Geometry of the Triangle

Paul Yiu

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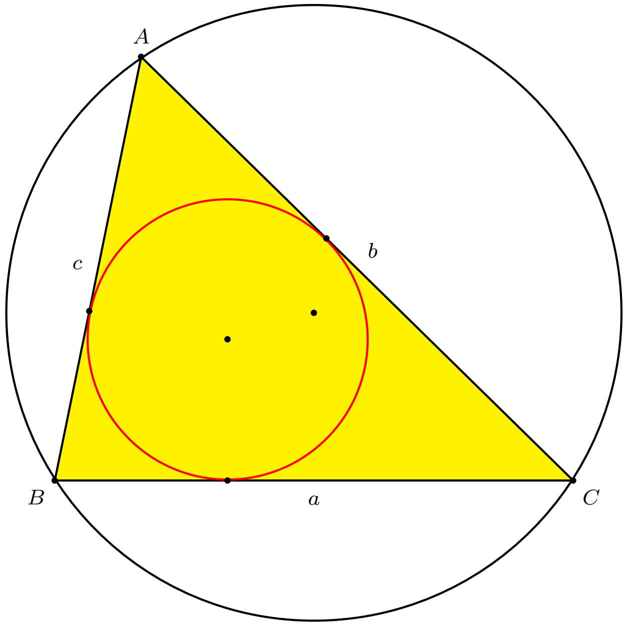
Department of Mathematics
Florida Atlantic University



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Chapter 1

Some basic notions and fundamental theorems

1.1 Menelaus and Ceva theorems

Let B and C be two fixed points on a line $\mathcal{L}$. Every point X on $\mathcal{L}$, apart from B and C, can be coordinatized by the ratio of division $\frac{BX}{XC}$, where BX and XC are signed lengths. We begin with two classical theorems for the collinearity of three points and concurrency of three lines. Consider a triangle $\mathbf{T}$ with points X, Y, Z on the side lines BC, CA, AB respectively.

Theorem (Menelaus)**.** The points X, Y, Z are collinear if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1. \tag{1.1}$$

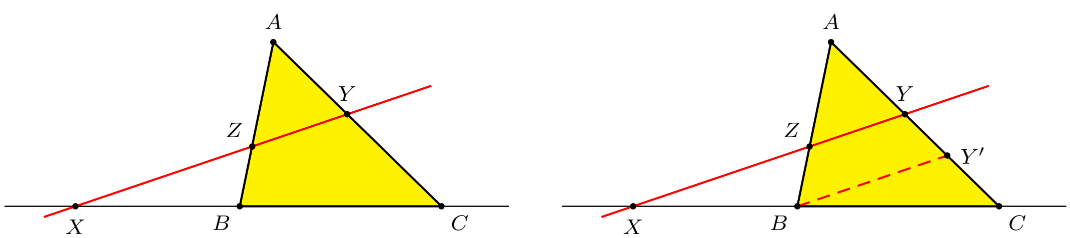


Figure 1.1: The Menelaus theorem

Proof. (⇒) Construct a parallel to the line XYZ through B, to intersect the line AC at Y'. It is clear that

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = \frac{Y'Y}{YC}\cdot\frac{CY}{YA}\cdot\frac{AY}{YY'} = \frac{Y'Y}{YY'}\cdot\frac{CY}{YC}\cdot\frac{AY}{YA} = (-1)(-1)(-1) = -1.$$

($\Leftarrow$) If the lines YZ and BC intersect at X', then

$$\frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$

Comparsion with (1.1) gives $\frac{BX}{XC} = \frac{BX'}{X'C}$. The points X and X' divide BC in the same ratio. They are necessarily the same point. This means that X, Y, Z are collinear. □

Theorem (Ceva)**.** The lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1. \tag{1.2}$$

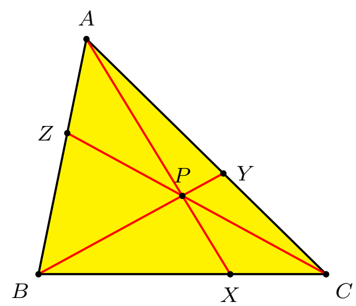


Figure 1.2: The Ceva theorem

Proof. ($\Rightarrow$) Suppose that the lines are concurrent at a point P. Applying the Menelaus theorem to triangle AXC with transversal BPY, we have

$$\frac{XB}{BC} \cdot \frac{CY}{YA} \cdot \frac{AP}{PX} = -1.$$

Likewise, for triangle ABX with transversal CPZ,

$$\frac{XP}{PA} \cdot \frac{AZ}{ZB} \cdot \frac{BC}{CX} = -1.$$

Combining the two relations, with appropriate reversal of signs, we obtain (1.2).

($\Leftarrow$) If the lines BY and CZ intersect at P, and AP intersects BC at X', then

$$\frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1.$$

Comparison with (1.2) gives $\frac{BX}{XC} = \frac{BX'}{X'C}$. The points X and X' divide BC in the same ratio. They are necessarily the same point. This shows means that AX, BY, CZ are concurrent at P. □

1.2 Harmonic conjugates

Given four points B, C, X, Y on a line, X and Y are said to divide B and C harmonically if

$$\frac{BX}{XC} = -\frac{BY}{YC}.$$

In this case, X and Y are *harmonic conjugates* of each other with respect to the segment BC.

Construction. Given a point X on the line BC, to construct the harmonic conjugate of X with respect to the segment BC,

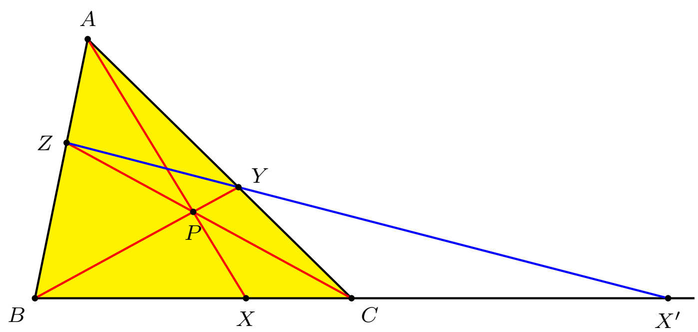


Figure 1.3: Harmonic conjugates

(1) take an arbitrary point A outside the line BC and construct the lines AB and AC;
(2) take an arbitrary point P on the line AX and construct the lines BP and CP to intersect the lines CA and AB at Y and Z respectively;
(3) construct the line YZ to intersect BC at X'.

Then X and X' divide B and C harmonically (see Figure 1.3).

If M is the midpoint of a segment BC, it is not possible to find a *finite* point N on the line BC so that M, N divide B, C harmonically. This is because $\frac{BN}{NC} = -\frac{BM}{MC} = -1$ requires $BN = -NC = CN$, and $BC = CN - BN = 0$, a contradiction. We shall agree to say that if M and N divide B, C harmonically, then N is the *infinite point* of the line BC.

1.3 Directed angles

A reference triangle $\mathbf{T}$ in a plane induces an *orientation* of the plane, with respect to which all angles are *signed*. For two given lines $\mathcal{L}$ and $\mathcal{L}'$, the directed angle $\angle(\mathcal{L},\ \mathcal{L}')$ between them is the angle of rotation from $\mathcal{L}$ to $\mathcal{L}'$ in the induced orientation of the plane. It takes values of modulo π. The following basic properties of directed angles make many geometric reasoning simple without the reference of a diagram.

Theorem. (1) $\angle(\mathcal{L}', \mathcal{L}) = -\angle(\mathcal{L}, \mathcal{L}')$.

(2) $\angle(\mathcal{L}_1, \mathcal{L}_2) + \angle(\mathcal{L}_2, \mathcal{L}_3) = \angle(\mathcal{L}_1, \mathcal{L}_3)$ for any three lines $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$.

(3) Four points P, Q, X, Y are concyclic if and only if $\angle(PX, XQ) = \angle(PY, YQ)$.

Remark. In calculations with directed angles, we shall slightly abuse notations by using the equality sign instead of the sign for congruence modulo π. It is understood that directed angles are defined up to multiples of π. For example, we shall write $\beta+\gamma = -\alpha$ even though it should be more properly $\beta + \gamma = \pi - \alpha$ or $\beta + \gamma \equiv -\alpha \mod \pi$.

In terms of the angles of $\mathbf{T}$ are the directed angles between the sidelines:

$$\angle(\mathsf{c},\ \mathsf{b}) = \alpha, \quad \angle(\mathsf{a},\ \mathsf{c}) = \beta, \quad \angle(\mathsf{b},\ \mathsf{a}) = \gamma.$$

Let ℓ be a line with $\angle(\mathsf{a},\ \ell) = \theta$. Then
$\angle(\mathsf{b},\ \ell) = \gamma + \theta$ and $\angle(\mathsf{c},\ \ell) = -\beta + \theta$.

The tangents to the circumcircle at the vertices of $\mathbf{T}$ are characterized by

$$\begin{aligned}
\angle(\mathsf{b},\ \mathsf{t}_A) &= \beta, & \angle(\mathsf{t}_A,\ \mathsf{c}) &= \gamma,\\
\angle(\mathsf{c},\ \mathsf{t}_B) &= \gamma, & \angle(\mathsf{t}_B,\ \mathsf{a}) &= \alpha,\\
\angle(\mathsf{a},\ \mathsf{t}_C) &= \alpha, & \angle(\mathsf{t}_C,\ \mathsf{b}) &= \beta.
\end{aligned}$$

Therefore, $\angle(\mathsf{t}_C,\ \mathsf{t}_B) = \angle(\mathsf{t}_C,\ \mathsf{a}) + \angle(\mathsf{a},\ \mathsf{t}_B) = -\alpha - \alpha = -2\alpha$. Similarly, $\angle(\mathsf{t}_A,\ \mathsf{t}_C) = -2\beta$ and $\angle(\mathsf{t}_B,\ \mathsf{t}_A) = -2\gamma$.

Theorem (Miquel)**.** Let X, Y, Z be points on the sidelines BC, CA, AB of $\mathbf{T}$ respectively. The circles AYZ, BZX, CXY are concurrent.

Proof. Let M be the intersection of the circles BZX and CXY. We prove that M also lies on the circle AYZ. This follows from

$$\begin{aligned}
\angle(YM,\ MZ) &= \angle(YM,\ MX) + \angle(XM,\ MZ)\\
&= \angle(YC,\ CX) + \angle(XB,\ BZ)\\
&= \angle(AC,\ BC) + \angle(BC,\ AB)\\
&= \angle(AC,\ AB)\\
&= \angle(YA,\ AZ).
\end{aligned}$$

□

M is called the Miquel point associated with X, Y, Z. If X, Y, Z are the traces of P, we call M the Miquel associate of P.

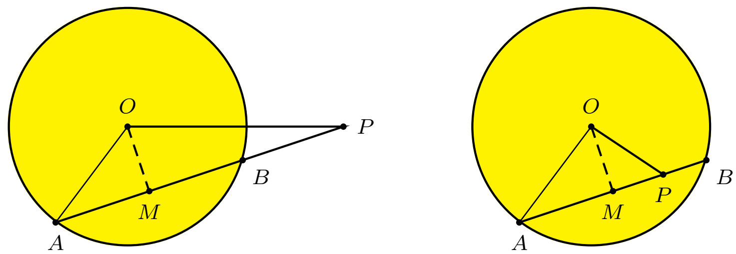
P	Miquel associate
$\mathtt{G}$	$\mathtt{O}$
$\mathtt{H}$	$\mathtt{H}$
$\mathtt{G_e}$	$\mathtt{I}$

Exercise

1. If a, b, c are the sidelines of triangle **T**, then $\angle(\mathsf{a},\ \mathsf{b}) = -\gamma$ etc.

1.4 The power of a point with respect to a circle

Theorem. Given a point P and a circle $O(r)$, if a line through P intersects the circle at two points A and B, then $PA \cdot PB = OP^2 - r^2$, independent of the line.



Proof. Let M be the midpoint of AB. Note that OM is perpendicular to AB. If P is outside the circle, then

$$\begin{aligned} PA \cdot PB &= (PM + MA)(PM - BM) \\ &= (PM + MA)(PM - MA) \\ &= PM^2 - MA^2 \\ &= (OM^2 + PM^2) - (OM^2 + MA^2) \\ &= OP^2 - r^2. \end{aligned}$$

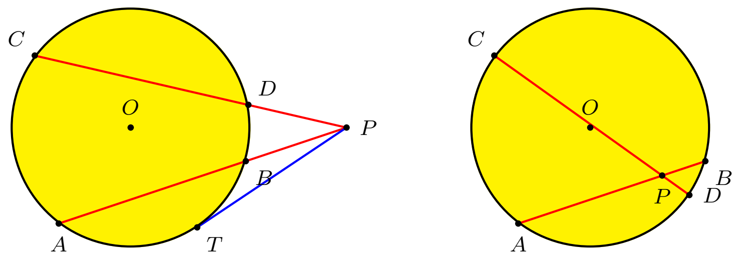
The same calculation applies to the case when P is inside or on the circle, provided that the lengths of the directed segments are signed. □

The quantity $OP^2 - r^2$ is called the **power** of P with respect to the circle. It is positive, zero, or negative according as P is outside, on, or inside the circle.

Corollary (Intersecting chords theorem)**.** If two chords AB and CD of a circle intersect, extended if necessary, at a point P, then $PA \cdot PB = PC \cdot PD$.

In particular, if the tangent at T intersects AB at P, then $PA \cdot PB = PT^2$.

The converse of the intersecting chords theorem is also true.



Theorem. Given four points A, B, C, D, if the lines AB and CD intersect at a point P such that $PA \cdot PB = PC \cdot PD$ (as signed products), then A, B, C, D are concyclic.

In particular, if P is a point on a line AB, and T is a point outside the line AB such that $PA \cdot PB = PT^2$, then PT is tangent to the circle through A, B, T.

1.4.1 Inversion formulas

Proposition. The inversive image of P in a circle $O(\rho)$ is the point P^{-1} which divides OP in the ratio $OP^{-1} : OP = \rho^2 : OP^2$.

Proof. The inversive image is the point P^{-1} on the half line OP such that $OP \cdot OQ = \rho^2$.

$$OP^{-1} : OP = \; OP \cdot OP^{-1} : OP^2 = \; \rho^2 : OP^2.$$

□

Proposition. The center of the inversive image of the circle $Q(r)$ in the circle $O(\rho)$ is the point Q' which divides OQ in the ratio

$$OQ' : OQ = \rho^2 : OQ^2 - r^2.$$

Proof. Let Q_- and Q_+ be the points of the diameter of (Q) through O.

$$\begin{aligned}
\overrightarrow{OQ_+} &= \frac{OQ + r}{OQ}\overrightarrow{OQ},\\
\overrightarrow{OQ_-} &= \frac{OQ - r}{OQ}\overrightarrow{OQ};\\
\overrightarrow{OQ_+^{-1}} &= \frac{\rho^2}{OQ_+^2} \cdot \overrightarrow{OQ_+} = \frac{\rho^2}{OQ_+^2} \cdot \frac{OQ + r}{OQ}\overrightarrow{OQ} = \frac{\rho^2}{OQ(OQ + r)}\overrightarrow{OQ},\\
\overrightarrow{OQ_-^{-1}} &= \frac{\rho^2}{OQ_-^2} \cdot \overrightarrow{OQ_-} = \frac{\rho^2}{OQ_-^2} \cdot \frac{OQ - r}{OQ}\overrightarrow{OQ} = \frac{\rho^2}{OQ(OQ - r)}\overrightarrow{OQ}.
\end{aligned}$$

The center of the image circle is the midpoint Q' of Q_+^{-1} and Q_-^{-1}.

$$\begin{aligned}
\overrightarrow{OQ'} &= \frac{1}{2}\left(\overrightarrow{OQ_+^{-1}} + \overrightarrow{OQ_-^{-1}}\right) \\
&= \frac{1}{2}\left(\frac{\rho^2}{OQ(OQ+r)} + \frac{\rho^2}{OQ(OQ-r)}\right)\overrightarrow{OQ} \\
&= \frac{1}{2}\cdot\frac{\rho^2}{OQ}\cdot\frac{2OQ}{(OQ+r)(OQ-r)}\overrightarrow{OQ} \\
&= \frac{\rho^2}{OQ^2-r^2}\overrightarrow{OQ}.
\end{aligned}$$

□

1.5 The 6 concyclic points theorem

The radical axis of two nonconcentric circles $\mathcal{C}_1$ and $\mathcal{C}_2$ is the locus of points of equal powers with respect to the circle. It is a straight line perpendicular to the line joining their centers. [1]

Proposition. Given three circles with mutually distinct centers, the radical axes of the three pairs of circles are either concurrent or are parallel.

Proof. If the centers of the circles are noncollinear, then two of the radical axes, being perpendiculars to two distinct lines with a common point, intersect at a point. This intersection has equal powers with respect to all three circles, and also lies on the third radical axis.

If the three centers are collinear, then the three radical axes three parallel lines, which coincide if any two of them do. This is the case if and only if the three circles two points in common, or at mutually tangent at a point. In this case we say that the circles are coaxial. □

If the three circles have non-collinear centers, the unique point with equal powers with respect to the circles is called the radical center.

Example: The radical center of the excircles

Consider the excircles of $\mathbf{T}$. The excenter $\mathtt{I}^a$ is the intersection of the external bisectors of angles B and C; similarly for the excenters $\mathtt{I}^b$ and $\mathtt{I}^c$.

[1] If the circles are concentric, there is no finite point with equal powers with respect to the circles.

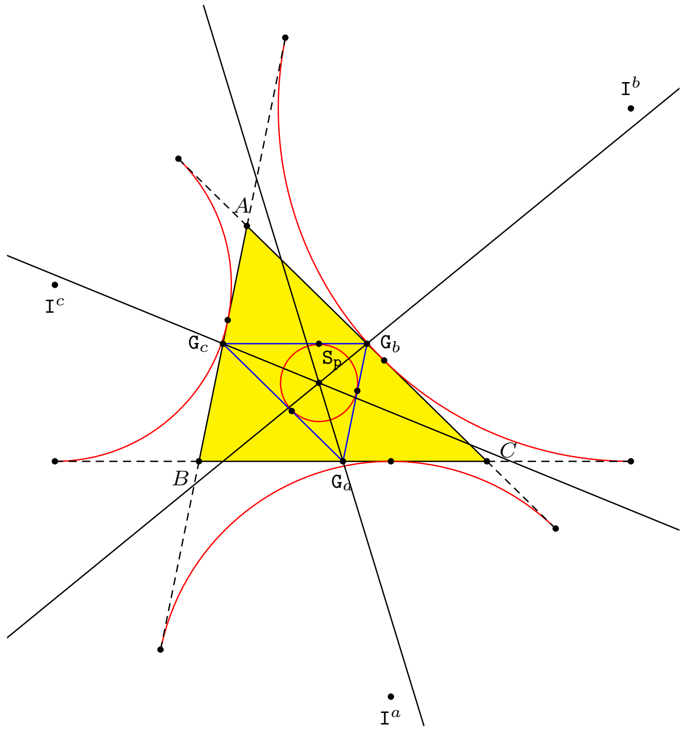


Figure 1.4: Radical center of the excircles

Since the tangents to the B- and C-excircles from the midpoint $\mathtt{G}_a$ of BC have equal lengths $\frac{b+c}{2}$, $\mathtt{G}_a$ is on the radical axis of the B- and C-excircle. This radical axis being a line perpendicular to $\mathtt{I}^b\mathtt{I}^c$, it is the bisector of angle $\mathtt{G}_b\mathtt{G}_a\mathtt{G}_c$, where $\mathtt{G}_b$ and $\mathtt{G}_c$ are the midpoints of CA and AB respectively; similarly for the other two radical axes. The radical center of the excircles is the incenter of the inferior triangle $\mathtt{G}_a\mathtt{G}_b\mathtt{G}_c$, also called the *Spieker center* $\mathtt{S}_\mathtt{p}$ of $\mathbf{T}$.

Theorem (The 6 concyclic points theorem)**.** Let X_1, X_2 be points on the sideline BC, Y_1, Y_2 on CA, and Z_1, Z_2 on AB of triangle ABC. If

$$AY_1{\cdot}AY_2 = AZ_1{\cdot}AZ_2, \quad BZ_1{\cdot}BZ_2 = BX_1{\cdot}BX_2, \quad CX_1{\cdot}CX_2 = CY_1{\cdot}CY_2,$$

then the six points X_1, X_2, Y_1, Y_2, Z_1, Z_2 are concyclic.

Proof. By the intersecting chords theorem, the points Y_1, Y_2, Z_1, Z_2 lie on a circle $\mathcal{C}_1$. Likewise, Z_1, Z_2, X_1, X_2 lie on a circle $\mathcal{C}_2$, and X_1, X_2, Y_1, Y_2 lie on a circle $\mathcal{C}_3$. If any two of these circles coincide, then all three circles coincide. If the circles are all distinct, then the sidelines of the triangle, being the three radical axes of the circles and nonparallel, should concur at a point, a contradiction. $\square$

Example: The nine-point circle

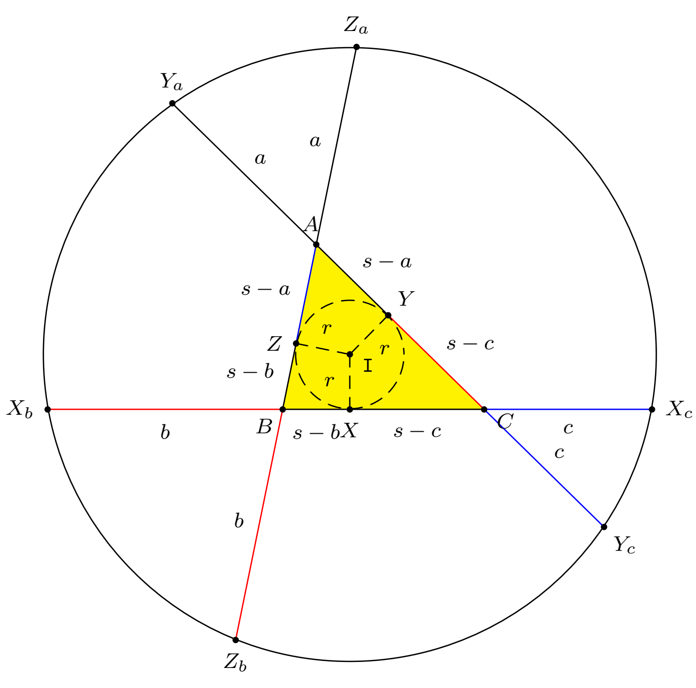
Let $\mathtt{G}_a$, $\mathtt{G}_b$, $\mathtt{G}_c$ be the midpoints of the sides BC, CA, AB of triangle ABC, and $\mathtt{H}_a$, $\mathtt{H}_b$, $\mathtt{H}_c$ the pedals of A, B, C on their opposite sides. It is easy to see that

$$\begin{aligned} A\mathtt{G}_b \cdot A\mathtt{H}_b = &\ A\mathtt{G}_c \cdot A\mathtt{H}_c = \frac{1}{2}bc\cos\alpha, \\ B\mathtt{G}_c \cdot B\mathtt{H}_c = &\ B\mathtt{G}_a \cdot B\mathtt{H}_a = \frac{1}{2}ca\cos\beta, \\ C\mathtt{G}_a \cdot C\mathtt{H}_a = &\ C\mathtt{G}_b \cdot C\mathtt{H}_b = \frac{1}{2}ab\cos\gamma. \end{aligned}$$

Therefore, the six points $\mathtt{G}_a$, $\mathtt{G}_b$, $\mathtt{G}_c$, $\mathtt{H}_a$, $\mathtt{H}_b$, $\mathtt{H}_c$ are on a circle. This is called the *nine-point circle* of triangle ABC.

Example: The Conway circle

Given triangle ABC, extend
(i) CA and BA to Y_a and Z_a such that $AY_a = AZ_a = a$,
(ii) AB and CB to Z_b and X_b such that $BZ_b = BX_b = b$,
(iii) BC and AC to X_c and Y_c such that $CX_c = CY_c = c$.



It is clear that

$$\begin{aligned} AY_a \cdot AY_c = &\ AZ_a \cdot AZ_b = -a(b+c), \\ BZ_b \cdot BZ_a = &\ BX_b \cdot BX_c = -b(c+a), \\ CX_c \cdot CX_b = &\ CY_c \cdot CY_a = -c(a+b). \end{aligned}$$

The six points X_b, X_c, Y_c, Y_a, Z_a, Z_b are concyclic. It is concentric with the incircle and has radius $\sqrt{r^2+s^2}$.

Chapter 2

Barycentric coordinates

2.1 Barycentric coordinates on a line

2.1.1 Absolute barycentric coordinates with reference to a segment

Consider a line defined by two distinct points B and C. Every finite point on the line is uniquely determined by the ratio of division $\frac{BX}{XC}$. If this is $\frac{q}{p}$, then we write $X = \frac{pB+qC}{p+q}$. Here $p + q \neq 0$.

If X is the point on the line BC dividing the segment in the ratio $BX : XC = t : 1 - t$, we write

$$X = (1 - t)B + tC$$

and call this the absolute barycentric coordinates of X with reference to BC. Thus, the midpoint of the segment BC is $\frac{B+C}{2}$, and the trisection points are $\frac{2B+C}{3}$ and $\frac{B+2C}{3}$ respectively. More generally, if $BX : XC = q : p$, then $X = \frac{pB+qC}{p+q}$, provided $p + q \neq 0$. [1]

Example. If X and Y divide B and C harmonically, then B and C divide X and Y harmonically.

Proof. Suppose $BX : XC = q : p$ and $BX' : X'C = -q : p$. Then

$$X = \frac{pB + qC}{p + q}, \qquad Y = \frac{pB - qC}{p - q}.$$

Solving for B and C in terms of X and Y, we have

$$B = \frac{(p + q)X + (p - q)Y}{2p}, \qquad C = \frac{(p + q)X - (p - q)Y}{2q}.$$

[1] Let B and C be distinct points. If $p + q = 0$, then $q : p = 1 : -1$. There is no finite point on the line BC satisfying this condition. We shall say that the condition $BX : XC = 1 : -1$ defines the infinite point of the line.

Therefore, $XB : BY = p - q : p + q$ and $XC : CY = -(p-q) : p + q$. The points B and C divide X and Y harmonically. $\square$

2.1.2 The circle of Apollonius

In a triangle, the two bisectors of an angle divide the opposite side harmonically. If X and X' are points on the sideline BC of triangle $\mathbf{T}$ such that AX and AX' are the internal and external bisectors of angle BAC, then

$$\frac{BX}{XC} = \frac{c}{b}, \qquad \frac{BX'}{X'C} = -\frac{c}{b}.$$

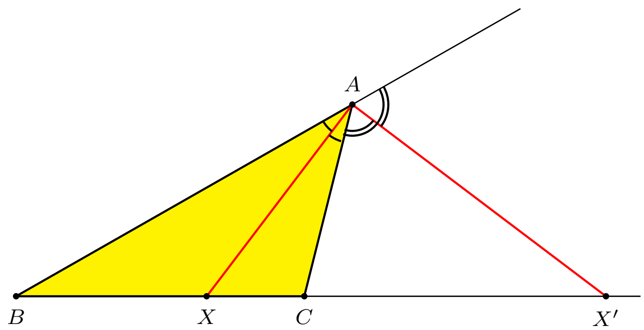


Figure 2.1: Harmonic division by angle bisectors

Theorem. Given two fixed points B, C, and a positive number $k \neq 1$, [2] the locus of points P satisfying $BP : PC = k : 1$ is the circle with diameter XY, where X and Y are points on the line BC such that $BX : XC = k : 1$ and $BY : YC = k : -1$.

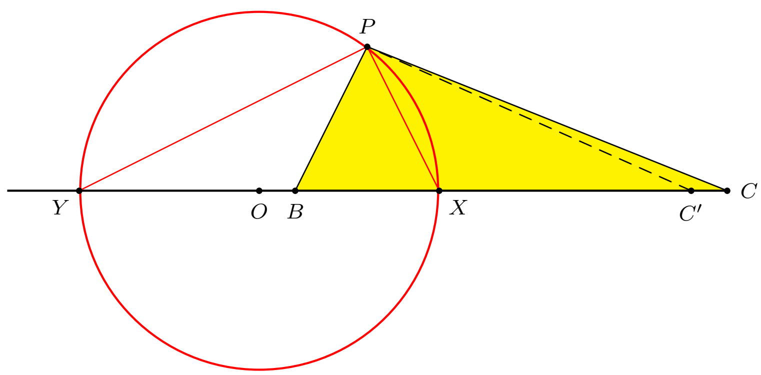


Figure 2.2: Circle of Apollonius

[2]If $k = 1$, the locus is clearly the perpendicular bisector of the segment AB.

Proof. Since $k \neq 1$, points X and Y can be found on the line BC satisfying the above conditions.

Consider a point P not on the line BC with $BP : PC = k : 1$. Note that PX and PY are respectively the internal and external bisectors of angle BPC. This means that angle XPY is a right angle, and P lies on the circle with XY as diameter.

Conversely, let P be a point on this circle. We show that $BP : CP = k : 1$. Let C' be a point on the line BC such that PX bisects angle BPC'. Since PB and PC are perpendicular to each other, the line PC is the external bisector of angle BPC', and

$$\frac{BY}{YC'} = -\frac{BX}{XC'} = \frac{XB}{XC'} = \frac{BY - XB}{YX}.$$

On the other hand,

$$\frac{BY}{YC} = -\frac{BX}{XC} = \frac{XB}{XC} = \frac{BY - XB}{YX}.$$

Comparison of the two expressions shows that C' coincides with C, and PX is the bisector of angle BPC. It follows that $\frac{PB}{PC} = \frac{BX}{XC} = k$. □

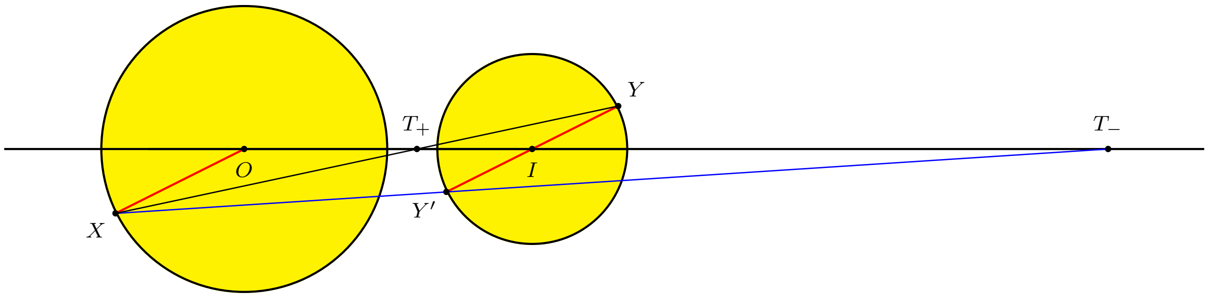
Remark. If $BC = d$, and $k \neq 1$, the radius of the Apollonius circle is $\left|\frac{k}{k^2-1}\right| d$.

2.1.3 The centers of similitude of two circles

Consider two circles $O(R)$ and $I(r)$, whose centers O and I are at a distance d apart. Animate a point X on $O(R)$ and construct a ray through I oppositely parallel to the ray OX to intersect the circle $I(r)$ at a point Y. You will find that the line XY always intersects the line OI at the same point T. This we call the *internal center of similitude*, or simply the *insimilicenter*, of the two circles. It divides the segment OI in the ratio $OT_+ : T_+I = R : r$. The absolute barycentric coordinates of P with respect to OI are

$$T_+ = \frac{R \cdot I + r \cdot O}{R + r}.$$

If, on the other hand, we construct a ray through I directly parallel to the ray OX to intersect the circle $I(r)$ at Y', the line XY' always intersects OI at another point T_-. This is the *external center of similitude*, or simply



the *exsimilicenter*, of the two circles. It divides the segment OI in the ratio $OT_- : T_-I = R : -r$, and has absolute barycentric coordinates

$$T_- = \frac{R \cdot I - r \cdot O}{R - r}.$$

Example: Points with equal views of two circles

Given two disjoint circles (B) and (C), find the locus of the point P such that the angle between the pair of tangents from P to (B) and that between the pair of tangents from P to (C) are equal.

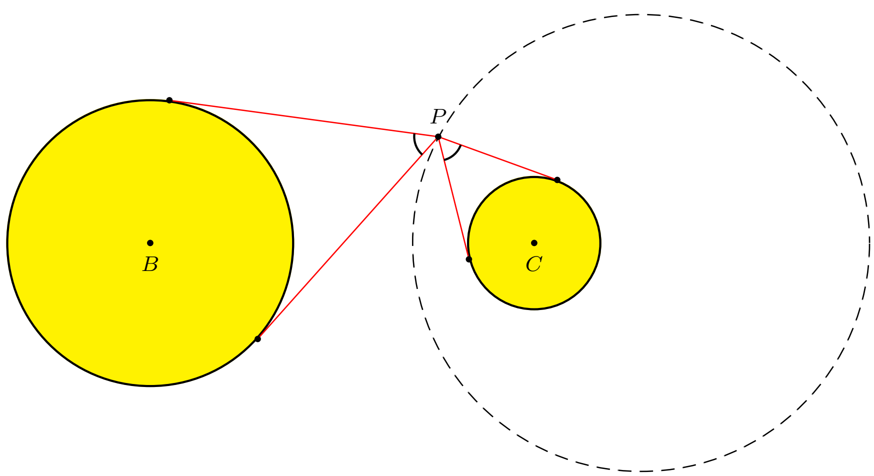
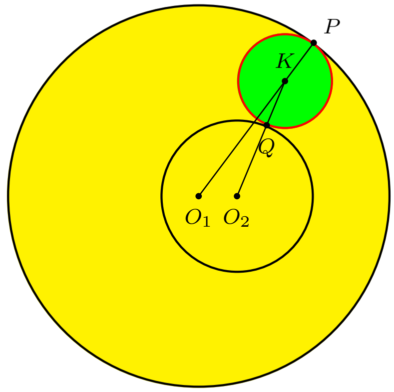


Figure 2.3: Circle of points of equal views of two circles

Let b and c be the radii of the circles. Suppose each of these angles is 2θ. Then $\frac{b}{BP} = \sin\theta = \frac{c}{CP}$, and $BP : CP = b : c$. From this, it is clear that the locus of P is the circle with the segment joining the centers of similitude of (B) and (C) as diameter.

Example: Circles tangent to two given circles



Given two circles $O_1(r_1)$ and $O_2(r_2)$, suppose there is a third circle $K(\rho)$ tangent to $O_1(r_1)$ internally at P and to $O_2(r_2)$ externally at Q. Note that K divides O_1P internally in the ratio $O_1K : KP = r_1 - \rho : \rho$, so that

$$K = \frac{\rho \cdot O_1 + (r_1 - \rho)P}{r_1}.$$

Similarly, the same point K divides O_2Q externally in the $O_2K : KQ = r_2 + \rho : -\rho$, so that

$$K = \frac{-\rho \cdot O_2 + (r_2 + \rho)Q}{r_2}.$$

Eliminating K from these two equations, and rearranging, we obtain

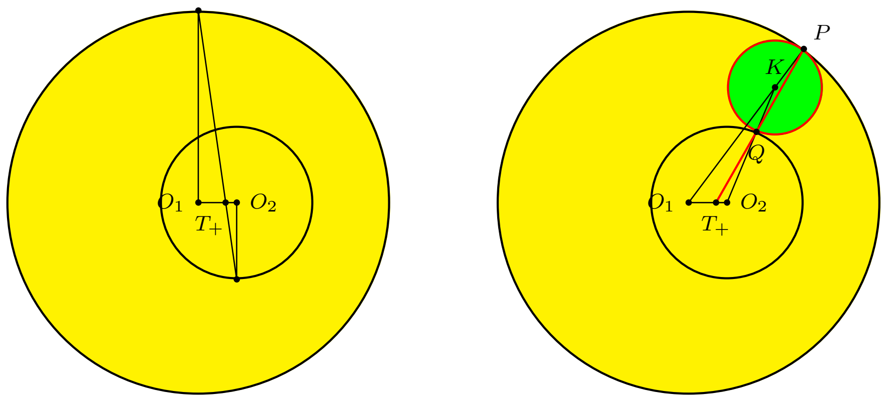
$$\frac{-r_2(r_1 - \rho)P + r_1(r_2 + \rho)Q}{(r_1 + r_2)\rho} = \frac{r_2 \cdot O_1 + r_1 \cdot O_2}{r_1 + r_2}.$$

This equation shows that a point on the line PQ is the same as a point on the line O_1O_2. This is the intersection of the lines PQ and O_1O_2. Note that the point on the line O_1O_2 depends *only* on the two circles $O_1(r_1)$ and $O_2(r_2)$. It is indeed the insimilicenter T_+ of the two circles, dividing O_1O_2 in the ratio $O_1T_+ : T_+O_2 = r_1 : r_2$.

From the equation

$$\frac{-r_2(r_1 - \rho)P + r_1(r_2 + \rho)Q}{(r_1 + r_2)\rho} = T_+,$$

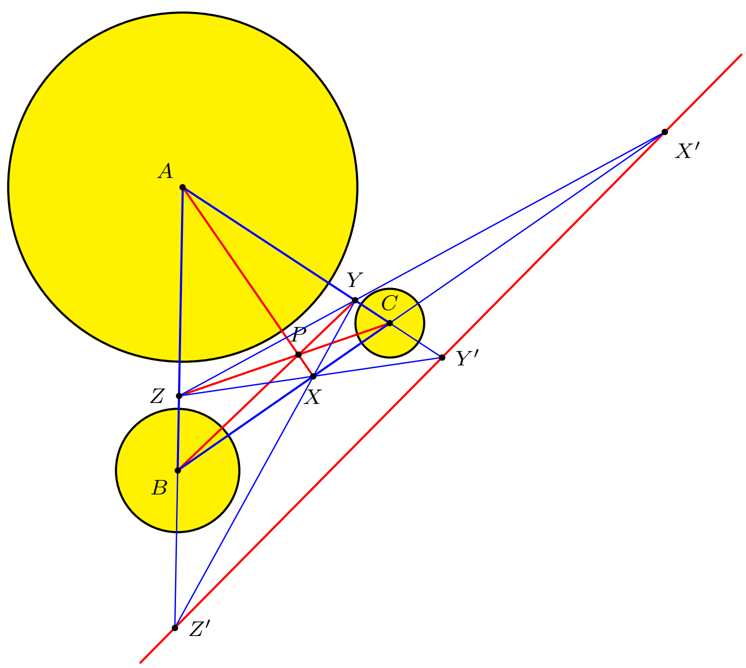
we note that each of P and Q determines the other, since the line PQ passes through T_+. This leads to an easy construction of the point K as the intersection of the lines O_1P and O_2Q. From this, the circle $K(\rho)$ can be constructed.



Example: Desargues Theorem

As a simple illustration of the use of the Menelaus and Ceva theorems, we prove the following Desargues Theorem.

Proposition. Given three circles, the exsimilicenters of the three pairs of circles are collinear. Likewise, the three lines each joining the insimilicenter of a pair of circles to the center of the remaining circle are concurrent.



Proof. We prove the second statement only. Given three circles $A(r_1)$, $B(r_2)$ and $C(r_3)$, the insimilicenters X of (B) and (C), Y of (C), (A), and Z of (A), (B) are the points which divide BC, CA, AB in the ratios

$$\frac{BX}{XC} = \frac{r_2}{r_3}, \qquad \frac{CY}{YA} = \frac{r_3}{r_1}, \qquad \frac{AZ}{ZB} = \frac{r_1}{r_2}.$$

It is clear that the product of these three ratios is $+1$, and it follows from the Ceva theorem that AX, BY, CZ are concurrent. $\square$

2.2 Absolute barycentric coordinates

We consider a nondegenerate triangle $\mathbf{T}$ with vertices A, B, C as the reference triangle, and set up a coordinate system for points in the plane of the triangle.

Theorem. Every finite point P of the plane can be expressed as $P = uA + vB + wC$ for unique real numbers u, v, w satisfying $u + v + w = 1$.

Proof. This is clearly true if P is one of the vertices A, B, C.

If P is a finite point other than the vertices, at most of one of the lines AP, BP, CP is parallel to its opposite sideline. We may assume AP intersecting BC at a finite point X. If $BX : XC = r : q$, and $AP : PX = s : t$ for q, r, s, t. Since $q + r \neq 0$ and $s \neq 0$, we may rewrite $AP : PX = q + r : p$. Then, $X = \frac{qB+rC}{q+r}$ and $P = \frac{pA+(q+r)X}{p+q+r}$. Substitution yields

$$P = \frac{pA + qB + rC}{p + q + r}.$$

With $u = \frac{p}{p+q+r}$, $v = \frac{q}{p+q+r}$, and $w = \frac{r}{p+q+r}$, we have $P = uA + vB + wC$ for $u + v + w = 1$.

Uniqueness. Suppose also $P = u'A + v'B + w'C$ for $u' + v' + w' = 1$. If $u \neq u'$, then

$$uA + vB + wC = u'A + v'B + w'C \implies A = \frac{(v' - v)B + (w' - w)C}{u - u'},$$

and A is a point on the line BC, a contradiction. It follows that $u = u'$, and similarly $v = v'$, $w = w'$. $\square$

If $P = uA + vB + wC$ with $u + v + w = 1$, we shall say that P has *absolute barycentric coordinates* $uA + vB + wC$ with reference to $\mathbf{T}$.

Examples

(1) The centroid $\mathtt{G} = \frac{1}{3}(A + B + C)$.

(2) The incenter $\mathtt{I}$ (see Figure 2.4). The bisector $A\mathtt{I}$ intersects BC at $\mathtt{I}_a$. By the angle bisector theorem, $B\mathtt{I}_a : \mathtt{I}_aC = c : b$, so that $B\mathtt{I}_a = \frac{ca}{b+c}$. In triangle $AB\mathtt{I}_a$, $B\mathtt{I}$ is the bisector of angle $AB\mathtt{I}_a$, and

$$A\mathtt{I} : \mathtt{I}\mathtt{I}_a = BA : B\mathtt{I}_a = c : \frac{ca}{b + c} = b + c : a.$$

Therefore,

$$\mathtt{I} = \frac{aA + (b+c)\mathtt{I}_a}{a+b+c} = \frac{aA + bB + cC}{a+b+c}.$$

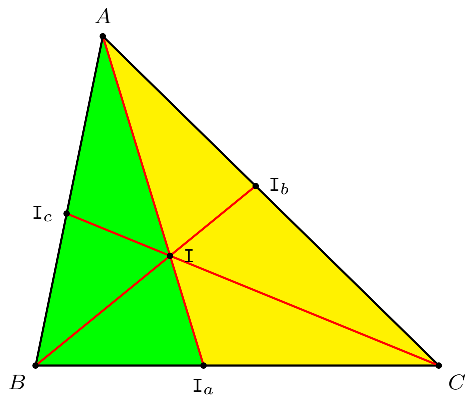


Figure 2.4: The incenter

2.2.1 Homotheties

Let P be a given point, and k a real number. The *homothety* with center P and ratio k is the transformation $\mathsf{h}(P,k)$ which maps a point X to the point Y such that $\overrightarrow{PY} = k \cdot \overrightarrow{PX}$. Equivalently, Y divides PX in the ratio $PY : YX = k : 1-k$, and

$$\mathsf{h}(P,k)(X) = (1-k)P + kX.$$

Figure 2.5: $Y = \mathsf{h}(P,\ k)(X)$

2.2.2 Superior and inferior

The homotheties $\mathsf{h}(\mathtt{G},-2)$ and $\mathsf{h}\left(\mathtt{G},-\frac{1}{2}\right)$ are called the *superior* and *inferior* operations respectively. Thus, $\mathsf{sup}(P)$ and $\mathsf{inf}(P)$ are the points dividing P and the centroid $\mathtt{G}$ according to the ratios

$$\begin{aligned} P\mathtt{G} : \mathtt{G}\mathsf{sup}(P) &= \ 1 : 2, \\ P\mathtt{G} : \mathtt{G}\mathsf{inf}(P) &= \ 2 : 1. \end{aligned}$$

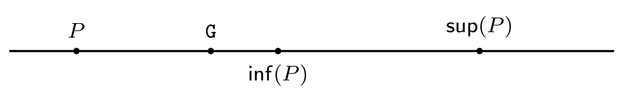


Figure 2.6: Superior and inferior

In absolute barycentric coordinates,

$$\begin{aligned} \mathsf{sup}(P) &= \ 3\mathtt{G} - 2P, \\ \mathsf{inf}(P) &= \ \frac{1}{2}(3\mathtt{G} - P). \end{aligned}$$

Remark. $\mathsf{inf}(P)$ is the midpoint between P and $\mathsf{sup}(P)$.

2.3 Homogeneous barycentric coordinates

If $P = uA + vB + wC$ in absolute barycentric coordinates with reference to $\mathbf{T}$, we say that P has *homogeneous barycentric coordinates* $(u : v : w)$

or $k(u:v:w)$ for any nonzero k. Homogeneous barycentric coordinates are often much simpler and easier to use than absolute coordinates. Thus, in homogeneous barycentric coordinates,

$$\begin{aligned} \mathtt{G} =& \ (1:1:1), \\ \mathtt{I} =& \ (a:b:c). \end{aligned}$$

Points on the line BC have homogeneous barycentric coordinates of the form $(0:v:w)$. If $BX:XC = q:p$, then $X = (0:p:q)$ in homogeneous barycentric coordinates. Likewise, those on CA are $(u:0:w)$, and those on AB are $(u:v:0)$ respectively.

Proposition. If $P = (u:v:w)$ in homogeneous barycentric coordinates, then

$$\begin{aligned} \mathsf{sup}(P) =& \ (v+w-u: \ w+u-v: \ u+v-w), \\ \mathsf{inf}(P) =& \ (v+w: \ w+u: \ u+v). \end{aligned}$$

Proof. In absolute barycentric coordinates,

$$\begin{aligned} \mathsf{sup}(P) =& \ 3\mathtt{G} - 2P \\ =& \ (A+B+C) - \frac{2(uA+vB+wC)}{u+v+w} \\ =& \ \frac{(u+v+w)(A+B+C)}{u+v+w} - \frac{2(uA+vB+wC)}{u+v+w} \\ =& \ \frac{(v+w-u)A+(w+u-v)B+(u+v-w)C}{u+v+w}. \end{aligned}$$

Therefore, $\mathsf{sup}(P) = (v+w-u: \ w+u-v: \ u+v-w)$ in homogeneous barycentric coordinates.

The case for inferior is similar. □

Example: the inferior triangle

Consider the midpoints $\mathtt{G}_a$, $\mathtt{G}_b$, $\mathtt{G}_c$ of the sides BC, CA, AB respectively. Each of these is the inferior of the opposite vertex. The triangle $\mathtt{G}_a\mathtt{G}_b\mathtt{G}_c$ is the image of $\mathbf{T}$ under the inferior operation; it is called the *inferior* triangle. The Spieker center $\mathtt{S}_\mathtt{p}$, being the incenter of the inferior triangle, is the inferior of the incenter $\mathtt{I} = (a:b:c)$. In homogeneous barycentric coordinates,

$$\mathtt{S}_\mathtt{p} = (b+c:c+a:a+b).$$

2.3.1 Euler line and the nine-point circle

The *superior* triangle is the image of $\mathbf{T}$ under the superior operation $\mathtt{h}(\mathtt{G}, -2)$. Its vertices are

$$\begin{aligned}\mathtt{G}^a &:= \mathsf{sup}(A) = (-1 : 1 : 1),\\ \mathtt{G}^b &:= \mathsf{sup}(B) = (1 : -1 : 1),\\ \mathtt{G}^c &:= \mathsf{sup}(C) = (1 : 1 : -1).\end{aligned}$$

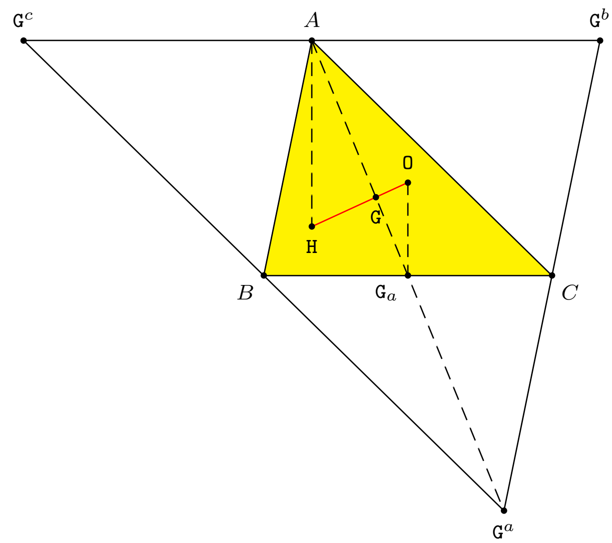


Figure 2.7: The superior triangle and the Euler line

Since $\mathtt{G}_a$ is the common midpoint of BC and $A\mathtt{G}^a$, $AB\mathtt{G}^aC$ is a parallelogram. Similarly, $BC\mathtt{G}^bA$ and $CA\mathtt{G}^cB$ are also parallelograms. It follows that $\mathtt{G}^b$, A, and $\mathtt{G}^c$ are collinear, and $\mathtt{G}^b\mathtt{G}^c$ is parallel to BC.

Since A is the midpoint of $\mathtt{G}^b\mathtt{G}^c$, the A-altitude of $\mathbf{T}$ is the perpendicular bisector of $\mathtt{G}^b\mathtt{G}^c$; similarly for the B- and C-altitudes. Therefore, the three altitudes of $\mathbf{T}$ are concurrent at a point $\mathtt{H}$ which is the circumcenter of the superior triangle. This is the *orthocenter* $\mathtt{H}$, which is the superior of the circumcenter $\mathtt{O}$. In particular, $\mathtt{H}$, $\mathtt{G}$, and $\mathtt{O}$ are collinear. The line containing them is the *Euler line* of $\mathbf{T}$.

The circumcenter of the inferior triangle is the point

$$\mathtt{N} := \mathsf{inf}(\mathtt{O}) = \frac{1}{2}(3\mathtt{G} - \mathtt{O}) = \frac{1}{2}(2\mathtt{O} + \mathtt{H} - \mathtt{O}) = \frac{\mathtt{O} + \mathtt{H}}{2},$$

the midpoint of $\mathtt{O}$ and $\mathtt{H}$.

The orthocenter of the superior triangle is the point

$$\mathtt{L}_{\text{O}} := \mathsf{sup}(\mathtt{H}) = 3\mathtt{G} - 2\mathtt{H} = (2\mathtt{O} + \mathtt{H}) - 2\mathtt{H} = 2\mathtt{O} - \mathtt{H},$$

the *reflection* of $\mathtt{H}$ in $\mathtt{O}$, and is called the deLongchamps point $\mathtt{L}_o$ of $\mathbf{T}$.

Proposition. The circumcircles of the following three triangles are identical:
(a) the inferior triangle,
(b) the orthic triangle,
(c) the image of $\mathbf{T}$ under the homothety $\mathsf{h}\left(\mathtt{H}, \frac{1}{2}\right)$.

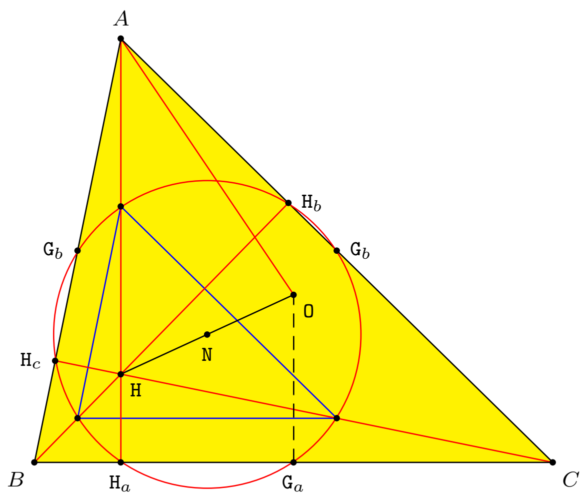


Figure 2.8: The nine-point circle

Proof. The fact that the traces of the centroid and the orthocenter are concyclic has been established in §1.5. The circle containing them has $\mathtt{N}$ and radius $\frac{1}{2}R$. Consider the circumcircle of the image of $\mathbf{T}$ under the homothety $\mathsf{h}\left(\mathtt{H}, \frac{1}{2}\right)$. This clearly also radius $\frac{1}{2}R$. Its center is the image of $\mathtt{O}$ under the homothety, i.e., $\frac{1}{2}(\mathtt{H}+\mathtt{O}) = \mathtt{N}$. $\square$

The common circumcircle of these three triangles is called the *nine-point circle* of $\mathbf{T}$.

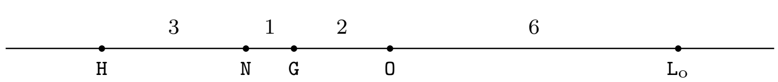


Figure 2.9: The Euler line

2.4 Barycentric coordinates as areal coordinates

Applying the Menelaus theorem to triangle ABP_a with transversal CPP_c, it is easy to see that P divides the segment AP_a in the ratio $PP_a : AP_a = u : u+v+w$. It follows that the areas of the oriented triangles PBC and ABC are in the ratio $\Delta(PBC) : \Delta(ABC) = u : u+v+w$. Similarly, $\Delta(PCA) : \Delta(ABC) = v : u+v+w$ and $\Delta(PAB) : \Delta(ABC) = w : u+v+w$.

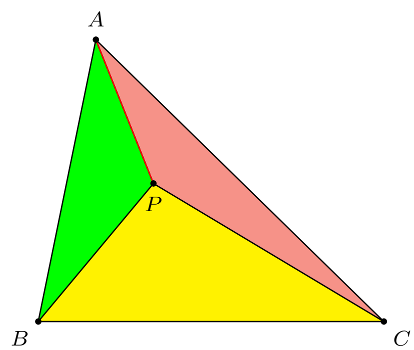
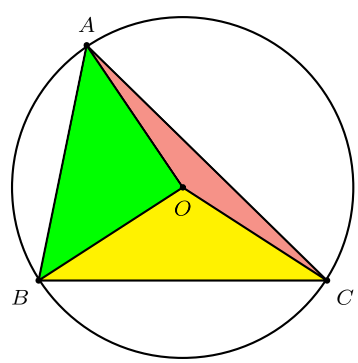


Figure 2.10:

This leads to the interpretation of the homogeneous barycentric coordinates of a point P as the proportions of (signed) areas of oriented triangles:

$$P = (\Delta(PBC) : \Delta(PCA) : \Delta(PAB)).$$

2.4.1 The circumcenter O



If R denotes the circumradius, the coordinates of the circumcenter O are

$$\begin{aligned} O = & \ \Delta OBC : \Delta OCA : \Delta OAB \\ = & \ \frac{1}{2}R^2 \sin 2\alpha : \frac{1}{2}R^2 \sin 2\beta : \frac{1}{2}R^2 \sin 2\gamma \\ = & \ \sin\alpha\cos\alpha : \sin\beta\cos\beta : \sin\gamma\cos\gamma \\ = & \ a \cdot \frac{b^2+c^2-a^2}{2bc} : b \cdot \frac{c^2+a^2-b^2}{2ca} : c \cdot \frac{a^2+b^2-c^2}{2ab} \\ = & \ a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2). \end{aligned}$$

2.4.2 The incenter and excenters

The homogeneous barycentric coordinates of the incenter $\mathtt{I}$ can be easily computed as areal coordinates. Let r be the inradius of the triangle.

$$\begin{aligned} \mathtt{I} = & \ (\Delta(\mathtt{I}BC) : \ \Delta(\mathtt{I}CA) : \ \Delta(\mathtt{I}AB)) \\ = & \ \left(\frac{1}{2}ra : \ \frac{1}{2}rb : \ \frac{1}{2}rc\right) \\ = & \ (a : b : c). \end{aligned}$$

The A-excenter $\mathtt{I}^a$ is the center of the excircle tangent to BC and the extensions of AC and AB respectively. Let r_a be the radius of the A-excircle. In homogeneous barycentric coordinates,

$$\begin{aligned} \mathtt{I}^a = & \ (\Delta(\mathtt{I}^aBC) : \ \Delta(\mathtt{I}^aCA) : \ \Delta(\mathtt{I}^aAB)) \\ = & \ \left(-\frac{1}{2}r_a \cdot a : \ \frac{1}{2}r_a \cdot b : \ \frac{1}{2}r_a \cdot c\right) \\ = & \ (-a : b : c). \end{aligned}$$

Similarly, the other two excenters are

$$\mathtt{I}^b = (a : -b : c) \quad \text{and} \quad \mathtt{I}^c = (a : b : -c).$$

2.5 The area formula

Let P_i, $i = 1, 2, 3$, be points given in absolute barycentric coordinates

$$P_i = x_iA + y_iB + z_iC,$$

with $x_i + y_i + z_i = 1$. The *oriented area* of triangle $P_1P_2P_3$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta,$$

where Δ is the area of ABC.

If the vertices are given in homogeneous coordinates, this area is given by

$$\frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}}{(x_1+y_1+z_1)(x_2+y_2+z_2)(x_3+y_3+z_3)} \Delta.$$

In particular, Since the area of a degenerate triangle whose vertices are collinear is zero, we have the following useful formula.

Example. The area of triangle `GIO` is

$$\begin{aligned}
&\frac{1}{3(a+b+c)\cdot 16\Delta^2}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2(b^2+c^2-a^2) & b^2(c^2+a^2-b^2) & c^2(a^2+b^2-c^2) \end{vmatrix}\cdot\Delta \\
=&\frac{1}{3(a+b+c)\cdot 16\Delta^2}\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ -(a-b)(a+b)(a^2+b^2-c^2) & -(b-c)(b+c)(b^2+c^2-a^2) & c^2(a^2+b^2-c^2) \end{vmatrix}\cdot\Delta \\
=&\frac{(a-b)(b-c)}{3(a+b+c)\cdot 16\Delta^2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ -(a+b)(a^2+b^2-c^2) & -(b+c)(b^2+c^2-a^2) & c^2(a^2+b^2-c^2) \end{vmatrix}\cdot\Delta \\
=&\frac{(a-b)(b-c)}{3(a+b+c)\cdot 16\Delta^2}\begin{vmatrix} 1 & 1 \\ -(a+b)(a^2+b^2-c^2) & -(b+c)(b^2+c^2-a^2) \end{vmatrix}\cdot\Delta \\
=&\frac{(a-b)(b-c)(c-a)(a+b+c)^2}{3(a+b+c)(a+b+c)(b+c-a)(c+a-b)(a+b-c)}\cdot\Delta \\
=&\frac{(a-b)(b-c)(c-a)}{3(b+c-a)(c+a-b)(a+b-c)}\cdot\Delta
\end{aligned}$$

2.5.1 Conway's notation

We shall make use of the following convenient notations introduced by John H. Conway. Instead of $\triangle$ for the area of triangle ABC, we shall find it more convenient to use

$$S := 2\triangle.$$

For a real number θ, denote $S \cdot \cot\theta$ by S_θ. In particular,

$$S_\alpha = \frac{b^2+c^2-a^2}{2}, \quad S_\beta = \frac{c^2+a^2-b^2}{2}, \quad S_\gamma = \frac{a^2+b^2-c^2}{2}.$$

For arbitrary θ and φ, we shall simply write $S_{\theta\varphi}$ for $S_\theta \cdot S_\varphi$.

We shall mainly make use of the following relations.

(1) $a^2 = S_\beta + S_\gamma$, $b^2 = S_\gamma + S_\alpha$, $c^2 = S_\alpha + S_\beta$.
(2) $S_\alpha + S_\beta + S_\gamma = \frac{a^2+b^2+c^2}{2}$.
(3) $S_{\beta\gamma} + S_{\gamma\alpha} + S_{\alpha\beta} = S^2$.

Proof. (1) and (2) are clear. For (3), since $\alpha+\beta+\gamma=\pi$, $\cot(\alpha+\beta+\gamma)$ is infinite. Its denominator

$$\cot\alpha\cdot\cot\beta + \cot\beta\cdot\cot\gamma + \cot\gamma\cdot\cot\alpha - 1 = 0.$$

From this, $S_{\alpha\beta}+S_{\beta\gamma}+S_{\gamma\alpha} = S^2(\cot\alpha\cdot\cot\beta+\cot\beta\cdot\cot\gamma+\cot\gamma\cdot\cot\alpha) = S^2$. $\square$

Example: The circumcenter and orthocenter

In Conway's notations,

$$\begin{aligned}\mathtt{O} =&\ (a^2S_\alpha : b^2S_\beta : c^2S_\gamma)\\ =&\ (S_\gamma S_\alpha + S_\alpha S_\beta : S_\alpha S_\beta + S_\beta S_\gamma : S_\beta S_\gamma + S_\gamma S_\alpha)\\ =&\ \mathsf{inf}(S_\beta S_\gamma : S_\gamma S_\alpha : S_\alpha S_\beta).\end{aligned}$$

Since $\mathtt{O} = \mathsf{inf}(\mathtt{H})$, it follows that

$$\mathtt{H} = (S_\beta S_\gamma : S_\gamma S_\alpha : S_\alpha S_\beta) = \left(\frac{1}{S_\alpha} : \frac{1}{S_\beta} : \frac{1}{S_\gamma}\right).$$

2.5.2 Conway's formula

The position of a point can be specified by its "compass bearings" from two vertices of the reference triangle. Given triangle ABC and a point P, the AB- *swing angle* of P is the oriented angle CBP, of magnitude φ reckoned positive if and only if it is away from the vertex A. Similarly, the AC-swing angle is the oriented angle BCP, of magnitude ψ reckoned positive if and only if it is away from the vertex A. The swing angles are chosen in the range $-\frac{\pi}{2} \leq \varphi, \psi \leq \frac{\pi}{2}$. The point P is uniquely determined by φ and ψ in this range. We shall denote this by $X(\varphi,\ \psi)$.

Theorem (Conway's formula)**.** In homogeneous barycentric coordinates,

$$X(\varphi,\ \psi) = (-a^2 : S_\gamma + S_\psi : S_\beta + S_\varphi).$$

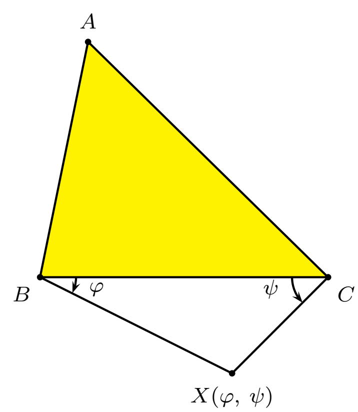


Figure 2.11: The point $X(\varphi,\ \psi)$

Proof. Let $X = X(\varphi,\ \psi)$. Its homogeneous barycentric coordinates are

$$\begin{aligned}
& \Delta(XBC) : \Delta(XCA) : \Delta(XAB) \\
= & -\frac{a^2 \sin\varphi \sin\psi}{2\sin(\varphi+\psi)} : \frac{b \cdot a \sin\varphi}{2\sin(\varphi+\psi)} \cdot \sin(\psi+\gamma) : \frac{c \cdot a \sin\psi}{2\sin(\varphi+\psi)} \cdot \sin(\varphi+\beta) \\
= & -a^2 : \frac{ab\sin(\psi+\gamma)}{\sin\psi} : \frac{ca\sin(\varphi+\beta)}{\sin\varphi} \\
= & -a^2 : ab\cos\gamma + ab\sin\gamma\cot\psi : ca\cos\beta + ca\sin\beta\cot\varphi \\
= & -a^2 : S_\gamma + S_\psi : S_\beta + S_\varphi.
\end{aligned}$$

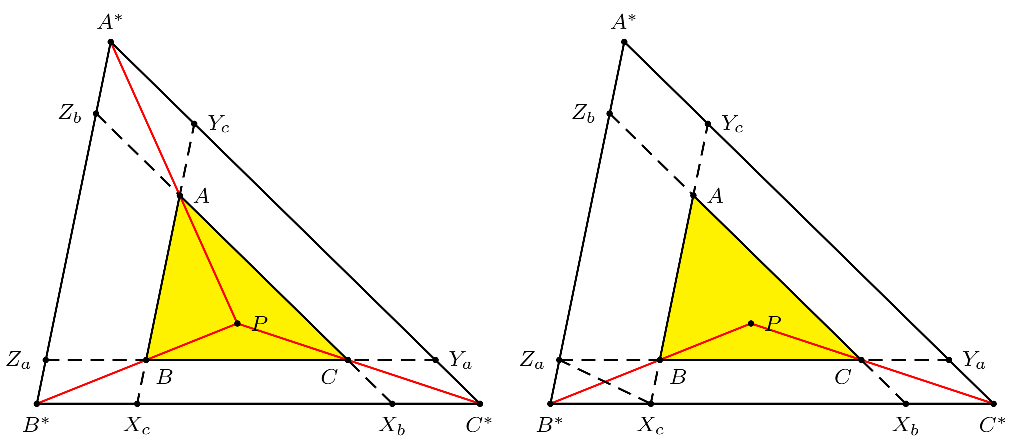
$\square$

2.6 Triangles bounded by lines parallel to the sidelines

Theorem (Homothetic center theorem)**.** If parallel lines X_bX_c, Y_cY_a, Z_aZ_b to the sides BC, CA, AB of triangle ABC are constructed such that

$$\begin{aligned} AB : BX_c = AC : CX_b =& 1 : t_1, \\ BC : CY_a = BA : AY_c =& 1 : t_2, \\ CA : AZ_b = CB : BZ_a =& 1 : t_3, \end{aligned}$$

these lines bound a triangle $A^*B^*C^*$ homothetic to ABC with homothety ratio $1+t_1+t_2+t_3$. The homothetic center is a point P with homogeneous barycentric coordinates $t_1 : t_2 : t_3$.



Proof. Let P be the intersection of B^*B and C^*C. Since

$$\begin{aligned} B^*C^* =& \ B^*X_c + X_cX_b + X_bC^* \\ =& \ t_3a + (1+t_1)a + t_2a = (1+t_1+t_2+t_3)a, \end{aligned}$$

we we have

$$PB : PB^* = PC : PC^* = 1 : 1 + t_1 + t_2 + t_3.$$

A similar calculation shows that AA^* and BB^* intersect at the same point P. This shows that $A^*B^*C^*$ is the image of ABC under the homothety $\mathsf{h}(P,\ 1+t_1+t_2+t_3)$.

Now we compare areas. Note that

(1) $\Delta(BZ_aX_c) = \frac{BX_c}{AB} \cdot \frac{BZ_a}{CB} \cdot \Delta(ABC) = t_1t_3\Delta(ABC)$,

(2) $\frac{\Delta(PBC)}{\Delta(BZ_aB^*)} = \frac{PB}{BB^*} \cdot \frac{CB}{BZ_a} = \frac{1}{t_1+t_2+t_3} \cdot \frac{1}{t_3} = \frac{1}{t_3(t_1+t_2+t_3)}$.

Since $\Delta(BZ_aB^*) = \Delta(BZ_aX_c)$, we have $\Delta(PBC) = \frac{t_1}{t_1+t_2+t_3} \cdot \Delta(ABC)$.

Similarly, $\Delta(PCA) = \frac{t_2}{t_1+t_2+t_3} \cdot \Delta(ABC)$ and $\Delta(PAB) = \frac{t_3}{t_1+t_2+t_3} \cdot \Delta(ABC)$. It follows that

$$\Delta(PBC) : \Delta(PCA) : \Delta(PAB) = t_1 : t_2 : t_3.$$

□

Corollary. Two triangles with corresponding sidelines parallel are homothetic.

2.6.1 The symmedian point

Consider the square erected externally on the side BC of triangle ABC, The line containing the outer edge of the square is the image of BC under the homothety $\mathsf{h}(A, 1+t_1)$, where $1 + t_1 = \frac{\frac{S}{a}+a}{\frac{S}{a}} = 1 + \frac{a^2}{S}$, i.e., $t_1 = \frac{a^2}{S}$. Similarly, if we erect squares externally on the other two sides, the outer edges of these squares are on the lines which are the images of CA, AB under the homotheties $\mathsf{h}(B, 1+t_2)$ and $\mathsf{h}(C, 1+t_3)$ with $t_2 = \frac{b^2}{S}$ and $t_3 = \frac{c^2}{S}$.

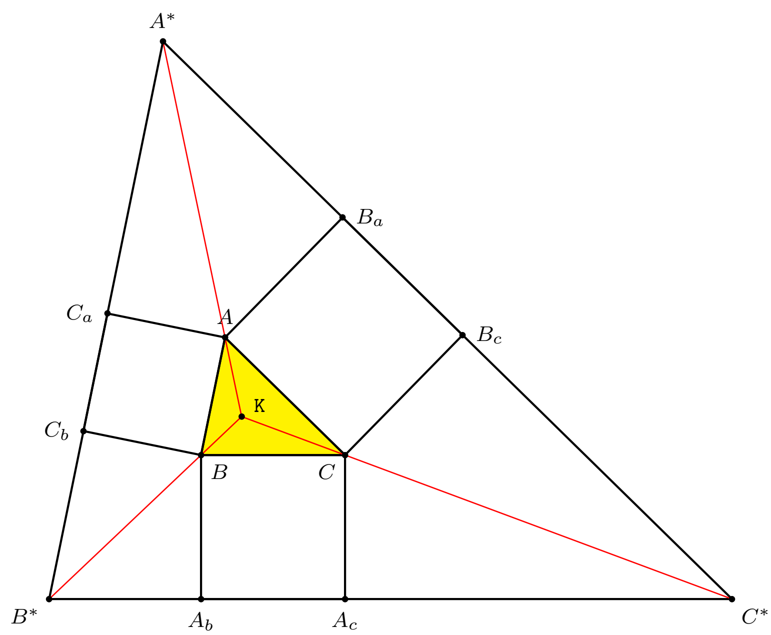


Figure 2.12: The Grebe triangle and the symmedian point

The triangle bounded by the lines containing these outer edges is called the *Grebe triangle* of ABC. It is homothetic to ABC at

$$\mathtt{K} = \left(\frac{a^2}{S} : \frac{b^2}{S} : \frac{c^2}{S}\right) = (a^2 : b^2 : c^2),$$

the symmedian point, and the ratio of homothety is

$$1 + (t_1 + t_2 + t_3) = \frac{S + a^2 + b^2 + c^2}{S}.$$

2.6.2 The first Lemoine circle

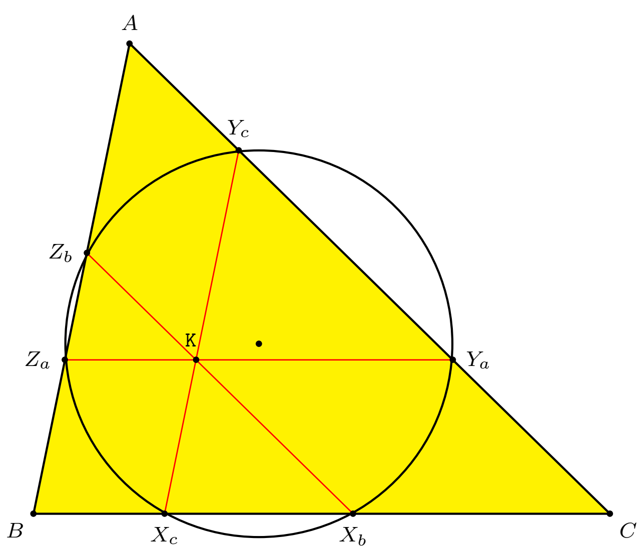
Given a point $P = (u : v : w)$, the parallel to BC through P intersects AC at Y_a and AB at Z_a. We call the segment Z_aY_a the A-parallelian through P. Likewise, the B- and C-parallelians are the segments X_bZ_b and Y_cX_c.

We locate the point(s) P for which the six parallelian endpoints lie on a circle. Note that

$$AY_a = \frac{v+w}{u+v+w} \cdot b \quad \text{and} \quad AZ_a = \frac{v+w}{u+v+w} \cdot c.$$

Also,

$$AY_c = \frac{w}{u+v+w} \cdot b \quad \text{and} \quad AZ_b = \frac{v}{u+v+w} \cdot c.$$



Therefore, $AY_a \cdot AY_c = AZ_a \cdot AZ_b$ if and only if $wb^2 = vc^2$, or $v : w = b^2 : c^2$. Similarly, $BZ_b \cdot BZ_a = BX_b \cdot BX_c$ if and only if $w : u = c^2 : a^2$,

and
$CX_c \cdot CX_b = CY_c \cdot CY_a$ if and only if $u : v = a^2 : b^2$.

It follows that the six parallelian endpoints are concyclic if and only if $u : v : w = a^2 : b^2 : c^2$, *i.e.*, P is the symmedian point K. The circle containing them is called the *first Lemoine circle*.

Remark. The center of this circle is the midpoint of OK.

Exercise

1. Show that for the first Lemoine circle,

$$\begin{array}{ccccccccccc} BX_c & : & X_cX_b & : & X_bC & = & c^2 & : & a^2 & : & b^2, \\ CY_a & : & Y_aY_c & : & Y_cA & = & a^2 & : & b^2 & : & c^2, \\ AZ_b & : & Z_bZ_a & : & Z_aB & = & b^2 & : & c^2 & : & a^2. \end{array}$$

2. Show that $Y_cZ_b = Z_aX_c = X_bY_a$.

Chapter 3

Straight lines

3.1 The two-point form

A straight line in the plane of $\mathbf{T}$ is represented by a homogeneous linear equation in the homogeneous barycentric coordinates of points the line. Given two points $P_1 = (x_1 : y_1 : z_1)$ and $P_2 = (x_2 : y_2 : z_2)$, a point $P = (x : y : z)$ lies on the line P_1P_2 if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x & y & z \end{vmatrix} = 0,$$

or

$$(y_1z_2 - y_2z_1)x + (z_1x_2 - z_2x_1)y + (x_1y_2 - x_2y_1)z = 0.$$

Proof. As vectors in $\mathbb{R}^3$, (x_1, y_1, z_1), (x_2, y_2, z_1), and (x, y, z) are linearly dependent. $\square$

Proposition. (a) The intersection of the two lines

$$\begin{aligned} p_1x + q_1y + r_1z &= 0, \\ p_2x + q_2y + r_2z &= 0 \end{aligned}$$

is the point

$$(q_1r_2 - q_2r_1 : r_1p_2 - r_2p_1 : p_1q_2 - p_2q_1).$$

(b) Three lines $p_ix + q_iy + r_iz = 0$, $i = 1, 2, 3$, are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0.$$

Examples. (1) The equations of the sidelines BC, CA, AB are respectively $x = 0$, $y = 0$, $z = 0$.

(2) The equation of the line joining the centroid and the incenter is

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ x & y & z \end{vmatrix} = 0,$$

or $(b - c)x + (c - a)y + (a - b)z = 0$.

(3) The equations of some important lines:

Euler line	OH	$\sum_{\text{cyclic}}(b^2 - c^2)(b^2 + c^2 - a^2)x = 0$
OI-line	OI	$\sum_{\text{cyclic}} bc(b - c)(b + c - a)x = 0$
Brocard axis	OK	$\sum_{\text{cyclic}} b^2c^2(b^2 - c^2)x = 0$
Soddy line	$\mathtt{IG_e}$	$\sum_{\text{cyclic}}(b - c)(b + c - a)^2x = 0$

3.1.1 Cevian and anticevian triangles of a point

Let $P = (u : v : w)$ be a given point.

(1) The equations of the lines AP, BP, CP are

$$\begin{array}{llcccl} AP: & & & +\dfrac{y}{v} & -\dfrac{z}{w} & = 0, \\ BP: & & -\dfrac{x}{u} & & +\dfrac{z}{w} & = 0, \\ CP: & & +\dfrac{x}{u} & -\dfrac{y}{v} & & = 0. \end{array}$$

(2) These lines intersect the sidelines at the points

$$\begin{aligned} P_a := \ & BC \cap AP = (0 : v : w), \\ P_b := \ & CA \cap BP = (u : 0 : w), \\ P_c := \ & AB \cap CP = (u : v : 0). \end{aligned}$$

The triangle $P_aP_bP_c$ is called the cevian triangle of P, and is denoted by $\mathsf{cev}(P)$.

(3) The equations of the lines P_bP_c, P_cP_a, P_aP_b are

$$\begin{array}{llcl}
P_bP_c: & -\dfrac{x}{u} + \dfrac{y}{v} + \dfrac{z}{w} & = 0, \\
P_cP_a: & +\dfrac{x}{u} - \dfrac{y}{v} + \dfrac{z}{w} & = 0, \\
P_aP_b: & +\dfrac{x}{u} + \dfrac{y}{v} - \dfrac{z}{w} & = 0.
\end{array}$$

(4) These lines intersect the sidelines at

$$\begin{aligned}
P_a' &:= BC \cap AP_a = (0 : v : -w), \\
P_b' &:= CA \cap BP_b = (-u : 0 : w), \\
P_c' &:= AB \cap CP_c = (u : -v : 0).
\end{aligned}$$

(5) The three points P_a', P_b', P_c' on the sidelines are collinear. The line containing them is

$$\mathscr{L}_P: \qquad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$

This line is called the trilinear polar of P.

(6) The equations of the lines AP_a', BP_b', CP_c' are

$$\begin{array}{llcl}
AP_a': & \phantom{\dfrac{x}{u}} + \dfrac{y}{v} + \dfrac{z}{w} & = 0, \\
BP_b': & \dfrac{x}{u} \phantom{+\dfrac{y}{v}} + \dfrac{z}{w} & = 0, \\
CP_c': & \dfrac{x}{u} + \dfrac{y}{v} \phantom{+\dfrac{z}{w}} & = 0.
\end{array}$$

(7) The lines AP_a', BP_b', CP_c' bound a triangle with vertices

$$\begin{aligned}
P^a &:= BP_b' \cap CP_c' = (-u : v : w), \\
P^b &:= CP_c' \cap AP_a' = (u : -v : w), \\
P^c &:= AP_a' \cap BP_b' = (u : v : -w).
\end{aligned}$$

The triangle $P^aP^bP^c$ is called the anticevian triangle of P, and is denoted by $\mathsf{cev}^{-1}(P)$.

3.2 Infinite points and parallel lines

3.2.1 The infinite point of a line

A line $px + qy + rz = 0$ contains the point $(q - r : r - p : p - q)$, as is easily verified. Since the sum of its coordinates is zero, this is not a finite point. We call it an infinite point. Thus, an infinite point $(x : y : z)$ is one which satisfies the equation $x + y + z = 0$, which we regard as defining the *line at infinity* $\mathcal{L}_\infty$. Now, it is easy to see that unless $p : q : r = 1 : 1 : 1$, a line $px + qy + rz = 0$ has a unique infinite point as given above. Therefore, the infinite point of a line determines its "direction".

Two lines are *parallel* if and only if they have the same infinite point. It follows that the line through $(u : v : w)$ parallel to $px + qy + rz = 0$ has equation

$$\begin{vmatrix} x & y & z \\ u & v & w \\ q - r & r - p & p - q \end{vmatrix} = 0.$$

Examples

(1) The infinite points of the sidelines of ABC are as follows.

Line	Equation	Infinite point
a	$x = 0$	$(0 : 1 : -1)$
b	$y = 0$	$(-1 : 0 : 1)$
c	$z = 0$	$(1 : -1 : 0)$

(2) If $P = (u : v : w)$, the line joining the centroid G to P has equation $(v - w)x + (w - u)y + (u - v)z = 0$. It has infinite point $(v + w - 2u : w + u - 2v : u + v - w)$.

(3) The Euler line, being the line joining G to $H = (S_{\beta\gamma} : S_{\gamma\alpha} : S_{\alpha\beta})$ has infinite point

$$E_\infty := (S_{\gamma\alpha} + S_{\alpha\beta} - 2S_{\beta\gamma} : +S_{\alpha\beta} + S_{\beta\gamma} - 2S_{\gamma\alpha} : 2S_{\beta\gamma}S_{\gamma\alpha} - 2S_{\alpha\beta}).$$

This is called the Euler infinity point.

Parametrization of a line

The finite points of the line $ux + vy + wz = 0$ can be parametrized as

$$((v-w)(vw+t) : \ (w-u)(wu+t) : \ (u-v)(uv+t)).$$

Exercise

1. Find the equations of the lines through $P = (u : v : w)$ parallel to the sidelines.

2. (a) Find the infinite point of the bisector of angle A. [1]

(b) Find the infinite point of the external bisector of angle A. [2]

3. Find the infinite point of the trilinear polar of $(u : v : w)$.

4. Let D, E, F be the midpoints of the sides BC, CA, AB of triangle ABC. For a point P with traces A_P, B_P, C_P, let X, Y, Z be the midpoints of B_PC_P, C_PA_P, A_PB_P respectively.

(a) Find P such that the lines DX, EY, FZ are parallel to the internal bisectors of angles A, B, C respectively. [3]

(b) Explain why the lines DX, EY, FZ are concurrent and identify the point of concurrency. [4]

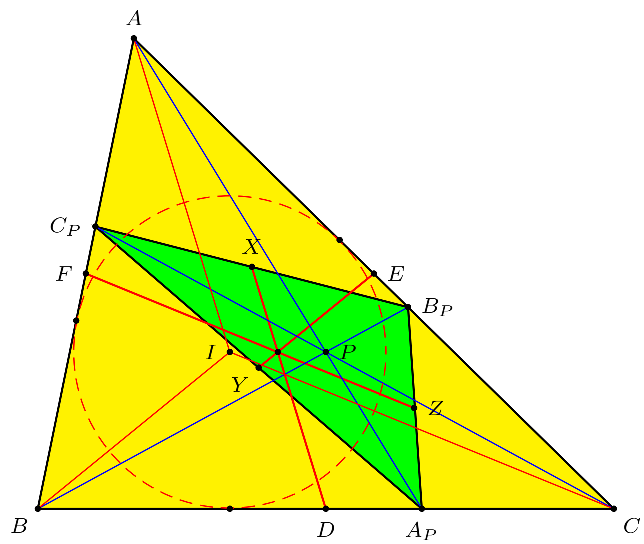
---

[1] $(-(b+c) : \underline{:}c)$.

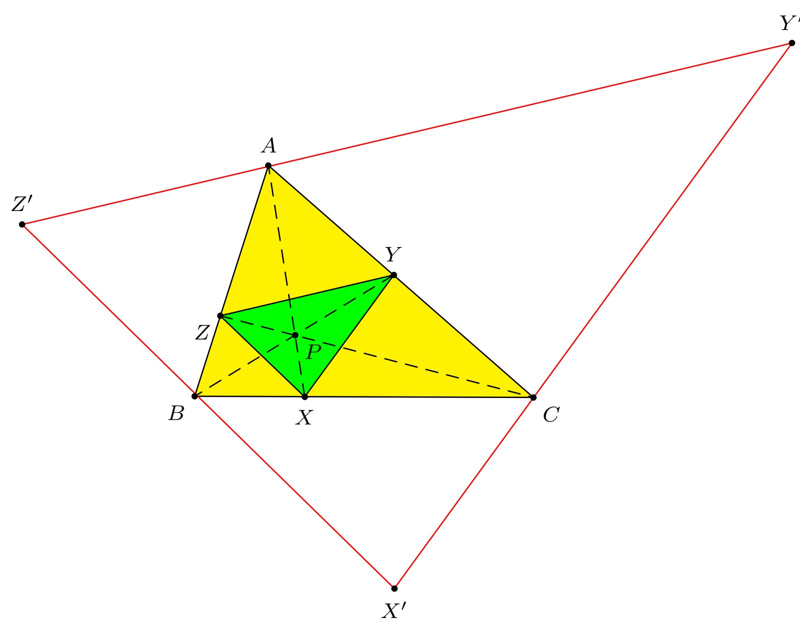
[2] $(b-c : \ -b : \ c)$.

[3] The Nagel point.

[4] Spieker center.



5. Let P be a given point with cevian triangle XYZ. Consider the triangle $X'Y'Z'$ bounded by the parallels to YZ, ZX, XY through A, B, C respectively. Show that $X'Y'Z'$ is an anticevian triangle, and find the homothetic center with cev(P). [5]



[5]If $P = (u : v : w)$, this is the anticevian triangle of $(u(v+w) : v(w+u) : w(u+v))$. The homothetic center is $(u^2(v+w) : v^2(w+u) : w^2(u+v))$.

3.2.2 Infinite point as vector

The infinite point of a line through two given points can be computed through a calculation of absolute barycentric coordinates. If $P = (u : v : w)$ and $Q = (u' : v' : w')$ are finite points, the infinite point of the line PQ can be computed from

$$Q-P = \left(\frac{u'}{u'+v'+w'} - \frac{u}{u+v+w}, \frac{v'}{u'+v'+w'} - \frac{v}{u+v+w}, \frac{w'}{u'+v'+w'} - \frac{w}{u+v+w}\right)$$

which we regard as the vector $\overrightarrow{PQ}$. From this, we obtain a simpler expression for the homogeneous barycentric coordinates of the infinite point, namely

$$(u'(v+w) - u(v'+w') : v'(w+u) - v(w'+u') : w'(u+v) - w(u'+v')).$$

The altitudes

The infinite point of the A-altitude is given by the vector

$$\frac{1}{S_\beta + S_\gamma}(0, S_\gamma, S_\beta) - (1,0,0) = \frac{1}{S_\beta + S_\gamma}(-(S_\beta + S_\gamma), S_\gamma, S_\beta).$$

From this, we obtain the homogeneous barycentric coordinates of the infinite points of the altitudes.

Line	Infinite point
$A-$altitude	$-(S_\beta + S_\gamma) : \ S_\gamma : \ S_\beta$
$B-$altitude	$S_\gamma : \ -(S_\gamma + S_\alpha) : \ S_\alpha$
$C-$altitude	$S_\beta : \ S_\alpha : \ -(S_\alpha + S_\beta)$

It follows that the equation of the perpendicular bisector of BC is

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 1 \\ -(S_\beta + S_\gamma) & S_\gamma & S_\beta \end{vmatrix} = 0,$$

or

$$(S_\beta - S_\gamma)x - (S_\beta + S_\gamma)(y-z) = 0.$$

Since the intouch triangle is homothetic to the excentral triangle, its altitudes are parallel to the angle bisectors of triangle ABC. Thus, the altitude through X has infinite point $(b+c, -b, -c)$; it is the line

$$\begin{vmatrix} 0 & a+b-c & c+a-b \\ b+c & -b & -c \\ x & y & z \end{vmatrix} = -((b-c)(b+c-a)x-(b+c)(c+a-b)y+(b+c)(a+b-c)$$

Similarly, the other two altitudes are

$$\begin{array}{rcrcrcl} (c+a)(b+c-a)x & + & (c-a)(c+a-b)y & - & (c+a)(a+b-c)z & = & 0, \\ -(a+b)(b+c-a)x & + & (a+b)(c+a-b)y & + & (a-b)(a+b-c)z & = & 0. \end{array}$$

The orthocenter of the intouch triangle is the intersection of these two lines, namely,

$$\begin{aligned} & (b+c-a)x : \ (c+a-b)y : \ (a+b-c)z \\ = \ & \begin{vmatrix} c-a & -(c+a) \\ a+b & a-b \end{vmatrix} : \ - \begin{vmatrix} c+a & -(c+a) \\ -(a+b) & a-b \end{vmatrix} : \ \begin{vmatrix} c+a & c-a \\ -(a+b) & a+b \end{vmatrix} \\ = \ & 2a(b+c) : 2b(c+a) : 2c(a+b). \end{aligned}$$

Therefore, the orthocenter of the intouch triangle is the point with homogeneous barycentric coordinates

$$\left(\frac{a(b+c)}{b+c-a} : \ \frac{b(c+a)}{c+a-b} : \ \frac{c(a+b)}{a+b-c}\right).$$

Exercise

1. Let XYZ be the intouch triangle of triangle ABC. Show that the pedal (orthogonal projection) of X on YZ is the point

$$X' = \left(\frac{b+c}{b+c-a} : \ \frac{b}{c+a-b} : \ \frac{c}{a+b-c}\right).$$

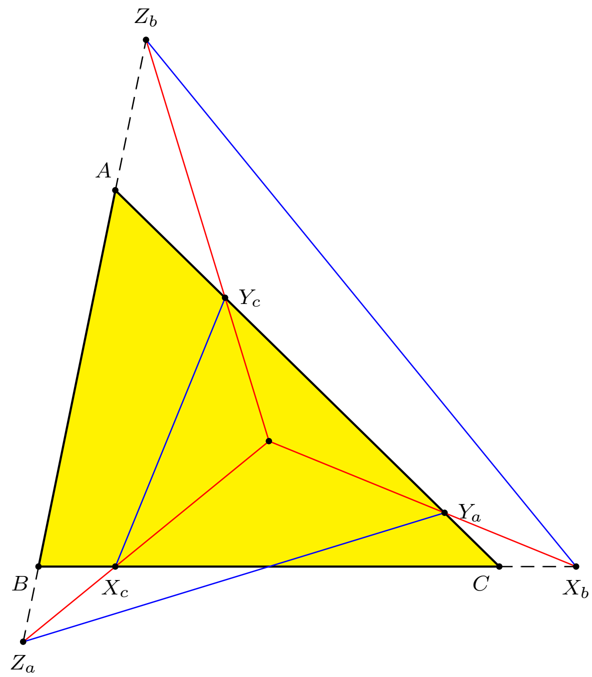
Similarly define Y' and Z'. The triangle $X'Y'Z'$ is homothetic to ABC at T.

2. Compute the coordinates of the infinite point of the OI-line. [6]

[6] $(a(a^2(b+c)-2abc-(b+c)(b-c)^2) : \cdots : \cdots)$.

Exercise

1. Given triangle ABC, extend, if necessary,
(i) AC and AB to Y_a and Z_a such that $AY_a = AZ_a = a$,
(ii) BA and BC to Z_b and X_b such that $BZ_b = BX_b = b$,
(iii) CB and CA to X_c and Y_c such that $CX_c = CY_c = c$.



(a) Find the coordinates of the points X_b, X_c, Y_c, Y_a, Z_a, Z_b. [7]

(b) Find the equations of the three lines Y_cZ_b, Z_aX_c, X_bY_a, and show that they are concurrent. [8]

(c) Let $X = BC \cap Y_cZ_b$, $Y = CA \cap Z_aX_c$ and $Z = AB \cap X_bY_a$. Show that XYZ is perspective with ABC and find the perspector. [9]

(d) Find the equations of the lines Y_aZ_a, Z_bX_b, X_cY_c, and show that they are perpendicular to Y_cZ_b, Z_aX_c, X_bY_a respectively. [10]

(e) Let $X' = BC \cap Y_aZ_a$, $Y' = CA \cap Z_bX_b$ and $Z' = AB \cap X_cY_c$. Show that $X'Y'Z'$ is perspective with ABC and find the perspector. [11]

[7] $X_b = (0 : a - b : b)$, $X_c = (0 : c : a - c)$ etc.

[8] $Y_cZ_b :\ (b-c)x + by - cz = 0$ etc. These are parallel to the angle bisectors. They are concurrent at the Nagel point.

[9] $X = (0 : c : b)$ etc. The triangle has perspector $\left(\frac{1}{a} : \frac{1}{b} : \frac{1}{c}\right)$.

[10] $Y_aZ_a :\ ax + (a-c)y + (a-b)z = 0$ etc. These are parallel to the external angle bisectors.

[11] $X' = (0 : a - b : c - a)$ etc., and $X'Y'Z'$ is perspective with ABC at $\left(\frac{1}{b-c} : \frac{1}{c-a} : \frac{1}{a-b}\right)$.

(f) Let $X'' = Y_cZ_b \cap Y_aZ_a$, $Y'' = Z_aX_c \cap Z_bX_b$ and $Z'' = X_bY_a \cap X_cY_c$. Show that $X''Y''Z''$ is perspective with ABC and find the perspector. [12]

[12] $X'' = (-a(b+c) + (b^2+c^2) : b(c+a-b) : c(a+b-c))$ etc. The triangle has perspector $M_1 = (a(b+c-a) : b(c+a-b) : c(a+b-c))$.

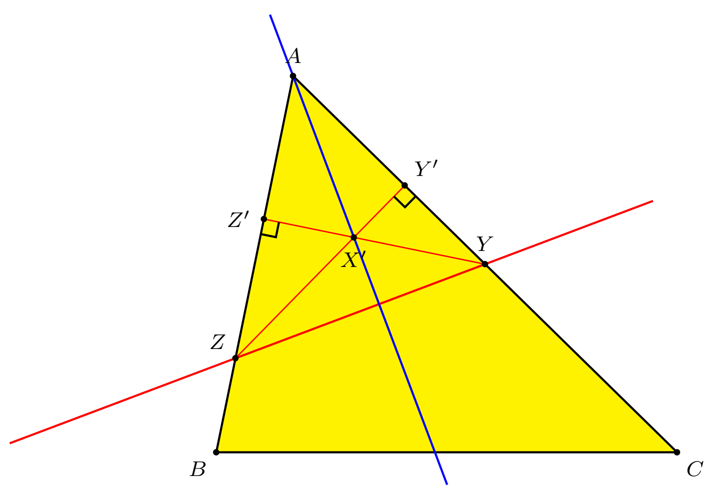
3.3 Perpendicular lines

Given a line $\mathcal{L}: px + qy + rz = 0$, we determine the infinite point of lines perpendicular to it. The line $\mathcal{L}$ intersects the side lines CA and AB at the points $Y = (-r : 0 : f)$ and $Z = (q : -p : 0)$. To find the perpendicular from A to $\mathcal{L}$, we first find the equations of the perpendiculars from Y to AB and from Z to CA. These are

$$\begin{vmatrix} S_\beta & S_\alpha & -c^2 \\ -r & 0 & f \\ x & y & z \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} S_\gamma & -b^2 & S_\alpha \\ q & -p & 0 \\ x & y & z \end{vmatrix} = 0$$

These are

$$\begin{aligned} S_\alpha f x + (c^2 h - S_\beta f) y + S_\alpha r z &= 0, \\ S_\alpha f x + S_\alpha q y + (b^2 g - S_\gamma p) z &= 0. \end{aligned}$$



These two perpendiculars intersect at the orthocenter of triangle AYZ, which is the point

$$\begin{aligned} X' &= (*** : S_\alpha p(S_\alpha r - b^2 q + S_\gamma p) : S_\alpha p(S_\alpha q + S_\beta p - c^2 r) \\ &\sim (*** : S_\gamma(p-q) - S_\alpha(q-r) : S_\alpha(q-r) - S_\beta(r-p)). \end{aligned}$$

The perpendicular from A to $\mathcal{L}$ is the line AX', which has equation

$$\begin{vmatrix} 1 & 0 & 0 \\ *** & S_\gamma(p-q) - S_\alpha(q-r) & -S_\alpha(q-r) + S_\beta(r-p) \\ x & y & z \end{vmatrix} = 0,$$

or

$$-(S_\alpha(q-r)-S_\beta(r-p))y+(S_\gamma(p-q)-S_\alpha(q-r))z=0.$$

This has infinite point

$$(S_\beta(r-p)-S_\gamma(p-q):S_\gamma(p-q)-S_\alpha(q-r):S_\alpha(q-r)-S_\beta(p-q)).$$

Note that the infinite point of $\mathcal{L}$ is $(q-r:r-p:p-q)$. We summarize this in the following theorem.

Theorem. If a line $\mathcal{L}$ has infinite point $(f:g:h)$ (satisfying $f+g+h=0$), the lines perpendicular to $\mathcal{L}$ have infinite point $(f':g':h')$, where

$$\begin{aligned} f' =& S_\beta g - S_\gamma h,\\ g' =& S_\gamma h - S_\alpha f,\\ h' =& S_\alpha f - S_\beta g. \end{aligned}$$

Equivalently, two lines with infinite points $(f:g:h)$ and $(f':g':h')$ are perpendicular to each other if and only if

$$S_\alpha ff' + S_\beta gg' + S_\gamma hh' = 0.$$

Corollary. The perpendicular from the point $(u:v:w)$ to the line with infinite point $(f:g:h)$ is the line

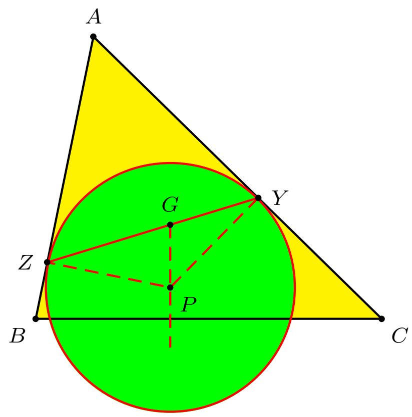
$$\begin{vmatrix} u & v & w\\ f' & g' & h'\\ x & y & z \end{vmatrix} = 0.$$

Orthogonal infinite points

Line	Infinite point	Orthogonal infinite p
Euler line	$E_\infty := (S_{\gamma\alpha}+S_{\alpha\beta}-2S_{\beta\gamma}:\cdots:\cdots)$	$(S_\beta - S_\gamma : \cdots : \cdots$
$\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=0$	$(u(v-w):\cdots:\cdots)$	$(S_Bv(w-u)-S_Cw(u-v)$
OI		$(a(b-c):b(c-a):c($

Exercise

1. Given triangle ABC, construct a circle tangent to AC at Y and AB at Z such that the line YZ passes through the centroid G.
(i) Show that $YG : GZ = c : b$, [13]
(ii) Calculate the coordinates of the center P of the circle. [14]
(iii) Show that the line GP is perpendicular to BC.



2. Given triangle ABC with orthic triangle DEF, let the parallels through H to BC, CA, AB intersect EF, FD, DE respectively at X', Y', Z'. The points X', Y', Z' are collinear on a line perpendicular to the Euler line. [15]

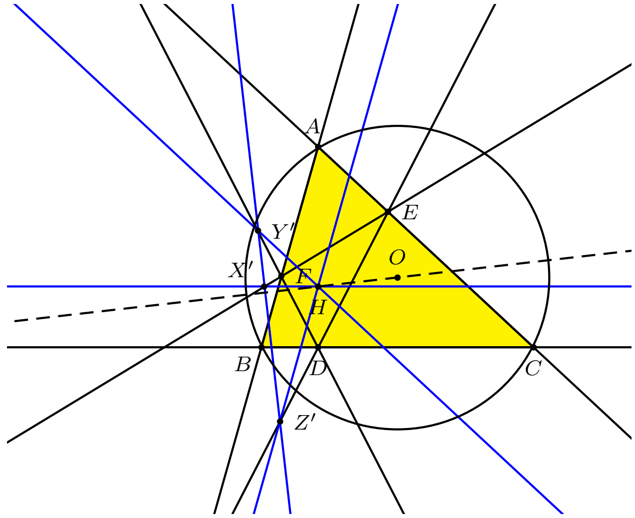
3. Given triangle ABC with orthic triangle DEF, let the parallels through H to EF, FD, DE intersect BC, CA, AB respectively at X, Y, Z. The points X, Y, Z are collinear on a line perpendicular to the Euler line. [16]

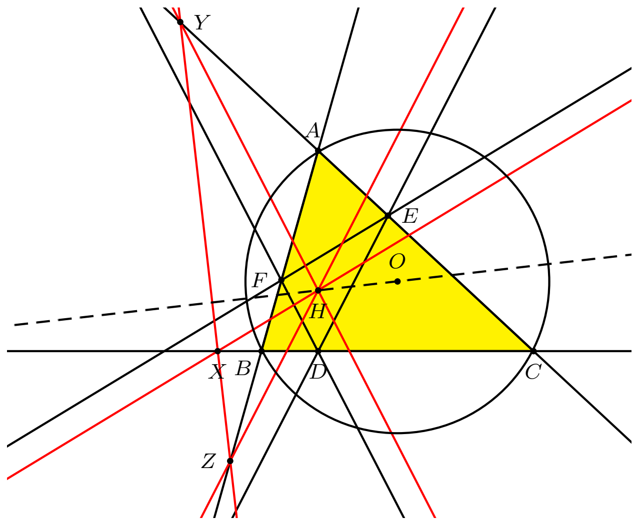
[13] If $AY = AZ = t$, $Y = (b-t : 0 : t)$ and $Z = (c-t : t : 0)$. If the line YZ contains G, then $\frac{\lambda}{b}(b-t, 0, t) + \frac{1-\lambda}{c}(c-t, t, 0) = \frac{1}{3}(1, 1, 1)$ for some λ. From this, we have $\lambda = \frac{b}{b+c}$ and $t = \frac{b+c}{3}$. Thus, $YG : GZ = c : b$.

[14] $(-3a^2 + (b+c)^2 : 2b(b+c) : 2c(b+c))$.

[15] Except for the perpendicularity, this is true if H and the orthic triangle are replaced by a generic point and its cevian triangle. If $P = (u : v : w)$, the line is the trilinear polar of $\left(\frac{u}{v+w-u} : \frac{v}{w+u-v} : \frac{w}{u+v-w}\right)$. This line has infinite point $(u(v-w) : v(w-u) : w(u-v)) = \left(\frac{1}{v} - \frac{1}{w} : \frac{1}{w} - \frac{1}{u} : \frac{1}{u} - \frac{1}{v}\right)$. In the case of the Euler line, this is $(S_\beta - S_\gamma : S_\gamma - S_\alpha : S_\alpha - S_\beta)$.

[16] Except for the perpendicularity, this is true if H and the orthic triangle are replaced by a generic point and its cevian triangle. If $P = (u : v : w)$, the line is the trilinear polar of $\left(\frac{u}{v+w} : \frac{v}{w+u} : \frac{w}{u+v}\right)$. This line has infinite point $(u(v-w) : v(w-u) : w(u-v)) = \left(\frac{1}{v} - \frac{1}{w} : \frac{1}{w} - \frac{1}{u} : \frac{1}{u} - \frac{1}{v}\right)$. In the case of the Euler line, this is $(S_\beta - S_\gamma : S_\gamma - S_\alpha : S_\alpha - S_\beta)$.





3.4 The distance formula

Let $P = uA + vB + wC$ and $Q = u'A + v'B + w'C$ be given in *absolute* barycentric coordinates. The distance between them is given by

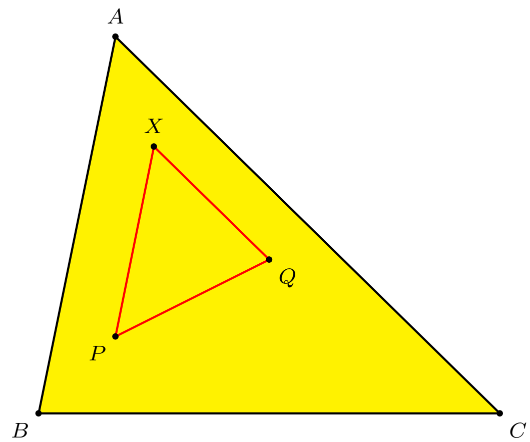
$$\boxed{PQ^2 = S_\alpha(u - u')^2 + S_\beta(v - v')^2 + S_\gamma(w - w')^2.}$$

Proof. Through P and Q draw lines parallel to AB and AC respectively, intersecting at X. The barycentric coordinates of X can be determined in two ways:

$$X = P + k(A - B) = Q + h(A - C)$$

for some h and k. It follows that

$$X = (u + k)A + (v - k)B + wC = (u' + h)A + v'B + (w' - h)C.$$



From these we have

$$h = -(w - w'), \qquad k = v - v',$$

and

$$X = (1 - w - v')A + v'B + wC.$$

Applying the law of cosines to triangle XPQ (in which $\angle PXQ = \alpha$, we have

$$\begin{aligned} PQ^2 &= (hb)^2 + (kc)^2 - 2(hb)(kc)\cos\alpha \\ &= h^2b^2 + k^2c^2 - 2hkS_\alpha \\ &= (w - w')^2(S_\gamma + S_\alpha) + (v - v')^2(S_\alpha + S_\beta) + 2(v - v')(w - w')S_\alpha \\ &= \big((w - w')^2 + (v - v')^2 + 2(v - v')(w - w')\big)\, S_\alpha + (v - v')^2 S_\beta + (w - w')^2 S_\gamma \\ &= ((w - w') + (v - v'))^2\, S_\alpha + (v - v')^2 S_\beta + (w - w')^2 S_\gamma \\ &= S_\alpha(u - u')^2 + S_\beta(v - v')^2 + S_\gamma(w - w')^2. \end{aligned}$$

□

In homogeneous barycentric coordinates, if $P = (x : y : z)$ and $Q = (u : v : w)$ are finite points, then the square distance between them is given by

$$\boxed{PQ^2 = \frac{1}{(u+v+w)^2(x+y+z)^2} \sum_{\text{cyclic}} S_\alpha((v+w)x - u(y+z))^2}$$

Exercise

1. Show that the distances from $P = (x : y : z)$ to the vertices of triangle ABC are given by

$$\begin{aligned} AP^2 &= \frac{c^2y^2 + 2S_\alpha yz + b^2z^2}{(x+y+z)^2}, \\ BP^2 &= \frac{a^2z^2 + 2S_\beta zx + c^2x^2}{(x+y+z)^2}, \\ CP^2 &= \frac{b^2x^2 + 2S_\gamma xy + a^2y^2}{(x+y+z)^2}. \end{aligned}$$

2. Show that the square distance between $P = (x : y : z)$ and $Q = (u : v : w)$ can be written as

$$PQ^2 = \frac{1}{x+y+z}\left(\sum_{\text{cyclic}} \frac{c^2v^2 + 2S_\alpha vw + b^2w^2}{(u+v+w)^2}x\right) - \frac{a^2yz + b^2zx + c^2xy}{(x+y+z)^2}.$$

3.4.1 The distance from a point to a line

Consider a line $\mathcal{L}$ with equation $px+qy+rz = 0$. We shall assume p, q, r not all equal so that $\mathcal{L}$ is not the line at infinity $x + y + z = 0$. For convenience, we write its infinite point as $(f : g : h) = (q - r : r - p : p - q)$. The orthogonal infinite point being

$$(f', g', h') = (S_\beta g - S_\gamma h,\ S_\gamma h - S_\alpha p,\ S_\alpha p - S_\beta q),$$

we seek a quantity k such that $A + k(f', g', h')$ lies on the line $\mathcal{L}$. Thus,

$$p(1 + kf') + q \cdot kg' + r \cdot kh' = 0,$$

and

$$k = -\frac{p}{pf' + qg' + rh'}.$$

The (signed) distance from A to $\mathcal{L}$ is k times the length of the vector (f', g', h').

Similar calculations give the signed distances from B and C, and we have the following simple characterization of the barycentric equation of a straight line.

Proposition. The signed distances from the vertices A, B, C to the line $px + qy + rz = 0$ are in the ratio $p : q : r$.

Note that

$$\begin{aligned} & -(pf' + qg' + rh') \\ = & -p(S_\beta(r-p) - S_\gamma(p-q)) - q(S_\gamma(p-q) - S_\alpha(q-r)) - r(S_\alpha(q-r) - S_\beta(r-p)) \\ = & \ S_\alpha(q-r)^2 + S_\beta(r-p)^2 + S_\gamma(p-q)^2, \end{aligned}$$

which is the square distance between the two finite points $(q : r : p)$ and $(r : p : q)$, assuming $p + q + r \neq 1$. [17] This is zero if and only if $p = q = r$, in which case $\mathcal{L}$ is the line at infinity.

Proposition. The signed distance from a finite point $(x : y : z)$ to the line $px + qy + rz = 0$ is

$$\frac{px + qy + rz}{x + y + z} \cdot \frac{S}{\sqrt{S_\alpha(q-r)^2 + S_\beta(r-p)^2 + S_\gamma(p-q)^2}}.$$

[17] This assumption is valid if $p + q + r \neq 0$. If $p + q + r = 0$, use instead $(q + s : r + s : p + s)$ and $(r + s : p + s : q + s)$ for some nonzero s.

Chapter 4

Cevian and anticevian triangles

4.1 Cevian triangles

We begin with a convenient reformulation of the Ceva theorem.

Theorem (Ceva)**.** For points X on BC, Y on CA, and Z on AB, the lines AX, BY, CZ are concurrent if and only if their homogeneous barycentric coordinates are

$$\begin{array}{rcccccc} X & = & 0 & : & v & : & w, \\ Y & = & u & : & 0 & : & w, \\ Z & = & u & : & v & : & 0, \end{array}$$

for some u, v, w. If this condition is satisfied, the point of concurrency is $P = (u : v : w)$.

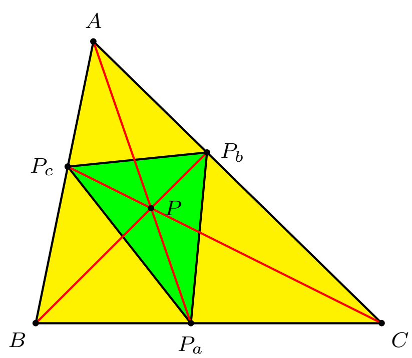


Figure 4.1: The traces and cevian triangle of P

The points X, Y, Z are called the *traces* of P. We also say that XYZ is the cevian triangle of P (with reference to $\mathbf{T}$). Sometimes, we shall adopt the more functional notation for the cevian triangle and its vertices:

$$\mathsf{cev}(P): \quad \boxed{P_a = (0 : v : w), \qquad P_b = (u : 0 : w), \qquad P_c = (u : v : 0).}$$

Examples.

P	$\mathsf{cev}(P)$
G	inferior triangle
I	incentral triangle
H	orthic triangle

4.1.1 The orthic triangle

The *orthic triangle* is the cevian triangle of the orthocenter H. Its vertices are

$$\mathtt{H}_a = (0 : S_\gamma : S_\beta), \quad \mathtt{H}_b = (S_\gamma : 0 : S_\alpha), \quad \mathtt{H}_c = (S_\beta : S_\alpha : 0).$$

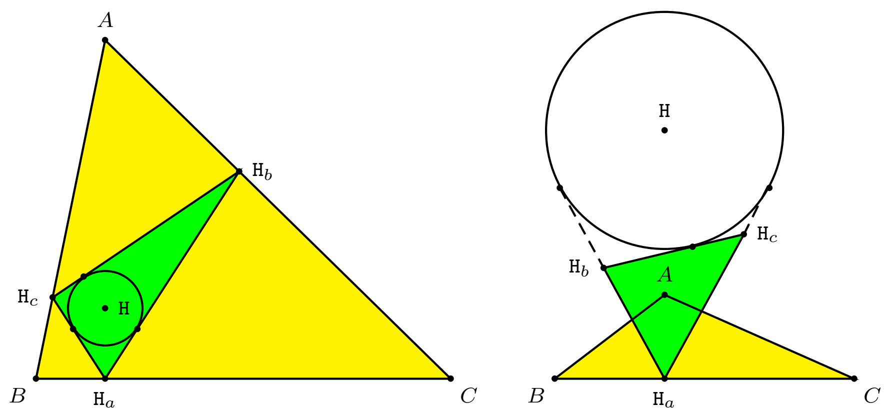


Figure 4.2: The orthic triangle and its ob-incicle

If triangle $\mathbf{T}$ is acute, then $\mathtt{H}$ is the incenter of the orthic triangle XYZ. If the triangle is obtuse, then $\mathtt{H}$ is an excenter of the orthic triangle. We say that $\mathtt{H}$ is the ob-incenter of the orthic triangle.

4.1.2 The intouch triangle and the Gergonne point

The incircle touches the sides of $\mathbf{T}$ at the pedals of the incenter $\mathtt{I}$. These define the Gergonne point $\mathtt{G_e}$ and the intouch triangle:

$$\begin{array}{cccccccc}
\mathtt{I}_{[a]} & = & 0 & : & \frac{1}{c+a-b} & : & \frac{1}{a+b-c}, \\
\mathtt{I}_{[b]} & = & \frac{1}{b+c-a} & : & 0 & : & \frac{1}{a+b-c}, \\
\mathtt{I}_{[c]} & = & \frac{1}{b+c-a} & : & \frac{1}{c+a-b} & : & 0, \\
\hline
\mathtt{G_e} & = & \frac{1}{b+c-a} & : & \frac{1}{c+a-b} & : & \frac{1}{a+b-c}.
\end{array}$$

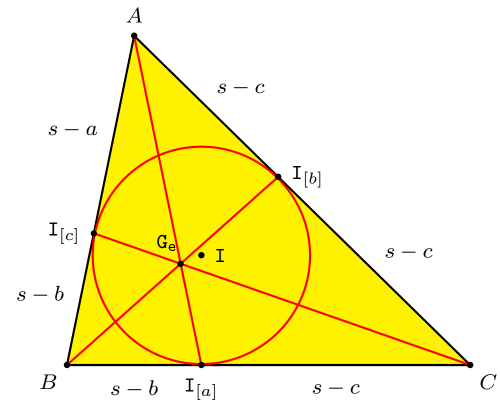


Figure 4.3: The Gergonne point and the intouch triangle

4.1.3 The Nagel point and the extouch triangle

The A-excircle touches the side BC at a point $\mathtt{I}^a_{[a]}$ such that $B\mathtt{I}^a_{[a]} = s-c$ and $\mathtt{I}^a_{[a]}C = s-b$. From this, we have the homogeneous barycentric coordinates of $\mathtt{I}^a_{[a]}$; similarly, for the points of tangency $\mathtt{I}^b_{[b]}$ and $\mathtt{I}^c_{[c]}$ of the B- and C-excircles:

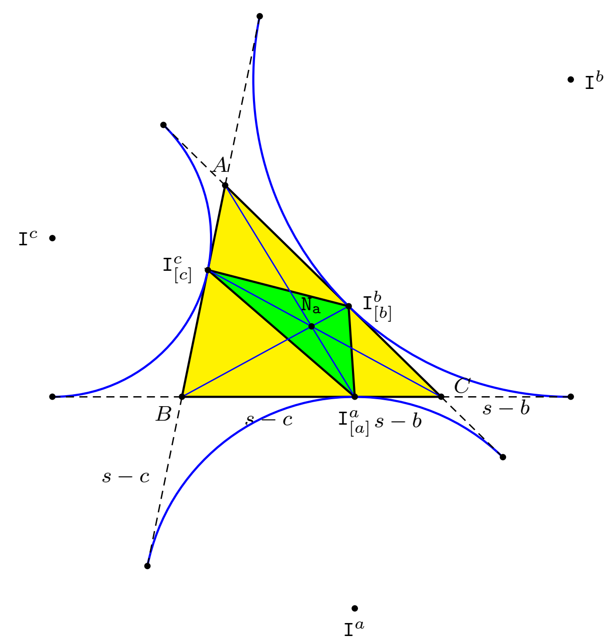


Figure 4.4: The Nagel point and the extouch triangle

$$\begin{array}{rcccccc}
\mathtt{I}^a_{[a]} & = & 0 & : & c+a-b & : & a+b-c, \\
\mathtt{I}^b_{[b]} & = & b+c-a & : & 0 & : & a+b-c, \\
\mathtt{I}^c_{[c]} & = & b+c-a & : & c+a-b & : & 0, \\
\hline
\mathtt{N_a} & = & b+c-a & : & c+a-b & : & a+b-c.
\end{array}$$

The cevian triangle of the Nagel point is called the extouch triangle. Its centroid is the point

$$\mathtt{G}[\mathsf{cev}(\mathtt{N_a})] = (a(b+c)(b+c-a) : \ b(c+a)(c+a-b) : \ c(a+b)(a+b-c)).$$

4.2 The trilinear polar

Consider the cevian triangle $\mathsf{cev}(P) = P_aP_bP_c$ for $P = (u : v : w)$. The line joining P_b and P_c has equation

$$-\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$

It intersects the sideline a at $X' = (0 : -v : w)$. Similarly, the lines P_cP_a and P_aP_b intersect b and c at $Y' = (u : 0 : -w)$ and $Z' = (-u : v : 0)$.

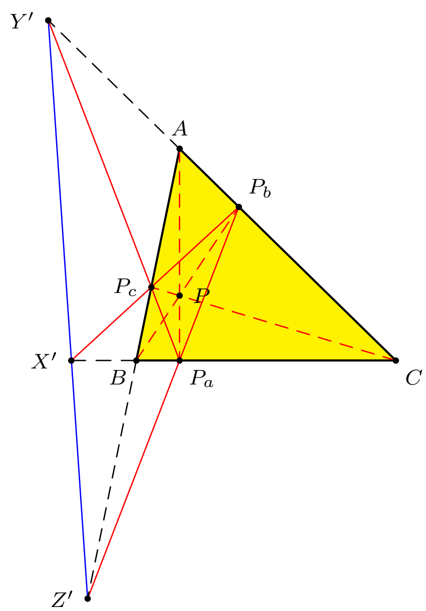


Figure 4.5:

The three points X', Y', Z' are collinear. The line containing them is called the trilinear polar of P:

$\mathscr{L}_P$: $\boxed{\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.}$

Examples.

P	Trilinear polar	barycentric equation
G	line at infinity	$x + y + z = 0$
H	orthic axis	$S_\alpha x + S_\beta y + S_\gamma z = 0$
$\mathtt{G_e}$	Gergonne axis	$(s-a)x + (s-b)y + (s-c)z = 0$

Remark. The orthic axis is perpendicular to the Euler line.

4.3 Anticevian triangles

A triangle XYZ is said to be an anticevian triangle of P with respect to triangle ABC if ABC is a cevian triangle of XYZ (with the same perspector P).

Suppose P has coordinates $(x : y : z)$ with respect to XYZ. We want to find its coordinates, and those of X, Y, Z, with respect to ABC.

Note that in absolute barycentric coordinates,

$$A = \frac{yY + zZ}{y+z}, \qquad B = \frac{xX + zZ}{x+z}, \qquad C = \frac{xX + yY}{x+y}.$$

It is easier to write X, Y, Z in terms of A, B, C:

$$\begin{aligned} X = & \frac{-(y+z)A + (z+x)B + (x+y)C}{2x}, \\ Y = & \frac{(y+z)A - (z+x)B + (x+y)C}{2y}, \\ Z = & \frac{(y+z)A + (z+x)B - (x+y)C}{2z}. \end{aligned}$$

From these we also obtain the coordinates of P:

$$P = \frac{xX + yZ + zZ}{x+y+z} = \frac{(y+z)A + (z+x)B + (x+y)C}{2(x+y+z)}.$$

We relabel X, Y, Z by P^a, P^b, P^c and call triangle $\mathsf{cev}^{-1}(P) := P^aP^bP^c$ the anticevian triangle of P.

If we write the coordinates of P with respect to ABC as $(u : v : w)$, then the coordinates of P^a, P^b, P^c with respect to ABC as as follows.

$$\begin{array}{ccccccc} P^a & = & -u & : & v & : & w \\ P^b & = & u & : & -v & : & w \\ P^c & = & u & : & v & : & -w \\ \hline P & = & u & : & v & : & w. \end{array}$$

Proposition (Construction of anticevian triangle)**.** Let the trilinear polar $\mathscr{L}_P$ intersect the sidelines a, b, c respectively at X', Y', Z'. The triangle bounded by the lines AX', BY', CZ' is the anticevian triangle of P.

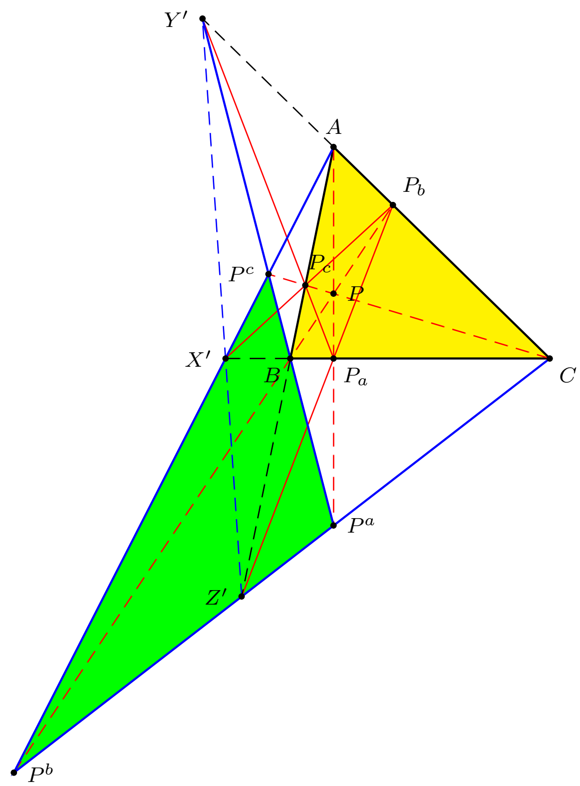


Figure 4.6:

Proof. Let $\mathsf{cev}^{-1}(P) = P^aP^bP^c$ be the anticevian triangle of P. The line joining $P^b = (u : -v : w)$ and $P^c = (u : v : -w)$ has equation $\frac{v}{y} + \frac{w}{z} = 0$. It clearly contains the vertex $\mathtt{A}$ and $X' = (0 : v : -w)$. Therefore, P^bP^c and AX' are the same line. Similarly, P^cP^a and P^aP^b are the same lines as BY' and CZ' respectively. $\square$

The superior triangle

The anticevian triangle of the centroid is the superior triangle.

4.3.1 The excentral triangle $\mathsf{cev}^{-1}(\mathtt{I})$

Consider $\mathsf{cev}(\mathtt{I}) = \mathtt{I}_a\mathtt{I}_b\mathtt{I}_c$. The harmonic conjugates of $\mathtt{I}_a$, $\mathtt{I}_b$, $\mathtt{I}_c$ on the respective sidelines are the points $\mathtt{I}'_a$, $\mathtt{I}'_b$, $\mathtt{I}'_c$ for which $A\mathtt{I}'_a$, $B\mathtt{I}'_b$, and $C\mathtt{I}'_c$ are the external bisectors of the $\mathbf{T}$. Their pairwise intersections are the excenters of $\mathbf{T}$. These are the points

$$\mathtt{I}^a = (-a : b : c), \qquad \mathtt{I}^b = (a : -b : c), \qquad \mathtt{I}^c = (a : b : -c).$$

They form the *excentral triangle* $\mathsf{cev}^{-1}(\mathtt{I})$. Its altitudes are the internal bisectors of the angles of $\mathbf{T}$. From this we deduce some basic facts about the excentral triangle:

Element in excentral triangle	Element in $\mathbf{T}$
orthocenter	$\mathtt{I}$
orthic triangle	$\mathbf{T}$
nine-point circle	circumcircle $\mathtt{O}(R)$
circumradius	$2R$
circumcenter	$\mathtt{O}^\dagger(\mathtt{I})$ = reflection of $\mathtt{I}$ in $\mathtt{O}$
Euler line	$\mathtt{OI}$

Since the pedal triangle of $\mathtt{I}$ is the intouch triangle $\mathtt{I}_{[a]}\mathtt{I}_{[b]}\mathtt{I}_{[c]}$, the pedal triangle of $\mathtt{O}^\dagger(\mathtt{I})$ is the extouch triangle $\mathtt{I}^a_{[a]}\mathtt{I}^b_{[b]}\mathtt{I}^c_{[c]}$ whose vertices are the points of tangency of the excircles and the corresponding sides of $\mathbf{T}$). It follows that $\mathtt{O}^\dagger(\mathtt{I})$ is the point of concurrency of the perpendiculars from the excenters to the sidelines of $\mathbf{T}$.

In homogeneous coordinates,

$$\mathtt{O}[\mathsf{cev}^{-1}(\mathtt{I})] = \mathtt{O}^\dagger(\mathtt{I}) = (a(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2) : \cdots : \cdots).$$

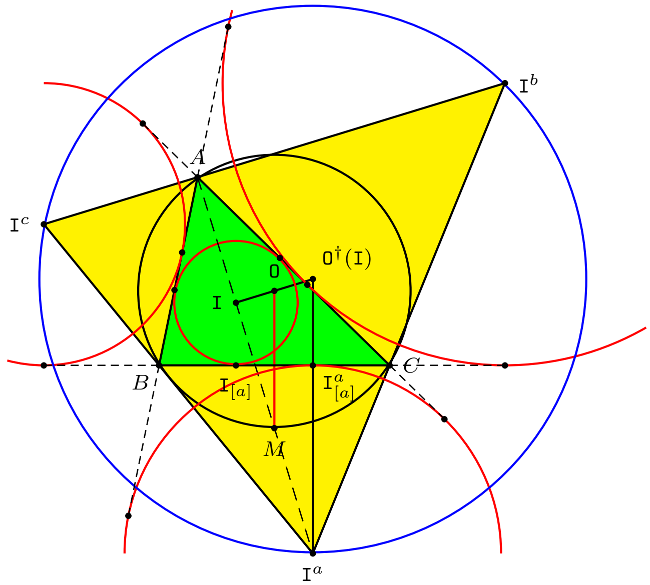


Figure 4.7: The excentral triangle and its circumcircle

Chapter 5

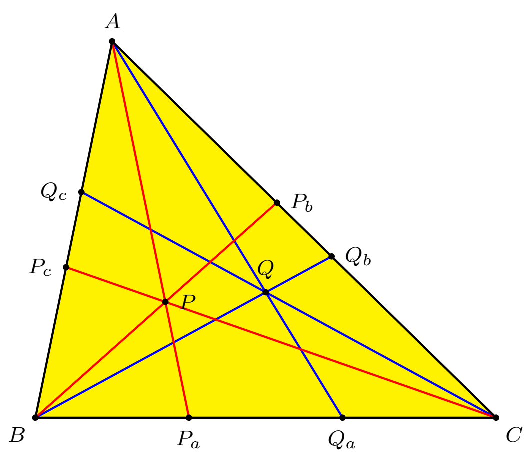
Isotomic and isogonal conjugates

5.1 Isotomic conjugates

Two points P and Q (not on any of the sidelines of the reference triangle) are said to be *isotomic conjugates* if their respective traces on the sidelines are symmetric with respect to the endpoints of the corresponding sides. Thus,

$$BP_a = Q_aC, \qquad CP_b = Q_bA, \qquad AP_c = Q_cB.$$

We shall denote the isotomic conjugate of P by $P^\bullet$.



If $P = (u : v : w)$, then $P_a = (0 : v : w)$, $BP_a : P_aC = w : v$, $BQ_a : Q_aC = P_aC : BP_a = v : w$, $Q_a = (0 : w : v) = \left(0 : \frac{1}{v} : \frac{1}{w}\right)$. Similarly, $Q_b = \left(\frac{1}{u} : 0 : \frac{1}{w}\right)$ and $Q_c = \left(\frac{1}{u} : \frac{1}{v} :\right)$. We conclude that the isotomic conjugate of P is the point

$$P^\bullet = \left(\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right) = (vw : wu : uv).$$

5.1.1 Example: the Gergonne and Nagel points

$$\begin{aligned}
\mathtt{G_e} &= \left(\frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right),\\
\mathtt{N_a} &= (b+c-a : c+a-b : a+b-c).
\end{aligned}$$

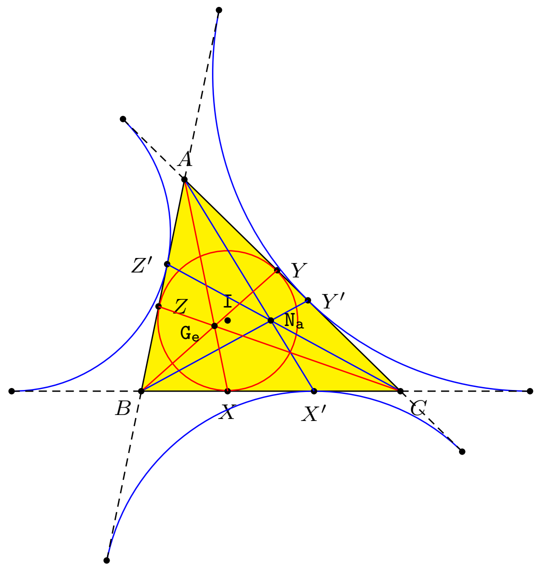


Figure 5.1: The Gergonne and Nagel points

5.1.2 Example: isotomic conjugate of the orthocenter

The isotomic conjugate of the orthocenter is the point

$$\mathtt{H}^{\bullet} = (S_\alpha : S_\beta : S_\gamma).$$

Its traces are the pedals of the deLongchamps point $\mathtt{L_o}$, the reflection of $\mathtt{H}$ in $\mathtt{O}$.

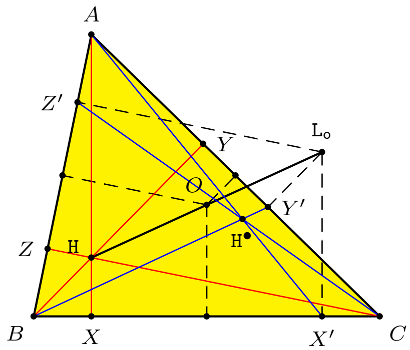


Figure 5.2: The orthocenter and its isotomic conjugate

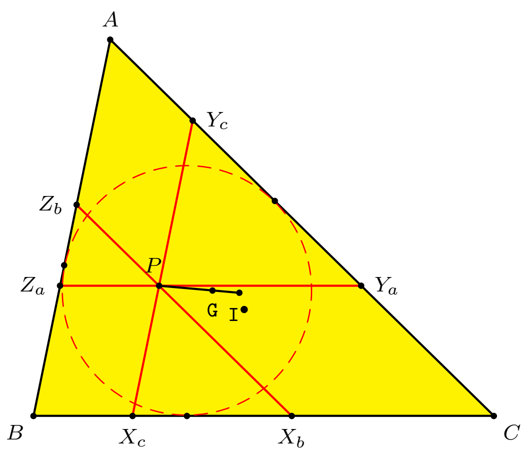
5.1.3 The equal-parallelians point

We want to find the point P for which the three parallelian segments Z_aY_a, X_bZ_b and Y_cX_c have equal lengths. Note that $Z_aY_a = \frac{v+w}{u+v+w} \cdot a$ etc. Equality of the three lengths follows if and only if $(v+w)a = (w+u)b = (u+v)c$. Equivalently,

$$v + w : w + u : u + v = \frac{1}{a} : \frac{1}{b} : \frac{1}{c},$$

or $\mathsf{inf}(P) = I^\bullet$, $P = \mathsf{sup}(\mathtt{I}^\bullet)$. This is called the *equal-parallelians point* of triangle ABC. It has coordinates

$$(ca + ab - bc : \; ab + bc - ca : \; bc + ca - ab).$$



Remark. The common length of the equal parallelians is $\frac{2abc}{bc+ca+ab}$.

5.1.4 Crelle-Yff points

Consider a point P satisfying $BP_a = CP_b = AP_c = t$. By Ceva's theorem,

$$\frac{t}{a-t} \cdot \frac{t}{b-t} \cdot \frac{t}{c-t} = 1.$$

The solutions of this equation are the roots of the polynomial

$$f(t) := 2t^3 - (a+b+c)t^2 + (ab+bc+ca)t - abc.$$

Since $f'(t) = 6t^2 - 2(a+b+c)t + (ab+bc+ca) > 0$, $f(t)$ is increasing in t. It has a unique positive root $\mu < \min(a,b,c)$ since $f(0) < 0$ and $f(a)$, $f(b)$, $f(c)$ are all positive.

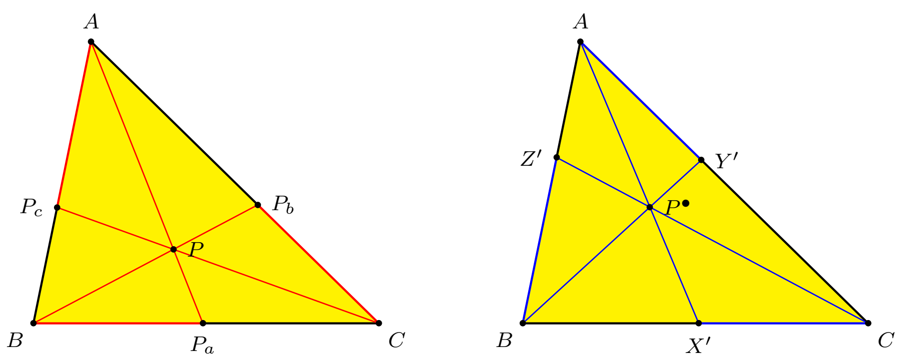


Figure 5.3: The Crelle-Yff points

From this, we have

$$\begin{aligned} BP_a : P_aC &= \mu : a-\mu \\ &= (\mu^3)^{\frac{1}{3}} : ((a-\mu)^3)^{\frac{1}{3}} \\ &= ((a-\mu)(b-\mu)(c-\mu))^{\frac{1}{3}} : ((a-\mu)^3)^{\frac{1}{3}} \\ &= ((b-\mu)(c-\mu))^{\frac{1}{3}} : (a-\mu)^{\frac{2}{3}} \\ &= \left(\frac{b-\mu}{a-\mu}\right)^{\frac{1}{3}} : \left(\frac{a-\mu}{c-\mu}\right)^{\frac{1}{3}}, \end{aligned}$$

and analogous expressions for $CP_b : P_bA$ and $AP_c : P_cB$. Hence, the coordinates of P:

$$\begin{array}{rcccccc}
P_a & = & 0 & : & \left(\frac{a-\mu}{c-\mu}\right)^{\frac{1}{3}} & : & \left(\frac{b-\mu}{a-\mu}\right)^{\frac{1}{3}}, \\
P_b & = & \left(\frac{c-\mu}{b-\mu}\right)^{\frac{1}{3}} & : & 0 & : & \left(\frac{b-\mu}{a-\mu}\right)^{\frac{1}{3}}, \\
P_c & = & \left(\frac{c-\mu}{b-\mu}\right)^{\frac{1}{3}} & : & \left(\frac{a-\mu}{c-\mu}\right)^{\frac{1}{3}} & : & 0; \\
\hline
P & = & \left(\frac{c-\mu}{b-\mu}\right)^{\frac{1}{3}} & : & \left(\frac{a-\mu}{c-\mu}\right)^{\frac{1}{3}} & : & \left(\frac{b-\mu}{a-\mu}\right)^{\frac{1}{3}}.
\end{array}$$

The isotomic conjugate

$$P^{\bullet} = \left(\left(\frac{b-\mu}{c-\mu}\right)^{\frac{1}{3}} : \left(\frac{c-\mu}{a-\mu}\right)^{\frac{1}{3}} : \left(\frac{a-\mu}{b-\mu}\right)^{\frac{1}{3}}\right)$$

has traces X', Y', Z' that satisfy $X'C = Y'A = Z'B = \mu$. These points are called the *Crelle-Yff points*. [1] They were briefly considered by A. L. Crelle. [2]

[1] P. Yff, An analogue of the Brocard points, *Amer. Math. Monthly*, 70 (1963) 495 – 501.
[2] A. L. Crelle, 1815.

5.2 Isogonal conjugates

Two cevian lines ℓ and ℓ' through A are isogonal in angle A if $\angle(\mathsf{c},\ \ell) = \angle(\ell',\ \mathsf{b})$.

Lemma. Let ℓ and ℓ' be isogonal lines through A intersecting the sideline BC at X and X' respectively. If $X = (0 : y : z)$ in homogeneous barycentric coordinates, then $X' = \left(0 : \frac{b^2}{y} : \frac{c^2}{z}\right)$.

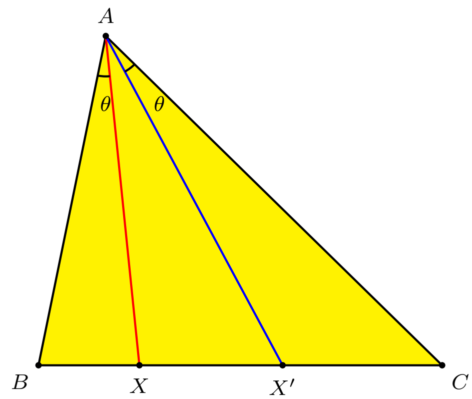


Figure 5.4:

Proof. Let $\angle(\mathsf{c}, AX) = \angle(AX', \mathsf{b}) = \theta$. By Conway's formula,

$$\begin{aligned} X &= (0 : S_\alpha - S_\theta : -c^2), \\ X' &= (0 : -b^2 : S_\alpha - S_\theta). \end{aligned}$$

Therefore, if $X = (0 : y : z)$ and $X' = (0 : y' : z')$, then

$$\frac{y}{z} \cdot \frac{y'}{z'} = \frac{S_\alpha - S_\theta}{-c^2} \cdot \frac{-b^2}{S_\alpha - S_\theta} = \frac{b^2}{c^2} \implies y' : z' = \frac{b^2}{y} : \frac{c^2}{z}.$$

$\square$

An application of this lemma shows that for a point $P = (x : y : z)$, the isogonal lines of AP, BP, CP (in the respective angles) are concurrent at a point P^* which we call the *isogonal conjugate* of P. If $P = (x : y : z)$, and if the isogonal lines of AP, BP, CP intersecting the sidelines BC, CA, AB at X', Y', Z' respectively, then

$$\begin{array}{rcccccc} X' & = & 0 & : & \frac{b^2}{y} & : & \frac{c^2}{z}, \\ Y' & = & \frac{a^2}{x} & : & 0 & : & \frac{c^2}{z}, \\ Z' & = & \frac{a^2}{x} & : & \frac{b^2}{y} & : & 0; \\ \hline P^* & = & \frac{a^2}{x} & : & \frac{b^2}{y} & : & \frac{c^2}{z}. \end{array}$$

Examples. (1) The incenter is the isogonal conjugate of itself: $\mathtt{I}^* = \mathtt{I}$; so is each of the excenters.

(2) The circumcenter and the orthocenter H are isogonal conjugates, since at each vertex, the altitude and the circumradius are isogonal in the corresponding angle:

$$\angle(AB, A\mathtt{H}) = \frac{\pi}{2} - \beta = \angle(A\mathtt{O}, AC)$$

etc.

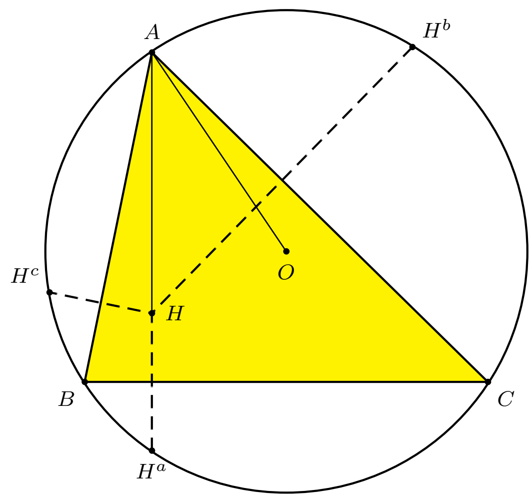
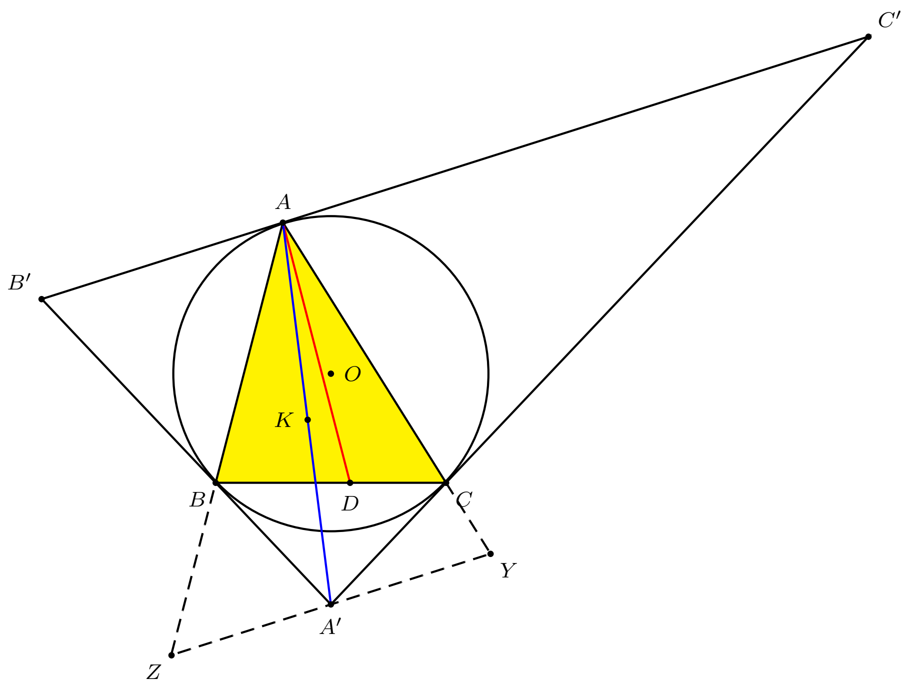


Figure 5.5:

Since $\mathtt{OO}^a = A\mathtt{H}$, $A\mathtt{OO}^a\mathtt{H}$ is a parallelogram, and $\mathtt{HO}^a = A\mathtt{O}$. This means that the circle of reflections of $\mathtt{O}$ is congruent to the circumcircle. Therefore, the circle of reflections of $\mathtt{H}$ is the circumcircle, and the reflections of $\mathtt{H}$ lie on the circumcircle.

5.2.1 The symmedian point and the centroid

Consider triangle ABC together with its *tangential triangle* $A'B'C'$, the triangle bounded by the tangents of the circumcircle at the vertices. A simple application of the Ceva theorem shows that the lines AA', BB', CC' are concurrent at a point K. [3] We call this point K the *symmedian point* of triangle ABC for the reason below.



Since A' is equidistant from B and C, we construct the circle $A'(B) = A'(C)$ and extend the sides AB and AC to meet this circle again at Z and Y respectively. Note that

$$\angle(A'Y, A'B') = \pi - 2(\pi - \alpha - \gamma) = \pi - 2\beta,$$

and similarly, $\angle(A'C', A'Z') = \pi - 2\gamma$. Since $\angle(A'B', A'C') = \pi - 2\alpha$, we have

$$\begin{aligned}
\angle(A'Y, A'Z) = &\ \angle(A'Y, A'B') + \angle(A'B', A'C') + \angle(A'C', A'Z) \\
= &\ (\pi - 2\beta) + (\pi - 2\alpha) + (\pi - 2\gamma) \\
= &\ \pi \\
\equiv &\ 0 \bmod \pi.
\end{aligned}$$

[3] If triangle ABC is acute, this is the Gergonne point of $A'B'C'$.

This shows that Y, A' and Z' are collinear, so that
(i) AA' is a median of triangle AYZ,
(ii) AYZ and ABC are similar.
It follows that AA' is the isogonal line of the A-median. We say that it is a *symmedian* of triangle ABC. Similarly, the BB' and CC' are the symmedians isogonal to B- and C-medians. The lines AA', BB', CC' therefore intersect at the isogonal conjugate of the centroid $\mathtt{G}$. This we call the *symmedian point* $\mathtt{K}$ of triangle ABC.

5.2.2 The tangential triangle $\mathsf{cev}^{-1}(\mathtt{K})$

The circle with diameter $\mathtt{I}_a\mathtt{I}'_a$ is the A-Apollonian circle (containing points whose distances from B and C are in the ratio $c : b$). The center of the circle is the midpoint of $\mathtt{I}_a\mathtt{I}'_a$, namely, $X' = (0 : b^2 : -c^2)$. [4] We claim that AX' is tangent to the circumcircle at A. For this, it is enough to show that $\angle(AX',\ \mathsf{c}) = \angle(\mathsf{b},\ \mathsf{a})$.

$$\begin{aligned}\angle(AX',\ \mathsf{c}) &= \angle(AX',\ A\mathtt{I}_a) + \angle(A\mathtt{I}_a, \mathsf{c})\\ &= \angle(A\mathtt{I}_a,\ \mathsf{a}) + \angle(\mathsf{b},\ A\mathtt{I}_a)\\ &= \angle(\mathsf{b},\ \mathsf{a}).\end{aligned}$$

Similarly, if Y' and Z' are the midpoints of $\mathtt{I}_b\mathtt{I}'_b$ and $\mathtt{I}_c\mathtt{I}'_c$ respectively, the lines BY' and CZ' are tangents to the circumcircle. The lines AX', BY', CZ' bound the tangential triangle of $\mathbf{T}$.

The harmonic conjugates of X' in BC, Y' in CA, Z' in AB are the traces of a point $\mathtt{K}$ with coordinates $(a^2 : b^2 : c^2)$. This is called the symmedian point $\mathtt{K}$ of $\mathbf{T}$.

$$\begin{array}{ccccccc} X & = & 0 & : & b^2 & : & c^2,\\ Y & = & a^2 & : & 0 & : & c^2,\\ Z & = & a^2 & : & b^2 & : & 0,\\ \hline \mathtt{K} & = & a^2 & : & b^2 & : & c^2. \end{array}$$

The tangential triangle is therefore the anticevian triangle $\mathsf{cev}^{-1}(\mathtt{K})$ and

[4]Proof: $\frac{1}{2}(\mathtt{I}_a + \mathtt{I}'_a) = \frac{1}{2}\left(\frac{(0,b,c)}{b+c} + \frac{(0,b,-c)}{b-c}\right) \approx (0, (b-c)b + (b+c)b, (b-c)c + (b+c)(-c)) = (0, 2b^2, -2c^2) \approx (b^2, -c^2)$.

has vertices

$$\mathtt{K}^a = (-a^2 : b^2 : c^2), \quad \mathtt{K}^b = (a^2 : -b^2 : c^2), \quad \mathtt{K}^c = (a^2 : b^2 : -c^2).$$

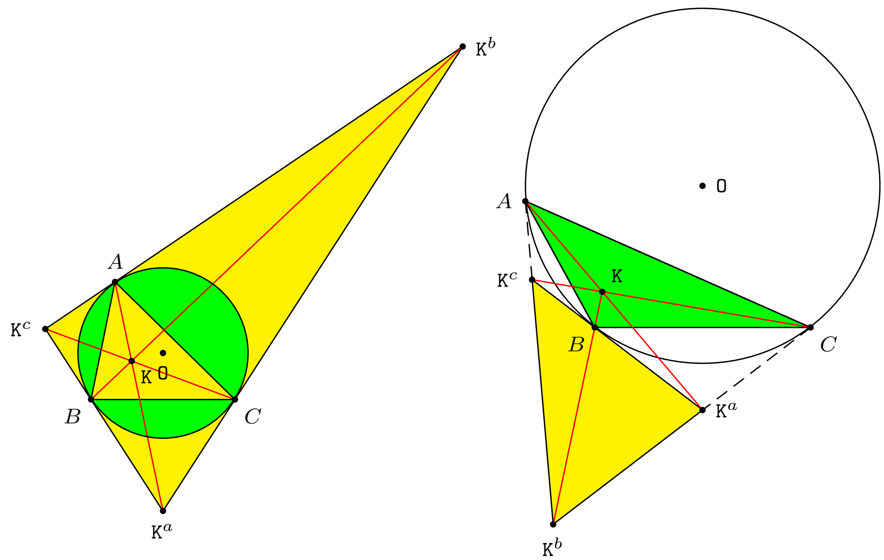


Figure 5.6: The tangential triangle

The circumcircle of $\mathbf{T}$ is the incircle of the tangential triangle, provided that $\mathbf{T}$ is acute. Note that the tangential triangle degenerates when $\mathbf{T}$ contains a right angle. When $\mathbf{T}$ is obtuse, the circumcircle of $\mathbf{T}$ is no longer the incircle, but the excircle on the opposite side of the obtuse angle. We describe this situation by saying that the circumcircle of $\mathbf{T}$ is the *ob-incircle* of triangle $\mathbf{T}$.

Remark. If the A-Apollonian circle intersects the circumcircle again at X'', then AX'' is the A-symmedian. It follows that the symmedian point $\mathtt{K}$ has equal powers with respect to the three Apollonian circles (which is the *negative* power of $\mathtt{K}$ in the circumcircle). On the other hand, the circumcenter $\mathtt{O}$ also has equal powers (R^2) with respect to the Apollonian circles. Therefore, the three Apollonian circles are coaxial, with two real common points on the line $\mathtt{OK}$. These are called the *isodynamic points* $\mathtt{J}_{\pm}$. These are the points satisfying

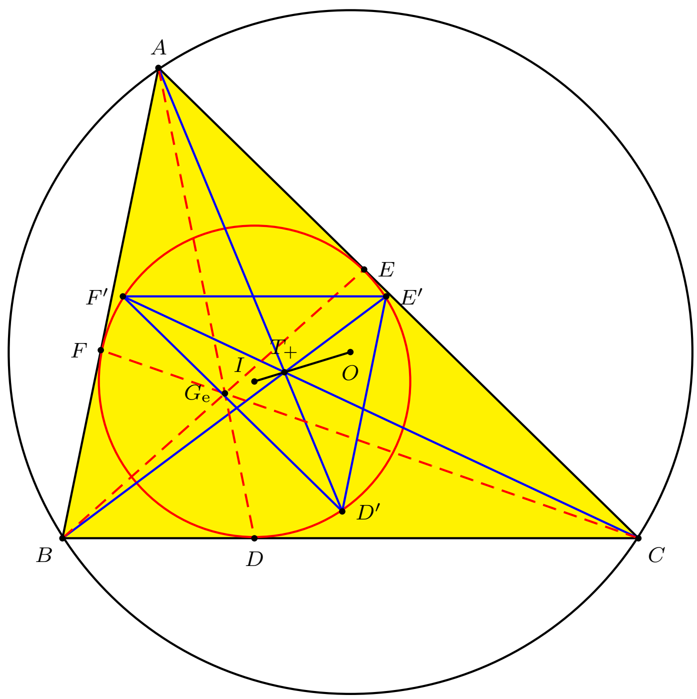
$$AJ_\varepsilon : BJ_\varepsilon : CJ_\varepsilon = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$$

for $\varepsilon = \pm 1$.

5.2.3 The Gergonne point and the insimilicenter T_+

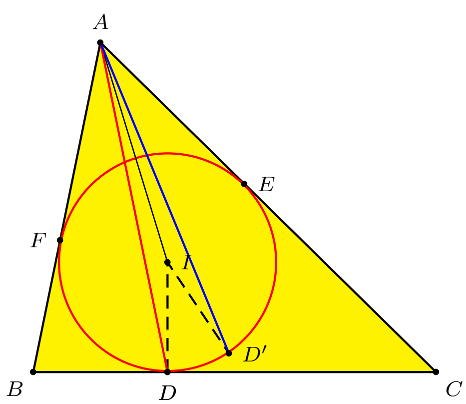
Theorem. The isogonal conjugate of the Gergonne point is the insimilicenter of the circumcircle and the incircle:

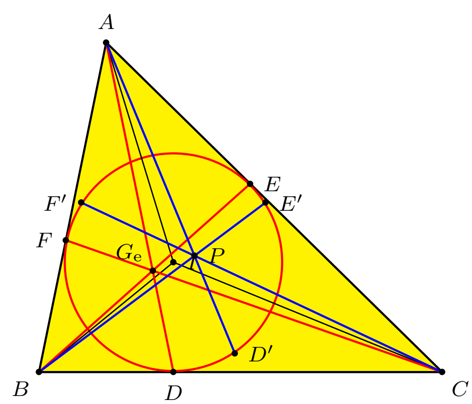
$$\boxed{T_+ = G_e^*.}$$



Proof. Consider the intouch triangle DEF of triangle ABC.

(1) If D' is the reflection of D in the bisector AI, then
(i) D' is a point on the incircle, and
(ii) the lines AD and AD' are isogonal with respect to A.

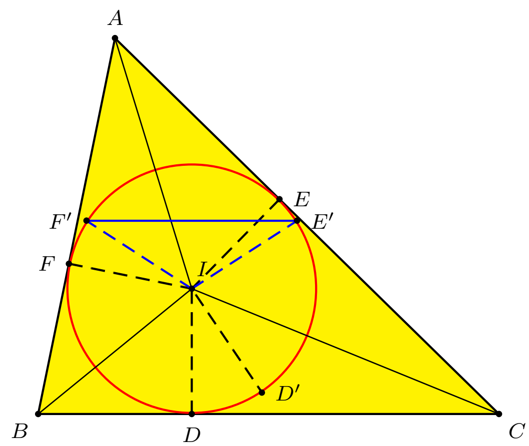




(2) Likewise, E' and F' are the reflections of E and F in the bisectors BI and CI respectively, then

(i) these are points on the incircle,
(ii) the lines BE' and CF' are isogonals of BE and CF with respect to angles B and C.

Therefore, the lines AD', BE', and CF' concur at the isogonal conjugate of the Gergonne point.



(3) In fact, $E'F'$ is parallel to BC. This follows from

$$\begin{aligned}
(ID,\ IE') &= (ID,\ IE) + (IE,\ IE') \\
&= (ID,\ IE) + 2(IE,\ IB) \\
&= (ID,\ IE) + 2((IE,\ AC) + (AC, IB)) \\
&= (ID,\ IE) + 2(AC,\ IB) \qquad \text{since } (IE, AC) = \frac{\pi}{2} \\
&= (\pi - C) + 2\left(C + \frac{B}{2}\right) \\
&= B + C = -A \pmod{\pi}
\end{aligned}$$

Similarly,

$$\begin{aligned}
(ID,\ IF') &= (ID,\ IF) + (IF,\ IF') \\
&= (ID,\ IF) + 2(IF,\ IC) \\
&= (ID,\ IF) + 2((IF,\ AB) + (AB, IC)) \\
&= (ID,\ IF) + 2(AB,\ IC) \qquad \text{since } (IF, AB) = \frac{\pi}{2} \\
&= -(\pi - B) - 2\left(B + \frac{C}{2}\right) \\
&= -(B + C) = A \pmod{\pi}
\end{aligned}$$

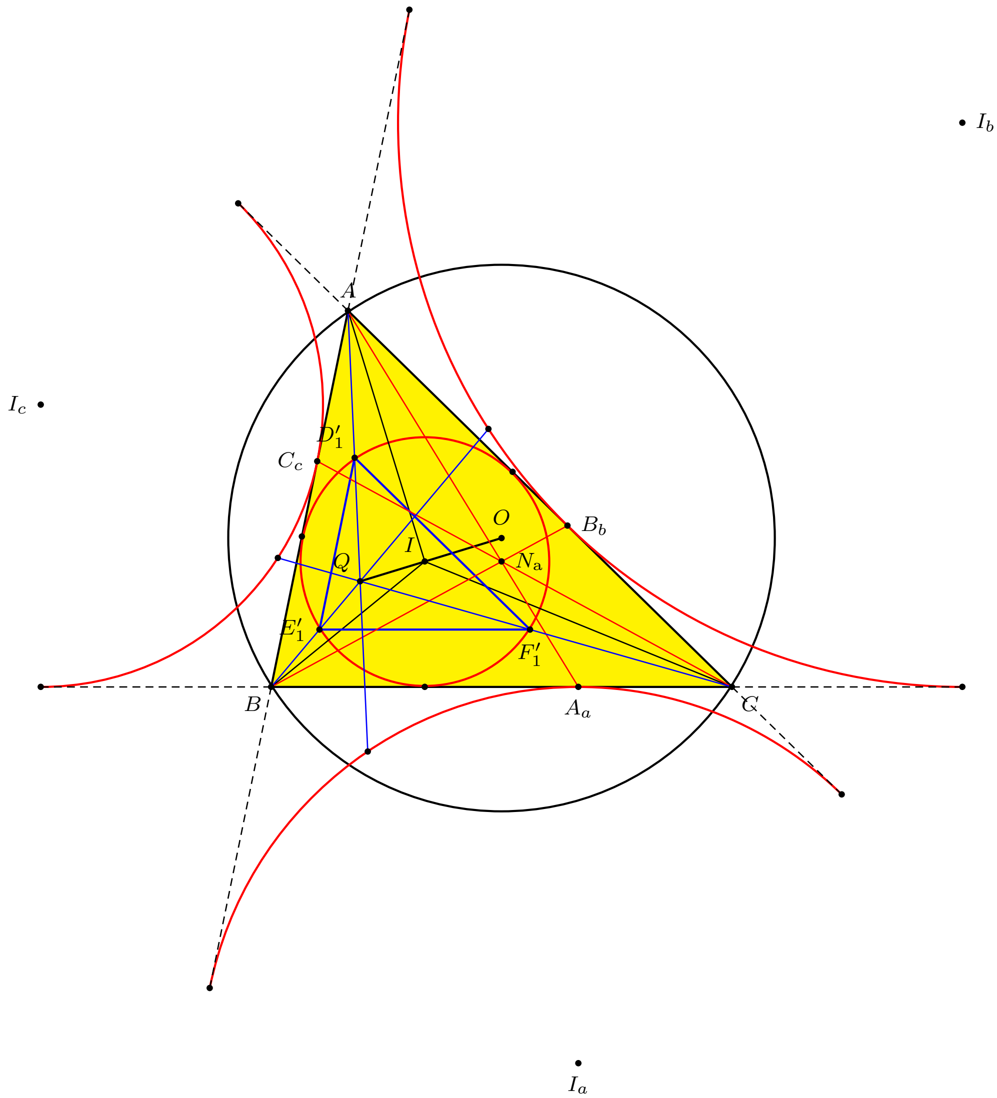
(4) Similarly, $F'D'$ and $D'E'$ are parallel to CA and AB respectively. It follows that $D'E'F'$ is homothetic to ABC.

The ratio of homothety is $r : R$. Therefore, the center of homothety is the point which divides OI in the ratio $R : r$. This is the insimilicenter of (O) and (I). $\square$

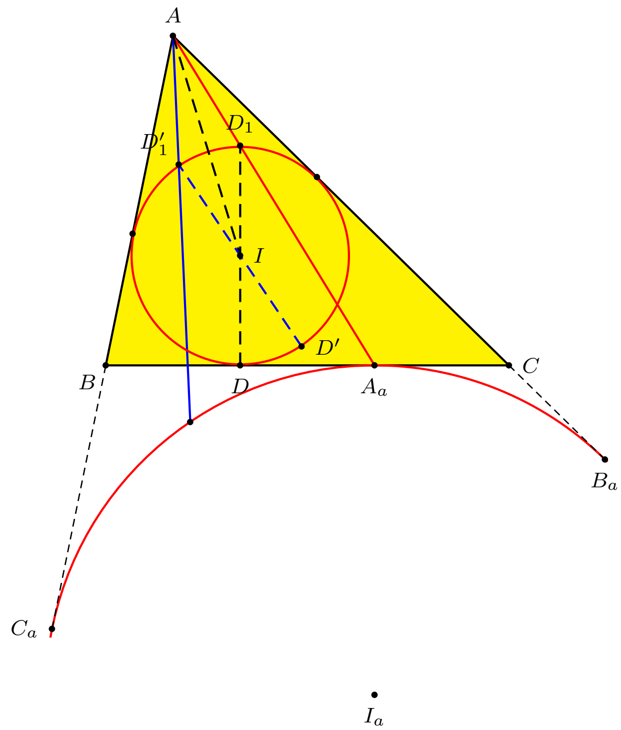
5.2.4 The Nagel point and the exsimilicenter T_-

Theorem. The isogonal conjugate of the Nagel point is the exsimilicenter of the circumcircle and the incircle:

$$\boxed{T_- = N_a^*.}$$



Proof. (1) Consider the A-cevian of the Nagel point, which joins the vertex A to the point of tangency A_a of the excircle with BC. This contains the antipode D_1 on the incircle of the point of tangency D with BC. Therefore, the reflection of AA_a in the bisector AI contains the reflection D_1' of D_1. D_1' is clearly on the incircle. Indeed, it is the antipode of D', the reflection of D in AI.



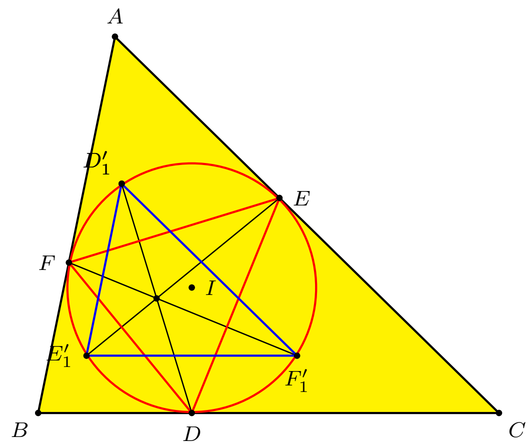
(2) Likewise, if we consider the isogonal lines of the BB_b and CC_c in the respective angle bisectors, these contains E_1' and F_1' which are the antipodes of E' and F' on the incircle.

The lines AD_1', BE_1', and CF_1' concur at the isogonal conjugate of the Nagel point.

Clearly, $D_1'E_1'F_1'$ and $D'E'F'$ are oppositely congruent at the O. Since $D'E'F'$ is homothetic to ABC, so are $D_1'E_1'F_1'$ and ABC, with ratio of homothety $-r : R$. The center of homothety is the point which divides OI in the ratio $R : -r$. This is the exsimilicenter of (O) and (I). $\square$

Exercise

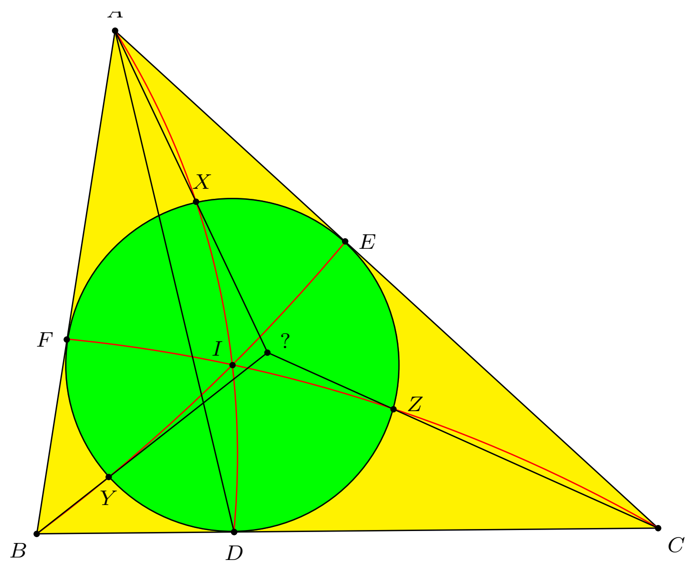
1. The triangles $D_1'E_1'F_1'$ is perspective with the intouch triangle DEF. The perspector is the orthocenter of DEF. (Hint: Show that DD_1' is parallel to the bisector IA).



2. Let DEF be the intouch triangle of ABC. The circumcircles of AID, BIE, CIF intersect the incircle again at X, Y, Z respectively.

(a) Prove that the lines AX, BY, CZ are concurrent and identify the point of concurrence. [5]

(b) Show that DX, EY, CZ are also concurrent, and identify the point of concurrence. [6]



[5] Problem 1864, *Math. Mag.*, 84 (2011) 64. Solution: Since IX and ID are equal chords in the circle AID, $\angle DAI = \angle XAI$. The lines AX and AD are isogonal with respect to angle A. Similarly, BY and BE are isogonal, so are CZ and C. The lines AX, BY, CZ concur at the isogonal conjugate of G_{e}, which is T_{+}.

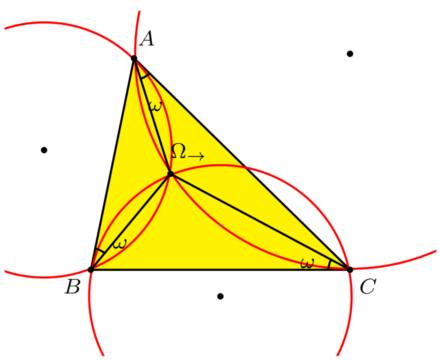
[6] The centroid of the intouch triangle.

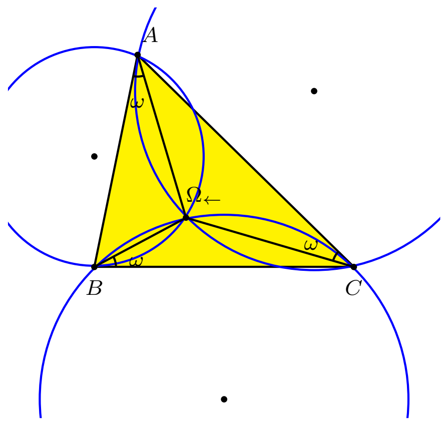
5.2.5 The Brocard points

Given triangle ABC, there is an interior point $\Omega_{\rightarrow}$ satisfying

$$\angle(A\Omega_{\rightarrow},\ \mathsf{b}) = \angle(B\Omega_{\rightarrow},\ \mathsf{c}) = \angle(C\Omega_{\rightarrow},\ \mathsf{a}).$$

It can be constructed as the common point of the three circles:
$\mathcal{C}_{AAB}$: through B, tangent to CA at A,
$\mathcal{C}_{BBC}$: through C, tangent to AB at B, and
$\mathcal{C}_{CCA}$: through A, tangent to BC at C.





The center of the circle $\mathcal{C}_{AAB}$, for example, is the intersection of the perpendicular bisector of AB and the perpendicular to CA at A. This circle has radius $\frac{c}{\sin\alpha}$; similarly for the other two circles. If we denote the common angle by ω, then

$$A\Omega_{\rightarrow} = \frac{c}{\sin\alpha}\cdot\sin\omega, \quad B\Omega_{\rightarrow} = \frac{a}{\sin\beta}\cdot\sin\omega, \quad C\Omega_{\rightarrow} = \frac{b}{\sin\gamma}\cdot\sin\omega.$$

From these we easily obtain in homogeneous barycentric coordinates

$$\begin{aligned}
\Omega_{\rightarrow} &= (\Delta\Omega_{\rightarrow}BC:\ \Delta\Omega_{\rightarrow}CA:\Delta\Omega_{\rightarrow}AB)\\
&= \left(\frac{1}{2}a\cdot\frac{b}{\sin\gamma}\cdot\sin^2\omega:\ \frac{1}{2}b\cdot\frac{c}{\sin\alpha}\cdot\sin^2\omega:\ \frac{1}{2}c\cdot\frac{a}{\sin\beta}\cdot\sin^2\omega\right)\\
&= \left(\frac{1}{c^2}:\frac{1}{a^2}:\frac{1}{b^2}\right).
\end{aligned}$$

Similarly, there is another interior point $\Omega_{\leftarrow}$ satisfying

$$\angle(\mathsf{c},\ A\Omega_{\leftarrow}) = \angle(\mathsf{a},\ B\Omega_{\leftarrow}) = \angle(\mathsf{b},\ C\Omega_{\leftarrow}).$$

This is the common point of the three circles:
$\mathcal{C}_{ABB}$: through A, tangent to BC at B,
$\mathcal{C}_{BCC}$: through B, tangent to CA at C, and
$\mathcal{C}_{CAA}$: through C, tangent to AB at A.

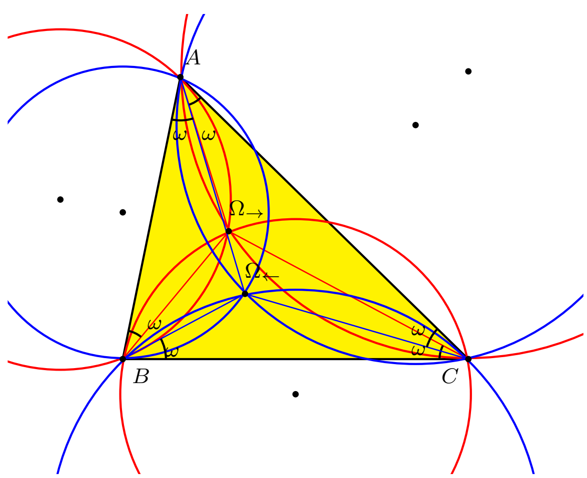
In homogeneous barycentric coordinates,

$$\Omega_{\leftarrow} = \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right).$$

From their coordinates, it is easy to see that $\Omega_{\leftarrow}$ and $\Omega_{\rightarrow}$ are isogonal conjugates. It follows that

$$\begin{aligned}\angle(\mathsf{c},\ A\Omega_{\leftarrow}) =&\ \angle(A\Omega_{\rightarrow},\ \mathsf{b}) = \omega,\\ \angle(\mathsf{a},\ B\Omega_{\leftarrow}) =&\ \angle(B\Omega_{\rightarrow},\ \mathsf{c}) = \omega,\\ \angle(\mathsf{b},\ C\Omega_{\leftarrow}) =&\ \angle(C\Omega_{\rightarrow},\ \mathsf{a}) = \omega.\end{aligned}$$

The points $\Omega_{\leftarrow}$ and $\Omega_{\rightarrow}$ are called the Brocard points, and ω the Brocard angle of triangle ABC.



Proposition. *(a)* $\csc^2\omega = \csc^2\alpha + \csc^2\beta + \csc^2\gamma$.
(b) $\cot\omega = \cot\alpha + \cot\beta + \cot\gamma$.

Proof. (a) Since the area of triangle ABC is the sum of the areas of $CA\Omega_{\rightarrow}$,

$AB\Omega_{\rightarrow}$, and $BC\Omega_{\rightarrow}$, we have

$$\begin{aligned} S = & \ (b \cdot A\Omega_{\rightarrow} + c \cdot B\Omega_{\rightarrow} + a \cdot C\Omega_{\rightarrow}) \sin\omega \\ = & \ \left(\frac{bc}{\sin\alpha} + \frac{ca}{\sin\beta} + \frac{ab}{\sin\gamma}\right) \sin^2\omega \\ = & \ S\left(\frac{1}{\sin^2\alpha} + \frac{1}{\sin^2\beta} + \frac{1}{\sin^2\gamma}\right) \sin^2\omega. \end{aligned}$$

From this we obtain

$$\csc^2\omega = \csc^2\alpha + \csc^2\beta + \csc^2\gamma.$$

(b) Now,

$$\begin{aligned} \cot^2\omega = & \ \csc^2\omega - 1 \\ = & \ \csc^2\alpha + \csc^2\beta + \csc^2\gamma - 1 \\ = & \ \cot^2\alpha + \cot^2\beta + \cot^2\gamma + 2 \\ = & \ \cot^2\alpha + \cot^2\beta + \cot^2\gamma + 2(\cot\alpha\cot\beta + \cot\beta + \cot\gamma + \cot\gamma\cot\alpha) \\ = & \ (\cot\alpha + \cot\beta + \cot\gamma)^2. \end{aligned}$$

Since ω is an acute angle, we have

$$\cot\omega = \cot\alpha + \cot\beta + \cot\gamma.$$

$\square$

Corollary. $S_\omega = S_\alpha + S_\beta + S_\gamma = \frac{a^2+b^2+c^2}{2}$.

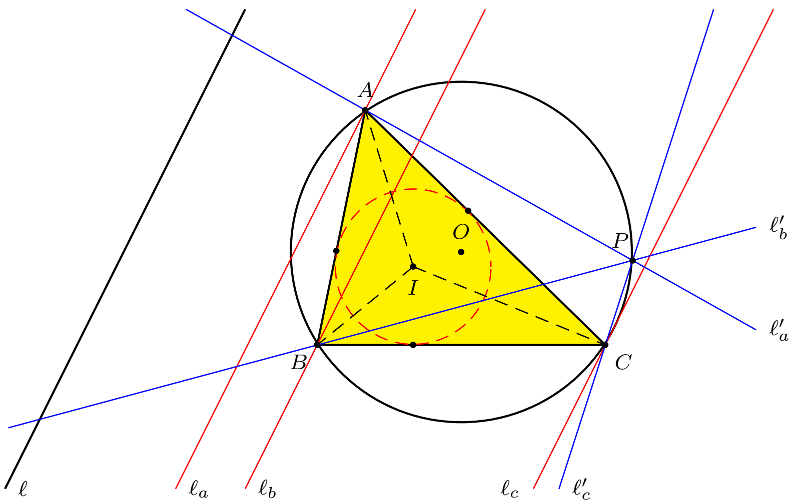
Exercise

1. Let XYZ be the intouch triangle and ℓ a line through the incenter I of triangle ABC. Construct the pedals X' of A, Y' of B, and Z' of C on ℓ. The lines XX', YY', ZZ' concur at a point P. The locus of the isogonal conjugate of P with respect to XYZ is the nine-point circle of XYZ.

2. Let XYZ be the intouch triangle and ℓ a line through the incenter I of triangle ABC. Construct the pedals X' of X, Y' of Y, and Z' of Z on ℓ. The lines AX', BY', CZ' concur at a point P. The locus of the isogonal conjugate of P with respect to ABC is a circle.

5.3 Isogonal conjugate of an infinite point

Proposition. Given a triangle ABC and a line ℓ, let ℓ_a, ℓ_b, ℓ_c be the parallels to ℓ through A, B, C respectively, and ℓ_a', ℓ_b', ℓ_c' their reflections in the angle bisectors AI, BI, CI respectively. The lines ℓ_a', ℓ_b', ℓ_c' intersect at a point on the circumcircle of triangle ABC.



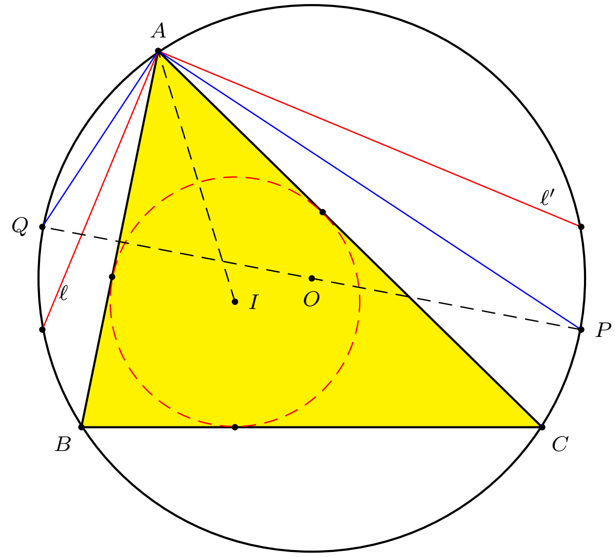
Proof. Let P be the intersection of ℓ_b' and ℓ_c'.

$$\begin{aligned}(BP,\ PC) =& (\ell_b',\ \ell_c')\\ =&(\ell_b',\ IB) + (IB,\ IC) + (IC,\ \ell_c')\\ =&(IB,\ \ell_b) + (IB,\ IC) + (\ell_c,\ IC)\\ =&(IB,\ \ell) + (IB,\ IC) + (\ell,\ IC)\\ =&2(IB,\ IC)\\ =&2\left(\frac{\pi}{2} + \frac{A}{2}\right)\\ =&(BA,\ AC) \pmod{\pi}.\end{aligned}$$

Therefore, ℓ_b' and ℓ_c' intersect at a point on the circumcircle of triangle ABC.

Similarly, ℓ_a' and ℓ_b' intersect at a point P' on the circumcircle. Clearly, P and P' are the same point since they are both on the reflection of ℓ_b in the bisector IB. Therefore, the three reflections ℓ_a', ℓ_b', and ℓ_c' intersect at the same point on the circumcircle. $\square$

Proposition. The isogonal conjugates of the infinite points of two perpendicular lines are antipodal points on the circumcircle.



Proof. If P and Q are the isogonal conjugates of the infinite points of two perpendicular lines ℓ and ℓ' through A, then AP and AQ are the reflections of ℓ and ℓ' in the bisector AI.

$$(AP,\ AQ) = (AP,\ IA) + (IA,\ AQ) = -(\ell,\ IA) - (IA,\ \ell') = -(\ell,\ \ell') = \frac{\pi}{2}.$$

Therefore, P and Q are antipodal points. $\square$

5.3.1 Homogeneous barycentric equation of the circumcircle

The interesting fact of Proposition 5.3 leads to the very simple equation of the circumcircle.

Theorem. A point with homogeneous barycentric coordinates $(x : y : z)$ lies on the circumcircle of triangle ABC if and only if

$$a^2yz + b^2zx + c^2xy = 0.$$

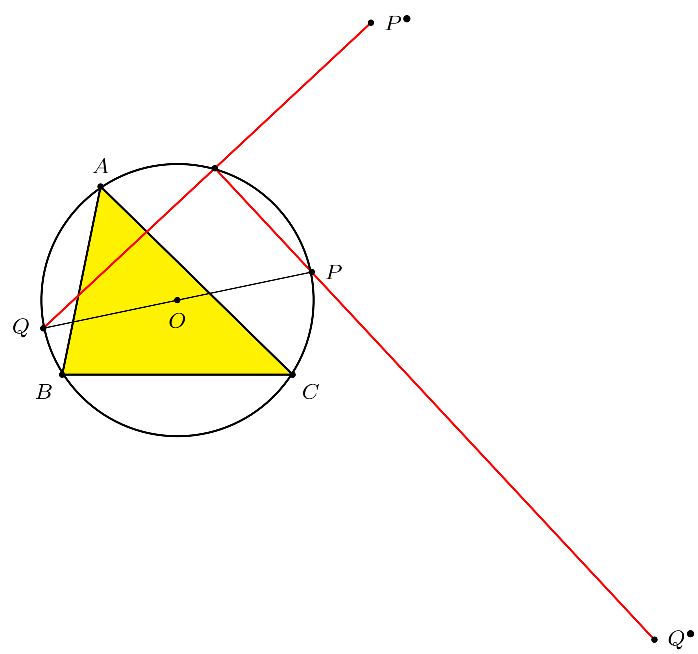
Corollary. If $P = \left(\frac{a^2}{f} : \frac{b^2}{g} : \frac{c^2}{h}\right)$ for an infinite point $(f : g : h)$, its antipodal point on the circumcircle is the point

$$\left(\frac{a^2}{S_\beta g - S_\gamma h} : \frac{b^2}{S_\gamma h - S_\alpha a} : \frac{c^2}{S_\alpha f - S_\beta g}\right).$$

Exercise

1. Find the locus of the isotomic conjugates of points on the circumcircle. [7]

2. Let P and Q be antipodal points on the circumcircle. The lines $PQ^\bullet$ and $QP^\bullet$ joining each of these points to the isotomic conjugate of the other intersect orthogonally on the circumcircle.



3. A transversal ℓ cuts the sidelines BC, CA, AB of triangle ABC at X, Y, Z respectively. The parallels to ℓ through A, B, C intersect the circumcircle at X', Y', Z' respectively. Show that XX', YY', ZZ' intersect on the circumcircle. [8]

---

[7]The line $a^2x + b^2y + c^2z = 0$ perpendicular to the Euler line.

[8]If $\ell : ux + vy + wz = 0$, then the intersection is $\left(\frac{a^2}{u(v-w)} : \frac{b^2}{v(w-u)} : \frac{c^2}{w(u-v)}\right)$ on the circumcircle, the isogonal conjugate of the infinite point of the isotomic line of ℓ.

5.4 The isotomic conjugates of infinite points

The isotomic conjugate of an infinite point cannot be an infinite point.

If $P = (u : v : w)$ is an infinite point, then so is $Q = (v - w : w - u : u - v)$. The isotomic conjugates of these two infinite points are symmetric with respect to $\mathtt{G}$.

Proof.

$$\begin{aligned} Q^\bullet &= ((w-u)(u-v), (u-v)(v-w), (v-w)(w-u)) \\ &= (-u^2 + u(v+w) - vw, \cdots, \cdots) \\ &= (2u(v+w) - vw, \cdots, \cdots) \\ &= (2(vw+wu+uv) - 3vw, \cdots, \cdots) \\ &= 2(vw+wu+uv)(1,1,1) - 3(vw, wu, uv) \\ &= 3(vw+wu+uv)(2\mathtt{G} - P^\bullet). \end{aligned}$$

is the symmetric of $(vw : wu : uv)$ in $G = (1 : 1 : 1)$. □

The Steiner circum-ellipse and the circumcircle intersect at the Steiner point

$$\mathtt{S_t} := \left(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2}\right).$$

A line through the Steiner point intersects the circumcircle and the Steiner circum-ellipse again at the isogonal and isotomic conjugates of the same infinite point.

The line joining the isogonal and isotomic conjugates of a point $P(u : v : w)$ has equation

$$(b^2-c^2)ux + (c^2-a^2)vy + (a^2-b^2)wz = 0.$$

This contains the Steiner point if and only if $u + v + w = 0$, i.e., P is an infinite point.

Chapter 6

Some basic constructions

6.1 Perspective triangles

Many interesting points and lines in triangle geometry arise from the *perspectivity* of triangles. We say that two triangles $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are perspective, $X_1Y_1Z_1 \barwedge X_2Y_2Z_2$, if the lines X_1X_2, Y_1Y_2, Z_1Z_2 are concurrent. The point of concurrency, $\bigwedge(X_1Y_1Z_1, X_2Y_2Z_2)$, is called the *perspector*. If one of the triangle is $\mathbf{T}$, we shall simply write $\bigwedge(XYZ)$ for $\bigwedge(\mathbf{T},\ XYZ)$. Along with the perspector, there is an *axis of perspectivity*, or the *perspectrix*, which is the line joining containing

$$Y_1Z_2 \cap Z_1Y_2, \quad Z_1X_2 \cap X_1Z_2, \quad X_1Y_2 \cap Y_1X_2.$$

We denote this line by $\mathscr{L}_\wedge(X_1Y_1Z_1, X_2Y_2Z_2)$.

Homothetic triangles are clearly prespective. If triangles $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are homothetic, their perspector is the homothetic center, which we denote by $\bigwedge_0(X_1Y_1Z_1, X_2Y_2Z_2)$.

Proposition. A triangle with vertices

$$\begin{array}{ccccccc} X & = & U & : & v & : & w, \\ Y & = & u & : & V & : & w, \\ Z & = & u & : & v & : & W, \end{array}$$

for some U, V, W, is perspective to ABC at $\bigwedge(XYZ) = (u : v : w)$. The perspectrix is the line

$$\frac{x}{u-U} + \frac{y}{v-V} + \frac{z}{w-W} = 0.$$

Proof. The line AX has equation $wy - vz = 0$. It intersects the sideline BC at the point $(0 : v : w)$. Similarly, BY intersects CA at $(u : 0 : w)$ and CZ intersects AB at $(u : v : 0)$. These three are the traces of the point $(u : v : w)$.

The line YZ has equation $-(vw-VW)x+u(w-W)y+u(v-V)z = 0$. It intersects the sideline BC at $(0 : v-V : -(w-W))$. Similarly, the lines ZX and XY intersect CA and AB respectively at $(-(u-U) : 0 : w-W)$ and $(u-U : -(v-V) : 0)$. It is easy to see that these three points are collinear on the line

$$\frac{x}{u-U} + \frac{y}{v-V} + \frac{z}{w-W} = 0.$$

$\square$

Examples. **The Conway configuration**

Given triangle ABC, extend
(i) CA and BA to Y_a and Z_a such that $AY_a = AZ_a = a$,
(ii) AB and CB to Z_b and X_b such that $BZ_b = BX_b = b$,
(iii) BC and AC to X_c and Y_c such that $CX_c = CY_c = c$.

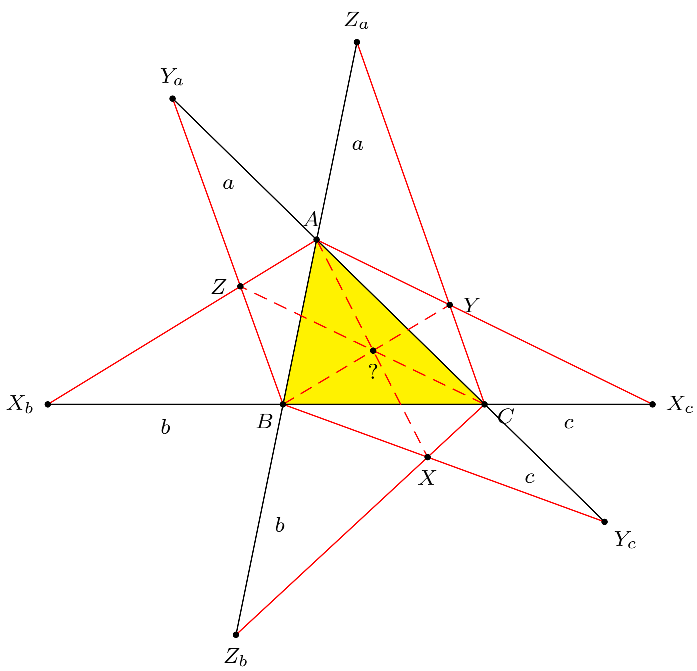


Figure 6.1: The Conway configuration

These points have coordinates

$$\begin{array}{ll} Y_a = (a+b:0:-a), & Z_a = (c+a:-a:0); \\ Z_b = (-b:b+c:0), & X_b = (0:a+b:-b); \\ X_c = (0:-c:c+a), & Y_c = (-c:0:b+c). \end{array}$$

From the coordinates of Y_c and Z_b, we determine easily the coordinates of $X = BY_c \cap CZ_b$:

$$\begin{array}{cccccccccccc} Y_c & = & -c & : & 0 & : & b+c & = & -bc & : & 0 & : & b(b+c) \\ Z_b & = & -b & : & b+c & : & 0 & = & -bc & : & c(b+c) & : & 0 \\ \hline X & = & & & & & & = & -bc & : & c(b+c) & : & b(b+c) \end{array}$$

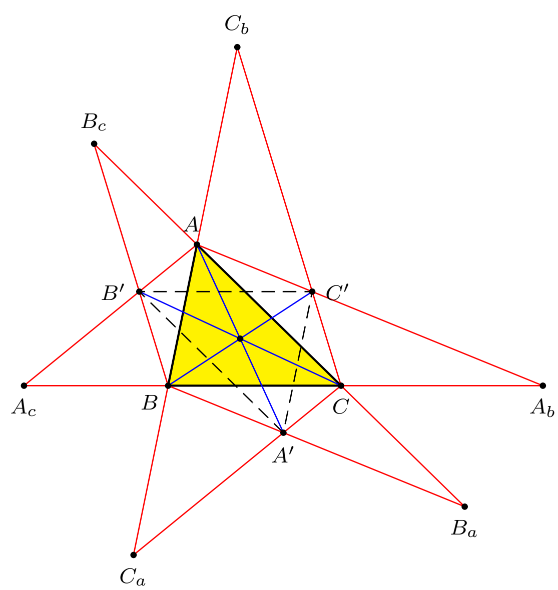
Similarly, the coordinates of $Y = CZ_a \cap AX_c$, and $Z = AX_b \cap BY_a$ can be determined. The following table shows that the perspector of triangles ABC and XYZ is the point with homogeneous barycentric coordinates $\left(\frac{1}{a} : \frac{1}{b} : \frac{1}{c}\right)$.

$$\begin{array}{ccccccccccccc}
X & = & -bc & : & c(b+c) & : & b(b+c) & = & \frac{-1}{b+c} & : & \frac{1}{b} & : & \frac{1}{c} \\
Y & = & c(c+a) & : & -ca & : & a(c+a) & = & \frac{1}{a} & : & \frac{-1}{c+a} & : & \frac{1}{c} \\
Z & = & b(a+b) & : & a(a+b) & : & -ab & = & \frac{1}{a} & : & \frac{1}{b} & : & \frac{-1}{a+b} \\
\hline
? & = & & & & & & = & \frac{1}{a} & : & \frac{1}{b} & : & \frac{1}{c}
\end{array}$$

Remark. The points X, Y, Z, together with the vertices of $\mathbf{T}$, lie on an ellipse with center G (the Steiner circum-ellipse).

Exercise

Given triangle ABC, extend the sides AC to B_a and AB to C_a such that $CB_a = BC_a = a$. Similarly define C_b, A_b, A_c, and B_c. Calculate the coordinates of the intersections A' of BB_a and CC_a, B' of CC_b and AA_b, C' of AA_c, BB_c. Show that AA', BB' and CC' are concurrent by identifying their common point. [1]



Exercise

Let $P = (u : v : w)$ and cev$(P) = P_aP_bP_c$ its cevian triangle. Prove that
(i) the inferior triangle of cev(P) is perspective with ABC,
(ii) if X, Y, Z are the midpoints of AP_a, BP_b, CP_c respectively, then XYZ is perspective with the inferior triangle of $\mathbf{T}$. [2]

[1] Spieker center.
[2] Answer: in each case, the perspector is $(u(v+w) : v(w+u) : w(u+v))$.

The perspectrix is the line $\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0$, the trilinear polar of P.

6.2 Jacobi's Theorem

Theorem (Jacobi)**.** Suppose X, Y, Z are points with swing angles

$$\angle CAY = \angle BAZ = \alpha,$$
$$\angle ABZ = \angle CBX = \beta,$$
$$\angle BCX = \angle ACY = \gamma.$$

The lines AX, BY, CZ are concurrent at the point

$$\left(\frac{1}{S_\alpha + S_\alpha} : \frac{1}{S_\beta + S_\beta} : \frac{1}{S_\gamma + S_\gamma}\right).$$

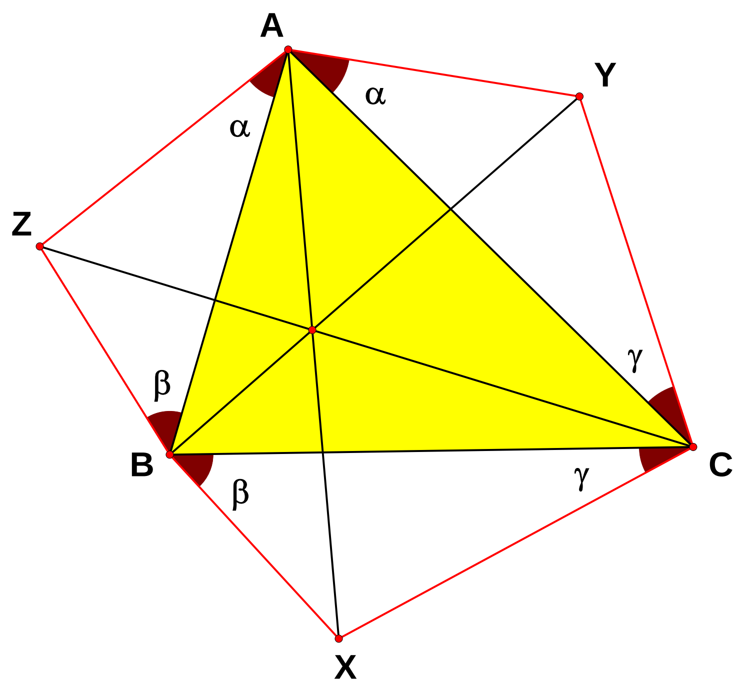


Figure 6.2: Jacobi's Theorem

Proof.

$$\begin{aligned}
X =& -a^2 : S_\gamma + S_\gamma : S_\beta + S_\beta \\
=& \frac{-a^2}{(S_\beta + S_\beta)(S_\gamma + S_\gamma)} : \frac{1}{S_\beta + S_\beta} : \frac{1}{S_\gamma + S_\gamma}, \\
Y =& \frac{1}{S_\alpha + S_\alpha} : \frac{-b^2}{(S_\gamma + S_\gamma)(S_\alpha + S_\alpha)} : \frac{1}{S_\gamma + S_\gamma}, \\
Z =& \frac{1}{S_\alpha + S_\alpha} : \frac{1}{S_\beta + S_\beta} : \frac{-c^2}{(S_\alpha + S_\alpha)(S_\beta + S_\beta)}.
\end{aligned}$$

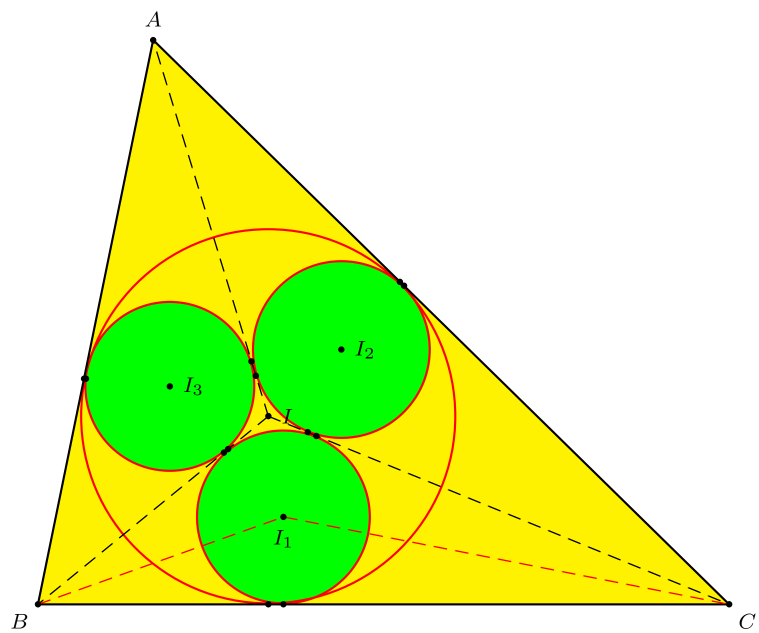
$\square$

Examples

(1) The *Morley center*. Let X, Y, Z be points such that AY, AZ trisect angle A, BZ, BX trisect angle B, and CX, CY trisect angle C. The famous *Morley's theorem* states that XYZ is an equilateral triangle. [3] Here, we note that XYZ is perspective with ABC. The perspector is the point

$$M_o = \left(\frac{1}{S_\alpha - S_{\frac{A}{3}}} : \frac{1}{S_\beta - S_{\frac{B}{3}}} : \frac{1}{S_\gamma - S_{\frac{C}{3}}}\right) = \left(\frac{a}{\cos\frac{A}{3}} : \frac{b}{\cos\frac{B}{3}} : \frac{c}{\cos\frac{C}{3}}\right).$$

(2) Consider triangle ABC with incenter I. Let I_1, I_2, I_3 be the incenters of triangles IBC, ICA, IAB respectively.



The homogeneous barycentric coordinates of I_1 can be easily written down:

$$I_1 = (-a^2 : S_C - S_{\frac{C}{4}} : S_B - S_{\frac{B}{4}}).$$

Now, for arbitrary θ, we have $\cot\frac{\theta}{2} - \cot 2\theta = \frac{1+2\cos\theta}{\sin 2\theta}$. Putting $\theta = \frac{B}{2}$ and $S = 2rs = \frac{4Rrs}{2R} = \frac{abc}{2R}$, we have

$$S_B - S_{\frac{B}{4}} = S\left(\cot B - \cot\frac{B}{4}\right) = -\frac{abc}{2R}\cdot\frac{1+2\cos\frac{B}{2}}{\sin B} = -ca\left(1+2\cos\frac{B}{2}\right).$$

[3]See § below.

Similarly, $S_C - S_{\frac{C}{4}} = -ab\left(1 + 2\cos\frac{C}{2}\right)$. It follows that

$$I_1 = \left(a : b\left(1 + 2\cos\frac{C}{2}\right) : c\left(1 + 2\cos\frac{B}{2}\right)\right).$$

The coordinates of the other two centers I_2 and I_3 can be written down by cyclic permutations of these coordinates. From these coordinates, we readily see that triangle $I_1I_2I_3$ is perspective with ABC at

$$\left(\frac{a}{1 + 2\cos\frac{A}{2}} : \frac{b}{1 + 2\cos\frac{B}{2}} : \frac{c}{1 + 2\cos\frac{C}{2}}\right).$$

Exercise

1. Let X', Y', Z' be respectively the pedals of X on BC, Y on CA, and Z on AB. Show that $X'Y'Z'$ is a cevian triangle. [4]

2. For $i = 1, 2$, let $X_iY_iZ_i$ be the triangle formed with given angles θ_i, φ_i and ψ_i. Show that the intersections

$$X = X_1X_2 \cap BC, \quad Y = Y_1Y_2 \cap CA, \qquad Z = Z_1Z_2 \cap AB$$

form a cevian triangle. [5]

3. Prove (a) $\cot\frac{\theta}{2} - \cot 2\theta = \frac{1+2\cos\theta}{\sin 2\theta}$; (b) $\cot 2\theta + \cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1+2\sin\theta}{\sin 2\theta}$.

4. Let I_a, I_b, I_c be the excenters of triangle ABC. Compute the coordinates of the incenter I_1' of triangle I_aBC, and show that if I_2' and I_3' are similarly defined, then triangle $I_1'I_2'I_3'$ is perspective with ABC at

$$\left(\frac{a}{1 + 2\sin\frac{A}{2}} : \frac{b}{1 + 2\sin\frac{B}{2}} : \frac{c}{1 + 2\sin\frac{C}{2}}\right).$$

[4] Floor van Lamoen.

[5] Floor van Lamoen. $X = (0 : S_{\psi_1} - S_{\psi_2} : S_{\varphi_1} - S_{\varphi_2})$.

6.2.1 The Kiepert perspectors

The Kiepert triangle $\mathcal{K}(\theta)$ is perspective with $\mathbf{T}$ at the Kiepert perspector $K(\theta)$:

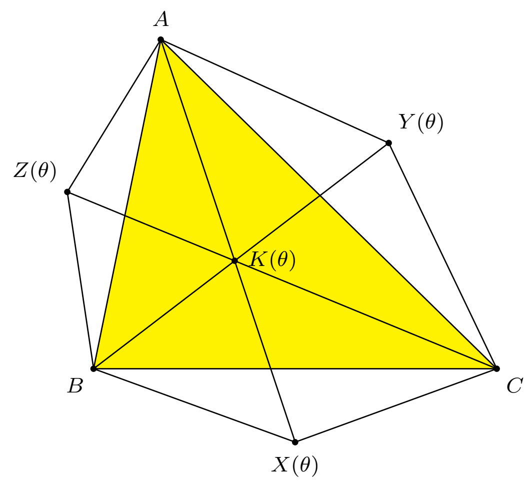


Figure 6.3: The Kiepert perspector

$$\begin{array}{rcccccc}
X(\theta) & = & ***** & : & \frac{1}{S_\beta+S_\theta} & : & \frac{1}{S_\gamma+S_\theta}, \\
Y(\theta) & = & \frac{1}{S_\alpha+S_\theta} & : & ***** & : & \frac{1}{S_\gamma+S_\theta}, \\
Z(\theta) & = & \frac{1}{S_\alpha+S_\theta} & : & \frac{1}{S_\beta+S_\theta} & : & *****; \\ \hline
K(\theta) & = & \frac{1}{S_\alpha+S_\theta} & : & \frac{1}{S_\beta+S_\theta} & : & \frac{1}{S_\gamma+S_\theta}.
\end{array}$$

Kiepert perspector	θ	homogeneous barycentric coordinates
centroid `G`	0	$(1:1:1)$
orthocenter `H`	$\frac{\pi}{2}$	$\left(\frac{1}{S_\alpha}:\frac{1}{S_\beta}:\frac{1}{S_\gamma}\right)$
Fermat points	$\pm\frac{\pi}{3}$	$\left(\frac{1}{\sqrt{3}S_\alpha\pm S}:\frac{1}{\sqrt{3}S_\beta\pm S}:\frac{1}{\sqrt{3}S_\gamma\pm S}\right)$
Napoleon points	$\pm\frac{\pi}{6}$	$\left(\frac{1}{S_\alpha\pm\sqrt{3}S}:\frac{1}{S_\beta\pm\sqrt{3}S}:\frac{1}{S_\gamma\pm\sqrt{3}S}\right)$
Vecten points	$\pm\frac{\pi}{4}$	$\left(\frac{1}{S_\alpha\pm S}:\frac{1}{S_\beta\pm S}:\frac{1}{S_\gamma\pm S}\right)$

6.3 Gossard's theorem

Proposition. Let $\mathcal{L}: \ ux + vy + wz = 0$ be a line intersecting the sidelines at finite points: BC at X, AC at Y, and AB at Z. The following statements are equivalent:
(a) $\mathcal{L}$ is parallel to the Euler line of triangle ABC.
(b) The Euler line of triangle AYZ is parallel to the sideline BC.
(c) The Euler line of triangle BZX is parallel to the sideline CA.
(d) The Euler line of triangle CXY is parallel to the sideline AB.

Proof. It is enough to prove the equivalence of (a) and (b). These intercepts are the points

$$X = (0 : w : -v), \qquad Y = (-w : 0 : u), \qquad Z = (v : -u : 0).$$

The centroid and orthocenter of triangle AYZ are the points

$$\begin{aligned}
G_a = &\ (u^2 - 2u(v+w) + 3vw : \ u(u-w) : \ u(u-v)),\\
H_a = &\ (S_{\beta\gamma}(u-v)(u-w) - S_{\gamma\alpha}w(u-v) - S_{\alpha\beta}v(u-w)\\
&\quad : \ S_\alpha u(S_\gamma(u-v) - S_\alpha(v-w) : \ S_\alpha u(S_\alpha(v-w) - S_\beta(w-u))).
\end{aligned}$$

The line containing, the Euler line of triangle AYZ, has equation

$$\begin{aligned}
&(S_{\alpha\alpha}u(v-w)(2u-v-w) - S_{\gamma\alpha}u(u-v)^2 + S_{\alpha\beta}u(u-w)^2)x\\
&\ + (-S_{\alpha\alpha}(v-w)(u^2+3vw-2wu-2uv) + S_{\beta\gamma}(u-v)^2(u-w)\\
&\qquad - S_{\gamma\alpha}w(u-v)^2 - S_{\alpha\beta}(u-w)(u^2-v^2+3vw-2wu-uv))y\\
&\ + (-S_{\alpha\alpha}(v-w)(u^2+3vw-2wu-2uv) - S_{\beta\gamma}(u-v)(u-w)^2\\
&\qquad + S_{\gamma\alpha}(u-v)(u^2-w^2+3vw-wu-2uv) + S_{\alpha\beta}v(u-w)^2)z\\
= &\ 0.
\end{aligned}$$

This is parallel to the sideline BC if and only if it contains the infinite point $(0 : -1 : -1)$. This condition reduces to

$$-(u-v)(u-w)((S_{\alpha\beta}+S_{\gamma\alpha}-2S_{\beta\gamma})u+(S_{\beta\gamma}+S_{\alpha\beta}-2S_{\gamma\alpha})v+(S_{\gamma\alpha}+S_{\beta\gamma}-2S_{\alpha\beta})w) = 0.$$

Since u, v, w are distinct, we must have

$$(S_{\alpha\beta} + S_{\gamma\alpha} - 2S_{\beta\gamma})u + (S_{\beta\gamma} + S_{\alpha\beta} - 2S_{\gamma\alpha})v + (S_{\gamma\alpha} + S_{\beta\gamma} - 2S_{\alpha\beta})w = 0;$$

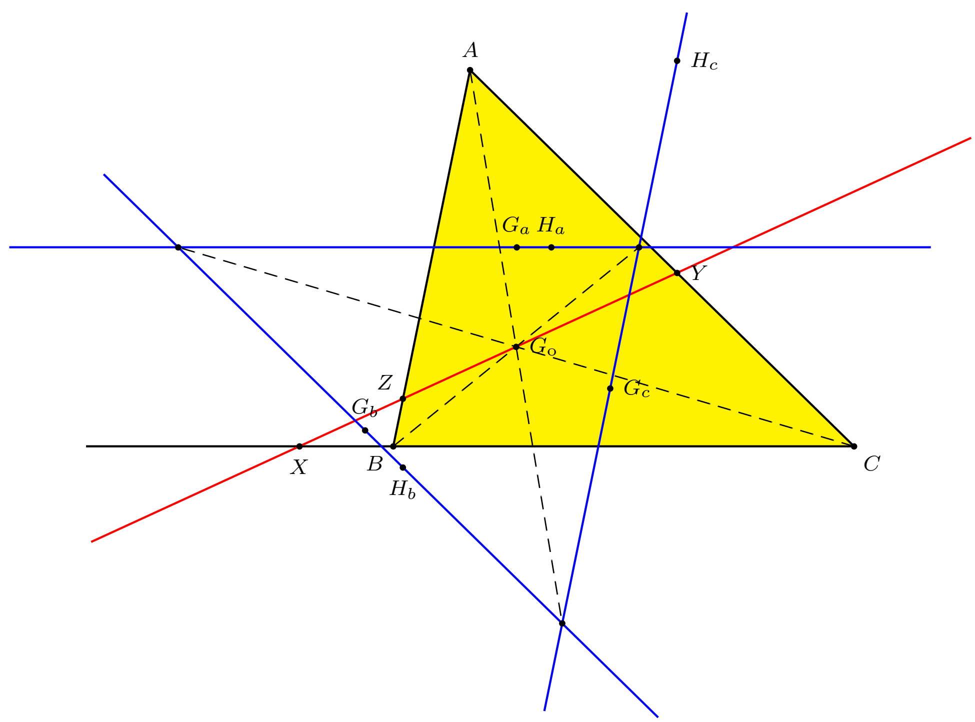
equivalently, the line $ux + vy + wz = 0$ containing the infinite point of the Euler line

$$E_\infty = (S_{\alpha\beta} + S_{\gamma\alpha} - 2S_{\beta\gamma} \ : \ S_{\beta\gamma} + S_{\alpha\beta} - 2S_{\gamma\alpha} \ : \ S_{\gamma\alpha} + S_{\beta\gamma} - 2S_{\alpha\beta}).$$

Therefore the Euler line of AYZ is parallel to BC if and only if $\mathcal{L}$ is parallel to the Euler line of triangle ABC. $\square$

From this we deduce the following theorem.

Theorem (Gossard)**.** Suppose the Euler line of triangle ABC intersects the side lines BC, CA, AB at X, Y, Z respectively. The Euler lines of the triangles AYZ, BZX and CXY bound a triangle homothetic to ABC at a point on the Euler line.



Let $\mathcal{L}$ be parallel to the Euler line through a point $(\mathbb{X},\ \mathbb{Y},\ \mathbb{Z})$. We write

$$p =: S_{\alpha\beta}+S_{\gamma\alpha}-2S_{\beta\gamma}, \quad q =: S_{\beta\gamma}+S_{\alpha\beta}-2S_{\gamma\alpha}, \quad r =: S_{\gamma\alpha}+S_{\beta\gamma}-2S_{\alpha\beta}.$$

The Euler line of AYZ is the line

$$S_\alpha(S_\beta-S_\gamma)(r\mathbb{Y}-q\mathbb{Z})x-(qr\mathbb{X}-S_\gamma(S_\alpha-S_\beta)r\mathbb{Y}+S_\beta(S_\gamma-S_\alpha)q\mathbb{Z})(y+z)=0$$

This line is the image of the sideline BC under the homothety $\mathsf{h}(A, 1+t_a)$, where

$$t_a=\frac{-qr\mathbb{X}+S_\gamma(S_\alpha-S_\beta)r\mathbb{Y}-S_\beta(S_\gamma-S_\alpha)q\mathbb{Z}}{qr(\mathbb{X}+\mathbb{Y}+\mathbb{Z})}.$$

Similarly, the Euler lines of triangles BZX and CXY are images of CA and AB under the homotheties $\mathsf{h}(B, t_b)$ and $\mathsf{h}(C, t_c)$, where t_b and t_c are obtained from t_a above by cyclic permutations of parameters. By the homothetic center theorem, the homethetic center is the point

$$\begin{aligned}
&(t_a,\ t_b,\ t_c)\\
=&\,(-pqr\mathbb{X} + S_\gamma(S_\alpha - S_\beta)rp\mathbb{Y} - S_\beta(S_\gamma - S_\alpha)pq\mathbb{Z},\\
&\quad - S_\gamma(S_\alpha - S_\beta)qr\mathbb{X} - pqr\mathbb{Y} + S_\alpha(S_\beta - S_\gamma)pq\mathbb{Z},\\
&\quad S_\beta(S_\gamma - S_\alpha)qr\mathbb{X} - S_\alpha(S_\beta - S_\gamma)rp\mathbb{Y} - pqr\mathbb{Z})\\
=&- pqr(\mathbb{X}, \mathbb{Y}, \mathbb{Z})\\
&\ + (S_\gamma(S_\alpha - S_\beta)rp\mathbb{Y} - S_\beta(S_\gamma - S_\alpha)pq\mathbb{Z},\\
&\quad - S_\gamma(S_\alpha - S_\beta)qr\mathbb{X} + S_\alpha(S_\beta - S_\gamma)pq\mathbb{Z},\\
&\quad S_\beta(S_\gamma - S_\alpha)qr\mathbb{X} - S_\alpha(S_\beta - S_\gamma)rp\mathbb{Y})
\end{aligned}$$

Proof. The intercepts of the Euler line

$$S_\alpha(S_\beta - S_\gamma)x + S_\beta(S_\gamma - S_\alpha)y + S_\gamma(S_\alpha - S_\beta)z = 0$$

are the points

$$\begin{aligned}
X =&(0 : -S_\gamma(S_\alpha - S_\beta) : S_\beta(S_\gamma - S_\alpha)),\\
Y =&(S_\gamma(S_\alpha - S_\beta) : 0 : -S_\alpha(S_\beta - S_\gamma)),\\
Z =&(-S_\beta(S_\gamma - S_\alpha) : S_\alpha(S_\beta - S_\gamma) : 0),
\end{aligned}$$

$\square$

In Conway's notation, these are

$$: (S_C - S_A)^2 S_B^2 (S_{BC} + S_{AB} - 2S_{AC}) : \cdots$$

and

$$\cdots : \frac{S_{BC} + S_{AB} - 2S_{AC}}{S_B(S_C - S_A)} : \cdots .$$

6.4 Cevian quotients

Theorem (The cevian nest theorem)**.** For arbitrary points $P = (u : v : w)$ and $Q = (u' : v' : w')$, the cevian triangle $\mathsf{cev}(P)$ and the anticevian triangle $\mathsf{cev}^{-1}(Q)$ are always perspective. (a) The perspector is the point

$$\bigwedge(\mathsf{cev}(P), \mathsf{cev}^{-1}(Q)) = \left(u'\left(-\frac{u'}{u}+\frac{v'}{v}+\frac{w'}{w}\right) : v'\left(-\frac{v'}{v}+\frac{w'}{w}+\frac{u'}{u}\right) : w'\left(-\frac{w'}{w}+\frac{u'}{u}+\frac{v'}{v}\right)\right),$$

(b) The perspectrix is the line $\mathcal{L}_\wedge(\mathsf{cev}(P), \mathsf{cev}^{-1}(Q))$ with equation

$$\sum_{\text{cyclic}} \frac{1}{u}\left(-\frac{u'}{u}+\frac{v'}{v}+\frac{w'}{w}\right)x = 0.$$

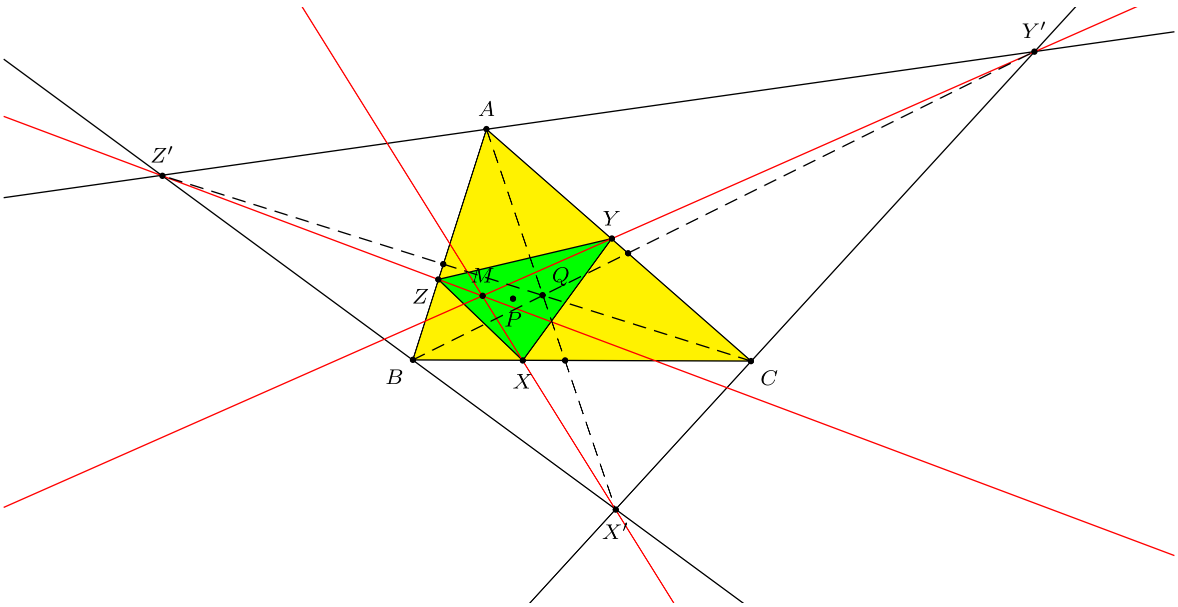


Figure 6.4:

Proof. (a) Let $\mathsf{cev}(P) = XYZ$ and $\mathsf{cev}^{-1}(Q) = X'Y'Z'$. Since $X = (0 : v : w)$ and $X' = (-u' : v' : w')$, the line XX' has equation

$$\frac{1}{u'}\left(\frac{w'}{w}-\frac{v'}{v}\right)x - \frac{1}{v}\cdot y + \frac{1}{w}\cdot z = 0.$$

The equations of YY' and ZZ' can be easily written down by cyclic permutations of (u, v, w), (u', v', w') and (x, y, z). It is easy to check that the line XX' contains the point

$$\left(u'\left(-\frac{u'}{u}+\frac{v'}{v}+\frac{w'}{w}\right) : v'\left(-\frac{v'}{v}+\frac{w'}{w}+\frac{u'}{u}\right) : w'\left(-\frac{w'}{w}+\frac{u'}{u}+\frac{v'}{v}\right)\right)$$

whose coordinates are invariant under the above cyclic permutations. This point therefore also lies on the lines YY' and ZZ'.

(b) The lines YZ and $Y'Z'$ have equations

$$\begin{array}{rcccccl} -\frac{x}{u} & + & \frac{y}{v} & + & \frac{z}{w} & = & 0, \\ & & \frac{y}{v'} & + & \frac{z}{w'} & = & 0. \end{array}$$

They intersect at the point

$$U' = (u(wv' - vw') : \ vwv' : \ -vww').$$

Similarly, the lines pairs ZX, $Z'X'$ and XY, $X'Y'$ have intersections

$$V' = (-wuu' : \ v(uw' - wu') : \ wuw')$$

and

$$W' = (uvu' : \ -uvv' : \ w(vu' - uv')).$$

The three points U', V', W' lie on the line with equation given above. $\square$

We shall simply write

$$\boxed{P/Q := \bigwedge(\mathsf{cev}(P), \mathsf{cev}^{-1}(Q))}$$

and call it the *cevian quotient* of P by Q. Clearly, $P/P = P$.

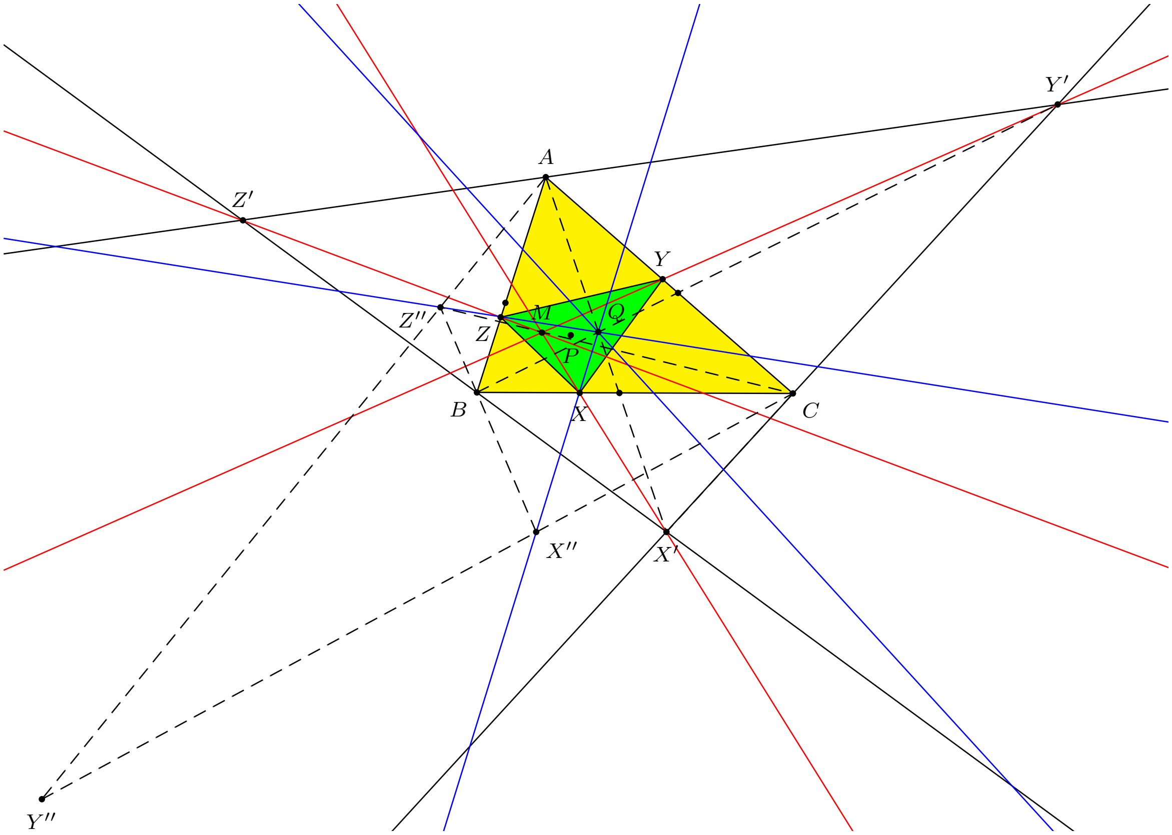
Proposition. $P/Q = M$ if and only if $Q = P/M$.

Proof. Let $P = (u : v : w)$, $Q = (u' : v' : w')$, and $M = (x : y : z)$. We have

$$\begin{aligned} \frac{x}{u} &= \frac{u'}{u}\left(-\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w}\right), \\ \frac{y}{v} &= \frac{v'}{v}\left(\frac{u'}{u} - \frac{v'}{v} + \frac{w'}{w}\right), \\ \frac{z}{w} &= \frac{w'}{w}\left(\frac{u'}{u} + \frac{v'}{v} - \frac{w'}{w}\right). \end{aligned}$$

From these,

$$\begin{aligned} -\frac{x}{u} + \frac{y}{v} + \frac{z}{w} &= \left(\frac{u'}{u} - \frac{v'}{v} + \frac{w'}{w}\right)\left(\frac{u'}{u} + \frac{v'}{v} - \frac{w'}{w}\right), \\ \frac{x}{u} - \frac{y}{v} + \frac{z}{w} &= \left(-\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w}\right)\left(\frac{u'}{u} + \frac{v'}{v} - \frac{w'}{w}\right), \\ \frac{x}{u} + \frac{y}{v} - \frac{z}{w} &= \left(-\frac{u'}{u} - \frac{v'}{v} + \frac{w'}{w}\right)\left(\frac{u'}{u} - \frac{v'}{v} + \frac{w'}{w}\right), \end{aligned}$$



and

$$\frac{x}{u}\left(-\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right) : \frac{y}{v}\left(\frac{x}{u}-\frac{y}{v}+\frac{z}{w}\right) : \frac{z}{w}\left(\frac{x}{u}+\frac{y}{v}-\frac{z}{w}\right)$$
$$= \frac{u'}{u} : \frac{v'}{v} : \frac{w'}{w}.$$

It follows that

$$u' : v' : w' = x\left(-\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right) : y\left(\frac{x}{u}-\frac{y}{v}+\frac{z}{w}\right) : z\left(\frac{x}{u}+\frac{y}{v}-\frac{z}{w}\right).$$

$\square$

6.4.1

Consider the cevian triangle $\mathsf{cev}(P) = P_aP_bP_c$ of $P = (u : v : w)$, and the anticevian triangle $\mathsf{cev}^{-1}(Q) = Q^aQ^bQ^c$ of $Q = (x : y : z)$.

Now, let $X = P_bQ^c \cap P_cQ^b$, $Y = P_cQ^a \cap P_aQ^c$, $Z = P_aQ^b \cap P_bQ^a$.

$$X = \left(x\left(\frac{x}{u} + \frac{y}{v} + \frac{z}{w}\right) : y\left(\frac{x}{u} + \frac{y}{v} - \frac{z}{w}\right) : z\left(\frac{x}{u} - \frac{y}{v} + \frac{z}{w}\right)\right)$$

etc.

The triangle XYZ is perspective with

(i) ABC at

$$\left(\frac{x}{-\frac{x}{u} + \frac{y}{v} + \frac{z}{w}} : \frac{y}{\frac{x}{u} - \frac{y}{v} + \frac{z}{w}} : \frac{z}{\frac{x}{u} + \frac{y}{v} - \frac{z}{w}}\right),$$

(ii) $\mathsf{cev}(P)$ at Q, and

(iii) $\mathsf{cev}^{-1}(Q)$ at $\left(\frac{x^2}{u} : \frac{y^2}{v} : \frac{z^2}{w}\right)$.

Example. If $Q = \mathtt{I} = (a : b : c)$, XYZ is perspective with

(i) ABC at $(\mathtt{I}/P)^*$,

(ii) $\mathsf{cev}(P)$ at I, and

(iii) the excentral triangle $\mathsf{cev}^{-1}(\mathtt{I})$ at P^*.

6.5 The cevian quotient $\mathtt{G}/P$

If $P = (u : v : w)$,

$$\boxed{\mathtt{G}/P = (u(-u+v+w) : v(-v+w+u) : w(-w+u+v)).}$$

Some common examples of G/P.

P	$\mathtt{G}/P$	coordinates
$\mathtt{I}$	$\mathtt{M_i}$	$(a(b+c-a) : b(c+a-b) : c(a+b-c))$
$\mathtt{O}$	$\mathtt{K}$	$(a^2 : b^2 : c^2)$
$\mathtt{K}$	$\mathtt{O}$	$(a^2S_\alpha : b^2S_\beta : c^2S_\gamma)$

Proposition. The cevian quotient G/P is the isotomic conjugate of P in the inferior triangle.

Proof. Let $\mathtt{G}_aP$ and $P^a\mathtt{G}_a$ intersect $\mathtt{G}_b\mathtt{G}_c$ at X and X' respectively. Note that X' is the trace of the cevian quotient $\mathtt{G}/P$ on $\mathtt{G}_b\mathtt{G}_c$.

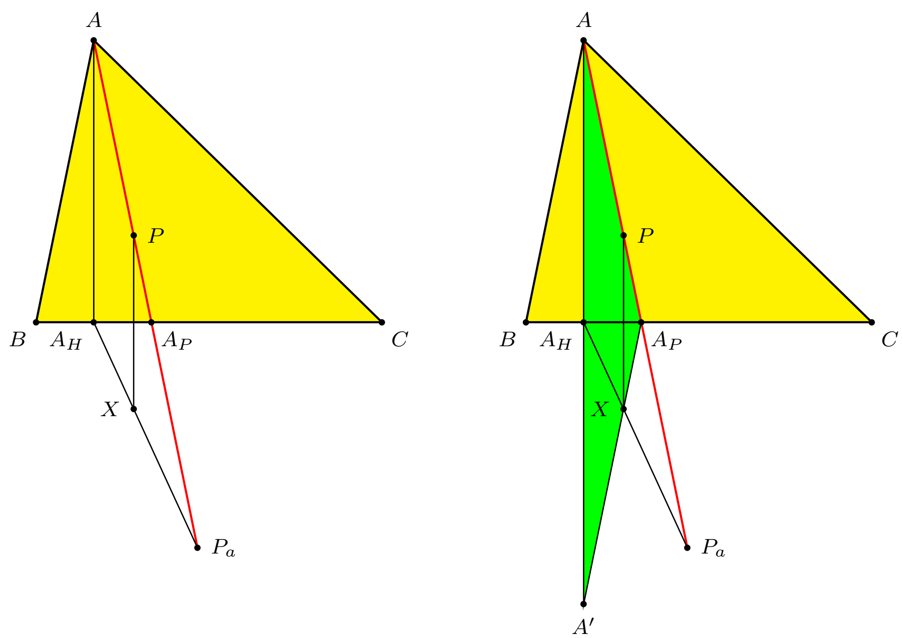
From $(u, v, w) = 2u(1, 0, 0) + (-u, v, w)$, we have
(i) $AP_a : P_aP^a = 2u : (v + w - u)$;
(ii) $\mathtt{G}_aP : PX = 2u : v + w - u$, since P has coordinates $(v + w - u : w + u - v : u + v - w)$ in the inferior triangle.

Therefore, $AX//P^a\mathtt{G}_a$. From this, the triangles $A\mathtt{G}_cX$ and $\mathtt{G}_a\mathtt{G}_bX'$ are congruent. This means that the traces of P and $\mathtt{G}/P$ on $\mathtt{G}_b\mathtt{G}_c$, namely, X and X', are isotomic points on $\mathtt{G}_b\mathtt{G}_c$. Analogous results hold for the traces of the other sides of the inferior triangle. Therefore, P and $\mathtt{G}/P$ are isotomic conjugates in the inferior triangle. $\square$

6.6 The cevian quotient $\mathtt{H}/P$

Proposition. Let $A_HB_HC_H$ be the orthic triangle, and X the reflection of P in a, then the intersection of the lines A_HX and AP is the harmonic conjugate P_a of P in AA_P:

$$\frac{AP_a}{P_aA_P} = -\frac{AP}{PA_P}.$$



Proof. Let A' be the reflection of A in BC. Applying Menelaus' theorem to triangle A_PAA' with transversal A_HXP_a, we have

$$\frac{AP_a}{P_aA_P}\cdot\frac{A_PX}{XA'}\cdot\frac{A'A_H}{A_HA} = -1.$$

This gives

$$\frac{AP_a}{P_aA_P} = -\frac{XA'}{A_PX} = -\frac{PA}{A_PP} = -\frac{AP}{PA_P},$$

showing that P_a and P divide AA_P harmonically. $\square$

Corollary. The anticevian triangle and the reflection triangle of P are perspective with the orthic triangle at $\mathtt{H}/P$.

1. Construction of anticevian triangle $\text{cev}^{-1}(P)$: $P_a = A_HX \cap AP$ etc.
2. If $P = (u : v : w)$, then $P_a = (-u : v : w)$.

3. Let M_a be the midpoint of the altitude AA_H. The line M_aP_a intersects a at the pedal of P on a.

4. The reflection triangle of P, the anticevian triangle of P, and the orthic triangle are pairwise perspective at the same point (H/P).

5. The pedal of P on a has coordinates

$$\begin{aligned} & u(S_\beta + S_\gamma,\ S_\gamma,\ S_\beta) + (S_\beta + S_\gamma)(-u, v, w) \\ = \ & (0, uS_\gamma + v(S_\beta + S_\gamma), uS_\beta + w(S_\beta + S_\gamma)) \\ = \ & (0, vS_\beta + (u+v)S_\gamma,\ (w+u)S_\beta + wS_\gamma). \end{aligned}$$

Note coordinate sum $= (S_\beta + S_\gamma)(u + v + w)$.

6. The reflection of P in a:

$$\begin{aligned} X \sim \ & 2(0, uS_\gamma + v(S_\beta + S_\gamma), uS_\beta + w(S_\beta + S_\gamma)) - (S_\beta + S_\gamma)(u, v, w) \\ \sim & (-(S_\beta + S_\gamma)u, 2uS_\gamma + v(S_\beta + S_\gamma), 2uS_\beta + w(S_\beta + S_\gamma)). \end{aligned}$$

Note coordinate sum $= (S_\beta + S_\gamma)(u + v + w)$.

For $P = (u : v : w)$,

$$\boxed{\mathtt{H}/P = (u(-S_\alpha u + S_\beta v + S_\gamma w) : v(-S_\beta v + S_\gamma w + S_\alpha u) : w(-S_\gamma w + S_\alpha u + S_\beta v)).}$$

Examples

(1) $\mathtt{H}/\mathtt{G} = (S_\beta + S_\gamma - S_\alpha : S_\gamma + S_\alpha - S_\beta : S_\alpha + S_\beta - S_\gamma)$ is the superior of $H^\bullet$.

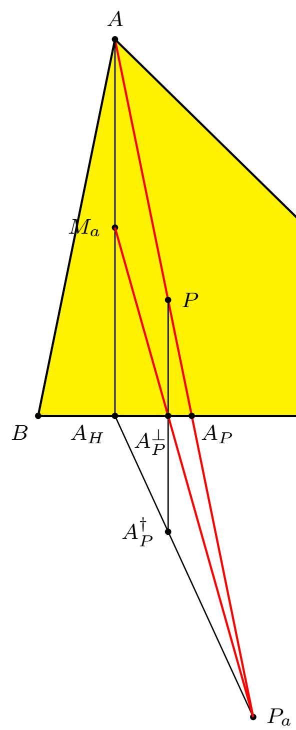
(2) $\mathtt{H}/\mathtt{I}$ is a point on the OI-line, dividing OI in the ratio $R + r : -2r$. [6]

$$\mathtt{H}/\mathtt{I} = (a(a^3 + a^2(b+c) - a(b^2+c^2) - (b+c)(b-c)^2) : \cdots : \cdots).$$

(3) $\mathtt{H}/\mathtt{K} = \left(\frac{a^2}{S_\alpha} : \frac{b^2}{S_\beta} : \frac{c^2}{S_\gamma}\right)$ is the homothetic center of the orthic and tangential triangle. [7] It is a point on the Euler line.

(4) $\mathtt{H}/\mathtt{O}$ is the orthocenter of the tangential triangle. [8]

(5) $\mathtt{H}/\mathtt{N}$ is the orthocenter of the orthic triangle. [9]



[6] This point appears as X_{46} in ETC.
[7] This appears as X_{25} in ETC.
[8] This appears as X_{155} in ETC.
[9] This appears as X_{52} in ETC.

6.7 Pedal and reflection triangles

6.7.1 Reflections and isogonal conjugates

Consider a point P with reflections $P^a_\dagger$, $P^b_\dagger$, $P^c_\dagger$ in the sidelines BC, CA, AB. Let Q be a point on the line isogonal to AP with respect to angle A, *i.e.*, the lines AQ and AP are symmetric with respect to the bisector of angle BAC.

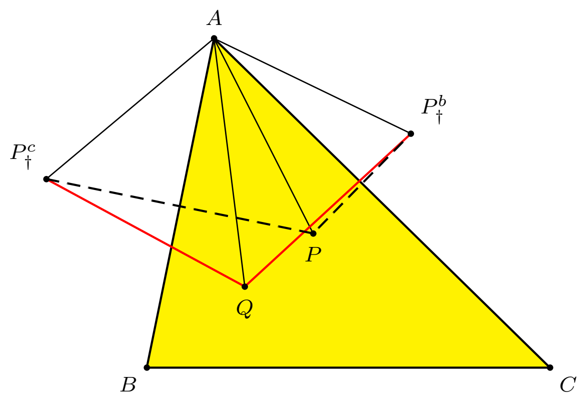


Figure 6.5:

Clearly, the triangles $AQP^b_\dagger$ and $AQP^c_\dagger$ are congruent, so that Q is equidistant from $P^b_\dagger$ and $P^c_\dagger$. For the same reason, any point on a line isogonal to BP is equidistant from $P^c_\dagger$ and $P^a_\dagger$. It follows that the intersection P^* of two lines isogonal to AP and BP is equidistant from the three reflections $P^a_\dagger$, $P^b_\dagger$, $P^c_\dagger$. Furthermore, P^* is on a line isogonal to CP. For this reason, we call P^* the *isogonal conjugate* of P. It is the center of the circle of reflections of P.

6.7.2 The pedal circle

Clearly, $P^* = P$. Moreover, the circles of reflections of P and P^* are congruent, since, in Figure 6.7, the trapezoid $PP^*P^*_aP^a_\dagger$ being isosceles, $PP^*_a = P^*P^a_\dagger$. It follows that the pedals of P and P^* on the sidelines all lie on the same circle with center the midpoint of PP^*. We call this the common *pedal circle* of P and P^*.

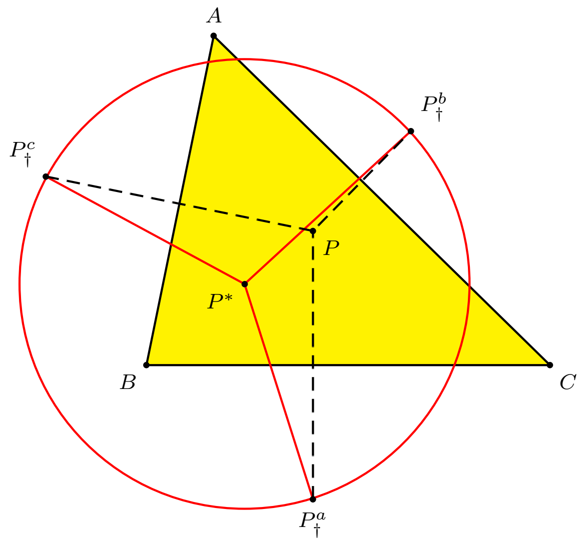


Figure 6.6:

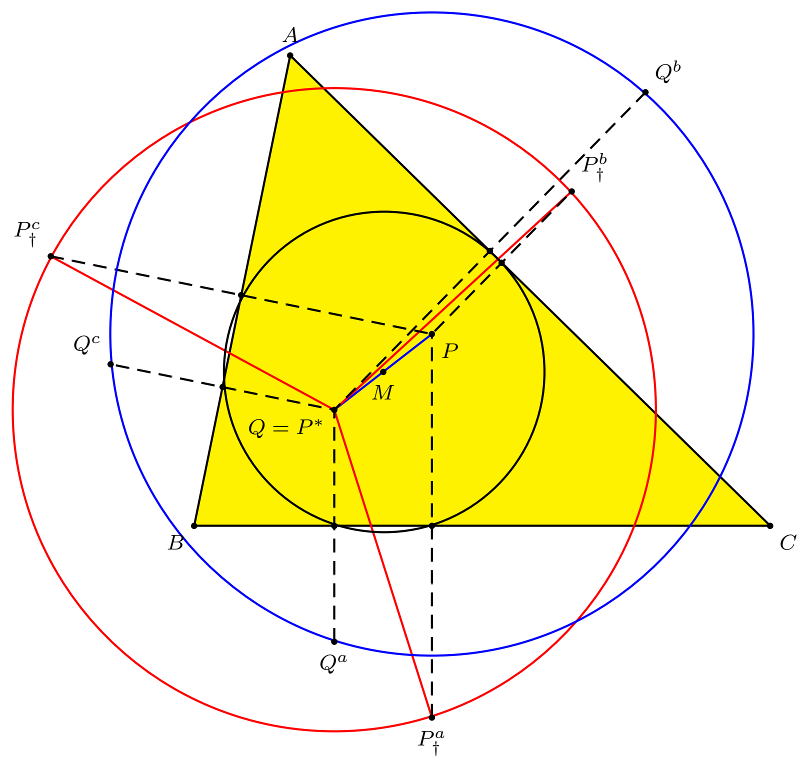


Figure 6.7:

Exercise

1. The perpendiculars from the vertices of ABC to the corresponding sides of the pedal triangle of a point P concur at the isogonal conjugate of P.

2. Given a point P with isogonal conjugate P^*, let X, Y, Z be the pedals of P on the sidelines BC, CA, AB of triangle ABC. If the circle $X(P^*)$ intersects BC at X_1, X_2, $Y(P^*)$ intersects CA at Y_1, Y_2, and $Z(P^*)$ intersects AB at Z_1, Z_2, then the six points X_1, X_2, Y_1, Y_2, Z_1, Z_2 are on a circle center P.

6.7.3 Pedal triangle

The pedals of a point $P = (u : v : w)$ are the intersections of the side lines with the corresponding perpendiculars through P. The $A-$altitude has infinite point $A_H - A = (0 : S_\gamma : S_\beta) - (S_\beta + S_\gamma : 0 : 0) = (-a^2 : S_\gamma : S_\beta)$. The perpendicular through P to BC is the line

$$\begin{vmatrix} -a^2 & S_\gamma & S_\beta \\ u & v & w \\ x & y & z \end{vmatrix} = 0,$$

or

$$-(S_\beta v - S_\gamma w)x + (S_\beta u + a^2 w)y - (S_\gamma u + a^2 v)z = 0.$$

Figure

This intersects BC at the point

$$A_{[P]} = (0 : S_\gamma u + a^2 v : S_\beta u + a^2 w).$$

Similarly the coordinates of the pedals on CA and AB can be written down. The triangle $A_{[P]}B_{[P]}C_{[P]}$ is called the *pedal triangle* of triangle ABC:

$$\begin{pmatrix} A_{[P]} \\ B_{[P]} \\ C_{[P]} \end{pmatrix} = \begin{pmatrix} 0 & S_\gamma u + a^2 v & S_\beta u + a^2 w \\ S_\gamma v + b^2 u & 0 & S_\alpha v + b^2 w \\ S_\beta w + c^2 u & S_\alpha w + c^2 v & 0 \end{pmatrix}$$

6.7.4 Examples

(1) The pedal triangle of the circumcenter is clearly the medial triangle.

(2) The pedal triangle of the orthocenter is called the *orthic* triangle. Its vertices are clearly the traces of H, namely, the points $(0 : S_\gamma : S_\beta)$, $(S_\gamma : 0 : S_\alpha)$, and $(S_\beta : S_\alpha : 0)$.

(3) Let L be the deLongchamps point, *i.e.*, the reflection of the orthocenter H in the circumcenter O. Show that the pedal triangle of L is the cevian triangle of some point P. What are the coordinates of P? [10]

Figure

[10] $P = (S_\alpha : S_\beta : S_\gamma)$ is the isotomic conjugate of the orthocenter. It appears in ETC as the point X_{69}.

(4) Let L be the de Longchamps point again, with homogeneous barycentric coordinates

$$(S_{\gamma\alpha}+S_{\alpha\beta}-S_{\beta\gamma}:S_{\alpha\beta}+S_{\beta\gamma}-S_{\gamma\alpha}:S_{\beta\gamma}+S_{\gamma\alpha}-S_{\alpha\beta}).$$

Find the equations of the perpendiculars to the side lines at the corresponding traces of L. Show that these are concurrent, and find the coordinates of the intersection.

The perpendicular to BC at $A_L=(0:S_{\alpha\beta}+S_{\beta\gamma}-S_{\gamma\alpha}:S_{\beta\gamma}+S_{\gamma\alpha}-S_{\alpha\beta})$ is the line

$$\begin{vmatrix} -(S_\beta+S_\gamma) & S_\gamma & S_\beta \\ 0 & S_{\alpha\beta}+S_{\beta\gamma}-S_{\gamma\alpha} & S_{\beta\gamma}+S_{\gamma\alpha}-S_{\alpha\beta} \\ x & y & z \end{vmatrix}=0.$$

This is

$$S^2(S_\beta-S_\gamma)x-a^2(S_{\beta\gamma}+S_{\gamma\alpha}-S_{\alpha\beta})y+a^2(S_{\beta\gamma}-S_{\gamma\alpha}+S_{\alpha\beta})z=0.$$

Similarly, we write down the equations of the perpendiculars at the other two traces. The three perpendiculars intersect at the point [11]

$$(a^2(S_\gamma^2S_\alpha^2+S_\alpha^2S_\beta^2-S_\beta^2S_\gamma^2):\cdots:\cdots).$$

$\boxed{\text{Figure}}$

Exercise

1. Let D, E, F be the midpoints of the sides BC, CA, AB, and A', B', C' the pedals of A, B, C on their opposite sides. Show that $X=EC'\cap FB'$, $Y=FA'\cap DC'$, and $Z=DB'\cap EC'$ are collinear. [12]

2. Let X be the pedal of A on the side BC of triangle ABC. Complete the squares AXX_bA_b and AXX_cA_c with X_b and X_c on the line BC. [13]

   (a) Calculate the coordinates of A_b and A_c. [14]

[11] This point appears in ETC as X_{1078}. Conway calls this point the *logarithm of the de Longchamps point*.

[12] These are all on the Euler line. See G. Leversha, Problem 2358 and solution, *Crux Mathematicorum*, 24 (1998) 303; 25 (1999) 371 –372.

[13] A.P. Hatzipolakis, Hyacinthos, message 3370, 8/7/01.

[14] $A_b=(a^2:-S:S)$ and $A_c=(a^2:S:-S)$.

(b) Calculate the coordinates of $A' = BA_c \cap CA_b$. [15]

(c) Similarly define B' and C'. Triangle $A'B'C'$ is perspective with ABC. What is the perspector? [16]

(d) Let A'' be the pedal of A' on the side BC. Similarly define B'' and C''. Show that $A''B''C''$ is perspective with ABC by calculating the coordinates of the perspector. [17]

6.7.5 Reflection triangle

The reflection triangle of $P = (u : v : w)$ have vertices

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} -a^2u & 2S_\gamma u + a^2v & 2S_\beta u + a^2w \\ 2S_\gamma v + b^2u & -b^2v & 2S_\alpha v + b^2w \\ 2S_\beta w + c^2u & 2S_\alpha w + c^2v & -c^2w \end{pmatrix}.$$

The construction of harmonic conjugates in §?? shows that the line containing the harmonic conjugate of P in AA_P and the reflection of P in a and passe through A_H. Therefore,

Proposition. The anticevian and reflection triangles of P are perspective at H/P.

Examples

(1) Since the reflection triangle of I is homothetic to the excentral triangle, with ratio $2r : 2R = r : R$, the homothetic center is the point dividing II' in the ratio $-r : R$ *i.e.*,

$$\frac{R \cdot I - r \cdot I'}{R - r} = \frac{(R + r)I - 2r \cdot O}{R - r}.$$

This shows that H/I is the point dividing OI in the ratio $R + r : -2r$.

---

[15] $A' = (a^2 : S : S)$.

[16] The centroid.

[17] $(\frac{1}{S_\alpha + S} : \frac{1}{S_\beta + S} : \frac{1}{S_\gamma + S})$. This is called the first Vecten point; it appears as X_{485} in ETC.

Exercise

1. Given a point P, construct a circle with center P whose reflections in the sidelines of triangle ABC are concurrent. [18]

2. The perspector of the intouch triangle and the tangential triangle is the point [19]

$$G_{\mathrm{e}}/K = (a^2(a^3 - a^2(b+c) + a(b^2+c^2) - (b+c)(b-c)^2) : \cdots : \cdots).$$

3. Let XYZ be the circumcevian triangle of P, and $X'Y'Z'$ be that of P^*. The lines XX', YY', ZZ' bound a triangle homothetic to ABC. What is the homothetic center? [20]

[18] The circumcircle of the reflection triangle of P^*.

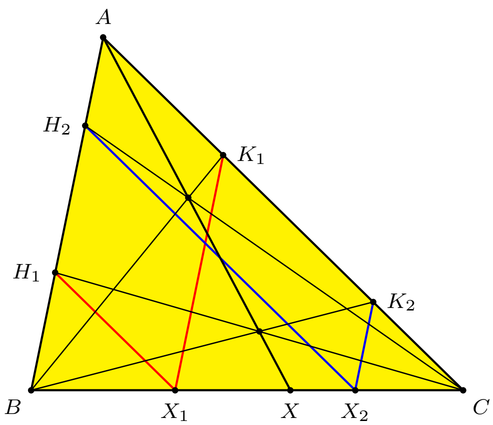
[19] This appears as X_{1486} in ETC.

[20] If $P = (u : v : w)$, the homothetic center is $\left(\frac{a^2}{u(S_\alpha(v+w)^2 + S_\beta v^2 + S_\gamma w^2)} : \cdots : \cdots\right)$.

6.8 Barycentric product

Let X_1, X_2 be two points on the line BC, distinct from the vertices B, C, with homogeneous coordinates $(0 : y_1 : z_1)$ and $(0 : y_2 : z_2)$. For $i = 1, 2$, complete parallelograms $AK_iX_iH_i$ with K_i on AB and H_i on AC. The coordinates of the points H_i, K_i are

$$H_1 = (y_1 : 0 : z_1), \qquad K_1 = (z_1 : y_1 : 0);$$
$$H_2 = (y_2 : 0 : z_2), \qquad K_2 = (z_2 : y_2 : 0).$$



From these, we easily find that

$$BK_1 \cap CH_2 = (y_1z_2 : y_1y_2 : z_1z_2),$$
$$BK_2 \cap CH_1 = (y_2z_1 : y_1y_2 : z_1z_2).$$

Both of these have A-trace

$$X = (0 : y_1y_2 : z_1z_2).$$

The line joining them passes through A.

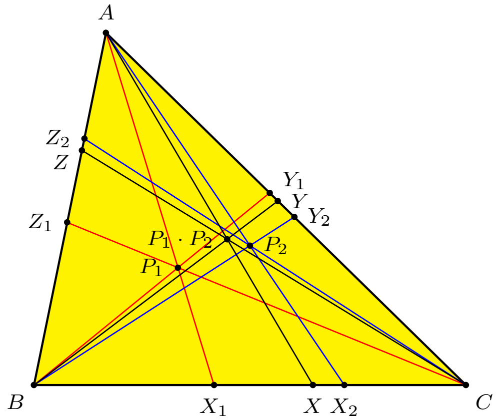
This simple observation leads to the notion of the *barycentric product*. Given two points $P_1 = (x_1 : y_2 : z_1)$ and $P_2 = (x_2 : y_2 : z_1)$, the above construction (applied to the traces on each side line) gives the traces of a point

$$\boxed{P_1 \cdot P_2 = (x_1x_2 : y_1y_2 : z_1z_2).}$$

6.8.1 Barycentric square

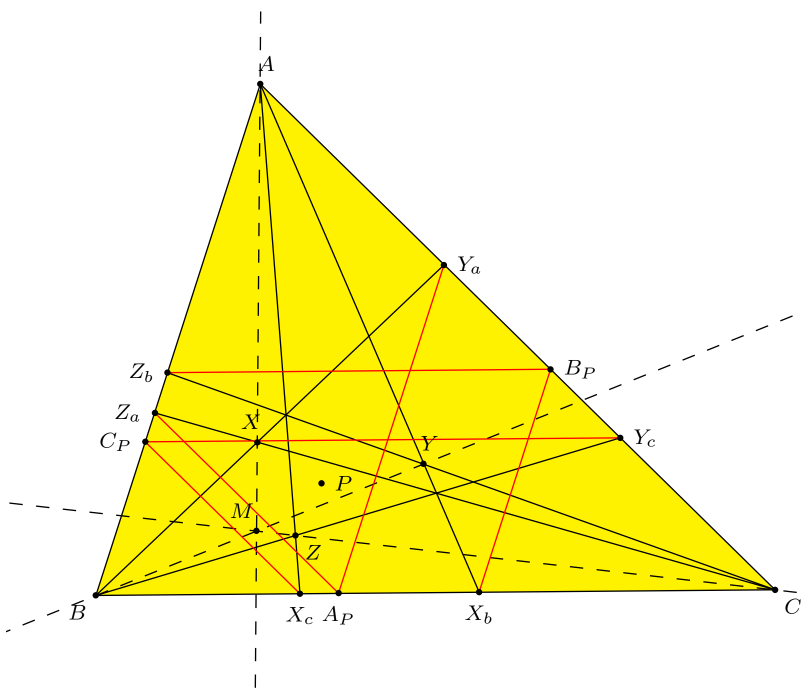
In particular, the *barycentric square* of a point $P = (x : y : z)$ is the point

$$\boxed{P \cdot P = (x^2 : y^2 : z^2)}$$



and can be constructed as follows.
(i) Complete the parallelograms $AY_aA_PZ_a$, $BZ_bB_PX_b$ and $CX_cC_PY_c$.
(ii) Let $X = BY_a \cap CZ_a$, $Y = CZ_b \cap AX_b$, and $Z = AX_c \cap BY_c$.

Then XYZ is perspective with ABC at the barycentric square of P.



Exercise

1. Find the equation of the circle through B and C, tangent (internally) to incircle. Show that the point of tangency has coordinates

$$\left(\frac{a^2}{s-a} : \frac{(s-c)^2}{s-b} : \frac{(s-b)^2}{s-c}\right).$$

Construct this circle by making use of the barycentric cube of the Gergonne point.

2. Three parallel lines with infinite point $(u : v : w)$ through the vertices of a triangle ABC intersect the sidelines BC, CA, AB at X, Y, Z respectively. Show that the centroid of the cevian triangle XYZ has coordinates $(u^3 : v^3 : w^3)$.

3. A circle is tangent to the side BC of triangle ABC at the $A-$trace of a point $P = (u : v : w)$ and internally to the circumcircle at A'. Show that the line AA' passes through the point $(au : bv : vw)$.

Make use of this to construct the three circles each tangent internally to the circumcircle and to the side lines at the traces of P.

4. Two circles each passing through the incenter I are tangent to BC at B and C respectively. A circle (J_a) is tangent externally to each of these, and to BC at X. Similarly define Y and Z. Show that XYZ is perspective with ABC, and find the perspector. [21]

5. Let $P_1 = (f_1 : g_1 : h_1)$ and $P_2 = (f_2 : g_2 : h_2)$ be two given points. Denote by X_i, Y_i, Z_i the traces of these points on the sides of the reference triangle ABC.

(a) Find the coordinates of the intersections $X_+ = BY_1 \cap CZ_2$ and $X_- = BY_2 \cap CZ_1$. [22]

(b) Find the equation of the line X_+X_-. [23]

(c) Similarly define points Y_+, Y_-, Z_+ and Z_-. Show that the three lines X_+X_-, Y_+Y_-, and Z_+Z_- intersect at the point

$$(f_1f_2(g_1h_2 + h_1g_2) : g_1g_2(h_1f_2 + f_1h_2) : h_1h_2(f_1g_2 + g_1f_2)).$$

6. The barycentric cube of an infinite point is the centroid of the cevian triangle of the point. It happens that the barycentric cube of the Euler line is again on the Euler line, and the Euler line is the only line through O with this property. For a given point $P \neq G$, only the line PG has this property. [24]

[21] The barycentric square root of $(\frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c})$. See Hyacinthos, message 3394, 8/9/01.
[22] $X_+ = f_1f_2 : f_1g_2 : h_1f_2$; $X_- = f_1f_2 : g_1f_2 : f_1h_2$.
[23] $(f_1^2g_2h_2 - f_2^2g_1h_1)x - f_1f_2(f_1h_2 - h_1f_2)y + f_1f_2(g_1f_2 - f_1g_2)z = 0$..
[24] 2/14/04.

6.8.2 Barycentric square root

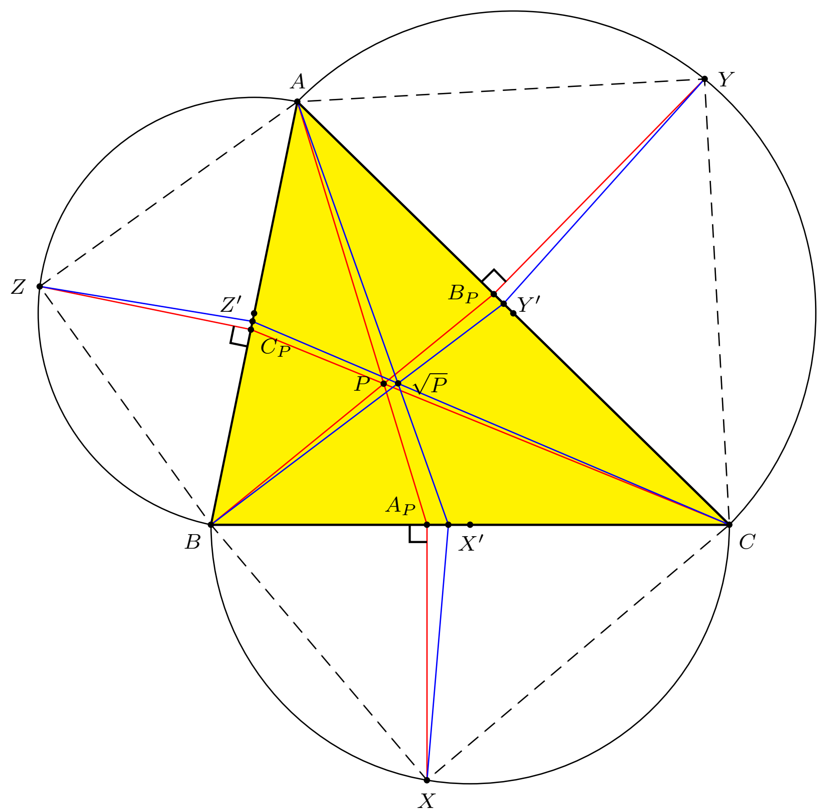
Let $P = (u : v : w)$ be a point in the interior of triangle ABC, the *barycentric square root* $\sqrt{P}$ is the point Q in the interior such that $Q^2 = P$. This can be constructed as follows.

(1) Construct the circle with BC as diameter.

(2) Construct the perpendicular to BC at the trace A_P to intersect the circle at X.[25] Bisect angle BXC to intersect BC at X'.

(3) Similarly obtain Y' on CA and Z' on AB.

The points X', Y', Z' are the traces of the barycentric square root $\sqrt{P}$.



Proof. $BX'^2 : X'C^2 = BX^2 : XC^2 = BA_P : A_PC$ etc. □

[25]It does not matter which of the two intersections is chosen.

Chapter 7

Orthology

7.1 Triangle determined by orthology centers

Given triangles ABC and a finite point $Q' = (u' : v' : w')$, consider a triangle $A'B'C'$ for which the perpendiculars from A' to BC, B' to CA, C' to AB are concurrent at Q'. We shall say that $A'B'C'$ is orthologic to ABC with orthology center $Q' = \perp (A'B'C',\ ABC)$.

In absolute barycentric coordinates

$$\begin{aligned} A' =&\ Q' + t_1(-a^2, S_\gamma, S_\beta), \\ B' =&\ Q' + t_2(S_\gamma, -b^2, S_\alpha), \\ C' =&\ Q' + t_3(S_\beta, S_\alpha, -c^2) \end{aligned}$$

for some t_1, t_2, t_3. These three points are collinear, i.e., triangle $A'B'C'$ is degenerate, if and only if

$$t_2t_3 + t_3t_1 + t_1t_2 = 0.$$

The infinite point of $B'C'$ is

$$\begin{aligned} & t_3(S_\beta, S_\alpha, -c^2) - t_2(S_\gamma, -b^2, S_\alpha) \\ =&\ (t_3S_\beta - t_2S_\gamma,\ t_3S_\alpha + t_2b^2,\ -(t_3c^2 + t_2S_\alpha)). \end{aligned}$$

Note that

$$\begin{aligned} & S_\alpha(t_2+t_3)(t_3S_\beta - t_2S_\gamma) + S_\beta(-t_3)(t_3S_\alpha + t_2b^2) + S_\gamma(-t_2)(-(t_3c^2 + t_2S_\alpha)) \\ =& S_\alpha(t_2+t_3)(t_3S_\beta - t_2S_\gamma) - S_\beta t_3(t_3S_\alpha + t_2(S_\gamma + S_\alpha)) + S_\gamma t_2(t_3(S_\alpha + S_\beta) + t_2S_\alpha)) \\ =& 0. \end{aligned}$$

It follows that the infinite point orthogonal to $B'C'$ is $(t_2+t_3, -t_3, -t_2)$, and the perpendicular from A to $B'C'$ is the line

$$0 = \begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ t_2+t_3 & -t_3 & -t_2 \end{vmatrix} = t_2y - t_3z.$$

Similarly, the perpendiculars from B to $C'A'$ and C to $A'B'$ are the lines $t_3z - t_1x = 0$ and $t_1x - t_2y = 0$.

These three lines intersect at $Q = (t_2t_3 : t_3t_1 : t_1t_2)$. This is a finite point if and only if $A'B'C'$ is nondegenerate. If we write $Q = (u : v : w)$ in homogeneous barycentric coordinates with reference to ABC, then $(t_1, t_2, t_3) = \left(\frac{t}{u}, \frac{t}{v}, \frac{t}{w}\right)$ for some t. It follows that

$$\begin{aligned} A' &= Q' + \frac{t}{u}(-a^2, S_\gamma, S_\beta), \\ B' &= Q' + \frac{t}{v}(S_\gamma, -b^2, S_\alpha), \\ C' &= Q' + \frac{t}{w}(S_\beta, S_\alpha, -c^2). \end{aligned}$$

From these,

$$uA' + vB' + wC' = (u+v+w)Q',$$

and Q' has the same barycentric coordinates with reference to $A'B'C'$ as does Q with reference to ABC.

We summarize these results in the following theorem.

Theorem. Let ABC and $A'B'C'$ be nondegenerate triangles.
(a) ABC is orthologic to $A'B'C'$ if and only if $A'B'C'$ is orthologic to ABC.
(b) If the triangles are orthologic, the orthology center $\perp\ (ABC,\ A'B'C')$ has the same barycentric coordinates with reference to ABC as the orthology center $\perp\ (A'B'C',\ ABC)$ with reference to $A'B'C'$.

7.1.1 Examples

(1) The tangential triangle is orthologic to $\mathbf{T}$; both orthology centers are $\mathtt{O}$.
(2) The excentral triangle is orthologic to $\mathbf{T}$:

$$\perp(\mathbf{T},\ \mathsf{cev}^{-1}(\mathtt{I})) = \mathtt{I}, \qquad \perp(\mathsf{cev}^{-1}(\mathtt{I}),\ \mathbf{T}) = \mathtt{N_a}.$$

(3) Every pedal triangle is orthologic to $\mathbf{T}$:

$$\perp(\mathrm{Ped}(P),\ \mathbf{T}) = P, \qquad \perp(\mathbf{T},\ \mathrm{Ped}(P)) = P^*.$$

(4) Every Kiepert triangle is orthologic to $\mathbf{T}$:

$$\perp(\mathcal{K}(\theta),\ \mathbf{T}) = \mathtt{O}, \qquad \perp(\mathbf{T},\ \mathcal{K}(\theta)) = K\left(\frac{\pi}{2} - \theta\right).$$

(5) The triangle of reflections $\mathbf{T}^\dagger$ has vertices

$$\begin{aligned} A^\dagger &= (-a^2, 2S_\gamma, 2S_\beta),\\ B^\dagger &= (2S_\gamma, -b^2, 2S_\alpha),\\ C^\dagger &= (2S_\beta, 2S_\alpha, -c^2). \end{aligned}$$

It is orthologic to $\mathbf{T}$:

$$\perp(\mathbf{T}^\dagger,\ \mathbf{T}) = \mathtt{H}, \qquad \perp(\mathbf{T},\ \mathbf{T}^\dagger) = \mathtt{N}^*.$$

(6) Let P be a point on a given circle $\mathscr{C}$. Construct perpendiculars from P to the sidelines of $\mathbf{T}$, to intersect $\mathscr{C}$ again at X, Y, Z respectively. The triangle XYZ is oppositely similar to $\mathbf{T}$. It is clearly orthologic to $\mathbf{T}$: $\perp(XYZ,\ \mathbf{T}) = P$. Therefore, $\perp(\mathbf{T},\ XYZ)$ is a point on the circumcircle of $\mathbf{T}$.
(7)Let $A'B'C'$ be the cevian triangle of $P = (u : v : w)$, and O_a the circumcenter of triangle $PB'C'$. Similarly define O_b and O_c. The line O_bO_c is the perpendicular bisector of PA'. Therefore, AP is perpendicular to O_bO_c. Similarly, $BP \perp O_cO_a$ and $CP \perp O_aO_b$. It follows that triangle $O_aO_bO_c$ is orthologic to ABC, with $P = \perp(\mathbf{T},\ O_aO_bO_c)$.

P	$\perp(O_aO_bO_c,\ \mathbf{T})$
$\mathtt{G}$	$\mathtt{N}$
$\mathtt{H}$	$\mathtt{H}$
$\mathtt{S_t}$	$\mathtt{L}$
$\mathtt{E}$	$X(6759) = \mathtt{EL} \cap \mathtt{HN}^*$
$X(107)$	$X(6523) = \mathtt{L}X(107) \cap \mathtt{HL}^*$

The perpendicular from O_a to BC is the line.
The three perpendiculars intersect at the point Q.
The barycentrics of Q in $O_aO_bO_c$ are the same as those of P in ABC.
Nikolaos Dergiades has given a wonderful proof.

7.2 Perspective orthologic triangles

Now t is nonzero if $A'B'C'$ is nondegenerate. If these two triangles are perspective, then the perspector is the second intersection of the line QQ' with the rectangular circum-hyperbola through Q. This is

$$P = \left(\frac{S_By - S_Cz}{wy - vz} : \cdots : \cdots\right),$$

corresponding to

$$t = \frac{\sum_{\text{cyclic}} S_A ux(wy - vz)}{\sum_{\text{cyclic}} S_{BC}(wy - vz)}.$$

7.2.1

Consider the functions

$$\begin{aligned}
p_a(t) =&\ S^2(v+w)t + u(S_\alpha(v+w)(v'+w') + S_\beta vv' + S_\gamma ww'),\\
p_b(t) =&\ S^2(w+u)t + v(S_\beta(w+u)(w'+u') + S_\gamma ww' + S_\alpha uu'),\\
p_c(t) =&\ S^2(u+v)t + w(S_\gamma(u+v)(u'+v') + S_\alpha uu' + S_\beta vv');\\
q_a(t) =&\ -S^2t + S_\alpha(v+w)(v'+w') + S_\beta vv' + S_\gamma ww',\\
q_b(t) =&\ -S^2t + S_\beta(w+u)(w'+u') + S_\gamma ww' + S_\alpha uu',\\
q_c(t) =&\ -S^2t + S_\gamma(u+v)(u'+v') + S_\alpha uu' + S_\beta vv'.
\end{aligned}$$

The circles with diameters $A'Q$, $B'Q$, $C'Q$ have equations

$$\begin{aligned}
u(u+v+w)(u'+v'+w')(a^2yz+b^2zx+c^2xy) - (x+y+z)(p_ax + u(q_by+q_cz)) = 0,\\
v(u+v+w)(u'+v'+w')(a^2yz+b^2zx+c^2xy) - (x+y+z)(p_by + v(q_ax+q_cz)) = 0,\\
w(u+v+w)(u'+v'+w')(a^2yz+b^2zx+c^2xy) - (x+y+z)(p_cz + w(q_ax+q_by)) = 0.
\end{aligned}$$

The circle with diameter $A'Q$ intersects BC at two points given by

$$(u+v+w)(u'+v'+w')\cdot a^2yz - (y+z)(q_by+q_cz) = 0;$$

similarly for the other two circles.

The six intercepts on the three sidelines, if real, lie on the circle

$$\begin{aligned}
&(u+v+w)(u'+v'+w')(a^2yz+b^2zx+c^2xy)\\
&- (x+y+z)(q_ax+q_by+q_cz) = 0.
\end{aligned}$$

This has center the midpoint of PQ. The square radius of the circle is

$$QQ'^2 + t \cdot \frac{S^2}{(u+v+w)(u'+v'+w')}.$$

This circle is imaginary for $\frac{S^2}{(u+v+w)(u'+v'+w')} \cdot t < -QQ'^2$.

The circle with diameter AQ' has equation

$$(u'+v'+w')(a^2yz+b^2zx+c^2xy) - (x+y+z)((S_\alpha u' + S_\beta(w'+u'))y + (S_\alpha u' + S_\gamma(u' +$$

The line $B'C'$ has equation

$$\begin{aligned} &q_a x - (S^2t + S_\alpha u'(v+w) + S_\beta v(w'+u') - S_\gamma ww')y \\ &\quad - (S^2t + S_\alpha u'(v+w) - S_\beta vv' + S_\gamma w(u'+v'))z = 0. \end{aligned}$$

Now, consider

$$\begin{aligned} &(u+v+w)(u'+v'+w')(a^2yz+b^2zx+c^2xy) \\ &- (x+y+z)(q_a x + q_b y + q_c z) \\ &+ (x+y+z)(q_a x - (S^2t + S_\alpha u'(v+w) + S_\beta v'(w+u) - S_\gamma ww')y \\ &- (S^2t + S_\alpha u'(v+w) - S_\beta vv' + S_\gamma w'(u+v))z). \end{aligned}$$

$$\begin{aligned} & q_b + (S^2t + S_\alpha u'(v+w) + S_\beta v(w'+u') - S_\gamma ww') \\ =\ & -S^2t + S_\beta(w+u)(w'+u') + S_\gamma ww' + S_\alpha uu' \\ & + (S^2t + S_\alpha u'(v+w) + S_\beta v(w'+u') - S_\gamma ww') \\ =\ & (u+v+w)(S_\alpha u' + S_\beta(w'+u')); \\ & q_c + (S^2t + S_\alpha u'(v+w) - S_\beta vv' + S_\gamma w(u'+v')) \\ =\ & (u+v+w)(S_\alpha u' + S_\gamma(u'+v')). \end{aligned}$$

Therefore, the circle of diameter AQ' is in the pencil generated by $\mathcal{C}$ and the line $B'C'$.

Chapter 8

The circumcircle

8.1 The circumcircle

The equation of the circumcircle of $\mathbf{T}$,

$$a^2yz + b^2zx + c^2xy = 0,$$

is derived from the fact that the circumcircle consists of the isogonal conjugate of infinite points. Here are a few triangle centers on the circumcircle with simple coordinates.

- The Euler reflection point $\mathtt{E} = \left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right)$ may be regarded as the isogonal conjugate of the infinite point of $a^2x + b^2y + c^2z = 0$, the isotomic line of the Lemoine axis $\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = 0$.

- The isogonal conjugate of the infinite point of the Lemoine axis is the Steiner point

$$\mathtt{S_t} = \left(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2}\right).$$

- The isogonal conjugate of the infinite point of the trilinear polar of the incenter $\mathtt{I}$:

$$(\mathscr{L}(\mathtt{I})_\infty)^* = \left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right).$$

- The isogonal conjugate of the infinite point of the line $\mathtt{IG}$:

$$(\mathtt{IG}_\infty)^* = \left(\frac{a^2}{b+c-2a} : \frac{b^2}{c+a-2b} : \frac{c^2}{a+b-2c}\right).$$

8.1.1 Tangents to the circumcircle

Proposition. A line $fx+gy+hz=0$ is tangent to the circumcircle if and only if

$$a^4f^2+b^4g^2+c^4h^2-2b^2c^2gh-2c^2a^2hf-2a^2b^2fg=0. \tag{8.1}$$

If this condition is satisfied, the point of tangency is

$$P=(a^2(-a^2f+b^2g+c^2h):b^2(a^2f-b^2g+c^2h):c^2(a^2f+b^2g-c^2h)).$$

Proof. Clearly at most one of f, g, h can be zero. Assume $g, h \neq 0$. Eliminating x from $fx+gy+hz=0$ and $a^2yz+b^2zx+c^2xy=0$, we obtain

$$c^2gy^2+(-a^2f+b^2g+c^2h)yz+b^2hz^2=0.$$

This is a quadratic in y, z with discriminant

$$\begin{aligned}&(-a^2f+b^2g+c^2h)^2-4b^2c^2gh\\=&a^4f^2+b^4g^2+c^4h^2-2b^2c^2gh-2c^2a^2hf-2a^2b^2fg.\end{aligned}$$

The line $fx+gx+hz=0$ is tangent to the circumcircle if and only if this discriminant vanishes. This is the condition given in (8.1).

The polynomial in (8.1) can be rewritten in two different ways, showing that when the condition (8.1) is satisfed,
(i) the point P lies on the line $fx+gy+hz=0$

$$\begin{aligned}&a^4f^2+b^4g^2+c^4h^2-2b^2c^2gh-2c^2a^2hf-2a^2b^2fg\\=&f\cdot a^2(-a^2f+b^2g+c^2h)+g\cdot b^2(a^2f-b^2g+c^2h)\\&+h\cdot c^2(a^2f+b^2g-c^2h);\end{aligned}$$

(ii) the point P also lies on the circumcircle:

$$\begin{aligned}&a^4f^2+b^4g^2+c^4h^2-2b^2c^2gh-2c^2a^2hf-2a^2b^2fg\\=&a^2\cdot b^2(a^2f-b^2g+c^2h)\cdot c^2(a^2f+b^2g-c^2h)\\&+b^2\cdot c^2(a^2f+b^2g-c^2h)\cdot a^2(-a^2f+b^2g+c^2h)\\&+c^2\cdot a^2(-a^2f+b^2g+c^2h)\cdot b^2(a^2f-b^2g+c^2h)=0.\end{aligned}$$

The point P is therefore the point of tangency of the line and the circumcircle. $\square$

Corollary. The trilinear polar of a point P is tangent to the circumcircle if and only if its isogonal conjugate P^* lies on the Steiner inellipse. If this condition is satisfied, the point of tangency is the cevian quotient $P/\mathtt{K}$.

Corollary. Let $(u : v : w)$ be an infinite point so that $\left(\frac{a^2}{u} : \frac{b^2}{v} : \frac{c^2}{w}\right)$ lies on the circumcircle. The tangent to the circumcircle at this point is the trilinear polar of the isogonal conjugate of $(u^2 : v^2 : w^2)$.

Example. The tangent to the circumcircle at the point $\left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right)$ is the line

$$(b-c)^2x + (c-a)^2y + (a-b)^2z = 0.$$

8.2 Simson lines

We begin with a fundamental theorem.

Theorem. The pedals of a point on the sidelines of $\mathbf{T}$ are collinear if and only if the point lies on the circumcircle of $\mathbf{T}$.

Proof. Let P be a point with pedals $P_{[a]}$, $P_{[b]}$, $P_{[c]}$ on the sidelines of $\mathbf{T}$.

$$\begin{aligned}
\angle(P_{[a]}P_{[b]},\ P_{[a]}P_{[c]}) &= \angle(P_{[a]}P_{[b]},\ PP_{[a]}) + \angle(PP_{[a]},\ P_{[a]}P_{[c]})\\
&= \angle(P_{[b]}P_{[a]},\ P_{[a]}P) + \angle(PP_{[a]},\ P_{[a]}P_{[c]})\\
&= \angle(P_{[b]}C,\ CP) + \angle(PB,\ BP_{[c]})\\
&= \angle(AC,\ CP) + \angle(PB,\ BA)\\
&= \angle(AC,\ CP) - \angle(AB,\ BP).
\end{aligned}$$

It follows from this that $\angle(P_{[a]}P_{[b]},\ P_{[a]}P_{[c]}) = 0$ if and only if $\angle(AB, BP) = \angle(AC, CP)$. In other words, $P_{[a]}$, $P_{[b]}$, $P_{[c]}$ are collinear if and only if A, B, C, P are concyclic. $\square$

For a point P on the circumcircle of $\mathbf{T}$, we denote by $\mathsf{s}(P)$ the line containing its pedals on the sidelines, and call this the Simson line of P.

If P is a vertex of $\mathbf{T}$, $\mathsf{s}(P)$ is the altitude through the vertex.

If P is the antipode of a vertex on the circumcircle, $\mathsf{s}(P)$ is the sideline opposite to the vertex.

Let P be the intersection of the circumcircle with the bisector of angle A. What is the Simson line of P?

The external bisector of $\mathtt{G}_a$ of the inferior triangle.

The bisectors of the inferior triangle are Simson lines. What are the corresponding points on the circumcircle?

Proposition. Let the line $PP_{[a]}$ intersect the circumcircle of $\mathbf{T}$ again at Q_a. The line AQ_a is parallel to $\mathsf{s}(P)$.

Proof.

$$\begin{aligned}
\angle(AQ_a,\ P_{[b]}P_{[a]}) &= \angle(AQ_a,\ Q_aP) + \angle(Q_aP,\ P_{[b]}P_{[a]})\\
&= \angle(AC,\ CP) + \angle(PP_{[a]},\ P_{[a]}P_{[b]})\\
&= \angle(AC,\ CP) + \angle(PC,\ CP_{[b]})\\
&= \angle(AC,\ CP) + \angle(CP,\ AC)\\
&= 0.
\end{aligned}$$

Therefore, AQ_a is parallel to $\mathsf{s}(P) = P_{[b]}P_{[a]}$. $\square$

Let P be a point on the circumcircle with Simson line $\mathsf{s}(P)$. is perpendicular to the isogonal lines of AP, BP, CP. Therefore, $\mathsf{s}(P)$ is perpendicular to the lines defining P on the circumcircle.

Proposition. The Simson line $\mathsf{s}(P)$ is orthogonal to the lines defining P as the isogonal conjugate of their (common) infinite point.

Proof. Let Q be the reflection of P in the bisector of angle A.

$$\begin{aligned}
\angle(P_{[b]}P_{[c]},\ AQ) &= \angle(P_{[b]}P_{[c]},\ AC) + \angle(AC,\ AQ)\\
&= \angle(P_{[c]}P_{[b]},\ P_{[b]}A) + \angle(AC,\ AQ)\\
&= \angle(P_{[c]}P,\ PA) + \angle(AC,\ AQ)\\
&= \frac{\pi}{2} - \angle(AP,\ AB) + \angle(AC,\ AQ)\\
&= \frac{\pi}{2}.
\end{aligned}$$

$\square$

Corollary. The Simson lines of antipodal points are orthogonal.

If $P = \left(\frac{a^2}{u} : \frac{b^2}{v} : \frac{c^2}{w}\right)$ for an infinite point $(u : v : w)$, then the Simson line $\mathsf{s}(P)$ has infinite point

$$(S_\beta v - S_\gamma w :\ S_\gamma w - S_\alpha u : S_\alpha u - S_\beta v).$$

Let P be a point on the circumcircle of $\mathbf{T}$. Consider the reflection $P_a^\dagger$ of P in the sideline BC.

Proposition. The line $\mathtt{H}P_a^\dagger$ is parallel to the Simson line $\mathsf{s}(P)$.

Proof. Let $\mathtt{H}_a$ be the reflection of $\mathtt{H}$ in BC. Since $\mathtt{H}_a$ lies on the circumcircle of $\mathbf{T}$, the circle $\mathscr{C}_a := \mathtt{H}BC$ is the reflection of the circumcircle in BC. The pedal $P_{[a]}$ is the midpoint of $PP_a^\dagger$. Let $X_a^\dagger$ be the reflection of X_a in BC. This is a point on $\mathscr{C}_a$.

Note that AP and $\mathtt{H}X_a^\dagger$ are reflections of $\mathtt{H}_a^\dagger X_a$ in two parallel lines, BC and the diameter of $\mathscr{C}$ parallel to BC. The equality $\mathbf{AP} = \mathtt{H}\mathbf{X}_\mathbf{a}^\dagger$ of vectors gives $\mathbf{A}\mathtt{H} = \mathbf{PX}_\mathbf{a}^\dagger = \mathbf{X_aP}_\mathbf{a}^\dagger$. Therefore, $\mathtt{H}\mathbf{P}_\mathbf{a}^\dagger = \mathbf{AX_a}$. In particular, $\mathtt{H}P_a^\dagger$ is parallel to AX_a, and to the Simson line $\mathsf{s}(P)$ by Proposition ?. $\square$

Corollary. The Simson line $\mathtt{s}(P)$ bisects the segment $P\mathtt{H}$.

Proof. Since $P_{[a]}$ is the midpoint of $PP_a^\dagger$, the parallel to $P_a^\dagger\mathtt{H}$ through $P_{[a]}$ bisects the segment $P\mathtt{H}$. Since $P_a^\dagger\mathtt{H}$ is parallel to the Simson line $\mathtt{s}(P)$, the parallel through $P_{[a]}$ is indeed the Simson line. □

Proposition. The Simson lines of antipodal points intersect orthogonally at a point on the nine-point circle.

Proof. Let P and P' be antipodal points on the circumcircle. The Simson lines $\mathtt{s}(P)$ and $\mathtt{s}(P')$ contain respectively the midpoints of the segments $\mathtt{H}P$ and $\mathtt{H}P'$. These midpoints are antipodal on the nine-point circle of **T**. Since the Simson lines are orthogonal to each other, their intersection also is a point on the nine-point circle. □

Examples. The Simson line of the Euler reflection point $\mathtt{E}$ is the parallel to the Euler line through the midpoint of $\mathtt{EH}$, which is

$$X_{113} = (-2a^4+a^2(b^2+c^2)+(b^2-c^2)^2)(a^4(b^2+c^2)-2a^2(b^4-b^2c^2+c^4)+(b^2-c^2)^2(b^2+c^2)$$

This is the line

$$\sum_{\text{cyclic}} \frac{S_B - S_C}{S_A(S_B+S_C) - 2S_{BC}} x = 0.$$

The other intersection with the nine-point circle is

$$X_{3258} = (b^2-c^2)^2((b^2+c^2-a^2)^2 - b^2c^2)(2a^4 - a^2(b^2+c^2) - (b^2-c^2)^2).$$

This line contains $X(n)$ for the following values of n:

$$30, 113, 1495, 1511, 1514, 1524, 1525, 1531, 1533, 1539, 1544, 1545, \ 1546, 1553, 1554,$$

The Simson line of the Steiner point $\mathtt{S_t}$ is

$$\sum_{\text{cyclic}} \frac{S_\beta - S_\gamma}{S_{\alpha\alpha} - S_{\beta\gamma}} x = 0.$$

This line contains
(i) $X(114)$: antipode of Kiepert center on the nine-point circle,
(ii) $X(325)$
(iii) $X(511)$: parallel to the Brocard axis
(iv) $X(1513)$ on the Euler line,

(v) $X(2679)$: center of the rectangular hyperbola with asymptotes parallel to the Brocard and Lemoine axes.

The line of reflections:

$$\sum_{\text{cyclic}} a^2(b^2-c^2)S_\alpha x = 0.$$

is the line $\texttt{HH}^\bullet$ containing $X(n)$ for the following values of n:

$4, 69, 76, 264, 286, 311, 314, 315, 317, 340, 511, 877, 1232, 1234, 1235, 1236, 1330, 1352, 1531,$

8.3 Line of reflections

Since the reflections of P in the sidelines of $\mathbf{T}$ are images of the pedals of P under the homothety $\mathsf{h}(P,2)$, we conclude that the reflections of P in the sidelines are collinear if and only if P lies on the circumcircle of $\mathbf{T}$. The line of reflections of P is the image of the Simson line $\mathsf{s}(P)$ under the same homothety.

Theorem. The line of reflections of a point on the circumcircle contains the orthocenter $\mathtt{H}$.

Proof. The lines $\mathtt{H}P_a^\dagger$, $\mathtt{H}P_b^\dagger$, $\mathtt{H}P_c^\dagger$ are all parallel to the Simson line $\mathsf{s}(P)$. It follows that the four points $\mathtt{H}$, $P_a^\dagger$, $P_b^\dagger$, $P_c^\dagger$ are collinear. □

Corollary. If $P = \left(\frac{a^2}{u} : \frac{b^2}{v} : \frac{c^2}{w}\right)$ for an infinite point $(u;v:w)$, the line of reflections of P is

$$S_\alpha ux + S_\beta vy + S_\gamma wz = 0.$$

Proof. This clearly contains the orthocenter $\mathtt{H} = \left(\frac{1}{S_\alpha} : \frac{1}{S_\beta} : \frac{1}{S_\gamma}\right)$ and has the same infinite point as $\mathsf{s}(P)$. □

Example. (The Euler reflection point $\mathtt{E}$) Since the equation of the Euler line is

$$S_\alpha(b^2-c^2)x + S_\beta(c^2-a^2)y + S_\gamma(a^2-b^2)z = 0,$$

by choosing the infinite point $(u:v:w) = (b^2-c^2 : c^2-a^2 : a^2-b^2)$, we obtain the point

$$\mathtt{E} := \left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right)$$

on the circumcircle, whose reflections in the sidelines lie on the Euler line. This is called the **Euler reflection point**.

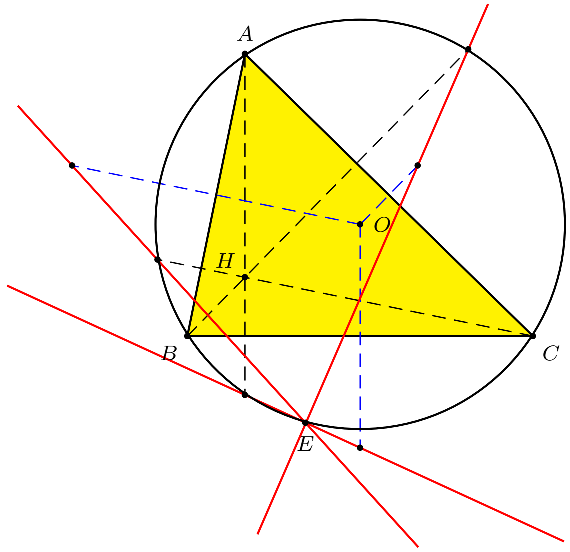


Figure 8.1: Not the right illustration

8.4 Reflections of a line in the sidelines of T

Let $\mathscr{L}$ be a given line. Consider its reflections $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ in the sidelines a, b, c.

$$\begin{aligned}\angle(\mathscr{L}_b,\mathscr{L}_c) &= \angle(\mathscr{L}_b,\mathscr{L})+\angle(\mathscr{L},\mathscr{L}_c)\\ &= 2\angle(\mathsf{b},\mathscr{L})+2\angle(\mathscr{L},\mathsf{c})\\ &= 2\angle(\mathsf{b},\mathsf{c}).\end{aligned}$$

Important note: $\angle(\mathsf{b},\mathsf{c})=\frac{1}{2}\angle(\mathscr{L}_b,\mathscr{L}_c)+\frac{\pi}{2}$.

Let $\mathbf{T}^\dagger(\mathscr{L})$ be the triangle bounded by these three reflection lines. This is oppositely similar to the tangential triangle since $\angle(\mathsf{t}_B,\mathsf{t}_C)=-2\angle(\mathsf{b},\mathsf{c})$ for the tangents t_B and t_C to the circumcircle of $\mathbf{T}$ at the vertices of $\mathbf{T}$.

Proposition. The triangle $\mathbf{T}^\dagger(\mathscr{L})$ is perspective with ABC at a point on the circumcircle of $\mathbf{T}$.

Proof. The vertex A is equidistant from $\mathscr{L}$, $\mathscr{L}_b$ and $\mathscr{L}_c$. Therefore it lies on the bisector of an angle between $\mathscr{L}_b$ and $\mathscr{L}_c$. Similarly, each of B and C is on the bisector of an angle between two of the reflection lines. The triangle ABC is perspective with $\mathbf{T}^\dagger(\mathscr{L})$, at a point Q which is the incenter or an excenter of $\mathbf{T}^\dagger(\mathscr{L})$.

$$\begin{aligned}
\angle(BQ, QC) = & \angle(BQ, \mathscr{L}_a) + \angle(\mathscr{L}_a, QC) \\
= & \frac{1}{2}\angle(\mathscr{L}_c, \mathscr{L}_a) + \frac{1}{2}\angle(\mathscr{L}_a, \mathscr{L}_b) \\
= & \angle(\mathsf{c}, \mathsf{a}) + \frac{\pi}{2} + \angle(\mathsf{a}, \mathsf{b}) + \frac{\pi}{2} \\
= & \angle(\mathsf{c}, \mathsf{b}) \\
= & \angle(BA, AC).
\end{aligned}$$

Therefore the perspector Q lies on the circumcircle of $\mathbf{T}$. □

Note that

$$\begin{aligned}
\angle(AQ, AB) = & \angle(AQ, \mathscr{L}_c) + \angle(\mathscr{L}_c, \mathsf{c}) \\
= & \frac{1}{2}\angle(\mathscr{L}_b, \mathscr{L}_c) + \angle(\mathscr{L}_c, \mathsf{c}) \\
= & \angle(\mathsf{b}, \mathsf{c}) + \frac{\pi}{2} + \angle(\mathsf{c}, \mathscr{L}) \\
= & \angle(\mathsf{b}, \mathscr{L}) + \frac{\pi}{2}.
\end{aligned}$$

$$\angle(AC, \mathscr{L}^{\perp}) = \angle(\mathsf{b}, \mathscr{L}) + \angle(\mathscr{L}, \mathscr{L}^{\perp}) = \angle(\mathsf{b}, \mathscr{L}) + \frac{\pi}{2}.$$

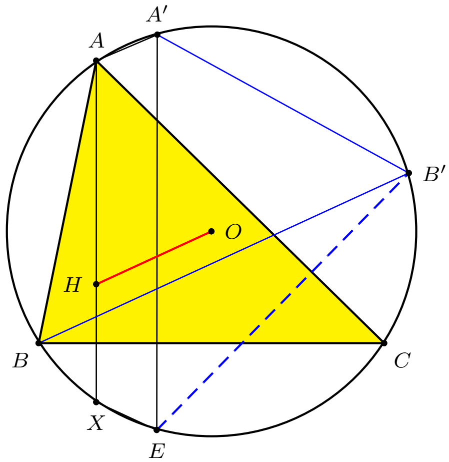
Therefore, Q is the isogonal conjugate of the infinite point of $\mathscr{L}^{\perp}$.

When are the reflection lines concurrent?
If the reflection lines of $\mathscr{L}$ are concurrent, the point of concurrency must be the perspector Q on the circumcircle. The reflections of Q in the sidelines are all on the line $\mathscr{L}$. This shows that $\mathscr{L}$ must contain the orthocenter H.

Conversely, if the line $\mathscr{L}$ contains the orthocenter H, then the lines of reflections are concurrent.

Theorem. The triangle $\mathbf{T}^{\dagger}(\mathscr{L})$ is degenerate, i.e., the reflection lines $\mathscr{L}_a^{\dagger}$, $\mathscr{L}_b^{\dagger}$, $\mathscr{L}_c^{\dagger}$ are concurrent, if and only if $\mathscr{L}$ contains the orthocenter.

Proof. Construct the parallel to $\mathscr{L}$ through A to intersect the circumcircle at A', and the perpendiculars to BC from A and A', to intersect the circumcircle again at X and Q respectively. The line XQ is the reflection of $\mathscr{L}$ in BC.

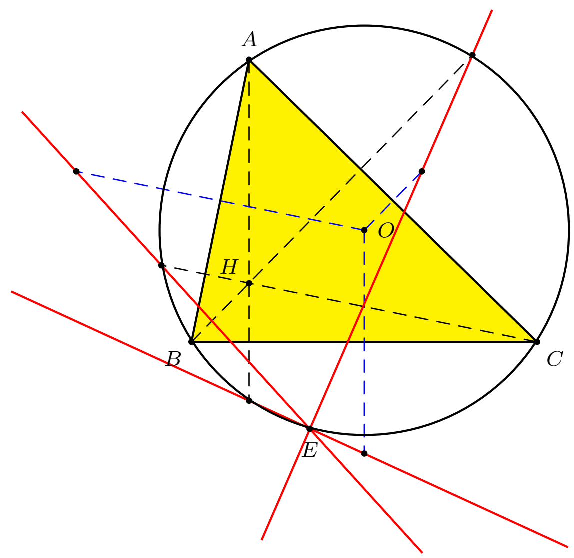


Now let the parallel of $\mathscr{L}$ through B intersect the circumcircle at B'. We show that $B'Q \perp CA$:

$$\begin{aligned}
(B'Q,\ AC) &= (B'Q,\ B'C) + (B'C,\ AC) \\
&= (AQ,\ AC) + (A'B',\ BB') \\
&= (AQ,\ AC) + (A'B',\ AB') + (AB',\ BB') \\
&= (AQ,\ AC) + (A'B',\ AB') + (AC,\ BC) \\
&= (AQ,\ AC) + (A'X,\ AX) + (AC,\ BC) \\
&= (AQ,\ AC) + (AX,\ AQ) + (AC,\ BC) \\
&= (AX,\ BC) \\
&= \frac{\pi}{2}.
\end{aligned}$$

Therefore, the reflection line $\mathscr{L}_b^{\dagger}$ contains the point Q, so does the reflection line $\mathscr{L}_c^{\dagger}$ from a similar calculation. □

The Euler reflection point



8.4.1

We give a simple algebraic proof of Theorem ?.

A line $\mathscr{L}$ through H has equation

$$uS_\alpha x + vS_\beta y + wS_\gamma z = 0$$

for an infinite point $(u : v : w)$. The reflection line $\mathsf{a}^\dagger(\mathscr{L})$ has equation

$$px + uS_\alpha x + vS_\beta y + wS_\gamma z = 0$$

for some p. Since this contains the point

$$\mathtt{H}_a^\dagger = (-a^2 S_{\beta\gamma} : (S^2 + S_{\beta\gamma})S_\gamma : (S^2 + S_{\beta\gamma})S_\beta),$$

we have

$$-(p + uS_\alpha) \cdot a^2 S_{\beta\gamma} + (S^2 + S_{\beta\gamma})(vS_\beta \cdot S_\gamma + wS_\gamma S_\beta) = 0.$$

$$-(p + uS_\alpha) \cdot a^2 - (S^2 + S_{\beta\gamma})u = 0.$$

$$p = -u(a^2 S_\alpha + S_\beta S_\gamma) = -u \cdot S^2.$$

Therefore, $\mathsf{a}^\dagger(\mathscr{L})$ (or $\mathscr{L}_a^\dagger$) is the line

$$-(S^2 + S_{\beta\gamma})ux + a^2 S_\beta vy + a^2 S_\gamma wz = 0.$$

It is easy to see that this contains the point $\left(\frac{a^2}{u}:\frac{b^2}{v}:\frac{c^2}{w}\right)$ (on the circumcircle).

Similar calculations show that the reflection lines $\mathscr{L}_b^\dagger$ and $\mathscr{L}_c^\dagger$ contain the same point on the circumcircle.

Example. If $P=(u:v:w)$, the line $\mathtt{H}P$ has equation

$$S_\alpha(S_\beta v-S_\gamma w)x+S_\beta(S_\gamma w-S_\alpha u)+S_\gamma(S_\alpha u-S_\beta v)z=0.$$

The reflections of $\mathtt{H}P$ in the sidelines of **T** are concurrent at

$$\left(\frac{a^2}{S_\beta v-S_\gamma w}:\frac{b^2}{S_\gamma w-S_\alpha u}:\frac{c^2}{S_\alpha u-S_\beta v}\right).$$

on the circumcircle.

$P_b^\dagger P_b{}^\dagger P^\dagger{}_b$.

8.4.2 Perspectivity of reflection triangles

Theorem. The reflection triangles of $P = (u : v : w)$ and $Q = (x : y : z)$ are perspective if and only if the line PQ contains the orthocenter H.

Proof. The condition of perspectivity is

$$\left(\sum_{\text{cyclic}} S_A(S_Bv - S_Cw)x\right) Q = 0,$$

where

$$Q = (u+v+w)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)\left(\sum_{\text{cyclic}}(c^2v^2 + (b^2 + c^2 - a^2)vw + b^2u\right.$$

This defines the square of the distance between P and Q. □

If the line PQ contains H, the perspector is the intersection of the reflections of the line PQ in the sidelines.

8.5 Circumcevian triangles

Let XYZ be the circumcevian triangle of P, and $X'Y'Z'$ be that of P^*. The lines XX', YY', ZZ' bound a triangle homothetic to ABC. What is the homothetic center? [1]

Lemma. The vertices of the circumcevian triangle cev$^{\circ}(P)$ are the isogonal conjugates of the infinite points of the cevian lines of P^*.

More generally, for $P = (u : v : w)$ and $Q = (x : y : z)$, the lines joining the corresponding vertices of the circumcevian triangles of P and Q bound a triangle perspective with ABC at

$$\left(\frac{a^2}{\left(\frac{b^2}{v}+\frac{c^2}{w}\right)\left(\frac{b^2}{y}+\frac{c^2}{z}\right)-\frac{a^2}{u}\cdot\frac{a^2}{x}} : \cdots : \cdots\right).$$

For the Gergonne and Nagel points, this yields

$$\left(\frac{a^2}{(b+c-a)(3a^2+(b-c)^2)} : \cdots : \cdots\right)$$

with ETC (6-9-13)-search number $0.682731527705\cdots$

[1] If $P = (u : v : w)$, the homothetic center is $\left(\frac{a^2}{u(S_\alpha(v+w)^2+S_\beta v^2+S_\gamma w^2)} : \cdots : \cdots\right)$.

Proposition. The circumcevian triangle of $P = (u : v : w)$ is perspective with the tangential triangle at

$$\left(a^2\left(-\frac{a^4}{u^2}+\frac{b^4}{v^2}+\frac{c^4}{w^2}\right) : \cdots : \cdots\right).$$

When are the lines joining the corresponding vertices of the circumcevian triangles of P and Q concurrent?

This is the case if and only if one of the points lies on the circumcircle (trivial) or they are inverse in the circumcircle.

Proposition. Let $P = (u : v : w)$. The tangents of the circumcircle at the vertices of $\mathrm{cev}^{\circ}(P)$ bound a triangle with perspector

$$\left(\frac{a^2}{-a^2vw+b^2wu+c^2uv} : \cdots : \cdots\right).$$

This is the isogonal conjugate of the superior of the isogonal conjugate of P.

8.5.1 Circumcevian triangle

Let $P = (u : v : w)$. The lines AP, BP, CP intersect the circumcircle again at the vertices of the *circumcevian triangle* $\mathsf{ocev}(P)$ of P:

$$\boxed{\mathsf{ocev}(P) = \begin{pmatrix} A^{(P)} \\ B^{(P)} \\ C^{(P)} \end{pmatrix} = \begin{pmatrix} \frac{-a^2vw}{c^2v+b^2w} & : & v & : & w \\ u & : & \frac{-b^2wu}{a^2w+c^2u} & : & w \\ u & : & v & : & \frac{-c^2uv}{b^2u+a^2v} \end{pmatrix}}.$$

The circumcevian triangle $\mathsf{ocev}(P)$ is similar to the pedal triangle $\mathsf{ped}(P)$ and the reflection triangle $\mathsf{rfl}(P)$.

Theorem. The circumcevian triangle $\mathsf{ocev}(P)$ is perspective with the tangential triangle $\mathsf{cev}^{-1}(K)$ at

$$\boxed{\wedge(\mathsf{ocev}(P),\ \mathsf{cev}^{-1}(K))\left(a^2(-\frac{a^4}{u^2}+\frac{b^4}{v^2}+\frac{c^4}{w^2}) : b^2(\frac{a^4}{u^2}-\frac{b^4}{v^2}+\frac{c^4}{w^2}) : c^2(\frac{a^4}{u^2}+\frac{b^4}{v^2}-\frac{c^4}{w^2})\right)}.$$

Proof. The line joining $A^{(P)}$ to $K_a = (-a^2 : b^2 : c^2)$ has equation

$$(b^4w^2 - c^4v^2)x + a^2b^2w^2y - c^2a^2v^2z = 0.$$

Similarly, the lines $B^{(P)}P_b$ and $C^{[P]}P_c$ have equations

$$\begin{array}{rcrcrcl} -a^2b^2w^2x & + & (c^4u^2 - a^4w^2)y & + & b^2c^2u^2z & = & 0, \\ c^2a^2v^2x & - & b^2c^2u^2y & + & (a^4v^2 - b^4u^2)z & = & 0. \end{array}$$

These lines are concurrent since

$$\begin{aligned} & a^2((b^4w^2 - c^4v^2)x + a^2b^2w^2y - c^2a^2v^2z) \\ & +b^2(-a^2b^2w^2x + (c^4u^2 - a^4w^2)y + b^2c^2u^2z) \\ & +c^2(c^2a^2v^2x - b^2c^2u^2y + (a^4v^2 - b^4u^2)z) \\ & =0. \end{aligned}$$

These lines intersect at

$$\begin{aligned} & x : y : z \\ =& \begin{vmatrix} (c^4u^2 - a^4w^2) & b^2c^2u^2 \\ -b^2c^2u^2 & (a^4v^2 - b^4u^2) \end{vmatrix} : - \begin{vmatrix} -a^2b^2w^2 & b^2c^2u^2 \\ c^2a^2v^2 & (a^4v^2 - b^4u^2) \end{vmatrix} : \begin{vmatrix} -a^2b^2w^2 & (c^4u^2 - a^4w^2) \\ c^2a^2v^2 & -b^2c^2u^2 \end{vmatrix} \\ =&(c^4u^2 - a^4w^2)(a^4v^2 - b^4u^2) + b^2c^2u^2 \cdot b^2c^2u^2 \\ & \quad : a^2b^2((a^4v^2 - b^4u^2)w^2 + c^4u^2v^2)\ : a^2c^2(b^4w^2u^2 - (c^4u^2 - a^4w^2)v^2) \\ =&a^4(-a^4v^2w^2 + b^4w^2u^2 + c^4u^2v^2) \\ & \quad : a^2b^2(a^4v^2w^2 - b^4w^2u^2 + c^4u^2v^2) : a^2c^2(a^4v^2w^2 + b^4w^2u^2 - c^4u^2v^2). \end{aligned}$$

$\square$

P	$\wedge(\mathsf{ocev}(P),\ \mathsf{cev}^{-1}(K))$
G	$(a^2(b^4+c^4-a^4) : b^2(c^4+a^4-b^4) : c^2(a^4+b^4-c^4))$
H	$\left(\frac{a^2(S^2-S_{\alpha\alpha})}{S_\alpha} : \frac{b^2(S^2-S_{\beta\beta})}{S_\beta} : \frac{c^2(S^2-S_{\gamma\gamma})}{S_\gamma}\right)$

Note that this perspector is the same for the points in the harmonic quadruple of P. Indeed, if X', Y', Z' are the second intersections of the circumcircle with the sides of the anticevian triangle of P, then the line XX' passes through the point $(-a^2 : b^2 : c^2)$, and the circumcevian triangle of A^P is $XY'Z'$.

8.5.2 The circumcevian triangle of H

The circumcevian triangle of H is homothetic to the tangential triangle, with ratio $r : R$. The homothetic center is

The circumcevian triangles of P and Q are perspective if and only if one of them lies on the circumcircle or they are inversive. In the latter case, the two triangles are oppositely congruent about the line containing the two points.

Proposition. The lines joining corresponding vertices of the circumcevian triangles of $(u : v : w)$ and $(x : y : z)$ bound a triangle perspective with ABC at

$$\begin{aligned} & \left(\frac{a^2}{(b^2wu+c^2uv)(b^2zx+c^2xy)-a^4vwxy} : \cdots : \cdots\right) \\ = & \left(\frac{a^2}{\left(\frac{b^2}{v}+\frac{c^2}{w}\right)\left(\frac{b^2}{y}+\frac{c^2}{z}\right)-\frac{a^2}{u}\cdot\frac{a^2}{x}} : \cdots : \cdots\right). \end{aligned}$$

8.5.3 The circum-tangential triangle

The circum-tangential triangle of P is bounded by the tangents to the circumcircle at the vertices of $\mathsf{ocev}(P)$. These tangents are the lines

$$\frac{1}{a^2}\left(\frac{b^2}{v}+\frac{c^2}{w}\right)^2 x+\frac{b^2}{v^2}y+\frac{c^2}{w^2}z=0,$$

etc. They bound a triangle with perspector

$$\left(\frac{a^2}{-a^2vw+b^2wu+c^2uv}:\cdots:\cdots\right).$$

Examples

(1) The circumtangential triangle of I is homothetic to ABC with ratio of homothety $\frac{R}{r}$. The homothetic center is T_-, the exsimilicenter of (O) and (I)

(2) The circumtangential triangle of K has perspector K.

(3) The circumtangential triangle of G has perspector $\left(\frac{a^2}{S_\alpha}:\frac{b^2}{S_\alpha}:\frac{c^2}{S_\gamma}\right)$. [2]

(4) For $P=T$, $Q=T_+$.

Corollary. The circum-tangential triangle of $(u:v:w)$ is perspective with ABC at

$$\left(\frac{a^2}{b^2wu+c^2uv-a^2vw}:\frac{b^2}{c^2uv+a^2vw-b^2wu}:\frac{c^2}{a^2vw+b^2wu-c^2uv}\right).$$

Exercise

1. Let DEF be the circumcevian triangle of G, then [3]

$$\frac{AG}{GD}+\frac{BG}{GE}+\frac{CG}{GF}=3.$$

2. The locus of P with circumcevian triangle DEF such that the sum of the ratios is 3 is the circle with diameter OG.

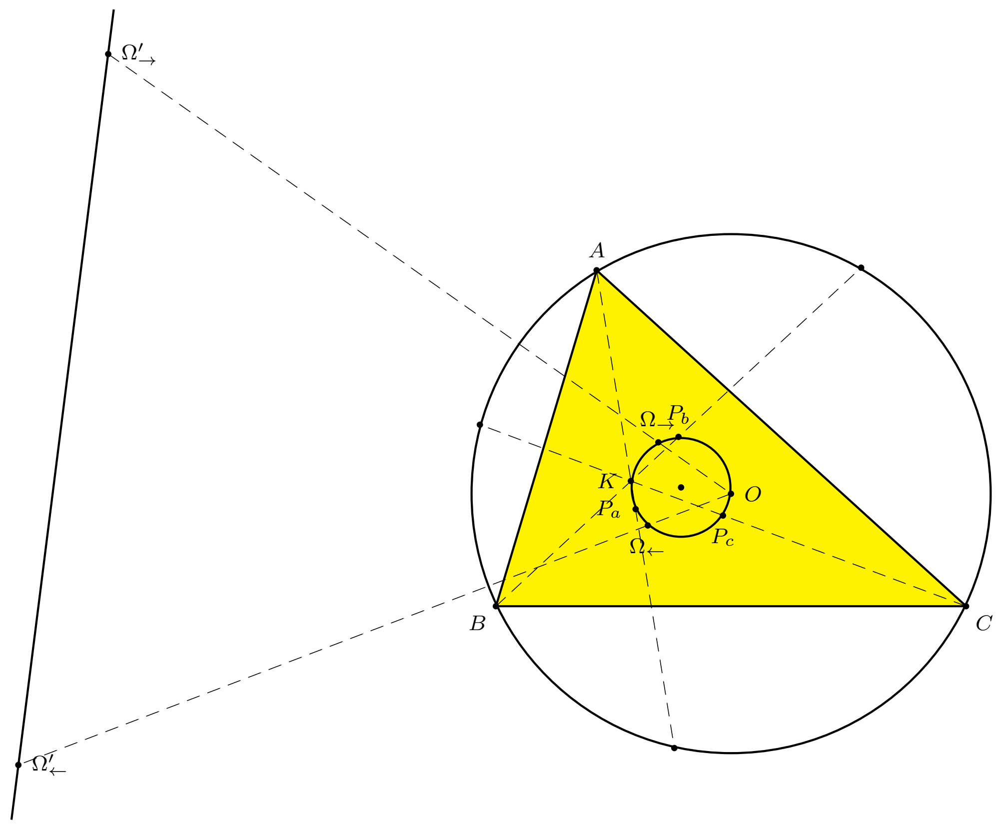
[2] This is X_{25} in ETC.

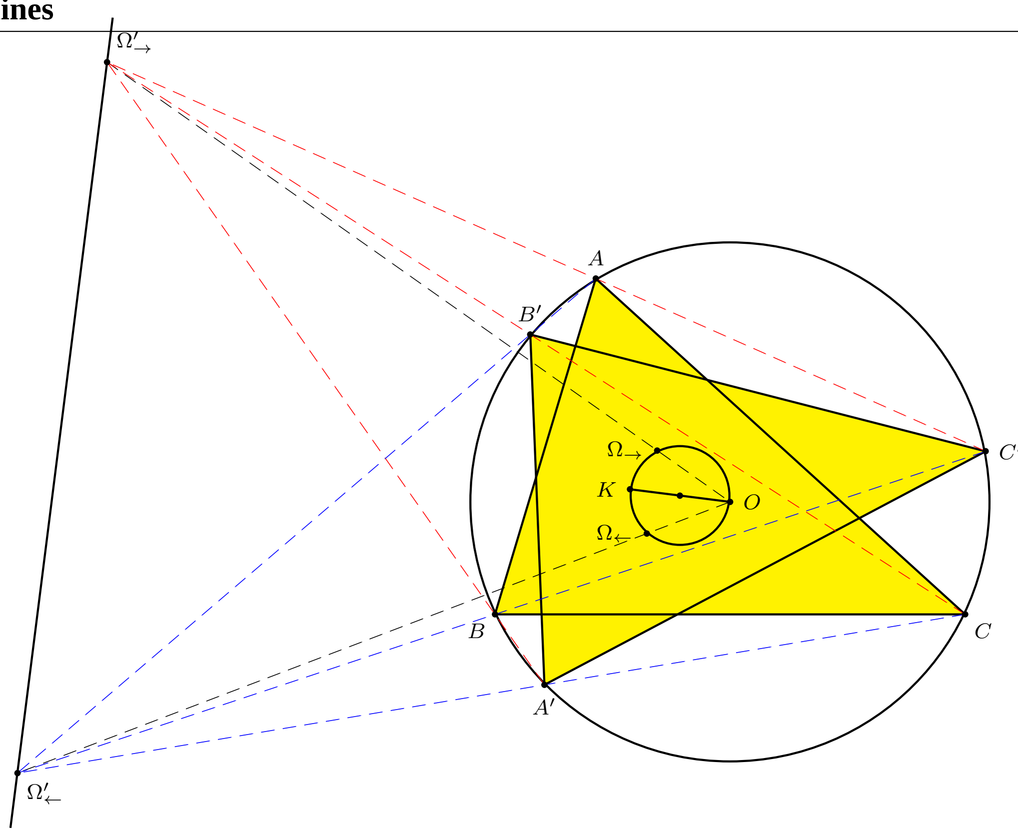
[3] Sastry, Problem 1119, Math. Magazine, (Solution, 55:3 (1982) 180–182). This is true also for the circumcenter. See also Problem 1120.

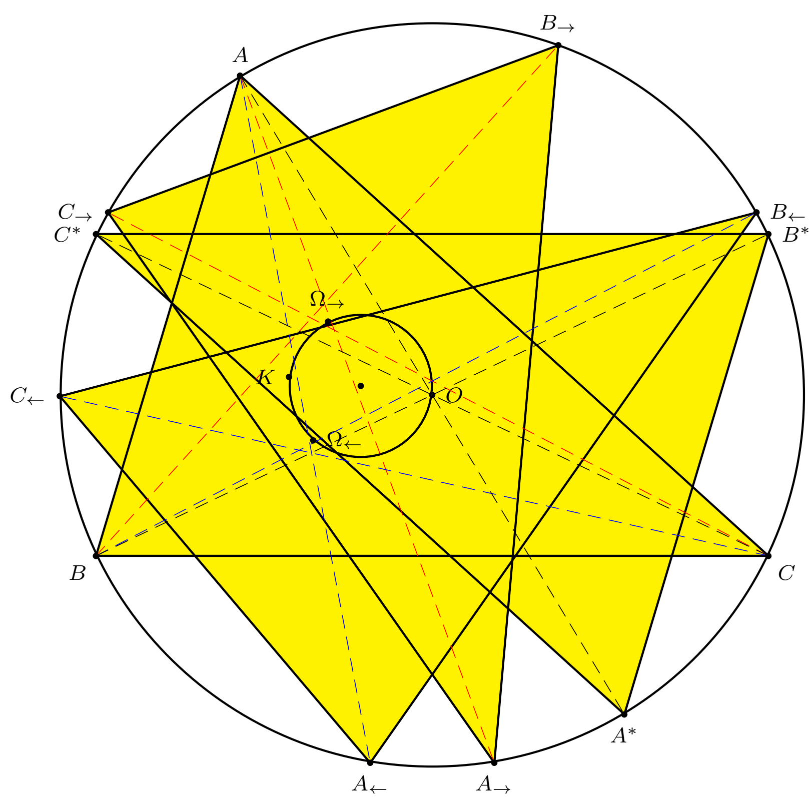
8.5.4 Circumcevian triangles congruent to the reference triangle

The circumcevian triangle of P is congruent to ABC if and only if P is one of the following points on the Brocard circle and their inverse on the Lemoine axis: the circumcenter, the Brocard points, and the midpoints of the symmedian chords.

Theorem. The circumcevian triangle of P is congruent to ABC if and only if P is one of the following points:
(i) the circumcenter O,
(ii) the Brocard points $\Omega_{\rightarrow}$ and $\Omega_{\leftarrow}$,
(iii) their inversive images $\Omega'_{\rightarrow}$ and $\Omega'_{\leftarrow}$ in the circumcircle, and
(iv) the intersections P_a, P_b, P_c of the symmedians with the Brocard circle,
(iv) the inversive image of P_a, P_b, P_c in the circumcircle.

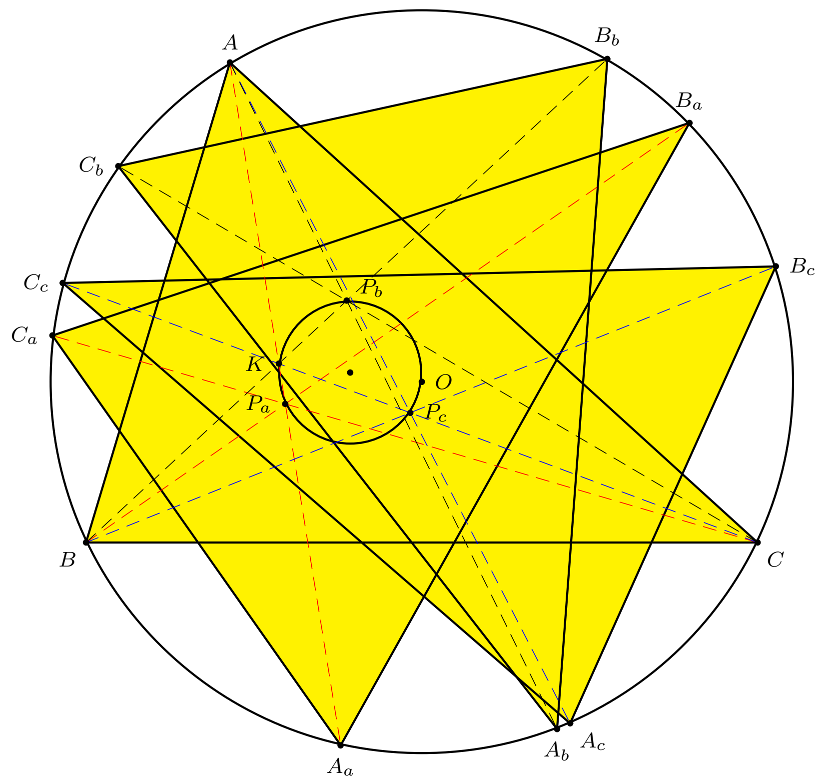






8.6 Triangle bounded by the reflections of a tangent to the circumcircle in the sidelines

Given a point P on the circumcircle of triangle $\mathbf{T}$, consider the triangle bounded by the reflections of the tangent at P in the sidelines.



(1) These reflections bound a triangle $A'B'C'$ which is perspective with ABC at a point Q on the circumcircle.

(2) The circumcircles of ABC and $A'B'C'$ are tangent internally at a point T which is the intersection of the reflections of the line $\mathtt{H}P$ in the sidelines of ABC.

(3) The circumcenter of $A'B'C'$ is the intersection of $\mathtt{O}T$ and $\mathtt{E}Q$.

Chapter 9

Circles

9.1 Generic circles

Proposition. A circle $\mathcal{C}$ is represented by an equation of the form

$$a^2yz + b^2zx + c^2xy - (x + y + z)(fx + gy + hz) = 0$$

in homogeneous barycentric coordinates.

Proof. Every circle $\mathcal{C}$ is homothetic to the circumcircle by a homothety, say $\mathsf{h}(T, k)$, where $T = uA + vB + wC$ (in absolute barycentric coordinate) is a center of similitude of $\mathcal{C}$ and the circumcircle. This means that if $P(x : y : z)$ is a point on the circle $\mathcal{C}$, then

$$\mathsf{h}(T, k)(P) = (1 - k)T + kP \sim (x + tu(x + y + z) : y + tv(x + y + z) : z + tw(x + y + z)),$$

where $t = \frac{(1-k)(u+v+w)}{k}$, lies on the circumcircle. In other words,

$$\begin{aligned} 0 &= \sum_{\text{cyclic}} a^2(y + tv(x + y + z))(z + tw(x + y + z)) \\ &= \sum_{\text{cyclic}} a^2(yz + t(wy + vz)(x + y + z) + t^2vw(x + y + z)^2) \\ &\quad + t^2(a^2vw + b^2wu + c^2uv)(x + y + z)^2. \end{aligned}$$

Note that the last two terms factor as the product of $x + y + z$ and another *linear form*. It follows that every circle can be represented by an equation of the form

$$a^2yz + b^2zx + c^2xy - (x + y + z)(fx + gy + hz) = 0.$$

□

Example (1). *The nine-point circle*.
Under the homothety $\mathtt{h}(\mathtt{G}, -2)$, the nine-point circle is mapped onto the circumcircle. This means that if $(x : y : z)$ is a point on the nine-point circle, then its superior $(y+z-x : z+x-y : x+y-z)$ lies on the circumcircle. It follows that

$$a^2(z+x-y)(x+y-z)+b^2(x+y-z)(y+z-x)+c^2(y+z-x)(z+x-y) = 0.$$

Simplifying this equation, we have

$$a^2yz + b^2zx + c^2xy - \frac{1}{2}(x+y+z)(S_\alpha x + S_\beta y + S_\gamma z) = 0.$$

Example (2). *The circumcircle of the superior triangle*.
The equation of the circle can be obtained from that of the circumcircle by substituting $(x : y : z)$ by $(y+z : z+x : x+y)$. Thus,

$$a^2yz + b^2zx + c^2xy + (x+y+z)(a^2x + b^2y + c^2z) = 0.$$

The center of the circle is the orthocenter $\mathtt{H}$.

Example (3). *Circumcircle of the excentral triangle*.
To find the equation of the circle through the excenters, we solve the system of linear equations

$$\begin{aligned}
-fa + gb + hc &= \frac{a^2bc - ab^2c - abc^2}{-a+b+c} = -abc,\\
fa - gb + hc &= \frac{-a^2bc + ab^2c - abc^2}{a-b+c} = -abc,\\
fa + gb - hc &= \frac{-a^2bc - ab^2c + abc^2}{a+b-c} = -abc.
\end{aligned}$$

From these,

$$fa = gb = hc = -abc \implies f = -bc, g = -ca, h = -ab.$$

It follows that the circumcircle of the excentral triangle is the circle

$$a^2yz + b^2zx + c^2xy + (x+y+z)(bcx + cay + abz) = 0.$$

Exercise

1. Show that a circle passing through A has equation of the form

$$a^2yz + b^2zx + c^2xy - (x + y + z)(qy + rz) = 0$$

for some q and r.

2. Show that a circle passing through B and C has equation of the form

$$a^2yz + b^2zx + c^2xy - px(x + y + z) = 0$$

for some p.

3. Show that the superior of the circle $\mathcal{C}$ is the circle

$$(a^2yz + b^2zx + c^2xy) - (x + y + z)\left(\sum_{\text{cyclic}}(2g + 2h - a^2)x\right) = 0.$$

4. Show that the inferior of the circle $\mathcal{C}$ is the circle

$$\begin{aligned} &4(a^2yz + b^2zx + c^2xy) \\ &-(x + y + z)\left(\sum_{\text{cyclic}}(b^2 + c^2 - a^2 + g + h - f)x\right) = 0. \end{aligned}$$

9.2 Power of a point with respect to a circle

Consider a circle $\mathcal{C} := O(\rho)$ and a point P. By the theorem on intersecting chords, for any line through P intersecting $\mathcal{C}$ at two points X and Y, the product $PX \cdot PY$ of *signed* lengths is constant. We call this product the *power* of P with respect to $\mathcal{C}$. By considering the diameter through P, we obtain $|OP|^2 - \rho^2$ for the power of a point P with respect to $O(\rho)$.

Proposition. With respect to the circle

$$\mathcal{C}: \qquad a^2yz + b^2zx + c^2xy - (x+y+z)(fx+gy+hz) = 0,$$

the powers of the vertices A, B, C of $\mathbf{T}$ are f, g, h respectively. More generally, the power of a finite point $(x : y : z)$ is $\frac{fx+gy+hz}{x+y+z}$.

Proof. Let the circle intersect the sideline a at the points $X = (0 : v : w)$ and $X' = (0 : v' : w')$. These coordinates satisfy the equation

$$a^2yz - (y+z)(gy+hz) = 0 \implies gy^2 + (g+h-a^2)yz + hz^2 = 0.$$

Note that $\frac{v}{w} \cdot \frac{v'}{w'} = \frac{h}{g}$ and $\frac{v}{w} + \frac{v'}{w'} = -\frac{g+h-a^2}{g}$.

The power of B with respect to the circle is

$$\begin{aligned} BX \cdot BX' &= \frac{wa}{v+w} \cdot \frac{w'a}{v'+w'} = \frac{ww'}{(v+w)(v'+w')} \cdot a^2 \\ &= \frac{1}{\left(\frac{v}{w}+1\right)\left(\frac{v'}{w'}+1\right)} \cdot a^2 = \frac{1}{\frac{vv'}{ww'} + \frac{v}{w} + \frac{v'}{w'} + 1} \cdot a^2 \\ &= \frac{1}{\frac{h}{g} - \frac{g+h-a^2}{g} + 1} \cdot a^2 = g. \end{aligned}$$

Similarly, the powers of A and C are f and h. □

9.2.1 Radical axis and radical center

Corollary. (a) The radical axis of

$$\mathcal{C}: \qquad a^2yz + b^2zx + c^2xy - (x+y+z)(fx+gy+hz) = 0$$

and the circumcircle is the line $fx + gy + hz = 0$.

(b) The radical axis of the circles $\mathcal{C}$ and

$$\mathcal{C}': \qquad a^2yz + b^2zx + c^2xy - (x+y+z)(f'x+g'y+h'z) = 0$$

is the line

$$(f - f')x + (g - g')y + (h - h')z = 0.$$

Proposition. The circle $\mathcal{C}$ is tangent to the circumcircle if and only if the line $fx + gy + hz = 0$ is tangent to any one of them.

Proposition. If, for $i = 1, 2, 3$, the centers of the three circles

$$\mathcal{C}_i : \qquad a^2yz + b^2zx + c^2xy - (x + y + z)(f_ix + g_iy + h_iz) = 0$$

are noncollinear, the radical center is the point given by

$$f_1x + g_1y + h_1z = f_2x + g_2y + h_2z = f_3x + g_3y + h_3z.$$

Explicitly,

$$x : y : z = \begin{vmatrix} 1 & g_1 & h_1 \\ 1 & g_2 & h_2 \\ 1 & g_3 & h_3 \end{vmatrix} : \begin{vmatrix} f_1 & 1 & h_1 \\ f_2 & 1 & h_2 \\ f_3 & 1 & h_3 \end{vmatrix} : \begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_3 & g_3 & 1 \end{vmatrix}.$$

Exercise

1. Find the radical center of the circumcircles of $\mathbf{T}$, the superior triangle, and the excentral triangles. [1]

2. Find the radical center of the circumcircles of $\mathbf{T}$, the inferior triangle, and the excentral triangles. [2]

3. Find the radical axis of the circumcircles of the superior and inferior triangles. [3]

4. Find the radical axis of the circumcircles of the superior and excentral triangles. [4]

5. Find the radical center of the circumcircles of the superior, inferior, and excentral triangles. [5]

---

[1] $X(1491) = (a(b^3 - c^3) : b(c^3 - a^3) : c(a^3 - b^3))$.

[2] $X(650) = (a(b - c)(b + c - a) : b(c - a)(c + a - b) : c(a - b)(a + b - c))$.

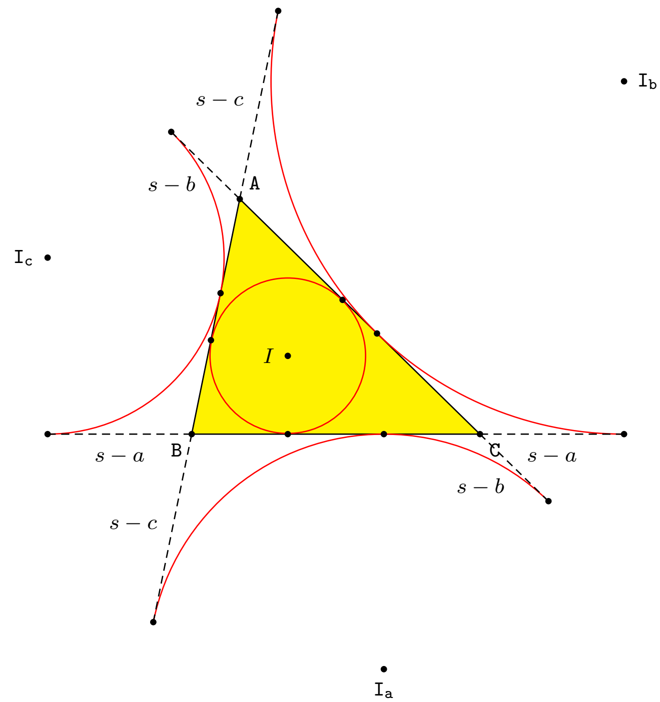
[3] $(3a^2 + b^2 + c^2)x + (a^2 + 3b^2 + c^2)y + (a^2 + b^2 + 3c^2)z = 0$.

[4] $(a^2 - bc)x + (b^2 - ca)y + (c^2 - ab)z = 0$.

[5] $((b - c)(a^3 + a^2(b + c) + a(3b^2 + 2bc + 3c^2) + (b + c)(b^2 + c^2)) : \cdots : \cdots)$.

9.3 The tritangent circles

By the tritangent circles of a triangle we mean the circles each tangent to the three sides of the triangle. These include the incircle and the three excircles.



The powers of the vertices with respect to each of these circles can be easily written found, leading to the equations of the circles.

$$
\begin{aligned}
(I):& \quad a^2yz+b^2zx+c^2xy-(x+y+z)((s-a)^2x+(s-b)^2y+(s-c)^2z)=0;\\
(I_a):& \quad a^2yz+b^2zx+c^2xy-(x+y+z)(s^2x+(s-c)^2y+(s-b)^2z)=0,\\
(I_b):& \quad a^2yz+b^2zx+c^2xy-(x+y+z)((s-c)^2x+s^2y+(s-a)^2z)=0,\\
(I_c):& \quad a^2yz+b^2zx+c^2xy-(x+y+z)((s-b)^2x+(s-a)^2y+s^2z)=0.
\end{aligned}
$$

9.3.1 The Feuerbach theorem

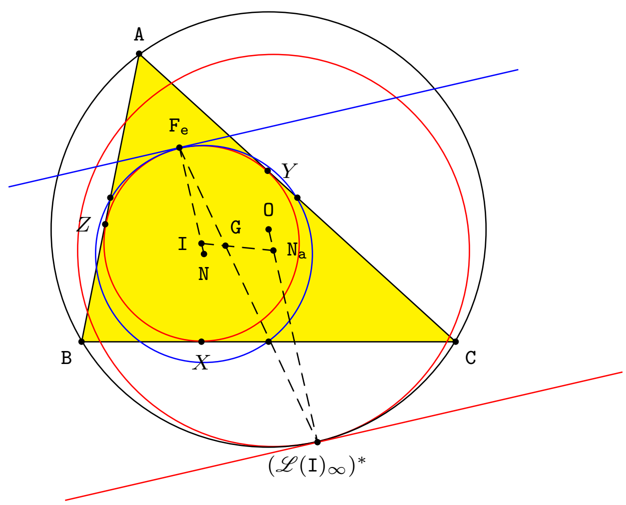
The superior of the incircle is the image of the incircle under the homothety $\mathsf{h}(\mathtt{G}, -2)$. Applying Proposition **??**(b) with $(f, g, h) = ((s-a)^2, (s-b)^2, (s-c)^2)$, we obtain the equation

$$a^2yz + b^2zx + c^2xy - (x+y+z)((b-c)^2x + (c-a)^2y + (a-b)^2z) = 0.$$

This is tangent to the circumcircle at the point

$$(\mathscr{L}(\mathtt{I})_\infty)^* = \left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right),$$

the isogonal conjugate of the infinite point of the trilinear polar of the incenter.



Likewise, the superior of the A-excircle is the circle

$$a^2yz + b^2zx + c^2xy - (x+y+z)((b-c)^2x + (c+a)^2y + (a+b)^2z) = 0.$$

This is tangent to the circumcircle at the point

$$\left(\frac{a^2}{a(b-c)} : \frac{b^2}{-b(c+a)} : \frac{c^2}{c(a+b)}\right) = \left(\frac{a}{b-c} : \frac{b}{-(c+a)} : \frac{c}{a+b}\right).$$

Similarly, the superiors of the B-and C-excircles are also tangent to the circumcircle. From these, we deduce the famous Feuerbach theorem.

Theorem (Feuerbach)**.** The nine-point circle is tangent internally to the incircle at the Feuerbach point

$$\mathtt{F_e} := ((b-c)^2(b+c-a) : (c-a)^2(c+a-b) : (a-b)^2(a+b-c)),$$

and externally to the excircles respectively at

$$\begin{aligned}
F_a &= \ (-(b-c)^2(a+b+c) : \ (c+a)^2(a+b-c) : \ (a+b)^2(c+a-b)),\\
F_b &= \ ((b+c)^2(a+b-c) : \ -(c-a)^2(a+b+c) : \ (a+b)^2(b+c-a)),\\
F_c &= \ ((b+c)^2(c+a-b) : \ (c+a)^2(b+c-a) : \ -(a+b)^2(a+b+c)).
\end{aligned}$$

Proof. The tangency follows from the fact the nine-point circle is the inferior of the circumcircle. The point of tangency is the Feuerbach point

$$\begin{aligned}
\mathsf{F_e} := \ \mathsf{Inf}\left(\left(\frac{a}{b-c} : \ \frac{b}{c-a} : \ \frac{c}{a-b}\right)\right)\\
= \ \left(\frac{b}{c-a}+\frac{c}{a-b} : \ \frac{c}{a-b}+\frac{a}{b-c} : \ \frac{a}{b-c}+\frac{b}{c-a}\right)\\
= \ \left((b-c)^2(b+c-a) : (c-a)^2(c+a-b) : (a-b)^2(a+b-c)\right).
\end{aligned}$$

On the other hand, the point of tangency with the A-excircle is the inferior of the point

$$\left(\frac{a}{b-c} : \frac{b}{-(c+a)} : \ \frac{c}{a+b}\right)$$

on the circumcircle. This is F_a given above; similarly for F_b and F_c. □

Proposition. (a) The points of the tangency of the nine-point circle with the excircles form a triangle perspective with $\mathbf{T}$ at the **outer Feuerbach point**

$$\left(\frac{(b+c)^2}{b+c-a} : \ \frac{(c+a)^2}{c+a-b} : \ \frac{(a+b)^2}{a+b-c}\right).$$

(b) The common tangents of the nine-point circle with the excircles bound a triangle perspective with the inferior triangle at the Spieker center $\mathtt{S_p}$.

Proof. The common tangents of the circumcircle with the superiors of the excircles are the lines

$$\begin{aligned}
(b-c)^2x+(c+a)^2y+(a+b)^2z &= \ 0,\\
(b+c)^2x+(c-a)^2y+(a+b)^2z &= \ 0,\\
(b+c)^2x+(c+a)^2y+(a-b)^2z &= \ 0.
\end{aligned}$$

These bound a triangle with vertices

$$(-(a^2+bc) : b(b+c) : c(b+c)), \quad (a(c+a) : -(b^2+ca) : c(c+a)), \quad (a(a+b) : b(a+b)$$

perspective with $\mathbf{T}$ at the incenter $\mathtt{I}$. □

Remark. The common tangents of the nine-point circle with the incircle and the excircles are the lines

$$\begin{aligned}
\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b} &= 0;\\
\frac{x}{b-c}+\frac{y}{c+a}-\frac{z}{a+b} &= 0,\\
-\frac{x}{b+c}+\frac{y}{c-a}+\frac{z}{a+b} &= 0,\\
\frac{x}{b+c}-\frac{y}{c+a}+\frac{z}{a-b} &= 0.
\end{aligned}$$

9.3.2 Radical axes of the circumcircle with the excircles

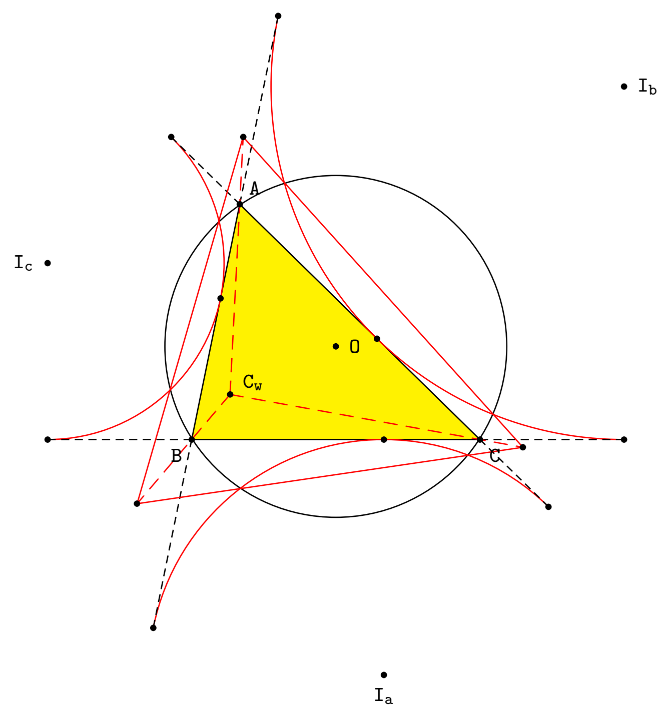
Clearly the incircle and the circumcircle do not intersect at real points. From the equation of the incircle, the line

$$(s-a)^2x + (s-b)^2y + (s-c)^2z = 0$$

is the radical axis with the circumcircle. It is the trilinear polar of the barycentric square of the Gergonne point.

On the other hand, the radical axes of the circumcircle with the excircles are the lines

$$\begin{aligned} s^2x+(s-c)^2y+(s-b)^2z &= 0,\\ (s-c)^2x+\qquad s^2y+(s-a)^2z &= 0,\\ (s-b)^2x+(s-a)^2y+\qquad s^2z &= 0. \end{aligned}$$



These lines bound a triangle with vertices

$$\begin{array}{rcrcl} (-(b+c)(a^2+(b+c)^2) & : & b(a^2+b^2-c^2) & : & c(c^2+a^2-b^2)),\\ (a(a^2+b^2-c^2) & : & -(c+a)(b^2+(c+a)^2) & : & c(b^2+c^2-a^2)),\\ (a(c^2+a^2-b^2) & : & b(b^2+c^2-a^2) & : & -(a+b)(c^2+(a+b)^2) \end{array}$$

perspective with $\mathbf{T}$ at the Clawson point

$$\mathtt{C_w} := \left(\frac{a}{S_\alpha} : \frac{b}{S_\beta} : \frac{c}{S_\gamma}\right).$$

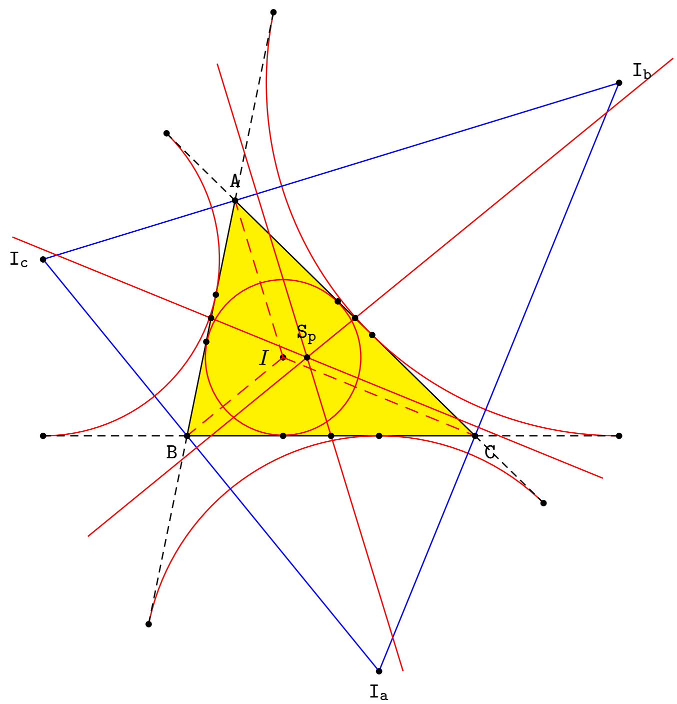
Exercise

1. The radical axis of the incircle and the A-excircle is the perpendicular to the bisector of angle A through the midpoint of BC. It is the image of the external bisector of angle A under the homothety $\mathsf{h}\left(\mathsf{G}, -\frac{1}{2}\right)$.

2. The same conclusion applies to the radical axes of the incircle with the B- and C-excircles. The triangle bounded by the three radical axes is therefore the image of the excentral triangle under the homothety $\mathsf{h}\left(\mathsf{G}, -\frac{1}{2}\right)$.

9.3.3 The radical center of the excircles

The radical axis of the B- and C-excircles contains the midpoint $\mathtt{G_a}$ of BC, and is perpendicular to the line joining the excenters $\mathtt{I_b}$ and $\mathtt{I_c}$. This is parallel to the bisector of angle $\mathtt{A}$. It is indeed the A-bisector of the inferior triangle $\mathsf{cev}(\mathtt{G})$. Similarly, the radical axis of the C- and A-excircles is the B-bisector. It follows that the radical center of the excircles is the incenter of the inferior triangle. This is the Spieker center $\mathtt{S_p}$.



9.3.4 The Spieker radical circle

The Spieker radical circle is the circle orthogonal to the three excircles. Its center is the radical center $\mathtt{S_p}$. Its squared radius is the common power of $\mathtt{S_p}$ in the excircles. This is

$$\begin{aligned} &\frac{s^2(b+c)+(s-c)^2(c+a)^2+(s-b)(a+b)}{b+c+c+a+a+b}\\ &-\frac{a^2(c+a)(a+b)+b^2(a+b)(b+c)+c^2(b+c)(c+a)}{(b+c+c+a+a+b)^2}\\ =&\ \frac{a^2(b+c)+b^2(c+a)+c^2(a+b)+abc}{4(a+b+c)}. \end{aligned}$$

We compute the equation of the Spieker radical circle through its image under the homothety $\mathtt{h}(\mathtt{G}, -2)$. This is a circle with center $\mathtt{I}$. The square radius of the circle is 4 times the square radius of the radical circle. The power of A in this superior circle is

$$\begin{aligned} &\mathtt{IA}^2-\frac{a^2(b+c)+b^2(c+a)+c^2(a+b)+abc}{a+b+c}\\ =&\ \frac{bc(b+c-a)}{a+b+c}-\frac{a^2(b+c)+b^2(c+a)+c^2(a+b)+abc}{a+b+c}\\ =&\ -a(b+c). \end{aligned}$$

Similarly, the powers of B and C with respect to the superior of the spieker radical circle are $-b(c+a)$ and $-c(a+b)$. This superior has barycentric equation

$$a^2yz+b^2zx+c^2xy+(x+y+z)(a(b+c)x+b(c+a)y+c(a+b)z)=0.$$

Replacing x, y, z by $y+z-x$, $z+x-y$, $x+y-z$ respectively, we obtain the barycentric equation of the Spieker radical circle:

$$a^2yz+b^2zx+c^2xy+(x+y+z)((s-b)(s-c)x+(s-c)(s-a)y+(s-a)(s-b)z)=0.$$

Exercise

1. Find the radical axis of the Spieker radical circle and the A-excircle. [6]

[6] $(a(a+b+c)+2bc)x+b(a+b-c)y+c(c+a-b)z=0.$

2. Show that the triangle bounded by the radical axes of the Spieker radical circle and the excircles bound a triangle perspective with ABC and find the perspector. [7]

Here is an interesting property of the Spieker radical circle.

Theorem. The locus of P whose polars with respect to the excircles are concurrent is the Spieker radical circle. The point of concurrency is the antipode of P on the circle.

[7] $X(2051) = \left(\frac{1}{a^3-a(b^2-bc+c^2)-bc(b+c)} : \cdots : \cdots\right)$.

9.4 The Conway circle

If Y_a, Z_a are points on the extensions of CA and BA such that $AY_a = AZ_a = a$, and similarly define Z_b, X_b, X_c, Y_c, the six points lie on the Conway circle (see §**??**). The coordinates of these points are

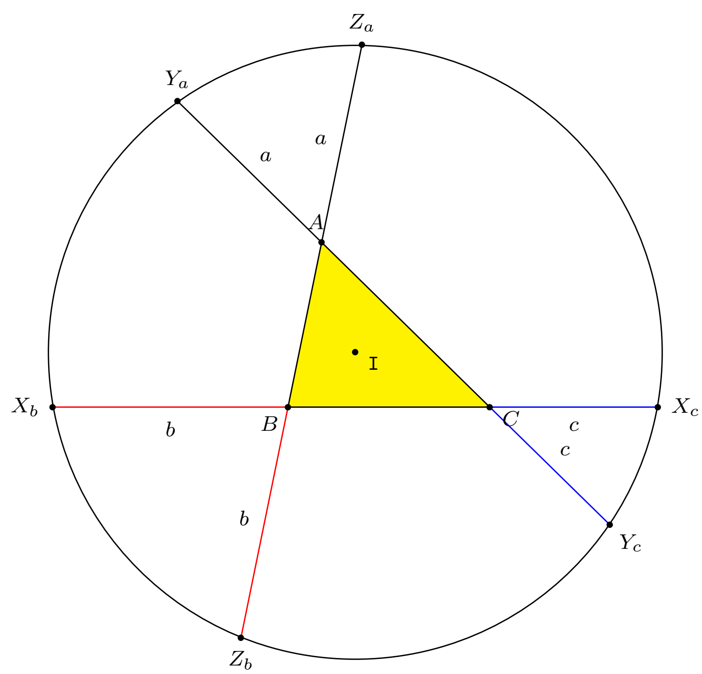
$$\boxed{\begin{array}{lll} & X_b = (0 : a+b : -b), & X_c = (0 : -c : c+a); \\ Y_a = (a+b : 0 : -a), & & Y_c = (-c : 0 : b+c); \\ Z_a = (c+a : -a : 0), & Z_b = (-b : b+c : 0). & \end{array}}$$

The power of A with respect to the circle is

$$\mathtt{A}Y_a \cdot \mathtt{A}Y_c = \mathtt{A}Z_a \cdot \mathtt{A}Z_b = -a(b+c).$$

Similarly, the powers of B and C are $-b(c+a)$ and $-c(a+b)$. Therefore, the equation of the Conway circle is

$$a^2yz + b^2zx + c^2xy + (x+y+z)(a(b+c)x + b(c+a)y + c(a+b)z) = 0.$$



Exercise

1. Find the radical axis of the Conway circle with the circumcicle of the excentral triangle. [8]

[8] $(a(b+c) - bc)x + (b(c+a) - ca)y + (c(a+b) - ab)z = 0.$

2. Find the radical axis of the Conway circle with the circumcircle of the superior triangle. [9]

3. Find the radical center of the Conway circle and the circumcircles of the excentral and superior triangles. [10]

4. Show that the circle AY_aZ_a has equation

$$a^2yz + b^2zx + c^2xy + (x+y+z)(c(c+a)y + b(a+b)z) = 0;$$

similarly for the circles BZ_bX_b and CX_cY_c. The radical center of the three circles is the Schiffler point

$$\mathtt{S_c} = \left(\frac{a(b+c-a)}{b+c} : \cdots : \cdots \right).$$

5. Show that the circle AY_cZ_b has equation

$$a^2yz + b^2zx + c^2xy + bc(x+y+z)(y+z) = 0;$$

similarly for the circles BZ_aX_c and CX_bY_a. The centers of these circles are on the circumcircle of $\mathbf{T}$. Find the radical center of the three circles. [11]

---

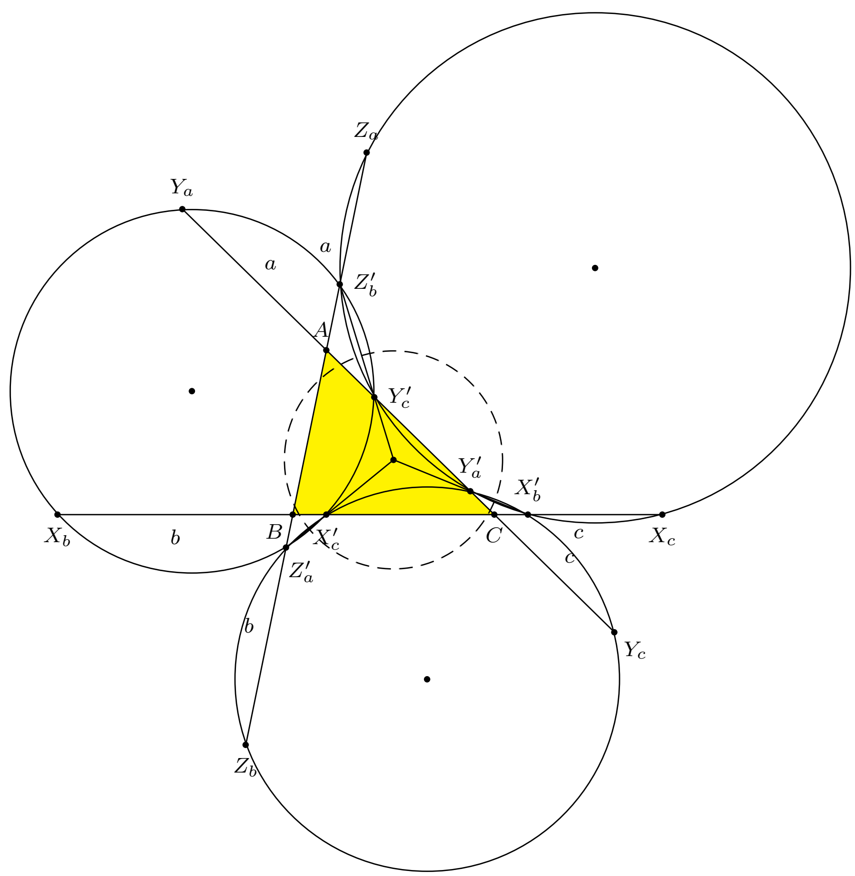
[9] $a(b+c-a)x + b(c+a-b)y + c(a+b-c)z = 0$.
[10] $((b-c)(a^2(b+c) - a(b^2 - bc + c^2) + bc(b+c)) : \cdots : \cdots)$.
[11] The Nagel point.

9.4.1 Sharp's triad of circles

We extend the Conway configuration by considering the reflections Y'_a, Z'_a of Y_a, Z_a in the vertex A, and similarly for the other four points.

	$X_b = (0 : a+b : -b)$, $X'_b = (0 : a-b : b)$,	$X_c = (0 : -c : c+a)$, $X'_c = (0 : c : -c+a)$;
$Y_a = (a+b : 0 : -a)$, $Y'_a = (-a+b : 0 : a)$,		$Y_c = (-c : 0 : b+c)$, $Y'_c = (c : 0 : b-c)$;
$Z_a = (c+a : -a : 0)$, $Z'_a = (c-a : a : 0)$,	$Z_b = (-b : b+c : 0)$. $Z'_b = (b : -b+c : 0)$.	

This results in a triad of circles
$\mathcal{C}_a$ through X'_b, X'_c, Y_c, Y'_a, Z'_a, Z_b;
$\mathcal{C}_b$ through X'_b, X_c, Y'_c, Y'_a, Z_a, Z'_b;
$\mathcal{C}_c$ through X_b, X'_c, Y'_c, Y_a, Z'_a, Z'_b.



The equations of the circles are

$$\begin{aligned}
\mathcal{C}_a: &\quad a^2yz + b^2zx + c^2xy + (x+y+z)(-a(b+c)x + b(c-a)y + c(-a+b)z) = 0,\\
\mathcal{C}_b: &\quad a^2yz + b^2zx + c^2xy + (x+y+z)(a(-b+c)x - b(c+a)y + c(a-b)z) = 0,\\
\mathcal{C}_c: &\quad a^2yz + b^2zx + c^2xy + (x+y+z)(a(b-c)x + b(-c+a)y - c(a+b)z) = 0.
\end{aligned}$$

Exercise

1. Find the centers and the radii of the circles in the triad. [12]
2. Find the radical center of the triad. [13]
3. Find the radius of the radical circle. [14]
4. Establish the barycentric equation of the Sharp radical circle, the radical circle of Sharp's triad of circles:

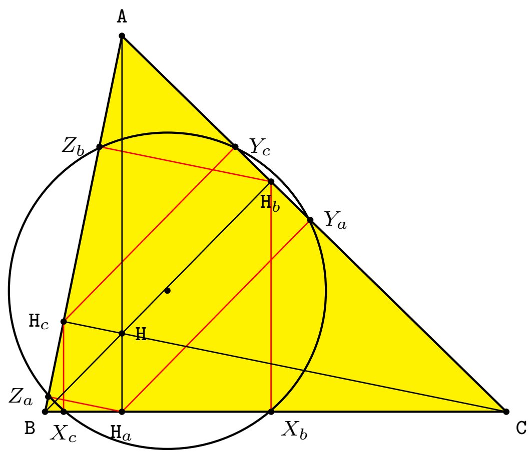
$$\begin{aligned}
&(a+b+c)(a^2yz + b^2zx + c^2xy)\\
&-(x+y+z)\left(\sum_{\text{cyclic}}((a+b+c)(2b+2c-a) - 10bc)x\right) = 0.
\end{aligned}$$

[12] $\mathcal{C}_c$ has center $\mathtt{I_a}$ and radius $\sqrt{r_a^2 + (s-a)^2}$; similarly for the other two circles.
[13] $\mathtt{N_a}$.
[14] The square of the radius is the (common) power of the radical center in the circles; it is $\frac{2abc}{a+b+c} = 4Rr$.

9.5 The Taylor circle

Consider the orthic triangle $\mathtt{H}_a\mathtt{H}_b\mathtt{H}_c$, and the pedals of each of the points $\mathtt{H}_a$, $\mathtt{H}_b$, $\mathtt{H}_c$ on the two sides not containing it. Thus,

Line	Pedal of $\mathtt{H}_a$	Pedal of $\mathtt{H}_b$	Pedal of $\mathtt{H}_c$
a		X_b	X_c
b	Y_a		Y_c
c	Z_a	Z_b	



It is easy to write down the lengths of various segments. For example,

$$\begin{aligned}\mathtt{A}Y_a &= b\sin\gamma\sin\gamma = b\sin^2\gamma,\\ \mathtt{A}Y_c &= b\cos\alpha\cos\alpha = b\cos^2\alpha;\\ \mathtt{A}Z_a &= c\sin\beta\sin\beta = c\sin^2\beta,\\ \mathtt{A}Z_b &= c\cos\alpha\cos\alpha = c\cos^2\alpha.\end{aligned}$$

Since $b\sin\gamma = c\sin\beta$, $\mathtt{A}Y_a\cdot\mathtt{A}Y_c = \mathtt{A}Z_a\cdot\mathtt{A}Z_b$. More precisely,

$$\begin{aligned}\mathtt{A}Y_a\cdot\mathtt{A}Y_c &= \mathtt{A}Z_a\cdot\mathtt{A}Z_b = \frac{S^2\cdot S_{\alpha\alpha}}{a^2b^2c^2},\\ \mathtt{B}Z_b\cdot\mathtt{B}Z_a &= \mathtt{B}X_b\cdot\mathtt{B}X_c = \frac{S^2\cdot S_{\beta\beta}}{a^2b^2c^2},\\ \mathtt{C}X_c\cdot\mathtt{C}X_b &= \mathtt{C}Y_c\cdot\mathtt{C}Y_a = \frac{S^2\cdot S_{\gamma\gamma}}{a^2b^2c^2}.\end{aligned}$$

We conclude that the six pedals of pedals are concyclic. The circle containing them is called the **Taylor circle**, and has barycentric equation

$$a^2b^2c^2(a^2yz + b^2zx + c^2xy) - S^2(x+y+z)(S_{\alpha\alpha}x + S_{\beta\beta}y + S_{\gamma\gamma}z) = 0.$$

Remark. The coordinates of the pedals of pedals are as follows.

Line	Pedals of $\mathtt{H}_a$	Pedals of $\mathtt{H}_b$	Pedals of $\mathtt{H}_c$
a		$X_b = (0 : S_{\gamma\gamma} : S^2)$	$X_c = (0 : S^2 : S_{\beta\beta})$
b	$Y_a = (S_{\gamma\gamma} : 0 : S^2)$		$Y_c = (S^2 : 0 : S_{\alpha\alpha})$
c	$Z_a = (S_{\beta\beta} : S^2 : 0)$	$Z_b = (S^2 : S_{\alpha\alpha} : 0)$	

Exercise

1. Show that $Y_aZ_a = \frac{S^2}{abc} = \frac{\triangle}{R}$.

2. Find the equations of the lines Y_aZ_a, Z_bX_b, X_cY_c, and show that they bound a triangle perspective to ABC at the symmedian point. [15]

3. Show that the Euler lines of the triangles $\mathtt{H}_aY_aZ_a$, $X_bH_bZ_b$, $X_cY_cH_c$ are concurrent and find the coordinates of the intersection. [16]

[15] The line Y_aZ_a has equation $-S^2x + S_{\beta\beta}y + S_{\gamma\gamma}z = 0$

[16] J.-P. Ehrmann, Hyacinthos 3695. The common point of the Euler lines is $X_{973} = (a^2(S^2 + S_{\beta\gamma})((S_\alpha + S_\beta + S_\gamma)S^4 + S_{\alpha\beta\gamma}(2S^2 - S_{\alpha\alpha})) : \cdots : \cdots)$.

9.5.1 The Taylor circle of the excentral triangle

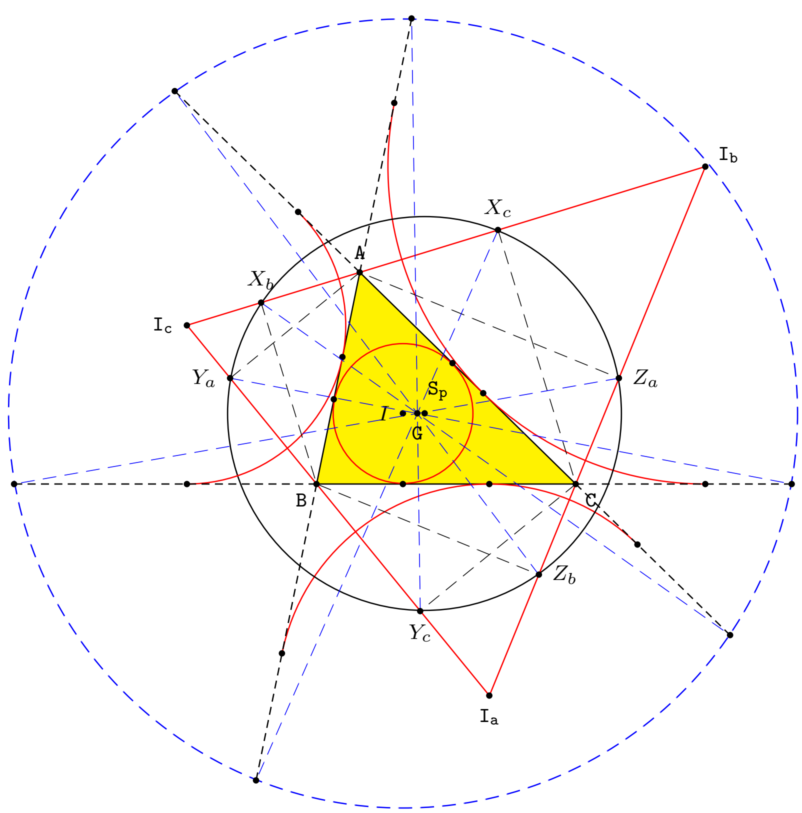
Triangle $\mathbf{T} := \mathtt{ABC}$ is the orthic triangle of its excentral triangle $\text{cev}^{-1}(\mathtt{I}) = \mathtt{I}^a\mathtt{I}^b\mathtt{I}^c$. The orthocenter is $\mathtt{I}$ since $\mathtt{AI}^a$ is perpendicular to $\mathtt{I}^b\mathtt{I}^c$. The pedals of pedals have very simple coordinates.

Line	Pedal of A	Pedal of B	Pedal of C
$\mathtt{I}^b\mathtt{I}^c$		$X_b = (b+c : b : -c)$	$X_c = (b+c : -b : c)$
$\mathtt{I}^c\mathtt{I}^a$	$Y_a = (a : c+a : -c)$		$Y_c = (-a : c+a : c)$
$\mathtt{I}^a\mathtt{I}^b$	$Z_a = (a : -b : a+b)$	$Z_b = (-a : b : a+b)$	

For example, BZ_b is parallel to the bisector of angle C, and has infinite point $(a : b : -(a+b))$. It is the line $(a+b)x + az = 0$; it intersects the external bisector of angle C, namely, $\frac{x}{a} + \frac{y}{b} = 0$ at $Z_b = (-a : b : a+b)$.

The superior of these points are the point in the Conway configuration:

Line	superior of pedal of A	superior of pedal of B	superior of pedal of C
a		$(-c : 0 : b+c)$	$(-b : b+c : 0)$
b	$(0 : -c : c+a)$		$(c+a : -a : 0)$
c	$(0 : a+b : -b)$	$(a+b : 0 : -a)$	



Therefore, the Taylor circle of the excentral triangle is the inferior of the Conway circle, and is the same as the Spieker radical circle.

9.6 Some triads of circles

9.6.1 Circles with sides as diameters

Consider the circle with diameter $\mathtt{BC}$. The radius of the circle is $\frac{a}{2}$. The power of $\mathtt{A}$ with respect to the circle is $m_a^2 - \left(\frac{a}{2}\right)^2 = \frac{1}{4}(2b^2+2c^2-a^2) - \frac{1}{4}a^2 = \frac{b^2+c^2-a^2}{2} = S_\alpha$. From this we obtain the equation of the circle:

$$a^2yz + b^2zx + c^2xy - S_\alpha x(x+y+z) = 0.$$

Similarly, the circles with diameters $\mathtt{CA}$ and $\mathtt{AB}$ are

$$a^2yz + b^2zx + c^2xy - S_\beta y(x+y+z) = 0,$$
$$a^2yz + b^2zx + c^2xy - S_\gamma z(x+y+z) = 0.$$

Consider the orthocenter $\mathtt{H}$ with the orthic triangle $\mathtt{H}_a\mathtt{H}_b\mathtt{H}_c$. The circle $\mathcal{C}_a$ contains the pedals $\mathtt{H}_b$ and $\mathtt{H}_c$. Similarly, $\mathcal{C}_b$ contains $\mathtt{H}_c$ and $\mathtt{H}_a$, and $\mathcal{C}_c$ contains $\mathtt{H}_a$ and $\mathtt{H}_b$. It follows that $A\mathtt{H}_a$ is the radical axis of $\mathcal{C}_b$ and $\mathcal{C}_c$. Similarly, $B\mathtt{H}_b$ is the radical axis of $\mathcal{C}_c$ and $\mathcal{C}_a$, and $C\mathtt{H}_c$ that of $\mathcal{C}_a$ and $\mathcal{C}_b$. The radical center is the orthocenter $\mathtt{H}$, and

$$\mathtt{H}A \cdot \mathtt{HH}_a = \mathtt{H}B \cdot \mathtt{HH}_b = \mathtt{H}C \cdot \mathtt{HH}_c.$$

9.6.2 Circles with cevians as diameters

Let $P = (u : v : w)$, with cevian triangle cev$(P) = XYZ$. Consider the circle with diameter AX. The circle intersects the sideline $\mathtt{BC}$ at X and the pedal $\mathtt{H}_a$. The powers of $\mathtt{B}$ and $\mathtt{C}$ in this circle are $\frac{S_\beta}{a} \cdot \frac{aw}{v+w} = \frac{S_\beta w}{v+w}$ and $\frac{S_\gamma v}{v+w}$. This circle has equation

$$a^2yz + b^2zx + c^2xy - \frac{1}{v+w}\left(S_\beta wy + S_\gamma vz\right)(x+y+z) = 0.$$

Note that the power of $\mathtt{H}$ in this circle is $\mathtt{H}A \cdot \mathtt{HH}_a$. Since this is equal to $\mathtt{H}B \cdot \mathtt{HH}_b$ and $\mathtt{H}C \cdot \mathtt{HH}_c$, the orthocenter $\mathtt{H}$ has equal powers with respect to the three circles. It is the radical center of the triad of circles.

We rewrite this in a slightly different form as follows, along with the circles with diameters BY and CZ:

$$\begin{aligned} a^2yz + b^2zx + c^2xy - \frac{vw}{v+w}\left(\frac{S_\beta}{v}y + \frac{S_\gamma}{w}z\right)(x+y+z) =&\ 0, \\ a^2yz + b^2zx + c^2xy - \frac{wu}{w+u}\left(\frac{S_\gamma}{w}z + \frac{S_\alpha}{u}x\right)(x+y+z) =&\ 0, \\ a^2yz + b^2zx + c^2xy - \frac{uv}{u+v}\left(\frac{S_\alpha}{u}x + \frac{S_\beta}{v}y\right)(x+y+z) =&\ 0. \end{aligned}$$

The radical center of the three circles is the point given by

$$\frac{\frac{S_\beta}{v}y + \frac{S_\gamma}{w}z}{\frac{1}{v} + \frac{1}{w}} = \frac{\frac{S_\gamma}{w}z + \frac{S_\alpha}{u}x}{\frac{1}{w} + \frac{1}{u}} = \frac{\frac{S_\alpha}{u}x + \frac{S_\beta}{v}y}{\frac{1}{u} + \frac{1}{v}}.$$

From these,

$$S_\alpha x = S_\beta y = S_\gamma z \implies x : y : z = \frac{1}{S_\alpha} : \frac{1}{S_\beta} : \frac{1}{S_\gamma},$$

and the radical center is the orthocenter $\mathtt{H}$.

Exercise

The perpendiculars from $\mathtt{H}$ to the three cevians intersect the corresponding circle at two points. The six intersections with the three circles lie on a circle with center P. Find the equation of the circle. [17]

[17] $(u+v+w)(a^2yz + b^2zx + c^2xy) - (x+y+z)\left(\sum_{\text{cyclic}} S_\alpha(v+w-u)x\right) = 0.$

9.6.3 Circles with centers at vertices and altitudes as radii

Consider the circle $\mathcal{C}_a$ with center A and radius $A\mathtt{H}_a$. Since the length of $A\mathtt{H}_a$ is $\frac{S}{a}$, the powers of A, B, C with respect to the circle are $-\frac{S^2}{a^2}$, $\left(\frac{S_\beta}{a}\right)^2$, and $\left(\frac{S_\gamma}{a}\right)^2$, the equation of the circle is

$$a^2yz + b^2zx + c^2xy - \frac{-S^2x + S_\beta^2y + S_\gamma^2z}{a^2}(x + y + z) = 0.$$

Similarly, we have the equations of the circles $B(\mathtt{H}_b)$ and $C(\mathtt{H}_c)$. The radical center of the three circles is the point given by

$$\frac{-S^2x + S_\beta^2y + S_\gamma^2z}{a^2} = \frac{S_\alpha^2x - S^2y + S_\gamma^2z}{b^2} = \frac{S_\alpha^2x + S_\beta^2y - S^2z}{c^2}.$$

Exercise

1. This radical center has coordinates

$$(a^2(S^4 - S_{\alpha\beta\gamma} \cdot S_\alpha) : b^2(S^4 - S_{\alpha\beta\gamma} \cdot S_\beta) : c^2(S^4 - S_{\alpha\beta\gamma} \cdot S_\gamma)).$$

9.6.4 Circles with altitudes as diameters

Consider a circle tangent to BC at the point $(0:v:w)$. The powers of B and C in the circle are respectively $\left(\frac{aw}{v+w}\right)^2$ and $\left(\frac{av}{v+w}\right)^2$. If this circle also passes through A, then its equation is

$$a^2yz+b^2zx+c^2xy-\frac{a^2}{(v+w)^2}(w^2y+v^2z)(x+y+z)=0.$$

Similarly, we have the equations of the circles through the vertices and tangent to the opposite sides at the traces of $P=(u:v:w)$.

These are

$$\begin{aligned}
a^2yz+b^2zx+c^2xy-\frac{a^2}{(v+w)^2}(w^2y+v^2z)(x+y+z) &= 0,\\
a^2yz+b^2zx+c^2xy-\frac{b^2}{(w+u)^2}(u^2z+w^2x)(x+y+z) &= 0,\\
a^2yz+b^2zx+c^2xy-\frac{c^2}{(u+v)^2}(v^2x+u^2y)(x+y+z) &= 0.
\end{aligned}$$

The radical center of the three circles is the point given by

$$\frac{a^2(w^2y+v^2z)}{(v+w)^2}=\frac{b^2(u^2z+w^2x)}{(w+u)^2}=\frac{c^2(v^2x+u^2y)}{(u+v)^2}.$$

$$\frac{\frac{y}{v^2}+\frac{z}{w^2}}{\frac{u^2(v+w)^2}{a^2}}=\frac{\frac{z}{w^2}+\frac{x}{u^2}}{\frac{v^2(w+u)^2}{b^2}}=\frac{\frac{x}{u^2}+\frac{y}{v^2}}{\frac{w^2(u+v)^2}{c^2}}.$$

Therefore,

$$\mathsf{inf}\left(\frac{x}{u^2}:\frac{y}{v^2}:\frac{z}{w^2}\right)=\left(\frac{u^2(v+w)^2}{a^2}:\frac{v^2(w+u)^2}{b^2}:\frac{w^2(u+v)^2}{c^2}\right).$$

$$x:y:z=\left(u^2\left(-\frac{u^2(v+w)^2}{a^2}+\frac{v^2(w+u)^2}{b^2}+\frac{w^2(u+v)^2}{c^2}\right):\cdots:\cdots\right).$$

9.6.5 Excursus: A construction problem

Given a finite point P in the plane of triangle ABC, to construct three circles $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$, each passing through P and one vertex of triangle ABC, such that their second intersections $X = \mathcal{C}_b \cap \mathcal{C}_c$, $Y = \mathcal{C}_c \cap \mathcal{C}_a$, and $Z = \mathcal{C}_a \cap \mathcal{C}_b$ lie on AP, BP, CP respectively. Equivalently, construct three points X, Y, Z (distinct from P) on the lines AP, BP, CP respectively, such that the circles (AYZ), (BZX) and (CXY) are concurrent at P.

Analysis. Let $P = (u : v : w)$ and the circles be

$$\begin{aligned}
\mathcal{C}_a: &\quad a^2yz + b^2zx + c^2xy - (x+y+z)(q_1y + r_1z) = 0,\\
\mathcal{C}_b: &\quad a^2yz + b^2zx + c^2xy - (x+y+z)(p_2x + r_2z) = 0,\\
\mathcal{C}_c: &\quad a^2yz + b^2zx + c^2xy - (x+y+z)(p_3x + q_3y) = 0,
\end{aligned}$$

for undetermined coefficients q_1, r_1, p_2, r_2, p_3, q_3.

Since we require the line $AP: \frac{y}{v} - \frac{z}{w} = 0$ to contain the point X, it is the radical axis of the circles $\mathcal{C}_b$ and $\mathcal{C}_c$, we must have
$p_2 = p_3$ and $q_3 : r_2 = \frac{1}{v} : \frac{1}{w}$. Similarly,
$q_1 = q_3$ and $r_1 : p_3 = \frac{1}{w} : \frac{1}{u}$, and
$r_1 = r_2$ and $p_2 : q_2 = \frac{1}{u} : \frac{1}{v}$.
From these, we rewrite the equations of the circles as

$$\begin{aligned}
\mathcal{C}_a: &\quad a^2yz + b^2zx + c^2xy - k(x+y+z)\left(\frac{y}{v} + \frac{z}{w}\right) &= 0,\\
\mathcal{C}_b: &\quad a^2yz + b^2zx + c^2xy - k(x+y+z)\left(\frac{x}{u} + \frac{z}{w}\right) &= 0,\\
\mathcal{C}_b: &\quad a^2yz + b^2zx + c^2xy - k(x+y+z)\left(\frac{x}{u} + \frac{y}{v}\right) &= 0,
\end{aligned}$$

for some k. Since these circles contain the point P, we must have

$$k = \frac{a^2vw + b^2wu + c^2uv}{2(u+v+w)},$$

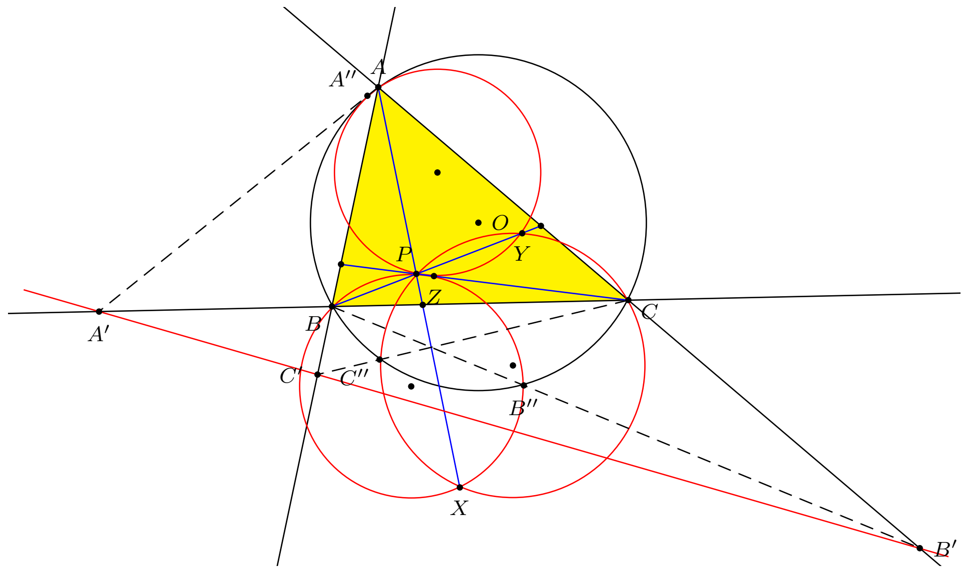
with $a^2vw + b^2wu + c^2uv \neq 0$, i.e., P not lying on the circumcircle.

Consider the trilinear polar of P:

$$\mathscr{L}: \quad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0,$$

intersecting the sidelines at A', B', C' respectively. The line AA' has equation $\frac{y}{v} + \frac{z}{w} = 0$. It is the radical axis of $\mathcal{C}_a$ and the circumcircle. Therefore,

$\mathcal{C}_a$ is the circle through A, P, and the intersection of AA' with the circumcircle; similarly for $\mathcal{C}_b$ and $\mathcal{C}_c$. The points X, Y, Z are now the pairwise intersections of the circles C_a, C_b, C_c.



9.6.6 Excursus: Radical center of a triad of circles

Given a point $P = (u : v : w)$, consider the triad of circles each through one vertex of $\mathbf{T}$ and tangent to opposite side at the trace of P. These are the circles

$$\begin{aligned}
\mathcal{C}_a : &\qquad a^2yz + b^2zx + c^2xy - \frac{a^2(w^2y + v^2z)}{(v+w)^2}(x+y+z) = 0,\\
\mathcal{C}_b : &\qquad a^2yz + b^2zx + c^2xy - \frac{b^2(u^2z + w^2x)}{(w+u)^2}(x+y+z) = 0,\\
\mathcal{C}_c : &\qquad a^2yz + b^2zx + c^2xy - \frac{c^2(v^2x + u^2y)}{(u+v)^2}(x+y+z) = 0,
\end{aligned}$$

We determine the radical center Q of the triad of circles. For example, if $P = \mathtt{H}$, then $Q = \mathtt{H}$. More generally, $Q = (x : y : z)$ is given by

$$\frac{a^2(w^2y + v^2z)}{(v+w)^2} = \frac{b^2(u^2z + w^2x)}{(w+u)^2}(x+y+z) = \frac{c^2(v^2x + u^2y)}{(u+v)^2}.$$

Equivalently,

$$\frac{a^2\left(\frac{y}{v^2} + \frac{z}{w^2}\right)}{\left(\frac{1}{v} + \frac{1}{w}\right)^2} = \frac{b^2\left(\frac{z}{w^2} + \frac{x}{u^2}\right)}{\left(\frac{1}{w} + \frac{1}{u}\right)^2} = \frac{c^2\left(\frac{x}{u^2} + \frac{y}{v^2}\right)}{\left(\frac{1}{u} + \frac{1}{v}\right)^2}.$$

Therefore,

$$\left(\frac{x}{u^2} : \frac{y}{v^2} : \frac{z}{w^2}\right) = \mathsf{sup}\left(\frac{1}{a^2}\left(\frac{1}{v} + \frac{1}{w}\right)^2 : \frac{1}{b^2}\left(\frac{1}{w} + \frac{1}{u}\right)^2 : \frac{1}{c^2}\left(\frac{1}{u} + \frac{1}{v}\right)^2\right).$$

From this, the coordinates of the radical center Q can be easily computed. Here are some examples.

P	Q	
G	$\left(-\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} : \cdots : \cdots\right)$	$X(194)$
I	$(a^2(a(a+b+c) - bc) : \cdots : \cdots)$	$X(595)$
H	H	
$\mathtt{G_e}$	$\left(\frac{1}{(b+c-a)^2} : \frac{1}{(c+a-b)^2} : \frac{1}{(a+b-c)^2}\right)$	$X(279)$
$\mathtt{N_a}$	$((b+c-a)^2((a+b+c)^2 - 8bc) : \cdots : \cdots)$	$X(6552)$

9.6.7 Concurrency of three Euler lines

We have computed the equation of the line BZ_b: $(a+b)x+az=0$. Likewise, CY_c is the line $(c+a)x+ay=0$. It intersects the line BZ_b at the orthocenter of triangle $\mathtt{I}_aBC$. This is the point $(-a:c+a:a+b)$.

On the other hand, the centroid of triangle $\mathtt{I}_aBC$ is the point $(-a:2b+c-a:b+2c-a)$. With these we compute the equation of the Euler line of triangle $\mathtt{I}_aBC$, and similarly those of triangles $\mathtt{I}_bCA$ and $\mathtt{I}_cAB$. These are the lines

$$\begin{aligned}
-(b+c)(b-c)x+\quad a(c-a)y+\quad a(a-b)z&=0,\\
b(b-c)x-(c+a)(c-a)y+\quad b(a-b)z&=0,\\
c(b-c)x+\quad c(c+a)y-(a+b)(a-b)z&=0.
\end{aligned}$$

These lines are concurrent at a point $(x:y:z)$ given by

$$\begin{aligned}
&(b-c)x:(c-a)y:(a-b)z\\
=&\begin{vmatrix}-(c+a) & b\\ c & -(a+b)\end{vmatrix}:-\begin{vmatrix}b & b\\ c & -(a+b)\end{vmatrix}:\begin{vmatrix}b & -(c+a)\\ c & c\end{vmatrix}\\
=&\ a:b:c.
\end{aligned}$$

Therefore,

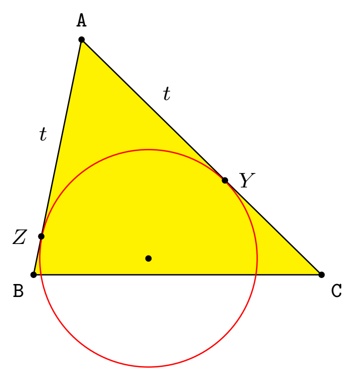
$$x:y:z=\frac{a}{b-c}:\frac{b}{c-a}:\frac{c}{a-b}.$$

Proposition. The Euler lines of triangles $\mathtt{I}_aBC$, $\mathtt{I}_bCA$ and $\mathtt{I}_cAB$ are concurrent at the superior of the Feuerbach point (on the circumcircle).

We begin with the equation of the circle tangent to two sides of the reference triangle $\mathbf{T}$.

Lemma. The circle tangent to AC and AB with common tangent length t from A (measured along AC and AB) is

$$\mathscr{C}_a(t): \quad a^2yz+b^2zx+c^2xy-(x+y+z)(t^2x+(c-t)^2y+(b-t)^2z)=0.$$



Proof. If the two tangents from A have length t (measured along AC and AB), the points of tangency are

$$Y=(b-t:0:t), \qquad Z=(c-t:t:0).$$

The circle $\mathcal{C}_a(t)$ intersects the line $\mathbf{b}:\ y=0$ at $(x:0:z)$ satisfying

$$b^2zx-(x+z)(t^2x+(b-t)^2z)=0 \implies (tx-(b-t)z)^2=0.$$

Therefore, $x:z=b-t:t$. The circle is tangent to the sideline $\mathbf{b}$ at Y given above. A similar calculation shows that it is tangent to $\mathbf{c}$ at Z. $\square$

Chapter 10

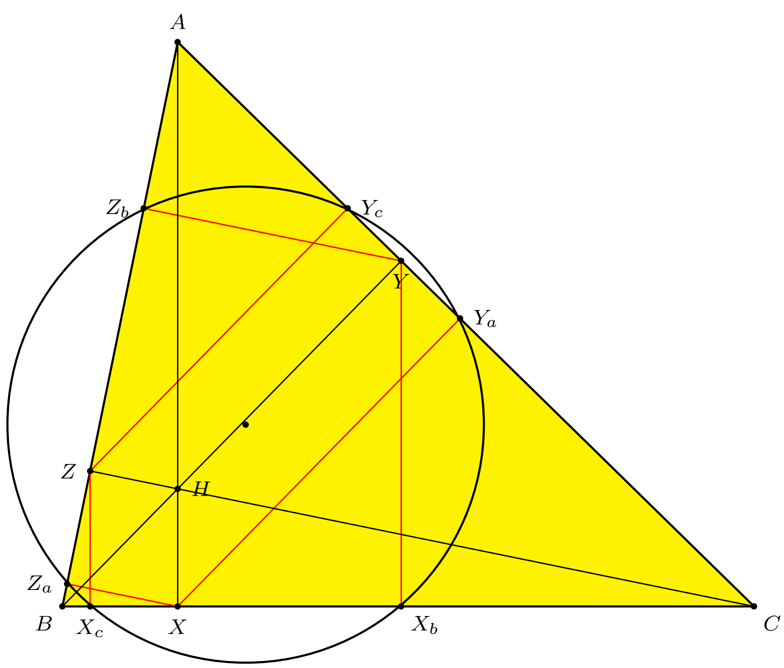
Tucker circles

10.1 The Taylor circle

10.1.1 The pedals of pedals

Consider the orthic triangle XYZ, and the pedals of each of the points X, Y, Z on the two sides not containing it. Thus,

Line	Pedals of X	Pedals of Y	Pedals of Z
a		X_b	X_c
b	Y_a		Y_c
c	Z_a	Z_b	



It is easy to write down the lengths of various segments. From these we easily determine the coordinates of these pedals. For example, from $Y_aC = b\cos^2\gamma$, we have $AY_a = b - b\cos^2\gamma = b\sin^2\gamma$. In homogeneous coordinates,

$$Y_a = (\cos^2\gamma : 0 : \sin^2\gamma) = (\cot^2\gamma : 0 : 1) = (S_{\gamma\gamma} : 0 : S^2).$$

Similarly we obtain the coordinates of the remaining pedals.

Line	Pedals of X	Pedals of Y	Pedals of Z
a		$X_b = (0 : S_{\gamma\gamma} : S^2)$	$X_c = (0 : S^2 : S_{\beta\beta})$
b	$Y_a = (S_{\gamma\gamma} : 0 : S^2)$		$Y_c = (S^2 : 0 : S_{\alpha\alpha})$
c	$Z_a = (S_{\beta\beta} : S^2 : 0)$	$Z_b = (S^2 : S_{\alpha\alpha} : 0)$	

10.1.2 The Taylor circle

Note that

$$\begin{aligned} AY_a \cdot AY_c =&(b - b\cos^2\gamma)(b\cos^2\alpha) = b^2\cos^2\alpha\sin^2\gamma = 4R^2\cos^2\alpha\sin^2\beta\sin^2\gamma, \\ AZ_a \cdot AZ_b =&(c - c\cos^2\beta)(c\cos^2\alpha) = c^2\cos^2\alpha\sin^2\beta = 4R^2\cos^2\alpha\sin^2\beta\sin^2\gamma, \end{aligned}$$

giving $AY_a \cdot AY_c = AZ_a \cdot AZ_b = \frac{S_{\alpha\alpha}}{4R^2} = \frac{S^2 \cdot S_{\alpha\alpha}}{a^2b^2c^2}$. Similarly, $BX_b \cdot BX_c = BZ_b \cdot BZ_a = \frac{S^2 \cdot S_{\beta\beta}}{a^2b^2c^2}$ and $CY_c \cdot CY_a = CX_c \cdot CX_b = \frac{S^2 \cdot S_{\gamma\gamma}}{a^2b^2c^2}$. By Proposition **??**, the six points X_b, X_c, Y_c, Y_a, Z_a, Z_b are concyclic. The circle containing them is called the *Taylor circle* with equation

$$a^2b^2c^2(a^2yz + b^2zx + c^2xy) - S^2(x + y + z)(S_{\alpha\alpha}x + S_{\beta\beta}y + S_{\gamma\gamma}z) = 0.$$

10.1.3 The Taylor center

Proposition. The center of the Taylor circle [1] has homogeneous barycentric coordinates

$$(a^2(S^4 - S_{\alpha\beta\gamma}S_\alpha) : b^2(S^4 - S_{\alpha\beta\gamma}S_\beta) : c^2(S^4 - S_{\alpha\beta\gamma}S_\gamma)).$$

[1] The Taylor center appears as X_{389} in ETC.

Proof. We compute the center of the Taylor circle by Proposition **??**. This is

$$\begin{aligned}
&a^2S_\alpha+\frac{S^2}{a^2b^2c^2}\left(S_\beta(S_{\gamma\gamma}-S_{\alpha\alpha})-S_\gamma(S_{\alpha\alpha}-S_{\beta\beta})\right)\ :\ \cdots :\ \cdots\\
=&a^2S_\alpha-\frac{S^2}{a^2b^2c^2}\cdot a^2(S_{\alpha\alpha}-S_{\beta\gamma})\ :\ \cdots :\ \cdots\\
=&a^2(a^2b^2c^2S_\alpha-S^2(S_{\alpha\alpha}-S_{\beta\gamma}))\ :\ \cdots :\ \cdots\\
=&a^2((S^2+S_{\alpha\alpha})(S^2-S_{\beta\gamma})-S^2(S_{\alpha\alpha}-S_{\beta\gamma}))\ :\ \cdots :\ \cdots\\
=&a^2(S^4-S_{\alpha\beta\gamma}\cdot S_\alpha)\ :\ \cdots :\ \cdots .
\end{aligned}$$

□

This is a point on the Brocard axis, namely,

$$K(\theta)^*=(a^2(S_\alpha+S_\theta):b^2(S_\beta+S_\theta):c^2(S_\gamma+S_\theta))$$

for $\cot\theta=-\tan\alpha\tan\beta\tan\gamma$.

Exercise

1. Show that the centroid of the *perimeter of the orthic triangle* is the center of the Taylor circle.

2. (a) Let XYZ be the orthic triangle of ABC. Show that the orthocenter of the residual triangle AYZ is

$$H_a=(S^4-S_{\alpha\beta\gamma}S_\alpha:b^2S_{\alpha\alpha\beta}:c^2S_{\alpha\alpha\gamma}).$$

(b) Similarly define H_b and H_c. Show that triangle $H_aH_bH_c$ is oppositely congruent to the orthic triangle at the Taylor center. [2]

[2]The midpoint between H_a and X is

$$\begin{aligned}
&b^2c^2S^2(0:S_\gamma:S_\beta)+a^2(S^4-S_{\alpha\beta\gamma}\cdot S_\alpha:b^2S_{\alpha\alpha\beta}:c^2S_{\alpha\alpha\gamma})\\
=&(a^2(S^4-S_{\alpha\beta\gamma}\cdot S_\alpha):b^2(c^2S^2S_\gamma+a^2S_{\alpha\alpha\beta}):c^2(b^2S^2S_\beta+a^2S_{\alpha\alpha\gamma}))\\
=&(a^2(S^4-S_{\alpha\beta\gamma}\cdot S_\alpha):b^2(c^2S^2S_\gamma+S_{\alpha\beta}(S^2-S_{\beta\gamma})):c^2(b^2S^2S_\beta+S_{\alpha\gamma}(S^2-S_{\beta\gamma}))\\
=&(a^2(S^4-S_{\alpha\beta\gamma}\cdot S_\alpha):b^2(S^2(c^2S_\gamma+S_{\alpha\beta})-S_{\alpha\beta\gamma}\cdot S_\beta)):c^2(S^2(b^2S_\beta+S_{\alpha\gamma})-S_{\alpha\beta\gamma}\cdot S_\gamma))\\
=&(a^2(S^4-S_{\alpha\beta\gamma}\cdot S_\alpha):b^2(S^4-S_{\alpha\beta\gamma}\cdot S_\beta)):c^2(S^4-S_{\alpha\beta\gamma}\cdot S_\gamma)).
\end{aligned}$$

This is the Taylor center. Similarly the midpoints of YH_b and ZH_c are the same point. Therefore, the triangles are oppositely congruent at the Taylor center.

10.1.4 The Taylor circle of the excentral triangle

Triangle ABC is the orthic triangle of its excentral triangle. To find the Taylor center of the excentral triangle, we find the orthocenter of triangle I_ABC. It is the point $(-a : c+a : a+b)$. The midpoint of this orthocenter and A is the point

$$(-a : c+a : a+b) + (a+b+c)(1 : 0 : 0) = (b+c : c+a : a+b),$$

the Spieker center.

The pedals themselves are very simple:

$$\begin{array}{ll} (-a : b : a+b), & (-a : c+a : c), \\ (b+c : -b : c), & (a : -b : a+b), \\ (a : c+a : -c), & (b+c : b : -c). \end{array}$$

The midpoint between $(a : -b : a+b)$ and $(a : c+a : -c)$ is $(2a : c+a-b : a+b-c)$, and the distance is s. This midpoint is precisely the point of tangency of the incircle with the corresponding side of the medial triangle. This shows that the radius of the Taylor circle is $\frac{1}{2}\sqrt{r^2+s^2}$.

This proves that the Taylor circle of the excentral triangle is the Spieker radical circle of the excircles.

Exercise

1. Show that $X_bX_c = \frac{S^2}{abc} = \frac{\triangle}{R}$.
2. Find the equations of the lines X_bX_c, Y_cY_a, Z_aZ_b, and show that they bound a triangle perspective to ABC at the symmedian point. [3]
3. Show that the Euler lines of the triangles XX_bX_c, YY_cY_a, ZZ_aZ_b are concurrent and find the coordinates of the intersection. [4]

10.1.5 A triad of Taylor circles

Consider the Taylor circle of HBC. This passes through the points

[3] The line X_bX_c has equation $-S^2x + S_{\beta\beta}y + S_{\gamma\gamma}z = 0$

[4] J.-P. Ehrmann, Hyacinthos 3695. The common point of the Euler lines is X_{973}.

10.2 The Taylor circle

The pedals of the vertices of the orthic triangle on the side lines are

$$\begin{aligned}
X_b &= (S_{CC} : 0 : S^2), \\
X_c &= (S_{BB} : S^2 : 0), \\
\\
Y_c &= (S^2 : S_{AA} : 0), \\
Y_a &= (0 : S_{CC} : S^2), \\
\\
Z_a &= (0 : S^2 : S_{BB}), \\
Z_b &= (S^2 : 0 : S_{AA}).
\end{aligned}$$

The orthocenter of triangle AZ_bY_c is the point

$$H_a = (S^4 - S_{AABC} : b^2S_{AAB} : c^2S_{AAC}).\ ^5$$

These orthocenters form a triangle perspective with ABC at the circumcenter $O = (a^2S_\alpha : b^2S_\beta : c^2S_\gamma)$.

It is more interesting to note that $H_aH_bH_c$ is homothetic to the orthic triangle.

The midpoint of H_aX is on the perpendicular bisector of Z_bX_b, and also of X_cY_c. It is necessarily the Taylor center. Similarly for the other two segments H_bY and H_cZ.

The midpoint between H_a and X is

$$\begin{aligned}
&b^2c^2S^2(0 : S_\gamma : S_\beta) + a^2(S^4 - S_{AABC} : b^2S_{AAB} : c^2S_{AAC}) \\
=&(a^2(S^4 - S_{AABC}) : b^2c^2S^2S_\gamma + a^2b^2S_{AAB} : b^2c^2S^2S_\beta + a^2c^2S_{AAC}) \\
=&(a^2(S^4 - S_{AABC}) : b^2(c^2S^2S_\gamma + a^2S_{AAB}) : c^2(b^2S^2S_\beta + a^2S_{AAC}))
\end{aligned}$$

Now

$$\begin{aligned}
c^2S^2S_\gamma + a^2S_{AAB} =&c^2S^2S_\gamma + a^2S_{AAB} + S_{BBCA} - S_{BBCA} \\
=&c^2S^2S_\gamma + S_{AABB} + S_{AABC} + S_{BBCA} - S_{BBCA} \\
=&c^2S^2S_\gamma + S_{AB}(S_{AB} + +S_{AC} + S_{BC}) - S_{BBCA} \\
=&c^2S^2S_\gamma + S_{AB}S^2 - S_{BBCA} \\
=&S^4 - S_{BBCA}
\end{aligned}$$

[5] The sum of the coordinates is $b^2c^2S^2$.

Similarly, the third coordinate is $c^2(S^4 - S_{CCAB})$. The triangles $H_aH_bH_c$ and XYZ therefore are homothetic at the triangle center

$$(a^2(S^4 - S_{AABC}) : b^2(S^4 - S_{BBCA}) : c^2(S^4 - S_{CCAB}))$$

with ratio of homothety -1. This homothetic center is the Taylor center.

The Taylor center is a point on the Brocard axis. Its coordinates can be written as

$$(a^2(S_\alpha + S_\theta) : b^2(S_\beta + S_\theta) : c^2(S_\gamma + S_\theta))$$

for

$$S_\theta = \frac{-S^4}{S_\alpha S_\beta S_\gamma}.$$

In other words,

$$\tan\theta = -\tan A \tan B \tan C.$$

The radius of the Taylor circle is

$$\frac{R \sin\omega}{\sin(\theta + \omega)}$$

10.2.1

Triangle ABC is the orthic triangle of its excentral triangle. To find the Taylor center of the excentral triangle, we find the orthocenter of triangle $I_\alpha BC$. It is the point $(-a : c+a : a+b)$. The midpoint of this orthocenter and A is the point

$$(-a : c+a : a+b) + (a+b+c)(1:0:0) = (b+c : c+a : a+b),$$

the Spieker center.

The pedals themselves are very simple:

$$\begin{array}{ll} (-a : b : a+b), & (-a : c+a : c), \\ (b+c : -b : c), & (a : -b : a+b), \\ (a : c+a : -c), & (b+c : b : -c). \end{array}$$

The midpoint between $(a : -b : a+b)$ and $(a : c+a : -c)$ is $(2a : c+a-b : a+b-c)$, and the distance is s. This midpoint is precisely the point of tangency of the incircle with the corresponding side of the medial triangle. This shows that the radius of the Taylor circle is $\frac{1}{2}\sqrt{r^2+s^2}$.

This proves that the Taylor circle of the excentral triangle is the radical circle of the excircles.

Exercise

(i) $X_bX_c = \frac{S^2}{abc} = \frac{\triangle}{R}$.

(ii) The line X_bX_c has equation

$$-S^2x + S_{\beta\beta}y + S_{\gamma\gamma}z = 0.$$

(iii) The three lines X_bX_c, Y_cY_a, Z_aZ_b bound a triangle with perspector K.

(iv) The Euler lines of XX_bX_c, YY_cY_a, ZZ_aZ_b are concurrent. (Ehrmann, Hyacinthos 3695).

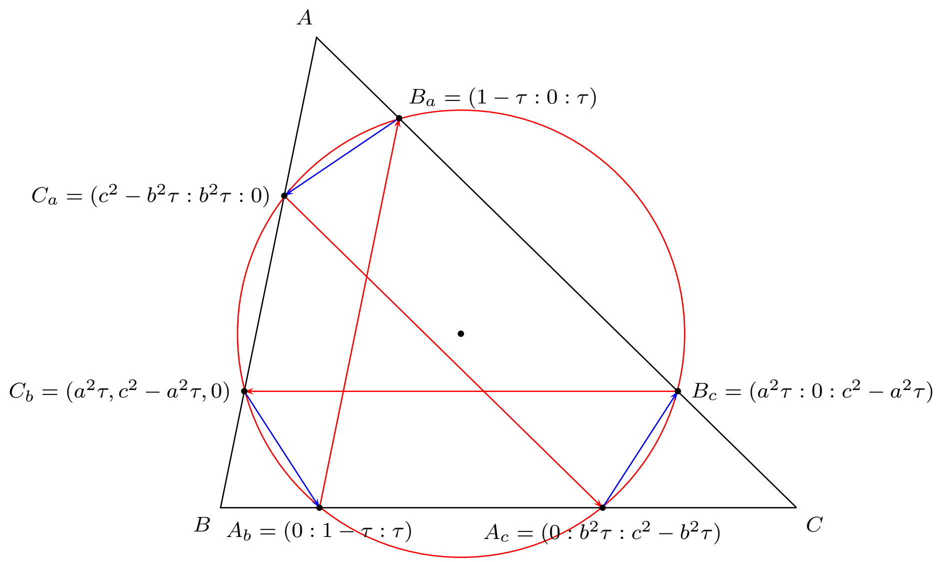
$$a^2(S^2 + S_{\beta\gamma})(S^2(S^2(S_\alpha + S_\beta + S_\gamma) + S_{\alpha\beta\gamma}) - 2S_{\alpha\beta\gamma} \cdot S_{\alpha\alpha})$$
$$: \cdots : \cdots).$$

(v) The Euler lines of AX_bX_c, BY_cY_a, CZ_aZ_b are concurrent.

These are the triangle centers $X(973)$ and $X(974)$.

10.3 Tucker circles

Given a triangle, through an arbitrary point on any one sideline, a point on a second sideline is obtained by alternately constructing parallels and antiparallels to a third sideline. The process terminates in six steps resulting in three pairs of points on the sidelines which lie on a Tucker circle. If the first point is $A_b = (0 : 1-\tau : \tau)$ on a, then by constructing alternatively parallels and antiparallels to c, a, b, c, a, b, we obtain the 6 points with coordinates indicated in the figure below.



These coordinates can be put in a symmetric form if we rewrite the coordinates of the first point as $A_b = (0 : S_\gamma + t : c^2)$ for some t. The above construction leads to the six points

$$\begin{array}{lll} A_b = (0 : S_\gamma + t : c^2), & B_a = (S_\gamma + t : 0 : c^2), & \\ A_c = (0 : b^2 : S_\beta + t), & & C_a = (S_\beta + t : b^2 : 0), \\ & B_c = (a^2 : 0 : S_\alpha + t), & C_b = (a^2 : S_\alpha + t : 0). \end{array}$$

The Tucker circle through these six points has barycentric equation

$$\begin{aligned} \mathcal{C}_{\text{Tucker}}(t): \qquad & (S_\alpha + S_\beta + S_\gamma + t)^2(a^2yz + b^2zx + c^2xy) \\ & - (x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(S_\alpha + t)x\right) = 0. \end{aligned}$$

10.3.1 The center of Tucker circle

Proposition. The center of the Tucker circle $\mathscr{C}_t$ is the point

$$K^*(t) := (a^2(S^2 + t \cdot S_\alpha) : \ b^2(S^2 + t \cdot S_\beta) : \ c^2(S^2 + t \cdot S_\gamma))$$

on the Brocard axis.

Proof. The perpendicular bisectors of the segments A_bA_c, B_cB_a, C_aC_b are the lines

$$\begin{aligned}
(b^2 - c^2)(S_\alpha + t)x \qquad +a^2(c^2 + S_\beta)y \qquad -a^2(b^2 + S_\gamma)z &= 0,\\
-b^2(c^2 + \alpha)x + (c^2 - a^2)(S_\beta + t)y \qquad +b^2(a^2 + S_\gamma)z &= 0,\\
c^2(b^2 + S_\alpha)x \qquad -c^2(a^2 + S_\beta)y + (a^2 - b^2)(S_\gamma + t)z &= 0.
\end{aligned}$$

These lines are concurrent at the point $K^*(t)$ given above, which clearly lies on the Brocard axis `OK`. □

Remark. $K^*(t)$ is the isogonal conjugate of the Kiepert perspector $K(\arctan t)$.

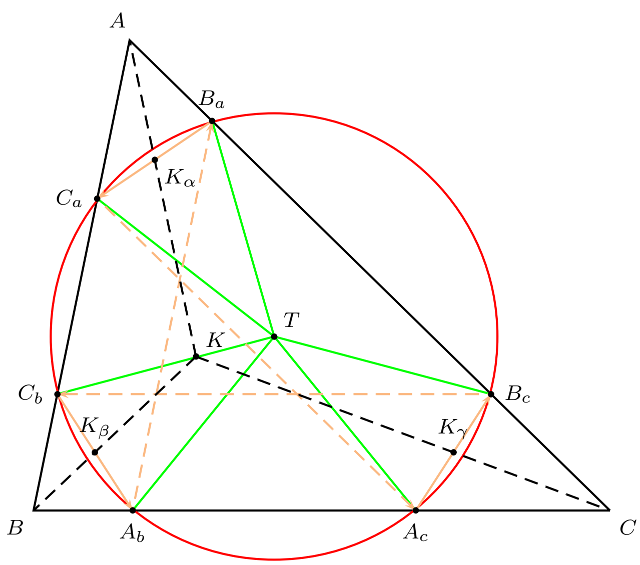


Figure 10.1:

The radius of the circle is

$$\frac{R \sin \omega}{\sin(\phi + \omega)},$$

where R is the circumradius of ABC.

The square radius of the circle is

$$\frac{S^2+t^2}{(S_\alpha+S_\beta+S_\gamma+t)^2}\cdot R^2=\frac{R^2\cdot S^2\csc^2\phi}{S^2(\cot\omega+\cot\phi)^2}=\frac{R^2\sin^2\omega}{\sin^2(\omega+\phi)}.$$

10.3.2 Construction of Tucker circle with given center

Let P be a point on the Brocard axis. To construct the Tucker circle with center P, draw parallels through P to OA, OB, OC to intersect the symmedians at X, Y, Z. The antiparallels through X, Y, Z give the Tucker hexagon.

10.3.3 Dao's construction of the Tucker circles

Let A' be the point dividing the A-altitude in the ratio $AA' : A\mathtt{H}_a = \tau : 1$.

The circle with diameter AA' intersects AC and AB at

$$\begin{aligned} B_a = &\ (S^2(1-\tau) + S_{\gamma\gamma} : 0 : S^2\tau),\\ C_a = &\ (S^2(1-\tau) + S_{\beta\beta} : S^2\tau : 0). \end{aligned}$$

The line B_aC_a is antiparallel to BC.

Similarly,

$$\begin{aligned} C_b = &\ (S^2\tau : S^2(1-\tau) + S_{\alpha\alpha} : 0),\\ A_b = &\ (0 : S^2(1-\tau) + S_{\gamma\gamma} : S^2\tau);\\ A_c = &\ (0 : S^2\tau : S^2(1-\tau) + S_{\beta\beta}),\\ B_c = &\ (S^2\tau : 0 : S^2(1-\tau) + S_{\alpha\alpha}). \end{aligned}$$

Note that B_cC_b is parallel to BC. Therefore the six points are concyclic on a Tucker circle. The equation of the circle is

$$\begin{aligned} &a^2b^2c^2(a^2yz + b^2zx + c^2xy)\\ &-S^2\tau(x+y+z)\left(\sum_{\text{cyclic}}(b^2c^2 - S^2\tau)x\right) = 0 \end{aligned}$$

with center

$$(a^2(a^2b^2c^2S_\alpha - S^2\tau(S_{\alpha\alpha} - S_{\beta\gamma})) : \cdots : \cdots).$$

This divides $\mathtt{OK}$ in the ratio

$$\tau S^2(S_\alpha + S_\beta + S_\gamma) : a^2b^2c^2 - \tau S^2(S_\alpha + S_\beta + S_\gamma).$$

This is the Tucker circle $\mathcal{C}(t)$ with

$$t = \frac{a^2b^2c^2 - \tau S_{\alpha\beta\gamma}}{\tau S^2}.$$

The three circles have radical center

$$\left(\frac{S_\alpha}{b^2c^2 - \tau S^2} : \cdots : \cdots\right).$$

This is the isogonal conjugate of the point

$$(a^2 S_{\beta\gamma}(b^2c^2 - \tau S^2) : \cdots : \cdots)$$

on the Euler line which divides OH in the ratio

$$(1-\tau)a^2b^2c^2 : 2\tau S_{\alpha\beta\gamma}.$$

The locus of the radical center is the Jerabek hyperbola.

10.4 The Lemoine circles

10.4.1 The first Lemoine circle

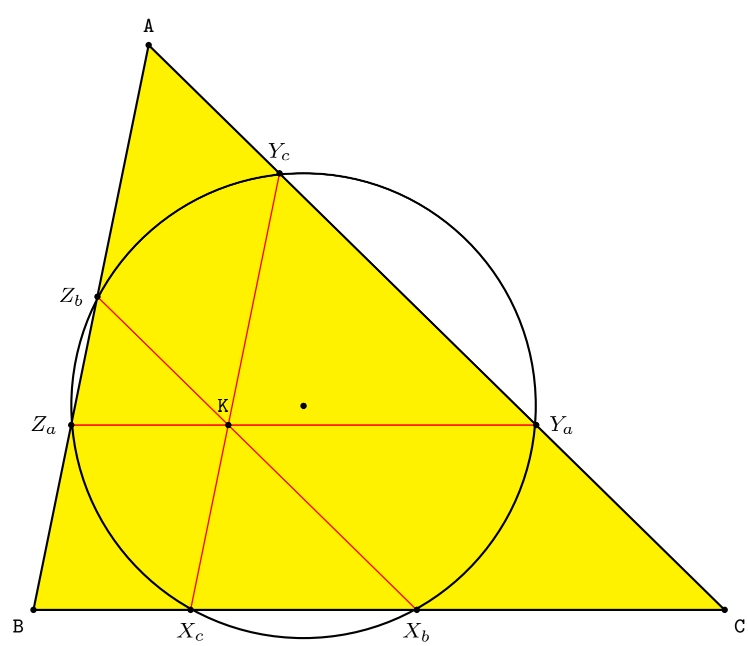
We find a point P the parallelians through which intersect the sidelines at 6 concyclic points. If the A-parallelian intersects b and c respectively at Y_a and Z_a, then

$$\mathtt{A}Y_a = \frac{v+w}{u+v+w}\cdot b \quad \text{and} \quad \mathtt{A}Z_a = \frac{v+w}{u+v+w}\cdot c.$$

Also, $\mathtt{A}Y_c = \frac{w}{u+v+w}\cdot b$ and $\mathtt{A}Z_b = \frac{v}{u+v+w}\cdot c$. Therefore, $\mathtt{A}Y_a\cdot\mathtt{A}Y_c = \mathtt{A}Z_a\cdot\mathtt{A}Z_b$ if and only if

$$\frac{v+w}{u+v+w}\cdot b\cdot\frac{w}{u+v+w}\cdot b = \frac{v+w}{u+v+w}\cdot c\cdot\frac{v}{u+v+w}\cdot c.$$

This reduces to $\frac{v}{b^2} = \frac{w}{c^2}$. Similarly, $\mathtt{B}Z_b\cdot\mathtt{B}Z_a = \mathtt{B}X_b\cdot\mathtt{B}X_c$ if and only if $\frac{w}{c^2} = \frac{u}{a^2}$, and $\mathtt{C}X_c\cdot\mathtt{C}X_b = \mathtt{C}Y_c\cdot\mathtt{C}Y_a$ if and only if $\frac{u}{a^2} = \frac{v}{b^2}$. It follows that the six points are on a circle if and only if $u:v:w = a^2:b^2:c^2$, *i.e.*, P is the symmedian point $\mathtt{K}$. This gives the *first Lemoine circle*.



The powers of $\mathtt{A}$, $\mathtt{B}$, $\mathtt{C}$ with respect to the circle are

$$\begin{aligned}
\mathtt{A}Y_a \cdot \mathtt{A}Z_a &= \frac{b^2c^2(b^2+c^2)}{(a^2+b^2+c^2)^2},\\
\mathtt{B}Z_b \cdot \mathtt{B}X_b &= \frac{c^2a^2(c^2+a^2)}{(a^2+b^2+c^2)^2},\\
\mathtt{C}X_c \cdot \mathtt{C}Y_c &= \frac{a^2b^2(a^2+b^2)}{(a^2+b^2+c^2)^2}.
\end{aligned}$$

From these we obtain the equation of the first Lemoine circle:

$$(a^2+b^2+c^2)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(b^2+c^2)x\right)=0.$$

Remark. The center of this circle is the midpoint of $\mathtt{OK}$. We shall establish this by showing that the circle is a Tucker circle. Clearly, $Y_cZ_b = Z_aX_c = X_bY_a$.

Exercise

1. Show that for the first Lemoine circle,

$$\begin{array}{ccccccccccc}
\mathtt{B}X_c & : & X_cX_b & : & X_b\mathtt{C} & = & c^2 & : & a^2 & : & b^2,\\
\mathtt{C}Y_a & : & Y_aY_c & : & Y_c\mathtt{A} & = & a^2 & : & b^2 & : & c^2,\\
\mathtt{A}Z_b & : & Z_bZ_a & : & Z_a\mathtt{B} & = & b^2 & : & c^2 & : & a^2.
\end{array}$$

10.4.2 The second Lemoine circle

10.4.3 Ehrmann's third Lemoine circle

Construct the circles $\mathtt{K}BC$, $\mathtt{K}CA$, $\mathtt{K}AB$ and note the intersections of these circles with the sidelines.

circle	a	b	c
$\mathtt{K}BC$:		$B_a = (a^2 + b^2 - 2c^2 : 0 : 3c^2)$,	$C_a = (c^2 + a^2 - 2b^2 : 3b^2 : 0)$;
$\mathtt{K}CA$:	$A_b = (0 : a^2 + b^2 - 2c^2 : 3c^2)$,		$C_b = (3a^2 : b^2 + c^2 - 2a^2 : 0)$;
$\mathtt{K}AB$:	$A_c = (0 : 3b^2 : c^2 + a^2 - 2b^2)$,	$B_c = (3a^2 : 0 : b^2 + c^2 - 2a^2)$.	

The six points are on a circle,

$$\begin{aligned}&(a^2 + b^2 + c^2)^2(a^2yz + b^2zx + c^2xy)\\ -3(x + y + z)&\left(\sum_{\text{cyclic}} b^2c^2(b^2 + c^2 - 2a^2)x\right) = 0.\end{aligned}$$

The center of the circle is the point

$$X_{576} = ((S_\alpha + S_\beta + S_\gamma)a^2S_A - 3a^2 \cdot S^2 : \cdots : \cdots),$$

dividing $\mathtt{OK}$ in the ratio $3 : -1$.

10.4.4 Bui's fourth Lemoine circle

Construct the three circles $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$ each through the symmedian point K and tangent to the circumcircle at the vertices A, B, C respectively. The intersections of these circles with the sidelines are the points

circle	a	b	c
$\mathcal{C}_a$		$B_a = (3a^2 : 0 : 2b^2 + 2c^2 - a^2)$,	$C_a = (3a^2 : 2b^2 + 2c^2 - a^2 : 0)$;
$\mathcal{C}_b$	$A_b = (0 : 3b^2 : 2c^2 + 2a^2 - b^2)$,		$C_b = (2c^2 + 2a^2 - b^2 : 3b^2 : 0)$;
$\mathcal{C}_c$	$A_c = (0 : 2a^2 + 2b^2 - c^2 : 3c^2)$,	$B_c = (2a^2 + 2b^2 - c^2 : 3c^2 : 0)$.	

These six points are concyclic. The circle containing them is called Bui's fourth Lemoine circle. It has barycentric equation

$$4(a^2+b^2+c^2)^2(a^2yz+b^2zx+c^2xy)$$
$$-(x+y+z)\left(\sum_{\text{cyclic}} 3b^2c^2(2b^2+2c^2-a^2)x\right) = 0,$$

and center

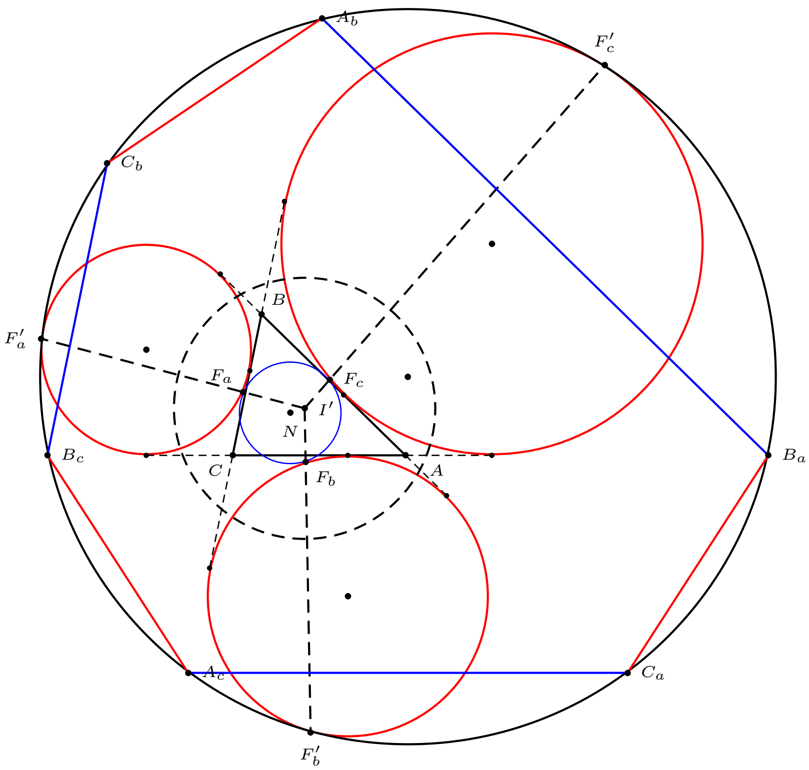
$$X(575) = ((S_\alpha + S_\beta + S_\gamma)a^2S_A + 3a^2 \cdot S^2 : \cdots : \cdots),$$

which divides OK in the ratio $3:1$.

10.5 The Apollonius circle

00907, 0210003 The Apollonius circle is the circular hull of the excircles. It is the inversive image of the nine-point circle in the Spieker radical circle. The touch points are the (second) intersections of the lines $\mathtt{S_pF}_a$, $\mathtt{S_pF}_b$, $\mathtt{S_pF}_c$ with the excircles. These are the points

$$\begin{aligned}
F'_a &= \ (-a^2(a(b+c)+(b^2+c^2))^2 : \ 4b^2(c+a)^2s(s-c) : \ 4c^2(a+b)^2s(s-b)),\\
F'_b &= \ (4a^2(b+c)^2s(s-c) : \ -b^2(b(c+a)+(c^2+a^2))^2 : \ 4c^2(a+b)^2s(s-a)),\\
F'_c &= \ (4a^2(b+c)^2s(s-b) : \ 4b^2(c+a)^2s(s-a) : \ -c^2(c(a+b)+(a^2+b^2))^2).
\end{aligned}$$



The equation of the Apollonius circle is

$$4abc(a^2yz+b^2zx+c^2xy)+(a+b+c)(x+y+z)\sum_{\text{cyclic}} bc(a(a+b+c)+2bc)x = 0.$$

The intersections with the sidelines are as follows.

$$\begin{aligned}
&A_b = (0 : cs+ab : -cs), \quad B_a = (cs+ab : 0 : -cs);\\
&A_c = (0 : -bs : bs+ca), \qquad\qquad\qquad C_a = (bs+ca : -bs : 0);\\
&\qquad\qquad B_c = (-as : 0 : as+bc), \quad C_b = (-as : as+bc : 0).
\end{aligned}$$

Clearly, B_cC_b, C_aA_c, and A_bB_a are parallel to the sidelines. Note that

$$AB_a \cdot AC = \left(\frac{-cs}{ab} \cdot b\right) b = \left(\frac{-bs}{ca} \cdot c\right) c = AC_a \cdot AB.$$

This means that B, C, C_a, B_a are concyclic, and B_aC_a is antiparallel to BC. Similarly, B_aC_a, C_bA_b, A_cB_c are antiparallel to CA and AB respectively. Therefore, the Apollonius circle is a Tucker circle $\mathcal{C}_{\text{Tucker}}(t)$. The parameter is indeed

$$t = -\left(\frac{a^2+b^2+c^2}{2} + \frac{2abc}{a+b+c}\right).$$

10.6 Tucker circles which are pedal circles

Consider a Tucker circle which is the common pedal triangle of an isogonal conjugate pair P and Q. Denote these pedals by $P_{[a]}$ and $Q_{[a]}$ etc.

Suppose the two endpoints of one of the "antiparallel" sides of the Tucker hexagon are both pedals of P. P must lie on an altitude of $\mathbf{T}$, and the endpoint of this altitude (a pedal of the orthocenter $\mathtt{H}$) is a vertex of the hexagon. One of the two sides of the hexagon containing this pedal of $\mathtt{H}$ must be another "antiparallel" side of the hexagon, and is parallel to a side of the orthic triangle of $\mathbf{T}$. This means that one side of the Tucker hexagon is a side of the orthic triangle. This is enough to determine the Tucker circle.

There are three such circles. Their centers are the intersections of the Brocard axis with the perpendicular bisectors of the sides of the orthic triangle.

P	Q	A_b	B_a	C_a	A_c	B_c	C_b
$B\mathtt{H} \cap C\mathtt{O}$	$B\mathtt{O} \cap C\mathtt{H}$	P_a	$P_b = \mathtt{H}_{[b]}$	$Q_c = \mathtt{H}_{[c]}$	Q_a	Q_b	P_c
$C\mathtt{H} \cap A\mathtt{O}$	$C\mathtt{O} \cap A\mathtt{H}$	$Q_a = \mathtt{H}_{[a]}$	Q_b	Q_c	P_a	P_b	$P_c = \mathtt{H}_{[c]}$
$A\mathtt{H} \cap B\mathtt{O}$	$A\mathtt{O} \cap B\mathtt{H}$	Q_a	P_b	P_c	$P_a = \mathtt{H}_{[a]}$	$Q_b = \mathtt{H}_{[b]}$	Q_c

$$
\begin{aligned}
b^2c^2(a^2yz + b^2zx + c^2xy) - S_\alpha(x+y+z)((S_{\alpha\alpha} + S_{\beta\gamma})x + a^2S_\beta y + a^2S_\gamma z) &= 0,\\
c^2a^2(a^2yz + b^2zx + c^2xy) - S_\beta(x+y+z)(b^2S_\alpha x + (S_{\beta\beta} + S_{\gamma\alpha})y + b^2S_\gamma z) &= 0,\\
a^2b^2(a^2yz + b^2zx + c^2xy) - S_\gamma(x+y+z)(c^2S_\alpha x + c^2S_\beta + (S_{\gamma\gamma} + S_{\alpha\beta})z) &= 0.
\end{aligned}
$$

Centers:

$$
\begin{aligned}
&S_{\beta\gamma}(a^2S_\alpha, b^2S_\beta, c^2S_\gamma) + S^2 \cdot S_\alpha(a^2, b^2, c^2),\\
&S_{\gamma\alpha}(a^2S_\alpha, b^2S_\beta, c^2S_\gamma) + S^2 \cdot S_\beta(a^2, b^2, c^2),\\
&S_{\alpha\beta}(a^2S_\alpha, b^2S_\beta, c^2S_\gamma) + S^2 \cdot S_\gamma(a^2, b^2, c^2).
\end{aligned}
$$

10.6.1 The Gallatly circle

We may assume the segments $P_{[b]}Q_{[c]}$, $P_{[c]}Q_{[a]}$, $P_{[a]}Q_{[b]}$ parallel to the side-line. Let $P = (x : y : z)$.

$$\begin{aligned} a^2yz + b^2zx + c^2xy - b^2x(x+y+z) = 0, \\ a^2yz + b^2zx + c^2xy - c^2y(x+y+z) = 0, \\ a^2yz + b^2zx + c^2xy - a^2z(x+y+z) = 0. \end{aligned}$$

These have $\Gamma_{\rightarrow} = \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right)$ as their common point. The isogonal conjugate is $\Gamma_{\leftarrow} = \left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right)$. The pedal circle is the Gallatly circle

$$\begin{aligned} &(S_{AA} + S_{BB} + S_{CC} + 3S^2)^2(a^2yz + b^2zx + c^2xy) \\ &- (S_A + S_B + S_C)(x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(S_{AA} + 2S_{AB} + 2S_{AC} + S_{BC})x\right) = 0. \end{aligned}$$

The radius of the circle is

$$\frac{S}{\sqrt{S^2 + (S_A + S_B + S_C)^2}} \cdot R = \frac{abc}{2\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}.$$

It is a Tucker circle with Kiepert parameter $\frac{\pi}{2} - \omega$.

The Gallatly circle is the smallest Tucker circle.

10.7 Tucker circles which are cevian circumcircles

Consider a Tucker circle which is the cevian circumcircle of P (and its cyclocevian conjugate $Q = P^{\mathrm{o}}$).

Suppose the two endpoints of one of the "parallel" sides of the Tucker hexagon are the traces of P. P must lie on a median of $\mathbf{T}$, and the endpoint of this median is a vertex of the hexagon. One of the two sides of the hexagon containing this midpoint must be another "parallel" side of the hexagon, and is parallel to a side of $\mathbf{T}$. This means that one side of the Tucker hexagon is a side of the inferior triangle of $\mathbf{T}$. This is enough to determine the Tucker circle.

$$\begin{array}{ll} \mathtt{G}_b\mathtt{G}_c & (-S_\alpha + S_\beta + S_\gamma)(a^2S_\alpha, b^2S_\beta, c^2S_\gamma) + S^2(a^2, b^2, c^2), \\ \mathtt{G}_c\mathtt{G}_a & (S_\alpha - S_\beta + S_\gamma)(a^2S_\alpha, b^2S_\beta, c^2S_\gamma) + S^2(a^2, b^2, c^2), \\ \mathtt{G}_a\mathtt{G}_b & (S_\alpha + S_\beta - S_\gamma)(a^2S_\alpha, b^2S_\beta, c^2S_\gamma) + S^2(a^2, b^2, c^2). \end{array}$$

These are the circles

$$\begin{aligned} 4a^2(a^2yz + b^2zx + c^2xy) - (x+y+z)(b^2c^2x + c^2(2a^2-b^2)y + b^2(2a^2-c^2)z) &= 0, \\ 4b^2(a^2yz + b^2zx + c^2xy) - (x+y+z)(c^2(2b^2-a^2)x + c^2a^2y + a^2(2b^2-c^2)z) &= 0, \\ 4c^2(a^2yz + b^2zx + c^2xy) - (x+y+z)(b^2(2c^2-a^2)x + a^2(2c^2-b^2)y + a^2b^2z) &= 0. \end{aligned}$$

These circles are the cevian circumcircles of the following cyclocevian conjugate pairs:

$$\begin{array}{ll} (2a^2-b^2 : b^2 : 2a^2-b^2), & (2a^2-c^2 : 2a^2-c^2 : c^2); \\ (2b^2-c^2 : 2b^2-c^2 : c^2), & (a^2 : 2b^2-a^2 : 2b^2-a^2); \\ (a^2 : 2c^2-a^2 : 2c^2-a^2), & (2c^2-b^2 : b^2 : 2c^2-b^2). \end{array}$$

Remark. There is a Tucker circle which is a cevian circumcircle of P and P^{o} the endpoints of whose "parallel" (and antiparallel) sides are not traces of the same point P or P^{o}.

10.7.1 Tucker circle congruent to the circumcircle

is the one with

$$t = \frac{(a^2+b^2+c^2)c^2}{a^2b^2+b^2c^2+c^2a^2}.$$

The Tucker hexagon has vertices

$$(0, -(c^4-a^2b^2), c^2(a^2+b^2+c^2)), \quad (-(c^4-a^2b^2), 0, c^2(a^2+b^2+c^2)), \quad (-(b^4-c$$
$$(0, , b^2(a^2+b^2+c^2), -(b^4-c^2a^2)), \quad (a^2(a^2+b^2+c^2), 0, -(a^4-b^2c^2)), \quad (a^2(a^2+$$

Its center is the point

$$X(3095) = (a^2(a^2(b^4+3b^2c^2+c^4)-(b^6+c^6)) : \cdots : \cdots).$$

The Tucker circles are symmetric about the line joining the Brocard points:

$$\frac{a^4-b^2c^2}{a^2}x + \frac{b^4-c^2a^2}{b^2}y + \frac{c^4-a^2b^2}{c^2}z = 0.$$

Envelope of the Tucker circles

From the equation of the Tucker circles, we obtain the envelope:

$$\begin{aligned} G_t &:= a^4(b^2+c^2)^2yz + b^4(c^2+a^2)^2zx + c^4(a^2+b^2)^2xy \\ &- (x+y+z)(b^4c^4x + c^4a^4y + a^4b^4z) = 0. \end{aligned}$$

This is the inscribed conic with perspector K and center $X(39)$. This is called the Brocard ellipse. Its foci are the Brocard points.

Now,

$$4a^2b^2F_t = -G_t + (b^2(c^2-2a^2t)x + a^2(c^2-2b^2t)y + a^2b^2(1-2t)z)^2$$

This line has infinite point $(a^2(b^2-c^2) : b^2(c^2-a^2) : c^2(a^2-b^2))$, and is perpendicular to the Brocard axis. The Tucker circle and the Brocard ellipse are bitangent if they intersect at real points.

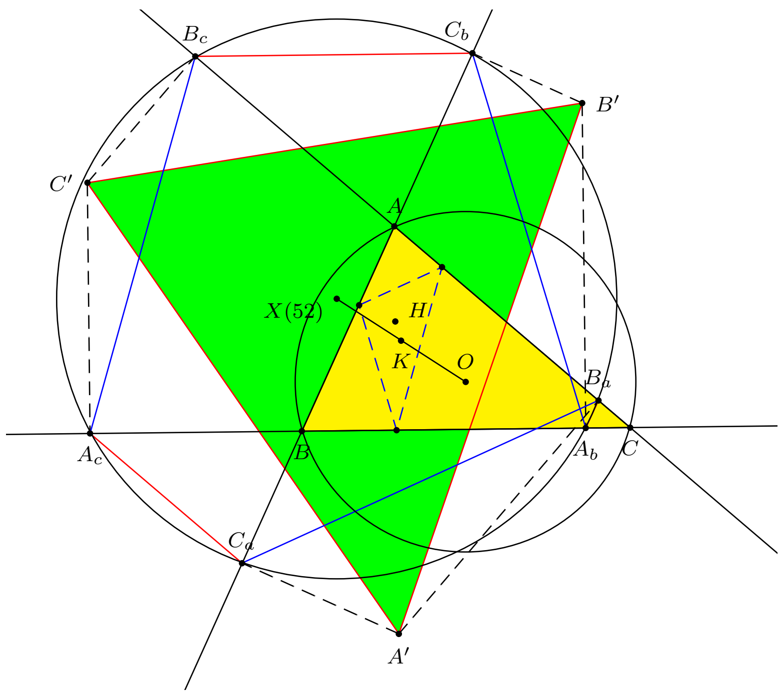
10.8 Torres' circle

Let $A'B'C'$ be the triangle of reflections. Consider the pedals of A', B', C' on the sidelines of $\mathbf{T}$. These are the points

	BC	CA	AB
A'		$B_a = (S_{CC} - S^2 : 0 : 2S^2)$,	$C_a = (S_{BB} - S^2 : 2S^2 : 0)$;
B'	$A_b = (0 : S_{CC} - S^2 : 2S^2)$,		$C_b = (2S^2 : S_{AA} - S^2 : 0)$;
C'	$A_c = (0 : 2S^2 : S_{BB} - S^2)$,	$B_c = (2S^2 : 0 : S_{AA} - S^2)$.	

The segments B_cC_b, C_aA_c, A_bB_a are parallel to BC, CA, AB respectively. The segments B_aC_a, C_bA_b, A_cB_c are antiparallel to BC, CA, AB respectively. Therefore, these six pedals define a Tucker circle $\mathcal{C}_{\text{Tucker}}(t)$. The parameter is

$$t = -\frac{1}{2S^2}\left(S^2(S_\alpha + S_\beta + S_\gamma) + S_{\alpha\beta\gamma}\right).$$



The equation of the circle containing these six pedals is

$$\begin{aligned}&a^2b^2c^2(a^2yz + b^2zx + c^2xy)\\ &- 2S^2(x+y+z)\left((S_{\alpha\alpha} - S^2)x + (S_{\beta\beta} - S^2)y + (S_{\gamma\gamma} - S^2)z\right) = 0.\end{aligned}$$

The center of the circle is the orthocenter of the orthic triangle:

$$X(52) = (a^2(S_{\alpha\alpha}-S^2)(S^2+S_{\beta\gamma}) : b^2(S_{\beta\beta}-S^2)(S^2+S_{\gamma\alpha}) : c^2(S_{\gamma\gamma}-S^2)(S^2+S_{\alpha\beta})).$$

10.9 Tucker circles tangent to the tritangent circles

10.9.1 The incircle

The radical axis of the Tucker circle $\mathscr{C}(t)$ and the incircle is the line

Proposition. The Tucker circle $\mathscr{C}(t)$ is tangent to the incircle if t is one of the following values:

$$t_0 = \frac{a^4+b^4+c^4-2(ab+bc+ca)(a^2+b^2+c^2-ab-bc-ca)}{2(2bc+2ca+2ab-a^2-b^2-c^2)},$$

$$t_a = \frac{a^3-a^2(b+c)+a(b^2+4bc+c^2)-(b+c)(b^2+c^2)}{2(b+c-a)},$$

and the two values t_b, t_c obtained by cyclic permutations of a, b, c.

With $t = t_0$, the equation of the Tucker circle is

$$4abc(a+b+c)^2(a^2yz+b^2zx+c^2xy)$$

$$-(2ab+2bc+2ca-a^2-b^2-c^2)(x+y+z)\left(\sum_{\text{cyclic}} bc(b+c-a)(2bc+a(b+c-\right.$$

The point of tangency is

$$X(1362) = \left(\frac{a^2(a(b+c)-(b^2+c^2))^2}{b+c-a} : \frac{b^2(b(c+a)-(c^2+a^2))^2}{c+a-b} : \frac{c^2(c(a+b)-(\right.}{a+b-}$$

The radius of the circle is

$$\frac{-(a^3b-2a^2b^2+ab^3+a^3c-a^2bc-ab^2c+b^3c-2a^2c^2-abc^2-2b^2c^2+ac^3+bc}{2abc(2s)}$$

$$= \frac{r((4R+r)^2+s^2)}{4s^2}.$$

$X(1362)$ can be constructed as the second intersection of the incircle with the line through F_e and $X(354)$, the centroid of the intouch circle.

For the value of t_a, the point of tangency with the incircle is

$$T_a = (a^2(a(b+c)-(b^2+c^2))^2 : b^2(c-a)^2(b+c-a)(c+a-b) : c^2(a-b)^2(b+c-a)(a+b-$$

Together with T_b and T_c, this forms a triangle perspective with ABC at

$$X(3271) = (a^2(b-c)^2(b+c-a) : b^2(c-a)^2(c+a-b) : c^2(a-b)^2(a+b-c)),$$

and with perspectrix the line

$$\frac{(b-c)(b+c-a)x}{a}+\frac{(c-a)(c+a-b)y}{b}+\frac{(a-b)(a+b-c)z}{c}=0$$

which is the line OI.

The circle corresponding to t_a:

$$4abc(a^2yz+b^2zx+c^2xy)-(b+c-a)(x+y+z)\cdot$$
$$(bc(2bc-a(b+c-a))x+ca(2ca-b(b+c-a))y+ab(2ab-c(b+c-a))z)=0.$$

It has radius

$$\begin{aligned}&\frac{a^2(b+c)-a(b^2+bc+c^2)+bc(b+c)}{2abc}\cdot R\\=&\ \frac{(s-a)^3+s(s-b)(s-c)}{4\Delta}\\=&\ \frac{(s-a)^2+r_a^2}{4r_a}\end{aligned}$$

This Tucker circle is also tangent to the B- and C-excircles. It is the inversive image of the line BC in the Spieker radical circle.

10.9.2 The excircles

(i)

$$t=\ -\frac{a^3+a^2(b+c)+a(b^2+4bc+c^2)+(b+c)(b^2+c^2)}{2(a+b+c)},$$

Point of tangency: $(-a^2(a(b+c)+(b^2+c^2))^2:\ b^2(c+a)^2(a+b+c)(a+b-c):c^2(a+b)^2(a+b+c)(c+a-b))$.

With two circles tangent to the B- and C-excircles, these points of tangency are perspective with ABC at

$$X(181)=\left(\frac{a^2(b+c)^2}{b+c-a}:\frac{b^2(c+a)^2}{c+a-b}:\frac{c^2(a+b)^2}{a+b-c}\right)$$

and with perspectrix $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$, the trilinear polar of I.

Note that $X(181)$ is the Apollonius point of the excircles. This gives the Apollonius circle.

(ii) With $t = -(a^3 + a^2b + ab^2 + b^3 - a^2c - 4abc - b^2c + ac^2 + bc^2 - c^3)/(2a + 2b - 2c)$, we have the circle

$$4abc(a^2yz + b^2zx + c^2xy) - (c + a - b)(x + y + z)\cdot$$
$$(bc(2bc - a(c + a - b))x + ca(2ca - b(c + a - b))y + ab(2ab - c(c + a - b))z) = 0.$$

with radius

$$\frac{-(a^2b - ab^2 - a^2c + abc - b^2c - ac^2 + bc^2)}{2abc}\cdot R = \frac{s(s-c)(s-a) + (s-b)^3}{4\Delta} = \frac{r_b^2 +$$

The point of tangency is

$$(-a^2(b-c)^2(a+b+c)(c+a-b) : b^2(c+a)^2(c+a-b)(a+b-c) : c^2(a^2+b^2+c(a-b))^2).$$

(iii) With $t = -(a^3 - a^2b + ab^2 - b^3 + a^2c - 4abc + b^2c + ac^2 - bc^2 + c^3)/(2a - 2b + 2c)$, we have the circle

$$4abc(a^2yz + b^2zx + c^2xy) - (a + b - c)(x + y + z)\cdot$$
$$(bc(2bc - a(a + b - c))x + ca(2ca - b(a + b - c))y + ab(2ab - c(a + b - c))z) = 0.$$

with radius

$$\frac{a^2b + ab^2 - a^2c - abc - b^2c + ac^2 + bc^2}{2abc}\cdot R = \frac{s(s-a)(s-b) + (s-c)^3}{4\Delta} = \frac{r_c^2 + (s -}{4r_c}$$

The point of tangency is

$$(-a^2(b-c)^2(a+b+c)(a+b-c) : b^2(c^2+a^2-b(c-a))^2 : c^2(a+b)^2(c+a-b)(a+b-c)).$$

(iv) With $t = -(a^4 + 2a^3b + 2a^2b^2 + 2ab^3 + b^4 + 2a^3c + 2a^2bc - 2ab^2c - 2b^3c + 2a^2c^2 - 2abc^2 + 2b^2c^2 + 2ac^3 - 2bc^3 + c^4)/(2(a^2 + b^2 + c^2 - 2bc + 2ca + 2ab))$, we have the circle

$$4abc(b + c - a)^2(a^2yz + b^2zx + c^2xy) + (a^2 + b^2 + c^2 - 2bc + 2ca + 2ab)(x + y + z$$
$$(bc(a + b + c)(2bc + a(a + b + c))x + ca(a + b - c)(2ca + b(a + b - c))y$$
$$+ ab(c + a - b)(2ab + c(c + a - b))z) = 0.$$

with radius

$$\frac{a^3b + 2a^2b^2 + ab^3 + a^3c + a^2bc - ab^2c - b^3c + 2a^2c^2 - abc^2 + 2b^2c^2 + ac^3 - bc^3}{2abc(b + c - a)}\cdot R =$$

The point of tangency is

$$\begin{aligned}
&(-a^2(c+a-b)(a+b-c)(b^2+c^2+a(b+c))^2\\
&:b^2(a+b+c)(c+a-b)(c^2+a^2-b(c-a))^2\\
&:c^2(a+b+c)(a+b-c)(a^2+b^2+c(a-b))^2).
\end{aligned}$$

This is

$$\left(\frac{u^2}{a+b+c} : \frac{v^2}{-a-b+c} : \frac{w^2}{-a+b-c}\right)$$

for the infinite point

$$(u:v:w) = (-a(b^2+c^2+a(b+c)) : b(c^2+a^2-b(c-a)) : c(a^2+b^2+c(a-b)).$$

The line through $I_a = (-a : b : c)$ with this infinite point intersects the sideline a at

$$(0 : b(a+b) : c(c+a)) = \left(0 : \frac{b}{c+a} : \frac{c}{a+b}\right).$$

The common length of the antiparallels is

$$\frac{(a+b+c)^2 - 4bc}{2(b+c-a)}.$$

Chapter 11

Some special circles

11.1 The Dou circle

Crux 1140: construction of a circle from which the chords cut out on the sidelines subtend right angles at their opposite vertices.

In the published solution [Crux 13 (1987) 232–234], it was established that if $P(\rho)$ is the circle, then

$$\rho^2 = PD^2 + h_a^2 = PE^2 + h_b^2 = PF^2 + h_c^2$$

where D, E, F are the vertices of the orthic triangle, and h_a, h_b, h_c the altitudes.

The locus of point P such that $PE^2 + h_b^2 = PF^2 + h_c^2$ is a line perpendicular to EF. This line contains the point $(-a^2 : b^2 : c^2)$, which is a vertex of the tangential triangle. It has equation

$$(S_B - S_C)x + \frac{S^2 - S_C^2}{b^2}y - \frac{S^2 - S_B^2}{c^2}z = 0.$$

Since the tangential triangle is homothetic to the orthic triangle, this line is indeed an altitude of the tangential triangle.

Nikolaos Dergiades [Hyacinthos 5815, 7/27/02] has given a simple verification of the fact that the A-vertex of the tangential triangle lies on this line.

It follows that the center of the circle we are seeking is the orthocenter of the tangential triangle. This is the point X_{155}. This is a finite point if and only if ABC does not contain a right angle.

The radius of the circle is the square root of

$$\frac{S^4(4R^6 - R^2S^2 + S_AS_BS_C)}{(S_AS_BS_C)^2}.$$

The equation of the circle is

$$2S_{ABC}(a^2yz+b^2zx+c^2xy)+(x+y+z)(\sum S_A(-a^2S_{AA}+b^2S_{BB}+c^2S_{CC})x)=0.$$

Suppose the center has homogeneous barycentric coordinates $(u:v:w)$. Its distance from BC is $\frac{uS}{(u+v+w)a}$, and its pedal on BC is the point

$$(0:S_Cu+a^2v:S_Bu+a^2w).$$

The square distance from this pedal to A is

$$\frac{S^2}{a^2}+\frac{(S_Bv-S_Cw)^2}{a^2(u+v+w)^2}.$$

The square radius of the circle is then

$$\frac{(u^2+(u+v+w)^2)S^2+(S_Bv-S_Cw)^2}{a^2(u+v+w)^2}$$

Theorem (Brisse)**.** The Dou circle is orthogonal to the circle through the centroid, X_{111}, and the anticomplement of X_{110}.

[This circle intersects the Euler line at X_{858}]. The center of this circle is the point [1]

$$[(b^2-c^2)(a^2(b^2+c^2)+(b^4-4b^2c^2+c^4)],$$

which is the superior of the point [2]

$$[(b^2-c^2)(b^2+c^2-2a^2)(b^2+c^2-3a^2)].$$

The equation of this circle is

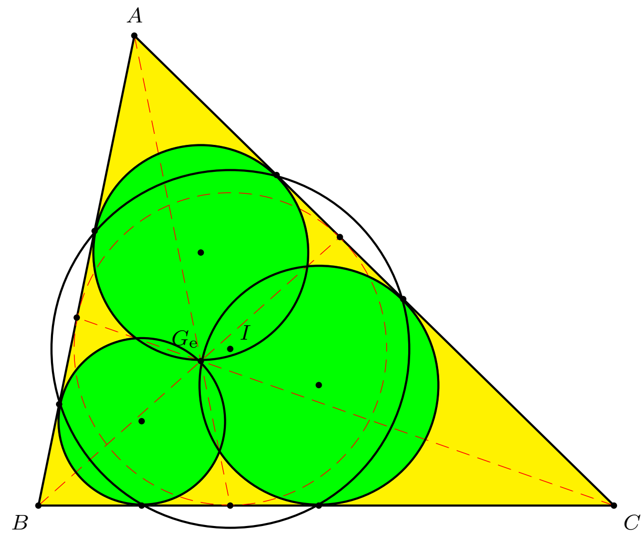
$$a^2yz+b^2zx+c^2xy+\frac{x+y+z}{3}\sum\frac{(S_B+S_C-2S_A)(S^2-S_A^2)}{(S_C-S_A)(S_A-S_B)}x=0.$$

[1]Identification number $2.33589872509\cdots$.

[2]Identification number $2.77610382619\cdots$.

11.2 The Adams circles

Construct the three circles each passing through the Gergonne point and tangent to two sides of triangle ABC. The 6 points of tangency lie on a circle. [3]



the Adams circle. It has radius

$$\frac{\sqrt{(4R+r)^2+s^2}}{4R+r}\cdot r.$$

Lemma. Every circle tangent to AB and AC has equation of the form

$$a^2yz+b^2zx+c^2xy-(x+y+z)(t^2x+(c-t)^2y+(b-t)^2z)=0.$$

Proof. The circle touches AB at $(c-t:t:0)$ and AC at $(b-t:0:t)$. $\square$

There are two values of t for which this circle passes through the Gergonne point. These are

$$t=\frac{-a(b+c-a)^2}{a^2+b^2+c^2-2ab-2bc-2ca},\quad \frac{-(b+c-a)(a^2+ab+ac-2b^2-2c^2+4bc)}{a^2+b^2+c^2-2ab-2bc-2ca}$$

The radius of the second circle is

$$\frac{\sqrt{(4R+r)^2+9s^2}}{4R+r}\cdot r.$$

[3]This is called the Adams circle. It is concentric with the incircle, and has radius $\frac{\sqrt{(4R+r)^2+s^2}}{4R+r}\cdot r$.

11.3 Hagge circles

Given a point $P = (u : v : w)$ with circumcevian triangle $A'B'C'$, let A'', B'', C'' be the reflections of A', B', C' in BC, CA, AB respectively. The circle $A''B''C''$ is called the Hagge circle of P. It has center

$$(-(a^2vw + b^2wu + c^2uv)(S^2 + S_{BC}) + 2a^2S^2vw : \cdots : \cdots),$$

which is the symmetric of the isogonal conjugate P^* in the nine-point center. This circle passes through the orthocenter H.

The reflections of A'' in the triangle HBC are on the circles ABC, AHC, and ABH respectively.

If the line HP intersects the circles HBC, AHC, ABH at X, Y, Z respectively, these are the reflections of a point Q on the circumcircle.

11.3.1

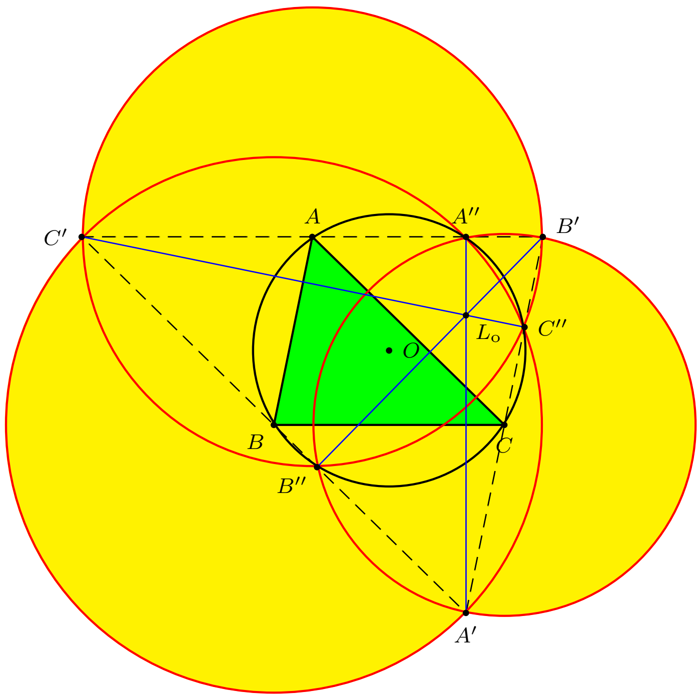
Theorem (Luong)**.** Let A', B', C' be points on the circumcircle of ABC, and A'', B'', C'' be their reflections in the respective sides. The circle $A''B''C''$ passes through H if and only if $A'B'C'$ is the circumcevian triangle of a point P.

If this condition is satisfied, the center O_P of the circle $A''B''C''$ is $\mathtt{N}^\dagger(P^*)$.

11.4 The deLongchamps triad of circles $(A(a), B(b), C(c))$

Consider the triad of circles $(A(a),\ B(b),\ C(c))$. Let $A'B'C$ be the superior triangle, and A'', B'', C'' the reflections of A, B, C respectively in the perpendicular bisectors of BC, CA, AB. These latter are the points

$$\begin{aligned}A'' =&(-a^2 : b^2 - c^2 : c^2 - b^2),\\ B'' =&(a^2 - c^2 : -b^2 : c^2 - a^2),\\ C'' =&(a^2 - b^2 : b^2 - a^2 : -c^2).\end{aligned}$$



From the obvious incidence relations

circle	points
$A(a)$	$B',\ C',\ B'',\ C''$
$B(b)$	$C',\ A',\ C'',\ A''$
$C(c)$	$A,\ B',\ A'',\ B''$

we obtain the radical axes of the circles:

circle	radical axis
$B(b)$ and $C(c)$	$A'A''$
$C(c)$ and $A(a)$	$B'B''$
$A(a)$ and $B(b)$	$C'C''$

These radical axes are the altitudes of the superior triangle. It follows that the radical center of the triad is the orthocenter of the superior triangle, which is the deLongchamps point L_o.

The equations of the circles are as follows.

circle	equation
$A(a)$:	$a^2yz + b^2zx + c^2xy + (x+y+z)(a^2x + (a^2-c^2)y + (a^2-b^2)z) = 0$
$B(b)$	$a^2yz + b^2zx + c^2xy + (x+y+z)((b^2-c^2)x + b^2y + (b^2-a^2)z) = 0$
$C(c)$	$a^2yz + b^2zx + c^2xy + (x+y+z)((c^2-b^2)x + (c^2-a^2)y + c^2z) = 0$

11.5 The orthial circles

Consider the A-orthial triangle in §**??**. Its circumcircle is tangent to that of ABC at A. Its equation can be written in the form

$$a^2yz + b^2zx + c^2xy - k(x+y+z)\left(\frac{y}{b^2}+\frac{z}{c^2}\right) = 0$$

for some k. Since it contains $X_b = (0 : S_\gamma + S_\alpha : -S_\alpha)$, we easily determine $k = -\frac{a^2b^2c^2S_\alpha}{S_{\beta\gamma}}$. From this, we have the triad of orthial circles:

$$\begin{aligned}
a^2yz + b^2zx + c^2xy + \frac{a^2b^2c^2S_\alpha}{S_{\beta\gamma}}(x+y+z)\left(\frac{y}{b^2}+\frac{z}{c^2}\right) =&0,\\
a^2yz + b^2zx + c^2xy + \frac{a^2b^2c^2S_\beta}{S_{\gamma\alpha}}(x+y+z)\left(\frac{y}{b^2}+\frac{z}{c^2}\right) =&0,\\
a^2yz + b^2zx + c^2xy + \frac{a^2b^2c^2S_\gamma}{S_{\alpha\beta}}(x+y+z)\left(\frac{y}{b^2}+\frac{z}{c^2}\right) =&0.
\end{aligned}$$

The radical center $(x : y : z)$ is given by

$$\frac{a^2b^2c^2S_\alpha}{S_{\beta\gamma}}\left(\frac{y}{b^2}+\frac{z}{c^2}\right) = \frac{a^2b^2c^2S_\beta}{S_{\gamma\alpha}}\left(\frac{z}{c^2}+\frac{x}{a^2}\right) = \frac{a^2b^2c^2S_\gamma}{S_{\alpha\beta}}\left(\frac{x}{a^2}+\frac{y}{b^2}\right),$$

$$\frac{y}{b^2}+\frac{z}{c^2} : \frac{z}{c^2}+\frac{x}{a^2} : \frac{x}{a^2}+\frac{y}{b^2} = \frac{1}{S_\alpha^2} : \frac{1}{S_\beta^2} : \frac{1}{S_\gamma^2}.$$

From these,

$$\begin{aligned}
&x : y : z\\
=&a^2(-S_\beta^2S_\gamma^2 + S_\gamma^2S_\alpha^2 + S_\alpha^2S_\beta^2) : b^2(S_\beta^2S_\gamma^2 - S_\gamma^2S_\alpha^2 + S_\alpha^2S_\beta^2) : c^2(S_\beta^2S_\gamma^2 + S_\gamma^2S_\alpha^2 - S_\alpha^2S_\beta^2)\\
=&a^2\left(-\frac{a^4}{(a^2S_\alpha)^2}+\frac{b^4}{(b^2S_\beta)^2}+\frac{c^4}{(c^2S_\gamma)^2}\right) : b^2\left(\frac{a^4}{(a^2S_\alpha)^2}-\frac{b^4}{(b^2S_\beta)^2}+\frac{c^4}{(c^2S_\gamma)^2}\right)\\
&\quad : c^2\left(\frac{a^4}{(a^2S_\alpha)^2}+\frac{b^4}{(b^2S_\beta)^2}-\frac{c^4}{(c^2S_\gamma)^2}\right).
\end{aligned}$$

This is the perspector of the tangential triangle and the circumcevian triangle of O.

Corollary. The radical axis of the B- and C-orthial circles contains the antipode of A on the circumcircle, and the intersection of the tangents of the circumcircle at B and C.

11.6 Bui's triad of circles

Let XYZ be the cevian triangle of O, and $A'B'C'$ the circumcevian triangle.

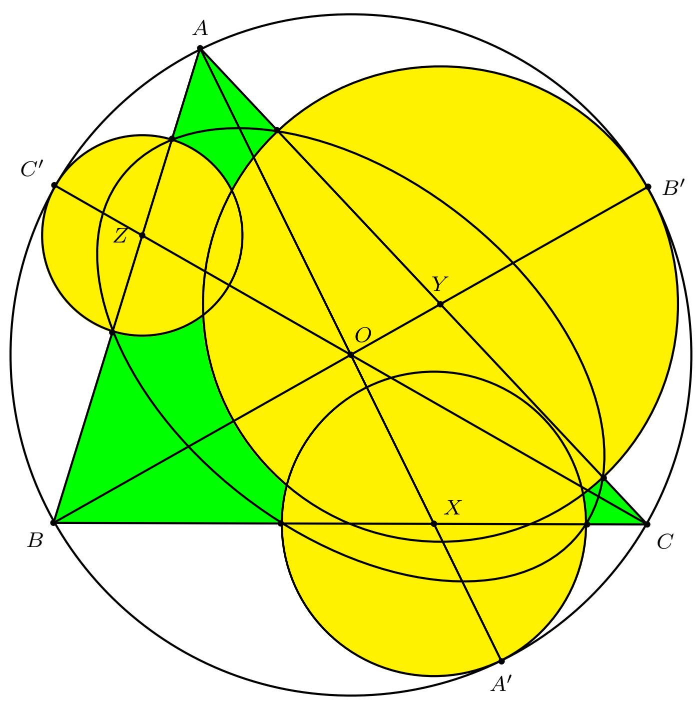
The circle $X(A')$ has equation

$$S^2(S^2+S_{BC})(a^2yz+b^12zx+c^2xy)$$
$$-a^2S_A(x+y+z)(a^2b^2c^2x+c^2S_{CC}y+b^2S_{BB}z)=0.$$

This circle intersects BC at two points. Similarly for the other two circles. The six points lie on the conic

$$S^2\sum_{\text{cyclic}}a^2S_A(S^2+S_{BC})yz-(x+y+z)\left(\sum_{\text{cyclic}}b^2c^2S_{BB}S_{CC}x\right)=0.$$

This conic has center O. It is homothetic to the circumconic with center N.



This is true for points on the Euler line.

11.7 Appendix: More triads of circles

11.7.1 The triad $\{A(A_H)\}$

The equation of the A-circle:

$$-a^2(a^2yz+b^2zx+c^2xy)+(x+y+z)(-S^2x+S_{BB}y+S_{CC}z)=0.$$

The radical center is therefore given by

$$\frac{-S^2x+S_{BB}y+S_{CC}z}{a^2}=\frac{S_{AA}x-S^2y+S_{CC}z}{b^2}=\frac{S_{AA}x+S_{BB}y-S^2z}{c^2}.$$

This is

$$x:y:z=a^2(S^4-S_{AABC}):b^2(S^4-S_{BBCA}:c^2(S^4-S_{CCAB}),$$

the center of the Taylor circle.

1. (a) Construct the circle tangent to the circumcircle *internally* at A and also to the side BC.

(b) Find the coordinates of the point of tangency with the side BC.

(c) Find the equation of the circle. [4]

(d) Similarly, construct the two other circles, each tangent internally to the circumcircle at a vertex and also to the opposite side.

(e) Find the coordinates of the radical center of the three circles. [5]

2. Construct the three circles each tangent to the circumcircle *externally* at a vertex and also to the opposite side. Identify the radical center, which is a point on the circumcircle. [6]

3. Let X, Y, Z be the traces of a point P on the side lines BC, CA, AB of triangle ABC.

(a) Construct the three circles, each passing through a vertex of ABC and tangent to opposite side at the trace of P.

(b) Find the equations of these three circles.

---

[4] $a^2yz+b^2zx+c^2xy-\frac{a^2}{(b+c)^2}(x+y+z)(c^2y+b^2z)=0.$

[5] $(a^2(a^2+a(b+c)-bc):\cdots:\cdots)$. This point appears as X_{595} in ETC.

[6] $\frac{a^2}{b-c}:\frac{b^2}{c-a}:\frac{c^2}{a-b}$. This point appears as X_{110} in ETC.

(c) The radical center of these three circles is a point independent of P. What is this point?

4. Find the equations of the three circles each through a vertex and the traces of the incenter and the Gergonne point on the opposite side. What is the radical center of the triad of circles? [7]

5. Let $P = (u : v : w)$. Find the equations of the three circles with the cevian segments AA_P, BB_P, CC_P as diameters. What is the radical center of the triad ? [8]

6. Given a point P. The perpendicular from P to BC intersects CA at Y_a and AB at Z_a. Similarly define Z_b, X_b, and X_c, Y_c. Show that the circles AY_aZ_A, BZ_bX_b and CX_cY_c intersect at a point on the circumcircle of ABC. [9]

Exercises

1. Consider triangle ABC with three circles $A(R_a)$, $B(R_b)$, and $C(R_c)$. The circle $B(R_b)$ intersects AB at $Z_{a+} = (R_b : c - R_b : 0)$ and $Z_{a-} = (-R_b : c + R_b : 0)$. Similarly, $C(R_c)$ intersects AC at $Y_{a+} = (R_c : 0 : b - R_c)$ and $Y_{a-} = (-R_c : 0 : b + R_c)$. [10]

(a) Show that the centers of the circles $AY_{a+}Z_{a+}$ and $AY_{a-}Z_{a-}$ are symmetric with respect to the circumcenter O.

(b) Find the equations of the circles $AY_{a+}Z_{a+}$ and $AY_{a-}Z_{a-}$. [11]

(c) Show that these two circles intersect at

$$Q = \left(\frac{-a^2}{bR_b - cR_c} : \frac{b}{R_b} : \frac{-c}{R_c}\right)$$

on the circumcircle.

---

[7]The external center of similitude of the circumcircle and incircle.

[8]Floor van Lamoen, Hyacinthos, message 214, 1/24/00.

[9]If $P = (u : v : w)$, this intersection is $(\frac{a^2}{vS_B - wS_C} : \frac{b^2}{wS_C - uS_A} : \frac{c^2}{uS_A - vS_B})$; it is the infinite point of the line perpendicular to HP. A.P. Hatzipolakis and P. Yiu, Hyacinthos, messages 1213, 1214, 1215, 8/17/00.

[10]A.P. Hatzipolakis, Hyacinthos, message 3408, 8/10/01.

[11]$a^2yz + b^2zx + c^2xy - \epsilon(x + y + z)(c \cdot R_by + b \cdot R_cz) = 0$ for $\epsilon = \pm 1$.

(d) Find the equations of the circles $AY_{a+}Z_{a-}$ and $AY_{a-}Z_{a+}$ and show that they intersect at

$$Q' = \left(\frac{-a^2}{bR_b + cR_c} : \frac{b}{R_b} : \frac{c}{R_c}\right)$$

on the circumcircle. [12]

(e) Show that the line QQ' passes through the points $(-a^2 : b^2 : c^2)$ and [13]

$$P = (a^2(-a^2R_a^2 + b^2R_b^2 + c^2R_c^2) : \cdots : \cdots).$$

(f) If W is the radical center of the three circles $A(R_a)$, $B(R_b)$, and $C(R_c)$, then $P = (1-t)O + t \cdot W$ for

$$t = \frac{2a^2b^2c^2}{R_a^2a^2S_A + R_b^2b^2S_B + R_c^2c^2S_C}.$$

(g) Find P if $R_a = a$, $R_b = b$, and $R_c = c$. [14]

(h) Find P if $R_a = s-a$, $R_b = s-b$, and $R_c = s-c$. [15]

(i) If the three circles $A(R_a)$, $B(R_b)$, and $C(R_c)$ intersect at $W = (u : v : w)$, then

$$P = (a^2(b^2c^2u^2 - a^2S_Avw + b^2S_Bwu + c^2S_Cuv) : \cdots : \cdots).$$

(j) Find P if W is the incenter. [16]

(k) If $W = (u : v : w)$ is on the circumcircle, then $P = Q = Q' = W$.

2. Given triangle ABC, construct a circle $\mathcal{C}_a$ tangent to AB at Z_a and AC at Y_a such that Y_aZ_a passes through the centroid G. Similarly construct the circles $\mathcal{C}_b$ and $\mathcal{C}_c$. What is the radical center of the three circles? [17]

[12] $a^2yz + b^2zx + c^2xy - \epsilon(x+y+z)(c \cdot R_by - b \cdot R_cz) = 0$ for $\epsilon = \pm 1$.

[13] $QQ' :\ (b^2R_b^2 - c^2R_c^2)x + a^2(R_b^2y - R_c^2z) = 0$.

[14] $(a^2(b^4+c^4-a^4) : b^2(c^4+a^4-b^4) : c^2(a^4+b^4-c^4))$. This point appears as X_{22} in ETC.

[15] $(\frac{a^2(a^2-2a(b+c)+(b^2+c^2))}{s-a} : \cdots : \cdots)$. This point appear in ETC as X_{1617}.

[16] $(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c})$.

[17] $\mathsf{h}(G, 2)(I)$. See Problem 2945, *Crux Math.*, 30 (2004) 233.

11.7.2 The triad of circles $(A_G(A_H), B_G(B_H), C_G(C_H))$

Consider the circle whose center is the midpoint A_G of BC, passing through the pedal A_H on BC. This circle has radius $\frac{1}{2a}|b^2-c^2|$. The power of A with respect to this circle

$$= m_a^2 - \frac{(b^2-c^2)^2}{4a^2} = S_A + \frac{a^2}{4} - \frac{(b^2-c^2)^2}{4a^2} = S_A + \frac{S_{BC}}{a^2}.$$

Those of B and C are each equal to

$$\frac{a^2}{4} - \frac{(b^2-c^2)^2}{4a^2} = \frac{S_{BC}}{a^2}.$$

The equation of this circle is therefore

$$S_A X + \frac{S_{BC}}{a^2}(X+Y+Z) = \frac{a^2YZ + b^2ZX + c^2XY}{X+Y+Z}.$$

Similarly, we write down the equations of the other two circles, and find the radical center of the three circles by solving

$$S_AX+\frac{S_{BC}}{a^2}(X+Y+Z) = S_BY+\frac{S_{CA}}{b^2}(X+Y+Z) = S_CZ+\frac{S_{AB}}{c^2}(X+Y+Z).$$

This gives

$$X:Y:Z = a^2S_A(b^2S_B^2+c^2S_C^2) : b^2S_B(c^2S_C^2+a^2S_A^2) : c^2(a^2S_A^2+b^2S_B^2)).$$

This is the point X_{185} in Kimberling's list. It is the Nagel point of the orthic triangle!

The radius of the orthogonal circle is

$$\frac{S_{ABC}}{abcS} = \frac{abc\cos A\cos B\cos C}{S} = 4R\cos A\cos B\cos C,$$

the diameter of the incircle of the orthic triangle.

This is the incircle of the antimedial triangle of the orthic triangle.

Therefore it touches the nine-point circle. The point of tangency is the Jerabek point.

11.8 The triad of circles $(A_G(A_H), B_G(B_H), C_G(C_H))$

Consider the circle whose center is the midpoint A_G of BC, passing through the pedal A_H on BC. This circle has radius $\frac{1}{2a}|b^2 - c^2|$. The power of A with respect to this circle

$$= m_a^2 - \frac{(b^2 - c^2)^2}{4a^2} = S_A + \frac{a^2}{4} - \frac{(b^2 - c^2)^2}{4a^2} = S_A + \frac{S_{BC}}{a^2}.$$

Those of B and C are each equal to

$$\frac{a^2}{4} - \frac{(b^2 - c^2)^2}{4a^2} = \frac{S_{BC}}{a^2}.$$

The equation of this circle is therefore

$$S_A X + \frac{S_{BC}}{a^2}(X + Y + Z) = \frac{a^2 YZ + b^2 ZX + c^2 XY}{X + Y + Z}.$$

Similarly, we write down the equations of the other two circles, and find the radical center of the three circles by solving

$$S_A X + \frac{S_{BC}}{a^2}(X+Y+Z) = S_B Y + \frac{S_{CA}}{b^2}(X+Y+Z) = S_C Z + \frac{S_{AB}}{c^2}(X+Y+Z).$$

This gives

$$X : Y : Z = a^2 S_A(b^2 S_B^2 + c^2 S_C^2) : b^2 S_B(c^2 S_C^2 + a^2 S_A^2) : c^2(a^2 S_A^2 + b^2 S_B^2)).$$

This is the point X_{185} in Kimberling's list. It is the Nagel point of the orthic triangle!

The radius of the orthogonal circle is

$$\frac{S_{ABC}}{abcS} = \frac{abc \cos A \cos B \cos C}{S} = 4R \cos A \cos B \cos C,$$

the diameter of the incircle of the orthic triangle.

This is the incircle of the antimedial triangle of the orthic triangle.

Therefore it touches the nine-point circle. The point of tangency is the Jerabek point.

11.9 The Lucas circles

Consider the square $A_bA_cA'_cA'_b$ inscribed in triangle ABC, with A_b, A_c on BC. Since this square can be obtained from the square erected externally on BC via the homothety $\mathsf{h}(A, \frac{S}{a^2+S})$, the equation of the circle $\mathcal{C}_A$ through A, A'_b and A'_c can be easily written down:

$$\mathcal{C}_A : \qquad a^2yz + b^2zx + c^2xy - \frac{a^2}{a^2+S} \cdot (x+y+z)(c^2y+b^2z) = 0.$$

Figure 11.1: Lucas circle

The center of the circle $\mathcal{C}_A$ is the point

$$O_a = (a^2(S_A + 2S) : b^2S_B : c^2S_C).$$

Figure 11.2: Radical center of Lucas circles

Figure 11.3: Radical center of Lucas circles

Likewise if we construct inscribed squares $B_cB_aB'_aB'_c$ and $C_aC_bC'_bC'_a$ on the other two sides, the corresponding Lucas circles are

$$\mathcal{C}_B: \qquad a^2yz + b^2zx + c^2xy - \frac{b^2}{b^2+S}\cdot(x+y+z)(c^2x+a^2z) = 0,$$

and

$$\mathcal{C}_C: \qquad a^2yz + b^2zx + c^2xy - \frac{c^2}{c^2+S}\cdot(x+y+z)(b^2x+a^2y) = 0.$$

The coordinates of the radical center satisfy the equations

$$\frac{a^2(c^2y+b^2z)}{a^2+S} = \frac{b^2(a^2z+c^2x)}{b^2+S} = \frac{c^2(b^2x+a^2y)}{c^2+S}.$$

Since this can be rewritten as

$$\frac{y}{b^2}+\frac{z}{c^2} : \frac{z}{c^2}+\frac{x}{a^2} : \frac{x}{a^2}+\frac{y}{b^2} = a^2+S : b^2+S : c^2+S,$$

it follows that

$$\frac{x}{a^2} : \frac{y}{b^2} : \frac{z}{c^2} = b^2+c^2-a^2+S : c^2+a^2-b^2+S : a^2+b^2-c^2+S,$$

and the radical center is the point

$$(a^2(2S_A+S) : b^2(2S_B+S) : c^2(2S_C+S)).$$

This is $K^*(\arctan 2)$ on the Brocard axis. [18]

The three Lucas circles are mutually tangent to each other, the points of tangency being

$$\begin{aligned} A' &= (a^2S_A : b^2(S_B+S) : c^2(S_C+S)), \\ B' &= (b^2(S_A+S) : b^2S_B : c^2(S_C+S)), \\ C' &= (a^2(S_A+S) : b^2(S_B+S) : c^2S_C). \end{aligned}$$

These point of tangency form a triangle perspective with ABC at the point [19]

$$K^*\left(\frac{\pi}{4}\right) = (a^2(S_A+S) : b^2(S_B+S) : c^2(S_C+S)).$$

Exercise

1. Show that the Lucas circles $\mathcal{C}_B$ and $\mathcal{C}_C$ also touch the A-Apollonian circle at A'.

[18] This appears as X_{1151} in ETC.

[19] This point appears in ETC as X_{371}, and is called the Kenmotu point.

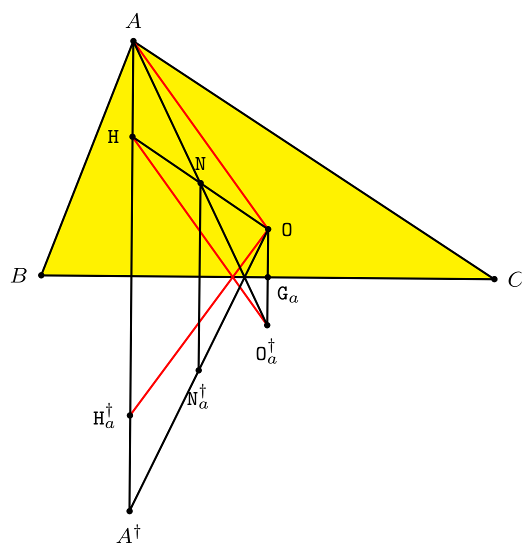
Chapter 12

The triangle of reflections

12.1 The triangle of reflections $\mathbf{T}^\dagger$

The triangle of reflections $\mathbf{T}^\dagger$ has vertices the reflections of A, B, C in their opposite sides:

$$A^\dagger := A_a^\dagger, \qquad B^\dagger := B_b^\dagger, \qquad C^\dagger := C_c^\dagger.$$

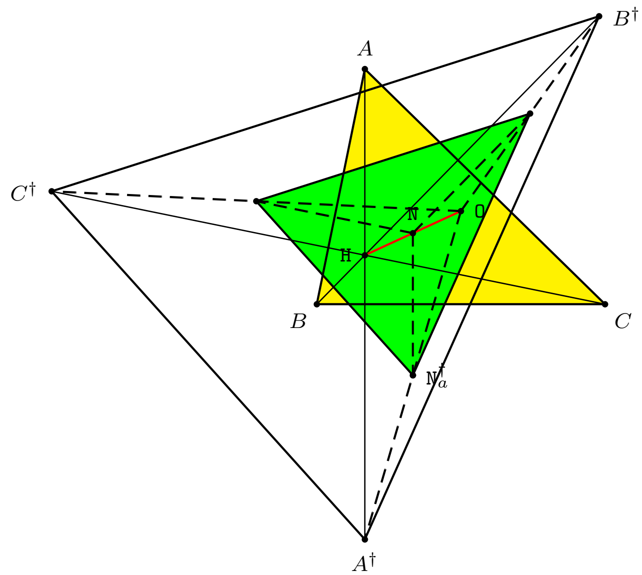


Since $\overrightarrow{A\mathtt{H}} = 2\overrightarrow{\mathtt{OG}_a} = \overrightarrow{\mathtt{OO}_a^\dagger}$, $\mathtt{OO}_a^\dagger\mathtt{H}A$ is a parallelogram. From this simple fact, we deduce several interesting results.

1. Note that $\mathtt{OO}_a^\dagger\mathtt{H}_a^\dagger\mathtt{H}$ is a symmetric trapezoid. This means that $\mathtt{OH}_a^\dagger = \mathtt{O}_a^\dagger\mathtt{H} = \mathtt{O}A$. Therefore, $\mathtt{H}_a^\dagger$ lies on the circumcircle of $\mathbf{T}$; similarly for $\mathtt{H}_b^\dagger$ and $\mathtt{H}_c^\dagger$. *The reflection triangle and the circumcevian triangle of* $\mathtt{H}$ *coincide*.

2. Since the midpoint of $\mathtt{OH}$ is the nine-point center $\mathtt{N}$, this is also the midpoint of $A\mathtt{O}_a^\dagger$. From this we conclude that *the reflection triangle of* $\mathtt{O}$ *is oppositely congruent to* $\mathbf{T}$ *at* $\mathtt{N}$.

3. Since $\mathtt{O}A^\dagger$ is the reflection of $\mathtt{O}_a^\dagger A$ in BC, $\mathtt{N}_a^\dagger$ is the midpoint of $\mathtt{O}A^\dagger$. Similarly, $\mathtt{N}_b^\dagger$ and $\mathtt{N}_c^\dagger$ are the midpoints of $\mathtt{O}B^\dagger$ and $\mathtt{O}C^\dagger$. From this we conclude that *the triangle of reflections* $\mathbf{T}^\dagger$ *is the image of the reflection triangle of* $\mathtt{N}$ *under the homothety* $\mathsf{h}(\mathtt{O}, 2)$.



4. *The circumcenter of* $\mathbf{T}^\dagger$ *is the reflection of* $\mathtt{O}$ *in* $\mathtt{N}^*$.

Proof. The circumcenter of the reflection triangle of a point P is the isogonal conjugate P^*. Since $\mathbf{T}^\dagger$ is the image of the reflection triangle of $\mathtt{N}$ under $\mathsf{h}(\mathtt{O}, 2)$, the circumcenter of $\mathbf{T}^\dagger$ is the point

$$\mathsf{h}(\mathtt{O}, 2)(\mathtt{N}^*) = 2\mathtt{N}^* - \mathtt{O}.$$

$\square$

5. $A\mathtt{N}^*$ is perpendicular to $B^\dagger C^\dagger$.

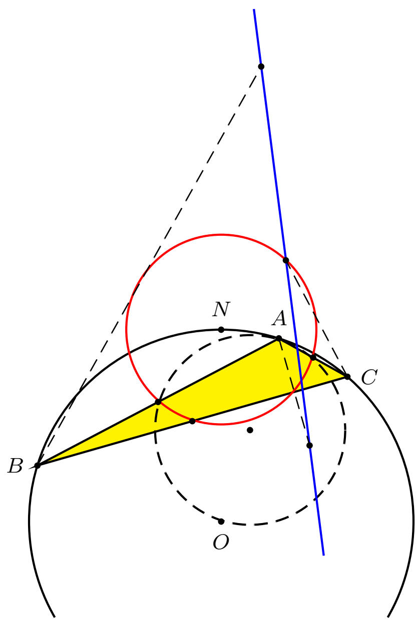
Proof. $B^\dagger C^\dagger$ is parallel to the a-side of the reflection triangle of $\mathtt{N}$, which is orthogonal to the line $A\mathtt{N}^*$. $\square$

Exercise

1. Show that the triangle of reflections is the image of the pedal triangle of N under the homothety $\mathsf{h}(G, 4)$.

12.2 Triangles with degenerate triangle of reflections

Proposition. The triangle of reflections is degenerate if and only if the nine-point center lies on the circumcircle.



Given a circle $O(R)$ and a point N on its circumference, let H be the reflection of O in N. For an arbitrary point P on the minor arc of the circle $N(\frac{R}{2})$ *inside* $O(R)$, let (i) A be the intersection of the *segment* HP with $O(R)$, (ii) the perpendicular to HP at P intersect $O(R)$ at B and C. Then triangle ABC has nine-point center N on its circumcircle $O(R)$. It is clear that $O(R)$ is the circumcircle of triangle ABC. Let M be the midpoint of BC so that OM is orthogonal to BC and parallel to PH. Thus, $OMPH$ is a (self-intersecting) trapezoid, and the line joining the midpoints of PM and OH is parallel to PH. Since the midpoint of OH is N and PH is orthogonal to BC, we conclude that N lies on the perpendicular bisector of PM. Consequently, $NM = NP = \frac{R}{2}$, and M lies on the circle $N(\frac{R}{2})$. This circle is the nine-point circle of triangle ABC, since it passes through the pedal P of A on BC and through the midpoint M of BC and has radius $\frac{R}{2}$.

Exercise

Suppose the nine-point center $\mathtt{N}$ of triangle ABC lies on the circumcircle. In this case, $\mathbf{T}^\dagger$ degenerates into a line $\mathcal{L}$.

1. If X, Y, Z are the centers of the circles $B\mathsf{O}C$, $C\mathsf{O}A$, $A\mathsf{O}B$, the lines AX, BY, CZ are all perpendicular to $\mathcal{L}$.

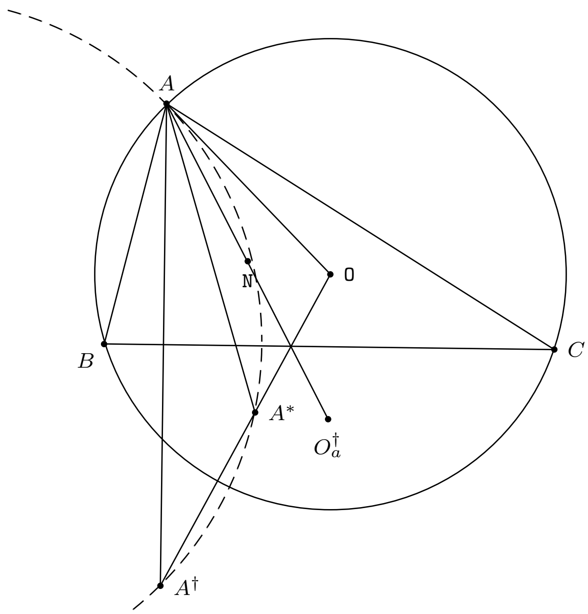
2. The circles $A\mathsf{O}A^\dagger$, $B\mathsf{O}B^\dagger$, $C\mathsf{O}C^\dagger$ are mutually tangent at O. The line joining their centers is the parallel to $\mathcal{L}$ through O.

3. The circles $AB^\dagger C^\dagger$, $BC^\dagger A^\dagger$, $CA^\dagger B^\dagger$ pass through O.

12.3 Triads of concurent circles

12.3.1 Musselman's theorem

Theorem (Musselman)**.** The circles $A\mathtt{O}A^\dagger$, $B\mathtt{O}B^\dagger$, and $C\mathtt{O}C^\dagger$ have a second common point, the inversive image of $\mathtt{N}^*$ in the circumcircle.



Proof. Let A^*, B^*, C^* be the inverses of $A^\dagger$, $B^\dagger$, $C^\dagger$ in the circumcircle. It is enough to show that the lines AA^*, BB^*, CC^* are concurrent in $\mathtt{N}^*$. [1] Since $\mathtt{O}A^\dagger \cdot \mathtt{O}A^* = \mathtt{O}A^2$, the line $\mathtt{O}A$ is tangent to the circle $AA^*A^\dagger$ at A.

$$\angle \mathtt{O}AA^* = \angle A^*A^\dagger A = \angle \mathtt{O}A^\dagger A = \angle \mathtt{O}_a^\dagger AA^\dagger.$$

It follows that

$$\angle CAA^* = \angle CA\mathtt{O} + \angle \mathtt{O}AA^* = \angle BAA^\dagger + \angle A^\dagger A\mathtt{O}_a^\dagger = \angle BA\mathtt{O}_a^\dagger.$$

Therefore, AA^* and $A\mathtt{O}_a^\dagger$ are isogonal lines with respect to AC and AB. Since $A\mathtt{O}_a^\dagger$ contains the nine-point center $\mathtt{N}$, the line AA^* contains its isogonal conjugate $\mathtt{N}^*$. Similarly, the lines BB^* and CC^* also contain $\mathtt{N}^*$. The three lines therefore are concurrent at $\mathtt{N}^*$. $\square$

[1] The isogonal conjugate of the nine-point center, $\mathtt{N}^*$, is usually called the Kosnita point.

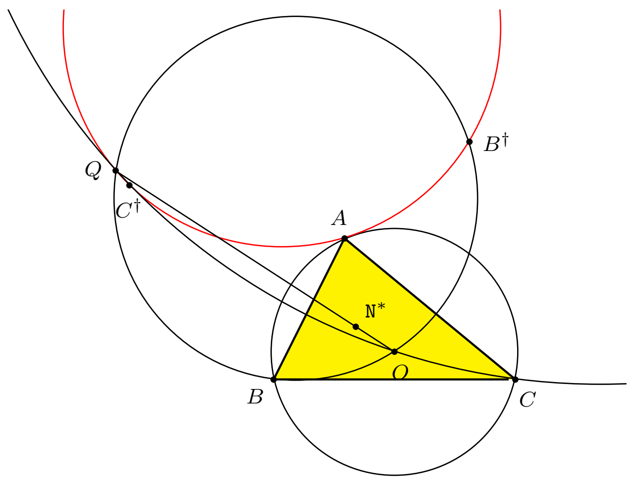
Corollary. Let A_0, B_0, C_0 be the centers of the circles $\mathtt{O}BC$, $\mathtt{O}CA$, $\mathtt{O}AB$ respectively. The lines AA_0, BB_0, CC_0 are concurrent at $\mathtt{N}^*$.

Proof. The center A_0 is the inverse of $O_a^\dagger$ in the circumcircle. By symmetry in BC, the point $O_a^\dagger$ lies on the circle $A\mathtt{O}A^\dagger$. Therefore, the line AA_0 is the inverse of this circle, and it contains the point $\mathtt{N}^*$ by Musselman's theorem. $\square$

12.3.2 The triad of circles $AB^\dagger C^\dagger$, $BC^\dagger A^\dagger$, $CA^\dagger B^\dagger$

Since the reflections of the orthocenter $\mathtt{H}$ in the sidelines of $\mathbf{T}$ lie on the circumcircle, the circumcircles of $A^\dagger BC$, $AB^\dagger C$, and $ABC^\dagger$ intersect at the orthocenter $\mathtt{H}$. By the dual triads of circles theorem, the circles $AB^\dagger C^\dagger$, $A^\dagger BC^\dagger$, $A^\dagger B^\dagger C$ also have a common point.

Theorem. The circles $AB^\dagger C^\dagger$, $BC^\dagger A^\dagger$, $CA^\dagger B^\dagger$ are concurrent at the inverse of $\mathtt{N}^*$ in the circumcircle.



Proof. Let Q be the common point of the circles $B\mathtt{O}B^\dagger$ and $C\mathtt{O}C^\dagger$ apart from $\mathtt{O}$. We prove that Q lies on the circle $AB^\dagger C^\dagger$, by calculating direct angles, with equalities modulo π:

$$\begin{aligned}
&\angle(QB^\dagger,\ QC^\dagger)\\
=\ &\angle(QB^\dagger,\ Q\mathtt{O}) + \angle(Q\mathtt{O},\ QC^\dagger)\\
=\ &\angle(BB^\dagger,\ B\mathtt{O}) + \angle(C\mathtt{O},\ CC^\dagger)\\
=\ &\angle(BB^\dagger,\ BC) + \angle(BC,\ B\mathtt{O}) + \angle(C\mathtt{O}, BC) + \angle(BC,\ CC^\dagger)\\
=\ &\angle(BB^\dagger,\ CC^\dagger) + \angle(C\mathtt{O}, B\mathtt{O})\\
=\ &\angle(B\mathtt{H}, C\mathtt{H}) + 2\angle(CA, BA)\\
=\ &(-\alpha) + 2(-\alpha)\\
=\ &(-\alpha) + (-\alpha) + (-\alpha)\\
=\ &\angle(AB^\dagger, AC) + \angle(AC, AB) + \angle(AB, AC^\dagger)\\
=\ &\angle(AB^\dagger, AC^\dagger).
\end{aligned}$$

Therefore, $B^\dagger$, Q, A, $C^\dagger$ are concyclic. Equivalently, Q lies on the circle $AB^\dagger C^\dagger$. By Musselman's theorem, Q is also lies on the circle $A\mathtt{O}A^\dagger$, and is the inverse of $\mathtt{N}^*$ in the circumcircle. The same reasoning shows that Q also lies on the circles $BC^\dagger A^\dagger$ and $CA^\dagger B^\dagger$, and is a common point of the three circles. $\square$

Exercise

1. The centers of the circles $A^\dagger BC$, $AB^\dagger C$ and $ABC^\dagger$ form a triangle homothetic to ABC. Identify the center of homothety. [2]

2. Let P be the center of the circle $A^\dagger B^\dagger C^\dagger$, and M the midpoint of OP. Identify the isogonal conjugate of M. [3]

---

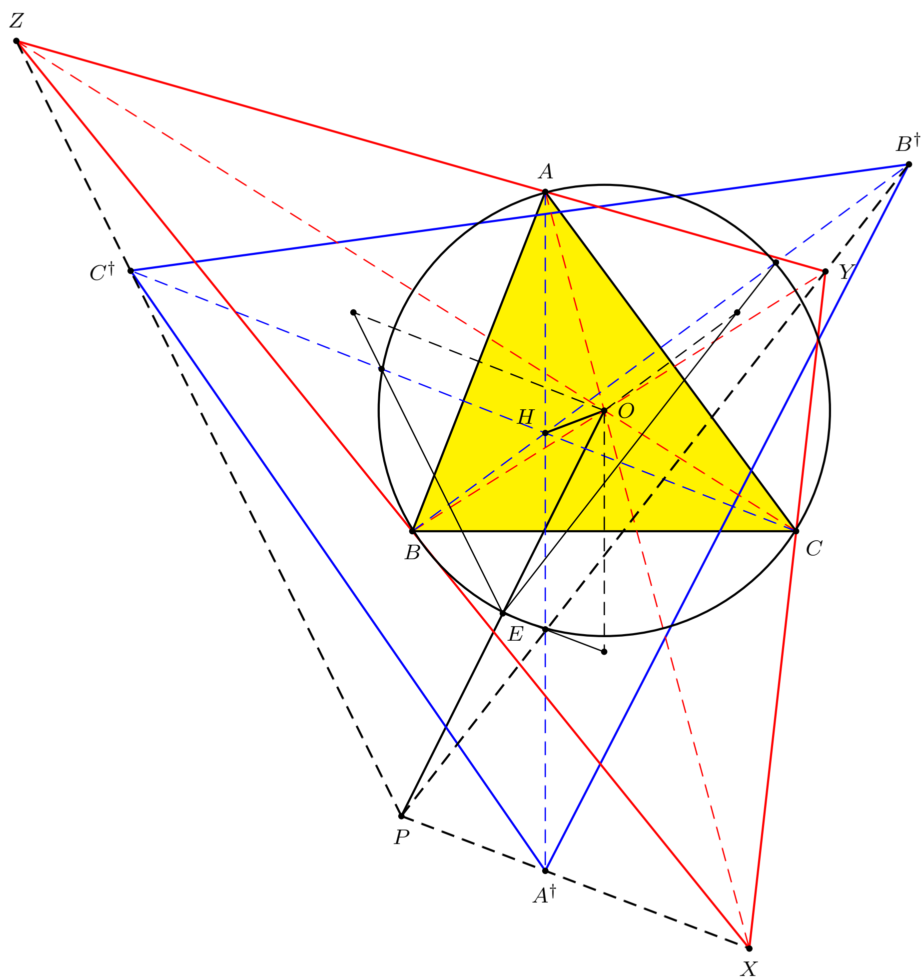
[2] X_5.

[3] The nine-point center. $M = \mathtt{N}^*$.

12.4 Triangle of reflections and $\mathsf{cev}^{-1}(\mathsf{O})$

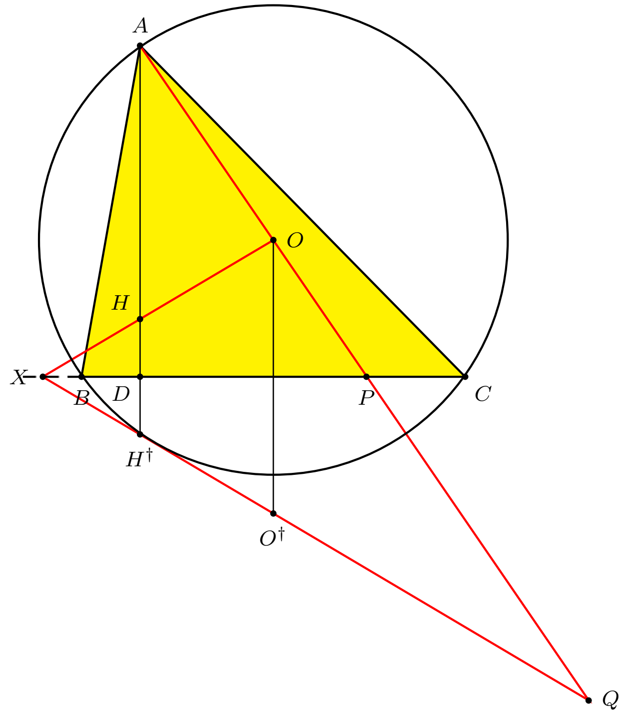
Theorem. The triangle of reflections and the anticevian triangle of the circumcenter are perspective at the reflection of the circumcenter in the Euler reflection point.

$$\wedge(\mathbf{T}^{\dagger},\ \mathsf{cev}^{-1}(\mathsf{O})) = \mathsf{O}^{\dagger}_{\mathsf{E}}.$$



Proposition. If the line OA intersects BC at P and the reflection of the Euler line in BC in Q, then $\frac{OQ}{QP} = \frac{OA}{OP}$.

Proof. Applying Menelaus' theorem to triangle APD with transversals XO



and XQ respectively, we have

$$\frac{PX}{XD} \cdot \frac{DH}{HA} \cdot \frac{AO}{OP} = -1, \tag{12.1}$$

$$\frac{DX}{XP} \cdot \frac{PQ}{QA} \cdot \frac{AH^\dagger}{H^\dagger D} = -1. \tag{12.2}$$

Note that $\frac{PX}{XD} \cdot \frac{DX}{XP} = 1$, and since $DH = H^\dagger D$,

$$\frac{DH}{HA} \cdot \frac{AH^\dagger}{H^\dagger D} = \frac{AH^\dagger}{HA} = \frac{AH^\dagger}{O^\dagger O} = \frac{QA}{OQ}.$$

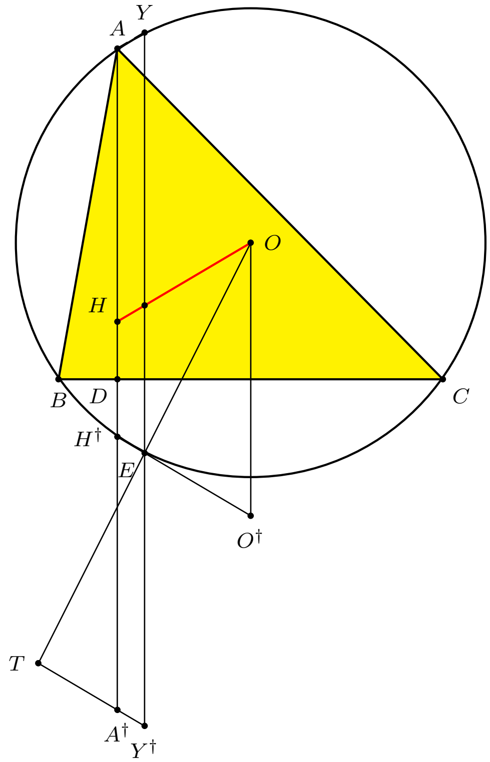
Multiplying the two equations (12.1) and (12.2), we have, after obvious cancellations,

$$\frac{PQ}{OQ} \cdot \frac{AO}{OP} = 1.$$

From this the result follows. □

Proposition. Let T be the reflection of O in the Euler reflection point E. The line $A^\dagger T$ is parallel to $H^\dagger E$.

Proof. $AH = OO^\dagger = H^\dagger A^\dagger = EY^\dagger$. $A^\dagger Y^\dagger$ is parallel to HE and intersects the extension of OE at T. □



Proposition. The lines $A^\dagger T$ and OA intersect at Z such that $\frac{AZ}{ZP} = -\frac{AP}{PO}$.

Proof. Since $A^\dagger T$ is parallel to $H^\dagger E$ and $OT = 2 \cdot OE$, $OZ = 2 \cdot OQ$. Since $\frac{AO}{OP} = \frac{OQ}{PQ}$,

$$\frac{AO}{OP} = \frac{OQ}{PQ} = \frac{AO + 2 \cdot OQ}{OP + 2 \cdot PQ} = \frac{AO + 2 \cdot OQ}{2 \cdot OQ - OP} = \frac{AZ}{PZ} = -\frac{AZ}{ZP}.$$

This shows that Z and O divide OP harmonically, and Z is the A-vertex of the anticevian triangle of O. $\square$

Theorem. The triangle of reflections is perspective to the anticevian triangle of the nine-point center. The line joining corresponding vertices are parallel to the Euler line.

$$\wedge(\mathbf{T}^\dagger, \mathsf{cev}^{-1}(\mathtt{N})) = \mathtt{E}_\infty.$$

12.4.1 The Parry reflection point $O_E^\dagger$

The point $O_E^\dagger$ is also called the Parry reflection point.

Theorem (C. F. Parry)**.** [4] Construct parallel lines through the vertices of ABC, and reflect them in the corresponding sidelines. The three reflection lines are concurrent if and only if they are reflections of lines parallel to the Euler line. In this case, the point of concurrency is $O_E^\dagger$.

Proof. Consider a line $ux+vy+wz=0$. The parallel through A is the line $(u-v)y+(u-w)z=0$. Its reflection in BC is the line

$$(2a^2u-(a^2+b^2-c^2)v-(c^2+a^2-b^2)w)x+a^2(u-v)y+a^2(u-w)z=0.$$

Similarly we find the equations of the reflections in CA, AB of the parallels through B and C respectively. These three reflections are concurrent if and only if

$$\left(\sum(2a^4-a^2(b^2+c^2)+(b^2-c^2)^2)u\right)\left(\sum(b^2+c^2-a^2)(v-w)^2\right)=0.$$

The second factor is always nonzero unless $u=v=w$. Thus, the line $ux+vy+wz=0$ contains the point

$$(2a^4-a^2(b^2+c^2)+(b^2-c^2)^2:\cdots:\cdots),$$

which is the infinite point of the Euler line. It is parallel to the Euler line. $\square$

Find a synthetic proof for the intersection to be the reflection of O in E.

[4] *Amer. Math. Monthly*, 105 (1998) 68; solution, ibid. 106 (1999) 779–780.

12.5 Triangle of reflections and the excenters

12.5.1 The Evans perspector

Theorem (Evans)**.** The triangle of reflections is perspective with the excentral triangle. The perspector is the point

$$(a(a^3 + a^2(b+c) - a(b^2+bc+c^2) - (b+c)(b-c)^2) : \cdots : \cdots)$$

Proof. The line joining $A^\dagger$ to the excenter $I_a = (-a : b : c)$ has equation

$$(b-c)(a+b+c)(b+c-a)x - a(c^2-ca+a^2-b^2)y + a(a^2-ab+b^2-c^2)z = 0.$$

This line intersects the OI line

$$bc(b-c)(b+c-a)x + ca(c-a)(c+a-b)y + ab(a-b)(a+b-c)z = 0$$

at

$$W = (a(a^3 + a^2(b+c) - a(b^2+bc+c^2) - (b+c)(b-c)^2) : \cdots : \cdots).$$

The coordinates are symmetric in a, b, c. Therefore the three lines are concurrent. □

This is called the Evans perspector $\mathtt{E_v}$. Note that the reflection of the incenter in the circumcenter is the point

$$\mathtt{I}_0^\dagger = 2O - I = (a(a^3 + a^2(b+c) - a(b+c)^2 - (b+c)(b-c)^2) : \cdots : \cdots)$$

The sum of the coordinates is $-4S^2$.

Proposition. $\mathtt{E_v}$ is the inverse of $\mathtt{I}$ in the circumcircle of the excentral triangle $\mathsf{cev}^{-1}(\mathtt{I})$.

Exercise

1. Let X be the reflection of the excenter I_a in BC; similarly define Y and Z. Calculate the coordinates of X, Y, Z. Show that XYZ is perspective with ABC.[5]

2. Let P be the perspector in the preceding exercise. Show that the cevian and the reflection triangles of P are perspective at the Evan perspector.

[5] The perspector is X_{80}.

Proposition. The circles $A^\dagger I_b I_c$, $I_a B^\dagger I_c$, $I_a I_b C^\dagger$ have the Parry reflection point as a common point.

Proposition. The circles $I_a B^\dagger C^\dagger$, $A^\dagger I_b C^\dagger$ and $A^\dagger B^\dagger I_c$ have a common point. Their centers are perspective with ABC at a point on the OI line.

Proposition. The reflections of the excenters in the respective sides of $\mathbf{T}$ form a triangle perspective with $\mathbf{T}$ at $\mathtt{I}^\dagger$.

Proposition. The triangle of reflections is perspective with the Kiepert triangle $\mathcal{K}(\theta)$ if and only if $\theta = \pm\frac{\pi}{3}$.

12.5.2 Triangle of reflections and the tangential triangle

Let XYZ be the tangential triangle.

The circles $XB^\dagger C^\dagger$, $A^\dagger YC^\dagger$, $A^\dagger B^\dagger Z$ have a common point, the Parry reflection point $\mathtt{O}_{\mathtt{E}}^\dagger$.

What is the common point of the triad of circles $A^\dagger YZ$, $B^\dagger ZX$, $C^\dagger XY$?

$$\begin{aligned}
A^\dagger =&(-a^2 : a^2+b^2-c^2 : c^2+a^2-b^2),\\
B^\dagger =&(a^2+b^2-c^2 : -b^2 : b^2+c^2-a^2),\\
C^\dagger =&(c^2+a^2-b^2 : b^2+c^2-a^2 : -c^2)
\end{aligned}$$

which are

The distance from $A^\dagger$ to $\mathtt{O}$ is the same as that of the reflection of $\mathtt{O}$ in BC to A. More directly, in triangle $\mathtt{O}BA^\dagger$, we have $\mathtt{O}B = R$, $BA^\dagger = BA = c$, and $\angle \mathtt{O}BA^\dagger = \beta + \left(\frac{\pi}{2} - \alpha\right) = \frac{\pi}{2} - (\alpha - \beta)$. This square distance is

$$\begin{aligned}
& R^2 + c^2 - 2cR\cos\left(\frac{\pi}{2} - (\alpha-\beta)\right)\\
= \;& R^2 + c^2 - 2cR\sin(\alpha-\beta)\\
= \;& R^2 + c^2 - c(a\cos\beta - b\cos\alpha)\\
= \;& R^2 + c^2 - \frac{1}{2}(c^2+a^2-b^2) + \frac{1}{2}(b^2+c^2-a^2)\\
= \;& R^2 + b^2 + c^2 - a^2.
\end{aligned}$$

Therefore the inverse is the point which divides $\mathtt{O}A^\dagger$ in the ratio

$$\mathtt{O}A^* : A^*A^\dagger = R^2 : (R^2+b^2+c^2-a^2) - R^2 = R^2 : b^2+c^2-a^2$$

$$A^* = \frac{1}{R^2 + b^2 + c^2 - a^2}\left((b^2 + c^2 - a^2)\mathsf{O} + R^2 \cdot A^\dagger\right).$$

$$A^\dagger = (-a^2 : 2S_\gamma : 2S_\beta) = (-a^2 : a^2 + b^2 - c^2 : c^2 + a^2 - b^2)$$

with coordinate sum a^2.

$$\frac{2S_\alpha}{2S^2}(a^2S_\alpha, b^2S_\beta, c^2S_\gamma) + \frac{R^2}{a^2}(-a^2, 2S_\gamma, 2S_\beta).$$

Since $2RS = abc$, the homogeneous coordinates of A^* can be taken as

$$\begin{aligned} & 4S_\alpha(a^2S_\alpha, b^2S_\beta, c^2S_\gamma) + b^2c^2(-a^2, 2S_\gamma, 2S_\beta) \\ = & (*** : 4b^2S_{\alpha\beta} + 2b^2c^2S_\gamma : 4c^2S_{\gamma\alpha} + 2b^2c^2S_\beta) \\ \sim & (*** : b^2(2S_{\alpha\beta} + c^2) : c^2(2S_{\gamma\alpha} + b^2S_\beta)) \\ \sim & (*** : b^2(S^2 + S_{\alpha\beta}) : c^2(S^2 + S_{\gamma\alpha})) \\ \sim & \left(*** : \frac{b^2}{S^2 + S_{\gamma\alpha}} : \frac{c^2}{S^2 + S_{\alpha\beta}}\right). \end{aligned}$$

It is clear that AA^*, BB^*, CC^* intersect at the isogonal conjugate of the nine-point center. The inverse in the circumcircle is the common point of the circles $A\mathsf{O}A^\dagger$, $B\mathsf{O}B^\dagger$, $C\mathsf{O}C^\dagger$.

Proposition. The triangle of reflections is perspective to the anticevian triangle of P if and only if P lies on the (Napoleon) isogonal cubic with pivot N, the nine-point center.

P	perspector
I	X_{484}Evans perspector
N	Euler infinity point
O	Parry reflection point
$X_{54} = N^*$	X_{1157} = inversive image of X_{54} in circumcircle
X_{195}	O

The excenters are also on the Napoleon cubic. Let $W = X_{484}$ be the Evans perspectors. For each of the excenters, the anticevian triangle is also

point	anticevian triangle	perspector
I_a	AI_cI_b	W_a^*
I_b	I_cBI_a	W_b^*
I_c	I_bI_aC	W_c^*

Chapter 13

Circumconics

In this chapter we introduce some basic examples of circumconics as isogonal (and isotomic) conjugates of lines. We show how to determine the center of a circumconic, and prove that among the circumconics, the rectangular hyperbolas are characterized as those passing through the orthocenter of the reference triangle, and that their centers are on the nine-point circle. One of the most remarkable results in this chapter is that a circum-hyperbola is uniquely determined by an asymptote. Throughout this chapter, we highlight specific examples and explicit geometric constructions. We give a large number of examples of circum-hyperbolas with two given infinite points, those with a specified asymptote, and the rectangular circum-hyperbola through a given point. We study a large number of concrete examples. Apart from the classical ones, the Steiner circum-ellipse and the Kiepert, Jerabek, and Feuerbach hyperbolas, we also study the rectangular circum-hyperbolas through the Steiner point, the Euler reflection points etc. We also study some important triangle centers assoicated with them. For example, we prove important properties of the Steiner point, the Kiepert center, and the Schiffler point. We also study the focus of a circum-parabola with a specified direction for the axis.

The last section on general conics actually belongs to another chapter, but is included here to give a flavor of the treatment of conics in the book.

13.1 The perspector of a circumconic

A homogeneous quadratic equation

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{13}zx + 2a_{12}xy = 0$$

in barycentric coordinates $(x : y : z)$ with reference to $\mathbf{T}$ represents a conic. The conic passes through the vertices of $\mathbf{T}$ if and only if $a_{11} = a_{22} = a_{33} = 0$. In this case we call it a circumconic and write its equation in the form

$$pyz + qzx + rxy = 0. \tag{13.1}$$

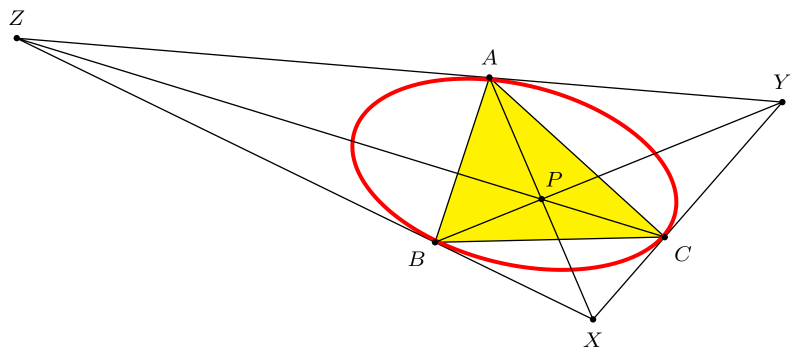


Figure 13.1: The circumconic $\mathscr{C}_{\mathrm{p}}(P)$ with perspector P

The tangents to the conic at the vertices of $\mathbf{T}$ are the lines

$$ry + qz = 0, \qquad rx + pz = 0, \qquad qx + py = 0.$$

They bound a triangle whose vertices are the points

$$X = (-p : q : r), \quad Y = (p : -q : r), \quad Z = (p : q : -r),$$

forming the anticevian triangle of $P = (p : q : r)$. For this reason we call P the perspector of the circumconic, and denote the conic by $\mathscr{C}_{\mathrm{p}}(P)$. For example, the circumcircle

$$a^2yz + b^2zx + c^2xy = 0$$

has perspector $\mathtt{K} = (a^2 : b^2 : c^2)$, the symmedian point, since the tangential triangle is the anticevian triangle $\mathrm{cev}^{-1}(\mathtt{K})$.

By rewriting (13.1) in the form

$$\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 0, \tag{13.2}$$

we note an interesting property of the perspector of a circumconic: *the trilinear polar for every point on $\mathscr{C}_{\mathrm{p}}(P)$ passes through the perspector P.*

13.2 Circumconics as isotomic and isogonal conjugates of lines

The circumconic $\mathscr{C}_{\mathrm{p}}(P)$ is the image of the line $px + qy + rz = 0$ under isotomic conjugation. We shall simply say that $\mathscr{C}_{\mathrm{p}}(P)$ is the isotomic conjugate of the line. For example, the circumconic

$$\mathscr{C}_{\mathrm{p}}(\mathtt{G}): \qquad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$$

is the isotomic conjugate of the line at infinity $\mathsf{L}_\infty : x + y + z = 0$. It is the Steiner circum-ellipse (see §13.5 below). This is an ellipse because it does not contain an infinite point. [1] The circumcircle

$$\mathscr{C}_{\mathrm{p}}(\mathtt{K}): \qquad a^2yz + b^2zx + c^2xy = 0$$

is the isotomic conjugate of the line $a^2x + b^2y + c^2z = 0$. [2]

More generally,

$$\boxed{\mathscr{C}_{\mathrm{p}}(P) = (\mathscr{L}(P^\bullet))^\bullet.}$$

We shall, however, more often regard circumconics as isogonal conjugates of lines instead of isotomic conjugates. Thus, $\mathscr{C}_{\mathrm{p}}(P)$ is the isogonal conjugate of the line

$$\frac{px}{a^2} + \frac{qy}{b^2} + \frac{rz}{c^2} = 0.$$

For example, the circumcircle is the isogonal conjugate of the line at infinity. More generally,

$$\boxed{\mathscr{C}_{\mathrm{p}}(P) = (\mathscr{L}(P^*))^*.}$$

We define a few basic circumconics as isogonal conjugates of lines.

---

[1] Proof: If $(u : v : w)$ is an infinite point on $\mathscr{C}_{\mathrm{p}}(\mathtt{G})$, then $u + v + w = 0$ and $uv + vw + wu = 0$. Eliminating w, we obtain $u^2 + uv + v^2 = 0$, an impossibility for $u, v \in \mathbb{R}$.

[2] The equation of the circumcircle is established in an earlier chapter.

Circumconic	as isogonal conjugate of	as $\mathscr{L}^\bullet$
Kiepert hyperbola $\sum_{\text{cyclic}} \frac{b^2-c^2}{x} = 0$	Brocard axis $\mathtt{OK}$ $\sum_{\text{cyclic}} b^2c^2(b^2-c^2)x = 0$	$\mathtt{GK}$
Jerabek hyperbola $\sum_{\text{cyclic}} \frac{a^2(b^2-c^2)(b^2+c^2-a^2)}{x} = 0$	Euler line $\mathtt{OH}$ $\sum_{\text{cyclic}} (b^2-c^2)(b^2+c^2-a^2)x = 0$	$\mathtt{HH}^\bullet$
Feuerbach hyperbola $\sum_{\text{cyclic}} \frac{a(b-c)(b+c-a)}{x} = 0$	$\mathtt{OI}$ line $\sum_{\text{cyclic}} bc(b-c)(b+c-a)x = 0$	$\mathtt{G_eN_a}$
through $\mathtt{G}$ and $\mathtt{K}$ $\sum_{\text{cyclic}} \frac{a^2(b^2-c^2)}{x} = 0$	$\mathtt{GK}$ $\sum_{\text{cyclic}} (b^2-c^2)x = 0$	$\mathtt{GK}^\bullet$

Each of these circumconics, being the isogonal conjugate of a line through the circumenter $\mathtt{O}$, is a hyperbola. It contains two infinite points which are the isogonal conjugates of the intersections of the line with the circumcircle (see §**??**). These hyperbolas can also be regarded as isotomic conjugates of lines indicated in the rightmost column in the table.

13.2.1 The fourth intersection of two circumconics

If we regard two circumconics as the isogonal conjugates of two lines $\mathscr{L}_1$ and $\mathscr{L}_2$, then, apart from the vertices of $\mathbf{T}$, they intersect at the isogonal conjugate of $\mathscr{L}_1 \cap \mathscr{L}_2$. We call this the fourth intersection of the two circumconics.

$$\boxed{\mathscr{L}_1^* \cap \mathscr{L}_2^* = \{A,\ B,\ C,\ (\mathscr{L}_1 \cap \mathscr{L}_2)^*\}.}$$

In particular, the fourth intersection of the circumconic $\mathscr{L}^*$ with the circumcircle is the isogonal conjugate of the infinite point of $\mathscr{L}$.

Circumconic	Fourth intersection with circumcircle	line
Steiner circumellipse	Steiner point $\mathtt{S_t} := \left(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2}\right)$	$\mathscr{L}(\mathtt{K})$
Kiepert hyperbola	Tarry point $\mathtt{T_a} := \left(\frac{1}{b^4+c^4-a^2(b^2+c^2)} : \cdots : \cdots\right)$	$\mathtt{OK}$
Jerabek hyperbola	$\mathtt{X}(74) := \left(\frac{a^2}{S_A(S_B+S_C)-2S_{BC}} : \cdots : \cdots\right)$	$\mathtt{OH}$
Feuerbach hyperbola	$\mathtt{X}(104) := \left(\frac{a}{a^2(b+c)-2abc-(b+c)(b-c)^2} : \cdots : \cdots\right)$	$\mathtt{OI}$
through $\mathtt{G}$ and $\mathtt{K}$	Parry point $\mathtt{P_a} := \left(\frac{a^2}{b^2+c^2-2a^2} : \cdots : \cdots\right)$	$\mathtt{GK}$

Analogous results hold if we treat the circumconics as isotomic conjugates of lines.

Example. (Steiner and Tarry points) These are the intersections of the circumcircle with the Steiner circum-ellipse and the Kiepert hyperbola respectively. Since the Lemoine axis $\mathscr{L}(\mathtt{K})$: $\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=0$ is the polar of $\mathtt{K}$ with respect to the circumcircle, it is orthogonal to the Brocard axis $\mathtt{OK}$. Therefore, the Steiner and the Tarry points are antipodal on the circumcircle (see Figure 13.2).

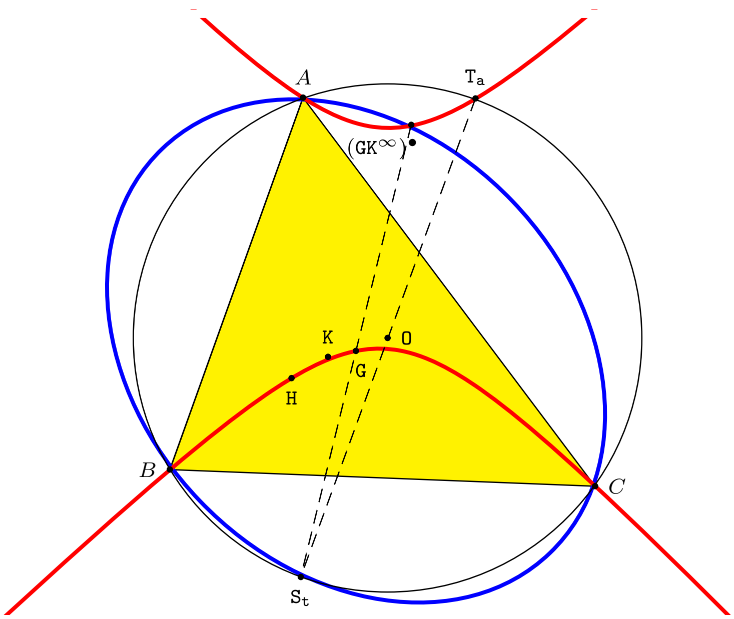


Figure 13.2: The Steiner circum-ellipse and the Kiepert hyperbola

The Kiepert hyperbola intersects the Steiner circum-ellipse at

$$(\mathtt{GK}^{\infty})^{\bullet}=\left(\frac{1}{b^2+c^2-2a^2}:\frac{1}{c^2+a^2-2b^2}:\frac{1}{a^2+b^2-2c^2}\right).$$

13.3 Circumconic with a given center

Given a (finite) point $Q = (u : v : w)$, we compute the equation of the circumconic which has Q as center. We denote this conic by $\mathscr{C}_{\rm o}(Q)$. It clearly contains the reflections of A, B, C in Q, namely, the points

$$\begin{aligned} Q^{\dagger}(A) &= \ (-(v+w-u) : \ 2v : \ 2w), \\ Q^{\dagger}(B) &= \ (2u : \ -(w+u-v) : \ 2w), \\ Q^{\dagger}(C) &= \ (2u : \ 2v : \ -(u+v-w)). \end{aligned}$$

The isotomic conjugates of these points are collinear. The line containing them is

$$u(v+w-u)x + v(w+u-v)y + w(u+v-w)z = 0,$$

as is easily verified. Therefore, $\mathscr{C}_{\rm o}(Q)$ is the conic

$$\frac{u(v+w-u)}{x} + \frac{v(w+u-v)}{y} + \frac{w(u+v-w)}{z} = 0.$$

Note that the perspector is the cevian quotient $\mathtt{G}/Q$. Therefore, we have established

$$\boxed{\mathscr{C}_{\rm o}(Q) = \mathscr{C}_{\rm p}(\mathtt{G}/Q).} \tag{13.3}$$

Proposition. For $Q = (u : v : w)$, the fourth intersection of $\mathscr{C}_{\rm o}(Q)$ with
(a) the circumcircle is the point

$$(\mathscr{L}((\mathtt{G}/Q)^*)^{\infty})^* = \left(\frac{1}{b^2w(u+v-w) - c^2v(w+u-v)} : \ \cdots : \ \cdots\right),$$

(b) the circumconic $\mathscr{C}_{\rm p}(Q)$ is $\left(\frac{u}{v-w} : \frac{v}{w-u} : \frac{w}{u-v}\right)$.

Example. (The superior of the Feuerbach point) The circum-ellipse $\mathscr{C}_{\rm o}(\mathtt{I})$ intersects the circumcircle and $\mathscr{C}_{\rm p}(\mathtt{I})$ at the same point

$$\left(\frac{a}{b-c} : \ \frac{b}{c-a} : \ \frac{c}{a-b}\right).$$

The inferior of this point has coordinates

$$\begin{aligned} &\left(\frac{b}{c-a} + \frac{c}{a-b} : \frac{c}{a-b} + \frac{a}{b-c} : \ \frac{a}{b-c} + \frac{b}{c-a}\right) \\ &= (b(a-b)(b-c) + c(b-c)(c-a) : c(b-c)(c-a) + a(c-a)(a-b) \\ &\qquad : a(c-a)(a-b) + b(a-b)(b-c)) \\ &= ((b-c)^2(b+c-a) : (c-a)^2(c+a-b) : (a-b)^2(a+b-c)). \end{aligned}$$

This is the Feuerbach point $\mathtt{F_e}$, the point of tangency of the nine-point circle and the incircle. Therefore, $\mathscr{C}_\mathrm{o}(\mathtt{I})$ intersects the circumcircle at $\mathsf{sup}(\mathtt{F_e})$. Since $\mathscr{C}_\mathrm{o}(\mathtt{O})$ is the nine-point circle of the superior triangle $\mathsf{cev}^{-1}(\mathtt{G})$, this point is the point of tangency with the incircle of the superior triangle, which has center $\mathtt{N_a}$ (see Figure 13.3).

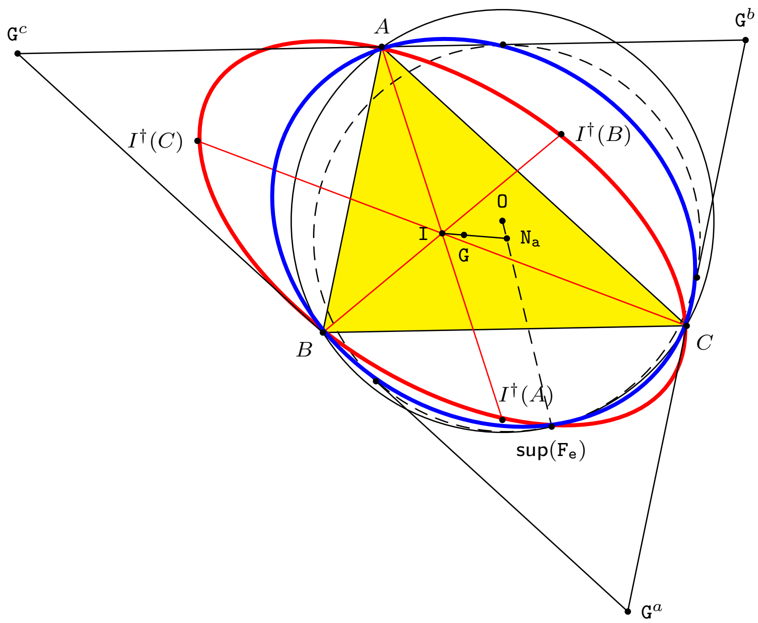


Figure 13.3: $\mathscr{C}_\mathrm{o}(\mathtt{I})$ and $\mathscr{C}_\mathrm{p}(\mathtt{I})$ intersect at $\mathsf{sup}(\mathtt{F_e})$ on the circumcircle

Remark. If $Q = (u : v : w)$ is an infinite point, *i.e.*, $u + v + w = 0$, the points $Q^\dagger(A)$, $Q^\dagger(B)$, $Q^\dagger(C)$ all coincide with Q. Nevertheless, we may still speak of the circumconic $\mathscr{C}_\mathrm{o}(Q)$ with equation

$$\frac{u^2}{x} + \frac{v^2}{y} + \frac{w^2}{z} = 0.$$

Since this conic has a unique infinite point, namely, $Q = (u : v : w)$, it is a parabola. [3] We shall continue to speak of Q as the center of the parabola, though more accurately it is the infinite point of the axis of the parabola. Circum-parabolas are discussed in more details in §**??**.

[3]Proof: Let $(x : y : z)$ be an infinite point on the conic. With $x = -(y+z)$, we have $0 = u^2yz + x(v^2z + w^2y) = u^2yz - (y+z)(v^2z + w^2y) = -w^2y^2 + (u^2 - v^2 - w^2)yz - v^2z^2 = -(w^2y^2 - 2vwyz + v^2z^2) = -(wy - vz)^2$. Therefore, $y : z = v : w$ and $x : y : z = -(v + w) : v : w = u : v : w$.

13.4 The center of a circumconic

Since $P = \mathtt{G}/Q$ if and only if $Q = \mathtt{G}/P$, [4] we conclude from (13.3) that *the center of the circumconic $\mathscr{C}_{\mathrm{p}}(P)$ is the cevian quotient* $\mathtt{G}/P$. Figure 13.4 illustrates

$$\boxed{\mathscr{C}_{\mathrm{p}}(P) = \mathscr{C}_{\mathrm{o}}(\mathtt{G}/P).}$$

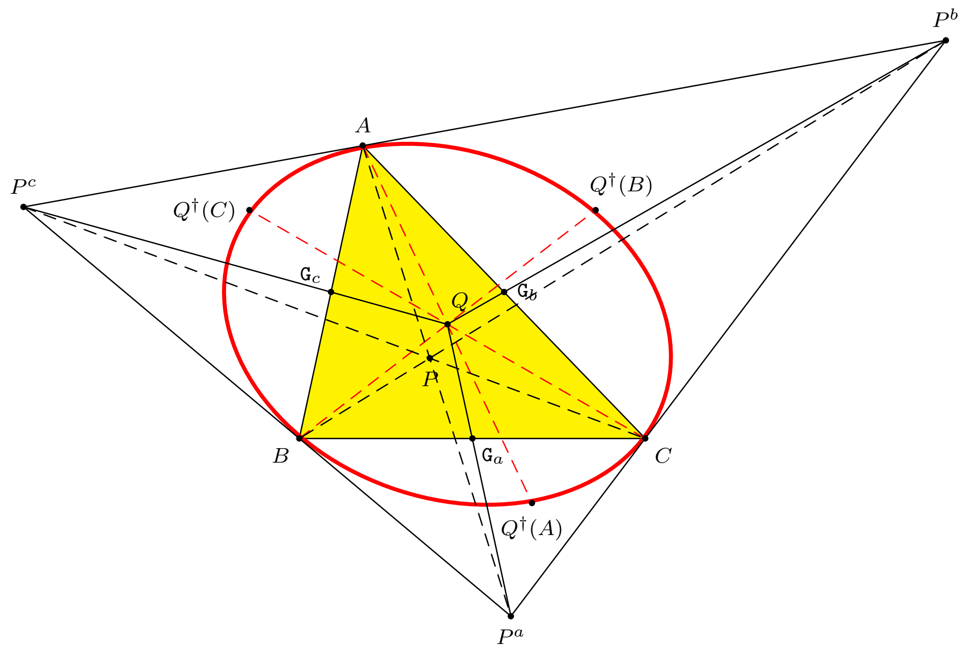


Figure 13.4: $\mathscr{C}_{\mathrm{p}}(P) = \mathscr{C}_{\mathrm{o}}(Q)$ with $Q = \mathtt{G}/P$

More explicitly, the center of the circumconic $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 0$ is the point

$$(p(q + r - p) : \ q(r + p - q) : \ r(p + q - r)).$$

Here are the centers of some common circumconics.

Circumconic	Perspector P	Center $\mathtt{G}/P$
Steiner circumellipse	$\mathtt{G}$	$\mathtt{G}$
circumcircle	$\mathtt{K}$	$\mathtt{O}$
Kiepert hyperbola	$(b^2 - c^2 : \cdots : \cdots)$	$\mathtt{K_i} := ((b^2 - c^2)^2 : \cdots : \cdots)$
Jerabek hyperbola	$((b^2 - c^2)S_A : \cdots : \cdots)$	$\mathtt{J_e} := ((b^2 - c^2)^2 S_A : \cdots : \cdots)$
Feuerbach hyperbola	$(a(b - c)(b + c - a) : \cdots : \cdots)$	$\mathtt{F_e} := ((b - c)^2(b + c - a) : \cdots : \cdots)$
through $\mathtt{G}$ and $\mathtt{K}$	$(a^2(b^2 - c^2) : \cdots : \cdots)$	$(a^4(b^2 - c^2)^2 : \cdots : \cdots)$

[4] A basic theorem on cevian quotients established in an earlier chapter.

13.5 The Steiner circum-ellipse

The Steiner circum-ellipse $\mathscr{C}_{\rm p}(\texttt{G}) = \mathscr{C}_{\rm o}(\texttt{G})$ consists of isotomic conjugates of infinite points. It intersects the circumcircle at the Steiner point

$$\texttt{S}_\texttt{t} = \left(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2}\right).$$

A line through $\texttt{S}_\texttt{t}$ intersects the Steiner circum-ellipse and the circumcircle each at one other point. These are the isotomic and isogonal conjugates of the same infinite point P. Figure 13.5 shows three parallel lines through A, B, C, intersecting their opposite sides at X, Y, Z respectively.

(1) The isotomic lines AX', BY', CZ' intersect at $P^\bullet$ on the Steiner circum-ellipse.

(2) The isogonal lines, which are the reflections of AX, BY, CZ in the respective bisectors of angles A, B, C, intersect at P^* on the circumcircle.

(3) The line through $P^\bullet$ and P^* passes through the Steiner point $\texttt{S}_\texttt{t}$.

In fact, for *arbitrary* $P = (u : v : w)$, the line containing $P^\bullet$ and P^* has equation

$$(b^2-c^2)ux + (c^2-a^2)vy + (a^2-b^2)wz = 0.$$

If $u + v + w = 0$, then this line clearly contains the Steiner point $\texttt{S}_\texttt{t}$.

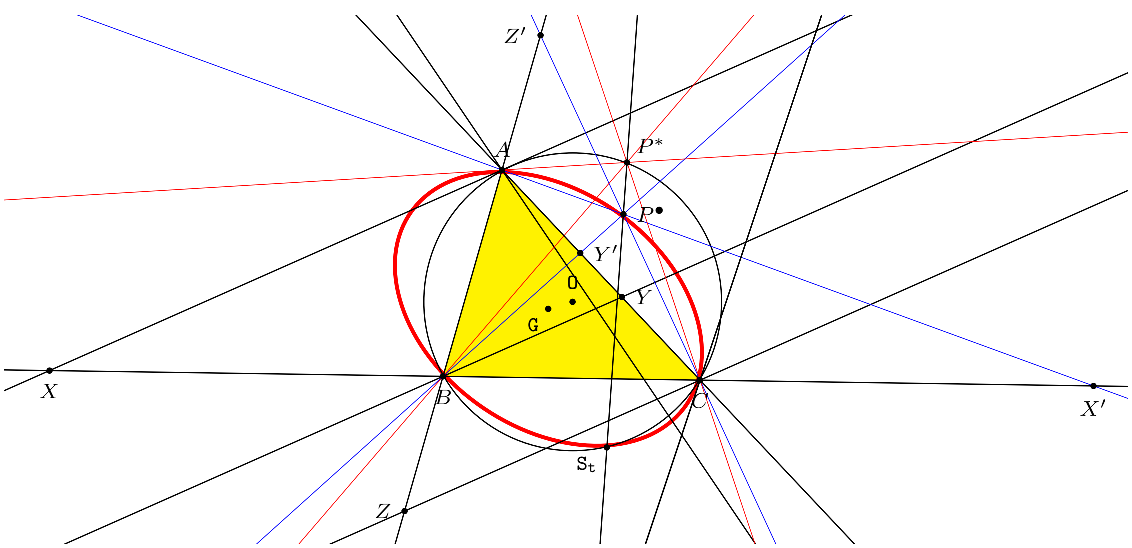


Figure 13.5: The Steiner circum-ellipse and the Steiner point

Example. (Antipodes on the Steiner circum-ellipse) If $(u : v : w)$ is an infinite point, $Q^\bullet$ is a point on the Steiner circum-ellipse. The antipode of

$Q^\bullet$ is the isotomic conjugate of $(v-w : w-u : u-v)$, the infinite point of the line $ux+vy+wz=0$. The antipode of $\mathtt{S_t}$ on the Steiner circum-ellipse is the point $(\mathtt{GK}^\infty)^\bullet$ on the Kiepert hyperbola (see Example 13.2.1).

13.6 The Kiepert hyperbola

The Kiepert hyperbola

$$\mathscr{H}(\mathtt{K}): \qquad \frac{b^2-c^2}{x}+\frac{c^2-a^2}{y}+\frac{a^2-b^2}{z}=0$$

is the isogonal conjugate of the Brocard axis $\mathtt{OK}$. Its center is the point

$$\mathtt{K_i} = ((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2).$$

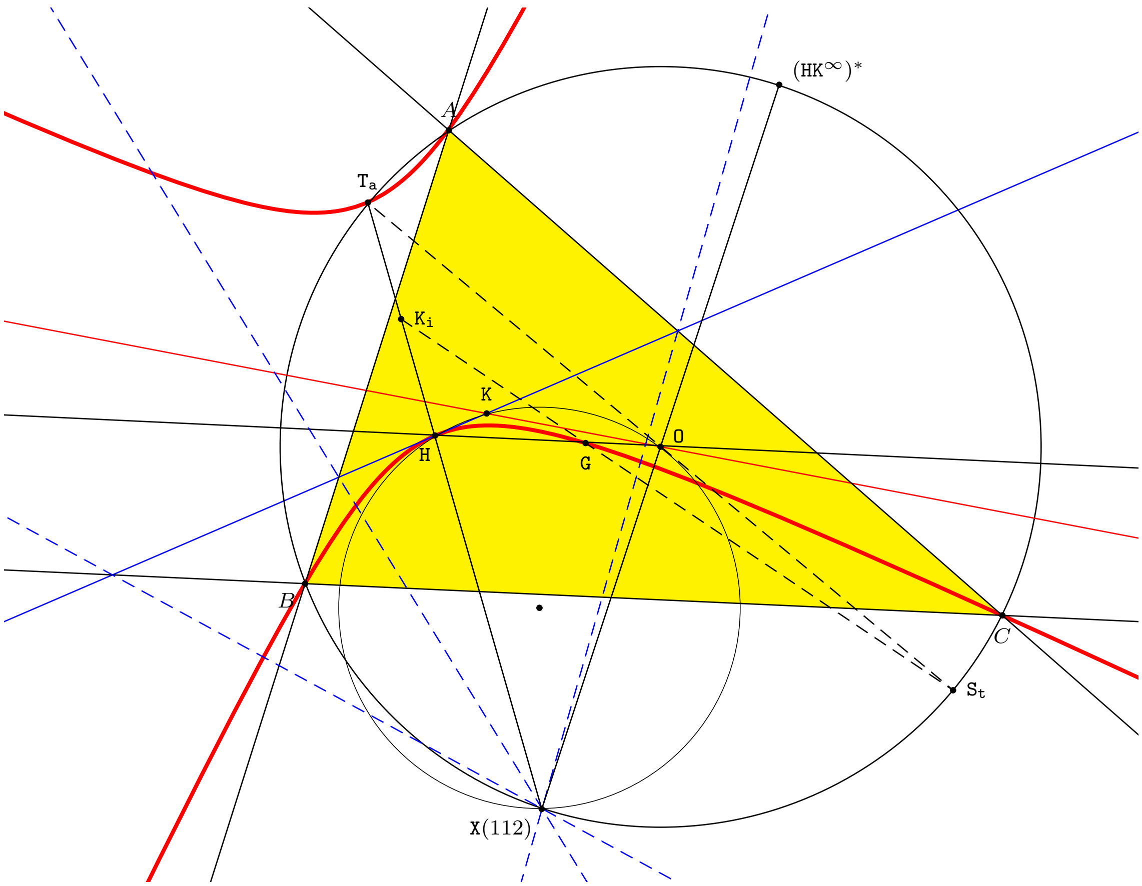


Figure 13.6: The Kiepert hyperbola

The tangent at $\mathtt{H}$ is the line $\mathtt{HK}$. The reflections of $\mathtt{HK}$ in the sidelines are concurrent at

$$\mathtt{X}(112) = \left(\frac{a^2}{(b^2-c^2)S_A} : \frac{b^2}{(c^2-a^2)S_B} : \frac{c^2}{(a^2-b^2)S_C}\right)$$

which also lies on the circle $\mathtt{OHK}$.

13.6.1 The Kiepert center

Proposition. *The Kiepert center is the inferior of the Steiner point.*

Proof. It is enough to determine the superior of the Kiepert center.

$$\begin{aligned}
\mathsf{sup}(\mathtt{K_i}) &= ((c^2-a^2)^2+(a^2-b^2)^2-(b^2-c^2)^2 : \cdots : \cdots)\\
&= ((c^2-a^2)^2+(a^2-c^2)(a^2+c^2-2b^2) : \cdots : \cdots)\\
&= ((c^2-a^2)(c^2-a^2+a^2+c^2-2b^2) : \cdots : \cdots)\\
&= ((c^2-a^2)(a^2-b^2) : \cdots : \cdots)\\
&= \left(\frac{1}{b^2-c^2} : \cdots : \cdots\right)\\
&= \mathtt{S_t}.
\end{aligned}$$

□

Proposition. *The Kiepert center is the point of concurrency of the Brocard axes of the residuals of the orthic triangle.*

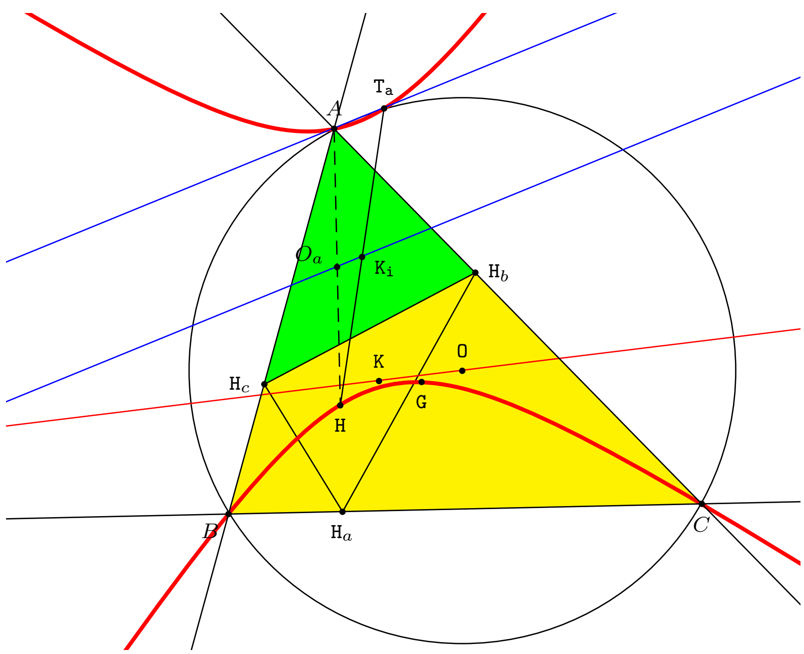


Figure 13.7: Brocard axis of residual of orthic triangle

Proof. Since $\mathtt{H}_b\mathtt{H}_c$ is antiparallel to BC, the Brocard axis of the residual triangle $A\mathtt{H}_b\mathtt{H}_c$ is the parallel to $A\mathtt{T_a}$ through the midpoint of $A\mathtt{H}$, the circumcenter O_a. (The symmedian point is the intersection of this line with the median $A\mathtt{G}$). This parallel clearly passes through the midpoint of $\mathtt{T_aH}$, which is the Kiepert center $\mathtt{K_i}$ (Corollary 16.1). The same reasoning applies to the other two residuals, showing that the three Brocard axes are concurrent at $\mathtt{K_i}$. □

13.7 The Jerabek hyperbola

The Jerabek hyperbola

$$\mathscr{H}(\mathtt{O}): \qquad \frac{a^2(b^2-c^2)S_A}{x}+\frac{b^2(c^2-a^2)S_B}{y}+\frac{c^2(a^2-b^2)S_C}{z}=0$$

is the isogonal conjugate of the Euler line. It clearly contains $\mathtt{H}$, $\mathtt{O}$, $\mathtt{K}=\mathtt{G}^*$, $\mathtt{N}^*$, and $\mathtt{H}^\bullet=(S_A:S_B:S_C)$. Its center is the point

$$\mathtt{J_e}:=((b^2-c^2)^2S_A:\ (c^2-a^2)^2S_B:\ (a^2-b^2)^2S_C)$$

called the Jerabek center. This is also the point of concurrency of the Euler lines of the "residuals" of the orthic triangle cev$(\mathtt{H})$ (see Figure 13.9).

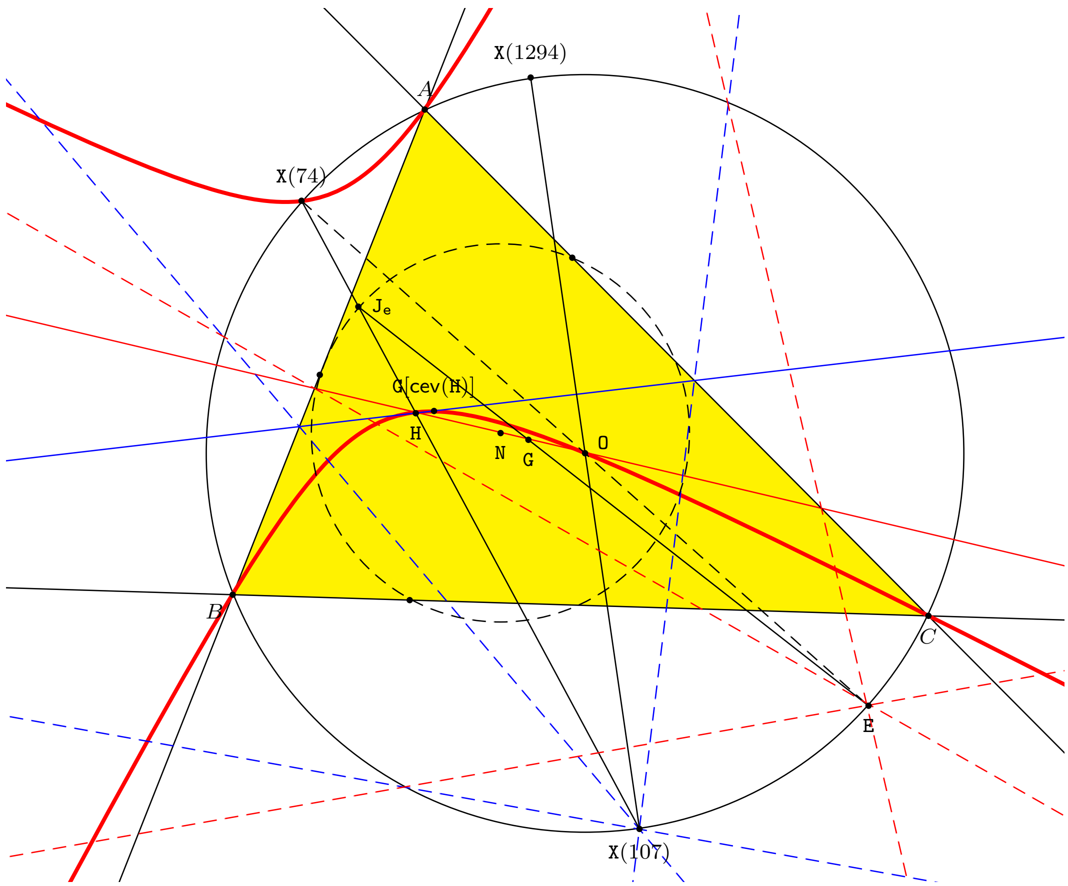


Figure 13.8: The Jerabek hyperbola

The tangent at $\mathtt{H}$ is the line

$$\sum_{\text{cyclic}} a^2(b^2-c^2)S_A^3x=0,$$

which contains the centroid of the orthic triangle. The reflections of this tangent in the sidelines are concurrent at

$$\mathtt{X}(107) = \left(\frac{1}{(b^2-c^2)S_{AA}} : \frac{1}{(c^2-a^2)S_{BB}} : \frac{1}{(a^2-b^2)S_{CC}}\right).$$

Proposition. (a) *The Jerabek center* $\mathtt{J_e}$ *is the inferior of the Euler reflection point* $\mathtt{E}$.
(b) *The Jerabek center is the point of concurrency of the Euler lines of the residuals of the orthic triangle.*

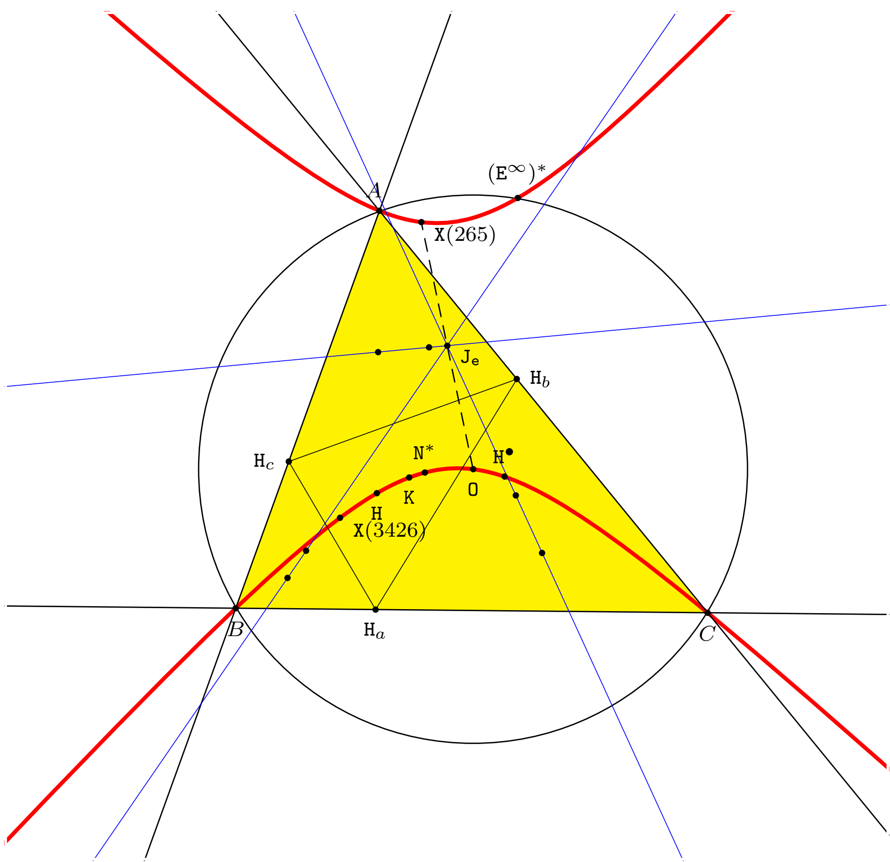


Figure 13.9: The Jerabek hyperbola and three Euler lines

13.8 The Feuerbach hyperbola

The Feuerbach hyperbola

$\mathscr{H}(\mathtt{I})$:

$$\frac{a(b-c)(b+c-a)}{x}+\frac{b(c-a)(c+a-b)}{y}+\frac{c(a-b)(a+b-c)}{z}=0$$

is the isogonal conjugate of the line $\mathtt{OI}$. Its center is the Feuerbach point $\mathtt{F_e}$, the point of tangency of the incircle and the nine-point circle. It intersects the Euler line at the Schiffler point

$$\mathtt{S_c}:=\left(\frac{a(b+c-a)}{b+c}:\frac{b(c+a-b)}{c+a}:\frac{c(a+b-c)}{a+b}\right).$$

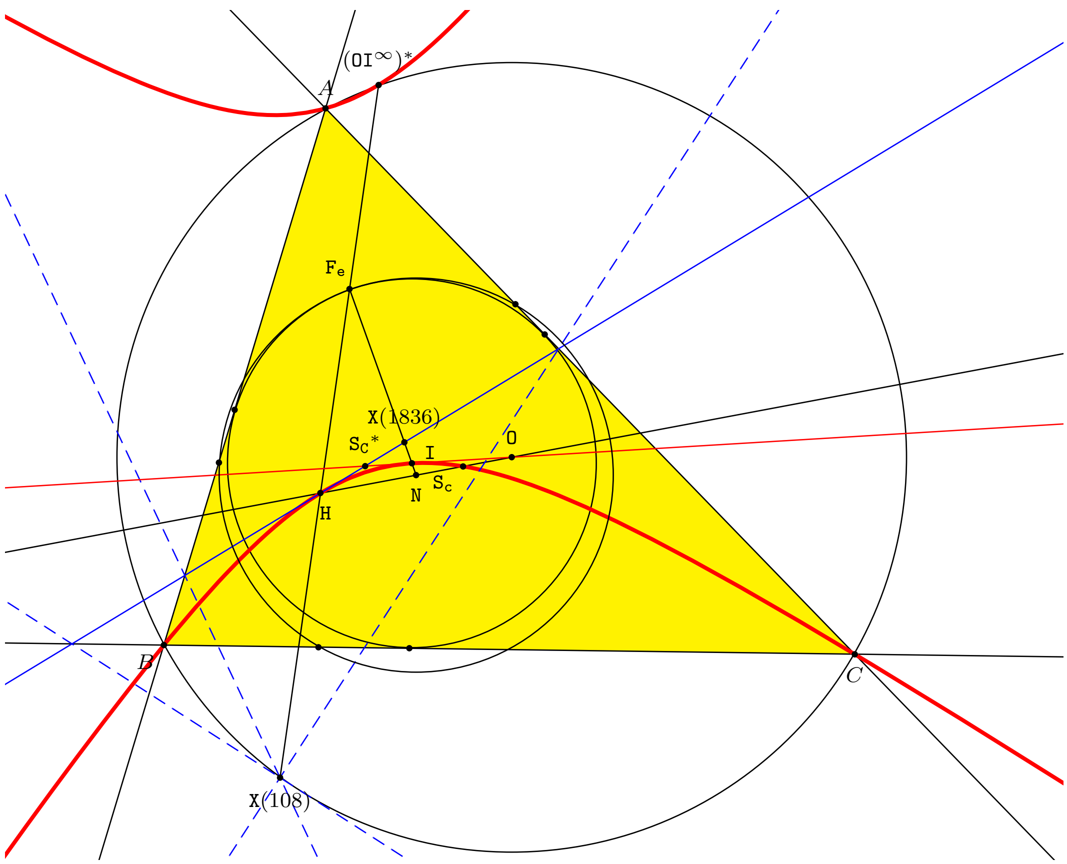


Figure 13.10: The Schiffler point on the Feuerbach hyperbola and the Euler line

Note that the $\mathtt{OI}$ line is tangent to the Feuerbach hyperbola at $\mathtt{I}$. The tangent at $\mathtt{H}$ is the line

$$\sum_{\text{cyclic}} a(b-c)(b+c-a)S_{AA}x=0.$$

It intersects the $\mathtt{OI}$ line at $\mathtt{S_c}^*$. The reflections of this tangent in the sidelines are concurrent at

$$\mathtt{X}(108) = \left(\frac{a}{(b-c)(b+c-a)S_A} : \frac{b}{(c-a)(c+a-b)S_B} : \frac{c}{(a-b)(a+b-c)S_C}\right).$$

Chapter 14

The Steiner circumellipse

14.1 The Steiner circum-ellipse

The Steiner circum-ellipse is the isotomic conjugate of the infinite line. It contains $X(n)$ for the following values of n:

$$\begin{array}{l} 99, 190, 290, 648, 664, 666, 668, 670, 671, 886, \\ 889, 892, 903, 1121, 1494, 2481, 2966, 3225, 3226, 3227, \\ 3228, 4555, 4562, 4569, 4577, 4586, 4597, \dots \end{array}$$

1. $X(190) = \frac{1}{b-c}$: Yff parabolic point, perspector of inscribed parabola with focus $X(101)$.

2. $X_{648} = \frac{1}{(b^2-c^2)S_\alpha}$

- Trilinear pole of the Euler line.
- Orthocorrespondent of X_{107} and X_{125}.
- "Third" orthoassociate of Steiner point. See 7/29/04.

3. $X_{671} = \frac{1}{b^2+c^2-2a^2}$

- Reflection conjugate of centroid X_2. [1]
- Reflection of centroid in Kiepert center.

[1]Also in ED's file.

14.2 The Steiner point

The Steiner point $\mathtt{S_t}$ is the fourth intersection of the circumcircle and the Steiner ellipse $yz + zx + xy = 0$. It has coordinates

$$\left(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2}\right).$$

14.2.1 Bailey's theorem on the Steiner point

Theorem (Bailey)**.** Let P be a point with cevian triangle XYZ, and the line $\mathscr{L}$ joining P^* and $P^\bullet$. If $\mathscr{L}$ intersects the sidelines of $\mathbf{T}$ at X', Y', Z' respectively, then the circles AXX', BYY', CZZ' are concurrent at the Steiner point $\mathtt{S_t}$.

Proof. Let $P = (u : v : w)$. The line joining P^* and $P^\bullet$ is

$$(b^2-c^2)ux + (c^2-a^2)vy + (a^2-b^2)wz = 0.$$

It intersects the sidelines of $\mathbf{T}$ at

$$X' = (0 : -(a^2-b^2)w : (c^2-a^2)v), \quad Y' = ((a^2-b^2)w : 0 : -(b^2-c^2)u), \quad Z' = (-(c$$

The circle AXX' has equation

$$\begin{aligned}&(v+w)((c^2-a^2)v-(a^2-b^2)w)(a^2yz+b^2zx+c^2xy)\\ &-a^2vw(x+y+z)((c^2-a^2)y-(a^2-b^2)z)=0.\end{aligned}$$

This clearly contains the Steiner point; so do the other two circles BYY' and CZZ'. □

14.3 Construction of the Steiner point

Let $\mathcal{C}$ be a circumconic with center Q. Denote by A', B', C' the symmetrics of A, B, C in Q. Then the fourth intersection of $\mathcal{C}$ with the circumcircle is the common point of the circles $AB'C'$, $BC'A'$ and $CA'B'$.

Let $P = (u : v : w)$ (in barycentrics), with pedal triangle $A_{[P]}B_{[P]}C_{[P]}$. Let A', B', C' be the reflections of A, B, C in the respective sides of the pedal triangle. Then $A'B'C'$ is perspective with ABC at the isogonal conjugate of P^*. (This is well known).

Now the circles $A_{[P]}B'C'$, $B_{[P]}C'A'$ and $C_{[P]}A'B'$ concur at a point.

This is a corollary of the following.

Let $P = (u : v : w)$ (in barycentrics), with *antipedal* triangle $A^{[P]}B^{[P]}C^{[P]}$. Let A', B', C' be the reflections of $A^{[P]}$, $B^{[P]}$, $C^{[P]}$ in the lines BC, CA, AB respectively.

Now the circles $AB'C'$, $BC'A'$ and $CA'B'$ concur at a point. The first component of the barycentric coordinates of this common point can be taken as $a^2F_1F_2G$ where

- $F_1 = (S^2 - S_{BB})vw + (S^2 - S_{AA})wu - c^4uv$,
- $F_2 = (S^2 - S_{CC})vw - b^4wu + (S^2 - S_{AA})uv$, and
- $G = -a^6S_Av^2w^2 + b^4S_{BC}w^2u^2 + c^4S_{BC}u^2v^2 + 2_SAABCu^2vw + a^2(b^2S^2 - 2S_{ACC})uvw^2 + a^2(c^2S^2 - 2S_{ABB})uv^2w$.

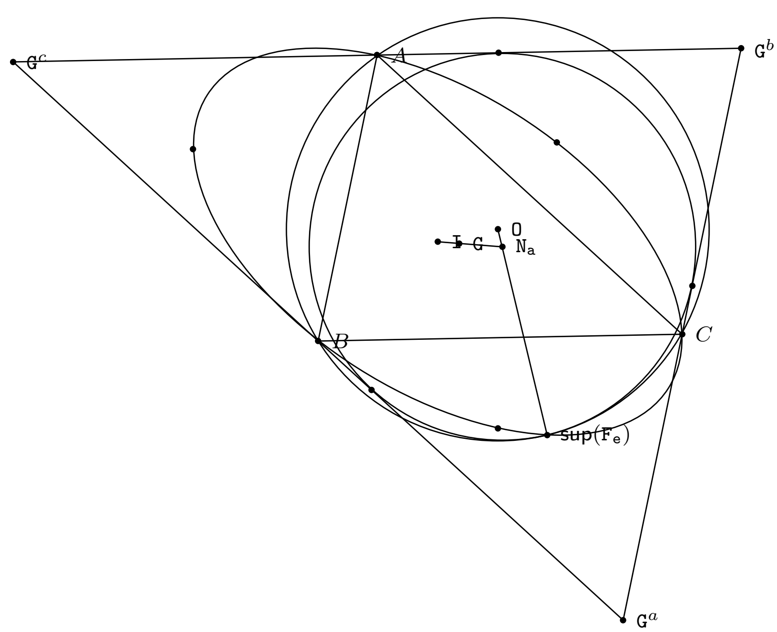


Figure 14.1:

14.3.1 Steiner point and the nine-point center

Let X be the circumcenter of the AA_+A_-, where $A_\pm$ are the apices of the Fermat triangles. Similarly define Y and Z. Darij Grinberg has noted that XYZ has the Steiner point as perspector.

Simply, $X = (0 : S_A - S_B : S_C - S_A)$. The circle has equation

$$a^2yz + b^2zx + c^2xy = \frac{a^2}{S_B - S_C}(x+y+z)((S_A - S_B)y + (S_C - S_A)z).$$

The radical center of the three circles is the point given by

$$\frac{(S_A - S_B)y + (S_C - S_A)z}{\frac{S_B - S_C}{a^2}} = \cdots = \cdots .$$

This means that $\left(\frac{x}{S_B - S_C} : \cdots : \cdots\right)$ is the superior of $\left(\frac{(S_B - S_C)^2}{a^2} : \cdots : \cdots\right)$, and the radical center is the nine-point center.

Chapter 15

Circum-hyperbolas

15.1 Circum-hyperbolas with given infinite points

We begin with the simple fact that a hyperbola has two asymptotes.

A circum-hyperbola is uniquely determined by its two infinite points (giving the directions of its two asympotes). If these two infinite points have homogeneous barycentric coordinates $(u_1 : v_1 : w_1)$ and $(u_2 : v_2 : w_2)$, then the equation of the hyperbola is simply

$$\frac{u_1u_2}{x} + \frac{v_1v_2}{y} + \frac{w_1w_2}{z} = 0. \tag{15.1}$$

From this we make the following simple conclusions.

1. The perspector of the circumhyperbola is the point $P = (u_1u_2 : v_1v_2 : w_1w_2)$.

2. The center of the hyperbola is the cevian quotient G/P:

$$\begin{aligned} G/P = & \ (u_1u_2(-u_1u_2 + v_1v_2 + w_1w_2) : \cdots : \cdots) \\ = & \ (u_1u_2(v_1w_2 + v_2w_1) : \cdots : \cdots). \end{aligned}$$

3. The asymptote with infinite point $(u_1 : v_1 : w_1)$ is the line

$$\begin{aligned} 0 = & \begin{vmatrix} x & y & z \\ u_1 & v_1 & w_1 \\ u_1u_2(v_1w_2 + v_2w_1) & v_1v_2(w_1u_2 + w_2u_1) & w_1w_2(u_1v_2 + u_2v_1) \end{vmatrix} \\ = & \begin{vmatrix} v_1 & w_1 \\ v_1v_2(w_1u_2 + w_2u_1) & w_1w_2(u_1v_2 + u_2v_1) \end{vmatrix} x + \cdots \\ = & \ v_1w_1(w_2(u_1v_2 + u_2v_1) - v_2(w_1u_2 + w_2u_1))x + \cdots \\ = & \ u_2v_1w_1(v_1w_2 - w_1v_2)x + v_2w_1u_1(w_1u_2 - u_1w_2)y + w_2u_1v_1(u_1v_2 - v_1u_2)z. \end{aligned}$$

Now, since

$$v_1w_2-w_1v_2 = -(w_1+u_1)w_2+w_1(w_2+u_2) = w_1u_2-u_1w_2 = u_1v_2-v_1u_2,$$

the equation of the asymptote can be simplified as

$$u_2v_1w_1x + v_2w_1u_1y + w_2u_1v_1z = 0$$

or

$$\frac{u_2}{u_1}x + \frac{v_2}{v_1}y + \frac{w_2}{w_1}z = 0.$$

4. Similarly the asymptote with infinite point $(u_2 : v_2 : w_2)$ is the line

$$\frac{u_1}{u_2}x + \frac{v_1}{v_2}y + \frac{w_1}{w_2}z = 0.$$

Proposition. The asymptotes of the circum-hyperbola with infinite points $(u_1 : v_1 : w_1)$ and $(u_2 : v_2 : w_2)$ are the isotomic lines

$$\begin{aligned}\frac{u_1}{u_2}x + \frac{v_1}{v_2}y + \frac{w_1}{w_2}z =& \ 0,\\ \frac{u_2}{u_1}x + \frac{v_2}{v_1}y + \frac{w_2}{w_1}z =& \ 0.\end{aligned}$$

15.1.1 The circum-hyperbola with perspector $\mathtt{S_t}$

The circumconic

$$\mathscr{C}_p(\mathtt{S_t}): \qquad \frac{yz}{b^2-c^2}+\frac{zx}{c^2-a^2}+\frac{xy}{a^2-b^2}=0.$$

has center

$$\begin{aligned}\mathtt{G/S_t} &= \left(\frac{a^4-a^2(b^2+c^2)-(b^4-3b^2c^2+c^4)}{b^2-c^2}:\cdots:\cdots\right)\\ &= \left(\frac{(c^2-a^2)(a^2-b^2)+(b^2-c^2)^2}{b^2-c^2}:\cdots:\cdots\right).\end{aligned}$$

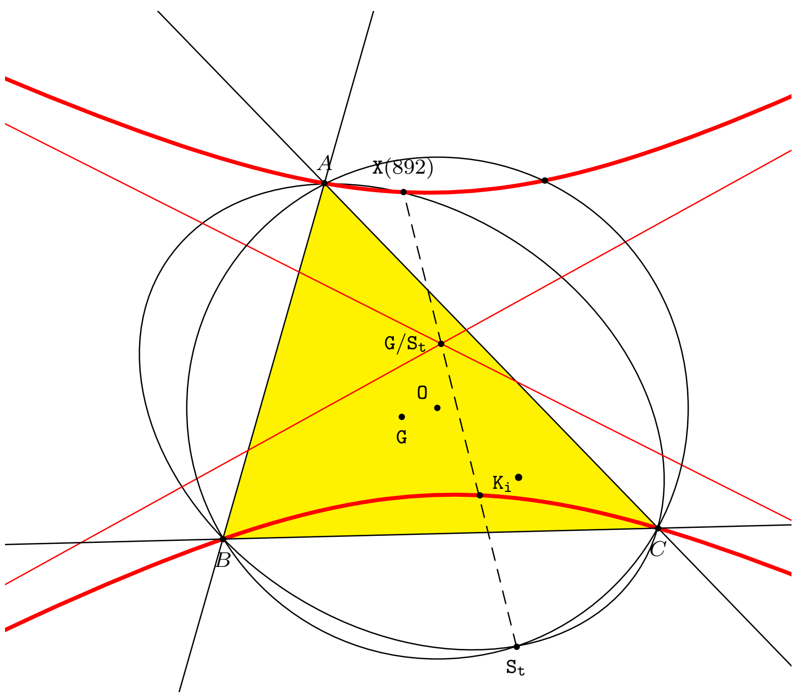


Figure 15.1: The circumconic $\mathscr{C}_p(\mathtt{S_t})$

It contains the two infinite points

$$(a^2-b^2:\ b^2-c^2:\ c^2-a^2) \quad \text{and} \quad (c^2-a^2:\ a^2-b^2:\ b^2-c^2).$$

Therefore, the asymptotes are parallel to the lines

$$c^2x+a^2y+b^2z=0 \quad \text{and} \quad b^2x+c^2y+a^2z=0.$$

This hyperbola also contains

$$\mathtt{K_i}^{\bullet}=\left(\frac{1}{(b^2-c^2)^2}:\frac{1}{(c^2-a^2)^2}:\frac{1}{(a^2-b^2)^2}\right).$$

It intersects
(i) the circumcircle at

$$\left(\frac{1}{(b^2-c^2)(a^2(b^2+c^2)-2b^2c^2)} : \cdots : \cdots\right),$$

(ii) the Steiner circum-ellipse at

$$\mathtt{X}(892) = \left(\frac{1}{(b^2-c^2)(b^2+c^2-2a^2)} : \cdots : \cdots\right),$$

the antipode of $\mathtt{K_i}^\bullet$ (see Figure 15.1).

15.2 Circum-hyperbola with a prescribed asymptote

Since the two asymptotes of a circum-hyperbola are isotomic lines, the hyperbola is uniquely determined by an asymptote. If it has one asymptote $fx+gy+hz=0$, then the other asymptote is the line $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0$. Their intersection is the center of the hyperbola:

$$Q=(f(g^2-g^2):\ g(h^2-f^2):\ h(f^2-g^2)).$$

The perspector is the cevian quotient

$$G/Q=(f(g-h)^2:\ g(h-f)^2:\ h(f-g)^2).$$

The equation of the hyperbola is therefore

$$\frac{f(g-h)^2}{x}+\frac{g(h-f)^2}{y}+\frac{h(f-g)^2}{z}=0.$$

15.2.1 The Euler asymptotic hyperbola

The Euler asymptote hyperbola is the circum-hyperbola with the Euler line as an asymptote:

$$\sum_{\text{cyclic}} \frac{S_A(S_B-S_C)(S_A(S_B+S_C)-2S_{BC})^2}{x} = 0.$$

Its center is the point

$$X_{1650} = (S_{AA}(S_B-S_C)^2(S_A(S_B+S_C)-2S_{BC}) : \cdots : \cdots).$$

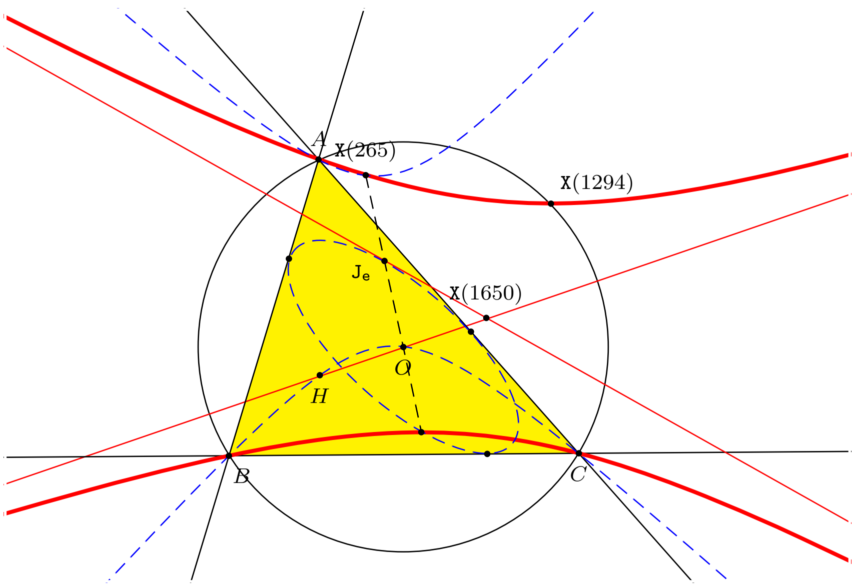


Figure 15.2: The Euler asymptotic hyperbola

This hyperbola also contains the following triangle centers:

- $X_{1494} = \left(\frac{1}{S_A(S_B+S_C)-2S_{BC}} : \cdots : \cdots\right)$, intersection with the Steiner circum-ellipse,
- $X_{1294} = \left(\frac{1}{S_{AA}(S_{BB}-S_{BC}+S_{CC})-S_{BB}S_{CC}} : \cdots : \cdots\right)$, intersection with circumcircle,
- $X_{265} = \left(\frac{S_A}{3S_{AA}-S^2} : \cdots : \cdots\right)$, which is the reflection conjugate of the circumcenter O. [1]

[1] If O_a, O_b, O_c are the reflections of O in a, b, c respectively, the three circles O_aBC, O_bCA, O_cAB intersect at the reflection conjugate of O.

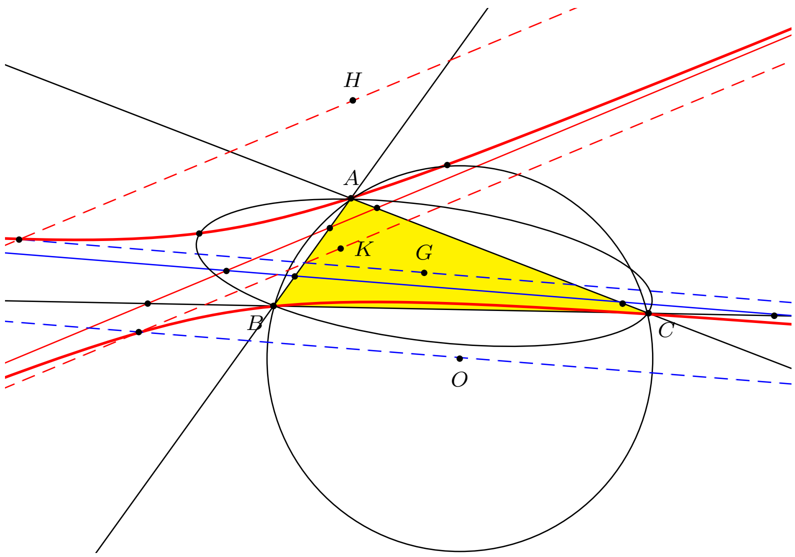
15.2.2 The orthic asymptotic hyperbola

The orthic asymptote hyperbola is the circum-hyperbola with the orthic axis $S_Ax + S_By + S_Cz = 0$ as an asymptote:

$$\sum_{\text{cyclic}} \frac{S_A(S_B - S_C)^2}{x} = 0.$$

Its center is the point

$$X_{647} = (S_A(S_{BB} - S_{CC}) : S_B(S_{CC} - S_{AA}) : S_C(S_{AA} - S_{BB})).$$



This hyperbola also contains the following triangle centers:

- $X_{2966} = \left(\frac{1}{((S_B-S_C)(S_{AA}-S_{BC})} : \cdots : \cdots\right)$, intersection with the Steiner circum-ellipse,
- $X_{935} = \left(\frac{1}{S_A(S_B-S_C)(4S_{BC}-(S_C+S_A)(S_A+S_B))} : \cdots : \cdots\right)$, intersection with circumcircle,
- $X_{879} = \left(\frac{S_A(S_B-S_C)}{S_{AA}-S_{BC}} : \cdots : \cdots\right)$, which is the intersection with the Jerabek hyperbola. [2]
- $X_{2394} = \left(\frac{S_B-S_C}{S_A(S_B+S_C)-2S_{BC}} : \cdots : \cdots\right)$, which is the intersection of the parallels to the orthic axis through the orthocenter and to the isotomic line through the centroid.

[2]Also of the parallels to the orthic axis through the symmedian point and to the isotomic line through the circumcenter.

15.2.3 The Lemoine asymptotic hyperbola

The Lemoine aymptotic hyperbola has the Lemonie axis

$$\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=0$$

and its isotomic line

$$a^2x+b^2y+c^2z=0$$

for asymptotes. It is the circumconic

$$\frac{a^2(b^2-c^2)^2}{x}+\frac{b^2(c^2-a^2)^2}{y}+\frac{c^2(a^2-b^2)^2}{z}=0$$

with center

$$\mathtt{X}(3005)=(a^2(b^4-c^4):b^2(c^4-a^4):c^2(a^4-b^4)),$$

and intersects the circumcircle at the reflection of $\mathtt{E}$ in the Brocard axis, namely,

$$\mathtt{X}(691)=\left(\frac{a^2}{(b^2-c^2)(b^2+c^2-2a^2)}:\frac{b^2}{(c^2-a^2)(c^2+a^2-2b^2)}:\frac{c^2}{(a^2-b^2)(a^2+b^2-2c^2)}\right).$$

15.3 Pencil of hyperbolas with parallel asymptotes

Suppose a circum-hyperbola $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 0$ has a prescribed *infinite* point $(u : v : w)$, *i.e.*, an asymptote with prescribed direction. Then the other infinite point is $\left(\frac{p}{u} : \frac{q}{v} : \frac{r}{w}\right)$. It follows that $\frac{p}{u}+\frac{q}{v}+\frac{r}{w} = 0$, and the perspector lies on the trilinear polar of the given infinite point $(u : v : w)$.

A typical point on the trilinear polar of $(u : v : w)$ has coordinates $(u(v-w+tu) : v(w-u+tv) : w(u-v+tw))$. The hyperbola is one in the pencil [3]

$$\frac{u(v-w)}{x} + \frac{v(w-u)}{y} + \frac{w(u-v)}{z} + t\left(\frac{u^2}{x} + \frac{v^2}{y} + \frac{w^2}{z}\right) = 0.$$

The center of the hyperbola, being the cevian quotient $G/(p : q : r)$, lies on the conic

$$\frac{x(y+z-x)}{u} + \frac{y(z+x-y)}{v} + \frac{z(x+y-z)}{w} = 0.$$

This is the (bicevian) conic which intersects the sidelines at the midpoints of the sides and the traces of $(u : v : w)$. This is a parabola.

[3]The circumconic $\frac{u^2}{x} + \frac{v^2}{y} + \frac{w^2}{z} = 0$ has only one infinite point; it is a parabola.

15.4 The circum-hyperbola $\frac{u}{x}+\frac{v}{y}+\frac{w}{z}=0$ for an infinite point $(u:v:w)$

If $(u:v:w)$ is an infinite point, the circumconic $\frac{u}{x}+\frac{v}{y}+\frac{w}{z}=0$ contains the centroid and is a hyperbola. The center of the hyperbola is the point $(u^2:v^2:w^2)$ on the Steiner inellipse.

Let $((v-w)(u+t):(w-u)(v+t):(u-v)(w+t))$ be an infinite point of the hyperbola. Then

$$\frac{u}{(v-w)(u+t)}+\frac{v}{(w-u)(v+t)}+\frac{w}{(u-v)(w+t)}=0.$$

Exercise

1. The circumconic through the Brocard points is

$$(a^4-b^2c^2)yz+(b^4-c^2a^2)zx+(c^4-a^2b^2)xy=0.$$

It contains
(i) X_{99} the Steiner point,
(ii) $X_{83}=\frac{1}{b^2+c^2}$,
(iii) $X_{880}=\frac{a^4-b^2c^2}{a^2(b^2-c^2)}$.

2. Given a point $P=(u:v:w)$, consider the circumconic $\mathcal{C}$ through P and its isotomic conjugate. For every point Q on $\mathcal{C}$ with cevian triangle XYZ, let X', Y', Z' be the midpoints of AX, BY, CZ respectively. $X'Y'Z'$ is perspective with cev(P).

If $Q=\left(\frac{1}{tu+vw}:\cdots:\cdots\right)$, then this perspector is $(t+u:t+v:t+w)$.
For $P=$ the Gergonne point with $u=1/(b+c-a)$, with $t=\frac{-1}{a+b+c}$, $Q=H$, the orthocenter. The perspector is X_{57}.

If we take P to be the Nagel point, and $Q=H$, the perspector is the incenter I.

The locus of the perspector is the line GP.

3. $X(1648)$ **and** $X(1649)$

Let $P=(u:v:w)$ be an infinite point. Consider the point

$$Q=(u^2(v-w):v^2(w-u):w^2(u-v)).$$

(a) Q is the intersection of the three lines:
(i) $ux+vy+wz=0$,
(ii) $\frac{u}{x}+\frac{v}{y}+\frac{w}{z}=0$,
(iii) $\frac{x}{u(v-w)}+\frac{y}{v(w-u)}+\frac{z}{w(u-v)}=0$.

(b) The cevian quotient of Q is the point $Q'=(u(v-w)^2:v(w-u)^2:w(u-v)^2)$.

(c) The line QQ' is the trilinear polar of $(u(v-w):v(w-u):w(u-v))$.

Therefore, for the circumconic with perspector Q:

$$\frac{u^2(v-w)}{x}+\frac{v^2(w-u)}{y}+\frac{w^2(u-v)}{z}=0,$$

(a) the center is Q',

(b) the infinite points are $P=(u:v:w)$ and $(u(v-w):v(w-u):w(u-v))$,

(c) the asymptotes are the isotomic lines
$(v-w)x+(w-u)y+(u-v)z=0$ and
$\frac{x}{v-w}+\frac{y}{w-u}+\frac{z}{u-v}=0$.

These are defined as the tripolar centroids of $X(523)$ and $X(524)$ respectively. They have barycentric coordinates

$$\begin{aligned}X_{1648}=&((b^2-c^2)^2(2a^2-b^2-c^2):\cdots:\cdots),\\ X_{1649}=&((b^2-c^2)(2a^2-b^2-c^2)^2:\cdots:\cdots).\end{aligned}$$

$G/X_{1648}=X_{1649}$.

The hyperbola with perspector X_{1648} have asymptotes which are the tripolars of the points $X_{671}=\left(\frac{1}{2a^2-b^2-c^2}:\cdots:\cdots\right)$ and $X_{524}=(2a^2-b^2-c^2:\cdots:\cdots)$.

According to ETC, X_{1648} can be constructed as the intersection of GK and the line joining the Kiepert and Jerabek centers.

The other hyperbola with perspector X_{1649} has asymptotes which are the trilinear polars of X_{99} and X_{523}.

X_{2398} and X_{2400} are isotomic conjugates. Their trilinear polars are the asymptotes of the hyperbola with center X_{1566} and perspector X_{676}.

These two hyperbolas have a common point

$$X_{690}=((b^2-c^2)(2a^2-b^2-c^2):\cdots:\cdots).$$

15.4.1 Perspective and orthologic triangles

Theorem. If $\mathbf{T}$ and a triangle XYZ is both perspective and orthologic to $\mathbf{T}$, then $\perp(\mathbf{T},\ XYZ)$ is the second intersection of the line joining $Q :=\perp(XYZ,\ \mathbf{T})$ to $P := \bigwedge(XYZ,\ \mathbf{T})$ and the rectangular circum-hyperbola $\mathscr{H}(P)$ through the perspector.

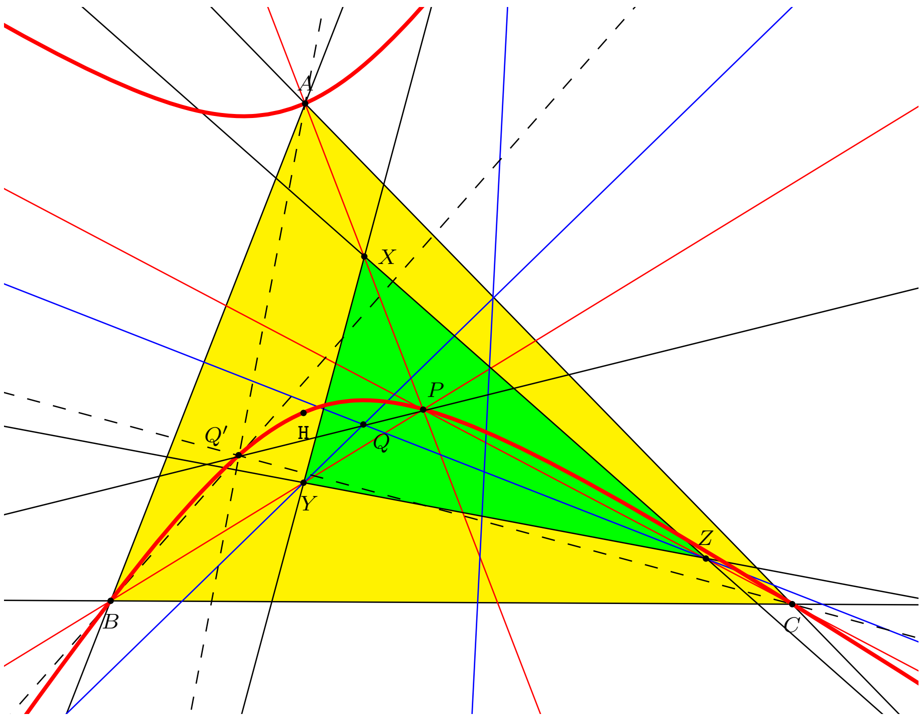


Figure 15.3: Perspectivity and orthology of two triangles

Proof. Let $\bigwedge(XYZ,\mathbf{T}) = P = (u:v:w)$ and $\perp(XYZ,\ \mathbf{T}) = Q = (u':v':w')$. The point X must lie on the line AP and the perpendicular from Q to BC. on the line AQ; similarly for Y and Z. These are the points

$$
\begin{aligned}
X &= (u'+v'+w')(S_Bv-S_Cw)(1,0,0)+(S_Bv'-S_Cw')(-(v+w),v,w),\\
Y &= (u'+v'+w')(S_Cw-S_Au)(0,1,0)+(S_Cw'-S_Au')(u,-(w+u),w),\\
Z &= (u'+v'+w')(S_Au-S_Bv)(0,0,1)+(S_Au'-S_Bv')(u,v,-(u+v)).
\end{aligned}
$$

The perpendiculars from A, B, C to YZ, ZX, XY respectively are concurrent at

$$Q' = \left(\frac{S_Bv-S_Cw}{vw'-wv'} : \frac{S_Cw-S_Au}{wu'-uw'} : \frac{S_Au-S_Bv}{uv'-vu'}\right).$$

Note that this point lies on the rectangular circum-hyperbola

$$\mathscr{H}(P): \qquad \sum_{\text{cyclic}} \frac{u(S_B v - S_C w)}{x} = 0,$$

since $u(vw' - wv') + v(wu' - uw') + w(uv' - vu') = 0$. Therefore this second orthology center Q' is the intersection of the line PQ with the circum-hyperbola $\mathscr{H}(P)$. $\square$

Chapter 16

Rectangular circum-hyperbolas

A hyperbola is rectangular if its asymptotes are perpendicular to each other.

Theorem. *A circumconic is a rectangular hyperbola if and only if it contains the orthocenter.*

Proof. Two lines with infinite points $(u_1 : v_1 : w_1)$ and $(u_2 : v_2 : w_2)$ are perpendicular to each other if and only if [1]

$$S_A u_1 u_2 + S_B v_1 v_2 + S_C w_1 w_2 = 0. \tag{16.1}$$

Thus, the orthocenter $\mathtt{H}$ lies on the hyperbola (15.1). Conversely, if a circumconic contains $\mathtt{H}$, its isogonal conjugate is a line containing the circumcenter $\mathtt{O}$, an interior point of the circumcircle. Therefore, the conic is a hyperbola, and we may assume its equation in the form (15.1). Now the condition (16.1) guarantees that the two asymptotes are perpendicular to each other. □

The Kiepert, Jerabek, and Feuerbach hyperbolas are all rectangular. On the other hand, the circumconic through $\mathtt{G}$ and $\mathtt{K}$ is a hyperbola (since these are interior points of the cirumcircle), albeit not rectangular (since the line $\mathtt{GK}$ does not contain $\mathtt{O}$). Nor is the hyperbola $\mathscr{C}_{\mathrm{p}}(\mathtt{S_t})$ in Example **??**.

[1] This condition of perpendicularity is established in an earlier chapter.

16.1 The center of a rectangular hyperbola

Theorem. *The center of a rectangular hyperbola is a point on the nine-point circle.*

Proof. From the equations of the asymptotes in (**??**) and (**??**), if $P_1 = (u_1 : v_1 : w_1)$ and $P_2 = (u_2 : v_2 : w_2)$ are orthogonal, then these are the Simson lines of the points P_1^* and P_2^* on the circumcircle. [2] Furthermore, these points are antipodal. Their Simson lines are orthogonal and intersect on the nine-point circle. □

Corollary. *The fourth intersection of a rectangular circum-hyperbola with the circumcircle is the antipode of the orthocenter* $\mathtt{H}$.

Proof. Since the circumcircle is the image of the nine-point circle under the homothety $\mathtt{h}(\mathtt{H}, 2)$, the antipode of $\mathtt{H}$ must be a point on the circumcircle. It is the fourth intersection of the rectangular circum-hyperbola and the circumcircle. □

The coordinates of the fourth intersections of the basic circumconics can be found in §13.2.1

[2]This is a nontrivial result. It follows from the equations of Simson lines established in the chapter on the circumcircle.

16.2 Construction of asymptotes

Given the center W of a rectangular hyperbola, and the tangent at a point P, the asymptotes can be easily constructed. Construct
(1) the circle $P(W)$ to intersect the tangent at two points Q and Q',
(2) the lines WQ and WQ'.

These are the asymptotes. [3]

Example. The case of the Feuerbach hyperbola is strikingly easy. Since the OI is the tangent at I, the lines joining the Feuerbach point $\mathtt{F_e}$ to the intersections of the incircle with OI are the asymptotes.

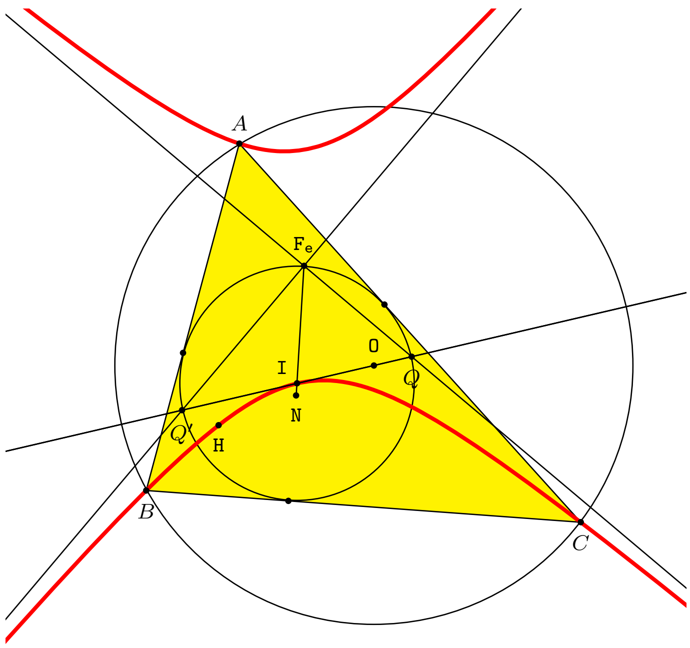


Figure 16.1: Construction of asymptotes of the Feuerbach hyperbola

16.2.1 Antipodes on rectangular circum-hyperbola

Theorem. *Two points on a rectangular circum-hyperbola are antipodal if and only if their isogonal conjugates are inverse with respect to the circumcircle.*

[3]In Cartesian coordinates, let P be the point $\left(t, \frac{c^2}{t}\right)$ on the rectangular hyperbola $xy = c^2$. The tangent at P is the line $\frac{1}{2}\left(\frac{c^2}{t}x + ty\right) = c^2$. This tangent intersects the axes (asymptotes) at $(2t, 0)$ and $\left(0, \frac{2c^2}{t}\right)$. These two points are clearly on the circle $P(O)$.

16.2.2 The Huygens hyperbola

The Huygens hyperbola (Alperin chose this name for convenience only) is the isogonal transform of the tangent to the Jerabek hyperbola at O. It is tangent to the Euler line at H.

$$\sum_{\text{cyclic}} S_{BC}(S_B - S_C)yz = 0.$$

It contains X_{1826} (on line joining H to Mittenspunkt), X_{225} (on HI), X_{264} (isotomic conjugate of O, on $HH^\bullet$), X_{393} (square of orthocenter, on HK), X_{1093}, X_{93}, X_{847}.

It intersects the circumcircle at X_{1300}.

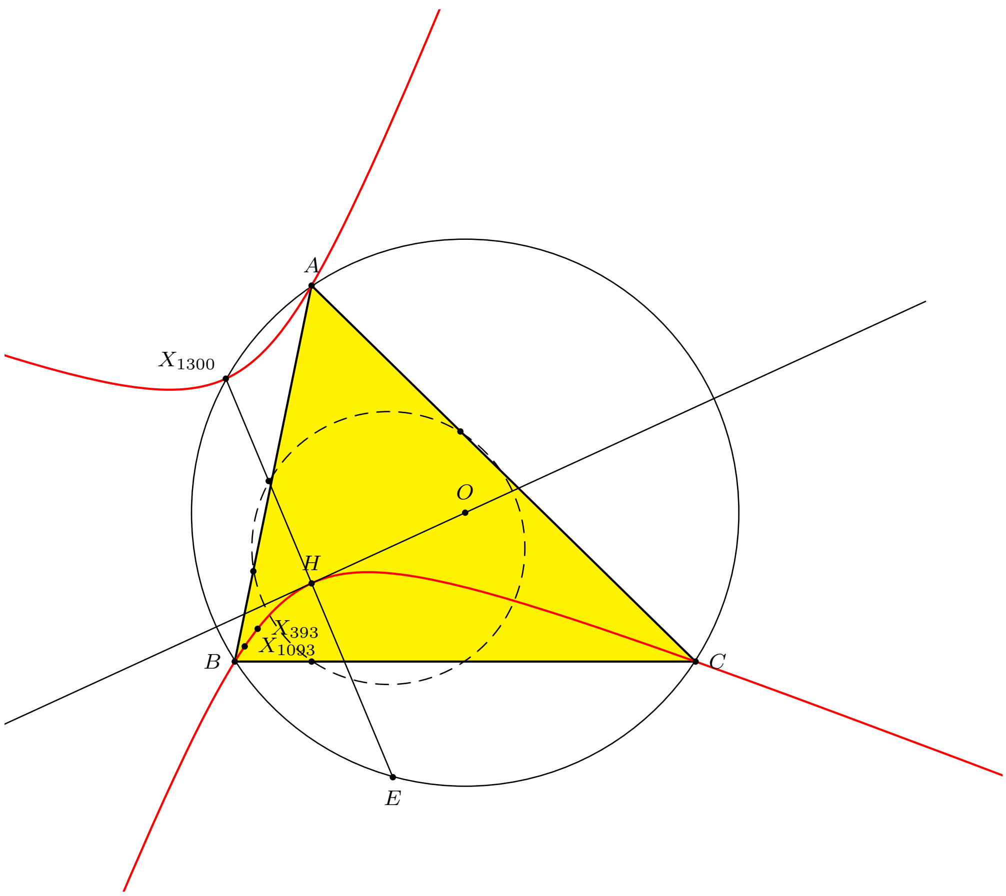


Figure 16.2: The Huygens hyperbola

16.3 The rectangular circum-hyperbola through a given point

Let $P = (u : v : w)$ be a given point other than the vertices and orthocenter $\mathtt{H}$ of $\mathbf{T}$. The rectangular circum-hyperbola through P is the isogonal conjugate of the line $\mathtt{O}P^*$. It is the hyperbola

$$\mathscr{H}(P) : \qquad \sum_{\text{cyclic}} \frac{u(S_B v - S_C w)}{x} = 0.$$

Apart from the orthocenter $\mathtt{H}$, it intersects the Euler line at the point

$$\left(\frac{u(S_B v - S_C w)}{S_B - S_C} : \frac{v(S_C w - S_A u)}{S_C - S_A} : \frac{w(S_A u - S_B v)}{S_A - S_B}\right).$$

16.3.1 The center

The center of $\mathscr{H}(P)$ is the cevian quotient

$$\begin{aligned}
W(P) :=\ & \mathtt{G}/(u(S_B v - S_C w) : v(S_C w - S_A u) : w(S_A u - S_B v)) \\
=\ & (u(S_B v - S_C w)(-u(S_B v - S_C w) + v(S_C w - S_A u) + w(S_A u - S_B v) \\
& \quad : \cdots : \cdots) \\
=\ & (u(S_B v - S_C w)(b^2(u+v)w - c^2(w+u)v) : \cdots : \cdots)
\end{aligned}$$

on the nine-point circle.

16.3.2 The fourth intersection with the circumcircle

The rectangular circum-hyperbola $\mathscr{H}(P)$ intersects the circumcircle at the point

$$((\mathtt{O}P^*)^\infty)^* = \left(\frac{1}{c^2 v(S_C w - S_A u) - b^2 w(S_A u - S_B v)} : \cdots : \cdots\right).$$

16.3.3 The tangent at $\mathtt{H}$ and P

The tangent to $\mathscr{H}(P)$ at the orthocenter is the line

$$\sum_{\text{cyclic}} S_{AA} u(S_B v - S_C w)x = 0.$$

Proposition. The following three points on the circumcircle coincide and have coordinates

$$\left(\frac{a^2}{S_Au(S_Bv-S_Cw)}:\frac{b^2}{S_Bv(S_Cw-S_Au)}:\frac{c^2}{S_Cw(S_Au-S_Bv)}\right):$$

(1) the antipode of the isogonal conjugate of the infinite point of the tangent at H,
(2) the point of concurrency of the reflections of the tangent in the sideline,
(3) the second intersection of the circumcircle of the line joining its intersection with $\mathscr{H}(P)$ and the orthocenter H.

16.3.4 The rectangular hyperbola through the Euler reflection point

$$\sum_{\text{cyclic}} a^2(-a^2S_{AA} + b^2S_{BB} + c^2S_{CC} - 2S_{ABC})yz = 0.$$

Center of conic X_{113}.

Intersection with the Euler line

$$\left(\frac{a^2((-a^2S_{AA} + b^2S_{BB} + c^2S_{CC} - 2S_{ABC}))}{b^2 - c^2} : \cdots : \cdots\right).$$

Tangent at H:

$$\sum_{\text{cyclic}} a^2S_{AA}(-a^2S_{AA} + b^2S_{BB} + c^2S_{CC} - 2S_{ABC})x = 0.$$

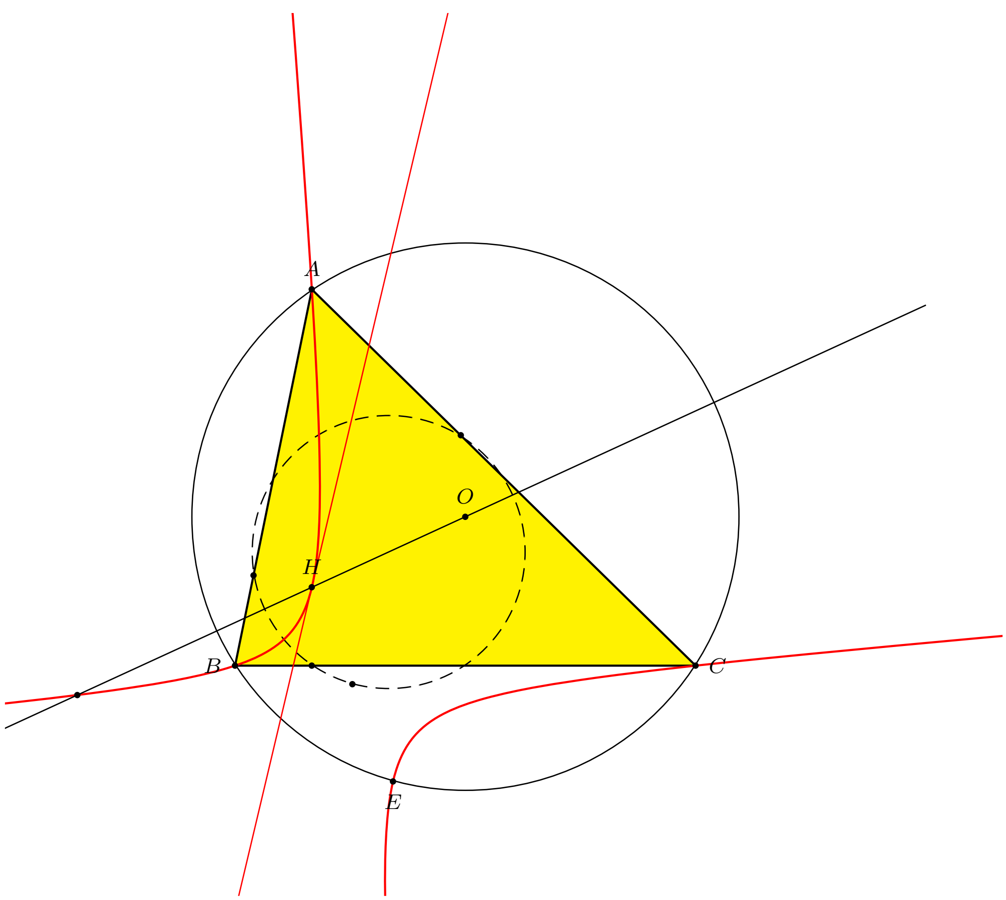


Figure 16.3: The rectangular circumhyperbola through the Euler reflection point

Exercise

1. Consider the rectangular circum-hyperbola $\mathcal{H}(P)$ through a given point P. Denote by Q the fourth intersection with the circumcircle.

The reflections of the tangent to $\mathcal{H}(P)$ at H in the sidelines intersect on the circumcircle at its second intersection with the line HQ.

Jerabek hyperbola	$X(107)$
Kiepert hyperbola	X_{112}

16.4 Reflection conjugates as antipodal points on a rectangular circum-hyperbola

Let $Q = (u : v : w)$ be a finite point with reflection triangle $Q_a^\dagger Q_b^\dagger Q_c^\dagger$. These are the points

$$\begin{aligned} Q_a^\dagger &= \ (-a^2u :\ 2S_Cu + a^2v :\ 2S_Bv + a^2w), \\ Q_b^\dagger &= \ (2S_Cv + b^2u :\ -b^2v :\ 2S_Av + b^2w), \\ Q_c^\dagger &= \ (2S_Bw + c^2u :\ 2S_Aw + c^2v :\ -c^2w). \end{aligned}$$

The reflection conjugate of Q is the common point $Q^\dagger$ of the circles $Q_a^\dagger BC$, $Q_b^\dagger CA$, $Q_c^\dagger AB$.

Theorem. *(a)* $Q^\dagger$ is the antipode of Q on the rectangular circum-hyperbola $\mathscr{H}(Q)$.
(b) $Q^\dagger$ lies on the circumconic $\mathscr{C}_\mathrm{o}(\mathsf{inf}(Q))$.

The three circles $AQ_b^\dagger Q_c^\dagger$, $BQ_c^\dagger Q_a^\dagger$,$CQ_a^\dagger Q_b^\dagger$ are concurrent at the the fourth intersection of $\mathscr{C}_\mathrm{o}(Q)$ and the circumcircle.

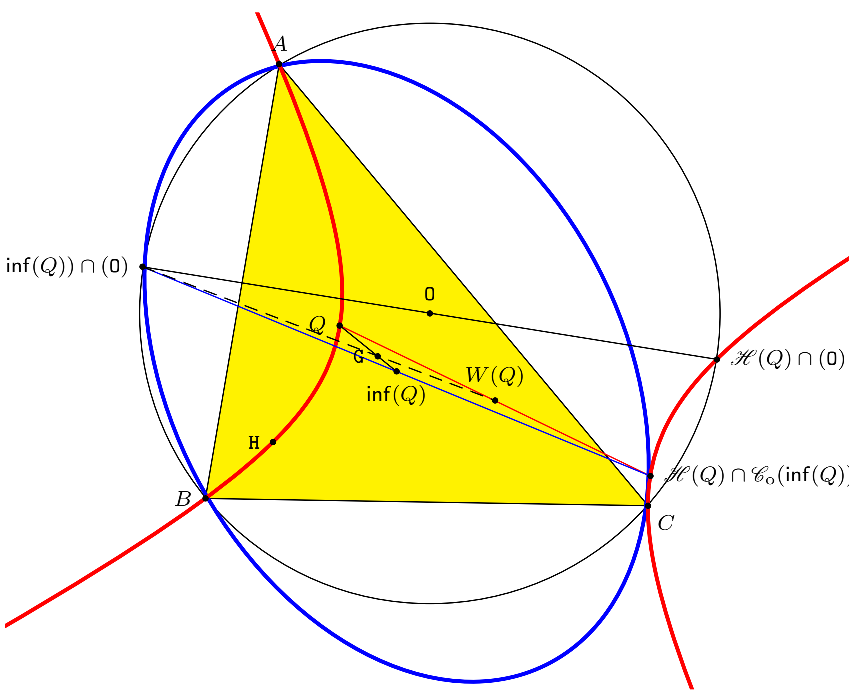


Figure 16.4: Intersections of $\mathscr{H}(Q)$ and $\mathscr{C}_\mathrm{o}(\mathsf{inf}(Q))$ and the circumcircle

16.5 Rectangular circum-hyperbola with a prescribed infinite point

If a rectangular circum-hyperbola has an infinite point $(u : v : w)$, the other infinite point is $(S_Bv - S_Cw : S_Cw - S_Au : S_Au - S_Bv)$. The hyperbola is

$$\sum_{\text{cyclic}} \frac{u(S_Bv - S_Cw)}{x} = 0.$$

The asymptote with the given infinite point is the line

$$\frac{S_Bv - S_Cw}{u}x + \frac{S_Cw - S_Au}{v}y + \frac{S_Au - S_Bv}{w}z = 0.$$

16.5.1 The Euler (rectangular) circum-hyperbola

The Euler (rectangular) circum-hyperbola has asymptotes parallel and perpendicular to the Euler line. The two infinite points are [4]

$$E_\infty = (S_A(S_B+S_C)-2S_{BC} : S_B(S_C+S_A)-2S_{CA} : S_C(S_A+S_B)-2S_{AB}),$$
$$(\mathcal{L}_*(H))_\infty = (S_B-S_C : S_C-S_A : S_A-S_B).$$

It follows that the Euler hyperbola has equation

$$\sum_{\text{cyclic}} \frac{(S_B-S_C)(S_A(S_B+S_C)-2S_{BC})}{x} = 0.$$

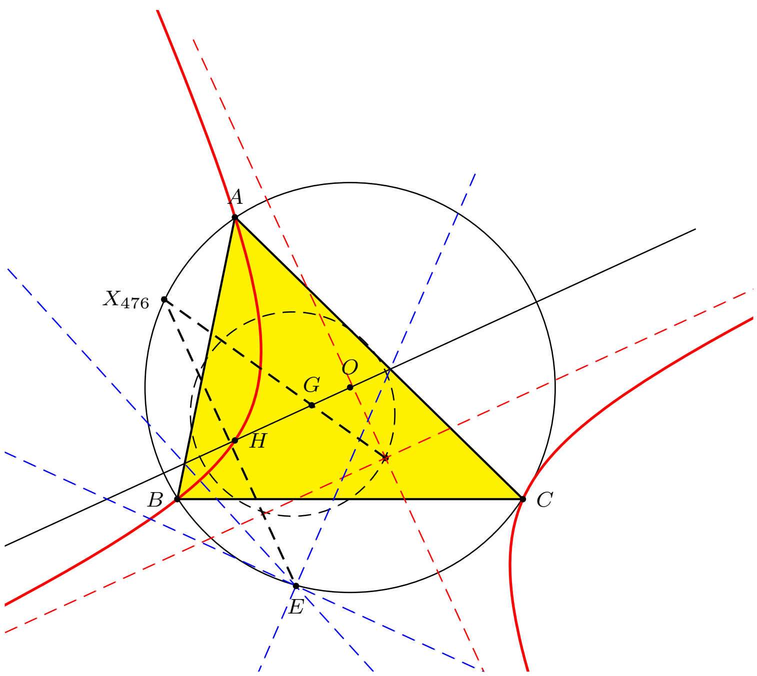


Figure 16.5: The Euler hyperbola

Its center is the point

$$X_{3258} = ((S_B-S_C)^2(S_A(S_B+S_C)-2S_{BC})(S^2-3S_{AA}) : \cdots : \cdots).$$

This is the reflection of the Jerabek center in the Euler line.

[4]The second infinite point is the infinite point of the orthic axis $S_Ax+S_By+S_Cz=0$. Its orthogonal infinite point is $(S_A(S_B-S_C)-S_B(S_C-S_A) : \cdots : \cdots) = (S_A(S_B+S_C)-2S_{BC} : \cdots : \cdots) = E_\infty$.

16.5.2 The circum-hyperbola with asymptotes parallel to the Brocard and Lemoine axes

This is the hyperbola

$$\boxed{\sum_{\text{cyclic}} \frac{a^4(S_B-S_C)(S_{AA}-S_{BC})}{x} = 0}$$

with center

$$\mathtt{X}(2679) = (a^2(S_B-S_C)^2(S_{AA}-S_{BC})(S_A(S_A+S_B+S_C)-(S_{BB}+S_{BC}+S_{CC})) : \cdots : \cdots$$

It contains the point $\mathtt{X}(32) = (a^4 : b^4 : c^4)$ on the Brocard axis, which is the isogonal conjugate of the Kiepert perspector $\mathtt{K}(-\omega)$.

16.5.3 The circum-hyperbola with asymptotes parallel and perpendicular to the OI line

This is the hyperbola

$$\boxed{\sum_{\text{cyclic}} \frac{a^2(b-c)(a^2(b+c)-2abc-(b+c)(b-c)^2)}{x} = 0}$$

with center

$$\mathtt{X}(3259) = ((b+c-2a)(b-c)^2(a^2(b+c)-2abc-(b+c)(b-c)^2) : \cdots : \cdots).$$

It contains the exsimilicenter $\mathtt{T}_-$ of the circumcircle and incircle.

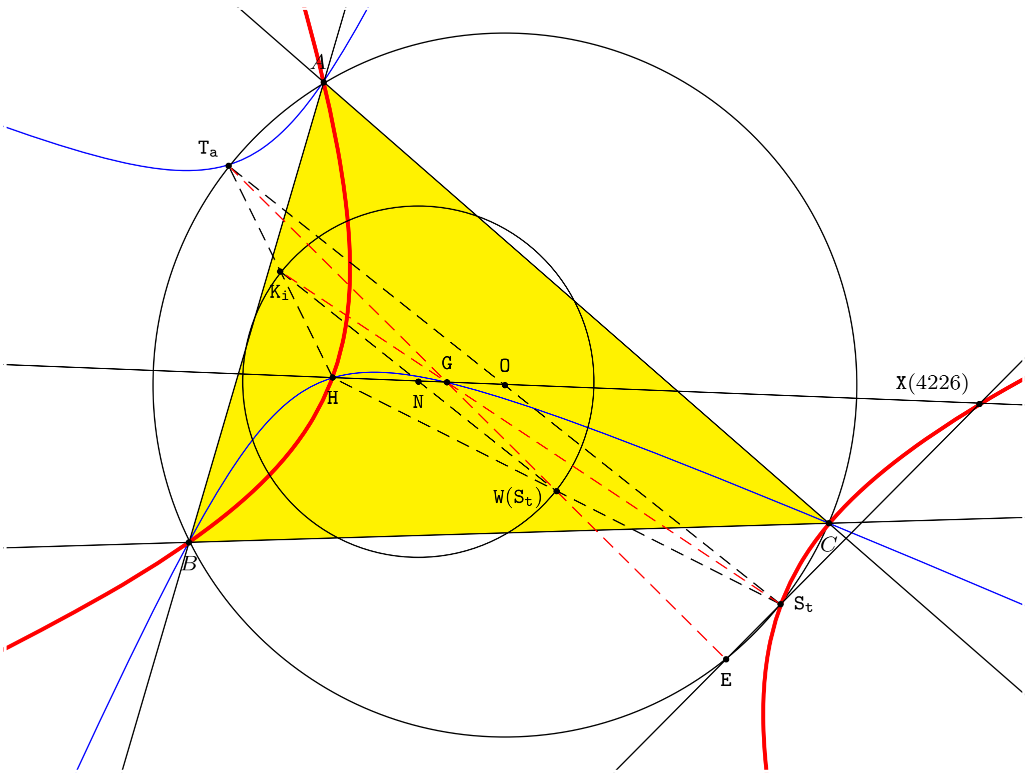


Figure 16.6: The rectangular circum-hyperbola through the Steiner poinrt

Example. Since the Steiner point $\mathtt{S_t} = \left(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2}\right)$ is the antipode of the Tarry point (the fourth intersection with the Kiepert hyperbola) on the circumcircle, the center of the rectangular circum-hyperbola

$$\mathscr{H}(\mathtt{S_t}) : \qquad \sum_{\text{cyclic}} \frac{2a^4 - a^2(b^2+c^2) + (b^2-c^2)^2}{x} = 0$$

is the antipode of the Kiepert center on the nine-point circle. This is the inferior of the Tarry point:

$$\mathtt{W}(\mathtt{S_t}) = \mathsf{inf}(\mathtt{T_a}) = \big((a^2(b^2+c^2)-(b^4+c^4))(2a^4-a^2(b^2+c^2)+(b^2-c^2)^2) : \cdots : \cdots$$

The hyperbola intersects the Euler line again at

$$\mathtt{X}(4226) = \left(\frac{2a^4-a^2(b^2+c^2)+(b^2-c^2)^2}{b^2-c^2} : \cdots : \cdots\right).$$

More generally, let P_1 and P_2 be antipodes on the circumcircle. The centers $Q_1 = W(P_1)$ and $Q_2 = W(P_2)$ of the hyperbolas $\mathscr{H}(P_1)$ and $\mathscr{H}(P_2)$ are antipodes on the nine-point circles, since these are the midpoints of the segments HP_1 and HP_2. Furthermore, $Q_1 = \mathsf{inf}(P_2)$ and $Q_2 = \mathsf{inf}(P_1)$.