Graph Theory: problems, in full bloom

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1 Introduction

Okay, so what's this title, you ask. Well, some time back I wrote an article on Graph Theory, didn't I? Here I discuss a few more examples of Graph Theory, some of which are somewhat unexpected. I know I oughtta be writing articles on new and RMO-related stuff, but due to APMO the time was cut short and I chose my favorite topic, combinatorics. So let's get on.

2 Here Come the Problems!

So let us start with a problem which was a actually a research problem; and was solved in 1993. It also happens to be a problem of WO 3 :) .

1. (Karger 1993) In a connected, undirected graph, a collection of edges is called a *cut* if removing those edges will disconnect the graph. A cut is called a min-cut if there is no cut with fewer edges. Prove that the maximum number of min-cuts on a graph with n vertices is exactly $\binom{n}{2}$.

Solution. What to do here? All primary attempts fail; the only thing that catches our eye is '2': why not any other integer? We for the moment we divert ourselves to consider connectedness. We would like to find an algorithm that reduces the number of vertices but preserves this property. Hence we come up with this: a *merge* $\mathcal{M}(A, B)$ is defined to be the procedure which merges A, B into a vertex V such that all the edges are preserved (note that this means we can have multiple edges between two vertices). We continue a series of merges until we have only two vertices left. We claim that the set of edges left is a *min-cut*. It is easy to realize this after a moment of thought. Conversely, note that each min-cut is achievable after a series of such merges.

And now I must say this: I found this problem in a Probabilistic Method Handout by David Arthur, so I *knew* that probabilistic method is the trick. Now consider an arbitrary min-cut. The probability that this is the min-cut left after the merges is exactly $\frac{1}{\binom{n}{2}}$ (prove it; it's easy). Note that finally one of the min-cuts is left, and each of the events of a min-cut being left are mutually exclusive. Hence follows the result.

Remark : It finally becomes clear why that '2' was important. Actually I was trying to induct when I found this solution; anyway, I did find a solution, and that is what I wanted, so yay!

So next comes this problem from APMO 2010.

2. (APMO 2010/3) Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

Solution. We translate everything into graph theory: the n people are denoted by vertices $A_1, ..., A_n$, and A_i, A_j are connected by an edge if and only if i, j are acquainted.

Well, did you get the idea? ;) We shall prove that the answer is $\binom{n-1}{2}$. Consider a graph G on $n+1$ vertices. Now, define a *merge* as in problem 1, but with the change that multiple edges are *not* allowed; hence we shall lose some edges: $n-1$ at most. Also, note that the number of pairs reduces by at most $n-1$!! We keep merging vertices. Hence, the answer for $n+1$ is at most $1+2+...+(n-2) = \binom{n-1}{2} + n - 1 = \binom{n}{2}$.

It remains to show that $\binom{n-1}{2}$ is achievable for all n. This is by induction. For the inductive step, just add a person acquainted with exactly one of the already present people. This does it.

We continue with a generalization of a problem of St. Petersburg 2000, and reproduce the solution found in the book *Problem from the book.* Note that at first this problem seems to have not even the remotest connections with graph theory. Alas, appearances can be deceptive!

3. (Generalization of St. Petersburg 2000) In each square of an $n \times n$ chessboard is written a positive real number, such that the sum of the numbers in each row is 1. Now we nominate one of the long diagonals as the *main diagonal.* It is known that for any n squares, no two in the same row or column, their product does not exceed the product of the numbers on the main diagonal. Prove that the sum of the numbers on the main diagona is at least 1.

Solution. (Adapted from [1]) We label the rows and columns $1, 2, ..., n$ consecutively, in increasing order. Let us assume the contrary of the result we are asked to prove. Then on row k there must be a cell, say (k, j), such that the number written on it is greater than the number in the cell (j, j). Color this cell red and draw an arrow from row k to row j. Some of these arrows must form a loop. From each row belonging to the loop we choose the red cell and from other rows we choose the cell on the main diagonal. Note that these cells give us a contradiction, and hence the result must be true.

The following is a nice but easy problem; I saw it in [1].

4. (St. Petersburg 2000) Does there exist a 3-regular graph (i.e. every vertex has degree 3) such that any cycle has length at least 30?

Solution. There exists a solution in [1], but here we present a simpler one found by the author. We have a lot of freedom since the number of vertices is not specified. So we do this: generate two cycles $A_1A_2...A_{900}$ and $B_1B_2...B_{900}$. Each vertex has degree 2. Now, join A_1B_1. Further, if A_i is joined to B_j, then join $A_{i+1}B_{j+29}$, where the indices are modulo 900. As $g.c.d(29, 900) = 1$, every vertex has exactly 3 neighbors; moreover the length of the shortest cycle is 32.

Some intuition. So how did all this come about? What I did was this: first I constructed two cycles, those of A_i's and the other one. Now, I wished to pair the vertices in the mentioned way, but for small $k << 900$, I faced a problem: A_i's and A_j's neighbors came very close as j became like $i + 10$ or something. Hence the reason to take a number as large as 900. The number 900 has another advantage: it is the square of 30, so $gcd(29, 900) = 1$. Hence I had a perfect matching.

Remark. There is no role played by 30 here. It can be replaced by any integer, e.g 2016 and the answer still remains same; just that the values $900, 29$ will have to be changed.

So we continue with this problem from Czech-Slovak Match 1997.

5. (Czech-Slovak Match 1997) In a society of at least 7 people each member communicates with three other members of the society. Prove that we can divide this society into two nonempty groups such that each member communicates with at least 2 members of his own group.

Solution. Just pour everything into graph theory language: we have a graph G with $n \geq 7$ vertices, with each vertex having degree exactly 3. We wish to partition G into two sub-graphs A, B such that each vertex has at least 2 neighbors in its own sub-graph.

We start with this: include in A an arbitrary cycle of length at least 3 (clearly there must exist one; basically we want to just let A have only 'good' vertices) and let the rest of the vertices be in B. Now, if there is a vertex b in B connected to at least two of the vertices in A, then bring it over to A. Keep doing this . We infer that not all vertices are transferred to A; by some double counting (exercise). Now look at this : A has only 'good' vertices; and every vertex in B has at most one neighbor in A; hence B too contains only good vertices. Hence we have reached an admissible partition.

Remark. Seeing this problem, I was reminded of the IMO 2007 problem which stumped all but a one-digit number of contestants. Fortunately, a similar strategy worked.

Now look at this Russian problem which was again procured from the problem section of [1].

6. (Russia) In a country, some pairs of towns are connected by a road. At least 3 roads leave each town. Show that there is a cycle containing a number of towns which is not a multiple of 3.

Solution. Put in graph theory language. What follows is a long argument, but it is fully backed up by intuition.

First step. Clearly, the graph G has a cycle, say $A_1...A_mA_1$; and let $|G| = n+m$. Characterize the other vertices by $B_1, ..., B_n$. We may assume that $3|m$. Now, if B_i is connected to A_j, A_k with $j < k$, then consider the cycles $B_iA_j...A_kB_i$ and $B_iA_kA_{k+1}...A_nA_1...A_j$. The sum of their lengths is congruent to $1 \pmod 3$, so at least one of them suffices. Hence we may assume that each B_i is connected to at most one A_j. Also, if $\{B_i\}$ has a subset in which each vertex has degree at least 3 in the same subset, then we may just prove the result for that, and then induct. Hence assume that no such subset exists.

Second step Now, just considering the B_i's, we see that each of them has degree at least two among themselves. Hence there must exist another cycle in it, say $C_1...C_p$.

Now see this: due to each vertex having degree at least 3, n, p are both *at most* $\frac{n}{2}$ (why?) and it is easy to see that equality cannot occur without impying the result(why?). Now let the rest of the vertices be $K_1, K_2, ..., K_a$. If each K_i (or each belonging to a fixed nonempty subset of them) is connected to at least two other K_j's (or the vertices in the nonempty subset) we repeat the above arguments and show the existence of a third cycle, and so on. This can't go on forever, so suppose that each K_i is connected to at most one other K_j.

What next? We leave it to the reader to finish: the rest is easy.

Observation. Russia has the ability to propose easy-looking problems which are actually very, very dangerous.

As usual, here follow a few problems for you to try.

3 Problems :D

1. (Gabriel Caroll, found in [1] as well as [2]) A building consists of 4004001 rooms arranged in a 2001×2001 square grid. Is it possible for each room to have exactly two doors to adjacent rooms?

2. (Moscow 1996) A math contest with 8 problems was written by 8 students. Each problem was solved by 5 students. Prove that there exist two students such that each problem was solved by at least one of these students.

3. (Russia 1993) In a country with 1993 cities, at least 93 roads exit out of each city. It is possible to travel from any city to any other city along the roads. Prove that it is possible to do this while visiting at most 62 other cities along the way (e.g. a route from city A to city E that includes cities B, C, D visits 3 cities along the way.)

4. (Russia 2000) (a) In a country with 2000 cities, some cities are connected by roads. It is known that through every city there are at most N non-self-intersecting cycles of odd length. Prove that the country can be divided into $2N + 2$ provinces so that no two cities from the same province are connected by a road.

(b) Same problem statement as above, but now prove that the country can be divided into $N + 2$ provinces so that no two cities from the same province are connected by a road.

5. (Japan 1997) Prove that one can write 2n numbers around a circle, each equal to 0 or 1 so that any string of n 0's and 1's can be obtained by starting somewhere on the circle and reading the next n digits in clockwise order.

6. (Iran TST 2006) Let G be a tournament on n vertices, with its edges colored either red or blue. Prove that there exists a vertex v such that for any other vertex u, there exists a monochromatic path from v to u.

7. (IMO 2005) In a mathematics competition with 6 problems, every two of the problems was solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all 6 problems. Show that there are at least 2 contestants who solved 5 problems each.

8. (Russia 1998) A connected graph has 1998 vertics and each vertex has degree 3. If 200 vertices, no two of which are neighbors, are deleted, then prove that the resultant graph is still connected.

4 References

[1] *Problems from the book*, Titu Andreescu and Gabriel Dospinescu

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[3] *Graph Theory*, Alexander Remorov
http://www.mit.edu/ alexrem/MC$_{G}raphTheory.pdf$

[4] *Various posts at Artofproblemsolving*
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