that

$$\lim_{k\to\infty} y_k^{(n)} = y_k$$

for all k. We claim $y = (y_1, y_2, \ldots) \in E$ and that $\lim_{n\to\infty} y^{(n)} = y$.

Fix an integer N. Then

$$\sum_{k=1}^{N} k|y_k| = \lim_{n\to\infty} \sum_{k=1}^{N} k|y_k^{(n)}| \leq 1,$$

since $y_k^{(n)} \in E$. Letting $N \to \infty$, it follows that $y \in E$.

Let $\varepsilon > 0$. Select N large enough so that $\frac{1}{N} < \varepsilon$. Then choose $N_1 \geq N$ such that

$$||y_k^{(n)} - y_k|| < \varepsilon$$

for $k = 1, \ldots, N$ and all $n \geq N_1$. This is valid since $\lim_{n\to\infty} y_k^{(n)} = y_k$ and $\{1, \ldots, N\}$ is finite. Then for $n \geq N_1$ and $k \geq N$, we have that

$$|y_k^{(n)} - y_k| \leq |y_k^{(n)}| + |y_k| \leq 1/N + 1/N = 2/N < 2\varepsilon,$$

whereas for $k < N$ we have that $|y_k^{(n)} - y_k| < \varepsilon$ by construction. Thus for $n \geq N$ we have that

$$||y^{(n)} - y|| < 2\varepsilon$$

so that $y^{(n)}$ is a subsequence of $x^{(n)}$ which converges to y. Hence E is sequentially compact and thus compact.

Spring 2012 #2. Recall that $f : [a, b] \to \mathbb{R}$ is convex if for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$. Let $f_n : [a, b] \to \mathbb{R}$ be convex functions and suppose that $f(x) := \lim_{n\to\infty} f_n(x)$ exists at all $x \in [a, b]$ and is continuous on $[a, b]$. Prove that $f_n \to f$ uniformly.

Assume for the sake of contradiction that f_n does not converge uniformly to f. Let $\varepsilon > 0$. Pass to a subsequence where $|f_n - f|_{\sup} \geq \varepsilon$. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ with elements in $[a, b]$ such that $|f_n(x_n) - f(x_n)| \geq \varepsilon/2$. By compactness of $[a, b]$, there exists a subsequence (also denoted by $(x_n)_{n=1}^{\infty}$) of $(x_n)_{n=1}^{\infty}$ which converges to some $x \in [a, b]$. We may choose x_n to be monotone and without loss of generality, assume they are non-decreasing.

Now we localize the problem to $[a_1, b_1]$, where $b_1 = x$. Since f is continuous, there exists c such that

$$|f(z) - c| < \varepsilon/20$$

for all $z \in [a_1, b_1]$ (choose a_1 such that this holds).

We will use convexity of the f_n to show pointwise convergence fails somewhere. We are given that $f_n(a_1) \to f(a_1)$ and $f_n(b_1) \to f(b_1)$ as $n \to \infty$. Choose N large enough that

$$|f_n(a_1) - c| \leq \varepsilon/10 \text{ and } |f_n(b_1) - c| \leq \varepsilon/10.$$

From the construction of x_n, for all n,

$$f_n(x_n) \leq f(x_n) - \varepsilon/2.$$

Consider the line from $(a_1, c + \varepsilon/10)$ to $(b_1, c - \varepsilon/4)$. This line intersects the horizontal line at $c - \varepsilon/10$ at some point $z \in (a_1, b_1)$. By convexity of f_n, $f_n(z)$ must lie below this line on $[a_1, x_n]$, so choosing n large enough that $z < x_n$,

$$f_n(z) < c - \varepsilon/10.$$

Since this holds for each n and $f_n(z) \to f(z)$, taking $n \to \infty$ yields

$$f(z) \leq c - \varepsilon/10.$$

But this contradicts $|f(z) - c| < \varepsilon/20$ above. Hence f_n does converge uniformly to f.

Fixed point

Spring 2008 #1. Let $g \in C([a,b])$, with $a \leq g(x) \leq b$ for all $x \in [a,b]$. Prove the following:

(i) g has at least one fixed point p in the interval $[a,b]$.

(ii) If there is a value $\gamma < 1$ such that

$$|g(x) - g(y)| \leq \gamma |x - y|$$

for all $x, y \in [a,b]$, then the fixed point is unique, and the iteration

$$x_{n+1} = g(x_n)$$

converges to p for any initial guess $x_0 \in [a,b]$.

(i) Note that $g(x) - x$ is continuous on $[a,b]$, non-negative at $x = a$ and non-positive at $x = b$. Thus by the intermediate value theorem, there exists some $p \in [a,b]$ such that $g(p) - p = 0$, or $g(p) = p$.

(ii) Suppose there exists $\gamma < 1$ such that

$$|g(x) - g(y)| \leq \gamma |x - y|$$

for all $x, y \in [a,b]$. Suppose both $p, q \in [a,b]$ were fixed points of g. Then if $p \neq q$,

$$|p - q| = |g(p) - g(q)| \leq \gamma |p - q| < |p - q|,$$

a contradiction. Hence $p = q$, so the fixed point of g is unique.

Let $x_0 \in [a,b]$ be an initial guess and consider the iterated sequence

$$x_{n+1} = g(x_n)$$

defined for all $n \geq 0$. We first show that $(x_n)_{n=0}^{\infty}$ converges. Follow the procedure below to show this is a Cauchy sequence and converges to a fixed point of g. Since the fixed point of g is unique, it must be that $(x_n)_{n=0}^{\infty}$ converges to p.

Fall 2009 #2. (i) Let X be a complete metric space with respect to a distance function d. We say that a map $T : X \to X$ is a *contraction* if for some $0 < \lambda < 1$ and all $x, y \in X$:

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

Prove that if T is a contradiction then it has a fixed point, i.e., there is an $x \in X$ such that $T(x) = x$.

(ii) Using (i) show that given a differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose first derivative satisfies $f'(x) = e^{-x^2} - e^{-x^4}$ there exists $\alpha \in \mathbb{R}$ with $f(\alpha) = \alpha$.

(i) Let $T : X \to X$ be a contraction. We assume X is non-empty, so there exists $x_0 \in X$. Inductively define $x_{n+1} = T(x_n)$ for all $n \geq 0$. A simple induction shows that for any natural number n,

$$d(x_{n+1}, x_n) = d(T^{n+1}(x_0)), T^n(x_0)) \leq \lambda^n (T(x_0), x_0).$$

Thus for any natural numbers n, m,

$$d(x_n, x_{n+m}) \leq \sum_{i=0}^{m-1} d(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{m-1} \lambda^{n+i} d(x_0, T(x_0))$$

$$= \lambda^n d(x_0, T(x_0)) \sum_{i=0}^{m-1} \lambda^i = \lambda^n d(x_0, T(x_0)) \frac{1-\lambda^m}{1-\lambda} \le \frac{\lambda^n}{1-\lambda} d(x_0, T(x_0)).$$

Let $\varepsilon > 0$. Choose N so that $\frac{\lambda^N}{1-\lambda} d(x_0, T(x_0)) < \varepsilon$. Then for any $n, m \ge N$, we have shown

$$d(x_n, x_m) \le \frac{\lambda^n}{1-\lambda} d(x_0, T(x_0)) \le \frac{\lambda^N}{1-\lambda} d(x_0, T(x_0)) < \varepsilon.$$

Thus the sequence $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence. Because X is complete, this sequence converges to some $x^* \in X$.

Since T is Lipschitz continuous on X, T is continuous on X. Proof: Let $\delta = \varepsilon/\lambda$.

Recall for all $n \ge 0$ we defined

$$x_{n+1} = T(x_n).$$

Thus taking the limit as $n \to \infty$ on both sides and using the continuity of T,

$$x^* = \lim_{n\to\infty} T(x_n) = T(x^*).$$

Hence x^* is a fixed point of T.

(ii) Note that e^{-x^2} and e^{-x^4} are even functions and decreasing on $[0, \infty)$. Thus for any $x \in [0, 1]$,

$$e^{-x^2} - e^{-x^4} \le e^{-x^2} - e^{-1^4} = e^{-x^2} - \frac{1}{e} \le e^{-0^2} - \frac{1}{e} = 1 - 1/e.$$

Likewise, for any $x \in [0, 1]$,

$$-(e^{-x^2} - e^{-x^4}) = e^{-x^4} - e^{-x^2} \le 1 - 1/e.$$

Thus for any $x \in [0, 1]$,

$$|e^{-x^2} - e^{-x^4}| \le 1 - 1/e.$$

For $x > 1$, $x^4 > x^2$, hence $-x^4 < -x^2$, so $e^{-x^2} - e^{-x^4} > 0$.

$$|e^{-x^2} - e^{-x^4}| = e^{-x^2} - e^{-x^4} \le e^{-1^2} = 1/e.$$

Thus for all $x \in \mathbb{R}$,

$$|f'(x)| = |e^{-x^2} - e^{-x^4}| \le 1 - 1/e.$$

Let $x, y \in \mathbb{R}$. By the mean value theorem, there exists $z \in (x, y)$ such that

$$f(x) - f(y) = f'(z)(x - y).$$

Thus

$$|f(x) - f(y)| \le |f'(z)||x - y| \le (1 - 1/e)|x - y|.$$

Thus f is a contraction, so by part (a), there exists $\alpha \in \mathbb{R}$ with $f(\alpha) = \alpha$.

Fall 2011 #1. Let (X, d) be a compact metric space and let $f : X \to X$ be a map satisfying

$$d(f(x), f(y)) < d(x, y), \qquad \text{for all } x, y \in X \text{ with } x \ne y.$$

Prove that there is a unique point $x \in X$ so that $f(x) = x$.

Define $\delta : X \to \mathbb{R}$ by $\delta(x) = d(x, f(x))$. For any $x, y \in X$,

$$\delta(y) = d(y, f(y)) \le d(y, x) + d(x, f(x)) + d(f(x), f(y)) \le 2d(x, y) + \delta(x),$$

thus

$$\delta(y) - \delta(x) \le 2d(x, y).$$

By symmetry, this implies

$$|\delta(y) - \delta(x)| \leq 2d(x, y).$$

Let $\varepsilon > 0$. Then for any $x, y \in X$ with $d(x, y) < \varepsilon/2$.

$$|\delta(x) - \delta(y)| \leq \varepsilon,$$

so δ is a continuous function.

Since X is compact and δ is continuous, there exists $x_0 \in X$ such that $\delta(x_0) \leq \delta(x)$ for all $x \in X$. Suppose for the sake of contradiction that $\delta(x_0) \neq 0$. Then $x_0 \neq f(x_0)$, so

$$\delta(f(x_0)) = d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = \delta(x_0),$$

a contradiction. Hence $\delta(x_0) = 0$, so $d(x_0, f(x_0)) = 0$, and $f(x_0) = x_0$. Thus f has a fixed point.

Suppose there exist $x, y \in X$ with $f(x) = x$ and $f(y) = y$. If $x \neq y$, then

$$d(x, y) = d(f(x), f(y)) < d(x, y),$$

a contradiction. Hence $x = y$, so there is a unique point $x \in X$ so that $f(x) = x$.

Spring 2011 #12. Fall 2004 #5; Fall 2003 #7; Fall 2001 #3. (see text). Given a metric space M, and a constant $0 < r < 1$, a continuous function $T : M \to M$ is said to be an *r-contraction* if it is a continuous map and $d(T(x), T(y)) < rd(x, y)$ for all $x \neq y$. A well-known fixed point theorem states that if M is complete and T an r-contraction, then it must have a unique fixed point (don't prove this). This result is often used to prove the existence of solutions of differential equations with initial conditions.

1. Illustrate this technique for the (trivial) case

$$f'(t) = f(t), f(0) = 1$$

by letting M be the space of continuous functions $C([0, c])$ for $0 < c < 1$ with the uniform distance

$$d(f, g) = \sup\{|f(t) - g(t)|\},$$

and defining $(Tf)(x) = 1 + \int_0^x f(t)dt$. Carefully explain your steps.

2. What approximations do you obtain from the sequence

$$T(0), T^2(0), T^3(0), \ldots?$$

1. Let $0 < c < 1$ and M be the space of continuous functions $C([0, c])$ with the distance

$$d(f, g) = \sup_{t \in [0,c]} \{|f(t) - g(t)|\}.$$

Define $T : M \to M$ by $(Tf)(x) = 1 + \int_0^x f(t)dt$. For some $f \in M$, since f is continuous on a compact set there exists $B > 0$ such that $|f| \leq B$ on $[0, c]$. Let $\varepsilon > 0$. Then for any $x, y \in [0, c]$ with $x \leq y$ and $|x - y| \leq \varepsilon/B$,

$$|(Tf)(y) - (Tf)(x)| = |\int_x^y f(t)dt| \leq \int_x^y Bdt = B(y - x) \leq B(\varepsilon/B) = \varepsilon.$$

Hence Tf is continuous, so $Tf \in M$, thus T is well-defined.

It is well known that M is complete.

Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in (M, d). Let $\varepsilon > 0$. Then there exists an integer $N \geq 1$ such that for all $n, m \geq N$,

$$d(f_n, f_m) \leq \varepsilon.$$

In particular, for any $t \in [0, c]$,

$$|f_n(t) - f_m(t)| \leq \varepsilon.$$

Hence $(f_n(t))_{n=1}^{\infty}$ is a Cauchy sequence for each $t \in [0, c]$. Since $\mathbb{R}$ is complete, we can define $f : [0, c] \to \mathbb{R}$ by

$$f(t) = \lim_{n \to \infty} f_n(t).$$

We now show that $(f_n)_{n=1}^{\infty}$ converge to f uniformly with respect to d. Then since the limit of a sequence of continuous functions that converges uniformly is a continuous function, f is continuous, so $f \in M$. Thus we will have shown that M is complete.

Let $\varepsilon > 0$. Choose N so that for all $n, m \geq N$, $d(f_m, f_n) \leq \varepsilon$. Then for any $t \in [0, c]$,

$$|f(t), f_n(t)| \leq |f(t) - f_m(t)| + |f_m(t) - f_n(t)| \leq |f(t) - f_m(t)| + d(f_m, f_n) \leq |f(t) - f_m(t)| + \varepsilon.$$

Letting $m \to \infty$,

$$|f(t), f_n(t)| \leq \varepsilon.$$

Taking the sup of both sides over all $t \in [0, c]$,

$$d(f, f_n) \leq \varepsilon.$$

Hence $(f_n)_{n=1}^{\infty}$ converges to f uniformly with respect to d.

Let $\varepsilon > 0$. Select N so that $d(f_N, f) \leq \varepsilon/3$. Since f_N is continuous, there exists $\delta > 0$ so that if $|x - y| \leq \delta$, then $|f_N(x) - f_N(y)| \leq \varepsilon/3$. For any $x, y \in [0, c]$ with $|x - y| \leq \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \leq 2d(f, f_n) + |f_n(x) - f_n(y)| \leq 2\varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Hence f is continuous.

We now show T is continuous. Let $\varepsilon > 0$ and suppose $f, g \in M$ with $d(f, g) \leq \varepsilon/c$. Then

$$d(Tf, Tg) = \sup_{t \in [0,c]} |(Tf)(t) - (Tg)(t)| = \sup_{t \in [0,c]} |\int_0^t f(t) - g(t)dt| \leq c \sup_{t \in [0,c]} (f - g) = c(d(f, g)) \leq c(\varepsilon/c) = \varepsilon.$$

Thus T is continuous.

Let $f, g \in M$ with $f \neq g$. It follows that

$$d(Tf, Tg) = \sup_{t \in [0,c]} \{|(Tf)(t) - (Tg)(t)|\} = \sup_{t \in [0,c]} \left\{ \left| \int_0^t f(z) - g(z)dz \right| \right\} \leq c \sup_{z \in [0,c]} |f(z) - g(z)| = c(d(f, g)).$$

Thus T is a $(c + \varepsilon)$-contraction mapping. (Choose ε so that $c + \varepsilon < 1$.) By the given fixed point theorem, there exists a fixed point F of T.

We now show F satisfies the differential equation and initial condition. Clearly

$$F(0) = (TF)(0) = 1 + \int_0^0 F(t)dt = 1.$$

By the fundamental theorem of calculus, for any $x \in [0, c]$,

$$F'(x) = (TF)'(x) = \frac{d}{dx}(1 + \int_0^x F(t)dt) = F(x).$$

Thus $F'(x) = F(x)$ for all $x \in [0, c]$. Hence F is a solution to the given differential equation on $[0, c]$.

2. Here we start with the constant 0 function, which is in M, and iterate T. The proof of the given fixed point theorem shows that iterating T on any function in M will converge to the fixed point. Hence the sequence $T(0), T^2(0), T^3(0), \ldots$ converges to the solution F found in part 1.

As part of the proof, we obtain that for any natural numbers n, m,

$$d(T^n(0), T^{n+m}(0)) \leq c^n \frac{1 - c^m}{1 - c} d(T(0), 0).$$

Thus

$$d(F, T^n(0)) \leq d(F, T^{n+m}(0)) + d(T^{n+m}(0), T^n(0)) \leq d(F, T^{n+m}(0)) + c^n \frac{1-c^m}{1-c} d(T(0), 0).$$

Note that $d(T(0), 0) = d(1, 0) = 1$. Letting $m \to \infty$,

$$d(F, T^n(0)) \leq \frac{c^n}{1-c} d(T(0), 0) = \frac{c^n}{1-c}.$$

Thus for any $t \in T$,

$$|F(t) - (T^n(0))(t)| \leq \frac{c^n}{1-c}.$$

Spring 2007 #7. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function with f'' uniformly bounded, and with a simple root at x^* (i.e., $f(x^*) = 0$, $f'(x^*) \neq 0$). Consider the fixed point iteration

$$x_n = F(x_{n-1}) \qquad \text{where} \qquad F(x) = x - \frac{f(x)}{f'(x)}.$$

Show that if x_0 is sufficiently close to x^*, then there exists a constant C so that for all n,

$$|x_n - x^*| \leq C|x_{n-1} - x^*|^2.$$

Note f' is continuous, so it is bounded away from zero in some open neighborhood U. Say $0 < C < f'(x) < D$ there, without loss of generality. Now suppose $x_0 \in U$ and by induction that $x_1, \ldots, x_{n-1} \in U$.

By the mean value theorem, there exists y_{n-1} between x^* and x_{n-1} such that

$$|x_n - x^*| = |F(x_{n-1}) - x^*| = |F(x_{n-1} - F(x^*)| = |x_{n-1} - x^*||F'(y_{n-1})|.$$

Now

$$F'(x) = \frac{f(x)f''(x)}{(f'(x))^2}.$$

Thus

$$|x_n - x^*| = |x_{n-1} - x^*|\frac{|f(y_{n-1})||f''(y_{n-1})|}{|f'(y_{n-1})|^2}.$$

Apply the mean value theorem again to obtain z_{n-1} between x^* and y_{n-1} such that $f(y_{n-1}) - f(x^*) = (y - x^*)f'(z_{n-1})$. Since $f(x^*) = 0$, we obtain

$$|x_n - x^*| = |x_{n-1} - x^*|\frac{|y_{n-1} - x^*||f'(z_{n-1})||f''(y_{n-1})|}{|f'(y_{n-1})|^2}.$$

Note $y_{n-1}, z_{n-1} \in U$. Thus

$$\frac{|f'(z_{n-1})|}{|f'(y_{n-1})|^2} \leq \frac{D}{C^2}.$$

Say the second derivative is uniformly bounded by M. It follows that

$$|x_n - x^*| \leq \frac{DM}{C^2}|x_{n-1} - x^*|^2.$$

If we forced $|x_{n-1} - x^*| \leq \frac{C^2}{MD}$, then $|x_n - x^*| \leq |x_{n-1} - x^*|$, hence $x_n \in U$, completing the induction.

Inverse Function Theorem, Implicit Function Theorem

Inverse Function Theorem. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function which has invertible derivative at a point p (the Jacobian determinant of f at p is non-zero). Then the inverse function f^{-1} exists and is continuously differentiable in some neighborhood of $f(p)$. Also,

$$J_{f^{-1}}(f(p)) = [J_f(p)]^{-1}.$$

Implicit Function Theorem. Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function, and let $\mathbb{R}^{n+m}$ have coordinates $(\mathbf{x}, \mathbf{y})$. Fix a point $(\mathbf{a}, \mathbf{b}) = (a_1, \dots, a_n, b_1, \dots, b_m)$ with $f(\mathbf{a}, \mathbf{b}) = \mathbf{c}$, where $\mathbf{c} \in \mathbb{R}^m$. If the matrix $[(\partial f_i/\partial y_j)(\mathbf{a}, \mathbf{b})]$ is invertible, then there exists an open set U containing $\mathbf{a}$, an open set V containing $\mathbf{b}$, and a unique continuously differentiable function $g : U \to V$ such that

$$\{(\mathbf{x}, g(\mathbf{x})) : \mathbf{x} \in U\} = \{(\mathbf{x}, \mathbf{y}) \in U \times V : f(\mathbf{x}, \mathbf{y}) = \mathbf{c}\}.$$

Fall 2001 #6. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuously differentiable function with $F((0,0)) = (0,0)$ and with the Jacobian of F at $(0,0)$ equal to the identity matrix (i.e., if $F = (f_1, f_2)$ then $\frac{\partial f_i}{\partial x_j}|_{(0,0)} = 1$ if $i = j$ and $= 0$ if $i \neq j$). Outline a proof that there exists $\delta > 0$ such that if $a^2 + b^2 < \delta$, then there is a point (x, y) in $\mathbb{R}^2$ with $F(x, y) = (a, b)$. (Your argument will be part of the proof of the Inverse Function Theorem. You may use any basic estimation you need about the change in F being approximated by the differential of F without proof.)

See standard proof of the Inverse Function Theorem.

Fall 2004 #7. Observe that the point $P = (1, 1, 1)$ belongs to the set S of points in $\mathbb{R}^3$ satisfying the equations

$$x^4y^2 + x^2z + yz^2 = 3.$$

Explain carefully how, in this case, the Implicit Function Theorem allows us to conclude that there exists a differentiable function $g(x, y)$ such that $(x, y, g(x, y))$ lie in S for all (x, y) in a small open set containing $(1, 1)$.

Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x^2y^3 + x^3z + 2yz^4$. Here $n = 2$ and $m = 1$. Here $\mathbf{a} = (1, 1)$ and $b = 1$. Note that $f(\mathbf{a}, b) = (1, 1, 1) = 3$ and f is continuously differentiable. The relevant matrix of partial derivatives is

$$[(\partial f/\partial z)(\mathbf{a}, b)] = [a_1^3 + 8a_2b^3] = [9]$$

which is clearly invertible. Thus by the Implicit Function Theorem, there exists an open set U containing $\mathbf{a} = (1, 1)$, an open set V containing $b = 1$ and a unique continuously differentiable function $g : U \to V$ such that

$$\{(x, y, g(x, y)) : (x, y) \in U\} = \{(x, y, z) \in U \times V : f(x, y, z) = 3\} \subset \{(x, y, z) : f(x, y, z) = 3\} =: S.$$

Thus $(x, y, g(x, y))$ lie in S for all $(x, y) \in U$, where U is a small open set containing $(1, 1)$.

Spring 2006 #4. Instead use

$$x^2y^3 + x^3z + 2yz^4 = 4.$$

Prove that there exists a differentiable function $g(x, y)$ defined in an open neighborhood N of $(1, 1)$ in $\mathbb{R}^2$ such that $g(1, 1) = 1$ and $(x, y, g(x, y))$ lies in S for all $(x, y) \in N$.

Define $f(x, y, z) = x^2y^3 + x^3z + 2yz^4$. Note $f(1, 1) = 4$ and f is continuously differentiable. The matrix of partial derivatives under consideration is

$$[(\partial f/\partial z)(1, 1, 1)] = [[(x, y, z) \mapsto (x^3 + 6yz^3)](1, 1, 1)] = [7],$$

which is clearly invertible. Hence the Implicit Function Theorem guarantees the existence of an open set N containing $(1,1)$, an open set V containing 1, and a unique continuously differentiable function $g(x,y)$ defined in U such that

$$\{(x,y,g(x,y)) : (x,y) \in N\} = \{(x,y,z) \in N \times V : f(x,y,z) = 4\} \subset \{(x,y,z) : f(x,y,z) = 4\} =: S.$$

Since $f(1,1,1) = 4$, then $g(1,1) = 1$.

Spring 2007 #11. (a) Consider the equations

$$u^3 + xv - y = 0, \qquad v^3 + yu - x = 0.$$

Can these equations be solved uniquely for u, v in terms of x, y in a neighborhood of $x = 0$, $y = 1$, $u = 1$, $v = -1$? Explain your answer.

(b) Give an example in which the conclusion of the implicit function theorem is true, but the hypothesis is not.

(a) Define $f : \mathbb{R}^4 \to \mathbb{R}^2$ by $f_1(x,y,u,v) = u^3 + xv - y$ and $f_2(x,y,u,v) = v^3 + yu - x$. Note that $f(0,1,1,-1) = (0,0)$ and f is continuously differentiable. The matrix of partial derivatives under consideration is

$$[(\partial f_i/\partial u_j)(0,1,1,-1)] = \begin{bmatrix} \frac{\partial f_1}{\partial u}(0,1,1,-1) & \frac{\partial f_1}{\partial v}(0,1,1,-1) \\ \frac{\partial f_2}{\partial u}(0,1,1,-1) & \frac{\partial f_2}{\partial v}(0,1,1,-1) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix},$$

which has determinant 9 and is thus invertible. Hence the implicit function theorem guarantees the existence of an open set U containing $(0,1)$, an open set V containing $(1,-1)$, and a uniquely continuously differentiable function $g(x,y)$ such that

$$\{(x,y,g_1(x,y),g_2(x,y)) : (x,y) \in U\} = \{(x,y,u,v) \in U \times V : f(x,y,u,v) = 0\}.$$

Thus the equations can be solved uniquely in a neighborhood of $x = 0$, $y = 1$, $u = 1$, $v = -1$.

(b) Consider solving $F(x,y) = y^3 - x$ for y near $(0,0)$. The known solution is $y = x^{1/3}$, but the implicit function theorem fails since the generated matrix is singular.

Spring 2010 #9. Assume that $f(x,y,z)$ is a real-valued, continuously differentiable function such that $f(x_0,y_0,z_0) = 0$. If $\nabla f(x_0,y_0,z_0) \neq 0$, show that there is a differentiable surface, given parametrically by $(x(s,t), y(s,t), z(s,t))$ with $(x(0,0), y(0,0), z(0,0)) = (x_0,y_0,z_0)$, on which $f = 0$.

Suppose without loss of generality that $\partial f/\partial z(x_0,y_0,z_0) \neq 0$. Then the Implicit Function Theorem guarantees the existence of an open set $U \subset \mathbb{R}^2$ containing (x_0,y_0), an open set $V \subset \mathbb{R}$ containing z_0, and a continuously differentiable function $g : U \to V$ such that

$$\{(x,y,g(x,y)) : (x,y) \in U\} = \{(x,y,z) : f(x,y,z) = 0\}.$$

Set $s = x$ and $t = y$. Thus taking $x(s,t) = s$, $y(s,t) = t$, and $z(s,t) = g(s,t) = g(x,y)$, it follows that

$$\{(x(s,t), y(s,t), z(s,t)) : (s,t) \in U\}$$

is a differentiable surface with $(x(0,0), y(0,0), z(0,0)) = (x_0,y_0,z_0)$ on which $f = 0$.

Fall 2002 #6. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ is continuously differentiable. Suppose for some $v_0 \in \mathbb{R}^3$ and $x_0 \in \mathbb{R}^2$ that $F(v_0) = x_0$ and $F'(v_0) : \mathbb{R}^3 \to \mathbb{R}^2$ is onto. Show that there is a continuously differentiable function $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ for some $\varepsilon > 0$, such that

(i) $\gamma'(0) \neq 0 \in \mathbb{R}^3$, and

(ii) $F(\gamma(t)) = x_0$ for all $t \in (-\varepsilon, \varepsilon)$.

This problem is an immediate consequence of the implicit function theorem. Without loss of generality, assume $x_0 = (0,0)$. Let $v_0 = (v_1, v_2, v_3)$. Let M be the matrix representation of $F'(v_0)$. Since $F'(v_0)$ is surjective, two columns of M are linearly independent. Assume without loss of generality that these are the last two columns of M. Then the matrix consisting of the last two columns of M is invertible, so we can apply the inverse function theorem. Thus there are continuously differentiable functions $f, g : (v_1 - \varepsilon, v_1 + \varepsilon) \to \mathbb{R}$ such that

$$F(t, f(t), g(t)) = 0$$

for all $t \in (v_1 - \varepsilon, v_1 + \varepsilon)$. Defining $\gamma(t) = (v_1 + t, f(v_1 + t), g(v_1 + t))$ for all $t \in (-\varepsilon, \varepsilon)$. Then $F(\gamma(t)) = (0,0) = x_0$ for all $t \in (-\varepsilon, \varepsilon)$. Also, the final entry of $\gamma'(0)$ is 1, so it is non-zero.

Spring 2002 #6. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function with grad $f \neq 0$ at $0 \in \mathbb{R}^3$. Show that there are two other continuously differentiable functions $g : \mathbb{R}^3 \to \mathbb{R}$, $h : \mathbb{R}^3 \to \mathbb{R}$ such that the function

$$(x, y, z) \to (f(x, y, z), g(x, y, z), h(x, y, z))$$

from $\mathbb{R}^3$ to $\mathbb{R}^3$ is one-to-one on some neighborhood of 0.

Since $\nabla f(0)$ is nonzero, $\{\nabla f(0)\}$ is linearly independent and may be extended with two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ to a basis for $\mathbb{R}^3$. Define $g(x, y, z) = (x, y, z) \cdot (v_1, v_2, v_3)$ and $h(x, y, z) = (x, y, z) \cdot (w_1, w_2, w_3)$ so that $\nabla g = v$ and $\nabla h = w$. In particular, the derivative of (f, g, h) at 0 has linearly independent columns $\nabla f(0)$, v, and w, so it is invertible. The Inverse Function Theorem then guarantees that (f, g, h) is invertible in a neighborhood of 0 (hence one-to-one).

Fall 2005 #4. Suppose $F : [0,1] \to [0,1]$ is a C^2 function with $F(0) = 0, F(1) = 0$, and $F''(x) < 0$ for all $x \in [0,1]$. Prove that the arc length of the curve $\{(x, F(x)) : x \in [0,1]\}$ is less than 3. (Suggestion: Remember that $\sqrt{a^2 + b^2} < |a| + |b|$ when you are looking at the arc length formula - and at a picture of what $\{(x, F(x)\}$ could look like.)

The arc length of the curve $\{(x, F(x)) : x \in [0,1]\}$ is given by

$$\int_0^1 \sqrt{1 + (F'(x))^2} dx < \int_0^1 1 + |F'(x)| dx = 1 + \int_0^1 |F'(x)| dx.$$

Since $F''(x) < 0$ for all $x \in [0,1]$, F' is decreasing on $[0,1]$. From $F(0) = F(1) = 0$, we conclude that F increases until it reaches its maximum, then decreases back to 0. Thus splitting the integral into two parts, to the left and right of the maximum occuring at $c \in (0,1)$,

$$\int_0^1 |F'(x)| dx = \int_0^c |F'(x)| dx + \int_c^1 |F'(x)| dx = \int_0^c F'(x) dx - \int_c^1 F'(x) dx = F(c) - F(0) - (F(1) - F(c)) = 2F(c).$$

By hypothesis, $F(c) \leq 1$. Thus the arc length is less than 3.

Infinite sequences and series

Spring 2009 #1. Set $a_1 = 0$ and define a sequence $\{a_n\}$ via the recurrence

$$a_{n+1} = \sqrt{6 + a_n} \text{ for all } n \geq 1.$$

Show that this sequence converges and determine the limiting value.

We first show by induction that $0 \leq a_n \leq a_{n+1} \leq 3$ for all $n \geq 0$. For the base case $n = 0$, $a_0 = 0$ and $a_1 = \sqrt{6}$, so $0 \leq a_0 \leq a_{n+1} \leq 3$. Now assume inductively that $0 \leq a_n \leq a_{n+1} \leq 3$. It follows that $a_{n+2} = \sqrt{6 + a_{n+1}} \leq \sqrt{6+3} = 3$. Also, $a_{n+2} \geq \sqrt{6+0} = \sqrt{6} \geq 0$. Finally,

$$a_{n+2} = \sqrt{6 + a_{n+1}} \geq \sqrt{6 + a_n} = a_{n+1}.$$

Thus $0 \leq a_{n+1} \leq a_{n+2} \leq 3$, completing the induction. Hence the sequence $(a_n)_{n=0}^{\infty}$ is increasing and bounded above by 3, so $L := \lim_{n\to\infty} a_n$ exists.

Sending $n \to \infty$ in

$$a_{n+1} = \sqrt{6 + a_n}$$

and using that the square root function is continuous,

$$L = \sqrt{6 + L},$$

hence $L^2 - L - 6 = 0$, and $L = 3$ or $L = -2$. Clearly $L \geq 0$, so we conclude $L = 3$.

Fall 2011 #4. Compute

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

We start from the well-known sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This implies that the given series is absolutely convergent. We can then derive its sum.

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{3\pi^2}{24}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{24} - \frac{3\pi^2}{24} = -\frac{\pi^2}{12}.$$

Spring 2013 #11. Define the Fibonacci sequence F_n by $F_0 = 0$, $F_1 = 1$, and recursively, $F_n = F_{n-1} + F_{n-2}$ for $n = 2, 3, 4, \ldots$.

(a) Show that the limit as n goes to infinity of F_n/F_{n-1} exists and find its value.

(b) Prove that $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1$ for all $n \geq 1$.

(a) The Fibonacci recurrence is $F_{n+2} - F_{n+1} - F_n = 0$ for all $n \geq 0$, $F_0 = 0$, $F_1 = 1$.

We guess that a solution takes the form $F_n = \lambda^n$. This implies $\lambda^2 - \lambda - 1 = 0$, so we obtain solutions

$$\lambda_1 = \frac{1}{2}(1+\sqrt{5}) \text{ and } \lambda_2 = \frac{1}{2}(1-\sqrt{5}).$$

The general solution is a linear combination of these. Using the initial conditions, we derive

$$F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n).$$

This implies

$$F_n/F_{n-1} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1^{n-1} - \lambda_2^{n-1}}.$$

As $n \to \infty$, since $\lambda_2 < 1$, $\lambda_2^n \to 0$, thus

$$F_n/F_{n-1} \to \lambda_1^n/\lambda_1^{n-1} = \lambda_1 = \frac{1}{2}1 + \sqrt{5}.$$

(b) This can be shown by a simple induction (easiest in the form $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$).

Winter 2006 #1. Show that for each $\varepsilon > 0$ there exists a sequence of intervals (I_n) with the properties

$$\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n \text{ and } \sum_{n=1}^{\infty} |I_n| < \varepsilon.$$

Enumerate the rationals with the sequence $(r_n)_{n=1}^{\infty}$. Define $I_n = [r_n, r_n + \varepsilon/2^{n+1}]$. Then $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n$ and

$$\sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} \varepsilon/2^{n+1} = \varepsilon/2 < \varepsilon,$$

as desired.

Spring 2003 #2. Prove: If $a_1, a_2, a_3, \ldots$ is a sequence of real numbers with

$$\sum_{j=1}^{+\infty} |a_j| < +\infty,$$

then $\lim_{N\to+\infty} \sum_{j=1}^{N} a_j$ exists.

For each $j \geq 1$, define $p_j = \frac{1}{2}(|a_j| - a_j)$ and $q_j = \frac{1}{2}(|a_j| - a_j)$. Then $p_j + q_j = |a_j|$ and $p_j - q_j = a_j$ for each j. Note also that $0 \leq p_j \leq |a_j|$ and $0 \leq q_j \leq |a_j|$ for each j. Thus by the squeeze theorem, since $\sum_{j=1}^{\infty} |a_j|$ converges, $\sum_{j=1}^{\infty} p_j$ and $\sum_{j=1}^{\infty} q_j$ converge. It follows that

$$\sum_{j=1}^{\infty} a_n = \sum_{j=1}^{\infty} (p_j - q_j)$$

converges to

$$\sum_{j=1}^{\infty} p_j - \sum_{j=1}^{\infty} q_j,$$

so $\lim_{N\to+\infty} \sum_{j=1}^{N} a_j = \sum_{j=1}^{\infty} a_j$ exists.

Spring 2010 #11. Suppose $\sum_{n=1}^{\infty} |a_n| < \infty$. Let σ be a one-to-one mapping of $\mathbb{N}$ onto $\mathbb{N}$. The series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is called a "rearrangement" of $\sum_{n=1}^{\infty} a_n$. Prove that all rearrangements of $\sum_{n=1}^{\infty} a_n$ are convergent and have the same sum.

By the above exercise, $\sum_{n=1}^\infty a_n$ is convergent. Define $S := \sum_{n=1}^\infty a_n$. Let $\varepsilon > 0$. It follows that $(\sum_{n=1}^N a_n)_{N=1}^\infty$ is a Cauchy sequence, thus there exists N such that for all $M \geq N$,

$$|\sum_{n=N+1}^M a_n| = |\sum_{n=1}^M a_n - \sum_{n=1}^N a_n| \leq \varepsilon/2.$$

In addition, increase N as necessary such that

$$|\sum_{n=1}^N a_n - S| \leq \varepsilon/2.$$

Now choose N' sufficiently large that $\{1, \dots, N\} \subset \{a_{\sigma(1)}, \dots, a_{\sigma(N')}\}$. Fix $m \geq N'$. Let

$$M = \max\{\sigma(k) : 1 \leq k \leq m\}.$$

It follows that

$$|\sum_{n=1}^N a_n - \sum_{k=1}^m a_{\sigma(k)}| \leq |\sum_{n=N+1}^M |a_n|| \leq \varepsilon/2.$$

Then

$$|\sum_{k=1}^m a_{\sigma(k)} - S| \leq |\sum_{k=1}^m a_{\sigma(k)} - \sum_{n=1}^N a_n| + |\sum_{n=1}^N a_n - S| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $\sum_{k=1}^\infty a_{\sigma(k)}$ exists and equals S.

Fall 2001 #2; Fall 2008 #5. Let N denote the positive integers, let $a_n = (-1)^n \frac{1}{n}$, and let α be any real number. Prove there is a one-to-one and onto mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^\infty a_{\sigma(n)} = \alpha.$$

We proceed for a general series which is conditionally convergent but not absolutely convergent. This series is conditionally convergent by the alternating series test, but not absolutely convergent. This is clear from the comparsion

$$\sum_{n=1}^\infty |a_n| = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots,$$

since the sum on the right is $1 + \frac{1}{2} + \frac{1}{2} + \cdots$, which does not converge.

Define $p_j = \frac{1}{2}(|a_j| + a_j)$ and $q_j = \frac{1}{2}(|a_j| - a_j)$. Then $p_j = a_j$ if a_j is non-negative, and $q_j = -a_j$ if a_j is negative. If both $\sum_{j=1}^\infty p_j$ and $\sum_{j=1}^\infty$ converge, then $\sum_{j=1}^\infty a_j = \sum_{j=1}^\infty (p_j - q_j)$ converges, a contradiction. Thus either $\sum_{j=1}^\infty p_j$ or $\sum_{j=1}^\infty q_j$ diverges. Suppose the former. If $\sum_{j=1}^\infty q_j$ converges, then $\sum_{j=1}^\infty p_j = \sum_{j=1}^\infty (a_j + q_j)$ converges, a contradiction. Hence both $\sum_{j=1}^\infty p_j$ and $\sum_{j=1}^\infty q_j$ converge.

Reindex p_j and q_j to eliminate the 0 terms which do not correspond with some a_j and preceed a term corresponding to some a_j. Clearly $\sum_{j=1}^\infty p_j$ and $\sum_{j=1}^\infty q_j$ still diverge.

Suppose $\alpha > 0$ (the other cases are similar). Let P_j and Q_j, $j \geq 1$ be the partial sums of $\sum_{j=1}^\infty p_j$ and $\sum_{j=1}^\infty q_j$ respectively. Select the smallest N_1 such that $P_{N_1} > \alpha$. (Such an N_1 exists since $\sum_{j=1}^\infty p_j$ is divergent and positive.) Then select the smallest N_2 such that $P_{N_1} - Q_{N_2} < \alpha$. Next, select $N_3 > N_1$ such that $P_{N_3} - Q_{N_2} > \alpha$. By the zero test for sequences, p_j and q_j approach 0 as $j \to \infty$. Thus it follows that when continuing this procedure,

$$P_{N_k} - Q_{N_{k+1}}$$

approaches α. Define the bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $a_{\sigma(1)} = p_1, \dots, a_{\sigma(N_1)} = p_{N_1}, a_{\sigma(N_1+1)} = q_1, \dots, a_{\sigma(N_2)} = q_{N_2}, \dots$. Then by construction, $\sum_{j=1}^N a_{\sigma(j)}$ converges to α as $N \to \infty$, so $\sum_{n=1}^\infty a_{\sigma(n)} = \alpha$.

Fall 2003 #3. Prove that the sequence $a_1, a_2, \ldots$ with

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

converges as $n \to \infty$.

By comparing $\sum_{k=1}^{\infty} \frac{1}{k!}$ to $\sum_{k=1}^{\infty} 2^{-k+1}$, which converges, we see that the former series converges. By the binomial theorem,

$$\begin{aligned} a_n = \left(1 + \frac{1}{n}\right)^n &= \sum_{k=1}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \\ &= \sum_{k=1}^{n} \frac{1}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} \\ &= \sum_{k=1}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{k-1}{n}\right) \leq \sum_{k=1}^{n} \frac{1}{k!}. \end{aligned}$$

Taking the limit as $n \to \infty$, we see that $\lim_{n\to\infty} a_n$ exists and does not exceed $\sum_{k=1}^{\infty} \frac{1}{k!}$.

Fall 2004 #1; Spring 2005 #9. Consider the following two statements:
(A) The sequence (a_n) converges.
(B) The sequence $((a_1 + a_2 + \cdots + a_n)/n)$ converges.
Does (A) imply (B)? Does (B) imply (A)? Prove your answers. (Summer 2005 #9 indicates that (A) implies (B) but (B) does not imply (A) in general).

The sequence $a_n = (-1)^n$ converges in the sense of (B) but not in the sense of (A). Thus (B) does not imply (A).

We show that (A) implies (B). Suppose the sequence $(a_n)_{n=1}^{\infty}$ converges to s. Let $\varepsilon > 0$. Select N such that for all $n \geq N$, $|a_n - s| \leq \varepsilon/2$. Select $N' > N$ such that

$$\frac{|a_1 - s| + \cdots + |a_N - s|}{N'} \leq \varepsilon/2.$$

Then for any $n \geq N'$,

$$\begin{aligned} \left|\frac{a_1 + \cdots + a_n}{n} - s\right| &= \frac{|a_1 - s| + \cdots + |a_N - s|}{n} + \frac{|a_{N+1} - s| + \cdots + |a_n - s|}{n} \\ &\leq \frac{|a_1 - s| + \cdots + |a_N - s|}{N'} + \frac{(n-N)\varepsilon}{2n} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus $\frac{a_1 + \cdots + a_n}{n}$ converges to s.

Spring 2012 #4. For a sequence $\{a_n\}$ of *non-negative* numbers, let $s_n := \sum_{n=1}^{N} a_n$ and suppose that s_n tends to a number $s \in \mathbb{R}$ in the Cesaro sense:

$$s = \lim_{n\to\infty} \frac{s_1 + \cdots + s_n}{n}.$$

Show that $\sum_{k=1}^{\infty} a_k$ exists and equals s.

Suppose for the sake of contradiction that $\sum_{k=1}^{\infty} a_k$ diverges. Then since all a_k are positive, it must diverge to $+\infty$. Thus there exists N such that for all $n \geq N$, $s_n = \sum_{k=1}^{n} a_n \geq 10s$. Then for any positive integer M,

$$\frac{s_1 + \cdots + s_{N+M}}{N+M} = \frac{s_1 + \cdots + s_N}{N+M} + \frac{s_{N+1} + \cdots + s_{N+M}}{N+M} \geq \frac{(10s)M}{N}.$$

Letting $M \to \infty$, this means s is as large as desired, a contradiction.

Thus $\sum_{k=1}^{\infty} a_k$ converges. By the previous problem,

$$s = \lim_{n\to\infty} \frac{s_1 + \cdots + s_n}{n} = \sum_{k=1}^{\infty} a_k.$$

Spring 2004 #1. Let S denote the set of sequences $a = (a_1, a_2, \ldots)$, with $a_k = 0$ or 1. Show that the mapping $\Theta : S \to \mathbb{R}$ defined by

$$\Theta((a_1, a_2, \ldots)) = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots$$

is an injection. Include an explanation of why the infinite series converges. Hint: if $a \neq b$, you may assume that

$$a = (a_1, \ldots, a_{n-1}, 0, a_{n+1}, \ldots),$$
$$b = (b_1, \ldots, b_{n-1}, 1, b_{n+1}, \ldots).$$

For any a,

$$\Theta(a) = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots \leq \frac{1}{10} + \frac{1}{10}^2 = \frac{1}{10}\Big/\frac{9}{10} = \frac{1}{9},$$

thus $\Theta(a)$ is finite.

Suppose $a \neq b$. Assume without loss of generality that

$$a = (a_1, \ldots, a_{n-1}, 0, a_{n+1}, \ldots),$$
$$b = (b_1, \ldots, b_{n-1}, 1, b_{n+1}, \ldots).$$

Then

$$|\Theta(b) - \Theta(a)| \geq 10^{-n} + \sum_{i=n+1}^{\infty} \frac{b_i - a_i}{10^i} \geq 10^{-n} - \sum_{i=n+1}^{\infty} \frac{1}{10^i} = 10^{-n} - \frac{1}{10^{n+1}}\Big/\frac{1}{9} = \frac{1}{10^{n+1}}.$$

Hence $\Theta(b) \neq \Theta(a)$, so Θ is injective.

Spring 2006 #2. Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with $a_n \in \mathbb{R}$. Show that there exists a unique number $\rho \geq 0$ such that $F(x)$ converges if $|x| < \rho$ and $F(x)$ diverges if $|x| > \rho$.

Suppose $\rho \in \mathbb{R}$ such that $F(\rho)$ converges. We first show that $F(x)$ converges for all $|x| < |\rho|$. Let $x \in \mathbb{R}$ with $|x| < \rho$. Since a convergent sequence is bounded, there exists B such that $|a_n \rho^n| \leq B$ for all n. Define $\alpha = |x|/|\rho|$. Then

$$\sum_{n=0}^{\infty} a_n |x|^n \leq \sum_{n=0}^{\infty} a_n \alpha^n |\rho|^n \leq B \sum_{n=0}^{\infty} \alpha^n,$$

which converges, thus $F(x)$ converges.

Let $\rho = \sup\{x \geq 0 : F(x) \text{ converges}\}$. Clearly 0 is in this set, so the supremum is non-negative. By the above result, $F(x)$ converges if $|x| < \rho$. If $\rho = +\infty$, then F converges everywhere. Otherwise, ρ is some finite non-negative real number. Suppose for the sake of contradiction that $F(x)$ converges for some x with $|x| > \rho$. Then $|x|$ is a member of $\{x \geq 0 : F(x) \text{ converges }\}$, but $|x| > \rho$, a contradiction. Hence $F(x)$ diverges if $|x| > \rho$.

Winter 2006 #2. Let $(a_n)_{n\geq 1}$ be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n = \infty$. Under what condition(s) is the function

$$f(x) = \sum_{n=1}^{\infty} (-1)^n a_n x^n$$

well-defined and left-continuous at $x = 1$? Carefully prove your assertion.

Suppose $\lim_{n\to\infty} a_n = 0$. Then by the alternating series test, $f(1)$ converges. Since $f(1) = \sum_{n=1}^{\infty}(-1)^n a_n$ converges, this series is Abel-summable, so f is left-continuous at $x = 1$.

Fall 2007 #8. Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. Does it follow that

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} = \infty?$$

Prove your answer.

Yes. Suppose first that $L := \limsup_{n\to\infty} a_n \neq 0$. Since the a_n are non-negative, $L > 0$. Let N be a positive integer. Then there exists $n > N$ such that $a_n > L/2$. It follows that

$$\frac{a_n}{1+a_n} \geq a_n > L/2.$$

Hence the limit of $\frac{a_n}{1+a_n}$ as $n \to \infty$ either does not exist, or does not equal 0, so the sum in question does not converge. Since the summands are positive, the series must diverge to $+\infty$.

Otherwise, $L := \limsup_{n\to\infty} a_n = 0$. Let $\varepsilon > 0$. Then there exists N such that for all $n \geq N$, $a_n < \varepsilon$. Thus for all $n \geq N$,

$$\frac{a_n}{1+a_n} > \frac{a_n}{1+\varepsilon}.$$

Note that

$$\sum_{n=N}^{\infty} \frac{a_n}{1+\varepsilon} = \frac{1}{1+\varepsilon}\sum_{n=N}^{\infty} a_n = +\infty,$$

thus $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ must also diverge to $+\infty$, as desired.

Fall 2010 #4. (a) Show that given a real-valued continuous function f on $[0,1]\times[0,1]$ and an $\varepsilon > 0$, there exist real-valued continuous functions $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$ on $[0,1]$ for some finite $n \geq 1$ so that

$$\left| f(x,y) - \sum_{i=1}^{n} g_i(x)h_i(y) \right| \leq \varepsilon, \qquad 0 \leq x, y \leq 1.$$

(b) If $f(x,y) = f(y,x)$ for all $0 \leq x, y \leq 1$, can this be done with $h_i = g_i$ for each i? Explain.

(a) Let $\varepsilon > 0$. Since f is continuous on a compact set, it is uniformly continuous. Thus there exists a positive integer N such that if $|(x,y) - (x',y')| \leq \frac{1}{N}$, then $|f(x,y) - f(x',y')| \leq \varepsilon/2$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(x) = \begin{cases} 1+x & x \in [-1,0], \\ 1-x & x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Define the real-valued continuous function $g_i(x) = \phi(Nx + i)$ for $0 \leq i \leq N$. Consider the function

$$\tilde{f}(x,y) = \sum_{i=0}^{N} g_i(x) f(i/N, y).$$

For any $(x,y) \in [0,1]$, it is straightforward to show from continuity of f that

$$|f(x,y) - \tilde{f}(x,y)| \leq \varepsilon/2.$$

By the Weierstrass Approximation Theorem in 1 dimension, there exists polynomials (real-valued continuous functions) $h_0, \ldots, h_N$ such that for each $1 \le i \le N$ and any $y \in [0,1]$,

$$|h_i(y) - f(i/N, y)| \le \varepsilon/4N.$$

It follows that

$$|f(x,y) - \sum_{i=0}^{N} g_i(x)h_i(y)| \le |f(x,y) - \tilde{f}(x,y)| + |\tilde{f}(x,y) - \sum_{i=0}^{N} g_i(x)h_i(y)|$$

$$= |f(x,y) - \tilde{f}(x,y)| + |\sum_{i=0}^{N} g_i(x)(f(i/N,y) - h_i(y))|$$

$$\le \varepsilon/2 + N(\varepsilon/2N) = \varepsilon.$$

Adjust indices as necessary to obtain the desired expression.

(b) No. Take some f which is negative on $(1,1)$. Then if $g_i = h_i$, $\sum_{i=1}^n g_i(1)h_i(1) = \sum_{i=1}^n g_i^2(1)$ is always non-negative, so it cannot come arbitrarily close to $f(1,1)$. Hence such an approximation is not possible.

Fall 2012 #1. Let $\{b_n\}_{n=1}^\infty$ be a sequence of real numbers with bounded partial sums, i.e., there is $M < \infty$ such that for all N, $|\sum_{n=1}^N b_n| \le M$, and let $\{a_n\}_{n=1}^\infty$ be a sequence of positive numbers decreasing to 0. Prove the series $\sum a_n b_n$ converges.

We use summation by parts. Let s_n denote the partial sums of $\sum_{j=1}^n b_j$. For any positive integers N, M,

$$\sum_{j=N}^{M} a_j b_j = \sum_{j=N}^{M} a_j(s_j - s_{j-1}) = \sum_{j=N}^{M} a_j s_j - \sum_{j=N}^{M} a_j s_{j-1}$$

$$= \sum_{j=N}^{M} a_j s_j - \sum_{j=N-1}^{M-1} a_{j+1} s_j$$

$$= a_M s_M - a_N s_{N-1} + \sum_{j=N}^{M-1} (a_j - a_{j+1}) s_j.$$

Thus

$$|\sum_{j=N}^{M} a_j b_j| \le M(a_M + a_N) + M \sum_{j=N}^{M-1} (a_j - a_{j+1})$$

$$= M(a_M + a_N + a_N - a_M) \le 2Ma_N.$$

Letting $N \to \infty$, since a_N decreases to 0,

$$\sum_{j=N}^{M} a_j b_j \to 0$$

as $N, M \to \infty$. Hence the partial sums of $\sum a_n b_n$ form a Cauchy sequence, so the series $\sum_{n=1}^\infty a_n b_n$ converges.

Spring 2013 #4. Denote by h_n the n-th harmonic number:

$$h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Prove that there is a limit

$$\gamma = \lim_{n \to \infty} (h_n - \ln n).$$

We show $x_n := h_n - \ln n$ is decreasing and bounded below, thus the desired limit exists.

Recall $\ln x = \int_1^x \frac{1}{y} dy$ for all $x \geq 1$. Thus h_n is an upper Riemann sum for $\ln n$, so $h_n - \ln n \geq 0$. Thus x_n is bounded below by 0.

For all $n \geq 1$, let $x_n = h_n - \ln n$. Fix $n \geq 1$. Applying the mean value theorem to ln, there exists $c \in (\frac{1}{n+1}, \frac{1}{n})$ such that

$$\ln(n+1) - \ln(n) = \frac{1}{c}((n+1) - n) = \frac{1}{c}.$$

Thus

$$x_{n+1} - x_n = \frac{1}{n+1} - (\ln(n+1) - \ln(n)) = \frac{1}{n+1} - \frac{1}{c} < 0.$$

Hence x_n is decreasing.

Spring 2006 #3. Prove that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{5/2}}$$

converges for all $x \in \mathbb{R}$ and that $f(x)$ is a continuous function on $\mathbb{R}$ with a continuous derivative. State clearly any facts you assume.

We first show that $\sum_{j=1}^{\infty} \frac{\sin(jx)}{j^{5/2}}$ converges uniformly. Note that the terms are bounded by $\frac{1}{j^{5/2}}$, and $\sum_{j=1}^{\infty} \frac{1}{j^{5/2}} < \infty$ by the integral test. Thus the series converges by the Weierstrass M-test to a continuous function. Also, the derivatives are continuous and the sum of the derivatives converge uniformly by the Weierstrass M-test (again using the integral test).

Partial Derivatives

Fall 2001 #5; Spring 2002 #5; Winter 2002 # 6; Spring 2003 #6; Fall 2005 #2. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere and are continuous everywhere, and $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ also exist, and are continuous everywhere. Prove that

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

at every point of $\mathbb{R}^2$.

Here I only assume $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ exists. Define

$$\delta(h,k) := f(h,k) - f(h,0) - f(0,k) + f(0,0).$$

Also let $g(x) := f(x,k) - f(x,0)$. Applying the mean value theorem twice, there exist $a_h \in (0,h)$ and $b_k \in (0,k)$ such that

$$\delta(h,k) = g(h) - g(0) = g'(a_h)h = (\frac{\partial f}{\partial x}(a_h,k) - \frac{\partial f}{\partial x}(a_h,0))h = \frac{\partial^2 f}{\partial y \partial x}(a_h,b_k)hk.$$

Thus since $\frac{\partial^2 f}{\partial y \partial x}$ is continuous,

$$\lim_{h\to 0}\lim_{k\to 0}\frac{\delta(h,k)}{hk} = \lim_{h\to 0}\lim_{k\to 0}\frac{\partial^2 f}{\partial y\partial x}(a_h,b_k) = \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

Hence $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ exists and equals $\frac{\partial^2 f}{\partial y\partial x}(0,0)$.

Spring 2008 #5. Fall 2012 #6. (a) Let $F(x,y)$ be a continuous function on the plane such that for every square S having its sides parallel to the axes,

$$\int\int_S F(x,y)dxdy = 0.$$

Prove $F(x,y) = 0$ for all (x,y).

(b) Assume $f(x,y)$, $\frac{\partial f(x,y)}{\partial x}$, $\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right)$, and $\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right)$ are all continuous in the plane. Use part (a) to prove that

$$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right).$$

Hint: you may assume the double integral in (a) equals the iterated integral $\int(\int F(x,y)dx)dy$ and equals the iterated integral $\int(\int F(x,y)dy)dx$.

Suppose for the sake of contradiction that $f(x^*,y^*) > 0$. There exists $\delta > 0$ such that if $|(x,y)-(x',y')| \leq \delta$, then $|f(x,y) - f(x',y')| \leq |f(x^*,y^*)|/2$. Take S with opposite corners $[x - delta/\sqrt{2}, y - delta/\sqrt{2}]$ and $[x+\delta/\sqrt{2}, y+\delta/\sqrt{2}]$. Then within S, $|(x,y)-(x',y')| = \sqrt{|x-x'|^2 + |y-y'|^2} \leq \sqrt{\delta^2} = \delta$, hence $|f(x,y) - f(x^*,y^*)| \leq |f(x^*,y^*)|/2$, so

$$|f(x,y)| \geq f(x^*,y^*) - \frac{f(x^*,y^*)}{2} = f(x^*,y^*)/2.$$

It follows that

$$\int\int_S f(x,y) \geq \int\int_S f(x^*,y^*)/2 > 0,$$

a contradiction. Thus $f(x,y)=0$ for all (x,y).

(b) Let S be a square with its sides parallel to the axes. We compute

$$\int\int_S \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} dxdy = \int\int_S \frac{\partial^2 f}{\partial x \partial y} dxdy - \int\int_S \frac{\partial^2 f}{\partial y \partial x} dxdy = 0$$

where the final step follows from the fundamental theorem of calculus. Thus applying part (a) to the continuous function $\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}$, we find $\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} = 0$.

Fall 2002 #5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives at every point bounded by $A > 0$.

(a) Show that there is an $M > 0$ such that

$$|f((x,y)) - f((x_1,y_1))| \leq M((x-x_1)^2 + (y-y_1)^2)^{1/2}.$$

(b) What is the smallest value of M (in terms of A) for which this always works?

(c) Give an example where that value of M makes the inequality an equality.

(a) Using the mean value theorem,

$$|f(x,y) - f(x_1,y_1)| \leq |f(x,y) - f(x_1,y)| + |f(x_1,y) - f(x_1,y_1)| \leq A|(x,y) - (x_1,y)| + A|(x_1,y) - (x_1,y_1)|$$
$$\leq A\sqrt{2}\sqrt{(x-x_1)^2 + (y-y_1)^2}.$$

Here we use the inequality

$$a + b \leq \sqrt{2}\sqrt{a^2+b^2}$$

for the last inequality. So setting $M = A\sqrt{2}$, we have the desired inequality.

(b) & (c) Use the example $f(x,y) = x + y$. Here $A = 1$ and taking $(x,y) = (1,1)$, $(x',y') = (0,0)$, we obtain the bound $A\sqrt{2}$. Thus this is strict.

Spring 2003 #4. Consider the following equation for a function $F(x,y)$ on $\mathbb{R}^2$:

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2}.$$

(a) Show that if a function F has the form $F(x,y) = f(x+y) + g(x-y)$ where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are twice differentiable, then F satisfies this equation.

(b) Show that if $F(x,y) = ax^2 + bxy + cy^2$, $a, b, c \in \mathbb{R}$, satisfies this equation, then $F(x,y) = f(x+y) + g(x-y)$ for some polynomials f and g in one variable.

(a) Straightforward with chain rule.

(b) Then $2a = 2c$, so $a = c$. Write $F(x,y) = a(x^2+y^2) + bxy = A(x+y)^2 + B(x-y)^2$. Then

$$a = A + B, b = 2(A - B)$$

can be inverted. Hence we can take $f(x+y) = A(x+y)^2$ and $g(x-y) = B(x-y)^2$.

Differentiation

Spring 2004 #5. Suppose that G is an open set in $\mathbb{R}^n$, $f : G \to \mathbb{R}^m$ is a function, and that $x_0 \in G$.

(a) Carefully define what is meant by $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^m$.

(b) Suppose that I is a line segment in G connecting points p and q such that $f'(x)$ is defined for all $x \in I$. Show that if f is differentiable at all the points of I, then for some point $c \in I$,

$$||f(q) - f(p)||_2 \leq ||f'(c)||||q - p||_2.$$

Hint: let w be a unit vector with $||f(q) - f(p)||_2 = (f(q) - f(p)) \cdot w$.

(a) $f'(x_0)$ is the unique linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that

$$\lim_{x \to x_0; x \in G \setminus \{x_0\}} \frac{||f(x) - (f(x_0) + (f'(x_0))(x - x_0))||}{||x - x_0||} = 0.$$

(b) If $f(q) = f(p)$, the desired inequality is obvious. Otherwise, let

$$w = \frac{f(q) - f(p)}{||f(q) - f(p)||_2}.$$

Then

$$||f(q) - f(p)||_2 = (f(q) - f(p)) \cdot w.$$

Define $g : G \to \mathbb{R}$ by

$$g(x) := f(x) \cdot w.$$

Then

$$g(q) - g(p) = (f(q) - f(p)) \cdot w = ||f(q) - f(p)||_2$$

and

$$g'(x)(u) = (f'(x)(u)) \cdot w$$

for all $x \in I$, $u \in \mathbb{R}^n$. By the mean value theorem from $\mathbb{R}^n$ to $\mathbb{R}$, there exists $c \in I$ such that

$$g(q) - g(p) = g'(c)(q - p) = (f'(c)(q - p)) \cdot w.$$

Thus by the Cauchy-Schwarz inequality,

$$||f(q) - f(p)||_2 = g(q) - g(p) = (f'(c)(q - p)) \leq ||f'(c)||_2||q - p||_2.$$

Winter 2002 #5; Spring 2009 #7. (a) Let $f : U \to \mathbb{R}^k$ be a function on an open set $U \subset \mathbb{R}^n$. Define what it means for f to be differentiable at a point $x \in U$.

(b) State carefully the Chain Rule for the composition of differentiable functions of several variables.

(c) Prove the Chain Rule you stated in (b).

(a) f is differentiable at $x \in U$ if there exists a linear map $f'(x) : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$\lim_{y \to x; y \in U \setminus \{x\}} \frac{||f(y) - (f(x) + f'(x)(y - x))||_2}{||y - x||_2} = 0.$$

(b) Let $U \in \mathbb{R}^k$, $V \in \mathbb{R}^m$, and consider $f : U \to V$ and $g : V \to \mathbb{R}^n$. Let $x_0 \in U$ and suppose that f is differentiable at x_0 and g is differentiable at $g(x_0)$. Then $g \circ f$ is differentiable at x_0 and as a composition of linear maps,

$$(g \circ f)'(x_0) = g'(f(x_0)) \circ f'(x_0).$$

(c) Let $L_f : \mathbb{R}^k \to \mathbb{R}^m$ and $L_g : \mathbb{R}^m \to \mathbb{R}^n$ be the functions

$$L_f(x) = f(x) + f'(x_0)(x - x_0), \qquad L_g(x) = g(x) + g'(x_0)(x - x_0).$$

We must show that

$$||(g \circ f)(x) - (L_g \circ L_f)(x)|| \leq \varepsilon ||x - x_0||$$

for all x in some neighborhood of x_0. By the triangle inequality,

$$||(g \circ f)(x) - (L_g \circ L_f)(x)|| \leq ||(g \circ f)(x) - (L_g \circ f)(x)|| + ||(L_g \circ f)(x) - (L_g \circ L_f)(x)||.$$

We shall handle the two terms on the right separately.

For the first term, fix some $M > ||f'(x_0)||$. Since g is differentiable at $f(x_0)$, there exists r such that if $||y - f(x_0)|| \leq r$, then

$$||g(y) - L_g(y)|| \leq \frac{\varepsilon}{2M} ||y - f(x_0)||.$$

Since f is differentiable at x_0, it is continuous at x_0, so there exists $\delta_1 > 0$ such that if $||x - x_0|| \leq \delta_1$, then $||f(x) - f(x_0)|| \leq r$. Finally, since f is differentiable at x_0, there exist $M > 0$ and $\delta_2 > 0$ such that if $||x - x_0|| \leq \delta_2$, then

$$||f(x) - f(x_0)|| \leq M||x - c||.$$

Thus for any x with $||x - x_0|| \leq \min(\delta_1, \delta_2)$, taking $y = f(x)$ above,

$$||(g \circ f)(x) - (L_g \circ f)(x)|| \leq \frac{\varepsilon}{2M} ||f(x) - f(x_0)|| \leq \frac{\varepsilon}{2} ||x - x_0||.$$

For the second term, since f is differentiable at x_0, there exists $\delta_3 > 0$ such that if $||x - x_0|| < \delta_3$, then

$$||f(x) - L_f(x)|| \leq \frac{\varepsilon}{2||L_g||} ||x - x_0||.$$

Thus

$$||(L_g \circ f)(x) - (L_g \circ f)(x_0)|| \leq ||L_g|| ||f(x) - L_f(x)|| \leq \frac{\varepsilon}{2} ||x - x_0||.$$

Combining the bounds yields the desired inequality to show that $g \circ f$ is differentiable at x_0 with derivative $L_g \circ L_f$.

Spring 2010 #10. Let $f(x, y)$ be the function defined by

$$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$$

when $(x, y) \neq (0, 0)$ with $f(0, 0) = 0$.

(a) Compute the directional derivatives of $f(x, y)$ at $(0, 0)$ in all directions where they exist.

(b) Is $f(x, y)$ differentiable at $(0, 0)$? Prove your answer.

(a) Let $u = (u_1, u_2)$ be a unit vector. Then the directional derivative of f at $(0, 0)$ in the direction of u, if it exists, is

$$\lim_{h \to 0} \frac{f(hu) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h^2 u_1 u_2}{h|h|} = \lim_{h \to 0} \frac{h}{|h|} u_1 u_2.$$

In particular, the directional derivatives of f along the x and y axes exist and equal 0. But since the limit for negative h does not equal the limit for positive h otherwise, all other directional derivatives of f do not exist.

(b) No. Suppose for the sake of contradiction that f is differentiable at $(0,0)$. Then since the x and y directional derivatives of f are 0, the derivative of f must be the 0 linear map. For any $x, y > 0$,

$$\frac{|f(x,y) - (f(0,0) + 0)|}{||(x,y) - (0,0)||_2} = \frac{|xy|}{x^2 + y^2}.$$

Thus approaching $(0,0)$ by (x,x) as $x \to 0$, this quotient is $\frac{1}{2}$, which does not approach 0. Hence f is not differentiable at $(0,0)$.

Spring 2011 #10. Suppose that f is a function defined on an open subset G of $\mathbb{R}^2$ and that $(x_0, y_0) \in G$.

1. Define what it means for f to be differentiable at (x_0, y_0).

2. Show that if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on an open set containing (x_0, y_0), then f is differentiable at $(x_0, y_0) \in G$.

1. f is differentiable at (x_0, y_0) if there exists a linear map Df such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{||f(x,y) - (f(x_0,y_0) + (Df)(x - x_0, y - y_0))||_2}{||(x,y) - (x_0,y_0)||_2} = 0.$$

2. We focus on the case where f maps into the real numbers. Let $\varepsilon > 0$. Using continuity of the first partials, select $\delta > 0$ such that if $||(x,y) - (x_0,y_0)|| \leq \delta$, then

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(x_0,y_0)\right| \leq \varepsilon \text{ and } \left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(x_0,y_0)\right| \leq \varepsilon.$$

Fix x, y such that $||(x,y) - (x_0,y_0)|| \leq \delta$. By the Mean Value Theorem, there exists some x^* lying between x and x_0 and y^* lying between y and y_0 such that

$$f(x,y) - f(x,y_0) = \frac{\partial f}{\partial y}(x,y^*)(y - y_0)$$

and

$$f(x,y_0) - f(x_0,y_0) = \frac{\partial f}{\partial x}(x^*,y_0)(x - x_0).$$

Thus

$$\begin{aligned} &f(x,y) - (f(x_0,y_0) + (x - x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0,y_0)) \\ &= (x - x_0)\left[\frac{\partial f}{\partial x}(x^*,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)\right] + (y - y_0)\left[\frac{\partial f}{\partial y}(x,y^*) - \frac{\partial f}{\partial y}(x_0,y_0)\right]. \end{aligned}$$

The expressions in braces have norm less than ε, hence by the Cauchy-Schwarz inequality, the norm of this expression is less than $\varepsilon\sqrt{2}||(x,y) - (x_0,y_0)||$. Thus f is differentiable at (x_0,y_0).

Fall 2003 #2. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely often differentiable function. Assume that for each element $x \in [0,1]$ there is a positive integer m such that the m-th derivative of f at x is not zero. Prove that there exists an integer M such that the following stronger statement holds: For each element $x \in [0,1]$, there is a positive integer m with $m \leq M$ such that the m-th derivative of f at x is not zero.

For each positive integer n, let

$$S_n = \{x \in [0,1] : \text{ there exists } 0 < m \leq n \text{ with } f^{(m)}(x) \neq 0\}.$$

Since f is infinitely differentiable, its derivatives are continuous, so the sets S_n are a union of open sets and hence open. By assumption, $\bigcup_{n\geq 1} S_n = [0,1]$. But $[0,1]$ is compact, so there exists some $n_1 < \cdots < n_k$ such that $[0,1] = \bigcup_{i=1}^k S_{n_i}$. Set $M = n_k$. Note the S_n are increasing with n, hence

$$[0,1] = S_M,$$

as desired.

Fall 2007 #2. Let $f : (a,b) \to \mathbb{R}$ be continuous and differentiable in $(a,b) \setminus \{c\}$. If $\lim_{x\to c} f'(x) = d \in \mathbb{R}$, show that f is differentiable at c, and $f'(c) = d$.

I assume $c \in (a,b)$. Let $\varepsilon > 0$. Select $\delta > 0$ such that if $x \in (a,b)$ with $|x-c| \leq \delta$, then

$$|f'(x) - d| \leq \varepsilon.$$

Let $0 < \delta' < \delta/2$. Choose $z \in (a,b)$ with $|z-c| \leq \delta' < \delta/2$. Since f is differentiable at z, there exists $\delta_1 > 0$ such that if $|x-z| \leq \delta_1$,

$$|f(x) - (f(z) + f'(z)(x-z)| \leq \varepsilon.$$

For any $x \in (a,b)$ with $|x-c| \leq \min(\delta'/2, \delta_1/2)$, we have $|x-z| \leq \delta_1$ and

$$|f(x)-(f(c)+d(x-c))| \leq |f(x)-(f(z)+f'(z)(x-z))| + |f(z)-f(c)| + |f'(z)(x-z)-d(x-z)| + |d(z-c)|$$

$$\leq \varepsilon + \varepsilon + d\delta'.$$

Choosing ε and δ' small enough, we see that f is differentiable at c with $f'(c) = d$.

Fall 2007 #9. Suppose $u_n : \mathbb{R} \to \mathbb{R}$ is differentiable and solves

$$u_n'(x) = F(u_n(x), x),$$

where F is continuous and bounded.

(a) Suppose $u_n \to u$ uniformly. Show that u is differentiable and solves

$$u'(x) = F(u(x), x).$$

(b) Suppose

$$u'(x) = F(u(x), x), \qquad u(x_0) = y_0$$

has a unique solution $u : \mathbb{R} \to \mathbb{R}$ and $u_n(x_0)$ converges to y_0 as $n \to \infty$. Show that u_n uniformly converges to u.

(a) The given ODE is equivalent, via the fundamental theorem of calculus to the following integral equation:

$$u_n(x) = u_n(x_0) + \int_{x_0}^x F(u_n(t), t)dt.$$

If we show that $u(x)$ also satisfies this integral equation, then by the fundamental theorem of calculus, it is differentiable with derivative satisfying the original ODE. Let $\varepsilon > 0$ and fix $x \neq x_0$. Since F is continuous, there exists $\delta < \varepsilon/3$ such that if $|u-v|_{\text{sup}} < \delta$, then $|F(u,x) - F(v,x)| \leq \varepsilon/(3(x-x_0))$. From uniform convergence, there exists N such that if $n \geq N$, $|u_n - u|_{\text{sup}} \leq \delta$. Thus for all $n \geq N$,

$$|F(u_n, x) - F(u, x)| \leq \varepsilon/3.$$

Also, for any x, $|u(x) - u_n(x)| \le \delta < \varepsilon$. Therefore,

$$|u(x) - u(x_0) - \int_{x_0}^{x} F(u(t),t)dt| = |u(x) - u_n(x) + u_n(x_0) - u(x_0) + \int_{x_0}^{x} (F(u_n(t),t) - F(u(t),t))dt|$$

$$\le |u(x) - u_n(x)| + |u(x_0) - u_n(x_0)| + \int_{x_0}^{x} |F(u_n(t),t) - F(u(t),t)|dt$$

$$\le \varepsilon/3 + \varepsilon/3 + (x - x_0)\frac{\varepsilon}{3(x - x_0)} = \varepsilon.$$

Thus $u(x)$ satisfies the integral equation and hence the given ODE.

(b) It suffices to show the u_n converge uniformly, since part (a) then implies the u_n converge to some solution u' which satisfies the given system. By the assumption of uniqueness, $u = u'$, so the u_n converge to u.

We want to use the Arzela-Ascoli Theorem on the u_n, so we need them to be equicontinuous and uniformly bounded. Consider the compact set $[-M, M]$. Since F is bounded (say by K),

$$|u_n(y) - u_n(x)| \le \int_{x}^{y} |F(u_n(t),t)|dt \le K|y - x|.$$

Thus the u_n are equicontinuous.

Let $\varepsilon > 0$. Since the $u_n(x_0)$ converge to y_0, there exists N such that for all $n \ge N$, $|u_n(x_0) - y_0| \le \varepsilon$. Then the above integral inequality is enough to guarantee that the u_n are uniformly bounded.

So the u_n, along with any subsequence u_{n_k}, satisfy the hypothesis of the Arzela-Ascoli theorem on $[-M, M]$, so every subsequence of the u_n has a uniformly convergent sub-subsequence. Therefore, the u_n are uniformly convergent on $[-M, M]$.

NOTE: This does not obviously imply they converge to u on $[-M, M]$. The statement to prove could be false. We would need the guarantee of a unique solution on every closed interval with x_0 as an endpoint.

Spring 2007 #8. Suppose the functions f_n are twice continuously differentiable on $[0, 1]$ and satisfy

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in [0, 1], \text{ and}$$

$$|f_n'(x)| \le 1, \qquad |f_n''(x)| \le 1 \text{ for all } x \in [0, 1], n \ge 1.$$

Prove that $f(x)$ is continuously differentiable on $[0, 1]$.

Mean Value Theorem + Arzela-Ascoli on f_n and f_n' implies uniform convergence of both of these sequences. Uniform convergence of continuous functions to a continuous function then implies f and $\lim_{n\to\infty} f_n'$ are continuous. Tao's Theorem 14.7.1 implies $f' = \lim_{n\to\infty} f_n'$, so f' is continuous and f is continuously differentiable.

Riemann integration

Fall 2003 #4; Fall 2010 #2; Spring 2007 #9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. State the definition of the Riemann integral

$$\int_0^1 f(x)dx$$

and prove that it exists.

First, we require that f is bounded on $[0,1]$ to be Riemann integrable. We define

$$\underline{\int_0^1} f = \sup\{\int_0^1 g : g \text{ a piecewise constant function on } [0,1] \text{ which minorizes } f\}$$

and

$$\overline{\int_0^1} f = \inf\{\int_0^1 g : g \text{ a piecewise constant function on } [0,1] \text{ which majorizes } f\},$$

integrate piecewise constant functions in the obvious way, and say that f is Riemann integrable if

$$\overline{\int_0^1} f = \underline{\int_0^1} f.$$

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let $\varepsilon > 0$. Since f is continuous on a compact set, f is bounded and uniformly continuous. Choose $\delta > 0$ such that if $|x - y| \leq \delta$, then $|f(x) - f(y)| \leq \varepsilon/2$. Choose N such that $\frac{1}{N} \leq \delta$. Form the equally spaced partition $x_0 = 0 < x_1 < \cdots < x_N = 1$ with spacing $1/N$. Define $\overline{f} : [0,1] \to \mathbb{R}$ so that $\overline{f}(x) = f(x_n) + \varepsilon/2$ for $x \in [x_n, x_{n+1})$ and $f(1) = 1$. Then $\overline{f}$ is piecewise constant and majorizes f.

Likewise, define $\underline{f}$. We compute

$$\int_0^1 \overline{f} - \int_0^1 \underline{f} \leq \sum_{n=0}^{N-1} \frac{1}{N}\varepsilon = \varepsilon.$$

Since ε was arbitrary, f is integrable.

Fall 2012 #2. Spring 2013 #1. Let f be a bounded, non-decreasing function on the closed interval $[0,1]$. Prove that $\int_0^1 f(x)dx$ exists.

Define the evenly-spaced partition $0 = x_0 < \cdots < x_N = 1$. Note that $\overline{f} : [0,1] \to \mathbb{R}$ given by $\overline{f}(x) = f(x_{n+1})$ on $(x_n, x_{n+1}]$ and $\overline{f}(0) = f(0)$ is piecewise constant and majorizes f. Likewise, define $\underline{f}$ by the left endpoint. Then the integral of $\overline{f} - \underline{f}$ is a telescoping series which tends to 0 as $N \to \infty$.

Fall 2007, #11. Let f be a bounded real function on $[0,1]$. Show that f is Riemann integrable if and only if f^3 is Riemann integrable.

Note f^3 is bounded on $[0,1]$. Use majoring piecewise constant functions of f to construct majorizing piecewise constants of f^3, and the identity

$$f^3(x) - f^3(y) = (f(x) - f(y))(f^2(x) + f(x)f(y) + f^2(y)).$$

Spring 2011 #9. Prove that if $f(x)$ is a continuous function on $[a,b]$ and $f(x) \geq 0$, then $\int_a^b f(x) = 0$ implies that $f = 0$.

Standard.

Fall 2011 #5. Give an example of a function $f(x)$ on $[0,1]$ with infinitely many discontinuities, but which is Riemann integrable. Include proof (don't just quote some theorem).

Just make something monotone increasing. Use $1/2^n$ - sized intervals.

Fall 2010 #11. Find the function $g(x)$ which minimizes

$$\int_0^1 |f'(x)|^2 dx$$

among smooth functions $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $f(1) = 1$. Is the optimal solution $g(x)$ unique?

In general, the Euler-Lagrange equation provides such optimizing functions.
Note that

$$\int_0^1 (f'(x) - 1)^2 = \int_0^1 (f'(x))^2 dx - 2\int_0^1 f'(x) dx + 1 = \int_0^1 (f'(x))^2 dx - 1,$$

and the left hand side is minimized only by $f(x) = x$, thus only $f(x) = x$ minimizes

$$\int_0^1 (f'(x))^2 dx.$$

Spring 2011 #8; Fall 2008 #3; Spring 2008 #2. Give examples:

1. A function $f(x)$ on $[0,1]$ which is not Riemann integrable, for which $|f(x)|$ is Riemann integrable.

2. Continuous functions f_n and f on $[0,1]$ such that $f_n(t) \to f(t)$ for all $t \in [0,1]$ but $\int_0^1 f_n(t)dt$ does not converge to $\int_0^1 f(t)dt$.

(a) -1 on rationals, 1 on irrationals.

(b) Small triangle near 0 with integral 1, length 1 / n, height n, converges to 0.

Fall 2002 #4; Spring 2009 #10; Spring 2013 #12. (a) Rigorously justify the following:

$$\int_0^1 \frac{dx}{1+x^2} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-1)^n}{2n+1}.$$

(b) Deduce the value of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

(a) For any $|x| < 1$, the series

$$\sum_{n=0}^{\infty} (-1)^n x^{2n}$$

is absolutely convergent (as a geometric series). Fix $a < 1$. The partial sums of this series converge on the compact set $[0,a]$, thus they converge uniformly on $[0,a]$. Thus for any $a \in [0,1)$,

$$\int_0^a \frac{dx}{1+x^2} = \int_0^1 \sum_{n=0}^{\infty} (-1)^n a^{2n} = \int_0^1 \lim_{N\to\infty} \sum_{n=0}^{N} (-1)^n a^{2n}$$

$$= \lim_{N\to\infty} \int_0^1 \sum_{n=0}^N (-1)^n a^{2n} = \lim_{N\to\infty} \sum_{n=0}^N \frac{(-1)^n}{2n+1} a^{2n+1}.$$

By the alternating series test,

$$\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}$$

converges, hence by applying Abel's Theorem in the final equality below,

$$\int_0^1 \frac{dx}{1+x^2} = \lim_{a\to 1-} \int_0^a \frac{dx}{1+x^2} = \lim_{a\to 1-} \lim_{N\to\infty} \sum_{n=0}^N \frac{(-1)^n}{2n+1} a^{2n+1} = \lim_{N\to\infty} \sum_{n=0}^N \frac{(-1)^n}{2n+1}.$$

(b) We know $\int_0^1 \frac{dx}{1+x^2} = \arctan x|_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}$, thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Winter 2002 #1. (a) State some reasonably general conditions under which this "differentiation under the integral sign" formula is valid:

$$\frac{d}{dx} \int_c^d f(x,y)dy = \int_c^d \frac{\partial f}{\partial x} dy.$$

(b) Prove that the formula is valid under the conditions you gave in part (a).

(a) Let $f : [a,b] \times [c,d] \to \mathbb{R}$. Suppose $\frac{\partial f}{\partial x}$ exists on $(a,b) \times [c,d]$ and extends to a continuous function on $[a,b] \times [c,d]$. Let

$$F(x) = \int_a^b f(x,y)dy.$$

Then

$$\frac{d}{dx} F(x) = \int_a^b \frac{\partial f}{\partial x}(x,y)dy.$$

(b) For h with $x+h \in [a,b]$, we estimate

$$\left| \frac{F(x+h) - F(x)}{h} - \int_a^b \frac{\partial f}{\partial x}(x,y)dy \right| = \left| \int_a^b \left(\frac{f(x+h,y) - f(x,y)}{h} - \frac{\partial f}{\partial x}(x,y) \right) dy \right|$$

$$\leq \int_a^b \left| \frac{f(x+h,y) - f(x,y)}{h} - \frac{\partial f}{\partial x}(x,y) \right| dy$$

By the mean value theorem, there exists $c \in (0,1)$ such that

$$\frac{f(x+h,y) - f(x,y)}{h} = \frac{\partial f}{\partial x}(x+ch, y).$$

Since $\frac{\partial f}{\partial x}$ is continuous on the compact set $[a,b] \times [c,d]$, it is uniformly continuous on this set. Choose δ such that $||(x,y) - (x',y')|| \leq \delta$ implies

$$\left| \frac{\partial f}{\partial x}(x',y') - \frac{\partial f}{\partial x}(x,y) \right| \leq \frac{\varepsilon}{b-a}.$$

Then using the above estimate, for $h < \delta$,

$$\left|\frac{F(x+h)-F(x)}{h} - \int_a^b \frac{\partial f}{\partial x}(x,y)dy\right| \le \int_a^b \left|\frac{f(x+h,y)-f(x,y)}{h} - \frac{\partial f}{\partial x}(x,y)\right| dy$$

$$= \int_a^b \left|\frac{\partial f}{\partial x}(x+ch,y) - \frac{\partial f}{\partial x}(x,y)\right| dy \le \int_a^b \frac{\varepsilon}{b-a} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have the desired equality.

Spring 2004 #3; Spring 2006 #1. Show that if f_n are Riemann integrable functions on $[0,1]$ and f_n converges to f uniformly, then f is Riemann integrable.
Fall 2004 #3. Show that if $f_n \to f$ uniformly on the bounded closed interval $[a,b]$, then

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

Let $\varepsilon > 0$. There exists N such that for all $n \ge N$ and all $x \in [a,b]$, $|f_n(x) - f(x)| \le \varepsilon/2$. By definition of the Riemann integral, there exists some $\overline{f_n}$ which is piecewise constant, majorizes f_n and $\int_a^b \overline{f_n} - \int_a^b f_n \le \varepsilon$. It follows that $\overline{f_n} + \varepsilon$ is piecewise constant and majorizes f. Thus for any $n \ge N$,

$$\overline{\int_a^b} f \le \int_a^b (\overline{f_n} + \varepsilon) = (\int_a^b \overline{f_n}) + (b-a)\varepsilon \le \int_a^b f_n + (b-a+1)\varepsilon.$$

Likewise,

$$\underline{\int_a^b} f \ge \int_a^b f_n - (b-a+1)\varepsilon.$$

Letting $\varepsilon \to 0$, we see that f is Riemann integrable and

$$\int_a^b f_n \to \int_a^b f.$$

Spring 2010 #12. Assume that $\{f_n\}$ is a sequence of nonnegative continuous functions on $[0,1]$ such that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$. Is it necessarily true that

(a) There is a B such that $f_n(x) \le B$ for $x \in [0,1]$ for all n?

(b) There are points x_0 in $[0,1]$ such that $\lim_{n\to\infty} f_n(x_0) = 0$?
Prove your answers.

(a) No, triangles to height n, width $1/n^2$.

(b) No, move triangles in (a) around.

Fall 2005 #3. **(a)** Prove that if $f_j : [0,1] \to \mathbb{R}$ is a sequence of continuous functions which converges uniformly on $[0,1]$ to a (necessarily continuous) function $F : [0,1] \to \mathbb{R}$ then

$$\int_0^1 F^2(x)dx = \lim_{j\to\infty} \int_0^1 f_j^2(x)dx.$$

(b) Give an example of a sequence $f_j : [0,1] \to \mathbb{R}$ of continuous functions which converges to a continuous function $F : [0,1] \to \mathbb{R}$ pointwise and for which

$$\lim_{j\to\infty} \int_0^1 f_j(x)^2 dx \text{ exists but}$$

$$\lim_{j\to\infty} \int_0^1 f_j^2(x)dx \neq \int_0^1 F^2(x)dx.$$

(f_j converges to F "pointwise" means that for each $x \in [0,1]$, $F(x) = \lim_{j\to\infty} f_j(x)$).

(a) Since F is continuous on a compact interval, it is bounded. Since the f_j converge to F uniformly, it follows that there exists J such that all f_j with $j \geq J$ and f are uniformly bounded (say in $[-M, M]$). Let $\varepsilon > 0$. Choose $J' > J$ such that $|f_j - F|_{\sup} \leq \varepsilon/(2M)$ for all $j \geq J'$. It follows that

$$|f_j^2 - F^2|_{\sup} = |f_j - F|_{\sup}|f_j + F|_{\sup} \leq (\varepsilon/(2M))(2M) = \varepsilon.$$

Hence $(f_j^2)_{j=1}^\infty$ converges uniformly to F^2. The result follows by previous exercises.

(b) Let $f_j(x) = j^{3/2}(1/j - x)$ for $x \in [0, 1/j]$ and 0 elsewhere. Then f_j is continuous for each j and

$$\int_0^1 f_j^2(x) = j^3 \int_0^{1/j} (1/j^2 - 2x/j + x^2)dx = j^3(1/j^3 - 1/j^3 + 1/(3j^3)) = 1/3.$$

Thus

$$\lim_{j\to\infty} \int_0^1 f_j^2(x)dx = \lim_{j\to\infty} 1/3 = 1/3,$$

but $F = 0$, so

$$\lim_{j\to\infty} \int_0^1 f_j^2(x)dx = 1/3 \neq 0 = \int_0^1 F^2(x)dx.$$

Spring 2005 #8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 (i.e., continuously differentiable). Show that

$$\lim_{n\to\infty} \sum_{j=1}^n \left| f\left(\frac{j-1}{n}\right) - f\left(\frac{j}{n}\right) \right|$$

is equal to

$$\int_0^1 |f'(t)|dt.$$

Note f' is uniformly compact on $[0,1]$. Let $\varepsilon > 0$. Let N be large enough that for all $x, y \in [0,1]$ with $|x - y| < \frac{1}{N}$, we have $||f'(x)| - |f'(y)|| \leq \varepsilon$. Fix $n \geq N$. By the mean value theorem, there exists $x_n \in (j-1/n, j/n)$ such that $f(\frac{j-1}{n}) - f(\frac{j}{n}) = \frac{1}{n} f'(x_n)$. Also let $|f'|$ achieve its maximum at $y_n \in [j-1/n, j/n]$ for each n. Then

$$\left| \int_0^1 |f'(t)|dt - \sum_{j=1}^n \left| f(\frac{j-1}{n}) - f(\frac{j}{n}) \right| \right| \leq \left| \frac{1}{n} \sum_{j=1}^n |f'(y_n)| - \frac{1}{n} \sum_{j=1}^n |f'(x_n)| \right|$$

$$\leq \frac{1}{n} \sum_{j=1}^n ||f'(y_n)| - |f'(x_n)|| \leq \frac{1}{n}(n\varepsilon) \leq \varepsilon.$$

Hence

$$\lim_{n\to\infty} \sum_{j=1}^n \left| f\left(\frac{j-1}{n}\right) - f\left(\frac{j}{n}\right) \right| = \int_0^1 |f'(t)|dt.$$

Fall 2007 #5. (a) Show that, given a continuous function $f : [0,1] \to \mathbb{R}$, which vanishes at $x = 1$, there is a sequence of polynomials vanishing at $x = 1$ which converges uniformly to f on $[0,1]$.

(b) If f is continuous on $[0,1]$, and

$$\int_0^1 f(x)(x-1)^k dx = 0 \text{ for each } k = 1, 2, \ldots,$$

show that $f(x)$ is identically 0.

(a) Fix $\varepsilon > 0$. By the Weierstrass Approximation Theorem, there exists a polynomial $P(x)$ such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0,1]$. Consider the sequence of polynomials $(P(x)(1-x^n))_{n=1}^{\infty}$. Clearly these polynomials vanish at $x = 1$. We show they converge uniformly to f on $[0,1]$. Since $f(1) = 0$, $|P(1)| < \varepsilon/2$. By continuity of P, there exists a $\delta > 0$ such that $|P(x)| < \varepsilon$ when $|1-x| < \delta$. Let $M = \sup_{[0,1]} P(x)$ and let n be large enough that $x^n < \varepsilon/M$ on $[0, 1-\delta]$. Then if $x \in [0,1]$, either $x > 1-\delta$, in which case $|x^n P(x)| \leq |P(x)| < \varepsilon$, or $x \leq 1 - \delta$, in which case $|x^n P(x)| < (\varepsilon/M)|P(x)| \leq \varepsilon$.

For any $x \in [0,1]$,

$$|P(x)(1-x^n) - f(x)| \leq |P(x) - f(x)| + |P(x)x^n| \leq 2\varepsilon.$$

Thus we have the desired uniform convergence.

(b) Let $\varepsilon > 0$. Select a sequence of polynomials $(P_n)_{n=1}^{\infty}$ vanishing at $x = 1$ which converges uniformly to f on $[0,1]$. Choose N such that $|P_n - f|_{\text{sup}} \leq \varepsilon$ for all $n \geq N$. Now each P_n can be written as a finite sum of polynomials of the form $(x-1)^k$ for $k \geq 1$, hence

$$\int_0^1 f(x)P_n(x)dx = 0.$$

Since f is bounded on $[0,1]$, fP_n also converges uniformly to f^2. Thus by uniform convergence,

$$\int_0^1 f^2(x)dx = \int_0^1 f(x)(\lim_{n\to\infty} P_n)dx = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)dx = \lim_{n\to\infty} 0 = 0.$$

From here it follows easily that f is identically 0.

Spring 2007 #6. Consider the integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s))ds$$

where $f(t,y)$ is continuous on $[0,T] \times \mathbb{R}$ and is Lipschitz in y with Lipschitz constant K. Assume that you have shown that the iterates defined by

$$y^n(t) = y_0 + \int_0^t f(s, y^{n-1}(s))ds, \qquad y^0(t) \text{ identically } y_0$$

converge uniformly to a solution $y(t)$ of the given integral equation. Show that if $Y(t)$ is a solution of the given integral equation and satisfies $|Y(t) - y_0| \leq C$ for some constant C and all $t \in [0,T]$, then $Y(t)$ agrees with $y(t)$ on $[0,T]$.

Let $\varepsilon > 0$. We estimate

$$|Y(t) - y(t)| \leq |Y(t) - y^n(t)| + |y(t) - y^n(t)|.$$

The second term on the right can be made arbitrarily small by uniform convergence, so we focus on the first. It follows that

$$|Y(t) - y^n(t)| \leq K^n \int_0^t \cdots \int_0^t |Y(s_n) - y_0| ds_1 \ldots ds_n \leq C(Kt)^n.$$

Choosing n arbitrarily large, this implies $Y(t) = y(t)$ for all $t \in [0, 1/K)$. Now repeating the previous argument with the initial condition at $t = i/K$ for each i necessary, we obtain $Y(t) = y(t)$ on $[0, T]$.

Fall 2010 #10. Suppose f is bounded and Lipschitz continuous. For $k \in \mathbb{N}$, define $x_k(t) : [0,1] \to \mathbb{R}$ by $x_k(0) = 0$ and

$$x_k(t) = x_k(n2^{-k}) + (t - n2^{-k})f(x_k(n2^{-k}))$$

for

$$n2^{-k} < t \le (n+1)2^{-k}, n \in \mathbb{N}.$$

Explain why $x_k(t)$ uniformly converges to a solution $x(t) : [0,1] \to \mathbb{R}$ of the ODE

$$x'(t) = f(x(t)), \qquad x(0) = 0,$$

as $k \to \infty$.

This is Euler's Method of solving ODEs. Petersen's solution in class is a less direct (and probably simpler) approach than this:

Let B be a bound for f and suppose $|f(x_1) - f(x_2)| \le L|x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$.

We now verify that $x_k(t)$ converges to some $x(t)$ as $k \to \infty$. Let $t \in [0,1]$. Fix $\varepsilon > 0$. Select K such that $2^{-K} \le \varepsilon$. If $t = 0$, then clearly $|x_k(t) - x_{k'}(t)| = 0 \le \varepsilon$. Now for $m \ge 0$ assume inductively that $(x_k(t))_{k=1}^{\infty}$ converges uniformly to some $x(t)$ for all $t \le m2^{-K}$. Increase K as necessary so that $|x_k(t) - x_{k'}(t)| \le \varepsilon$ for all $k, k' \ge K$. Let $k, k' \ge K$. Now there exists n, n' such that $n2^{-k} < t \le (n+1)2^{-k}$ and $n'2^{-k'} < t \le (n'+1)2^{-k}$. It follows that

$$|x_k(t) - x_{k'}(t)| = |x_k(n2^{-k}) - x_{k'}(n'2^{-k'}) + (t - n2^{-k})f(x_k(n2^{-k})) - (t - n'2^{-k'})f(x_{k'}(n'2^{-k'})|$$

$$\le |x_k(n2^{-k}) - x_{k'}(n'2^{-k'})| + B|n2^{-k} - n'2^{-k'}| + (t - n'2^{-k'})|f(x_k(n2^{-k})) - f(x_{k'}(n'2^{-k}))|$$

$$\le \varepsilon + B2^{-K} + 2^{-K}L|x_k(n2^{-k}) - x_{k'}(n'2^{-k'})| \le (1 + B + L\varepsilon)\varepsilon.$$

After finitely many such iterations, we cover all of $[0,1]$. Hence there exists $x(t)$ such that $x_k \to x$ pointwise. The above analysis also shows the sequence is uniformly Cauchy, hence its convergence to x is uniform.

Then since the x_k are continuous, x is continuous. We now verify that $x(t)$ is a solution to the ODE. Let $\varepsilon > 0$ and fix $t \in (0,1]$. Choose δ for continuity of f at $x(t)$. Clearly $x(0) = \lim_{k\to\infty} x_k(0) = \lim_{k\to\infty} 0 = 0$. Select K such that $|x_k - x|_{\sup} \le \delta$ for all $k \ge K$. By the continuity of f, we can adjust t by a tiny amount so that t does not take the form $n2^{-k}$. Let $k \ge K$ and select n such that $n2^{-k} < t \le (n+1)2^{-k}$. By the definition of x_k,

$$x_k'(t) = f(x_k(n2^{-k})).$$

Let $h < 2^{-k+1}$. Then the derivative quotient for x_k is ε-close to $f(x_k(n2^{-k}))$. Then

$$|\frac{x(t+h) - x(t)}{h} - f(x(t))| \le \frac{1}{2^{-k+1}}(|x(t+h) - x_k(t+h)| + (x(t) - x_k(t))|)$$

$$+|\frac{x_k(t+h) - x_k(t)}{h} - f(x_k(t))| + |f(x_k(t)) - f(x(t))|$$

$$\le 2^{k-1}2\varepsilon + \varepsilon + \varepsilon.$$

Hence the limit as $h \to 0$ of the left hand side is 0, so $x(t)$ is differentiable with $x'(t) = f(x(t))$, as desired.

Fall 2008 #10. Given $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we let $||v|| = (\sum |v_j|^2)^{1/2}$. If $f = (f_1, \ldots, f_n) : [a,b] \to \mathbb{R}^n$ is a continuous function, we define

$$\int_a^b f(t)dt = \left(\int_a^b f_1(t)dt, \ldots, \int_a^b f_n(t)dt \right).$$

Prove that

$$||\int_a^b f(t)dt|| \le \int_a^b ||f(t)||dt.$$

Let $y = (y_1, \dots, y_k)$, with $y_j = \int_a^b f_j(t)dt$. Then by definition $y = \int_a^b f(t)dt$. Note that

$$||y||^2 = \sum_{j=1}^n y_j^2 = \sum_{j=1}^n y_j \int_a^b f_j(t)dt = \int_a^b \left(\sum_{j=1}^n y_j f_j(t)dt \right).$$

By the Cauchy-Schwarz inequality,

$$\sum_{j=1}^n y_j f_j(t) \le ||y|| ||f(t)||,$$

thus since the integral is monotonic,

$$||y||^2 \le \int_a^b ||y|| ||f(t)||,$$

so

$$||y|| \le \int_a^b ||f(t)||.$$

Fall 2009 #10. (i) Let $I = [0, 2]$. If $f : I \to \mathbb{R}$ is a continuous function such that $\int_I f(x)dx = 36$, prove that there is an $x \in I$ such that $f(x) = 18$.

(ii) Consider $I^2 \subset \mathbb{R}^2$, and let $g : I^2 \to \mathbb{R}$ be a continuous function such that $\int_{I^2} g(x, y)dxdy = 36$. Prove that there is $(x, y) \in I^2$ such that $g(x, y) = 9$.

(a) Let $h(t) = \int_0^t f(x)dx$. Then by the fundamental theorem of calculus, h is differentiable with $h'(x) = f(x)$. The mean value theorem then ensures there exists $x \in [0, 2] = I$ with

$$36 = \int_I f(x)dx = h(2) - h(0) = h'(x)(2 - 0) = 2f(x).$$

Thus $f(x) = 18$.

(b) Suppose for the sake of contradiction that there is not $(x, y) \in I^2$ such that $g(x, y) = 9$. Then the intermediate value theorem implies that either $g(x, y) > 9$ for all $(x, y) \in I^2$ or $g(x, y) < 9$ for all $(x, y) \in I^2$. Either leads to a contradiction with $\int_{I^2} g(x, y)dxdy = 36$.

Spring 2009 #12. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $\rho : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions. Show that

$$\mathrm{div}(F) = \rho$$

for all points $(x, y, z) \in \mathbb{R}^3$ if and only if

$$\int\int_{\partial\Omega} F \cdot dS = \int\int\int_\Omega \rho dxdydz$$

for all balls Ω (with all radii $r > 0$ and all possible centers). [You may use the various standard theorems of vector calculus without proof.]

Write $F(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$. Since F is smooth, F_i is smooth. We verify the divergence theorem for one of the components, and the others follow similarly. Then by adding up the results on both sides, we get the desired equality. We can express $\Omega = \{(x, y, z) : f_1(x, y) \leq z \leq f_2(x, y), (x, y) \in D\}$, where D is the unit disc. We can split $\partial\Omega$ into $S_1 = \{(x, y, z) : z = f_1(x, y), (x, y) \in D\}$ and $S_2 = \{(x, y, z) : z = f_2(x, y), (x, y) \in D\}$. Then by the fundamental theorem of calculus and the definition of surface integrals,

$$\int\int\int_{\Omega} \frac{\partial F_3}{\partial z} dzdxdy = \int\int_D \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z} dzdxdy$$

$$= \int\int_D F_3((x, y, f_2(x, y))) - F_3((x, y, f_1(x, y)))dxdy$$

$$= \int\int_D F_3((x, y, f_2(x, y))) - \int\int_D F_3((x, y, f_1(x, y))dxdy$$

$$= \int\int_{S_2} (0, 0, F_3) \cdot dS + \int\int_{S_1} (0, 0, F_3) \cdot dS = \int\int_{\partial\Omega} (0, 0, F_3) \cdot dS.$$

Fall 2010 #12. Let us define $D(t) = \{x^2 + y^2 \leq r^2(t)\} \subset \mathbb{R}^2$, where $r(t) : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. For a given smooth, nonnegative function $u(x, t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, express the following quantity in terms of a surface integral:

$$\frac{d}{dt}\left(\int_{D(t)} u(x, t)dx\right) - \int_{D(t)} u_t(x, t)dx.$$

[You may use various theorems in Calculus without proof.]

We use Liebniz's Rule for differentiation under the integral sign, which requires the appropriate derivative to exist and be continuous.

First, switch to polar coordinates:

$$\int_{D(t)} u(x, t)dx = \int_0^{2\pi} \int_0^{r(t)} u(r, \theta, t)rdrd\theta.$$

Then by Liebniz's Rule,

$$\frac{d}{dt}\left(\int_0^{r(t)} u(r, \theta, t)rdr\right) = u(r(t), \theta, t)\frac{dr(t)}{dt}r + \int_0^{r(t)} \frac{\partial u}{\partial t}(r, \theta, t)dr.$$

Applying Liebniz's Rule again on the outer integral, we obtain

$$\frac{d}{dt}\int_{D(t)} u(x, t)dx = r\frac{dr(t)}{d\theta}\int_0^{2\pi} u(r(t), \theta, t)d\theta + \int_{D(t)} \frac{\partial u}{\partial t}(x, t)dx.$$

Thus the desired difference is

$$r\frac{dr(t)}{d\theta}\int_0^{2\pi} u(r(t), \theta, t)d\theta.$$

Taylor Series

Fall 2003 #5. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that all partial derivatives of order 3 exist and are continuous. Write down (explicitly in terms of partial derivatives of f) a quadratic polynomial $P(x, y)$ in x and y such that

$$|f(x, y) - P(x, y)| \leq C(x^2 + y^2)^{3/2}$$

for all (x, y) in some small neighborhood of $(0, 0)$, where C is a number that may depend on f but not on x and y. Then prove the above estimate.

Let

$$P(x, y) = f(0) + f_x(0)x + f_y(0)y + \frac{1}{2}[f_{xx}(0)x^2 + f_{xy}(0)xy + f_{yx}(0)xy + f_{yy}(0)y^2].$$

Since f is continuous, $f_{xy} = f_{yx}$ and

$$P(x, y) = f(0) + f_x(0)x + f_y(0)y + \frac{1}{2}[f_{xx}(0)x^2 + 2f_{xy}(0)xy + f_{yy}(0)y^2].$$

By Taylor's Theorem with Remainder, for any $v = (x, y) \in \mathbb{R}^2$, defining $g(t) = f((1 - t)0 + tv) = f(tv)$, there exists $t^* \in (0, 1)$ such that

$$R(v) := f(v) - P(v) = \frac{g^{(3)}(t^*)}{3!}.$$

It follows that

$$\frac{g^{(3)}(t^*)}{3!} = \sum_{|\alpha|=3} \frac{\partial^\alpha f(t^* v)v^\alpha}{\alpha!}.$$

Since the third partials of f are continuous, they are bounded on the compact set $\overline{B(0, 1)}$. Let M be this bound. Then for any $(x, y) \in B(0, 1)$,

$$|\frac{g^{(3)}(t^*)}{3!}| \leq M|x^3 + x^2y + y^2x + y^3| \leq 4M\max\{|x|, |y|\}^3 \leq 4M(x^2 + y^2)^{3/2}.$$

Note M depends solely on f, not x and y.

Winter 2006 #5. Consider a function $f(x, y)$ which is twice continuously differentiable. Suppose that f has its unique minimum at $(x, y) = (0, 0)$. Carefully prove that then at $(0, 0)$,

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} \geq \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

[You may use without proof that the mixed partials are equal for C^2 functions.]

Consider the Hessian matrix,

$$H = H(0, 0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0, 0) & \frac{\partial^2 f}{\partial x \partial y}(0, 0) \\ \frac{\partial^2 f}{\partial y \partial x}(0, 0) & \frac{\partial^2 f}{\partial y^2}(0, 0) \end{pmatrix}.$$

By Taylor's Theorem,

$$f(x, y) = f(0, 0) + \nabla f(0, 0) \cdot (x, y) + \frac{1}{2}(x, y) \cdot H(x, y) + R(x, y).$$

Since $(0, 0)$ is the unique minimum of f, it is a critical point, so $\nabla f(0, 0) = 0$. Suppose for the sake of contradiction that $\det(H) < 0$. Then H has a positive eigenvalue λ_1 and a negative eigenvalue λ_2. Choose v_2 with $Hv_2 = \lambda_2 v_2$. Then $f(tv_2) = f(0, 0) + \frac{1}{2}t^2\lambda_2|v_2|^2 + R(tv_2)$, so $f(tv_2)$ has a local maximum when $t = 0$. Thus, $(0, 0)$ cannot be the unique minimum point of f, a contradiction.

Fall 2010 #3. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ have continuous derivatives up to order three.

(a) State Taylor's Theorem with remainder for each of f and g.

(b) Using the statement for f, prove the statement for g.

(a) For any $x_0, x \in \mathbb{R}$, there exists λ between x and x_0 such that

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f^{(3)}(\lambda).$$

Likewise, for any $(x_0, y_0), (x, y) \in \mathbb{R}^2$, there exists a function $h_{(x_0,y_0)} : \mathbb{R}^2 \to \mathbb{R}$ such that

$$g((x, y)) = g((x_0, y_0)) + \frac{\partial g}{\partial x}(x - x_0) + \frac{\partial g}{\partial y}(y - y_0) + \frac{1}{2}(\frac{\partial^2 g}{\partial x^2}(x - x_0)^2 + 2\frac{\partial^2 g}{\partial x \partial y}(x - x_0)(y - y_0) + \frac{\partial^2 g}{\partial y^2}(y - y_0)^2)$$

$$+ \sum_{|\alpha|=3} h_{(x_0,y_0)}((x, y))((x, y) - (x_0, y_0))^\alpha,$$

and $\lim_{(x,y)\to(x_0,y_0)} h_{(x_0,y_0)}((x, y)) = 0$.

(b) It suffices to verify the statement for g when $(x_0, y_0) = 0$. Let $(x, y) \in \mathbb{R}^2$ and define $f(t) := g(tx, ty)$. Then f has continuous derivatives up to order three. By the chain rule,

$$f'(t) = \nabla f(tx, ty) \cdot (x, y),$$

and

$$f''(t) = (x, y) \cdot H_{(tx,ty)}(x, y)^t.$$

By Taylor's Theorem in 1 dimension, there exists a function $h_0 : \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to 0} h(t) = 0$ and

$$f(t) = f(0) + f'(t)t + \frac{1}{2}f''(t)t^2 + \frac{1}{3!}h_0(t)t^3.$$

Evaluating at $t = 1$, we reproduce Taylor's Theorem in 2 dimensions for g.

Spring 2008 #3. Assuming that $f \in C^4[a, b]$ is real, derive a formula for the error of approximation $E(h)$ when the second derivative is replaced by the finite-difference formula

$$f''(x) \approx \frac{f(x + h) - 2f(x) + f(x - h)}{h^2},$$

and h is mesh size. (Assume that $x, x + h, x - h \in (a, b)$).

By Taylor's Theorem, for some $\lambda \in (x, x + h)$,

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f^{(3)}(x) + \frac{h^4}{4!}f^{(4)}(\lambda).$$

Likewise, for some $\lambda' \in (x - h, x)$,

$$f(x + h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f^{(3)}(x) + \frac{h^4}{4!}f^{(4)}(\lambda').$$

Thus

$$f(x + h) - 2f(x) + f(x - h) = h^2 f''(x) + \frac{h^4}{4!}(f^{(4)}(\lambda) + f^{(4)}(\lambda')).$$

Since $f^{(4)}$ is continuous, it follows that the error in approximation is about $\frac{h^2}{12}|f^4(x)|$ for small h.

Fall 2007 #4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and its second derivative, f'' satisfies $|f''(x)| \le B$.

(a) Prove that

$$|2Af(0) - \int_{-A}^{A} f(x)dx| \le \frac{A^3}{3}B.$$

(b) Use the result of part (a) to justify the following estimate:

$$\left| \int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f(a + \frac{2k-1}{2n}(b-a)) \right| \le Cn^{-2},$$

where C is a constant that does not depend on n.

(a) Fix an $A > 0$. By Taylor's Theorem, there exists, for any $[-A, A]$, some $c \in [-A, A]$ such that

$$f(x) - f(0) - f'(0)x = \frac{f''(c)c^2}{2}.$$

Thus for any $x \in [-A, A]$, $|f(0) + f'(0)x - f(x)| \le A^2B/2$. We have

$$|2Af(0) - \int_{-A}^{A} f(x)dx| = |\int_{-A}^{A} f(0) + f'(0)x - f(x) - f'(0)xdx|$$

$$\le |\int_{-A}^{A} f(0) + f'(0)x - f(x)dx| + |\int_{-A}^{A} f'(0)xdx|$$

$$\le \int_{-A}^{A} |f(0) + f'(0)x - f(x)|dx$$

$$\le \int_{-A}^{A} \frac{A^2B}{2}dx = \frac{A^3}{3}B.$$

(b) Fix n. For $1 \le k \le n$, put $f_k(x) = f(x - a - \frac{2k-1}{2n}(b-a))$. Then

$$\int_a^b f(x)dx = \sum_{k=1}^{n} \int_{-\frac{1}{2n}(b-a)}^{\frac{1}{2n}(b-a)} f_k(x)dx.$$

By applying part (a) with $A = \frac{1}{2n}(b-a)$ to the $f_k(x)$, we obtain

$$|\int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f(a + \frac{2k-1}{2n}(b-a))| = |\sum_{k=1}^{n}\int_{-\frac{1}{2n}(b-a)}^{\frac{1}{2n}(b-a)} f_k(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f_k(0)|$$

$$\le \sum_{k=1}^{n} |\int_{-\frac{1}{2n}(b-a)}^{\frac{1}{2n}(b-a)} f_k(x)dx - \frac{b-a}{n} f_k(0)|$$

$$\le \sum_{k=1}^{n} \frac{B}{3}(\frac{b-a}{2n})^3 = \frac{B(b-a)^3}{24}n^{-2}.$$

Winter 2006 #3. Consider a function $f : [a, b] \to \mathbb{R}$ which is twice continuously differentiable (including the endpoints). Let $a = x_0 < x_1 < \cdots < x_n = b$ be the uniform partition of $[a, b]$, i.e., $x_{i+1} - x_i = (b - a)/n$ for all $0 \le i < n$. Show that there exists M such that for all $n \ge 1$,

$$\left| \frac{1}{n} \left(\frac{1}{2} f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right) - \int_a^b f(x)dx \right| \le \frac{M}{n^2}.$$

[Recall that the sum is an approximation of the integral in the Trapezoid Rule. It may be instructive to first solve the problem for $n = 1$ and then address the general case.]

In the case $n = 1$, consider the line $p(x) = f(a) + (x - a)(f(b))/(b - a)$. Note $p(a) = f(a)$, $p(b) = f(b)$, and $\int_a^b p(x) = (\frac{f(a)}{2} + \frac{f(b)}{2})(b - a)$. See the problem below for the general theory.

The general case follows easily from this case.

Spring 2013 #2. The approximation from "Simpson's Rule" for $\int_a^b f(x)dx$ is

$$S_{[a,b]}f = \left[\frac{2}{3} f\left(\frac{a+b}{2} \right) + \frac{1}{3} \left(\frac{f(a) + f(b)}{2} \right) \right] (b - a).$$

If f has continuous derivatives up to order three, prove that

$$\left| \int_a^b f(x)dx - S_{[a,b]}f \right| \le C(b - a)^4 \max_{[a,b]} |f^{(3)}(x)|,$$

where C does not depend on f.

We first want to find a quadratic polynomial $p(x)$ such that $p(a) = f(a)$, $p(b) = f(b)$, and $p((a + b)/2) = f((a + b)/2)$. This is

$$p(x) = f(a)\frac{(x - (a + b)/2)(x - b)}{(a - (a + b)/2)(a - b)} + f((a+b)/2)\frac{(x - a)(x - b)}{(a - (a + b)/2)(b - (a + b)/2)} + f(b)\frac{(x - a)(x - (a + b)/2)}{(b - a)(b - (a + b)/2)}.$$

Integrating p from a to b apparently gives the left hand side of the desired inequality.

Suppose p matches f in $N + 1$ places $x_0, \ldots, x_N$. For some $\eta \in [a, b]$,

$$f(x) - p_N(x) = \frac{(x - x_0) \cdots (x - x_N)}{(N + 1)!} f^{(N+1)}(\eta).$$

Thus

$$|\int_a^b f dx - \int_a^b p_N dx| \le \frac{(b - a)^{N+2}}{(N + 1)!} \max_{[a,b]}(|f^{(N+1)}(x)|).$$

Jacobian

Spring 2002 #7; Winter 2002 #7. Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and that the Jacobian matrix of F is everywhere nonsingular. Suppose also that $F(0) = 0$ and that $||F((x,y))|| \geq 1$ for all (x,y) with $||(x,y)|| = 1$. Prove that

$$\{(x,y) : ||(x,y)|| < 1\} \subset F(\{(x,y) : ||(x,y)|| < 1\}).$$

(Hint: Show, with $U = \{(x,y) : ||(x,y)|| < 1\}$, that $F(U) \cap U$ is both open and closed in U).

Let $V = F(U)$. Define $O_1 = V \cap U$. If O_1 is closed in U, then $O_2 = (V \cap U)^c \cap U$ is open. Thus if O_1 is also open, we have that $U = O_1 \cup O_2$ with O_1, O_2 disjoint open sets. But U is connected, and $0 \in O_1$, hence $O_2 = \emptyset$. Thus $U = O_1 = V \cap U$, so $F(U) \supset U$.

Thus it suffices to show that $O_1 = F(U) \cap U$ is both open and closed in U.

Let $(x_0, y_0) \in F(U) \cap U$. We are given that $F'((x_0, y_0))$ is nonsingular. Thus by the inverse function theorem, the range of F on $B((x_0, y_0), r)$ contains an open neighborhood of $F((x_0, y_0))$ for sufficiently small $r > 0$. Thus $F(U)$ is open, hence $U \cap F(U)$ is open in U.

Let $\{p_n\}$ be a sequence of points in $F(U) \cap U$ converging to some $p_\infty \in U$. Then there exist $q_n \in U$ with $p_n = F(q_n)$. By the Bolzano-Weierstrass Theorem, there exists a subsequence $\{q_{n_k}\}$ which converges to some $q_\infty \in \overline{U}$. By continuity of F,

$$p_\infty = F(q_\infty).$$

Now if $||q_\infty|| = 1$, then $||p_\infty|| = ||F(q_\infty)|| \geq 1$, so $p_\infty \notin U$, a contradiction. Hence $||q_\infty|| < 1$, so $q_\infty \in U$ and $p_\infty \in F(U) \cap U$. Hence $F(U) \cap U$ is closed relative to U.

Fall 2003 #6. Let $U = \{(x,y) : x^2 + y^2 < 1\}$ be the standard unit ball in $\mathbb{R}^2$ and let ∂U denote its boundary. Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and that the Jacobian determinant of F is everywhere non-zero. Suppose also that $F(x,y) \in U$ for some $(x,y) \in U$ and $F(x,y) \notin U \cup \partial U$ for all $(x,y) \in \partial U$. Prove that $U \subset F(U)$.

Same reasoning as the previous problem. These hypotheses are just a bit more general.

Lagrange multipliers

Spring 2003 #5. Consider the function $F(x,y) = ax^2 + 2bxy + cy^2$ on the set $A = \{(x,y) : x^2 + y^2 = 1\}$.

(a) Show that F has a maximum and minimum on A.
(b) Use Lagrange multipliers to show that if the maximum of F on A occurs at a point (x_0, y_0), then the vector (x_0, y_0) is an eigenvector of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

(a) A is a compact set, since $x^2 + y^2$ is continuous. Thus the continuous function F achieves its maximum and minimum on A.

(b) If a maximum occurs at (x,y,z), the method of Lagrange multipliers implies there exists λ such that $(\nabla F)(x,y,z) = \lambda(\nabla g)(x,y,z)$. In this case,

$$(2ax + 2by, 2bx + 2cy) = \lambda(2x, 2y).$$

Thus

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix},$$

so (x,y) is an eigenvector of the given matrix.

Miscellaneous

Spring 2005 #5. For a subset $X \subset \mathbb{R}$, we say that X is *algebraic* if there exists a family F of polynomials with rational coefficients, so that $x \in X$ if and only if $p(x) = 0$ for some $p \in F$.

(a) Show that the set $\mathbb{Q}$ of rational numbers is algebraic.

(b) Show that the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is not algebraic.

(a) Let $F = \{(x-q) : q \in \mathbb{Q}\}$. Then for every $q \in Q$, $(x-q)(q) = 0$, so $\mathbb{Q}$ is algebraic.

(b) The set of algebraic numbers is countable, but $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.

Fall 2009 #3. The purpose of this problem is to give a multi-variable calculus proof of the geometric and arithmetic means inequality along the concrete steps below. The inequality has numerous other proof and naturally you are not allowed to use it (or them) below.

Let $\mathbb{R}^n_+ \subset \mathbb{R}^n$ be the (open) subset of vectors all whose coordinates are *positive*, and $f : \mathbb{R}^n_+ \subset \mathbb{R}$ be defined by:

$$f(x_1, \ldots, x_n) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdot x_2 \cdots x_n}.$$

(i) Explain *carefully* why f attains a global (not necessarily unique) minimum at some $p \in \mathbb{R}^n_+$. (Hint: what happens when $x_i \to 0, \infty$?)

(ii) Find p.

(iii) Deduce that if all $x_i \in \mathbb{R}$ are positive and $\prod x_i = 1$ then$\sum x_i \geq n$, with equality iff $x_i = 1$ for all i. (This is a special case of the geometric and arithmetic means inequality, from which the general statement can be immediately deduced - no need to write down this part here.)

(i) As some $x_i \to 0$ or $x_i \to \infty$, $f(x_1, \ldots, x_n) \to \infty$. Thus we can restrict to a compact set, and f attains its minimum on the compact set.

(ii) The gradient of f is 0, hence $p = (1, 1, \ldots, 1)$.

(iii) Obvious from (ii).

Spring 2012 #5. Prove that there is a unique continuous function $y : [0,1] \to \mathbb{R}$ solving the equation

$$y(x) = e^x + \frac{y(x^2)}{2}, \qquad x \in [0,1].$$

Define $F(x, y(x)) = e^x + \frac{y(x^2)}{2}$. Then F is a contraction in y, so it has a unique fixed point. The contraction condition forces $F^n(x, y(x))$ to converge uniformly to the fixed point, so the fixed point is continuous.

Spring 2012 #6. Let γ be a smooth curve from $(1,0)$ to $(1,0)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ winding once around the origin in the clockwise direction. Compute the integral

$$I(\gamma) := \int_\gamma \frac{ydx - xdy}{x^2 + y^2}.$$

The problem does not specify we should show the integral is path independent, so we must merely choose a curve and compute. Let γ be the unit circle with clockwise orientation, given by the parametrization

$$\gamma(t) = (\cos(t), \sin(t))$$

for $t \in [0, 2\pi)$. Then

$$I(\gamma) := \int_\gamma \frac{ydx - xdy}{x^2 + y^2} = \int_0^{2\pi} \frac{\sin(t)(-\sin(t)) - \cos(t)\cos(t)dt}{1} = -2\pi.$$

Spring 2013 #5. Define the polynomials $U_n(x)$, $n = 0, 1, 2, \ldots$ as follows:

$$U_1(x) = 1, U_2(x) = 2x, U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

(a) Prove that

$$U_n(\cos(\theta)) = \frac{\sin(n\theta)}{\sin(\theta)}.$$

(b) Prove that the polynomials $U_n(x)$ satisfy:

$$\int_{-1}^1 U_m(x)U_n(x)\sqrt{1-x^2}dx = \begin{cases} 0 & \text{when } m \neq n, \\ \pi/2 & \text{when } m = n. \end{cases}$$

(a) We proceed by induction. For $n = 1$,

$$U_n(\cos(\theta)) = 1 = \sin(\theta)/\sin(\theta) = \sin(n\theta)/\sin(\theta).$$

Now assume inductively that

$$U_n(\cos(\theta)) = \frac{\sin(n\theta)}{\sin(\theta)}$$

for all non-negative integers $n \leq k$. Then

$$U_{k+1}(\cos(\theta)) = 2\cos(\theta)U_k(\cos(\theta)) - U_{k-1}(\cos(\theta)) = 2\cos(\theta)\frac{\sin(n\theta)}{\sin(\theta)} - \frac{\sin((n-1)\theta)}{\sin(\theta)}$$

$$= \frac{2\cos(\theta)\sin(n\theta) - (\sin(n\theta)\cos(\theta) - \sin(\theta)\cos(n\theta))}{\sin(\theta)}$$

$$= \frac{\sin(n\theta)\cos(\theta) + \sin(\theta)\cos(n\theta)}{\sin(\theta)} = \frac{\sin((n+1)\theta)}{\sin(\theta)},$$

which completes the induction.

(b) Make the change of variables $x = \cos(\theta)$, where θ ranges from π to 0. Then using part (a), the integral under consideration becomes

$$\int_\pi^0 \frac{\sin(m\theta)}{\sin(\theta)}\frac{\sin(n\theta)}{\sin(\theta)}\sin(\theta)(-\sin(\theta))d\theta = \int_0^\pi \sin(m\theta)\sin(n\theta)d\theta.$$

Use the identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

to finish the proof.

Spring 2005 #11. Let us make $M_n(\mathbb{C})$ into a metric space in the following fashion:

$$\text{dist}(A, B) = \left\{ \sum_{i,j} |A_{i,j} - B_{i,j}|^2 \right\}^{1/2}$$

(which is just the usual metric on $\mathbb{R}^{n^2}$.

(a) Suppose $F : \mathbb{R} \to M_n(\mathbb{C})$ is continuous. Show that the set

$$\{x \in \mathbb{R} : F(x) \text{ is invertible}\}$$

is open (in the usual topology on $\mathbb{R}$).

(b) Show that on the set given above, $x \mapsto |F(x)|^{-1}$ is continuous.

(a) The determinant function is continuous, thus $\{x \in \mathbb{R} : F(x) \text{ is invertible}\} = \{x \in \mathbb{R} : \det(F(x)) \neq 0\}$ is open as a pullback of an open set.

(b) I assume $|F(x)| = \det(F(x))$. Then this is the composition of the inverse function, the determinant, and F, which are all continuous on the given set, hence the composition is continuous.

Fall 2011 #2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *convex* if f satisfies

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y), \qquad \text{for all } x, y \in \mathbb{R}^n, 0 \leq \alpha \leq 1.$$

Assume that f is continuously differentiable and that, for some constant $c > 0$, the gradient ∇f satisfies

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \geq c(x-y) \cdot (x-y), \qquad \text{for all } x, y \in \mathbb{R}^n,$$

where $\cdot$ denotes the dot product. Show that f is convex.

Fix $x, y \in \mathbb{R}^n$. Let $\varphi(t) = f(tx + (1-t)y)$ and $\psi(t) = tf(x) + (1-t)f(y)$ for all $t \in [0,1]$. We want to show $\varphi - \psi \leq 0$ on $[0,1]$. Note $(\varphi - \psi)(0) = (\varphi - \psi)(1) = 0$. It is sufficient to show that $\frac{d}{dt}(\varphi - \psi)$ is monotone increasing in t. (The mean value theorem guarantees this by contradiction.) We compute

$$\frac{d}{dt}(\varphi(t) - \psi(t)) = \nabla f_{tx+(1-t)y} \cdot (x - y) - (f(x) - f(y)).$$

Suppose $t_1 \leq t_2$. Then by assumption,

$$(\nabla f_{t_2 x + (1-t_2)y} - \nabla f_{t_1 x + (1-t_1)y}) \cdot (t_2 - t_1)(x - y) \geq c(t_2 - t_1)^2 (x-y) \cdot (x-y).$$

Thus

$$(\nabla f_{t_2 x + (1-t_2)y}) \cdot (x-y) > (\nabla f_{t_1 x + (1-t_1)y}) \cdot (x-y),$$

so $\frac{d}{dt}(\varphi - \psi)$ is monotone increasing.

Linear Algebra

Important definitions and facts:

A monic polynomial $p \in \mathbb{F}[t]$ is said to be irreducible if the only monic polynomials from $\mathbb{F}[t]$ that divide p and 1 and p.

We say $v \neq 0$ is an eigenvector of $L : V \to V$ if there exists $\lambda \in \mathbb{F}$ with $Lv = \lambda v$. In this case λ is called an eigenvalue of L. The eigenspace corresponding to λ is

$$E_\lambda := \ker(L - \lambda 1_V).$$

The sum of eigenspaces for distinct eigenvalues is their direct sum.

The dual space of V is $V' = \hom(V, \mathbb{F})$.

The annihilator of a subspace $M \subset V$ is the subspace $M^\circ \subset V'$ such that

$$M^\circ = \{f \in V' : f(x) = 0 \text{ for all } x \in M\}.$$

The geometric multiplicity of an eigenvalue λ is $\dim(E_\lambda)$. The algebraic multiplicity of an eigenvalue λ is the number of times λ appears as a root of the characteristic polynomial. Geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.

The characteristic polynomial of L is $\chi_L(t) = \det(L - t1_V)$. Its roots are eigenvalues of L. Over $\mathbb{C}$, there are always n eigenvalues with $\lambda_1 \cdots \lambda_n = (-1)^n a_0$ and $\operatorname{tr}(L) = \lambda_1 + \cdots + \lambda_n = -a_{n-1}$. Complex roots always come in conjugate pairs.

An involution L is a linear operator such that $L^2 = 1_V$.

The minimal polynomial $\mu_L(t)$ of L is the monic polynomial of smallest rank such that $\mu_L(L) = 0$. All eigenvalues of L are roots of P. Note L is invertible if and only if $a_0 \neq 0$. Note μ_L divides any polynomial p with $p(L) = 0$. In particular, μ_L divides χ_L.

If two linear operators on an n-dimensional vector space have the same minimal polynomials of degree n, then they have the same Frobenius canonical form and are similar.

An operator $L : V \to V$ is said to be diagonalizable if we can find a basis for V that consists of eigenvectors of L. In other words, the matrix representation for L is a diagonal matrix. Note this depends on V as well as L.

A subspace $M \subset V$ is said to be L-invariant if $L(M) \subset M$.

The companion matrix of a monic polynomial $p(t) \in \mathbb{F}[t]$ is

$$C_p = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix},$$

when $p(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_1 t + \alpha_0$.

The characteristic and minimal polynomials of C_p are both $p(t)$ and all eigenspaces are one-dimensional. Thus C_p is diagonalizable if and only if all the roots of $p(t)$ are distinct and lie in $\mathbb{F}$.

The cyclic subspace corresponding to x is

$$C_x = \text{span}\{x, L(x), L^2(x), \dots, L^{k-1}(x)\},$$

where k is the smallest integer with

$$L^k(x) \in \text{span}\{x, L(x), L^2(x), \dots, L^{k-1}(x)\}.$$

An inner product on a vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) is an $\mathbb{F}$ valued pairing $(x|y)$ for $x, y \in V$, i.e., a map $(x|y) : V \times V \to \mathbb{F}$, that satisfies

(1) $(x|x) \geq 0$ and vanishes only when $x = 0$.
(2) $(x|y) = \overline{(y|x)}$.
(3) For each $y \in V$, the map $x \to (x|y)$ is linear.
The associated norm is $||x|| = \sqrt{(x|x)}$.

The adjoint is the transpose but with each entry conjugated, notation $A^* = \overline{A}^t$. This satisfies

$$(Ax|y) = (x|A^*y).$$

Note $(L_2L_1)^* = L_1^*L_2^*$ and if L is invertible, $(L^{-1})^* = (L^*)^{-1}$. Note λ is an eigenvalue for L if and only if $\overline{\lambda}$ is an eigenvalue for L^*. Moreover, these eigenvalue pairs have the same geometric multiplicity.

An orthogonal projection is a projection ($E^2 = E$) for which the range and the null space are orthogonal subspaces. E is orthogonal if and only if $E = E^*$ (E is self-adjoint, or Hermitian). Skew-adjoint: $E^* = -E$. When over a real space, this becomes symmetric and skew-symmetric.

A map is completely reducible or semi-simple if for each invariant subspace one can always find a complementary invariant subspace. Both self-adjoint and skew-adjoint maps are completely reducible (if $L(M) \subset M$, then $L(M^\perp) \subset M^\perp$.

Two inner product spaces V and W are isometric, if we can find an isometry $L : V \to W$, i.e., an isomorphism such that $(L(x)|L(y)) = (x|y)$.
Let $L : V \to W$ be an isomorphism. Then L is an isometry if and only if $L^* = L^{-1}$. When $V = W = \mathbb{R}^n$, isometries are called orthogonal matrices. When $V = W = \mathbb{C}^n$, isometries are called unitary matrices.

Gram-Schmidt procedure: Given a linearly independent set $\{x_1, \dots, x_m\}$, set $e_1 = x_1/||x_1||$ and inductively,

$$z_{k+1} = x_{k+1} - (x_{k+1}|e_1)e_1 - \cdots - (x_{k+1}|e_k)e_k$$

and

$$e_{k+1} = z_{k+1}/||z_{k+1}||.$$

The operator norm is

$$||L|| = \sup\{||Lx|| : ||x|| = 1\}.$$

This is finite in a finite dimensional inner product space. Note this implies $||L(x)|| \leq ||L||\,||x||$ for all $x \in V$.

The orthogonal complement to M in V is

$$M^\perp = \{x \in V : (x|z) = 0 \text{ for all } z \in M\}.$$

An operator $L : V \to V$ on an inner product space is normal if $LL^* = L^*L$. All self-adjoint, skew-adjoint, and isometric operators are clearly normal.

Two $n \times n$ matrices A and B are said to be unitarily equivalent if $A = UBU^*$, where $U \in U_n$ (U is unitary). If U is orthogonal, A and B are said to be orthogonally equivalent. If A and B are unitarily equivalent, A is normal / self-adjoint / skew-adjoint / unitary if and only if B is normal / self-adjoint / skew-adjoint / unitary. Two normal operators are unitarily equivalent if and only if they have the same eigenvalues (counted with multiplicities).

A unitary matrix U satisfies $U^*U = UU^* = I$. An orthogonal matrix O satisfies $OO^t = O^tO = I$.

$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Important Theorems:

The Fundamental Theorem of Algebra. Any complex polynomial of degree ≥ 1 has a root.

Characterizations of Diagonalizability. 1. V is n-dimensional, and the sum of the geometric multiplicities of all the eigenvalues is n. In particular, if all the eigenvalues are distinct, then the operator is diagonalizable.

2. There exists $p \in \mathbb{F}[t]$ such that $p(L) = 0$ and

$$p(t) = (t - \lambda_1) \cdots (t - \lambda_k)$$

where $\lambda_1, \ldots, \lambda_k$ are distinct.

Corollary: L is diagonalizable if and only if $\mu_L(t)$ factors as

$$\mu_L(t) = (t - \lambda_1) \cdots (t - \lambda_k),$$

with λ_i distinct. Since μ_L divides χ_L, all the λ_i must be eigenvalues of L.

Note a polynomial p has a multiple root if and only if p and Dp have a common root.

Cayley-Hamilton Theorem. Let L be a linear operator on a finite dimensional vector space. Then L is a root of its own characteristic polynomial:

$$\chi_L(L) = 0.$$

In particular, $\mu_L(t)$ divides $\chi_L(t)$.

Cyclic Subspace Decomposition. Let $L : V \to V$ be a linear operator on a finite dimensional vector space. Then V has a cyclic subspace decomposition

$$V = C_{x_1} \oplus \cdots \oplus C_{x_k}$$

where each C_x is a cyclic subspace. In particular, L has a block diagonal matrix representation where each block is a companion matrix:

$$[L] = C_{p_1} \oplus C_{p_2} \oplus \cdots \oplus C_{p_k}$$

and $\chi_L(t) = p_1(t) \cdots p_k(t)$. Moreover, the geometric multiplicity satisfies

$$\dim(\ker(L - \lambda 1_V)) = \text{ number of } p_i\text{s such that } p_i(\lambda) = 0.$$

In particular, we see that L is diagonalizable if and only if all of the companion matrices C_p individually have distinct eigenvalues. (In general this decomposition is not unique.)

Frobenius Canonical Form (Rational Canonical Form). Let $L : V \to V$ be a linear operator on a finite dimensional vector space. Then V has a cyclic subspace decomposition such that the block diagonal form of L,

$$[L] = C_{p_1} \oplus C_{p_2} \oplus \cdots \oplus C_{p_k}$$

has the property that p_i divides p_{i-1} for each $i = 2, \ldots, k$. Moreover, the monic polynomials $p_1, \ldots, p_k$ are unique. (The p_i are called the similarity invariants or invariant factors for L. Similar matrices have the same similarity invariants.)

Jordan-Chevalley decomposition. Let $L : V \to V$ be a linear operator on an n-dimensional complex vector space. Then $L = S + N$, where S is diagonalizable, $N^n = 0$, and $SN = NS$.

For $p(t) = (t-\lambda)^n$, C_p is similar to a Jordan block

$$[L] = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

Moreover the eigenspace for λ is 1-dimensional and is generated by the first basic vector.

Jordan Canonical Form Let $L : V \to V$ be a complex linear operator on a finite dimensional vector space. Then we can find L-invariant subspaces $M_1, \dots, M_s$ such that

$$V = M_1 \oplus \cdots \oplus M_s$$

and each $L|_{M_i}$ has a matrix representation of the form

$$[L] = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

where λ_i is an eigenvalue for L. Note each eigenvalue corresponds to as many blocks as the geometric multiplicity of the eigenvalue.

Cauchy-Schwarz Inequality

$$|(x|y)| \leq ||x|| \, ||y||$$

In $\mathbb{R}^n$, $(x|y)$ is usually $x \cdot y$.

Uniqueness of Inner Product Spaces. An n-dimensional inner product space over $\mathbb{R}$, respectively $\mathbb{C}$, is isometric to $\mathbb{R}^n$, respectively $\mathbb{C}^n$.

Orthogonal Sum Decomposition. Let V be an inner product space and $M \subset V$ a finite dimensional subspace. Then $V = M \oplus M^\perp$ and for any orthonormal basis $e_1, \dots, e_m$ for M, the projection onto M along $M^\perp$ is given by

$$\text{proj}_M(x) = (x|e_1)e_1 + \cdots + (x|e_m)e_m.$$

Also $\text{proj}_M(x)$ is the one and only point closest to x among all points in M.

Polarization. Let $L : V \to V$ be a linear operator on a complex inner product space. Then $L = 0$ if and only if $(L(x)|x) = 0$ for all $x \in V$.

Existence of Eigenvalues for Self-adjoint Operators. Let $L : V \to V$ be self-adjoint and V finite dimensional. Then L has a real eigenvalue.

The Spectral Theorem. Let $L : V \to V$ be a self-adjoint operator on a finite dimensional inner product space. Then there exists an orthonormal basis $e_1, \dots, e_n$ of eigenvectors of L. Moreover, all eigenvalues of L are real.

The Spectral Theorem for Normal Operators. Let $L : V \to V$ be a normal operator on a complex inner product space. Then there is an orthonormal basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of L.

Schur's Theorem. Let $L : V \to V$ be a linear operator on a finite dimensional complex inner product space. It is possible to find an orthonormal basis $e_1, \ldots, e_n$ such that the matrix representation $[L]$ is upper triangular in this basis.

Recurring Linear Algebra Problems

Fall 2001 #9; Fall 2002 #10; Spring 2006 #9. Let $A \in M_n(\mathbb{C})$ $(M_n(\mathbb{R}))$ be a normal (self-adjoint) matrix. Prove that there exists an orthonormal basis of $\mathbb{C}^n$ $(\mathbb{R}^n)$ such that the matrix of A is diagonal with respect to this basis.

Real case: Let S be a real n-dimensional inner product space. We proceed by induction on n to show that for each self-adjoint map $L : S \to S$, there exists an orthonormal basis for S consisting of eigenvectors of L. The case $n = 1$ is trivial. Let $n \geq 2$ and assume inductively that for all real $n-1$-dimensional real inner product spaces T and self-adjoint maps L', there exists an orthonormal basis of T consisting of eigenvectors of L'.

Lemma 1: If $L = L^*$ and $\lambda \in \mathbb{C}$ such that for some $v \in \mathbb{C}^n$ with $v \neq 0$, $Lv = \lambda v$, then λ is necessarily real.

Proof: Using that L is self-adjoint,

$$\lambda(v|v) = (\lambda v|v) = (T(v)|v) = (v|T^*(v)) = (v|T(v)) = (v|\lambda v) = \overline{\lambda}(v,v).$$

Thus since $v \neq 0$, $(v|v) \neq 0$, so $\lambda = \overline{\lambda}$, and λ is real.

Lemma 2: Let λ be an eigenvalue of L and V be the eigenspace corresponding to λ. Then $V^\perp$ is L-invariant.

Proof: For any $v \in V$ and $u \in V^\perp$,

$$(L(u)|v) = (v|L^*(u)) = (v|L(u)) = \overline{\lambda}(v|u) = 0.$$

Thus $L(u) \in V^\perp$, so $L(V^\perp) \subset V^\perp$ and $V^\perp$ is L-invariant.

Let A be a matrix representation of L. Let $\chi_A(t) = \det(A - t\mathrm{Id}_{n\times n})$ be the characteristic polynomial of L. The fundamental theorem of algebra guarantees there exists $\lambda \in \mathbb{C}$ such that $\chi_L(\lambda) = 0$. Thus $A - \lambda\mathrm{Id}_{n\times n}$ taken as an operator on $\mathbb{C}^n$ is not invertible, so $\ker(A - \lambda\mathrm{Id}_{n\times n}) \neq \{0\}$. Hence for some $v \in \mathbb{C}^n$ with $v \neq 0$, $L(v) := Av = \lambda v$. By Lemma 1, $\lambda \in \mathbb{R}$.

Thus $\chi_A(\lambda) = \det(A - \lambda\mathrm{Id}_{n\times n}) = 0$, so there exists $v \in \mathbb{R}^n$ with $v \neq 0$ and $L(v) = \lambda v$. Let $V = \mathrm{span}(v)$. Since V is 1-dimensional, it follows that $V^\perp$ is $n-1$ dimensional. By Lemma 2, $V^\perp$ is L invariant, thus $L|_{V^\perp} : V^\perp \to V^\perp$. Since L is self-adjoint, $L|_{V^\perp}$ is also self-adjoint. Applying the induction hypothesis to $L_{V^\perp}$, there exists an orthonormal basis $\{u_2, \ldots, u_n\}$ of $V^\perp$ consisting of eigenvectors of $L|_{V^\perp}$. Setting $u_1 = \frac{v}{||v||} \in V$, u_1 is a unit vector perpendicular to all u_i for $i \geq 2$ and an eigenvector of L. Hence $\{u_1, \ldots, u_n\}$ is an orthonormal basis for S consisting of eigenvectors of L. This completes the induction.

Let $L_A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map corresponding to A given by $L_A(v) := Av$. Applying the above result with $S = \mathbb{R}^n$, we see there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of L_A. Thus with respect to this basis, L_A is diagonal, as desired.

Complex case: Decompose $L = B + iC$, where B and C are self adjoint and $BC = CB$. Since B is self-adjoint, the above proof implies there exists an eigenvalue α such that $\ker(B - \alpha 1_V) \neq \{0\}$. Then $C : \ker(B - \alpha 1_V) \to \ker(B - \alpha 1_V)$. Then the restriction of C to $\ker(B - \alpha 1_V)$ is self-adjoint, so there exists $x \in \ker(B - \alpha 1_V)$ such that $C(x) = \beta x$. Then

$$L(x) = (\alpha + i\beta)x,$$

so we have found an eigenvalue and eigenvector of L. We also have

$$L^*(x) = B(x) - iC(x) = (\alpha - i\beta)x.$$

Thus $\mathrm{span}\{x\}$ is both L and L^* invariant. It follows that $M = (\mathrm{span}\{x\})^\perp$ is also L and L^* invariant. Hence $(L|_M)^* = L^*|_M$, so $L|_M : M \to M$ is also normal. We can use induction as in the real case above to finish the proof.

Spring 2002 #11; Spring 2006 #10; Spring 2008 #10; Spring 2010 #3. Let V be a complex (real) inner product space and let $\{T_\lambda\}_\Lambda$ be a collection (or two) normal (self-adjoint) operators. If $\{T_\lambda\}_\Lambda$ pairwise commute, prove that there exists an orthonormal basis for V consisting of vectors that are simultaneously eigenvectors of each T_λ.

Real case: S, T are self-adjoint commuting operators. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T and let $E_i = \ker(T - \lambda_i I)$. Since T is self-adjoint,

$$V = \bigoplus_{i=1}^{k} E_i$$

with E_i orthogonal to E_j for $i \neq j$. Suppose E_i has dimension d_i. Suppose $x \in E_i$. Then since S and T commute,

$$T(Sx) = S(Tx) = S(\lambda_i x) = \lambda_i(Sx).$$

Hence $Sx \in E_i$. Thus E_i is S-invariant. Hence $S|_{E_i}$ is self-adjoint. Thus there exists vectors $\{v_1^i, \ldots, v_{d_i}^i\}$ that constitute an orthonormal basis for E_i and are simultaneously eigenvectors of S and T. Then

$$\bigcup_{i=1}^{k} \bigcup_{j=1}^{d_i} v_j^i$$

is a basis of orthogonal vectors for V, since the E_i's are orthogonal, and it consists of simultaneous eigenvectors of S and T, as desired.

Complex Case: Induction on the number in the collection. Use the trick $B = \frac{1}{2}(A + A^*)$ and $C = \frac{1}{2i}(A - A^*)$, so $A = B + iC$, and B, C are self-adjoint. Since A is normal, $A^*A = AA^*$, thus $BC = CB$. Then proceed as above.

Fall 2005 #7. Let A be a real $n \times m$ matrix. Prove that the maximal number of linearly independent rows of A is equal to the maximum number of linearly independent columns of A.

Suppose the maximal number of linearly independent rows of A is r. Let $x_1, \ldots, x_r$ be a basis of the row space of A. Suppose the c_i are scalars such that

$$0 = c_1(Ax_1) + c_2(Ax_2) + \cdots + c_r(Ax_r) = A(c_1x_1 + \cdots + c_rx_r).$$

Then $v = c_1x_1 + \cdots + c_rx_r$ is in the row space of A, but since $Av = 0$, v is orthogonal to every vector in the row space of A. Thus v is orthogonal to itself, so $v = 0$. Thus

$$c_1x_1 + \cdots + c_rx_r = 0.$$

But $x_1, \ldots, x_r$ is a basis of the row space, hence $c_1 = \cdots = c_r = 0$. Thus $Ax_1, \ldots, Ax_r$ are linearly independent. Now each Ax_i is a vector in the column space of A, hence the dimension of the column space of A is at least r, the dimension of the row space of A. Applying the same argument to the transpose of A, we see that the dimension of the row space of A is at least the dimension of the column space of A. Hence these dimensions are equal.

Fall 2008 #12; Fall 2011 #12; Fall 2012 #9. Let A be an $m \times n$ real matrix and let $b \in \mathbb{R}^m$. Suppose Ax and Ay are both minimal distance to b (minimizing among members of $\mathrm{Im}(A)$). Prove $x - y \in \ker(A)$.

Then $||Ax - b|| = ||Ay - b||$. It follows that

$$||A((x+y)/2) - b|| \leq ||A(x)/2 - b/2|| + ||A(y)/2 - b/2|| = \frac{1}{2}(||A(x) - b|| + ||A(y) - b||) = ||A(x) - b||.$$

By the assumption, we conclude $||A((x+y)/2)-b|| = ||Ax-b||$. Suppose for the sake of contradiction that $x-y \notin \ker(A)$ so that $Ax \neq Ay$. It follows that $Ax \neq b$, $Ay \neq b$, and $A((x+y)/2) \neq b$. Then the triangles with points $Ax, A((x+y)/2), b$ and $Ay, A((x+y)/2), b$ are both isosceles and similar to one another. It follows that the two angles at $A((x+y)/2)$ are equal. Since A is linear, these angles add to 180 degrees, so they are both right angles. But then one leg of a right triangle has equal length to its hypotenuse, a contradiction. Hence $x-y \in \ker(A)$.

Fall 2003 #6; Fall 2011 #11; Fall 2008 #6; Fall 2007 #6. State and prove the Rank-Nullity Theorem.

Let $T : V \to W$ be a linear mapping, where V is finite dimensional. Then

$$\dim(V) = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)).$$

Proof: Note that the images of a basis of V will span $Im(T)$, hence $\mathrm{Im}(T)$ is finite dimensional. Choose a basis $w_1, \dots, w_n$ of $\mathrm{Im}(T)$. There exist preimages $v_1, \dots, v_n$ with $w_i = T(v_i)$ for $1 \leq i \leq n$. Select a basis $u_1, \dots, u_k$ of $\mathrm{Ker}(T)$. The result will follow once we show that $u_1, \dots, u_k, v_1, \dots, v_n$ is a basis of V.

Let $v \in V$. Since $T(v) \in \mathrm{Im}(T)$, there exist $b_1, \dots, b_n$ such that

$$T(v) = b_1 w_1 + \cdots + b_n w_n.$$

Then

$$T(b_1 v_1 + \cdots + b_n w_n - v) = 0,$$

so there exist scalars $a_1, \dots, a_k$ such that

$$b_1 v_1 + \cdots + b_n v_n - v = a_1 u_1 + \cdots + a_k u_k.$$

Thus $u_1, \dots, u_k, v_1, \dots, v_n$ span V.

Now let $a_1, \dots, a_k, b_1, \dots, b_n$ be scalars such that

$$a_1 u_1 + \cdots + a_k u_k + b_1 v_1 + \cdots + b_n v_n = 0.$$

Applying T,

$$b_1 w_1 + \cdots + b_n w_n = 0.$$

Since $w_1, \dots, w_n$ are linearly independent, $w_i = 0$ for $1 \leq i \leq n$. Then

$$a_1 u_1 + \cdots + a_k u_k = 0.$$

Since $u_1, \dots, u_n$ are linearly independent, $a_i = 0$ for $1 \leq i \leq k$. Thus $u_1, \dots, u_k, v_1, \dots, v_n$ are linearly independent and thus a basis for V.

Fall 2001 #7; Fall 2002 #7; Fall 2012 #12. Let $T : V \to W$ be a linear transformation of finite dimensional vector spaces. Define the transpose of T and then prove the following:

1. $(Im(T))^\circ = \ker(T^t)$
2. $\mathrm{Rank}(T) = \mathrm{Rank}(T^t)$, where the rank of a linear transformation is the dimension of its image.

The transpose of T is the linear map $T^t : W^* \to V^*$ defined by $T^t(f) = f \circ T$. In Petersen's book, this is called the dual map of T

1. Direction from definitions.

2. Use the Dimension Theorem and the fact that $\dim W + \dim W^\circ = \dim V$ for any subspace $W \subset V$.

Fall 2001 #10; Winter 2002 #9; Fall 2008 #7. Let V be a complex vector space and let $T : V \to V$ be a linear map. Let $v_1, \ldots, v_n$ be non-zero vectors in V, each an eigenvector for a different eigenvalue. Prove that $\{v_1, \ldots, v_n\}$ is linearly independent.

We proceed by induction. The statement is clear for $n = 1$. Now suppose $n \geq 2$ and the desired statement holds for $n - 1$. Suppose $a_1 v_1 + \cdots + a_n v_n = 0$. Applying T,

$$a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n = T(0) = 0 = \lambda_n (a_1 v_1 + \cdots + a_n v_n).$$

It follows that

$$a_1(\lambda_1 - \lambda_n)v_1 + \cdots + a_{n-1}(\lambda_{n-1} - \lambda_n)v_{n-1} = 0.$$

Since $v_1, \ldots, v_{n-1}$ are linearly independent, and $\lambda_1 - \lambda_i \neq 0$ for all $1 \leq i \leq n-1$, $a_i = 0$ for $1 \leq i \leq n-1$. Thus

$$a_n v_n = 0,$$

so since $v_n \neq 0$, $a_n = 0$. Hence $v_1, \ldots, v_n$ are linearly independent, which completes the induction.

Winter 2002 #11; Winter 2006 #10; Spring 2004 #10; Spring 2003 #8. Let V be a finite dimensional complex inner product space and let $T : V \to V$ be a linear map. Prove that there exists an orthonormal ordered basis for V such that the matrix representation of T with respect to this basis is upper triangular.

Suppose V has dimension n. We show by induction that there exists a flag of invariant subspaces

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V,$$

where $\dim(V_k) = k$ and each V_k is T-invariant with $V_k = \text{span}\{e_1, \ldots, e_k\}$. Clearly if $n = 1$, then $V_1 = V$ and we are finished.

Now assume inductively we have such a flag for all $n-1$ dimensional spaces V. Select an eigenvalue/vector pair $T^*(v) = \lambda v$ (using the fundamental theorem of algebra) and define $V_{n-1} = v^\perp = \{x \in V : (x|v) = 0\}$. Since V is n-dimensional, $v^\perp$ is $n-1$-dimensional and

$$(T(x)|y) = (x|T^*(y)) = (x|\lambda y) = \overline{\lambda}(x|y) = 0.$$

Thus V_{n-1} is T-invariant. Applying the induction hypothesis, there exists a flag $\{0\} \subset V_1 \subset \cdots \subset V_{n-1}$, where $\dim(V_k) = k$ and $T(V_k) \subset V_k$. Since $V_{n-1} \subset V$ and $T : V \to V$, we have the desired flag to V immediately, completing the induction.

Now choose unit vectors $e_k \in V_k \cap V_{k-1}^\perp$ for $1 \leq k \leq n$. Then these vectors must form an orthonormal basis of V. Then since $T(e_k) \in V_k$, we can express $T(e_k)$ as a linear combination of $e_1, \ldots, e_k$ for each k. Hence we obtain an upper triangular matrix representation for T with respect to the basis $\{e_1, \ldots, e_k\}$.

Fall 2002 #9; Winter 2006 #7. Let V be a complex inner product space. State and prove the Cauchy-Schwarz inequality.

The Cauchy-Schwarz inequality says for any $x, y \in V$,

$$|(x|y)| \leq ||x||\,||y||.$$

Proof: If $y = 0$, the inequality is obvious. Otherwise, define

$$z = x - \text{proj}_y(x) = x - \frac{(x|y)}{(y|y)} y.$$

Then

$$(z|y) = (x|y) - (x|y) = 0,$$

so z is orthogonal to y. Applying the pythagorean theorem to

$$x = \frac{(x|y)}{(y|y)}y + z,$$

$$||x||^2 = \frac{|(x|y)|^2}{||y||^2} + ||z||^2 \geq \frac{|(x|y)|^2}{||y||^2}.$$

Thus

$$||x||\,||y|| \geq |(x|y)|.$$

Spring 2003 #7; Spring 2004 #7. Let V be a finite dimensional real vector space. Let W_1 and W_2 be subspaces of V. Prove the following:

1. $W_1^\circ \cap W_2^\circ = (W_1 + W_2)^\circ$.
2. $(W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$.

1. Let $f \in W_1^\circ \cap W_2^\circ$. Then for any $x \in W_1$, $f(x) = 0$, and for any $y \in W_2$, $f(y) = 0$. Thus for any $x + y \in W_1 + W_2$, with $x \in W_1$ and $y \in W_2$,

$$f(x + y) = f(x) + f(y) = 0 + 0 = 0.$$

Hence $f \in (W_1 + W_2)^\circ$. Conversely, suppose $f \in (W_1 + W_2)^\circ$. Then for any $x \in W_1$, $x \in W_1 + W_2$, so $f(x) = 0$, hence $f \in W_1^\circ$. Likewise, $f \in W_2^\circ$, so $f \in W_1^\circ \cap W_2^\circ$. Thus we have shown part 1.

2. Let $f \in (W_1 \cap W_2)^\circ$. Then $f|_{W_1 \cap W_2} = 0$. Note that

$$f = f|_{V \setminus W_1} + f|_{W_1}.$$

Now $(f|_{V\setminus W_1})|_{W_1} = 0$ and $(f|_{W_1})|_{W_2} = 0$, thus extending these functions by 0 to the rest of the space, $f|_{V\setminus W_1} \in W_1^\circ$ and $f|_{W_1} \in W_2^\circ$. Hence $f \in W_1^\circ + W_2^\circ$. Conversely, suppose $f + g \in W_1^\circ + W_2^\circ$, where $f \in W_1^\circ$ and $g \in W_2^\circ$. Then for any $x \in W_1 \cap W_2$,

$$(f + g)(x) = f(x) + g(x) = 0 + 0 = 0,$$

hence $f + g \in (W_1 \cap W_2)^\circ$. Thus we have shown part 2.

Fall 2003 #8b. Let T be a linear transformation from a finite dimensional vector space V with an inner product to a finite dimensional vector space W also with an inner product. Show that the kernel (null space) of T^* is the orthogonal complement of the range of T.

By definition,

$$\ker(T) = \{x \in V : Lx = 0\},$$

and

$$\operatorname{im}(T^*)^\perp = \{x \in V : (x|T^*z) = 0 \text{ for all } z \in W\}.$$

Fix $x \in V$ and use that $(Tx|z) = (x|T^*z)$ for all $z \in W$. Then if $x \in \ker(T)$, then $x \in \operatorname{im}(T^*)^\perp$. Conversely, if $x \in \operatorname{im}(T^*)^\perp$, then $0 = (x|T^*z) = (Tx|z)$ for all $z \in W$. Picking $z = Tx$, we see that $Tx = 0$, so $x \in \ker(T)$. Thus

$$\ker(T) = \operatorname{im}(T^*)^\perp.$$

Replacing T in the above argument by T^* and using that $T^{**} = T$, we obtain

$$\ker(T^*) = \operatorname{im}(T)^\perp.$$

Spring 2006 #8; Fall 2009 #9. If $A \in M_{2n+1}(\mathbb{R})$ is such that $AA^t = \mathrm{Id}_{2n+1}$, then prove that one of 1 or -1 is an eigenvalue of A.

Note that $\det(T-\lambda I)$ is a real polynomial with degree $2n+1$. By the intermediate value theorem, it has a real root λ. Thus there exists $v \in \mathbb{R}^{2n+1}$ with $v \neq 0$ such that $Av = \lambda v$.

Now

$$(v|v) = (AA^*v|v) = (v|(AA^*)^*v) = (v|A^*Av) = (Av|Av) = (\lambda v|\lambda v) = |\lambda|^2(v|v).$$

Since $v \neq 0$, $(v|v) \neq 0$, hence $|\lambda|^2 = 1$. Since λ is real, $\lambda = \pm 1$.

Spring 2007 #1. Spring 2011 #6. Let V and W be finite dimensional real inner product spaces, and let $A : V \to W$ be a linear transformation. Let $w \in W$. Show that the elements $v \in V$ for which the norm $||Av - w||$ is minimal are exactly the solutions to the equations $A^*Ax = A^*w$.

Define $S = \mathrm{Im}(A)$ and let $\{e_1, \dots, e_n\}$ be an orthonormal basis for S. The vector in S closest to w is

$$s := \mathrm{proj}_S(w) := (w|e_1)e_1 + \cdots + (w|e_n)e_n.$$

A straightforward calculation shows that $w - s \in S^\perp$.

Using that $w - s \in S^\perp$, for any $z \in S$,

$$(z, w-s) = 0.$$

Let $x \in V$ such that $Ax = s$. Then for any $y \in V$, $Ay \in S$, thus

$$0 = (Ay, Ax - w) = (y, A^*(Ax - w)),$$

hence $A^*(Ax - w) = 0$. This implies $A^*Ax = A^*w$.

Conversely, suppose $A^*Ax = A^*w$. Let $s = Ax$. Then by the above calculation in reverse, $w - s \in S^\perp$. Thus for any $t \in S$, $t - s$ and $w - s$ are perpendicular, so

$$||s-w||^2 \leq ||s-w||^2 + ||s-t||^2 = ||t-w||^2,$$

hence $||s - w|| \leq ||t - w||$. Thus $v = x$ is a minimizer of $||Av - w||$.

Spring 2008 #11. Spring 2011 #2. Show that a positive power of an invertible matrix with complex entries is diagonalizable if and only if the matrix itself is diagonalizable.

Suppose A is an invertible $n \times n$ matrix and k be a positive integer. There exists an invertible matrix V such that VAV^{-1} is diagonal. It follows that $(VAV^{-1})^k = VA^kV^{-1}$ is also diagonal. Hence A^k is diagonalizable.

Conversely, suppose A^k is diagonalizable. Then $\mu_{A^k}(t) = (t-\lambda_1)\cdots(t-\lambda_k)$ with λ_i distinct. Since $\mu_{A^k}(A^k) = 0$,

$$(A^k - \lambda_1)\cdots(A^k - \lambda_k).$$

The λ_i are eigenvalues of A^k, thus since A is invertible, $\lambda_i =\neq 0$. It follows that the minimal polynomial of A divides

$$p(t) := (t^k - \lambda_1)\cdots(t^k - \lambda_k).$$

Suppose $p(r) = 0$, so there exists j with $r^k = \lambda_j$. Since $\lambda_i \neq 0$, $r \neq 0$. Note that

$$Dp(r) = (kr^{k-1})\sum_{i=1}^k (r^k - \lambda_1)\cdots(r^k \hat{-} \lambda_i)\cdots(r^k - \lambda_k)$$

$$= (kr^{k-1})\sum_{i=1}^k (\lambda_j - \lambda_1)\cdots(\lambda_j \hat{-} \lambda_i)\cdots(\lambda_j - \lambda_k)$$

$$= (kr^{k-1})(\lambda_j - \lambda_1)\cdots(\lambda_j \hat{-} \lambda_j)\cdots(\lambda_j - \lambda_k) \neq 0.$$

Hence p has no multiple roots, thus the minimal polynomial of A has no multiple roots. Since we are working over $\mathbb{C}$, the minimal polynomial of A necessarily factors as

$$\mu_A(t) = (t - \eta_1) \cdots (t - \eta_k),$$

with η_i distinct, so A is diagonalizable.

Winter 2011 #4. Fall 2012 #11. Show that an n by n matrix can be factored as $A = LU$ where L is a lower triangular matrix with ones along the diagonal and U is an upper triangular matrix provided each determinant $\det(A_j)$ (for $j \in \{1, \ldots, n-1\}$) is non-zero (where A_j is the submatrix of A consisting of the first j rows and first j columns of A).

Assume $\det(A_j) \neq 0$ for each j. We just need to show that Gaussian elimination does not need pivoting. We prove by induction on k that the kth step does not need pivoting. This holds for $k = 1$, since $A_1 = (a_11)$, so, $a_11 \neq 0$. Assume that no pivoting was necessary for the first k steps ($1 \leq k \leq n-1$). In this case, we have

$$E_k \cdots E_1 A = U_k,$$

where $L = E_k \cdots E_1$ is a unit lower-triangular matrix and $U_k[1..k, 1..k]$ is upper-triangular. Since $\det(A_{k+1}) \neq 0$, $(U_k)_{k+1,k+1} \neq 0$. Thus we can multiply U_k by unit lower-triangular matrices on the left to eliminate entries $(U_k)_{i,k+1}$ for $k+1 \leq i \leq n$. Thus the $(k+1)$st step of Gaussian elimination does not need pivoting, completing the induction.

Fall 2011 #10. Spring 2013 #9. Let A be a real orthogonal matrix. Show that A is similar to a block diagonal matrix, where each block is a scalar (which is a real eigenvalue of A) or of the form

$$T_{\alpha,\theta} = \alpha \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},$$

where $\alpha, \theta \in \mathbb{R}$.

Any complex eigenvalues of A come in conjugate pairs. And as an orthogonal matrix, all eigenvalues of A take the form $e^{i\theta}$ for some θ. Since A is diagonalizable, there exists an invertible $V \in M_n(\mathbb{C})$ such that VAV^{-1} is a diagonal matrix consisting of eigenvalues of A.

Suppose without loss of generality that $\lambda, \overline{\lambda}$ are the first eigenvalues of A in the diagonal form. Define

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

and note that

$$U^{-1} \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} U$$

has the desired form.

Also let V be a matrix with first column v and second column $\overline{v}$. It follows that UV is real, so A is similar to a block diagonal matrix of the desired form.

Other Linear Algebra Qualifying Exam Problems

Fall 2001 #8. Spring 2004 #8 (similar) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation by 60 degrees counterclockwise about the plane perpendicular to the vector $(1, 1, 1)$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection about the plane perpendicular to the vector $(1, 0, 1)$. Determine the matrix representation of $S \circ T$ in the standard basis $\{e_1, e_2, e_3\}$. You do not have to multiply the resulting matrices but you must determine any inverses that arise.

Let $v_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$, $v_2 = \frac{1}{\sqrt{2}}(1, 0, -1)$, and $v_3 = v_1 \times v_2 = \frac{1}{\sqrt{6}}(-1, 2, -1)$. Because v_1 is fixed under T and v_2 and v_3 are rotated counterclockwise by 60 degrees, the matrix representation of T in the basis $\beta = \{v_1, v_2, v_3\}$ is

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(60) & -\sin(60) \\ 0 & \sin(60) & \cos(60) \end{pmatrix}.$$

Let γ be the standard basis. The change of basis matrix is $[Id]_\beta^\gamma = (v_1, v_2, v_3)$. Since the columns are orthonormal, $[Id]_\beta^\gamma$ is orthogonal, so $[Id]_\gamma^\beta = ([Id]_\beta^\gamma)^{-1} = ([Id]_\beta^\gamma)^t$.

Do the same with $w_1 = \frac{1}{\sqrt{2}}[1, 0, 1]$, and two other perpendicular vectors and call this basis α. Then the reflection just takes $w_1 \to -w_1$ and leaves the others unchanged. Basis for S w.r.t. α is easy. The desired matrix representation is

$$[S \circ T]_\gamma^\gamma = [S]_\gamma^\gamma = [Id]_\alpha^\gamma [S]_\alpha^\alpha [Id]_\alpha^\gamma [Id]_\beta^\gamma [T]_\beta^\beta [Id]_\gamma^\beta.$$

Fall 2002 #8. Let T be the rotation of an angle 60 degrees counterclockwise about the origin in the plane perpendicular to $(1, 1, 2)$ in $\mathbb{R}^3$.

i. Find the matrix representation of T in the standard basis. Find all eigenvalues and eigenspaces of T.

ii. What are the eigenvalues and eigenspaces of T if $\mathbb{R}^3$ is replaced by $\mathbb{C}^3$?

[You do not have to multiply any matrices out but must compute any inverses.]

(i) Let $v_1 = \frac{1}{2}(1, 1, 2)$. Select $v_2 = \frac{1}{\sqrt{3}}(1, 1, -1)$. $v_3 = v_1 \times v_2$. Call this orthonormal basis α and the standard basis β.

$[T]_\alpha^\alpha = ...$

$[Id]_\alpha^\beta = [v_1 v_2 v_3]$.

Then $[T]_\beta^\beta = [Id]_\alpha^\beta [T]_\alpha^\alpha [Id]_\beta^\alpha = [Id]_\alpha^\beta [T]_\alpha^\alpha ([Id]_\alpha^\beta)^t$.

(ii) Find the eigenvalues and eigenspaces corresponding to $[T]_\alpha^\alpha$:

$$[T]_\alpha^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

Eigenvalues : 1, $e^{\pi i/3}, e^{2\pi i/3}$.

Spring 2002 #8. Let V be a finite dimensional real vector space. Let $W \subset V$ be a subspace and

$$W^\circ := \{f : V \to F \text{ linear } | f = 0 \text{ on } W\}.$$

Prove that

$$\dim(V) = \dim(W) + \dim(W^\circ).$$

Let $w_1, \ldots, w_k$ be a basis for W, and extend it to a basis $w_1, \ldots, w_k, v_{k+1}, \ldots, v_n$ for V. Define $f_i(v) = 1$ for $v = v_{k+i}$ and 0 otherwise. Then $f_i \in W^\perp$ and they are linearly independent. Also, any function in $W^\perp$ can be expressed as a linear combination of these. Hence $\{f_i\}_{i=1}^{n-k}$ is a basis for $W^\perp$, so we get the desired dimension equation.

Spring 2002 #9. Find the matrix representation in the standard basis for either rotation by an angle θ in the plane perpendicular to the subspace spanned by the vectors $(1,1,1,1)$ and $(1,1,1,0)$ in $\mathbb{R}^4$. [You do not have to multiply the matrices out but must compute any inverses.]

Additional perpendicular vectors are $(1,-1,0,0)$ and $(1,1,-2,0)$. Then follow previous problem.

Spring 2002 #10. Let V be a complex inner product space and W a finite dimensional subspace. Let $v \in V$. Prove that there exists a unique vector $v_W \in W$ such that

$$||v - v_W|| \leq ||v - w||$$

for all $w \in W$. Deduce that equality holds if and only if $w = v_W$.

Let $w_1, \dots, w_k$ be an orthogonal basis for W. Define

$$v_W = \text{proj}_W v := \frac{(v|w_1)}{(w_1|w_1)} w_1 - \cdots - \frac{(v|w_k)}{(w_k|w_k)} w_k.$$

Let $s = v - v_W$. Then $(s|w_i) = 0$ for all $1 \leq i \leq k$. Hence $s \in W^\perp$. Thus for any $w \in W$, s and $v_W - w$ are perpendicular. By the pythagorean theorem,

$$||v - w||^2 = ||v - v_W||^2 + ||w - v_W||^2 \geq ||v - v_W||^2,$$

hence

$$||v - w|| \geq ||v - v_W||$$

with equality if and only if $w = v_W$. This equality case forces uniqueness.

Winter 2002 #8. Let $T : V \to W$ and $S : W \to X$ be linear transformations of finite dimensional real vector spaces. Prove that

$$\text{rank}(T) + \text{rank}(S) - \dim(W) \leq \text{rank}(S \circ T) \leq \max\{\text{rank}(T), \text{rank}(S)\}.$$

From the Dimension Theorem,

$$\dim(V) = \text{rank}(T) + \text{nullity}(T),$$

$$\dim(W) = \text{rank}(S) + \text{nullity}(S),$$

and

$$\dim(V) = \text{rank}(S \circ T) + \text{nullity}(S \circ T).$$

Thus

$$\text{rank}(T) + \text{rank}(S) - \dim(W) = \text{rank}(S \circ T) + \text{nullity}(S \circ T) - \text{nullity}(T) - \text{nullity}(S).$$

Show that

$$\text{nullity}(S \circ T) \leq \text{nullity}(S) + \text{nullity}(T)$$

by appealing to a basis. This gives the first inequality.

Also, by appealing to a basis, we can check $\text{rank}(S \circ T) \leq \min\{\text{rank}(T), \text{rank}(S)\}$.

Winter 2002 #10. Let V be a finite dimensional complex inner product space and $f : V \to V$ a linear functional. Show that there exists a vector $w \in V$ such that $f(v) = (v|w)$ for all $v \in V$.

Let $\{v_1, \dots, v_n\}$ be an orthonormal basis for V. Let $\{f_1, \dots, f_n\}$ be the associated dual basis. In particular, this means $f_i(v) = (v, v_i)$ for all $v \in V$. There exists α_i with

$$f = \alpha_1 f_1 + \cdots + \alpha_n f_n.$$

Thus
$$f(v) = \sum_{i=1}^{n} \alpha_i f_i(v) = \sum_{i=1}^{n} \alpha_i (v, v_i) = (v, \sum_{i=1}^{n} \overline{\alpha_i} v_i).$$
Thus if we take
$$w = \overline{\alpha_1} v_1 + \cdots + \overline{\alpha_n} v_n,$$
then for any $v \in V$, $f(v) = (v, w)$ for all $v \in V$.

Fall 2003 #9. Consider a 3 by 3 real symmetric matrix with determinant 6. Assume $(1, 2, 3)$ and $(0, 3, -2)$ are eigenvectors with eigenvalues 1 and 2.

(a) Give an eigenvector of the form $(1, x, y)$ for some real x, y which is linearly independent of the two vectors above.

(b) What is the eigenvalue of this eigenvector?

(a) By the spectral theorem, the eigenspaces of the eigenvalues are orthogonal. Thus the cross product of the given vectors, scaled appropriately, is the answer.

(b) The eigenvalue must be 3, since the determinant is the product of the eigenvalues.

Fall 2003 #10. (a) Let $t \in \mathbb{R}$ such that t is not an integer multiple of π. For the matrix
$$A = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$
prove that there does not exist a real valued matrix B such that BAB^{-1} is a diagonal matrix.

(b) Do the same for the matrix
$$A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$
where $\lambda \in \mathbb{R} \setminus \{0\}$.

(a) If there were such a matrix, the diagonal entries would be the eigenvalues. Thus the eigenvalues would be real. However, the eigenvalues satisfy
$$(\cos(t) - \lambda)^2 + \sin^2(t) = 0,$$
so
$$\lambda^2 - 2\cos(t)\lambda + 1 = 0,$$
and
$$\lambda = \cos(t) \pm i \sin(t) = e^{\pm it},$$
which are not real.

(b) Just check directly. Or note that the geometric multiplicity of the only eigenvalue 1 is 1, so A can't be diagonalizable.

Spring 2003 #9. Let $A \in M_3(\mathbb{R})$ satisfy $\det(A) = 1$ and $A^t A = I = AA^t$ where I is the 3×3 identity matrix. Prove that the characteristic polynomial of A has 1 as a root.

Show complex eigenvalues come in conjugate pairs. Then since $\det(A) = 1$ is the product of the eigenvalues, and the product of conjugates is positive, there exists a real positive root. Then for a positive eigenvalue with eigenvector $v \neq 0$,
$$(v, v) = (A^t Av, v) = (Av, Av) = |\lambda|^2 (v, v).$$

Since $(v, v) \neq 0$, and λ is a positive real number, $\lambda = 1$.

Spring 2003 #10. Let V be a finite dimensional real inner product space and $T : V \to V$ a hermitian linear operator. Suppose the matrix representation of T^2 in the standard basis has trace zero. Prove that T is the zero operator.

Let A be the matrix representation of T. Since T is hermitian, A is diagonalizable. Thus there exists an invertible matrix P such that $A = PDP^{-1}$ with D a diagonal matrix consisting of eigenvalues of T. It follows that

$$A^2 = PD^2P^{-1},$$

thus the trace of A^2 is

$$\operatorname{tr}(PD^2P^{-1}) = \operatorname{tr}(PP^{-1}D^2) = \operatorname{tr}(D^2) = \sum_{i=1}^{n} \lambda_i^2.$$

Since this is assumed to be 0, $\lambda_i = 0$ for all i. It follows that $D = 0$, hence $A = 0$, so $T = 0$.

Fall 2004 #8. Let $A = (a_{ij})$ be a real, $n \times n$ symmetric matrix and let $Q(v) = v \cdot Av$ (ordinary dot product) be the associated quadratic form defined for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$.

1. Show that $\nabla Q_v = 2Av$ where ∇Q_v is the gradient at v of the function Q.
2. Let M be the minimum value of Q_v on the unit sphere $S^n = \{v \in \mathbb{R} : ||v|| = 1\}$ and let $u \in S^n$ be the vector such that $Q(u) = M$. Prove, using Lagrange multipliers, that u is an eigenvector of A with eigenvalue M.

Let A_i be the ith column of A. By the product rule,

$$\frac{\partial Q}{\partial v_i}(v) = v \cdot a_i + e_i \cdot Av.$$

Since A is symmetric,

$$e_i \cdot Av = e_i^t Av = e_i^t A^t v = (Ae_i)^t v = a_i^t v = a_i \cdot v,$$

thus

$$\frac{\partial Q}{\partial v_i}(v) = 2a_i \cdot v = 2a_i^t v,$$

and

$$\nabla Q_v = 2Av.$$

(b) Let $g(u) = ||u||^2 - 1$. Then we are trying to minimize Q_u subject to the constraint $g(u) = 0$. Lagrange multipliers yields the equation

$$\nabla Q_u = \lambda \nabla g(u).$$

From part (a), this becomes

$$2Au = \lambda(2u).$$

Thus $Au = \lambda u$, so u is an eigenvector of A. Then by definition of M,

$$M = Q_u = u \cdot Au = u \cdot \lambda u = \lambda |u|^2 = \lambda.$$

Fall 2004 #9. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation and $P(X)$ a polynomial such that $P(T) = 0$. Prove that every eigenvalue of T is a root of $P(X)$.

Let λ be an eigenvalue of T with an eigenvector v. A simple induction shows $(P(T))(v) = P(\lambda)v$. Since the expression on the left is zero and $v \neq 0$, $P(\lambda) = 0$.

Fall 2004 #10. Let $V = \mathbb{R}^n$ and let $T : V \to V$ be a linear transformation. For $\lambda \in \mathbb{C}$, the subspace

$$V(\lambda) = \{v \in V : (T - \lambda I)^N v = 0 \text{ for some } N \geq 1\}$$

is called a generalized eigenspace.

1. Prove that there exists a fixed number M such that $V(\lambda) = \ker((T - \lambda I)^M)$.

2. Prove that if $\lambda \neq \mu$, then $V(\lambda) \cap V(\mu) = \{0\}$. Hint: use the following equation by raising both sides to a high power:

$$\frac{T - \lambda I}{\mu - \lambda} + \frac{T - \mu I}{\lambda - \mu} = I.$$

1. Let $\{v_1, \ldots, v_n\}$ be a basis for $V(\lambda)$. Let N be the maximum over i of the N such that $(T - \lambda I)^N v_i = 0$ for each i. Then for any $v \in V(\lambda)$, $(T - \lambda I)^N v = 0$. Hence $V(\lambda) = \ker((T - \lambda I)^M)$.

2. Suppose $\lambda \neq \mu$. Suppose $v \in V(\lambda) \cap V(\mu)$. Note

$$\frac{T - \lambda I}{\mu - \lambda} + \frac{T - \mu I}{\lambda - \mu} = I.$$

Raising both sides to the power $2M$, note that every expression on the left has a factor of $(T - \lambda I)^M$ or $(T - \mu I)^M$. Thus when evaluating at v, the left hand side is 0, but the right hand side is v.

Spring 2004 #9. Let V be a finite dimensional real inner product space under (,) and $T : V \to V$ a linear operator. Show the following are equivalent:

a) $(Tx, Ty) = (x, y)$ for all $x, y \in V$,
b) $||T(x)|| = ||x||$ for all $x \in V$,
c) $T^*T = \mathrm{Id}_V$, where T^* is the adjoint of T.
d) $TT^* = \mathrm{Id}_V$.
[T is an orthogonal map.]

(a) $\implies$ (b). Then for any $x \in V$.

$$||T(x)||^2 = (Tx, Tx) = (x, x) = ||x||^2.$$

Since the norm is non-negative, this yields (b).

(b) $\implies$ (a). Then for any $x, y \in V$,

$$(Tx+Ty, Tx+Ty) = (Tx, Tx+Ty)+(Ty, Tx+Ty) = (Tx, Tx)+(Tx, Ty)+(Ty, Tx)+(Ty, Ty) = (Tx, Tx)+(Ty, Ty)+2(Tx, T$$

thus

$$2(Tx, Ty) = (T(x + y), T(x + y)) - (Tx, Tx) - (Ty, Ty) = (x + y, x + y) - (x, x) - (y, y) = 2(x, y).$$

Hence

$$(Tx, Ty) = (x, y).$$

(b) $\implies$ (c). For any $x \in V$,

$$((T^*T - Id_V)x, (T^*T - Id_V)x) = (T^*Tx, T^*Tx) - 2(T^*Tx, Id_V x) + (Id_V x, Id_V x)$$

$$= (Tx, TT^*Tx) - 2(Tx, Tx) + (x, x) = (x, T^*Tx) - 2(x, x) + (x, x) = 0.$$

Thus $(T^*T - Id_V)x = 0$ for all x, so $T^*T = Id_V$.

(c) $\implies$ (b). For any $x \in V$,

$$(T(x), T(x)) = (x, T^*Tx) = (x, x).$$

The equivalence of (b) and (d) follows analogously to the equivalence of (b) and (c).

Fall 2005 #6. (a) Prove that if P is a real-coefficient polynomial and if A is a real symmetric matrix, then the eigenvalues of $P(A)$ are exactly the numbers $P(\lambda)$, where λ is an eigenvalue of A.

(b) Use part (a) to prove that if A is a real symmetric matrix, then A^2 is non-negative definite.

(c) Check part (b) by verifying directly that $\det(A^2)$ and $\mathrm{trace}(A^2)$ are non-negative when A is real symmetric.

(a) Send A to diagonal matrix. $P(A)$ is similar to $P(D)$.

(b) A^2 is non-negative definite if for all v, v^tA^2v is non-negative.
All eigenvalues of A are real, so all eigenvalues of A^2 are non-negative.
Suppose $P^{-1}AP = D$. For any $v \in V$, there exists w with $v = Pw$. Then

$$v^tA^2v = w^t(P^{-1}A^2P)w = w^tD^2w = \sum_{i=1}^{n} \lambda_i^2 w_i^2 \geq 0.$$

(c) $\det(A^2) = (\det(A))^2 \geq 0$.

$$\mathrm{trace}(A^2) = \mathrm{trace}((\sum_{k=1}^{n} a_{ik}a_{kj})) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}^2 \geq 0.$$

Fall 2005 #9. Suppose V_1 and V_2 are subspaces of a finite-dimensional vector space V.

(a) Show that

$$\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - \dim(\mathrm{span}(V_1, V_2))$$

where $\mathrm{span}(V_1, V_2)$ is by definition the smallest subspace that contains both V_1 and V_2.

(b) Let $n = \dim(V)$. Use part (a) to show that, if $k < n$, then an intersection of k subspaces of dimension $n-1$ always has dimension at least $n-k$. (Suggestion: Do induction on k.)

(a) Let $v_1, \ldots, v_k$ be a basis for $V_1 \cap V_2$. Then there exists a basis $v_1, \ldots, v_k, w_1, \ldots, w_r$ for V_1 and $v_1, \ldots, v_k, w_1', \ldots, w_s'$ for V_2. It follows that $w_i \notin V_2$ and $w_i' \notin V_1$. Hence $v_1, \ldots, v_k, w_1, \ldots, w_r, w_1', \ldots, w_s'$ is a basis for $\mathrm{span}(V_1, V_2)$. This gives the desired equality.

(b) $k = 1$ obvious. Assume true for k. Let $k+1 < n$. Let $V_1, \ldots, V_{k+1}$ be subspaces of V of dimension $n-1$. Then since $\mathrm{span}(\bigcap_{i\leq k} V_i, V_{k+1}) \subset V$,

$$\dim(\bigcap_{i\leq k+1} V_i) = \dim(\bigcap_{i\leq k} V_i) + \dim(V_{k+1}) - \dim(\mathrm{span}(\bigcap_{i\leq k} V_i, V_{k+1}))$$

$$\geq (n-k) + (n-1) - n \geq n - (k+1),$$

which completes the induction.

Fall 2005 #10. (a) For each $n = 2, 3, 4, \ldots$, is there an $n \times n$ matrix A with $A^{n-1} \neq 0$ but $A^n = 0$?

(b) Is there an $n \times n$ upper triangular matrix A with $A^n \neq 0$ but $A^{n+1} = 0$?

(a). Yes, take the linear map which sends e_n to 0 and e_i to e_{i+1} for each $i < n$.
(b). I assume the field has characteristic 0.
No. Because A is upper triangular, A^{n+1} is diagonal and $(A^n)_{ii} = (A_{ii})^n$. Thus if $A^{n+1} = 0$, then $A_{ii} = 0$ for all i, in which case $A^2 = 0$. If $n \geq 2$, then $A^n = 0$. Otherwise, $n = 1$, and the statement is obvious.

Spring 2005 #1. Given $n \geq 1$, let $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ denote the trace of a matrix:

$$\mathrm{tr}(A) = \sum_{k=1}^{n} A_{k,k}.$$

(a) Determine a basis for the kernel (or null-space) of tr.

(b) For $X \in M_n(\mathbb{C})$, show that $\mathrm{tr}(X) = 0$ if and only if there exists an integer m and matrices $A_1, \ldots, A_m, B_1, \ldots, B_m \in M_n(\mathbb{C})$ so that

$$X = \sum_{i=1}^{m} A_j B_j - B_j A_j.$$

(a) E_{ij}, $i \neq j$ and $E_{11} - E_{ii}$ for $i \geq 2$.

Dimension of image of trace is 1. Thus the dimension of the kernel of the trace is $\dim(M_n(\mathbb{C})) - \dim(\mathrm{range(trace)}) = n^2 - 1$. Just need to show linear independence.

(b) $\Longleftarrow$ is obvious since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

$\Longrightarrow$: Suppose $A \in M_n(\mathbb{C})$ such that $\mathrm{tr}(A) = 0$. Then we can write A in terms of the basis from part (a). Note $E_{i,j} = [E_{i,i}, E_{i,j}]$ for $i \neq j$ and $E_{1,1} - E_{i,i} = [E_{1,j}, E_{j,1}]$ for all i.

Spring 2005 #2. Let V be a finite-dimensional vector space, and let V^* denote the dual space; that is, the space of linear maps $\phi : V \to \mathbb{C}$. For a set $W \subset V$, let

$$W^\perp = \{\phi \in V^* : \phi(w) = 0 \text{ for all } w \in W\}.$$

For a subset $U \subset V^*$, let

$$^\perp U = \{v \in V : \phi(v) = 0 \text{ for all } \phi \in U\}.$$

(a) Show that for any subset $W \subset V$, $^\perp(W^\perp) = \mathrm{span}(W)$. Recall that the span of a set of vectors is the smallest vector sub-space that contains these vectors.

(b) Let $W \subset V$ be a linear subspace. Give an explicit isomorphism between $(V/W)^*$ and $W^\perp$. Show that it is an isomorphism.

(a) Suppose $w \in W$. Then for all $\phi \in W^\perp$, $\phi(w) = 0$, hence $w \in^\perp (W^\perp)$. Since $^\perp(W^\perp)$ is a subspace and $\mathrm{Span}(W)$ is the smallest subspace containing W we have $^\perp(W^\perp) \supset \mathrm{Span}(W)$.

Conversely, select a basis $v_1, \ldots, v_k$ for $\mathrm{Span}(W)$ and then extend it to a basis $v_1, \ldots, v_n$ for W. Let ϕ_i be the corresponding dual basis vectors. Suppose $w \in^\perp (W^\perp)$. Then there exist a_i with $w = \sum_{i=1}^n a_i v_i$, with $a_i \in \mathbb{C}$. Since $\phi_j \in W^\perp$ for $k+1 \leq j \leq n$,

$$a_j = \phi_j(w) = 0.$$

Hence $w \in \mathrm{Span}(W)$, so $\mathrm{Span}(W) =^\perp (W^\perp)$.

(b) Define $\phi : (V/W)^* \to W^\perp$ by $\phi(T) = [v \mapsto T(v + W)]$. If $w \in W$, then $(\phi(T))(w) = T(0) = 0$, so $\phi(T) \in W^\perp$. Thus this map is well defined. If $\phi(T) = 0$, then $T(v + W) = 0$ for all $v \in V$, so $T = 0$. Hence ϕ is injective. Also,

$$\dim((V/W)^*) = \dim(V/W) = \dim(V) - \dim(W) = \dim(W^\perp).$$

Hence ϕ must be surjective as well. Hence ϕ is an isomorphism.

Spring 2005 #3. Let A be a Hermitian-symmetric $n \times n$ complex matrix. Show that if $(Av|v) \geq 0$ for all $v \in \mathbb{C}^n$, then there exists an $n \times n$ matrix T so that $A = T^*T$.

By the spectral theorem for Hermitian operators, we may select an orthonormal basis $v_1, \ldots, v_n$ and numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $Av_i = \lambda_i v_i$. We claim $\lambda_i \geq 0$ for all i. Note

$$\lambda_i(v_i|v_i) = (Av_i|v_i) \geq 0.$$

Define T by $T(v_i) = \sqrt{\lambda_i}v_i$. Then $T^*(v_i) = \sqrt{\lambda_i}v_i$, hence $A(v_i) = TT^*(v_i)$, so $A = TT^*$.

Spring 2005 #4. Let $A = M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices with complex entries. We say that $I \subset A$ is a two-sided ideal in A if

(i) For all $A, B \in I$, $A + B \in I$.

(ii) For all $A \in I$ and $B \in A$, AB and BA belong to I.

Show that the only two-sided ideals in A are $\{0\}$ and A itself.

Let $I \subset A$ be a two-sided ideal. Suppose there is a nonzero $A = (a_{ij}) \in I$. By multiplying A by a suitable permutation matrix and diagonal matrix we see by (ii) that the matrix $E_{ij} \in I$. Thus by (i), $M_n(\mathbb{C}) \in I$.

Spring 2006 #7. A matrix T (with entries, say, in the field $\mathbb{C}$ of complex numbers) is diagonalizable if there exists a non-singular matrix S such that STS^{-1} is diagonalizable. Prove that if $a, \lambda \in \mathbb{C}$ with $a \neq 0$, then the following matrix is not diagonalizable:

$$T = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & \lambda \end{pmatrix}.$$

Note the eigenvalues are 1 and λ and both have eigenspaces of dimension 1. Thus the geometric multiplicities of the eigenvalues do not add to 3, the dimension of the matrix, hence it is not diagonalizable.

Spring 2006 #10. Let Y be an arbitrary set of commuting matrices in $M_n(\mathbb{C})$ (i.e., $AB = BA$ for all $A, B \in Y$). Prove that there exists a non-zero vector $v \in \mathbb{C}^n$ which is a common eigenvector of all elements of Y.

We first work out the case of two matrices with $AB = BA$. Let λ be an eigenvalue of A and let V_λ be the eigenspace of λ. If $v \in V_\lambda$, then

$$A(Bv) = B(Av) = \lambda(Bv).$$

Hence $Bv \in V_\lambda$, so V_λ is invariant under B. Consider the restriction of B to V_λ. This restricted operator has at least one eigenvalue and eigenvector (say $w \in V_\lambda$ is an eigenvector). Then w is an eigenvector for both A and B.

Inductively, suppose v is a common eigenvector for $A_1, \ldots, A_{n-1}$ with eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. Then the commutative property implies the eigenspaces corresponding to each eigenvalue and matrix are invariant under A_n. Take the intersection of all such eigenspaces. This is a non-empty set since it contains v. Then the restriction of A_n to this intersection has an eigenvalue w, which is then a common eigenvalue for $A_1, \ldots, A_n$.

Winter 2006 #8. Let $T : V \to W$ be a linear transformation of finite dimensional real inner product spaces. Show that there exists a unique linear transformation $T^t : W \to V$ such that

$$(T(v)|w)_W = (v, T^t(w))_V$$

for all $v \in V$ and $w \in W$.

Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ be orthonormal bases for V and W respectively. Let A be the matrix representation of T relative to these bases. Define the map $T^t : W \to V$ to have matrix representation A^t with respect to these bases. It follows that

$$(v, T^t w)_V = (Tv, w)_W.$$

Suppose S and S' both satisfy

$$(Tv, w)_W = (v, Sw)_V = (v, S'w)_V$$

for all $v \in V$, $w \in W$.

Then

$$(v, Sw - S'w)_V = 0$$

for all $v \in V$, hence $Sw - S'w = 0$, so $S = S'$.

Fall 2007 #3. Let T be a linear transformation of the vector space V into itself. If Tv and v are linearly dependent for each $v \in V$, show that T must be a scalar multiple of the identity.

Take $v \in V$ and λ such that $Tv = \lambda v$. Then for any $w \in V$, $Tw = cw$ and $T(v+w) = c'(v+w)$. This implies $c'(v+w) = T(v+w) = Tv + Tw = \lambda v + cw$. Hence $(c' - \lambda)v = (c - c')w$. This is only generally true if V is 1-dimensional or $\lambda = c = c'$. In either case, $T(v) = \lambda v$ for all $v \in V$.

Fall 2007 #7. Let $A(x)$ be a function on $\mathbb{R}$ whose values are $n \times n$ matrices. Starting from the definition that the derivative $A'(x)$ is the matrix you get by differentiating the entries in $A(x)$, show that when $A(x)$ is invertible and differentiable for all x, $A^{-1}(x)$ is differentiable, and

$$(A^{-1})'(x) = -A^{-1}(x)A'(x)A^{-1}(x).$$

Suppose $A(x)$ is invertible and differentiable for all x. Note

$$A^{-1} = (\det(A))^{-1}\operatorname{adj}(A),$$

and since the adjugate is the transpose of the cofactor matrix, which consists of sums of products of elements of A, the adjugate is differentiable, hence A^{-1} is differentiable.

By the definition of $A^{-1}(x)$,

$$I_n = A^{-1}(x)A(x).$$

Supposing $A^{-1}(x)$ is differentiable, differentiating the i, j entry of each side and combining results,

$$0 = (A^{-1})'(x)A(x) + A^{-1}(x)A'(x).$$

Thus

$$(A^{-1})'(x) = -A^{-1}(x)A'(x)A^{-1}(x).$$

Fall 2007 #10. Suppose that $\{v_j\}_{j=1}^n$ is a basis for the complex vector space $\mathbb{C}^n$.

(a) Show that there is a basis $\{w_j\}_{j=1}^n$ such that $(w_j|v_k) = \delta_{jk}$. Here $(\cdot,\cdot)$ is the standard inner product, $(w, v) = \overline{w_1}v_1 + \cdots + \overline{w_n}v_n$.

(b) If the v_j's are eigenvectors for a linear transformation T of $\mathbb{C}^n$, show that the w_j's are eigenvectors for T^*, the adjoint of T with respect to the inner product.

(a) The matrix $A = (v_1 \ldots v_n)$ is invertible since the v_i are linearly independent. Let w_j be the jth row vector of $\overline{A^{-1}}$. Then

$$(w_j, v_k) = (A^{-1}A)_{jk} = (I_n)_{jk} = \delta_{jk}.$$

(b) Suppose $Tv_j = \lambda_j v_j$. Then $[T^*] = \overline{[T]}^t$. Let B be the diagonal matrix with entries λ_j. Then $[T]A = AB$, so $A^{-1}[T] = BA^{-1}$. Taking the transpose and conjugate of both sides,

$$[T^*]\overline{(A^{-1})^t} = \overline{[T]^t(A^{-1})^t} = \overline{(A^{-1})^t B^t} = \overline{(A^{-1})^t}\,\overline{B}.$$

Thus $T^*w_j = \overline{\lambda_j}w_j$.

Fall 2007 #12. (a) Suppose that $x_0 < x_1 < \cdots < x_n$ are points in $[a, b]$. Define linear functions on P^n, the vector space of polynomials of degree less than or equal to n, by setting

$$l_j(p) = p(x_j)$$

for $j = 0, \ldots, n$. Show that the set $\{l_j\}_{j=0}^n$ is linearly independent.

(b) Show that there are unique coefficients c_j such that

$$\int_a^b p(x)dx = \sum_{j=0}^n c_j l_j(p)$$

for all $p \in P^n$.

Set $a_0 l_0 + \cdots + a_n l_n = 0$. Then taking $p_i(x) = \prod_{j \neq i}(x - x_j)$,

$$0 = (a_0 l_0 + \cdots + a_n l_n)(p_i) = a_0 p(x_0) + \cdots + a_n p(x_n) = \alpha_i \prod_{j \neq i}(x_i - x_j).$$

Thus $a_i = 0$. This works for any i, hence the l_j are linearly independent.

(b) From part (a) and the fact that $(P_n)^*$ has dimension $n+1$, we see that $\{l_j\}_{j=0}^n$ forms a basis for $(P_n)^*$. Since the map which takes p to $\int_a^b p(x)dx$ is in $(P_n)^*$, then there are unique coefficients c_j with the desired equality.

Spring 2007 #2. Let V, W, Z be n-dimensional vector spaces and $T : V \to W$ and $U : W \to Z$ be linear transformations. Prove that if the composite transformation $UT : V \to Z$ is invertible, then both T and U are invertible. (Do not use determinants in your proof!)

Since UT is invertible, it is one-to-one and onto. Clearly this makes U onto. By the Dimension Theorem, it follows that U is one-to-one. If T is not one-to-one, then UT is not one-to-one, a contradiction. Hence T is one-to-one, and it follows by the Dimension Theorem that T is onto. Thus U and T are bijections, hence invertible.

Spring 2007 #3. Consider the space of infinite sequences of real numbers

$$S = \{(a_0, a_1, \ldots) : a_n \in \mathbb{R}, n = 0, 1, 2, \ldots\}$$

endowed with the standard operations of addition and scalar multiplication. For each pair of real numbers A and B, prove that the set of solutions $(x_0, x_1, \ldots)$ of the linear recursion

$$x_{n+2} = Ax_{n+1} + Bx_n$$

for $n \geq 0$ is a linear subspace of S of dimension 2.

Clearly the set of solutions forms a subspace. Show that $(1, 0, \ldots)$ and $(0, 1, \ldots)$ is a basis for the set of solutions. Clearly any solution is a linear combination of these two. Also, these are necessarily linearly independent vectors because of the first two components.

Spring 2007 #4. Suppose that A is a symmetric $n \times n$ real matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_l$, $(l \leq n)$. Find the sets

$$X = \{x \in \mathbb{R}^n : \lim_{k \to \infty} (x^t A^{2k} x)^{1/k} \text{ exists}\}$$

and

$$L = \{\lim_{k\to\infty} (x^t A^{2k} x)^{1/k} : x \in X\},$$

where $\mathbb{R}^n$ is identified with the set of real column vectors, and x^t denotes the transpose of x.

The answers are $X = \mathbb{R}^n$ and $L = \{\lambda_1, \dots, \lambda_l\}$.

By the spectral theorem, A is diagonalizable, so there exists an orthonormal basis $\{v_1, \dots, v_n\}$ such that $Av_i = \lambda_i v_i$ (different λ_i than the problem statement). Then for any $x \in X$, there exist a_i such that

$$x = a_1 v_1 + \cdots + a_n v_n.$$

We compute

$$x^t A^{2k} x = \lambda_1^2 a_1 v_1^2 + \cdots + \lambda_n^2 a_n v_n^2.$$

Suppose λ_i is the dominant eigenvalue. Then if $a_i \neq 0$,

$$(x^t A^{2k} x)^{1/k} = \lambda^2.$$

(More detail here on exam.)

Picking $x = v_i$ for each i, we can recover λ_i^2 in this way.

Spring 2007 #5. Let T be a normal linear operator on a finite dimensional complex inner product linear space V. Prove that if v is an eigenvector of T, then v is also an eigenvector of its adjoint T^*.

Since T is normal $TT^* - T^*T = 0$.

$$(Tv, Tv) = (v, T^*Tv) = (v, TT^*v) = (v, T^{**}T^*v) = (T^*v, T^*v).$$

Note that $(T - \lambda I)$ is also normal. Thus

$$0 = ||(T - \lambda I)v|| = ||(T - \lambda I)^* v|| = ||(T^* - \lambda^* I)v||.$$

Hence $T^* v = \lambda^* v$.

Fall 2008 #8. Must the eigenvectors of a linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ span $\mathbb{C}^n$? Prove your assertion.

No, take the nilpotent transformation $T(v_1) = 0$, $T(v_2) = v_1$. This only has 0 eigenvalues, and eigenvectors $(x, 0)$. Clearly these do not span $\mathbb{C}^2$.

Fall 2008 #9. (a) Prove that any linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ must have an eigenvector.

(b) Is (a) true for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$?

(a) Characteristic polynomial has a root.

(b) No, pick a non-trivial rotation.

Fall 2008 #11. Consider the Poisson equation with periodic boundary conditions on $[0, 1]$:

$$\frac{\partial^2 u}{\partial x^2} = f, x \in (0, 1)$$

$$u(0) = u(1).$$

A second order accurate approximation to the problem is given by the solution to the following system of equations

$$Au = \Delta x^2 f$$

where

$$A = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0\cdots & \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix},$$

$u = [u_0, u_1, \dots, u_{n-1}]$, $f = [f_0, f_1, \dots, f_{n-1}]$ and $u_i \approx u(x_i)$ with $x_i = i\Delta x$, $\Delta x = 1/n$, and $f_i = f(x_i)$ for $i = 0, \dots, n-1$.

(a) Show that the matrix A is singlar.

(b) What condition must f satisfy so that a solution exists?

(a) Apply it to $(1, \dots, 1)$ to get 0. If the matrix had an inverse, applying the inverse to 0 would yield 0, not $(1, \dots, 1)$, a contradiction.

(b) Based on part (a), f must satisfy $f_0 + \cdots + f_{n-1} = 0$. Then we can reduce the system to an $n-1$ by $n-1$ matrix that is not singular, hence a solution exists.

Spring 2008 #8. Assume V is an n-dimensional vector space over the rationals $\mathbb{Q}$, and T is a $\mathbb{Q}$-linear transformation $T : V \to V$ such that $T^2 = T$. Prove that every vector $v \in V$ can be written uniquely as $v = v_1 + v_2$ such that $T(v_1) = v_1$ and $T(v_2) = 0$.

Write $v = T(v) + (v - T(v))$. Then $T(T(v)) = T(v)$ and $T(v - T(v)) = T(v) - T(v) = 0$.

Note that if $v = v_1 + v_2$ with $T(v_1) = v_1$ and $T(v_2) = 0$, then $T(v) = T(v_1) = v_1$ and $(v - T(v)) = v - T(v_1) = v - v_1 = v_2$. Hence v_1 and v_2 are unique.

Spring 2008 #9. Let V be a vector space over $\mathbb{R}$.

(a) Prove that if V is odd dimension, and if T is an $\mathbb{R}$-linear transformation $T : V \to V$ of V, then T has a non-zero eigenvector $v \in V$.

(b) Show that for every even positive integer n, there is a vector space V over $\mathbb{R}$ of dimension n, and an $\mathbb{R}$-linear transformation $T : V \to V$ of V, such that there is no non-zero $v \in V$ satisfying $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$.

(a) See previous problems. (b) Take a block diagonal matrix consisting of 2 by 2 non-trivial rotations.

Spring 2008 #10. Suppose A is an $n \times n$ complex matrix such that A has n distinct eigenvalues. Prove that if B is an $n \times n$ complex matrix such that $AB = BA$, then B is diagonalizable.

Let v_i, λ_i be eigenvector/eigenvalue pairs for A. Note $\{v_1, \dots, v_n\}$ is linearly independent. It follows that $A(Bv_i) = B(Av_i) = \lambda Bv_i$. Thus Bv_i is an eigenvector of A with eigenvalue λ_i. The eigenspace of λ_i must be one-dimensional, hence $Bv_i = c_i v_i$ for some $c_i \in \mathbb{C}$. It follows that B is diagonal with respect to the basis $\{v_1, \dots, v_n\}$.

Spring 2008 #11. Assume A is an $n \times n$ complex matrix such that for some positive integer m the power $A^m = I_m$. Prove that A is diagonalizable.

Note that $x^m - 1$ has no repeated roots over $\mathbb{C}$. If $mx^{m-1} = 0$, then $x = 0$, but this is not a root of $x^m - 1$. Hence the minimal polynomial of A has no repeated roots. Thus the minimal polynomial factors into distinct monic linear factors with coefficients in $\mathbb{C}$. Hence A is diagonalizable.

Spring 2008 #12. Let A be an $n \times n$ real symmetric $(a_{i,j} = a_{j,i})$ matrix, and let S be the unit sphere of

$\mathbb{R}^n$. Let $x \in S$ be such that

$$(Ax, x) = \sup_S (Ay, y)$$

where $(z, y) = \sum z_j y_j$ is the usual inner product on $\mathbb{R}^n$. (By compactness, such x exists.)

(a) Prove that $(x, y) = 0$ implies $(Ax, y) = 0$. Hint: Expand

$$(A(x + \varepsilon y), x + \varepsilon y).$$

(b) Use (a) to prove x is an eigenvector for A.

(c) Use induction to prove $\mathbb{R}^n$ has an orthonormal basis of eigenvectors for A.

Note: If you use part (c) to prove part (a) or part (b), then your solution should include a proof of part (c) that does not use part (a) or part (b).

(a) Assume $(x, y) = 0$. Then

$$(A(x + \varepsilon y), x + \varepsilon y) = (Ax, x + \varepsilon y) + (\varepsilon y, x + \varepsilon y) = (Ax, x) + (Ax, \varepsilon y) + (\varepsilon y, x) + (\varepsilon y, \varepsilon y)$$

$$= (Ax, x) + \varepsilon(Ax, y) + 0 + \varepsilon^2 ||y||^2.$$

By the choice of x, we must have $\frac{1}{||x+\varepsilon y||^2}(A(x + \varepsilon y), x + \varepsilon y) \leq (Ax, x)$. Hence

$$(Ax, x) + \varepsilon(Ax, y) + \varepsilon^2 ||y||^2 \leq (||x||^2 + \varepsilon^2 ||y||^2)(Ax, x) = (Ax, x) + \varepsilon^2 ||y||^2 (Ax, x),$$

and

$$(Ax, y) \leq \varepsilon ||y||^2 ((Ax, x) - 1)$$

Replacing ε by $-\varepsilon$ in the analysis above,

$$(Ax, y) \geq -\varepsilon ||y||^2 ((Ax, x) - 1).$$

Thus taking $\varepsilon \to 0$, we find that $(Ax, y) = 0$.

(b) Form an orthonormal basis $x, y_2, \ldots, y_n$ for $\mathbb{R}^n$. Write $Ax = c_1 x + c_2 y_2 + \cdots + c_n y_n$. Then for all $i \geq 2$, $(x, y_i) = 0$, hence by part (a), $c_i = (Ax, y_i) = 0$. Hence $Ax = c_1 x$, so x is an eigenvector of A.

(c) This is the usual argument in the proof of the Spectral Theorem.

Fall 2009 #4. Let V be a finite dimensional $\mathbb{R}$-vector space equipped with an inner product. For a vector subspace $U \subset V$, denote by $U^\perp$ its orthogonal complement, i.e., the set of $v \in V$ such that $(v|u) = 0$ for all $u \in U$. Show that

$$\dim(U) + \dim(U^\perp) = \dim(V).$$

Let $u_1, \ldots, u_k$ be an orthonormal basis for U, and extend it to an orthonormal basis $u_1, \ldots, u_n$ for V. It follows that $u_{k+1}, \ldots, u_n$ is an orthonormal basis for $U^\perp$, hence we get the desired equality.

Fall 2009 #5. Show that if $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ are all different, and some $a_1, \ldots, a_n \in \mathbb{R}$ satisfy

$$\sum a_i e^{\alpha_i t} = 0$$

for all $t \in (-1, 1)$, then necessarily $a_i = 0$ for all $1 \leq i \leq n$. (Hint: you may use the differentiation operator and a theorem in Linear Algebra on distinct eigenvalues.)

Note $e^{\alpha_i t}$ are eigenvectors of the differentiation operator on smooth functions with support on $(-1, 1)$. Moreover these eigenvectors have distinct eigenvalues, thus they are linearly independent. (Same proof as in finite dimensional case for infinite dimensions and a finite number of eigenvectors.)

Or, repeatedly differentiate to get the Vandermonde matrix, which has non-zero determinant.

Fall 2009 #7. Let V isomorphic to $\mathbb{R}^n$ be an n-dimensional vector space over $\mathbb{R}$ and denote by $\mathrm{End}(V)$ the vector space of $\mathbb{R}$-linear transformations of V. (Note that $\dim(\mathrm{End}(V)) = \dim(V)^2 = n^2$.) Then for $T \in \mathrm{End}(V)$ show that the dimension of the subspace W of $\mathrm{End}(V)$ spanned by T^k for k running through non-negative integers, satisfies the inequality $\dim(W) \leq \dim(V) = n$.

Let $T \in \mathrm{End}(V)$. By the Cayley-Hamilton Theorem, $\chi_T(T) = 0$, and χ_T has degree n. This implies that $\{T^0, \ldots, T^{n-1}\}$ is a basis for W, hence we get the desired inequality.

Fall 2009 #8. For a matrix $A \in M_n(\mathbb{R})$, define $e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Let $v_0 \in \mathbb{R}^n$. Prove that the function $v : \mathbb{R} \to \mathbb{R}^n$ given by $v(t) = e^{At}v_0$ solves the linear differential equation $v'(t) = Av(t)$ with the initial condition $v(0) = v_0$. Explain precisely which theorems in calculus you are using in your proof and why they are applicable.

Note that

$$\frac{e^{A(t+h)} - e^{At}}{h} = \frac{e^{Ah} - 1_{\mathbb{F}^n}}{h} e^{At}.$$

By definition,

$$\frac{e^{Ah} - 1_{\mathbb{F}^n}}{h} = \sum_{n=1}^{\infty} \frac{1}{h} \frac{A^n h^n}{n!} = \sum_{n=1}^{\infty} \frac{A^n h^{n-1}}{n!} = A + \sum_{n=2}^{\infty} A^n h^{n-1} n!.$$

Now we estimate

$$||\sum_{n=2}^{\infty} \frac{A^n h^{n-1}}{n!}|| \leq \sum_{n=2}^{\infty} \frac{||A||^n |h|^{n-1}}{n!} = ||A|| \sum_{n=2}^{\infty} ||Ah||^{n-1} n! \leq ||A|| \sum_{n=1}^{\infty} ||Ah||^n = ||A|| ||Ah|| \frac{1}{1 - ||Ah||},$$

which approaches 0 as $|h| \to 0$. Thus

$$\lim_{|h| \to 0} \frac{e^{A(t+h)} - e^{At}}{h} = \left(\lim_{|h| \to 0} \frac{e^{Ah} - 1_{\mathbb{F}^n}}{h} \right) e^{At} = Ae^{At}.$$

Fall 2009 #11. Fall 2010 #6. (i) State the Cayley-Hamilton Theorem for matrices $A \in M_n(\mathbb{C})$.

(ii) Prove it directly for diagonalizable matrices.

(iii) Identify $M_n(\mathbb{C})$ isomorphic to $\mathbb{C}^{n^2}$ though some (say, the natural) linear isomorphism. Through this identification $M_n(\mathbb{C})$ becomes a metric space with the Euclidean metric. Fact: The subset of diagonalizable matrices in $M_n(\mathbb{C})$ (isomorphic to $\mathbb{C}^{n^2}$) is dense. Use this fact, together with part (ii), to prove the Cayley-Hamilton Theorem.

(i) $\chi_A(A) = 0$.

(ii) $A = P^{-1}DP$. $\chi_A(t) = \chi_D(t) = (t - \lambda_1) \cdots (t - \lambda_n)$. Note

$$\chi_A(A) = \chi_D(A) = \chi_D(D) = (D - \lambda_1) \cdots (D - \lambda_n) = 0,$$

since $D - \lambda_i$ has the ith row all 0.

(iii) The isomorphism comes by taking each entry. Take $A \in M_n(\mathbb{C})$. Let A_n be a sequence of diagonalizable matrices converging to A. It follows that A_n^k converges to A^k and thus $p(A_n)$ converges to $p(A)$ for any polynomial P. By (ii), $\chi_{A_n}(A_n) = 0$. We estimate

$$||\chi_A(A)|| \leq ||\chi_A(A) - \chi_A(A_n)|| + ||\chi_A(A_n) - \chi_{A_n}(A_n)|| + ||\chi_{A_n}(A_n)||$$

$$= ||\chi_A(A) - \chi_A(A_n)|| + ||\chi_A(A_n) - \chi_{A_n}(A_n)||.$$

The first term is small since χ_A is continuous. The second is small since det is continuous.

Fall 2009 #12. Let V be an n (≥ 2)-dimensional vector space over $\mathbb{C}$ with a set of basis vectors $e_1, \ldots, e_n$. Let T be a linear transformation of V satisfying $T(e_1) = e_2, \ldots, T(e_{n-1}) = e_n, T(e_n) = e_1$.

(i). Show that T has 1 as an eigenvalue and write down an eigenvector with eigenvalue 1. Show that up to scaling it is unique.

(ii) Is T diagonalizable? (Hint: calculate the characteristic polynomial.)

(i) $e_1 + \cdots + e_n$. Fairly straightforward to show unique up to scaling.

(ii) Yes. The characteristic polynomial is $t^n - 1$. Clearly the minimal polynomial is also $t^n - 1$. Both of these have n distinct complex eigenvalues, thus T is diagonalizable over $\mathbb{C}$.

Spring 2009 #2. Compute the norm of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}.$$

That is, determine the maximum value of the length of Ax over all unit vectors x.

We wish to maximize $f(x,y) = ||A[x,y]^t|| = \sqrt{(2x+y)^2 + (\sqrt{3}y)^2} = \sqrt{4x^2 + 4xy + 4y^2} = 2\sqrt{x^2 + xy + y^2} = 2\sqrt{1 + xy}$ given $x^2 + y^2 = 1$. Now $(x-y)^2 \geq 0$, so $2xy \leq x^2 + y^2 = 1$. Note that $x = y = \frac{1}{\sqrt{2}}$ achieves this maximum for xy, hence the maximum value of the length of Ax is

$$2\sqrt{3/2} = \sqrt{6}.$$

Spring 2009 #3. We wish to find a quadratic polynomial P obeying

$$P(0) = \alpha, P'(0) = \beta, P(1) = \gamma, \text{ and} P'(1) = \delta$$

where $'$ denotes differentiation.

(a) Find a minimal system of linear constraints on $(\alpha, \beta, \gamma, \delta)$ such that this is possible.

(b) When the constraints are met, what is P? Is it unique? Explain your answer.

(a) $P(t) = at^2 + bt + c$. The conditions imply $c = \alpha$, $b = \beta$. Thus

$$P(t) = at^2 + \beta t + \alpha.$$

From $P(1) = \gamma$,

$$\gamma = a + \beta + \alpha,$$

so $a = \gamma - \beta - \alpha$ and

$$P(t) = (\gamma - \beta - \alpha)t^2 + \beta t + \alpha.$$

Finally, the condition $P'(1) = \delta$ implies

$$2(\gamma - \beta - \alpha) + \beta = \delta.$$

Thus we get the constraint

$$2\alpha + \beta - 2\gamma + \delta = 0.$$

(b) If this constraint is met, P is unique and given by

$$P(t) = (\gamma - \beta - \alpha)t^2 + \beta t + \alpha.$$

The matrix equation for a, b, c in terms of α, β, γ involves an invertible matrix, so P is unique.

Spring 2009 #5. Compute e^{At} when

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}.$$

Recall that e^{At} is defined by the property that a smooth vector function $x(t)$ obeys:

$$\frac{dx}{dt}(t) = Ax(t) \text{ if and only if } x(t) = e^{At}x(0).$$

The characteristic polynomial of A is $\lambda^2(4-\lambda)$, so $A^3 = 4A^2$. Thus for all $k \geq 2$,

$$A^k = 4^{k-2}A^2.$$

Then by the series expansion for e^{At},

$$e^{At} = I_n + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{3!} + \cdots = I_n + tA + \frac{t^2A^2}{2} + \frac{t^3 4A^2}{3!} + \cdots$$

$$= I_n + tA + \frac{t^2A^2}{2} + \frac{A^2}{16}(e^{4t} - (1 + 4t + 8t^2)) = I_n + tA + \frac{A^2}{16}(e^{4t} - 1 - 4t).$$

We can use uniqueness of the solution to the differential equation with initial condition to verify the series formula for e^{tA}.

Spring 2009 #8. (a) Show that

$$(A|B) = \operatorname{tr}(AB^t)$$

defines an inner product on $M_{n\times n}(\mathbb{R})$. More precisely, show that it obeys the axioms of an inner product.

(b) Given $C \in M_{n\times n}(\mathbb{R})$, we define a linear transformation

$$\Phi_C : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$$

by

$$\Phi_C(A) = CA - AC.$$

Compute the adjoint of Φ_C. Check that when C is symmetric, then Φ_C is self-adjoint.

(c) Show that whatever the choice of C, the map Φ_C is not onto.

(a) Note

$$(A, A) = \operatorname{tr}(AA^t) = \operatorname{tr}((\sum_{k=1}^n a_{ik}a_{jk})_{ij})$$

$$= \sum_{k=1}^n a_{1k}a_{1k} + \cdots + \sum_{k=1}^n a_{nk}a_{nk} = \sum_{i,j=1}^n a_{ij}^2 \geq 0,$$

with equality if and only if $a_{ij} = 0$ for all i, j, in which case, $A = 0$.

Clearly $(A + B, C) = (A, C) + (B, C)$ and $(kA, B) = k(A, B)$.

Finally, $(A, B) = \operatorname{tr}(AB^t) = \operatorname{tr}((AB^t)^t) = \operatorname{tr}(BA^t) = (B, A)$. Thus $(\cdot, \cdot)$ is an inner product.

(b) We need that for all A, B,

$$\operatorname{tr}((CA - AC)B^t) = \operatorname{tr}(A(\Phi_C^* B)^t).$$

The left hand side is

$$\mathrm{tr}(CAB^t) - \mathrm{tr}(ACB^t) = \mathrm{tr}(AB^tC) - \mathrm{tr}(ACB^t) = \mathrm{tr}(A(B^tC - CB^t)) = \mathrm{tr}(A(C^tB - BC^t)^t).$$

Thus the adjoint is

$$\Phi_C^*(B) := C^tB - BC^t = \Phi_{C^t}(B).$$

Hence when C is symmetric, $\Phi_C^*(B) = \Phi_{C^t}(B) = \Phi_C(B)$, so Φ_C is self-adjoint.

(c) If $C = 0$, $\Phi_C = 0$, so it is clearly not onto. Otherwise, note $\Phi_C(C) = 0$, so the kernel of Φ_C is at least one-dimensional. Hence by the Dimension Theorem, Φ_C cannot be onto.

Spring 2009 #9. Let us say that a real symmetric $n \times n$ matrix A is a reflection if $A^2 = I_n$ and

$$\mathrm{rank}(A - I_n) = 1.$$

Given distinct unit vectors $x, y \in \mathbb{R}^n$ show that there is a reflection with $Ax = y$ and $Ay = x$. Moreover, show that the reflection A with these properties is unique.

For any orthogonal matrix P, A is a reflection if and only if PAP^{-1} is a reflection, thus we can shift to any orthonormal basis to solve the problem.

Let $\hat{w_1} = \frac{1}{2}(x+y)$ and $\hat{w_2} = \frac{1}{2}(x-y)$, then $w_1 = \hat{w_1}/||\hat{w_1}||$ and $w_2 = \hat{w_2}/||\hat{w_2}||$. It follows that $w_1 \cdot w_2 = 0$. Extend this to some orthonormal basis $w_1, \ldots, w_n$. Note $x = ||w_1||w_1 + ||w_2||w_2$ and $y = ||w_1||w_1 - ||w_2||w_2$. With respect to this basis, we take

For uniqueness, note that if $Rx = y$ and $Ry = x$, then $R(x-y) = -(x-y)$. Also, since $\mathrm{rank}(R - I_n) = 1$, $\ker(R - I_n) = (x-y)^\perp$. Thus for all $v \in (x-y)^\perp$, $Rv = v$. Thus $Rw_i = Aw_i$ for all i, so $R = A$.

Spring 2009 #11. (a) Explain the following (overly informal) statement:

Every matrix can be brought to Jordan normal form; moreover the normal form is essentially unique.

No proofs are required; however, all statements must be clear and precise. All required hypotheses must be included. The meaning of the phrases 'brought to', 'Jordan normal form', and 'essentially unique' must be defined explicitly.

(b) Define the minimal polynomial of a matrix. How may it be determined for a matrix in Jordan normal form?

(a) More precisely, every $n \times n$ with entries in $\mathbb{C}$ is similar to an essentially unique matrix in Jordan normal form.

Two matrices A and B are said to be similar if there exists an invertible matrix P such that $B = PAP^{-1}$.

A Jordan block J_λ^m is an $m \times m$ matrix with diagonal entries λ, superdiagonal entries 1, and all other entries 0. A matrix is in Jordan normal form if it has Jordan blocks along the diagonal, and is zero elsewhere.

Given a matrix A, its Jordan normal form is unique up to reordering of the Jordan blocks. In other words, two matrices in Jordan normal form are similar if and only if they are composed of the same Jordan blocks.

(b) The minimal polynomial of a matrix A is the unique monic polynomial p of minimal degree that satisfies $p(A) = 0$. Clearly the minimal polynomial of a Jordan block J_λ^m is $(t - \lambda)^m$. For a matrix in Jordan normal form, this implies the minimal polynomial is

$$p = \prod_{i=1}^{s} (t - \lambda_i)^{M_i},$$

where $\{\lambda_i\}$ is the set of distinct eigenvalues of A, and M_i is the maximal size of a λ_i-Jordan block in A.

Fall 2010 #5. Prove or disprove: For any two subsets S and S' of a vector space V,

(a) $\text{span}(S) \cap \text{span}(S') = \text{span}(S \cap S')$,
(b) $\text{span}(S) + \text{span}(S') = \text{span}(S \cup S')$.

(a) False. Take $V = \mathbb{R}$, $S = \{0, 1\}$ and $S' = \{0, 2\}$. Then

$$\text{span}(S) \cap \text{span}(S') = \mathbb{R} \cap \mathbb{R} = \mathbb{R} \neq \{0\} = \text{span}(S \cap S').$$

(b) True. For any $a \in \text{span}(S)$ and $b \in \text{span}(S')$, since $\text{span}(S \cup S')$ is a vector space, $a + b \in \text{span}(S \cup S')$. Thus $\text{span}(S) + \text{span}(S') \subset \text{span}(S \cup S')$. Conversely, note that $\text{span}(S) + \text{span}(S')$ is a vector space which contains $S \cup S'$, since $0 \in \text{span}(S)$ and $0 \in \text{span}(S')$. Thus as the smallest subspace containing $S \cup S'$, $\text{span}(S \cup S') \subset \text{span}(S) + \text{span}(S')$.

Fall 2010 #7. Let V and W be inner product spaces over $\mathbb{C}$ such that $\dim(V) \leq \dim(W) < \infty$. Prove that there is a linear transformation $T : V \to W$ satisfying

$$(T(v)|T(v'))_W = (v|v')_V$$

for all $v, v' \in V$.

We want to construct an isometry from V to a subspace of W. Let $v_1, \ldots, v_n$ be an orthogonal basis for V and $w_1, \ldots, w_m$ be a basis for W. By the assumptions, $n \leq m$. Define $T(v_i) = w_i$ for $1 \leq i \leq n$. Then for any $v, v' \in V$, we can write $v = a_1 v_1 + \cdots + a_n v_n$ and $v' = b_1 v_1 + \cdots + b_n v_n$ so that

$$(T(v)|T(v'))_W = (a_1 w_1 + \cdots + a_n w_n | b_1 w_1 + \cdots + b_n w_n)_W = a_1 \overline{b_1} + \cdots + a_n \overline{b_n}$$

$$= (a_1 v_1 + \cdots + a_n v_n | b_1 v_1 + \cdots + b_n)_V = (v|v')_V.$$

Fall 2010 #8. Let W_1 and W_2 be subspaces of a finite dimensional inner product space V. Prove that $(W_1 \cap W_2)^\perp = (W_1)^\perp + (W_2)^\perp$.

Let $v \in (W_1 \cap W_2)^\perp$. Write $v = v_1 + v_2$, where $v_1 = \text{proj}_{W_1^\perp} v$. Then $v_1 \in W_1^\perp$ and $v - v_1 \in W_1$. For any $w \in W_2$,

$$(v_2|w) = (v - v_1|w).$$

Write $w = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_1^\perp$. Then it follows that the right hand side is zero. Hence $v_2 \in W_2^\perp$, and $v \in (W_1)^\perp + (W_2)^\perp$.

Conversely, suppose $v_1 + v_2 \in (W_1)^\perp + (W_2)^\perp$. Then for any $w_1 \in W_1$, $(v_1|w_1) = 0$ and for any $w_2 \in W_2$, $(v_2|w_2) = 0$. Thus for any $w \in W_1 \cap W_2$, $w \in W_1$ and $w \in W_2$, hence

$$(v_1 + v_2|w) = (v_1|w) + (v_2|w) = 0 + 0 = 0.$$

Hence $v_1 + v_2 \in (W_1 \cap W_2)^\perp$.

Fall 2010 #9. Consider the following iterative method

$$x_{k+1} = A^{-1}(Bx_k + c)$$

where c is the vector $(1, 1)^t$ and A and B are the matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Assume the iteration converges; to what vector x does the iteration converge?
(b) Does this iteration converge for arbitrary initial vectors x_0?

(a) We must solve the matrix equation

$$L = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} L + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Thus $L_1 = L_1 + \frac{1}{2}L_2 + \frac{1}{2}$, so $L_2 = -1$ and likewise, $L_1 = -1$. Hence it converges to $L = (-1, -1)$.

(b) No, note if x_0 has positive entries, then x_k has positive entries for all k. Thus it cannot converge to -1.

Spring 2010 #1. Let $u_1, \ldots, u_n$ be an orthonormal basis of $\mathbb{R}^n$ and let $y_1, \ldots, y_n$ be a collection of vectors in $\mathbb{R}^n$ satisfying $\sum_i ||y_i||^2 < 1$. Prove that the vectors $u_1 + y_1, \ldots, u_n + y_n$ are linearly independent.

Define $T(u_k) = -y_k$ and extend it by linearity. Let $x \in \mathbb{C}^n$ and write $x = \sum_{j=1}^n \alpha_j u_j$. Then

$$||Tx|| = ||\sum_{j=1}^n \alpha_j y_j|| \leq \sum_{j=1}^n |\alpha_j|||y_j|| \leq \left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2} \left(\sum_{j=1}^n ||y_j||^2\right) = \left(\sum_{j=1}^n ||y_j||^2\right) ||x||.$$

Hence

$$||T|| \leq \sum_{j=1}^n ||y_j||^2 < 1.$$

It follows that $I - T$ is invertible with inverse $\sum_{i=0}^\infty T^i$. Since the u_k form a basis and $(I - T)(u_k) = u_k + y_k$, and invertible linear maps take bases to bases, we have that $u_1 + y_1, \ldots, u_n + y_n$ form a basis.

Spring 2010 #5. Let A, B be two $n \times n$ complex matrices which have the same minimal polynomial $M(t)$ and the same characteristic polynomial $P(t) = (t - \lambda_1)^{a_1} \cdots (t - \lambda_k)^{a_k}$, where the λ_i are distinct. Prove that if $P(t)/M(t) = (t - \lambda_1) \cdots (t - \lambda_k)$, then these matrices are similar.

Using Jordan canonical form, there must be a Jordan block $J_{\lambda_i}^{a_i - 1}$ and $J_{\lambda_i}^1$ for each i. Hence the two matrices have the same Jordan canonical form, up to rearranging the Jordan blocks, hence they are similar.

Spring 2010 #6. Let $A = \begin{pmatrix} 4 & -4 \\ 1 & 0 \end{pmatrix}$.

(i) Find Jordan form J of A and a matrix P such that $P^{-1}AP = J$.
(ii) Compute A^{100} and J^{100}.
(iii) Find a formula for a_n, when $a_{n+1} = 4a_n - 4a_{n-1}$ and $a_0 = a$, $a_1 = b$.

(i) A has minimal polynomial and characteristic polynomial $(t - 2)^2$. Thus its Jordan form is

$$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Since J is the representation of A in the basis consisting of columns of P, we need $A(v_1) = 2v_1$ and $A(v_2) = v_1 + 2v_2$. Picking $v_2 = (1\,0)^t$, we obtain

$$P = (v_1\, v_2) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

(ii) $J^n = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}$. Higher entries get $(nCk)\lambda^{n-k}$.

(iii) Straightforward.

Fall 2011 #7. Let $f : \mathbb{R} \to M_{n\times n}$ be a continuous function. Show that the function $g(t) = \text{rank}(f(t))$ is lower semi-continuous, meaning that if a sequence t_n converges to t then $g(t) \le \liminf_n g(t_n)$. Is g always continuous?

Recall that if $A \in M_n(\mathbb{R}) \setminus \{0\}$ then $\text{rank}(A)$ can be computed to be the largest k such that there exists a $k \times k$ submatrix B of A such that $\det(B) \neq 0$.

Let $f : \mathbb{R} \to M_n(\mathbb{F})$ be a continuous function. Therefore, for each $1 \le k \le n$ and selection $1 \le i_1 < i_2 < \cdots < i_k \le n$ and $1 \le j_1 < j_2 < \cdots < j_k \le n$, the function $f_{k,i_1,\dots,i_k,j_1,\dots,j_k} : \mathbb{R} \to M_k(\mathbb{R})$ defined as the $k \times k$ submatrix of $f(t)$ with rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$ is a continuous function. Therefore the composition $h_{k,i_1,\dots,i_k,j_1,\dots,j_k}$ of the determinant with this function is continuous.

Fix $t \in \mathbb{R}$ and let $(t_n)_{n\ge1} \subset \mathbb{R}$ be such that $\lim_{n\to\infty} t_n = t$. If $f(t) = 0$ there is nothing to prove. Otherwise, $f(t) \neq 0$. Then taking $k = \text{rank}(f(t))$ there exists a selection $1 \le i_1 < i_2 < \cdots < i_k \le n$ and $1 \le j_1 < j_2 < \cdots < j_k \le n$ such that $h_{k,t_1,\dots,t_k,j_1,\dots,j_k}(t) \neq 0$. Since $h_{\dots}$ is continuous, there exists an $N \in \mathbb{N}$ such that $h_{\dots}(t_n) \neq 0$ for all $n \ge N$. Hence

$$g(t_n) = \text{rank}(f(t_n)) \ge k = \text{rank}(f(t)) = g(t)$$

for all $n \ge N$. Hence $g(t) \le \liminf_n g(t_n)$, as desired.

To see that g is not always continuous, take $f(t) = tI_n$. Then for $t \neq 0$, $g(t) = n$, but $g(0) = 0$.

Fall 2011 #8. Assume that a complex matrix A satisfies $\ker((A - \lambda I)) = \ker((A - \lambda I)^2)$ for all $\lambda \in \mathbb{C}$. Show from first principles (i.e., without using the theory of canonical forms) that A must be diagonalizable.

Suppose for the sake of contradiction that $\mu_A(t)$ has a multiple root λ. Then

$$0 = \mu_A(A) = (A - \lambda I)^2(q(A))(v)$$

for all v. But since $\ker((A - \lambda I)^2) = \ker((A - \lambda I))$, this means

$$(A - \lambda I)(q(A))(v) = 0$$

for all v. But this contradicts the definition of the minimal polynomial. Hence the minimal polynomial of A has no repeated roots. This implies that A is diagonalizable.

Fall 2011 #10. Let A be a 3×3 real matrix with $A^3 = I$. Show that A is similar to a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

for some (real) θ. What values of θ are possible?

Note eigenvalues satisfy $\lambda^3 = 1$. Thus $\lambda = 1, e^{2\pi/3}$ or $e^{4\pi/3}$. And if λ is an eigenvalue, then so is $\overline{\lambda}$. Also, the product of the eigenvalues is $\det(A) = 1$. Hence all the eigenvalues are 1 or $\lambda_1 = 1$, $\lambda_2 = e^{2\pi/3}$, and $\lambda_3 = e^{4\pi/3}$. In the first case, the characteristic polynomial is $t^3 - 1$, so by the Cayley-Hamilton Theorem, $(A - I)^3 = 0$. Using $A^3 = I$, this implies $A^2 = A$, so $A = 0$ or $A = 1$. Clearly $A = 1$, and this is in the desired form for $\theta = 0$.

In the second case, defining $\theta = 2\pi/3$, the second and third eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$. Let v_1, v_2, v_3 be eigenvectors of A corresponding to $\lambda_1, \lambda_2, \lambda_3$. We can write

$$A = [v_1 v_2 v_3] \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} [v_1 v_2 v_3]^{-1}.$$

Now conjugate the lower two rows by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

and note UV is real.

Fall 2011 #11. Suppose V, W, and U are finite dimensional vector spaces over $\mathbb{R}$ and that $T : V \to W$ and $S : W \to U$ are linear operators. Suppose further that T is one-to-one, S is onto, and $S \circ T = 0$. Prove that $\ker(S) \supset \text{image}(T)$ and that

$$-\dim(V) + \dim(W) - \dim(U) = \dim(\ker(S)/\text{image}(T)).$$

From $S \circ T = 0$, we obtain $\ker(S) \supset \text{image}(T)$. We know

$$\dim(\ker(S)/\text{image}(T)) = \dim(\ker(S)) - \dim(\text{image}(T)).$$

By the dimension theorem, since T is one-to-one

$$\dim(V) = \dim(\ker(T)) + \dim(\text{image}(T)) = \dim(\text{image}(T)).$$

By the dimension theorem, since S is onto,

$$\dim(W) = \dim(\ker(S)) + \dim(\text{image}(S)) = \dim(\ker(S)) + \dim(U).$$

Putting these together directly yields the desired equation.

Spring 2011 #1. Let A be a 3 by 3 matrix with complex entries. Consider the set of such A that satisfy $\text{tr}(A) = 4$, $\text{tr}(A^2) = 6$, and $\text{tr}(A^3) = 10$. For each similarity (i.e. conjugacy) class of such matrices, give one member in Jordan normal form. The following identity may be helpful:

If $b_1 = a_1 + a_2 + a_3$, $b_2 = a_1^2 + a_2^2 + a_3^2$, and $b_3 = a_1^3 + a_2^3 + a_3^3$, then

$$6a_1a_2a_3 = b_1^3 + 2b_3 - 3b_1b_2.$$

We deduce from the algebra that the eigenvalues of A are $1, 1$, and 2. Thus the Jordan normal form must have a 1×1 block with eigenvalue 2 and a 2×2 block with diagonal entries 1 and either a 0 or 1 in the upper right.

Spring 2011 #3. Show that for any Hermitian (i.e. self-adjoint) operator H on a finite dimensional inner product space there exists a unitary operator U such that UHU^* is diagonal. (You may use a basis if you need to!)

The spectral theorem gives an orthonormal basis such that H is diagonal with respect to that basis; the change of basis matrix consists of the orthogonal basis vectors and hence is unitary.

Spring 2011 #4. Let A be an $n \times n$ real matrix. Define an LU decomposition of A. State a necessary and sufficient condition on A for the existence of such a decomposition. Suppose we normalize the decomposition by requiring that the diagonal entries of L are 1. Show that in this case, if the LU decomposition exists, then it is unique. Give the LU decomposition of the matrix

$$\begin{pmatrix} 4 & 3 \\ 6 & 3 \end{pmatrix}.$$

If the principal minors are invertible, the LU decomposition exists. Suppose $A = L_1U_1 = L_2U_2$. Then $L_2^{-1}L_1 = U_2U_1^{-1}$, where the left hand side is unit lower triangular and the right hand side is upper triangular. It follows that both sides are the identity, so $L_1 = L_2$ and $U_1 = U_2$.

$$\begin{pmatrix} 4 & 3 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 0 & -\frac{3}{2} \end{pmatrix}.$$

Fall 2012 #7. Let A be an invertible $n \times n$ matrix with entries in $\mathbb{C}$. Suppose that the set of powers A^n of A, for $n \in \mathbb{Z}$, is bounded. Show that A is diagonalizable.

Since $A^k = \lambda^k v$ for an eigenvector v, it is clear that if A^k is bounded for all integers k, then $|\lambda| = 1$. Suppose for the sake of contradiction that A is not diagonalizable. Then A has a Jordan normal form with a block of size at least 2, so there exist unit vectors v_1, v_2 such that $Av_1 = \lambda v_1$ and $Av_2 = v_1 + \lambda v_2$. It follows that $A^k v_2 = k\lambda^{k-1} v_1 + \lambda^k v_2$, so $||A^k|| \geq |A^k v_2| \geq k - 1$. Since k was arbitrary, the set of powers of A is not bounded, a contradiction.

Fall 2012 #8. Let H be an $n \times n$ Hermitian matrix with non-zero determinant. Use H to define an Hermitian form $[\cdot, \cdot]$ by the formula: for $x, y \in \mathbb{C}^n$ (column vectors!), $[x, y] = \overline{x}^t H y$. Let W be a complex subspace of $\mathbb{C}^n$ such that $[w_1, w_2] = 0$ for all $w_1, w_2 \in W$. Show that $\dim(W) \leq n/2$. Give also for each n an example of an H for which $\dim(W) = n/2$ if n is even or $\dim(W) = (n-1)/2$ if n is odd.

Select an orthonormal basis $e_1, \ldots, e_n$ so that H is diagonal with respect to this basis. Then the diagonal entries must be non-zero real numbers.

With respect to the orthonormal basis,

$$[w, w] = \lambda_1 |w_1|^2 + \lambda_2 |w_2|^2 + \cdots + \lambda_n |w_n|^2.$$

Pair up positive diagonal entries with negative to get the example H and W. We can always form such a basis, and then nothing can be added to it.

Fall 2012 #10. Let A be a linear operator on a four dimensional complex vector space that satisfies the polynomial equation $P(A) = A^4 + 2A^3 - 2A - I = 0$. Let $B = A + I$, and suppose $\dim(\text{range}(B)) = 2$. Finally, suppose that $|\text{tr}(A)| = 2$. Give a Jordan canonical form of A.

Factor $P(A) = (A - I)(A + I)^3$ and note this must be the characteristic polynomial of A. Thus the eigenvalues of A are 1 and -1. Since $\dim(\text{range}(B)) = 2$, the dimension theorem implies $\dim(\ker(A - (-1)I)) = \dim(\ker(B)) = 2$, thus the geometric multiplicity of -1 is 2. From the trace condition, -1 must have multiplicity 3 and 1 must have multiplicity 1. Hence the Jordan form has a 2×2 block with -1 on the diagonal, and 1 on the superdiagonal, as well as a 1×1 block with -1, and a 1×1 block with 1.

Spring 2012 #7. Let F be the finite field of p elements, let V be an n-dimensional vector space over F and let $0 \leq k \leq n$. Compute the number of invertible linear maps $V \to V$. It is acceptable if your solution is a lengthy algebraic expression, as long as you explain why it is correct.

The first column can be anything but the zero vector, the second can be anything but multiples of the first column, and generally the kth column can be any vector not in the span of the first $k-1$ columns. This gives

$$(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$$

invertible linear maps. Since the first $k-1$ columns are linearly independent, we are not overcounting the number of possibilities for the kth column.

Spring 2012 #8. Let A be an $n \times n$ complex matrix. Prove that there are two sequences of matrices $\{B_i\}$ and $\{L_i\}$, such that L_i are diagonal with distinct eigenvalues, and $B_i L_i B_i^{-1} \to A$ as $i \to \infty$. Here by convergence of matrices we mean convergence in all entries.

First, write A as VUV^{-1}, where U is upper triangular using Jordan canonical form. Then by changing the diagonal entries of U by less than $1/i$, we obtain a matrix T_i with distinct entries along the diagonal. Then T_i has distinct eigenvalues, so there exists some C_i with $T_i = C_i L_i C_i^{-1}$, where L_i is diagonal. It follows that

$$(VC_i)L_i(VC_i)^{-1} \to VUV^{-1} = A$$

as $i \to \infty$.

Spring 2012 #9. Let $a_1 = 1, a_2 = 4, a_{n+2} = 4a_{n+1} - 3a_n$ for all $n \geq 1$. Find a 2×2 matrix A such that

$$A^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$$

for all $n \geq 1$. Compute the eigenvalues of A and use them to determine the limit

$$\lim_{n\to\infty} (a_n)^{1/n}.$$

$$A = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of A are 1 and 3. Thus we can diagonalize A and $\lim_{n\to\infty}(a_n)^{1/n} = 3$ since it picks out the largest eigenvalue.

Spring 2012 #10. Let A be a complex $n \times n$ matrix. State and prove under which conditions on A the following identity holds:

$$\det(e^A) = \exp(\text{tr}(A)).$$

Here the matrix exponentiation is defined via the Taylor series. You can assume known that this sum converges (entrywise) for all complex matrices A.

This holds for any matrix A. Using Jordan normal form, there exists an invertible matrix V such that $U = VAV^{-1}$ is upper triangular. It is straightforward to show $e^{VAV^{-1}} = Ve^AV^{-1}$ using the definitions. Then

$$\det(e^A) = \det(e^{VUV^{-1}}) = \det(Ve^UV^{-1}) = \det(e^T)$$

and

$$e^{\text{tr}(A)} = e^{\text{tr}(VUV^{-1})} = e^{\text{tr}(UVV^{-1})} = e^{\text{tr}(T)}.$$

Thus it suffices to show that $\det(e^T) = e^{\text{tr}(T)}$ for upper triangular T. But this follows immediately since the determinant is the product of the diagonal entries of e^T, which are e^{λ_i}.

Spring 2012 #11. (a) Find a polynomial $P(x)$ of degree 2, such that $P(A) = 0$, for

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}.$$

(b) Prove that such $P(x)$ is unique, up to multiplication by a constant.

(i) $P(x) = x^2 - 3x - 10$.

(ii) Suppose there exist two monic quadratic polynomials P and Q with $P(A) = 0 = Q(A)$. Then $(P - Q)(A) = 0$, and $P - Q$ is either 0 or a first degree polynomial. But clearly no first degree polynomial evaluated at A is 0, hence $P - Q = 0$, so $P = Q$.

Spring 2012 #12. Recall that the quadratic forms $Q_1(x, y)$ and $Q_2(x', y')$ are said to be equivalent if they are related by a non-singular change of coordinates $(x, y) \mapsto (x', y')$. Decide whether $Q_1 = xy$ and $Q_2 = x^2 + y^2$ are equivalent over $\mathbb{C}$ and whether they are equivalent over $\mathbb{R}$. If not, give a proof. If yes, find the matrix for change of coordinates.

Send x to $x' = x + iy$ and y to $y' = x - iy$. Then $x'y' = x^2 + y^2$. This is a non-singular change of coordinates, so the quadratic forms are equivalent over $\mathbb{C}$. They are not equivalent over $\mathbb{R}$. Suppose $a, b, c, d \in \mathbb{R}$ with $(ax+by)(cx+dy) = acx^2 + bdy^2 + (ad+bc)xy$. If this equals x^2+y^2, then $ad+bc = 0$, and $ac = bd = 1$. Thus a and c are either both positive or both negative, and b and d are either both positive or both negative. But this implies ad and bc are both positive or both negative, so they cannot sum to 0, a contradiction.

Spring 2013 #6. (a) Prove that diagonalizable matrices are dense in the set of all $n \times n$ matrices with complex entries.

(b) Are diagonalizable matrices with real entries dense in the set of all $n \times n$ matrices with real entries?

(a) See Spring 2012 #8.

(b) No. For example, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which has eigenvalues $\pm i$ cannot be approximated by diagonalizable matrices. The roots of a polynomial are continuous functions of its coefficients, and the coefficients of the characteristic polynomial are continuous functions of the entries of the matrix, thus the eigenvalues of a matrix are continuous functions of its entries, so any far-enough-along approximating matrices cannot be diagonalizable in the reals, because the diagonal matrix would need to have complex entries.

Spring 2013 #7. (a) Show that the series $\exp(A) = I + A + A^2/2! + \cdots$ converges to a limit in the usual sense of convergence of matrices (converge entry by entry).

(b) Show that the series $\ln(I+A) = A - A^2/2 + A^3/3 + \cdots + (-1)^{n+1}A^n/n + \cdots$ converges if the operator norm of A is less than one.

(c) Show that $\exp(\ln(I + A)) = I + A$ if the operator norm of A is less than 1.

(a) Since $||AB|| \leq ||A||||B||$, we have $||A^k|| \leq ||A||^k$. By the ratio test, $\sum_{k=0}^{\infty} \frac{1}{k!}||A||^k$ converges, and the partial sums of e^A have norm less than the partial sums of this series, so e^A converges. This relies on the completeness of the normed space of matrices.

(b) Note

$$\sum_{k\geq 1} ||\frac{(-1)^{k+1}}{k}A^k|| \leq \sum_{k\geq 1} \frac{1}{k}||A^k|| \leq \sum_{k\geq 1} ||A||^k < \infty,$$

where the final series is geometric since $||A|| < 1$. Technically we should estimate the partial sums, then conclude that the series converges.

(c) Too long to work out - not worth it.

Spring 2013 #8. Let T be a linear transformation from a finite-dimensional vector space V with an inner product to a finite dimensional vector space W also with an inner product (the dimension of W can be different from the dimension of V here).

(a) Define the adjoint $T^* : W \to V$.

(b) Show that if matrices are written relative to orthonormal bases of V and W, then the matrix of T^* is the transpose of the matrix of T.

(a) Petersen takes a basis e_i for V and sets

$$T^* y = \sum_{i=1}^{n} (y|T(e_j))_W e_j.$$

Then it follows that

$$(Lx|y) = (x|L^* y).$$

Uniqueness: suppose $(x|K_1 y) = (x|K_2 y)$ for all x, y. Then

$$0 = (x|K_1 y - K_2 y).$$

Then taking $x = K_1 y - K_2 y$, we get $K_1 y = K_2 y$.

(b) Let $e_1, \ldots, e_n$ be an orthonormal basis of V and $f_1, \ldots, f_m$ be an orthonormal basis of W. Suppose T with respect to these bases has entries a_{ij}. Thus $T(e_i) = \sum_{i=1}^{m} a_{ij} f_i$. It follows that

$$T^* f_j = \sum_{i=1}^{n} (f_i|T(e_i)) e_i = \sum_{i=1}^{n} a_{ij} e_i.$$

Hence the matrix representation of T^* with respect to these bases has entries a_{ji}, so the matrix of T^* is the transpose of the matrix of T.

Spring 2013 #10. Denote by G the set of real 4×4 upper triangular matrices with 1's on the diagonal. Fix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Denote by C the set of matrices in G commuting with M.

(a) Prove that C is an affine subspace in the space $\mathbb{R}^{16}$ of all 4×4 real matrices. S is an "affine subspace" of a vector space V if there is a vector $w \in V$ such that $S^0 = \{v - w : v \in S\}$ is a subspace of V. The dimension of S is defined to be the dimension of S^0.

(b) Find the dimension of C.

By brute force, we check that matrices in C take the form

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where a, b, c are arbitrary. Taking $w = I_4$, $C - w$ is a subspace of $\mathbb{R}^{16}$, so C is an affine subspace of $\mathbb{R}^{16}$.

(b) Since a, b, c are arbitrary, $C - w$ has dimension 3, so C is an affine subspace of dimension 3.

Fall 2011 #9. Let V be a finite dimensional inner product space, and let $L : V \to V$ be a self-adjoint linear operator. Let μ and ε be given. Suppose there is a unit vector $x \in V$ such that

$$||L(x) - \mu x|| \leq \varepsilon.$$

Prove that L has an eigenvalue λ so that $|\lambda - \mu| \leq \varepsilon$.

By the spectral theorem, there exists an orthonormal basis (e_i) consisting of eigenvectors of L with eigenvalues λ_i. Write

$$x = \sum_{i=1}^{n} (x \cdot e_i) e_i.$$

Then
$$L(x) - \mu x = \sum((x \cdot e_i)\lambda_i - \mu(x \cdot e_i))e_i = \sum(\lambda_i - \mu)(x \cdot e_i)e_i.$$
Now
$$\sum |\lambda_i - \mu||x \cdot e_i|^2 = \sum ||(\lambda_i - \mu)(x - e_i)||^2 = ||L(x) - \mu x||^2 \leq \varepsilon^2.$$
Since $|x| = 1$, $\sum |x \cdot e_i|^2 = 1$, hence there must exist i with
$$|\lambda_i - \mu|^2 \leq \varepsilon^2.$$
Thus $|\lambda_i - \mu| \leq \varepsilon$.

Spring 2010 #2. Let A be an $n \times n$ real symmetric matrix and let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of A. Prove that
$$\lambda_k = \max_{U:\dim(U)=k} \min_{x \in U:||x||=1} (Ax|x),$$
where $(\cdot, \cdot)$ denotes the usual scalar product in $\mathbb{R}^n$ and the maximum is taken over all k-dimensional subspaces of $\mathbb{R}^n$.

As in Spring 2008 #12, we can show $\min_{x \in U:||x||=1}(Ax|x)$ is an eigenvalue for any subspace U. In fact it is the least eigenvalue with an eigenvector in U. Now any k-dimensional subspace contains eigenvectors for k eigenvalues λ_i. Hence the max over all U is λ_k.

Spring 2010 #4. (i) Let $A = (a_{i,j})$ be an $n \times n$ real symmetric matrix such that $\sum_{i,j} a_{i,j}x_ix_j \leq 0$ for every vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Prove that if $\mathrm{tr}(A) = 0$, then $A = 0$.

(ii) Let T be a linear transformation in the complex finite dimensional vector space V with a positive definite Hermitian inner product. Suppose that $TT^* = 4T - 3I$, where I is the identity transformation. Prove that T is positive definite Hermitian and find all possible eigenvalues of T.

(i) Suppose $\mathrm{tr}(A) = 0$. Using $x = e_i$, we see $a_{ii} \leq 0$. Thus the diagonal entries are 0. Then using $e_i + e_j$ and $e_i - e_j$ for $i \neq j$, as well as the symmetry of A, we see $a_{ij} \geq 0$ and $a_{ij} \leq 0$, so $A = 0$.

(ii) ?

Spring 2011 #5. Let A be an n by n matrix with real entries, and let b be an n by 1 column vector with real entries. Prove that there exists an n by 1 column vector solution x to the equation $Ax = b$ if and only if b is in the orthocomplement of the kernel of the transpose of A.

There exists x such that $Ax = b$ holds if and only if b is in the column space of A, which holds if and only if b is in the row space of A^T. Now b is in the row space of A^T if and only if every element in the kernel of A^T is orthogonal to b, which holds if and only if b is in the orthocomplement of the kernel of A^T.

Winter 2006 #9. Let $A \in \mathbb{M}_3(\mathbb{R})$ be invertible and satisfy $A = A^t$ and $\det A = 1$. Prove that A has one as an eigenvalue.

?