一.(15分).

(1). 记 $a=0$, $b=3$, 则插入的两点为:

$x_1=a+0.382(b-a)=0.382\times3=1.146$.

$x_2=a+0.618(b-a)=0.618\times3=1.854$

$\because f(x_1)=x_1^3-3x_1+1=1.146^3-3\times1.146+1=-0.933$

$f(x_2)=x_2^3-3x_2+1=1.854^3-3\times1.854+1=1.811$

$\because f(x_1)<f(x_2)$.

$\therefore$ 去掉最右边区间 $[x_2, b]$.

故新的搜索区间为 $[a, x_2]=[0, 1.854]$.

(2) $\because f(0)=1$, $f(1)=-1$, $f(3)=19$.

$\therefore$ 取 $x_1=0$, $x_2=1$, $x_3=3$.

构造二次多项式 $\phi(x)=a_0+a_1x+a_2x^2$.

由 $\phi(x_1)=a_0=f(x_1)=1$

$\phi(x_2)=a_0+a_1+a_2=f(x_2)=-1$

$\phi(x_3)=a_0+3a_1+9a_2=f(x_3)=19$.

可知, $a_0=1$, $a_1=-6$, $a_2=4$

$\therefore \phi$ 的极小点 $\bar{x}=-\frac{a_1}{2a_2}=\frac{3}{4}$

$\therefore f(\bar{x})=(\frac{3}{4})^3-3\times\frac{3}{4}+1=-\frac{53}{64}$.

$\because f(\bar{x})=-\frac{53}{64}>f(x_2)=-1$.

$\therefore x_1=\frac{3}{4}$, $x_2=1$, $x_3=3$. $f(x_1)=-\frac{53}{64}$, $f(x_2)=-1$, $f(x_3)=19$.

构造二次多项式 $\phi(x)=a_0+a_1x+a_2x^2$.

由 $\phi(x_1)=a_0+\frac{3}{4}a_1+\frac{9}{16}a_2=f(x_1)=-\frac{53}{64}$

$\phi(x_2)=a_0+a_1+a_2=f(x_2)=-1$.

$\phi(x_3)=a_0+3a_1+9a_2=f(x_3)=19$.

可知 $a_0=\frac{13}{4}$, $a_1=-9$, $a_2=\frac{19}{4}$

$\therefore \phi$ 的极小点 $\bar{x}=-\frac{a_1}{2a_2}=\frac{18}{19}$

$\therefore f(\bar{x})=(\frac{18}{19})^3-3\times\frac{18}{19}+1=-\frac{6803}{6859}$

$\because f(\bar{x})=-\frac{6803}{6859}>-1=f(x_2)$.

$\therefore x_1=\frac{18}{19}$, $x_2=1$, $x_3=3$.

二. (20分). $\nabla f(x)=\begin{pmatrix}2x_1+x_2-3\\ 2x_2+x_1\end{pmatrix}$. $A=\begin{pmatrix}2&1\\1&2\end{pmatrix}$

FR: $x^1=\begin{pmatrix}1\\1\end{pmatrix}$. $d^1=-\nabla f(x^1)=-\begin{pmatrix}2x_1+x_2-3\\ 2x_2+x_1\end{pmatrix}\Big|_{(1,1)^T}=\begin{pmatrix}0\\-3\end{pmatrix}$.

$$\lambda_1=-\frac{g_1^T d^1}{(d^1)^T A d^1}=-\frac{(0,3)\begin{pmatrix}0\\-3\end{pmatrix}}{(0,-3)\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}0\\-3\end{pmatrix}}=\frac{1}{2}$$

$x^2=x^1+\lambda_1 d^1=\begin{pmatrix}1\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0\\-3\end{pmatrix}=\begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix}$. $g_2=\nabla f(x^2)=\begin{pmatrix}-\frac{3}{2}\\0\end{pmatrix}$. 不满足精度,继续迭代:

$$\alpha_1=\frac{\|g_2\|^2}{\|g_1\|^2}=\frac{\frac{9}{4}}{9}=\frac{1}{4}$$

$\therefore d^2=-g_2+\alpha_1 d^1=-\begin{pmatrix}-\frac{3}{2}\\0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0\\-3\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\\-\frac{3}{4}\end{pmatrix}$.

$\lambda_2=-\dfrac{g_2^T d^2}{(d^2)^T A d^2}=\dfrac{2}{3}$, $x^3=x^2+\lambda_2 d^2=\begin{pmatrix}2\\-1\end{pmatrix}$. $g_3=\begin{pmatrix}0\\0\end{pmatrix}$.

$\therefore$ 最优解为 $\begin{pmatrix}2\\-1\end{pmatrix}$. 最优值为 -3.

DFP: $x^1=\begin{pmatrix}1\\1\end{pmatrix}$. $d^1=-g_1=\begin{pmatrix}0\\-3\end{pmatrix}$. $\lambda_1=-\dfrac{g_1^T d^1}{(d^1)^T A d^1}=\dfrac{1}{2}$.

$\therefore x^2=x^1+\lambda_1 d^1=\begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix}$. $g_2=\begin{pmatrix}-\frac{3}{2}\\0\end{pmatrix}$ 不满足精度,继续迭代.

$\Delta x^1=x^2-x^1=\begin{pmatrix}0\\-\frac{3}{2}\end{pmatrix}$. $\Delta g_1=g_2-g_1=\begin{pmatrix}-\frac{3}{2}\\-3\end{pmatrix}$.

$$H_2=H_1+\frac{\Delta x^1(\Delta x^1)^T}{(\Delta x^1)^T\Delta g_1}-\frac{H_1\Delta g_1(\Delta g_1)^T H_1}{(\Delta g_1)^T H_1\Delta g_1}$$

$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{\begin{pmatrix}0\\-\frac{3}{2}\end{pmatrix}(0,-\frac{3}{2})}{(0,-\frac{3}{2})\begin{pmatrix}-\frac{3}{2}\\-3\end{pmatrix}}-\frac{\begin{pmatrix}-\frac{3}{2}\\-3\end{pmatrix}(-\frac{3}{2},-3)}{(-\frac{3}{2},-3)\begin{pmatrix}-\frac{3}{2}\\-3\end{pmatrix}}$$

$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&0\\0&\frac{1}{2}\end{pmatrix}-\begin{pmatrix}\frac{1}{5}&\frac{2}{5}\\\frac{2}{5}&\frac{4}{5}\end{pmatrix}=\begin{pmatrix}\frac{4}{5}&-\frac{2}{5}\\-\frac{2}{5}&\frac{7}{10}\end{pmatrix}.$$

$d^2=-H_2 g_2=\begin{pmatrix}\frac{6}{5}\\-\frac{3}{5}\end{pmatrix}$. $\lambda^2=-\dfrac{g_2^T d^2}{(d^2)^T A d^2}=\dfrac{5}{6}$ $\therefore x^3=x^2+\lambda_2 d^2=\begin{pmatrix}2\\-1\end{pmatrix}$.

$g_3=\begin{pmatrix}0\\0\end{pmatrix}$. $\therefore$ 最优解为 $\begin{pmatrix}2\\-1\end{pmatrix}$, 最优值为 -3.

三．(10分).

(1) 设第i个产地运往第j个销地的量为 x_{ij}．($i=1,2$, $j=1,2,3$).

则优化模型为：$\min \ 3x_{11}+11x_{12}+3x_{13}+x_{21}+9x_{22}+2x_{23}$.

$$
\begin{aligned}
s.t.\quad & x_{11}+x_{12}+x_{13}=8 \\
& x_{21}+x_{22}+x_{23}=6 \\
& x_{11}+x_{21}=3 \\
& x_{12}+x_{22}=9 \\
& x_{13}+x_{23}=5 \\
& x_{ij}\geq 0,\ i=1,2,\ j=1,2,3
\end{aligned}
$$

(2) 对偶问题为：

$$
\begin{aligned}
\max\quad & 8y_1+6y_2+3y_3+9y_4+5y_5 \\
& y_1+y_3\leq 3. \\
& y_1+y_4\leq 11 \\
& y_1+y_5\leq 3. \\
& y_2+y_3\leq 1 \\
& y_2+y_4\leq 9. \\
& y_2+y_5\leq 2.
\end{aligned}
$$

四.(15分).

证明:(1).
$$H_{k+1}\Delta g_k = \left(H_k + \frac{(\Delta x^k - H_k\Delta g_k)\Delta g_k^T H_k}{\Delta g_k^T H_k \Delta g_k}\right)\Delta g_k$$

$$= H_k\Delta g_k + \frac{(\Delta x^k - H_k\Delta g_k)\Delta g_k^T H_k \Delta g_k}{\Delta g_k^T H_k \Delta g_k}$$

$$= H_k\Delta g_k + (\Delta x^k - H_k\Delta g_k)$$

$$= \Delta x^k.$$

(2). "$\Rightarrow$" ∵ f是正齐的. ∴ $\forall x, y \in \mathbb{R}^n$, 有

$$f(x+y) = f\left(2\cdot\tfrac{1}{2}(x+y)\right) = 2f\left(\tfrac{1}{2}x+\tfrac{1}{2}y\right)$$

又 f是凸函数.

$$\therefore f(x+y) = 2f\left(\tfrac{1}{2}x+\tfrac{1}{2}y\right) \le 2\left(\tfrac{1}{2}f(x)+\tfrac{1}{2}f(y)\right) = f(x)+f(y).$$

"$\Leftarrow$" $\forall x, y \in \mathbb{R}^n$, $\forall \lambda \in (0,1)$.

$$f(\lambda x + (1-\lambda)y) \le f(\lambda x) + f((1-\lambda)y)$$
$$= \lambda f(x) + (1-\lambda)f(y).$$

五 (15分)

解：(1) 化为标准型： $\min\ x_1+2x_2$

$$\text{s.t.}\ \begin{cases} x_1+2x_2-x_3=4 \\ x_1+x_4=5 \\ 3x_1+x_2-x_5=6 \\ x_i\geq 0,\ i=1,\cdots,5. \end{cases}$$

将第1个和第3个等式乘以 -1，得到 $\min\ x_1+2x_2$

$$\text{s.t.}\ \begin{cases} -x_1-2x_2+x_3=-4 \\ x_1+x_4=5 \\ -3x_1-x_2+x_5=-6 \\ x_i\geq 0,\ i=1\cdots,5. \end{cases}$$

取 x_3. x_4. x_5 为基变量，得到一个对偶可行的基本解 $(0,0,-4,5,-6)^T$

	C	1	2	0	0	0	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	b
0	x_3	-1	-2	1	0	0	-4
0	x_4	1	0	0	1	0	5
0	x_5	-3	-1	0	0	1	-6 $\rightarrow$
	σ_j	-1	-2	0	0	0	
		$\frac{1}{3}\uparrow$	2	—	—	—	
0	x_3	0	$-\frac{5}{3}$	1	0	$-\frac{1}{3}$	-2 $\rightarrow$
0	x_4	0	$-\frac{1}{3}$	0	1	$\frac{1}{3}$	3
1	x_1	1	$\frac{1}{3}$	0	0	$-\frac{1}{3}$	2
	σ_j	0	$-\frac{5}{3}$	0	0	$-\frac{1}{3}$	
		—	$1\uparrow$	—	—	1	
2	x_2	0	1	$-\frac{5}{3}$	0	$\frac{1}{5}$	$\frac{6}{5}$
0	x_4	0	0	$-\frac{1}{5}$	1	$\frac{2}{5}$	$\frac{17}{5}$
1	x_1	1	0	$\frac{1}{5}$	0	$-\frac{2}{5}$	$\frac{8}{5}$
	σ_j	0	0	-1	0	0	

所有的右端项 $b_i\geq 0$，得到最优解 $x^*=(\frac{8}{5},\frac{6}{5},0,\frac{17}{5},0)^T$，最优值 $f^*=4$.

(2). 在最优解 $x^*=(\frac{8}{5},\frac{6}{5},0,\frac{17}{5},0)^T$ 中，x_1. x_2. x_4 为基变量，对应的基为 $(P_1.\ P_2.\ P_4)$

$\therefore$ 最优基矩阵为 $(P_1.\ P_2.\ P_4)=\begin{pmatrix}1&2&0\\1&0&1\\3&1&0\end{pmatrix}$.

六. (10分).

令 $F(x, M_k) = 2x_1 + 3x_2 + M_k[\min\{0, -2x_1^2 - x_2^2 + 1\}]^2$

$$= \begin{cases} 2x_1 + 3x_2, & -2x_1^2 - x_2^2 + 1 \geq 0 \\ 2x_1 + 3x_2 + M_k(-2x_1^2 - x_2^2 + 1)^2, & -2x_1^2 - x_2^2 + 1 < 0 \end{cases}$$

$$\frac{\partial F}{\partial x_1} = \begin{cases} 2, & -2x_1^2 - x_2^2 + 1 \geq 0 \\ 2 + 2M_k(-2x_1^2 - x_2^2 + 1)\cdot(-4x_1), & -2x_1^2 - x_2^2 + 1 < 0 \end{cases}$$

$$\frac{\partial F}{\partial x_2} = \begin{cases} 3 & -2x_1^2 - x_2^2 + 1 \geq 0 \\ 3 + 2M_k(-2x_1^2 - x_2^2 + 1)\cdot(-2x_2). & -2x_1^2 - x_2^2 + 1 < 0 \end{cases}$$

令 $\frac{\partial F}{\partial x_1} = 0 = \frac{\partial F}{\partial x_2}$. 得. $x_2 = 3x_1$, $x_1(1 - 11x_1^2) = \frac{1}{4M_k}$

令 $M_k \to +\infty$. 得. $x_1 \to 0$ 或 $x_1 \to \pm\frac{\sqrt{11}}{11}$.

$\therefore \bar{x}^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\bar{x}^2 = \begin{pmatrix} \frac{\sqrt{11}}{11} \\ \frac{3\sqrt{11}}{11} \end{pmatrix}$, $\bar{x}^3 = \begin{pmatrix} -\frac{\sqrt{11}}{11} \\ -\frac{3\sqrt{11}}{11} \end{pmatrix}$.

经检验, 最优解为 $(-\frac{\sqrt{11}}{11}, -\frac{3\sqrt{11}}{11})^T$, 最优值为 $-\sqrt{11}$.

七.(15') 原问题化为

$$\begin{aligned} \min\ & 4x_1-3x_2 \\ s.t.\ & -(x_1-3)^2+x_2+1\geq 0 \\ & -x_1-x_2+4\geq 0 \\ & x_2+7\geq 0 \end{aligned}$$

(1) K-T条件为

$$\begin{pmatrix}4\\-3\end{pmatrix}-\lambda_1\begin{pmatrix}-2(x_1-3)\\1\end{pmatrix}-\lambda_2\begin{pmatrix}-1\\-1\end{pmatrix}-\lambda_3\begin{pmatrix}0\\1\end{pmatrix}=0$$

$$\begin{cases} \lambda_1\left[-(x_1-3)^2+x_2+1\right]=0 \\ \lambda_2(-x_1-x_2+4)=0 \\ \lambda_3(x_2+7)=0 \\ \lambda_1,\lambda_2,\lambda_3\geq 0 \\ -(x_1-3)^2+x_2+1\geq 0 \\ -x_1-x_2+4\geq 0 \\ x_2+7\geq 0 \end{cases}$$

(2) 在点 $x^1=\begin{pmatrix}1\\3\end{pmatrix}$ 处，第一个、第二个约束均为起作用约束，第三个约束为不起作用约束

故 $\lambda_3=0$

解方程组 $\begin{cases}4-4\lambda_1+\lambda_2=0\\-3-\lambda_1+\lambda_2=0\end{cases}\Rightarrow\begin{cases}\lambda_1=\frac{7}{3}\\\lambda_2=\frac{16}{3}\end{cases}$，故 $x^1=\begin{pmatrix}1\\3\end{pmatrix}$ 为K-T点

在点 $x^2=\begin{pmatrix}3\\1\end{pmatrix}$ 处，第二个约束为起作用约束，第一、三个约束为不起作用约束

故 $\lambda_1=\lambda_3=0$

解方程组 $\begin{cases}4+\lambda_2=0\\-3+\lambda_2=0\end{cases}\Rightarrow\begin{cases}\lambda_2=-4\\\lambda_2=3\end{cases}$ 故 $x^2=\begin{pmatrix}3\\1\end{pmatrix}$ 不是K-T点.

记 $f(x)=4x_1-3x_2$ $g_1(x)=-(x_1-3)^2+x_2+1\geq 0$. $g_2(x)=-x_1-x_2+4\geq 0$

$$g_3(x)=x_2+7\geq 0$$

$\because$ $\nabla f(x)=\begin{pmatrix}4\\-3\end{pmatrix}$ $\nabla^2 f(x)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$ $\nabla g_1(x)=\begin{pmatrix}-2(x_1-3)\\1\end{pmatrix}$ $\nabla^2 g_1(x)=\begin{pmatrix}-2&0\\0&0\end{pmatrix}$

$g_2(x)$, $g_3(x)$ 均为线性函数

$\therefore$ 所给规划为凸规划. 故 $x^1=\begin{pmatrix}1\\3\end{pmatrix}$ 为其最优解.

八.(15分)

解：(1) 化为标准型：

$$\begin{aligned} \min\ & x_1+2x_2 \\ \text{s.t.}\ & x_1+2x_2-x_3=4 \\ & x_1+x_4=5 \\ & 3x_1+x_2-x_5=6 \\ & x_i\geq 0,\ i=1,\cdots,5. \end{aligned}$$

将第1个和第3个等式乘以 -1，得到

$$\begin{aligned} \min\ & x_1+2x_2 \\ \text{s.t.}\ & -x_1-2x_2+x_3=-4 \\ & x_1+x_4=5 \\ & -3x_1-x_2+x_5=-6 \\ & x_i\geq 0,\ i=1\cdots,5. \end{aligned}$$

取 x_3. x_4. x_5 为基变量，得到一个对偶可行的基本解 $(0,0,-4,5,-6)^T$

	C	1	2	0	0	0	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
0	x_3	-1	-2	1	0	0	-4
0	x_4	1	0	0	1	0	5
0	x_5	-3	-1	0	0	1	-6 →
	σ_j	-1	-2	0	0	0	
		$\frac{1}{3}$↑	2	—	—	—	
0	x_3	0	$-\frac{5}{3}$	1	0	$-\frac{1}{3}$	-2 →
0	x_4	0	$-\frac{1}{3}$	0	1	$\frac{1}{3}$	3
1	x_1	1	$\frac{1}{3}$	0	0	$-\frac{1}{3}$	2
	σ_j	0	$-\frac{5}{3}$	0	0	$-\frac{1}{3}$	
		—	1↑	—	—	1	
2	x_2	0	1	$-\frac{5}{3}$	0	$\frac{1}{5}$	$\frac{6}{5}$
0	x_4	0	0	$-\frac{1}{5}$	1	$\frac{2}{5}$	$\frac{17}{5}$
1	x_1	1	0	$\frac{1}{5}$	0	$-\frac{2}{5}$	$\frac{8}{5}$
	σ_j	0	0	-1	0	0	

所有的右端项 $b_i\geq 0$，得到最优解 $x^*=(\frac{8}{5},\frac{6}{5},0,\frac{17}{5},0)^T$，最优值 $f^*=4$.

(2). 最优基矩阵为 $B=(P_1.P_2.P_4)=\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$.

$\therefore (y^*)^T=C_B^T B^{-1}=(1.2,0)\begin{pmatrix} -\frac{1}{5} & 0 & \frac{2}{5} \\ \frac{3}{5} & 0 & -\frac{1}{5} \\ \frac{1}{5} & 1 & -\frac{2}{5} \end{pmatrix}=(1.0,0)$

$\therefore$ 对偶问题的最优解为 $y^*=(1.0,0)^T$

九. (10分)

令 $G(x, r_k) = 2x_1 + 3x_2 + \dfrac{r_k}{-2x_1^2 - x_2^2 + 1}$

用解析法求 $\min G(x, r_k)$

$s.t.\ x \in \text{int} S.$

令 $\dfrac{\partial G}{\partial x_1} = 2 - \dfrac{r_k}{(-2x_1^2 - x_2^2 + 1)^2}(-4x_1) = 0$

$\dfrac{\partial G}{\partial x_2} = 3 - \dfrac{r_k}{(-2x_1^2 - x_2^2 + 1)^2}(-2x_2) = 0$

得 $x_2 = 3x_1$., 代入上面任一式可得. $2 = \dfrac{-4r_k x_1}{1 - 11x_1^2}$

即. $11x_1^2 - 2r_k x_1 - 1 = 0$.

$\therefore x_1 = \dfrac{2r_k \pm \sqrt{4r_k^2 + 44}}{2 \times 11} \longrightarrow \pm \dfrac{\sqrt{11}}{11}$

$\therefore \bar{x}^1 = (\frac{\sqrt{11}}{11}, \frac{3\sqrt{11}}{11})^T$, $\bar{x}^2 = (-\frac{\sqrt{11}}{11}, \frac{-3\sqrt{11}}{11})^T$

经检验, 最优解为 $(-\frac{\sqrt{11}}{11}, -\frac{3\sqrt{11}}{11})^T$. 最优值为 $-\sqrt{11}$.

十(15′) 原问题可化为 $\min\ (x_1-2)^2+(x_2-1)^2$

$$\begin{aligned} s.t.\quad & -x_1^2+x_2\geq 0 \\ & -x_1-x_2+2\geq 0 \end{aligned}$$

(1) K-T条件为

$$\begin{cases} \begin{pmatrix} 2(x_1-2) \\ 2(x_2-1) \end{pmatrix} - \lambda_1 \begin{pmatrix} -2x_1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 0 \\ \lambda_1(-x_1^2+x_2) = 0 \\ \lambda_2(-x_1-x_2+2) = 0 \\ \lambda_1, \lambda_2 \geq 0 \\ -x_1^2+x_2 \geq 0 \\ -x_1-x_2+2\geq 0 \end{cases}$$

分情况讨论

1° $\lambda_1=0$, $\lambda_2>0$ 解方程组可得 $x_1=\frac{3}{2}$, $x_2=\frac{1}{2}$, $\lambda_1=0$, $\lambda_2=1$. 但$(\frac{3}{2},\frac{1}{2})^T$不是可行点

2° $\lambda_1=0$, $\lambda_2=0$ 解方程组可得 $x_1=2$, $x_2=1$, $(2,1)^T$也为不可行点

3° $\lambda_1>0$ $\lambda_2=0$ 可分析知 $1<x_1<2$, $x_2>1$. 与 $-x_1-x_2+2\geq 0$矛盾

4° $\lambda_1>0$ $\lambda_2>0$ 解方程组可得 $x_1=1$, $x_2=1$, $\lambda_1=2/3$, $\lambda_2=2/3$. 故$(1,1)^T$为K-T点.

$x_1=-2$, $x_2=4$, $\lambda_1=-14/3<0$ 舍去

综上, K-T点为$(1,1)^T$.

(2) 记 $g_1(x)=-x_1^2+x_2\geq 0$, $g_2(x)=-x_1-x_2+2\geq 0$ $f(x)=(x_1-2)^2+(x_2-1)^2$

$\because \nabla g_1(x)=\begin{pmatrix} -2x_1 \\ 1 \end{pmatrix}$ $\nabla^2 g_1(x)=\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$ $g_2(x)$为線性函数

$\nabla f(x)=(2(x_1-2), 2(x_2-1))^T$, $\nabla^2 f(x)=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\therefore$ 所给问题为凸规划. 故(1)为其最优解.