LESSON 5: INVARIANTS

Invariants are very useful in problems concerning algorithms involving steps that are repeated again and again (games, transformations). While these steps are repeated, the "system" undergoes some transformations, going from one state to the next one. Sometimes these transformations are difficult to follow. There might be too many cases. In this situation, we should try to focus on **"what does not change"**, on things (facts, numbers associated to each state of the system) that remain **"invariant"** to the transformations the system is subjected to. So our advice is:

If something is repeated, look for what stays unchanged!

Typically, the system starts from an initial state S and undergoes a succession of transformations (steps, moves). Usually the questions associated to this phenomenon are:
1. Can a certain given state of the system be attained?
2. Find all the possible states of the system that can be attained.
3. Do the states of the system approach (in some sense) a certain "final state"?
4. If the states of the system repeat themselves, find all the periods.

Coloring is a type of invariant.

We have already seen how coloring could help us prove that a certain tiling was not possible. Take, for example, Problem 3 from Lesson 4. The steps are "laying a domino". We have seen that coloring provides us the solution to the problem. We color the board as a chessboard. Then we start laying dominoes one after the other. After each domino is placed, what stayed the same no matter how many dominoes we have placed and in what positions? Clearly, the fact that the NUMBER OF BLACK SQUARES AND THE NUMBER OF WHITE SQUARES COVERED BY THE DOMINOES ARE EQUAL does not change. This is an INVARIANT. Initially we have an equal number of blacks and whites covered by the dominoes: 0 of each. Since the two numbers will always remain equal, then the required configuration, one with 32 whites and 30 blacks, can not be obtained.
Generally, for tiling problems, the steps are "laying a tile". The coloring helps us see better what is the fact (number) that never changes throughout this process of laying tiles.

1. You have the numbers 1 through 1000 on the board. Erase two of them and replace them by their product. What is the number on the board at the end, and why?

Solution: Observe that whatever the order in which we choose the numbers to be multiplied, the PRODUCT OF THE NUMBERS WRITTEN ON THE BOARD does not change. In other words it is an INVARIANT. Indeed, instead of having the two factors a and b in the product, we shall have only the factor ab, but it is the same thing. After 999 such replacements, on the board there will be a single number. Since the product does not change throughout the 999 replacements, the final product (that is the last number standing on the board) will be equal to the initial value of the product, that is with $1 \cdot 2 \cdot ... \cdot 1000 = 1000!$. Hence, the last number on the board is 1000!

2. A little boy wrote the numbers 1,2,...,2011 on a blackboard. He picks any two numbers x, y, erases them with a sponge and writes the number $\mid x - y \mid$. This process continues until only one number is left. Prove that the number left is even. [Engel]

Solution 1: Observe what happens to the odd numbers during these replacements: if x, y are both even, their difference will be even, hence the number of the odd numbers does not change. If exactly one of the numbers x, y is even (and the other one odd), $\mid x - y \mid$ will give an odd number, the the total number of the odd numbers will, again, remain unchanged. Finally, if x, y are both odd, $\mid x - y \mid$ is even, so the total number of odd numbers will decrease by 2. Therefore THE PARITY OF THE NUMBER OF ODD NUMBERS does never change; it is an INVARIANT. Initially we have 1006 odd numbers on the board, that is we have an even number of odd numbers. Then at the end we shall still have an even number of odd numbers, therefore the last number on the board is even (it can not be odd, otherwise we would remain with an odd number of odd numbers).

Solution 2: When x, y are replaced with $\mid x - y \mid$ (that is, with one of the numbers $x - y$ or $y - x$), what happens to the sum of the numbers on the blackboard? Instead of $S = x + y +$ (the sum of the other numbers), we shall obtain either $(x - y) +$ (the sum of the other numbers), or $(y - x) +$ (the sum of the other numbers). From S, the sum becomes either $S - 2x$, or $S - 2y$. Since $2x$ and $2y$ is even, the PARITY OF THE SUM OF THE NUMBERS WRITTEN ON THE BOARD does not change; it is another INVARIANT. Initially this sum is $1+2+...+2011 = 2011 \cdot 1006$ (even), therefore the last number will also be even.

3. The numbers 256, 6561 and 390625 are written on a blackboard.
Step 1: Erase the three numbers and instead of each of them, write the geometric mean of the other two numbers.
Step 2: Apply step 1 to the numbers obtained after step 1.
....
Step n: Apply step 1 to the numbers obtained after the previous step.
Is it possible to obtain, after a finite number of steps, the numbers 3000, 2012, 7175 on the blackboard?

Solution: Each step consists in replacing the numbers a, b, c with the numbers

$\sqrt{bc}$, $\sqrt{ca}$, $\sqrt{ab}$. Notice that the PRODUCT OF THE THREE NUMBERS, $abc = \sqrt{bc} \cdot \sqrt{ca} \cdot \sqrt{ab}$ does not change when a step is made, therefore it is an INVARIANT. Initially, the product is $256 \cdot 6561 \cdot 390625 = 2^8 \cdot 3^8 \cdot 5^8$. But $3000 \cdot 2012 \cdot 7175 \neq 2^8 \cdot 3^8 \cdot 5^8$, therefore it is impossible to obtain these three numbers on the blackboard.

4. Three coins lie on integer points on the number line. A move consists of choosing and moving two coins, the first one 1 unit to the right and the second one 1 unit to the left. Under which initial conditions is it possible to move all coins to one single point?
(Swiss olympiad, 2010)

Solution: We shall prove that the condition is: "the sum of the coordinates of the three points is a multiple of 3" (i.e. "their average is an integer".)
The proof consists of two parts: first, prove that the above condition is necessary, then prove that it is also sufficient.
Necessity: The SUM OF THE COORDINATES OF THE THREE NUMBERS never changes; it is an INVARIANT. Indeed, by each move, the coordinate of the coin you move to the right increases by 1, while the coordinate of the coin that you move to the left decreases by 1, therefore their sum does not change. In the final position we need the three coordinates to be equal, therefore their sum must be a multiple of 3.
Sufficiency: Suppose the 3 coins have initial coordinates x, y, z respectively, with $x + y + z = 3k$ (where k is some integer). This means that the final configuration should be with all three coins in the point of coordinate k. All we need is to prove we can bring them there. Without loss of generality we may suppose that $x \leq k \leq y \leq z$ (all the other cases are similar). At first we shall move $y - k$ times the first coin to the right and the second one to the left. After that, the coins are in positions $x + y - k \leq k \leq z$. Then we move $z - k$ times the first coin to the right and the third coin to the left. After these moves the coins will be situated in $x + y + z - 2k$, k, k. But since $x + y + z = 3k$, this means all the three coins are in the point of coordinate k.
This concludes our proof.

5. In three piles there are 51, 49, and 5 stones, respectively. You can combine any two piles into one pile or divide a pile consisting of an even number of stones into two equal piles. Is it possible to get 105 piles with one stone each? (Tournament of the Towns, 2001)

Solution: There are three possibilities for the first operation: a) merge the first two piles; b) merge the last two piles; c) merge piles 1 and 3.
a) If we begin by merging the first two piles, we obtain two piles of 100 and 5 stones respectively. Notice that the number of stones in both piles is a multiple of

5. Next, also notice that when you merge two such piles, the number of the stones in the resulting pile will also be a multiple of 5. Also, when you split into two equal parts the stones from a pile in which the number of stones was a multiple of 5, you obtain two piles and in both of them the number of stones will be a multiple of 5. In conclusion, the FACT THAT THE NUMBER OF STONES IN EACH PILE IS A MULTIPLE OF 5 never changes; it is an INVARIANT. Therefore in this case it is impossible to obtain even a single pile with only one stone (1 is not a multiple of 5).
b) If we merge the last two piles as our first move, we obtain piles with 51 and 54 stones respectively. Notice that both are multiples of 3. As by a), the FACT THAT THE NUMBER OF STONES IN EACH PILE IS A MULTIPLE OF 3 never changes; it is an INVARIANT. Therefore in this case it is impossible to obtain even a single pile with only one stone (1 is not a multiple of 3).
c) If we merge the first and the last pile as our first move, we obtain piles with 49 and 56 stones, respectively. Notice that both are multiples of 7. As by a), the FACT THAT THE NUMBER OF STONES IN EACH PILE IS A MULTIPLE OF 7 never changes; it is an INVARIANT. Therefore in this case it is impossible to obtain even a single pile with only one stone (1 is not a multiple of 7).

6. 101 numbers are written on the blackboard: 1^2, $2^2, ..., 101^2$. Alex chooses any two numbers and replaces them by their positive difference. He repeats this operation until one number is left on the blackboard. Determine the smallest possible value of this number. (Tournament of the Towns, 2010)

Solution: There are 51 odd numbers and 50 even numbers on the blackboard. Each move either keeps the number of odd numbers unchanged, or reduces it by 2. Therefore the PARITY OF THE NUMBER OF ODD NUMBERS is an INVARIANT. It follows that the last number must be odd, and its minimum value is 1. The squares of four consecutive integers can be replaced by a 4 because $(n+3)^2 - (n+2)^2 - (n+1)^2 + n^2 = 4$. Hence the squares of eight consecutive integers can be replaced by 0. Taking the squares off from the end eight at a time, we may be left with 1, 4, 9, 16 and 25. However, the best we can get out of these five numbers is 3. Hence we must include 36, 49, 64, 81, 100, 121, 144 and 169. The sequence of combinations may be $169 - 144 = 25$, $25 - 25 = 0$, $100 - 0 = 100$, $100 - 36 = 64$, $64 - 64 = 0$, $121 - 0 = 121$, $121 - 81 = 40$, $49 - 40 = 9$, $16 - 9 = 7$, $9 - 7 = 2$, $4 - 2 = 2$, $2 - 1 = 1$.

7. On a square board divided into 15×15 little squares there are 15 rooks that do not attack each other. Then each rook makes one move like that of a knight. Prove that after this is done a pair of rooks will necessarily attack each other.

Solution: Let us number the rows and the columns of the board by numbers from 1 to 15. Then every square is represented by a pair of numbers (a, b), where a, b

are between 1 and 15. Let (a_k, b_k) represent the square on which the k-th rook is placed. Since at the beginning the rooks do not attack each other, in every row and in every column there is exactly one rook. Therefore the numbers a_1, a_2, ..., a_{15} are, in some order, precisely the numbers from 1 to 15. The same is true for b_1, b_2, ..., b_{15}. We conclude that if the rooks do not attack each other, then the sum of all their coordinates must be $S = a_1 + a_2 + ... + a_{15} + b_1 + b_2 + ... + b_{15} = 1 + 2 + ... + 15 + 1 + 2 + ... + 15 = 15 \cdot 16 = 240$. The fact that this sum is 240 is a necessary (but not sufficient) condition for the rooks not to attack each other.
Let us see what happens after each rook makes a move of a knight. When the k-th rook makes a move of a knight, either a_k changes (increases or decreases) by 1 and b_k changes by 2, or a_k changes by 2 and b_k by 1. Anyway, the sum $a_k + b_k$ changes by 1 or 3, which means it changes its parity. Since we have an odd number of rooks, the parity of the total sum S will change, so it will no longer be 240. But that was a necessary condition for the rooks not to attack each other. Since the condition is not fulfilled, at least two rooks will attack each other.
So, what is the invariant here? Call "a good position" a position in which 15 non-attacking rooks are placed on the board. Getting from one "good position" to another, the squares covered by the rooks can change significantly, but what stays INVARIANT, is the FACT THAT THE SUM OF ALL THE COORDINATES IS 240. In conclusion, the REPEATED MOVES that take the system from one state to another are not the moves of the rooks (those are not repeated) but that "getting from one good position to another".

8. You have the numbers 1 through 1000 on the board. Erase a and b and replace them by $ab + a + b$. What is the number on the board at the end? Is it always the same?

Solution: Call the number $x + 1$ the "successor" of the integer x.
An INVARIANT would then be the PRODUCT OF THE SUCCESSORS OF THE NUMBERS WRITTEN ON THE BOARD. Indeed, before a transformation applied to numbers a and b, the product of the successors is $(a + 1) \times (b + 1) \times$ the product of the successors of the other numbers written on the board. After the transformation, the product of the successors is $(ab+a+b+1)\times$ the product of the successors of the other numbers from the board. But $(a+1)(b+1) = ab+a+b+1$, therefore the product of the successors does not change. Initially the product of the successors was $(1 + 1) \cdot (2 + 1) \cdot (3 + 1) \cdot ... \cdot (1000 + 1) = 1001!$, therefore the product of the successors at the end, when there is only one number remaining, will still be $1001!$. Then the final number on the board is $1001! - 1$.

9. A circle is divided into six sectors. The numbers 1, 0, 1, 0, 0, 0 are written into the sectors (in counterclockwise order). At each step you may increase two neighboring numbers by 1. After a sequence of such steps is it possible for all the numbers to be equal? [Engel]

Solution: Color the sectors alternately in black and white, with the sectors initially containing the two 1-s being black. Let S be the difference between the sum of the numbers from the black sectors and the sum of the numbers from the white sectors. Initially this difference is $2 - 0 = 2$. Each time you increase two neighboring numbers by 1, you increase both the black sum and the white sum by 1, which means that the difference does not change. The ALTERNATED SUM OF THE NUMBERS is always 2; it is an INVARIANT. For the situation where all the numbers are equal this alternating sum is 0. But starting with a position with alternating sum equal to 2, we can obtain only positions with alternating sum equal to 2. Therefore we can not arrive to the position in which all the numbers are equal.

10. Start with the integer 7^{2011}. At each step, delete the leading digit, and add it to the remaining number. This is repeated until a number with exactly 10 digits remains. Prove that this number has al least two equal digits.

Solution: The operation of deleting the first digit and adding it to the remaining number means for an n-digit number $\overline{d_1d_2...d_n}$ the following transformation: $\overline{d_1d_2...d_n} \mapsto \overline{d_2...d_n} + d_1$. One can observe that this transformation does not change the remainder of the number when divided by 9. Let's see why :

$$\overline{d_1d_2\ldots d_n} = 10^{n-1} \times d_1 + \overline{d_2d_3\ldots d_n} = 999...9 \times d_1 + (d_1 + \overline{d_2d_3\ldots d_n})$$

So the initial number and the transformed one are congruent modulo 9. By this, we find that the invariant is the REMAINDER OF THE NUMBER WHEN DIVIDED BY 9. Initially, this number is the remainder of 7^{2011} when divided by 9. It can be proven that this remainder is 7, but what is obvious is that initially this remainder is not 0. Indeed, 7^{2011} is not a multiple of 9. Hence the last 10-digit number will be congruent with the initial number, 7^{2011}, modulo 9, which means it won't be a multiple of 9.
If our last number would have all its digits distinct, since it has 10 digits, it will contain each digit from 0 to 9 exactly once, arranged in some order. No matter how the digits are arranged, the sum of the digits will always be $45 = 0+1+\ldots+9$, which is divisible by 9. We know that all numbers are congruent to the sum of their digits modulo 9, so the last number will be divisible by 9. Recalling that this number was congruent with 7^{2011} modulo 9, we get that 7^{2011} should also be divisible by 9, contradiction. Thus, there are at least two equal digits in the final 10-digit number.

11. There is an integer in each square of an 8×8 chessboard. In one move you may choose any 3×3 or 4×4 square and add 1 to each integer of the chosen square. Can you always get a table with each entry divisible by: (a) 2, (b) 3 ?

Solution: (a) Let S be the sum of all the numbers except for those in the third and sixth row. We shall see that the PARITY OF S is INVARIANT. Color all the

unit squares in black, except for those in rows 3 and 6. Any 3×3 square covers exactly 6 black squares, so any move involving a 3×3 square will augment S by 6, which means it won't change the parity of S. A 4×4 square covers either 8 or 12 black squares, so a move involving a 4×4 square will augment S by 8 or by 12, so it won't change the parity of S. A final configuration in which each entry is even, means in the final configuration, S must be even. Such a configuration can only be obtained if the initial value of S is also even, which means it can not ALWAYS be obtained (not from ANY initial configuration).
(b) Proceed similarly. Color in black all the unit squares except for those in rows 4 and 8.

12. The term a_{n+1}, $n \in \mathbb{N}$, of a sequence of positive integers is defined as follows: the last digit of a_n is removed, multiplied by 4, then added to what is left of a_n. For example, if $a_n = 693$, then $a_{n+1} = 69 + 4 \cdot 3 = 81$, $a_{n+2} = 8 + 4 \cdot 1 = 12$, $a_{n+3} = 1 + 4 \cdot 2 = 9$, $a_{n+4} = 0 + 4 \cdot 9 = 36$, etc. Prove that if 1001 is a term of the sequence, then the sequence does not contain any prime numbers.

Solution: The numbers following 1001 in the sequence are: 104, 26, 26, 26,... . Indeed, $1001 \mapsto 100 + 4 \times 1 = 104 \mapsto 10 + 4 \times 4 = 26 \mapsto 2 + 4 \times 6 = 26$, and so on. The fact that all what the numbers 1001, 104 and 26 have in common is the fact that they are multiples of 13, leads us to the supposition that maybe we could prove that all the numbers in the sequence are multiples of 13. Let us indeed prove that if a_{n+1} is a multiple of 13, then a_n is also a multiple of 13. If we denote $a_n = \overline{d_1 d_2 ... d_k}$, then $a_{n+1} = \overline{d_1 d_2 ... d_{k-1}} + 4d_k$. If $13 \mid \overline{d_1 d_2 ... d_{k-1}} + 4d_k$, then $13 \mid 3 \cdot \overline{d_1 d_2 ... d_{k-1}} + 12d_k$ and, on the other hand, $13 \mid 13 \cdot \overline{d_1 d_2 ... d_{k-1}} + 13d_k$. Then $13 \mid 13 \cdot \overline{d_1 d_2 ... d_{k-1}} + 13d_k - \left(3 \cdot \overline{d_1 d_2 ... d_{k-1}} + 12d_k\right)$, that is $13 \mid 10\overline{d_1 d_2 ... d_{k-1}} + d_k$. But $10 \cdot \overline{d_1 d_2 ... d_{k-1}} + d_k = \overline{d_1 d_2 ... d_{k-1} 0} + d_k = \overline{d_1 d_2 ... d_{k-1} d_k} = a_n$, so we have proven that if 13 divides a term of the sequence, then it also divides the preceding term. Since $13 \mid 1001$, 13 will divide the term preceding 1001, and then the one before that, and so on. Finally, 13 divides all the terms of the sequence.
The fact that each term is a MULTIPLE OF 13 is our INVARIANT.
There is one final detail: we need to prove that the sequence does not contain any prime numbers. We also know that it only contains multiples of 13. We still need to prove that 13 is not in the sequence. We have seen that it can not appear in the sequence behind 1001 (only 104 and 26 can appear there). If 13 precedes 1001 in the sequence, let us notice that 13 is followed in the sequence by $1 + 4 \times 3 = 13$, so all the terms after 13 must be 13, therefore 1001 can not be behind 13 in the sequence.

13. Consider the points $A(0,0)$, $B(1,0)$, $C(0,1)$ painted in black. You can paint in black any point that is the reflection of a black point with respect to another black point. Can you paint in black the point $D(1,1)$?

Solution: The answer is no. One thing that A, B and C have in common is that all

three of them have at least one even coordinate. If we show that no matter how we reflect a black point over another black point we get a point with at least one even coordinate, then the problem is solved, since $D(1,1)$ has both coordinates odd.
Let's take two points which have at least one even coordinate, $X(a,b)$ and $Y(c,d)$. Suppose that if we reflect X over Y we get a point $Z(e,f)$. Since Y is the midpoint of XY, we have that $c = \dfrac{a+e}{2}$ and $d = \dfrac{b+f}{2}$, so $e = 2c - a$ and $f = 2d - b$. Since at least one of a and b is even, then at least one of e and f will also be even. Therefore, if we begin with points which have at least one even coordinate, we end up with points which also have at least one even coordinate. Since $D(1,1)$ has both coordinates odd, we can never reach it.
The invariant in this problem is the fact that all the black points have AT LEAST ONE EVEN COORDINATE.

14. If $x_1, x_2, ..., x_n \in \{-1, 1\}$ are such that $x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_nx_1 = 0$, prove that $4 \mid n$.

Solution 1: Let's consider the transformation in which we pick a number x_i and we change its sign, i.e. we turn a -1 into a 1 and vice-versa. We apply this transformation successively to all the terms equal to -1. (There is at least one transformation to be made). Changing the sign of x_i will also change the signs of the two terms that contain x_i. (Except for $i = 1$ and $i = n$ these terms are $x_{i-1} \cdot x_i$ and $x_i \cdot x_{i+1}$.) Each of the two terms will change (increase or decrease) by 2, so their sum will either change by 4, or not change at all. Therefore, mod 4, the sum will never change. The invariant is the REMAINDER OF THE SUM WHEN DIVIDED BY 4. Initially the sum is 0. In the end, when all the -1-s have been turned into $+1$-s, the sum will be equal to n. Therefore n must be congruent to 0 (mod 4).
Solution 2: (a more natural solution, without using invariants)
Each term of the sum is an odd number, the result is even, hence the number of terms, n, is even. Put $n = 2k$, $k \in \mathbb{Z}$. Half of the terms x_1x_2, x_2x_3, ..., x_nx_1 are equal to 1 and half of them are equal to -1. Hence the product of the terms is $(x_1x_2)(x_2x_3) \cdot ... \cdot (x_nx_1) = 1^k \cdot (-1)^k = (-1)^k$. On the other hand, $(x_1x_2)(x_2x_3) \cdot ... \cdot (x_nx_1) = (x_1x_2 \cdot ... \cdot x_n)^2 = 1$, hence $(-1)^k = 1$ which means k is even, i.e. $4 \mid n$.
Solution 3: Arrange the numbers x_1, x_2, ..., x_n (in this order) on a circle. As above, $n = 2k$, $k \in \mathbb{Z}$. When passing successively from x_1 to x_2, x_3, ... x_n and back to x_1 there are k changes of sign. Because at the end we are again at x_1, the number of times the sign has changed must be even.
Remark: Conversely, if $4 \mid n$, then one may find $x_1, x_2, ..., x_n \in \{-1, 1\}$ such that $x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_nx_1 = 0$. For example, put $x_k = -1$ if $4 \mid k$ and $x_k = 1$ otherwise.
Another similar problem: If $x_1, x_2, ..., x_n \in \{-1, 1\}$ are such that $x_1x_2x_3x_4 +$

$x_2x_3x_4x_5 + ... + x_nx_1x_2x_3 = 0$, prove that $4 \mid n$. [Engel]

15. The numbers $3, 4, 5$, and 6 are written on a blackboard. Any two numbers a, b from the blackboard can be replaced by the numbers $a + b + \sqrt{a^2 + b^2}$ and $a+b-\sqrt{a^2 + b^2}$. Is it possible to obtain on the blackboard a number smaller than 1?

Solution: It is easy to check that $\dfrac{1}{a + b + \sqrt{a^2 + b^2}} + \dfrac{1}{a + b - \sqrt{a^2 + b^2}} = \dfrac{1}{a} + \dfrac{1}{b}$.
This means, that the SUM OF THE RECIPROCALS OF THE NUMBERS WRITTEN ON THE BLACKBOARD is INVARIANT. Initially, the sum of the reciprocals is $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} < \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} = 1$. Then the sum of the reciprocals will always remain less then 1. Suppose that we could obtain a number x smaller than 1. Then the sums of the reciprocals of the 4 numbers on the blackboard will be larger than $\frac{1}{x}$ which is already larger than 1. Since the sum of the reciprocals stays always below 1, we cannot obtain a number smaller than 1.

16. In seven of the vertices of a cube we write the number 0, while in the eighth vertex we write 1. We can choose any edge and add 1 to the numbers written in the vertices of the chosen edge. Is it possible to obtain, after a few such transformations:
(a) eight equal numbers;
(b) eight multiples of 3?

Solution: Denote by $A, B, C, D, A', B', C', D'$ the vertices of the cube, such that 1 is in A. Paint the vertices A, B', C, D' in black and the vertices A', B, C', D in white. Let S be the difference between the sum of the numbers written in the black vertices and the sum of the numbers written in the white vertices. Initially this difference is 1. Since each edge contains exactly one black and one white vertex, the DIFFERENCE S will never change.
(a) If all the 8 numbers would have to be equal, then "the final value of S" should be 0. But the initial value of S is 1, and S does not change, hence S can not become 0, therefore the numbers can not become all equal.
(b) If all the 8 numbers would have to be multiples of 3, then "the final value of S" should be a multiple of 3. But this contradicts the fact that S is constantly 1, so we can not obtain eight multiples of 3 (in fact of any number $n > 1$).
Compare the idea used in solving this problem with the one used in Problem 9.
Alternative solution for (a): The total sum is always odd.

17. The polynomials $P(X) = X^2 + 2$ and $Q(X) = X + 1$ are written on a blackboard. You can write on the blackboard the sum, the difference or the product of two polynomials already written on the blackboard. Can the polynomial $R(X) = X^3 + 2$ appear on the blackboard?

Solution: Notice that $P(2) = 6$ and $Q(2) = 3$ are multiples of 3. The property that all the polynomials written on the blackboard TAKE, FOR $x = 2$, A VALUE THAT IS A MULTIPLE OF 3, never changes when one replaces two polynomials by either their sum, their difference or their product. (The difference and the product of two multiples of 3 is still a multiple of 3.) But for the polynomial R has $R(2) = 10$, which is not a multiple of 3. Therefore R can never appear on the blackboard.
Remark: Instead of looking at the value in $x = 2$ we could have looked at the value in $x = -1$ or in any other point that is $\equiv 2 \pmod 3$.

18. Consider a matrix whose entries are integers. An "operation" consists of adding a same integer to all the entries of a row or to all the entries of a column. It is given that, for infinitely many positive integers n, one can obtain, after performing a finite number of operations, a matrix having all its entries divisible by n. Prove that, through a finite number of operations, one can obtain the null matrix.
[Marius Cavachi, TST Romania, 2009]

Solution: Let the given matrix be (a_{st}). For any two rows i, j $(i \neq j)$ and columns k, ℓ $(k \neq \ell)$ the quantity $(a_{ik} - a_{jk}) - (a_{i\ell} - a_{j\ell})$ is INVARIANT to any operation. Since this quantity remains constant after any number of operations, it must be divisible by infinitely many values of n, hence it must be 0. This means that $a_{ik} - a_{jk} = a_{i\ell} - a_{j\ell}$, i.e. any column k differs from column 1 by a constant x_k. Similarly, it follows that any row j differs from row 1 by a constant y_j. Performing the operation with $x_k + a_{11}$ on each column $k \neq 1$ and with y_j on each row $j \neq 1$, one obtains the null matrix.

Conclusion:

Many problems involve configurations which are changing in complicated ways. An important technique is to look for an invariant, an auxiliary quantity which does not change when the configuration changes. The behavior of this auxiliary quantity can give you information about the achievable configurations.

As you have seen, finding an invariant is often quite difficult. When you are looking for an invariant, we advise you to:
- look for a property that is satisfied by the initial configuration (initial numbers, initial points), but is not satisfied by the given configuration (the one you are asked if it is attainable)
- look only for properties preserved by the moves, from one configuration to the next.

Usually, when a question starts with "Is it possible to..." the answer is NO, and the

proof is based on invariants. We have seen many examples in the previous lesson. To questions "Is it possible to tile..." the answer was generally "no", and the proof was based on exhibiting an appropriated coloring (an invariant).
However, not all such questions have negative answer. In order to prove that the answer is positive all we need is to provide an example. Here is such a problem:

The natural number n can be replaced by ab if $a + b = n$, where a and b are natural numbers. Can the number 2011 be obtained from 22 after a sequence of such replacements?

Answer: YES, it can!
An example: $22 = 11 + 11$, replace 22 by $11 \times 11 = 121$. $121 = 20 + 101$, replace 121 with $20 \times 101 = 2020$. Write $2020 = 1 + 2019$, replace 2020 with 2019; write $2019 = 2018 + 1$, replace 2019 by 2018, etc, until you obtain 2011.

PRACTICE PROBLEMS

1. Eli is fighting a mathematical hydra which has 9 heads. Every second, Eli can burn off exactly 5 heads with a torch. He can also slice exactly 7 heads off with a sword, in which the hydra grows 22 new heads. He can also rip off exactly 2002 of the heads with his hands, in which time 1337 new hydra heads grow. Can Eli ever kill the hydra?

Solution: When Eli employs the torch, he diminishes the number of heads by 5. When he uses the sword, the number of the heads of the hydra increases by $22 - 7 = 15$. When Eli uses his hands, the number of heads changes by $2002 - 1337 = 665$. Each of these numbers are multiples of 5, therefore, mod 5, Eli cannot produce any changes in the number of heads. It will always stay $\equiv 4 \pmod 5$ (because $9 \equiv 4 \pmod 5$). The NUMBER OF HEADS OF THE HYDRA, MOD 5 is an INVARIANT.
In conclusion, Eli cannot kill the hydra (he cannot bring it to 0 heads because 0 is not $\equiv 4 \pmod 5$).

2. A knight moves on a 2011×2011 board. He starts in the square just above the center. Can he land on every square of the board exactly once?

Solution: Color the squares in a chessboard pattern, with all the corners black. The center (generally, every square of the two diagonals) is black. The square just above the center is white. At each move the knight changes the color of the square it is positioned on. There are more black squares than white ones on the board, so starting with a white one means the knight will finish up the white squares while it has still to visit a black square, so such a tour is not possible.

3. 64 coins are arranged in an 8-by-8 square. Initially half the coins are heads-up and half the coins are tails-up, in an alternating pattern. At each step there are three possible moves: flip all eight coins in a single row; flip all eight coins in a single column; flip all four coins in a 2×2 square. Is it possible to reach a configuration where exactly one coin is heads-up and the rest are tails-up?

Solution: The PARITY OF THE HEADS (or that of the tails) never changes. When reversing the coins of a row or column, instead of x heads, we shall obtain $8 - x$ heads. Since $x + (8 - x) = 8$ is even, $8 - x$ has the same parity as x so the parity of the total number of the heads does not change. Similarly, when we reverse the coins from a 2×2 square, instead of y heads we shall have $4 - y$ which has the same parity as y. Initially we have 32 heads, an even number. It is asked if we can get one single head. The answer is: "no, because 1 is odd".

4. Each of the numbers 1 through 10^6 is repeatedly replaced by the sum of its digits until we obtain 10^6 one digit numbers. Which number is the most frequent among these numbers?

Solution: When we replace a number by the sum of its digits, we know that the REMAINDER AT THE DIVISION BY 9 does not change. All the numbers congruent to 1 (mod 9) will give in the end a digit that is congruent to 1 (mod 9). The only such digit is 1. From 1 through 10^6 there are 111112 numbers congruent to 1 (mod 9). Therefore the final result will be 1 for 111112 numbers.
All the numbers congruent to 2 (mod 9) will give in the end a digit that is congruent to 2 (mod 9). The only such digit is 2. From 1 through 10^6 there are 111111 numbers congruent to 2 (mod 9). Therefore the final result will be 2 for 111111 numbers.
Similarly, for 111111 numbers each, the final result will be 3,4,...,8. The same is true for 9, only a little bit of attention is needed:
All the numbers congruent to 0 (mod 9) will give in the end a digit that is congruent to 0 (mod 9). There are two such digits: 0 and 9. But 0 cannot be the sum of the digits of any of the given numbers. So all the 111111 numbers congruent to 0 (mod 9) will transform until they become the digit 9. Therefore the final result will be 1 for 111112 numbers, which makes 1 the most frequent final result.

SEMI-INVARIANTS

Sometimes we can not find such an invariant, i.e. such an unchanging quantity, but it is possible to find a quantity that changes in a way that is easy to understand (e.g. strictly increasing or strictly decreasing) and which, again, can deliver the desired information about the achievable configurations. Such quantities are called **"semi-invariants"**.

1. Some numbers are written on the blackboard. You can replace two numbers, a and b, by $a+b$ and $a-b$. Is it possible to obtain again, after a certain number of replacements, the numbers initially written on the blackboard?

Solution: THE SUM OF THE SQUARES OF THE NUMBERS WRITTEN ON THE BLACKBOARD is often a good invariant or semi-invariant. Here, the initial numbers a, b have the sum of their squares a^2+b^2. Then the sum of squares of all the numbers is $S = a^2 + b^2 +$ the sum of the squares of all the other numbers written on the blackboard. The numbers after the replacement have the sum of their squares $(a+b)^2 + (a-b)^2 +$ the sum of the squares of all the other numbers written on the blackboard $= a^2 + b^2 + S > S \; \forall \, a, b$, not both equal to 0. The sum of the squares is not a constant expression. It is not an invariant. But it is an expression that increases with every replacement, therefore it is a SEMI-INVARIANT. Except for the case when initially we have at least two 0-s on the board (then we can continue replacing two zeros by two "other" zeros), we cannot obtain again the numbers initially written on the blackboard.

2. Consider a row of $2n$ squares colored alternately black and white. A legal move consists of choosing a contiguous set of squares (one or more squares but they must be next to each other, no gaps are allowed) and inverting their colors. What is the minimum number of moves necessary to make the row entirely one color?

Solution 1: We shall prove that the minimum number of moves is n. It is clear that in n moves I can make all the squares to have a single color (just change the color of a single black square at a time). In order to prove it is not possible to obtain an entirely one color row in less than n moves, notice that If we change, by a move, the color of the square no. k, then by the same move we also change the color of square no. $k-1$, unless the selection of the squares to operate on stops right there, between squares $k-1$ and k. But squares $k-1$ and k have different color, and need to have at the end the same color, therefore there must be a move that stops right between $k-1$ and k. (This move might operate on either square $k-1$, or on square k, but not on both.) There are $2n$ squares, therefore $2n-1$ separators between two neighboring squares. Each move goes from one separating place to another (or uses one or both ends of the row), therefore each move uses up at most 2 separating places. Therefore we need at least n moves to use up all the $2n-1$ separating places.

Solution 2: The same as the above, but in terms of the semi-invariant:
Consider a "change of color" when in the row a square is immediately followed by a square of a different color. Initially, there are $2n-1$ "changes of color". Each move can reduce the "changes of color" by at most 2 (if both its ends are placed between squares of different color). Therefore, in order to make the "changes of color" to be 0, you need at least n moves.
The number of "changes of color" is semi-invariant.

3. There is a checker at the point (1; 1) in the plane. At each step, you may move the checker in one of two ways: by doubling one of the two coordinates, or (if the coordinates are unequal) by subtracting the smaller coordinate from the larger one. Is it possible for the checker to end up at the point (3; 3)? [Engel]

Solution: There are two types of moves allowed: the first type: $(x, y) \mapsto (2x, y)$ or $(x, y) \mapsto (x, 2y)$, and the second type: $(x, y) \mapsto (x - y, y)$ if $x > y$ or $(x, y) \mapsto (x, y - x)$ if $y > x$. Here we are lucky: the notation (x, y) brings in mind the GREATEST COMMON DIVISOR OF THE COORDINATES. Indeed, it is a SEMI-INVARIANT. The first type of move either doubles it or keeps it the same, while the second type of move leaves it unchanged. Then at all times the greatest common divisor of the coordinates should remain a power of 2. (This fact is an INVARIANT !) The checker can never end up at the point $(3, 3)$ because $\gcd(3, 3) = 3$ is not a power of 2.
In fact, by using Euclid's algorithm, you can prove the reciprocal of the statement above: the checker can get to any point (x, y) of positive integers with $\gcd(x, y)$ a power of 2.

4. 40 different natural numbers are put up on the blackboard. One can erase any two and replace them by their lcm and gcd. Prove that eventually the numbers will stop changing.

Solution: Suppose by contradiction that we can perform an infinite number of changes. Remark that if the "change" is applied to numbers a, b with $a \mid b$, then $gcd(a, b) = a$, $lcm(a, b) = b$ so there is no actual change. So these moves do not count (they do not produce any changes). So suppose we have an infinite number of "significant" changes. Let us examine such a change. Consider two numbers, x, y, and put $d = gcd(x, y)$. Then there are integers x' and y' such that $x = dx'$, $y = dy'$ and $gcd(x', y') = 1$. Then after the change will shall obtain the numbers d and $lcm(x, y) = dx'y'$ instead of $x = dx'$ and $y = dy'$. Now observe that the SUM OF THE NUMBERS FROM THE BLACKBOARD is a SEMI-INVARIANT. Indeed, $dx' + dy' +$the sum of the other terms becomes, after transformation, $d + dx'y' +$ the sum of the other terms. But the difference between the two sums is $d + dx'y' - dx' - dy' = d(x' - 1)(y' - 1) > 0$ (we have supposed that x is not a factor of y, which means $x' > 1$. Similarly, in order to have a "significative" change, we need to have $y' > 1$.) So we have proven that any significative change increases the total sum (by at least 1 because all the numbers are integers). Now we shall prove that we can not increase the sum indefinitely.
To this end we notice that the PRODUCT OF THE NUMBERS ON THE BOARD is INVARIANT. Indeed, before a change the product is $P = dx' \times dy' \times$ the product of the other numbers. After the change, the product is $d \times dx'y' \times$ the product of the other numbers. It is then obvious that the product is still P. As a consequence,

all the numbers on the board, after any number of changes, can not exceed P. In conclusion their sum is limited (it can not increase forever).

5. In the parliament of Sikinia, each member has at most three enemies. (Assume that being an enemy is symmetric: if the deputy X is an enemy of the deputy Y, then the deputy Y is an enemy of the deputy X.) Prove that the parliament can be separated into two houses so that each member has at most one enemy in his own house. [Engel]

Solution: We will try to increase the degree of satisfaction of the deputies from Sikinia by moving any unsatisfied deputy (one that has more then one enemy in his own house), into the other house. Let N be the total number of enemies, that is, the sum of the numbers of enemies each deputy has in his own house. Then each time a deputy with 2 or 3 enemies in his own house is moved into the other house, the number of his enemies he shares the house with, decreases either from 3 to 0, or from 2 to 0 or 1. Also, his two or three enemies he left behind lose an enemy, while at most the one he shares now room with gains an enemy. In conclusion, the total number of enemies, N, decreases by each move. This procedure can not continue infinitely because the semi-invariant N needs to remain non-negative. Therefore the procedure must stop, which means there are no more deputies to move. That happens when each member of the parliament has at most one enemy in his own house.

6. Nine 1×1 cells in a 10×10 square are infected. In one unit time, the cells with at least two infected neighbors (having a common side) also become infected. Can the infection spread to the whole square? [Engel]

Solution: By looking to a healthy cell with 2, 3, or 4 infected neighbors, we observe that the perimeter of the contaminated area does not increase. However, it can decrease, so the PERIMETER OF THE CONTAMINATED AREA is a SEMI-INVARIANT. Initially, the perimeter of the contaminated area is at most $4 \times 9 = 36$. Since its perimeter can never increase, the contaminated area cannot become the whole square because the perimeter of the square is 40, exceeding 36.