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Geometry

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Problems

Problem 1. Let $ABCD$ be a cyclic quadrilateral satisfying
$$AD^2 + BC^2 = AB^2.$$
The diagonals of $ABCD$ intersect at E. Let P be a point on side AB satisfying $\sphericalangle APD = \sphericalangle BPC$. Show that line PE bisects CD.

Problem 2. Let ABC be a triangle and let M and N denote the midpoints of AB and AC, respectively. Let X be a point such that AX is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to MX, and by ω_C the circle through N and C tangent to NX. Show that ω_B and ω_C intersect on line BC.

Problem 3. Let $ABCD$ be a trapezoid with bases AB, CD and with circumscribed circle Ω. Let M be the midpoint of arc CD of Ω, which does not contain A. Consider circle ω with centre M and tangent to line AD. Let X be an intersection point of ω and CD. Prove that tangent to ω at X bisects segment AB.

Problem 4. Let $ABCD$ be a circumscribed quadrilateral with $BC = 2AB$. Suppose that perpendicular bisector of BC and bisector of angle DCB intersect at X. Prove that AX and BD are perpendicular.

Problem 5. Let $ABCDEF$ be a convex hexagon in which $AB = AF$, $BC = CD$, $DE = EF$ and $\sphericalangle ABC = \sphericalangle EFA = 90°$. Prove that $AD \perp CE$.

Problem 6. Let $ABCD$ be a cyclic quadrilateral with AB and CD not parallel. Let M be the midpoint of CD. Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD and the perpendicular bisector of MP are concurrent.

Problem 7. Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to BC. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of AE_1F_1 and AE_2F_2 are congruent, and the line through their centres is parallel to the tangent to ω at A.

Problem 8. Let $ABCD$ be a trapezoid with the bases AD and BC. Let E and F be points on the segments AB and CD, respectively. Circumcircle of the triangle AEF intersects segment AD at point A_1. Circumcircle of the triangle CEF intersects segment BC at point C_1. Prove that A_1C_1, BD and EF concur.

Problem 9. Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of AB. Ray AI meets BC at D. Denote by ω and γ the circumcircles of BIC and BAD, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside ABC and $\measuredangle AQM = \measuredangle ACB$.
Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\measuredangle BAC \neq 60°$, prove that these four lines are concurrent on Γ.

Problem 10. Let $ABCD$ be a cyclic quadrilateral an let P be a point on the side AB. The diagonals AC meets the segments DP at Q. The line through P parallel to CD meets the extension of the side CB beyond B at K. The line through Q parallel to BD meets the extension of the side CB beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent.

Problem 11. Two circles, ω_1 and ω_2, centred at O_1 and O_2, respectively, meet at points A and B. A line through B meet ω_1 again at C, and ω_2 again at D. The tangents to ω_1 and ω_2 at C and D, respectively, meet at E, and the line AE meets the circle ω through A, O_1, O_2 again at F. Prove that the length of the segment EF is equal to the diameter of ω.

Problem 12. Let $ABCD$ be a quadrilateral inscribed in circle ω with $AC \perp BD$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of EPD meets ω at D and Q, and the circumcircle of FPD meets ω at D and R. Show that $EQ = FR$.

Problem 13. Let $ABCD$ be a circumscribed quadrilateral, and let I be its incircle. Points P and Q lies on segments AI and CI, respectively such that $\sphericalangle PBQ = \frac{1}{2}\sphericalangle ABC$. Prove that $\sphericalangle QDP = \frac{1}{2}\sphericalangle CDA$.

Problem 14. Let ABC be an acute triangle with orthocentre H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcentre of AEF, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to BC meet on Γ.

Problem 15. Let ABC be a triangle with incentre I. Points K and L are chosen on segment BC such that the incircles of ABK and ABL are tangent at P, and the incircles of ACK and ACL are tangent at Q. Prove that $IP = IQ$.

Problem 16. In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of ABE intersects line CB at B and P, and the circumcircle of ADE intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\sphericalangle MAC = 90°$.

Problem 17. The trapezoid $ABCD$ is inscribed in the circle ω ($AD \parallel BC$). The circles inscribed in the triangles ABC and ABD touch the base of the trapezoid BC and AD at points P and Q respectively. Points X and Y are the midpoints of the arcs BC and AD of circle ω that do not contain points A and B respectively. Prove that lines XP and YQ intersect on the circle ω.

Problem 18. Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\sphericalangle PXM = \sphericalangle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

Problem 19. Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

Problem 20. Let ABC be a triangle with $\sphericalangle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

Problem 21. In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

Problem 22. An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in AB$, $Q \in AC$, and $N, P \in BC$. Let S be the intersection of MN and PQ. Denote by ℓ the angle bisector of $\sphericalangle MSQ$. Prove that OI is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

Problem 23. There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$$\sphericalangle PAB + \sphericalangle PDC = \sphericalangle PBC + \sphericalangle PAD = \\ = \sphericalangle PCD + \sphericalangle PBA = \sphericalangle PDA + \sphericalangle PCB = 90°$$

if and only if the diagonals AC and BD are perpendicular.

Problem 24. Let the excircle of a triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Denote by γ the circumcircle of triangle $A_1B_1C_1$ and assume that γ passes through vertex A.

(a) Show that AA_1 is a diameter of γ.
(b) Show that the incenter of triangle ABC lies on line B_1C_1.

Problem 25. Let ABC be a triangle with altitude AE. The A-excircle touches BC at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

Problem 26. In triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\measuredangle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\measuredangle XSY = \measuredangle XTY = 90°$. Finally, let γ be the circumcircle of AST. Prove that γ is tangent to the circumcircle and incircle of ABC.

Problem 27. Let ω be the circumcircle of isosceles triangle ABC ($AB = AC$). Points P and Q lie on ω and BC respectively such that $AP = AQ$. Suppose AP and BC intersect at R. Prove that the tangents from B and C to the incircle of triangle AQR (different from BC) are concurrent on ω.

Problem 28. Let P be a point in the interior of quadrilateral $ABCD$ such that:

$$\measuredangle BPC = 2\measuredangle BAC, \quad \measuredangle PCA = \measuredangle PAD, \quad \measuredangle PDA = \measuredangle PAC$$

Prove that:

$$\measuredangle PBD = \left|\measuredangle BCA - \measuredangle PCA\right|.$$

Solutions

Problem 1 (USAMO 2019)**.** Let $ABCD$ be a cyclic quadrilateral satisfying

$$AD^2 + BC^2 = AB^2.$$

The diagonals of $ABCD$ intersect at E. Let P be a point on side AB satisfying $\sphericalangle APD = \sphericalangle BPC$. Show that line PE bisects CD.

Proof. Since $AD^2 + BC^2 = AB^2$, there exists a point P on AB satisfying

$$AD^2 = AP \cdot AB \quad \text{and} \quad BC^2 = BP \cdot BA.$$

Thus $APD \sim ADB$ and $BPC \sim BCA$, so

$$\sphericalangle APD = \sphericalangle BDA = \sphericalangle BCA = \sphericalangle CPB$$

as desired.

Now we prove that line PE bisects CD. Define $K := AC \cap PD$ and $L := BD \cap PC$. From earlier, $\sphericalangle CPB = \sphericalangle BDA$, so $APLD$ is cyclic. Likewise, $BPKC$ is cyclic.

Now the quadrilateral $AKLB$ is also cyclic, because

$$\sphericalangle AKB = \sphericalangle AKP + \sphericalangle PKB = \sphericalangle CBP + \sphericalangle PCB = \sphericalangle CPB$$

and similarly $\sphericalangle ALB = \sphericalangle APD$.

Hence by **Reim's theorem**, $CD \parallel KL$. Thus $CDKL$ is a trapezoid whose bases intersect at P and whose diagonals intersect at E, so line PE bisects the bases CD and KL, as desired. □

Another solution. As in the first solution, we find that

$$AD^2 = AP \cdot AB \quad \text{and} \quad BC^2 = BP \cdot BA.$$

In order to prove that PE is a median of triangle CDE it is enough to prove that PE is E-**symmedian** of triangle AEB. To do this, we need to check that

$$\frac{AP}{BP} = \frac{AE^2}{BE^2}.$$

But since triangles DEA and BEC are similar we have that

$$\frac{AE}{BE} = \frac{AD}{BC},$$

hence

$$\frac{AE^2}{BE^2} = \frac{AD^2}{BC^2} = \frac{AP \cdot AB}{BP \cdot BA} = \frac{AP}{BP}.$$

□

Another solution. By hypothesis, the circle ω_a centred at A with radius AD is orthogonal to the circle ω_b centred at B with radius BC. For brevity, we let $\mathbf{I}_a$ and $\mathbf{I}_b$ denote **inversion** with respect to ω_a and ω_b. We let P denote the intersection of AB with the radical axis of ω_a and ω_b; hence $P = \mathbf{I}_a(B) = \mathbf{I}_b(A)$. This already implies that

$$\measuredangle DPA \overset{\mathbf{I}_a}{=} \measuredangle ADB = \measuredangle ACB \overset{\mathbf{I}_b}{=} \measuredangle BPC$$

so P satisfies the angle condition.

Lemma 1. *The point $K = \mathbf{I}_a(C)$ lies on ω_b and DP. Similarly $L = \mathbf{I}_b(D)$ lies on ω_a and CP.*

Proof. The first assertion follows from the fact that ω_b is orthogonal to ω_a. For the other, since (BCD) passes through A, it follows $P = \mathbf{I}_a(B)$, $K = \mathbf{I}_a(C)$, and $D = \mathbf{I}_a(D)$ are collinear. □

Finally, since C, L, P are collinear, we get A is concyclic with $K = \mathbf{I}_a(C)$, $L = \mathbf{I}_a(L)$, $B = \mathbf{I}_a(B)$, i.e. that $AKLB$ is cyclic. So $KL \parallel CD$ by **Reim's theorem,** and hence PE bisects CD by **Ceva's theorem.** □

Discussion. https://artofproblemsolving.com/community/c5h1823553p12189455

Problem 2 (USA TST 2019)**.** Let ABC be a triangle and let M and N denote the midpoints of AB and AC, respectively. Let X be a point such that AX is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to MX, and by ω_C the circle through N and C tangent to NX. Show that ω_B and ω_C intersect on line BC.

Proof. Let XY be the other tangent from X to (AMN).

Lemma 2. *Line XM is tangent to (BMY); hence Y lies on ω_B.*

Proof. Let Z be the midpoint of AY. Then MX is the M-**symmedian** in triangle AMY. Since $MZ \parallel BY$, it follows that

$$\measuredangle AMX = \measuredangle ZMY = \measuredangle BYM.$$

We conclude that XM is tangent to the circumcircle of triangle BMY. □

Similarly, ω_C is the circumcircle of triangle CNY. As $AMYN$ is cyclic too, it follows that ω_B and ω_C intersect on BC, by **Miquel's theorem**. □

Another proof. Let Y be the **isogonal conjugate** of X in triangle AMN and Z be the reflection of Y in MN. As AX is tangent to the circumcircle of AMN, it follows that $AY \parallel MN$. Thus Z lies on BC since MN bisects the strip made by AY and BC.

Finally,

$$\measuredangle ZMX = \measuredangle ZMN + \measuredangle NMX = \measuredangle NMY + \measuredangle YMA = \measuredangle NMA = \measuredangle ZBM$$

so XM is tangent to the circumcircle of triangle ZMB, hence Z lies on ω_B. Similarly, Z lies on ω_C and we're done. □

Discussion. https://artofproblemsolving.com/community/c6h1751587p11419585

Problem 3. (Polish MO 2020) Let $ABCD$ be a trapezoid with bases AB, CD and with circumscribed circle Ω. Let M be the midpoint of arc CD of Ω, which does not contain A. Consider circle ω with centre M and tangent to line AD. Let X be an intersection point of ω and CD. Prove that tangent to ω at X bisects segment AB.

Proof. Notice that M is a midpoint of arc DC of Ω, so MB is bisector of angle CBD. Moreover ω is tangent to AD i BC, thus M incentre of triangle bounded by lines AD, BD i BC. In particular ω is tangent to BD – denote the point of tangency by K.

Let F be a tangency point of ω and AD, and let N be a midpoint of AB.

Lemma 3. *F, K and N are collinear.*

Proof. From **Menelaus theorem** for ADB and points F, K and N we see that it is enough to prove that

$$\frac{AN}{NB} \cdot \frac{BK}{DK} \cdot \frac{FD}{FA} = 1. \tag{1}$$

But $AN = NB$ (since N is midpoint of AB) and $FD = DK$, because they are tangents from D to ω.

Therefore (1) is equivalent to $AF = KB$. But MN is symmetry axis of $ABCD$ and ω. Therefore tangent segments to ω from points A and B are equal, so $AF = BK$. □

Proof. Note that points F, K and N are foots of M to lines AD, DB and AB, respectively. Therefore due to **Simson line theorem** applied for ADB and M which lies on Ω, we are done. □

From the above claim we see that N lies on **polar** FK of D wrt ω. Therefore by **La Hire theorem** D lies on polar of N wrt ω. But $MN \perp CD$, then CD is polar of N wrt ω. Therefore point X, which is a common point of ω and polar of N wrt ω, lies on tangent line from N to ω.

□

Discussion. https://om.mimuw.edu.pl/static/app_main/problems/om71_1r.pdf

Problem 4. (Polish MO 2020) Let $ABCD$ be a circumscribed quadrilateral with $BC = 2AB$. Suppose that perpendicular bisector of BC and bisector of angle DCB intersect at X. Prove that AX and BD are perpendicular.

Proof. Let M be a midpoint of BC and take N on segment CD such that $CN = CM$. Then $AB = BM = MC = CN$. Moreover $AB + CD = BC + AD$, by *described quadrilateral theorem*, so

$$2 \cdot AB + DN = AB + CN + DN = AB + CD = BC + AD = 2 \cdot AB + AD,$$

hence $AD = DN$.

Take circles $\odot(B, AB)$, $\odot(C, AB)$, $\odot(D, AD)$. From the above, circles in pairs $(\odot(B, AB), \odot(C, AB))$ and $(\odot(C, AB), \odot(D, AD))$ are externally tangent.

From the **radical axis theorem** if follows that X is a radical centre of $\odot(B, AB)$, $\odot(C, AB)$ and $\odot(D, AD)$. In particular AX is radical axis of $\odot(B, AB)$ and $\odot(D, AD)$, hence we are done. □

Discussion. https://om.mimuw.edu.pl/static/app_main/problems/om71_1r.pdf

Problem 5. (Baltic Way 2019) Let $ABCDEF$ be a convex hexagon in which $AB = AF$, $BC = CD$, $DE = EF$ and $\sphericalangle ABC = \sphericalangle EFA = 90^\circ$. Prove that $AD \perp CE$.

Proof. Draw circles $\odot(C, CD)$ and $\odot(E, ED)$. Clearly AB, AF are tangents, so AD is the **radical axis** of these circles. Hence we are done. □

Discussion. https://artofproblemsolving.com/community/c6h1954642p13500917

Problem 6. (Baltic Way 2016) Let $ABCD$ be a cyclic quadrilateral with AB and CD not parallel. Let M be the midpoint of CD. Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD and the perpendicular bisector of MP are concurrent.

Proof. Let $Q = AB \cap CD$. Note that $QA \cdot QB = QC \cdot QD$, so the power of Q to the circle centred at P with radius $PA = PB$ is equal to the power of Q to the circle centred at M with radius $MC = MD$. Since these circles are congruent and Q lies on their **radical axis**, Q lies on the perpendicular bisector of their centres, as desired. □

Discussion. https://artofproblemsolving.com/community/c6h1334580p7212380

Problem 7 (USA TST 2018)**.** Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to BC. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of AE_1F_1 and AE_2F_2 are congruent, and the line through their centres is parallel to the tangent to ω at A.

Proof. Let O be the centre of ω, and let $M = PQ \cap AB$ and $N = PQ \cap AC$ be the midpoints of AB and AC respectively.

The main idea is to prove two key claims involving O, which imply the result:

(i) quadrilaterals AOE_1F_1 and AOE_2F_2 are cyclic (giving the **radical axis** is AO),

(ii) $OE_1F_1 \cong OE_2F_2$ (giving the congruence of the circles).

We first note that above conditions are equivalent. Indeed, because $OP = OQ$, (ii) is equivalent to $OE_1F_1 \sim OE_2F_2$, and then by the **spiral similarity** we have (i) $\iff$ (ii).

- Proof of (i): Since P, M, N are collinear, we see that PMN is the **Simson line** of O with respect to AE_1F_1.
- Proof of (ii): By **butterfly theorem** on the three chords AC, PQ, PQ, it follows that $E_1N = NE_2$. Thus

  $$E_1P = \sqrt{E_1A \cdot E_1C} = \sqrt{E_2A \cdot E_2C} = E_2P.$$

  But also $OP = OQ$ and hence $OPE_1 \cong OQE_2$. Similarly for the other pair.
- Second proof of (ii): Let $T = PP \cap QQ$. Let S be on PQ with $ST \parallel AC$; then $TS \perp ON$, and it follows ST is the **polar** of N (it passes through T by **La Hire's theorem**).

  Now,

  $$-1 = (P, Q; N, T) \stackrel{T}{=} (E_1, E_2; N, \infty)$$

  with $\infty := AC \cap ST$ the point at infinity. Hence $E_1N = NE_2$ and we can proceed as in the previous solution.

$\square$

Discussion. https://artofproblemsolving.com/community/c6h1664170p10571000

Problem 8 (Kvant, RMM shortlist 2007)**.** Let $ABCD$ be a trapezoid with the bases AD and BC. Let E and F be points on the segments AB and CD, respectively. Circumcircle of the triangle AEF intersects segment AD at point A_1. Circumcircle of the triangle CEF intersects segment BC at point C_1. Prove that A_1C_1, BD and EF concur.

Proof. Let T be a common point of circumcircle of triangle BC_1E and line EF. Observe that

$$\measuredangle TC_1B = \measuredangle TEA = \measuredangle FEA = \measuredangle FA_1D.$$

Therefore $C_1T \parallel FA_1$. Similarly

$$\measuredangle TBC_1 = \measuredangle TEC_1 = \measuredangle FEC_1 = 180^\circ - \measuredangle FCC_1 = \measuredangle FDA_1,$$

hence $TB \parallel FD$. Since $BC \parallel A$ we see that triangles BC_1T and DA_1F are **homothetic**, and the centre of this homothety is the intersection point of A_1C_1, BD and EF. $\square$

Discussion. https://kvant.ras.ru/pdf/2019//2019-11.pdf, https://artofproblemsolving.com/community/c6h1610575p10055819

Problem 9 (USA TSTST 2018)**.** Let ABC be an acute triangle with incentre I, circumcentre O, and circumcircle Γ. Let M be the midpoint of AB. Ray AI meets BC at D. Denote by ω and γ the circumcircles of BIC and BAD, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside ABC and $\measuredangle AQM = \measuredangle ACB$.
Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\measuredangle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ.

Proof. Henceforth assume $\measuredangle A \neq 60^\circ$; we prove the concurrence. Let L denote the centre of ω, which is the midpoint of minor arc BC.

Lemma 4. *Let K be the point on ω such that $KL \parallel AB$ and $KC \parallel AL$. Then KA is tangent to γ, and we may put*

$$x = KA = LB = LC = LX = LY = KX = KY.$$

Proof. By construction, $KA = LB = LC$. Also, MO is the perpendicular bisector of KL (since the chords KL, AB of ω are parallel) and so $KXLY$ is a rhombus as well. Moreover, KA is tangent to γ as well since

$$\measuredangle KAD = \measuredangle KAL = \measuredangle KAC + \measuredangle CAL = \measuredangle KBC + \measuredangle ABK = \measuredangle ABC.$$

□

Up to now we have not used the existence of Q; we henceforth do so. Note that $Q \neq O$, since $\measuredangle A \neq 60^\circ \implies O \notin \omega$. Moreover, we have $\measuredangle AOM = \measuredangle ACB$ too. Since O and Q both lie inside $\triangle ABC$, this implies that A, M, O, Q are concyclic. As $Q \neq O$ we conclude $\measuredangle CQA = 90^\circ$.

Lemma 5. *Assuming Q exists, the rhombus $LXKY$ is a square. In particular, KX and KY are tangent to ω.*

Proof. Observe that Q lies on the circle with diameter AC, centered at N, say. This means that O lies on the radical axis of ω and (N), hence $NL \perp CO$ implying

$$NO^2 + CL^2 = NC^2 + LO^2 = NC^2 + OC^2 = NC^2 + NO^2 + NC^2$$
$$\implies x^2 = 2NC^2 \implies x = \sqrt{2}NC = \frac{1}{\sqrt{2}}AC = \frac{1}{\sqrt{2}}LK.$$

So $LXKY$ is a rhombus with $LK = \sqrt{2}x$. Hence it is a square. □

We finish by proving that

$$KD = KA$$

and hence line KD is tangent to γ. Let $E = BC \cap KL$. Then

$$LE \cdot LK = LC^2 = LX^2 = \frac{1}{2}LK^2$$

and so E is the midpoint of LK. Thus $MXOY$, BC, KL are concurrent at E. As $DL \parallel KC$, we find that $DLCK$ is a parallelogram, so $KD = CL = KA$ as well. Thus KD and KA are tangent to γ.

□

Discussion. https://artofproblemsolving.com/community/c6h1664164p10570988

Problem 10 (RMM 2018)**.** Let $ABCD$ be a cyclic quadrilateral an let P be a point on the side AB. The diagonals AC meets the segments DP at Q. The line through P parallel to CD meets the extension of the side CB beyond B at K. The line through Q parallel to BD meets the extension of the side CB beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent.

Proof. Denote by ω the circumcircle of $ABCD$. Let $T := DQ \cap \omega$. By converse of **Reim's Theorem** on the parallel lines $PK \parallel CD$ and circle ω we have that $BDTK$ is cyclic. By converse of **Reim's Theorem** on the parallel lines $LQ \parallel BD$ and circle ω we have that $CQTL$ is cyclic. Now because $\measuredangle ACT = \measuredangle ABT$ we have that the lines tangent to the circumcircles of QCT and BDT at T coincide, thus the circumcircles of the triangles BKP and CLQ are tangent at T. □

Discussion. https://artofproblemsolving.com/community/c6h1597669p9926981, https://kvant.ras.ru/pdf/2018/2018-07.pdf

Problem 11 (RMM 2016)**.** Two circles, ω_1 and ω_2, centred at O_1 and O_2, respectively, meet at points A and B. A line through B meet ω_1 again at C, and ω_2 again at D. The tangents to ω_1 and ω_2 at C and D, respectively, meet at E, and the line AE meets the circle ω through A, O_1, O_2 again at F. Prove that the length of the segment EF is equal to the diameter of ω.

Proof. Notice that since

$$\measuredangle ADO_2 = 90^\circ - \measuredangle EDA = 90^\circ - \measuredangle DBA = \measuredangle ABC - 90^\circ = \measuredangle ACE - 90^\circ = \measuredangle ACO_1,$$

the **spiral similarity** mapping O_1C to O_2B is centred at A, which implies that there is another **spiral similarity** also centred at A sending O_1O_2 to CD because spiral similarities come in pairs. As a result, we find that O_1C, O_2D, ω, and (ACD) concur at a point F'. Now it suffices to show that $\measuredangle F'AF = \measuredangle F'AE = 90^\circ$, which also follows from proving that $ACED$ is cyclic due to the fact that CE is tangent to ω_1. However, this is just angle chasing:

$$\measuredangle DEC = \measuredangle DCE + \measuredangle EDC = \measuredangle BAC + \measuredangle DAB = \measuredangle DAC$$

as desired. □

Discussion. https://artofproblemsolving.com/community/c6h1538017p9289488

Problem 12 (USAJMO 2018)**.** Let $ABCD$ be a quadrilateral inscribed in circle ω with $AC \perp BD$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of EPD meets ω at D and Q, and the circumcircle of FPD meets ω at D and R. Show that $EQ = FR$.

Proof. Let X, Y, be the feet from D to BA, BC, and let $Z := BD \cap AC$. By **Simson theorem,** the points X, Y, Z are collinear. Consequently, the point P is the reflection of D over Z, and so we conclude P is the orthocentre of ABC.

Suppose now we extend ray CP to meet ω again at Q'. Then BA is the perpendicular bisector of both PQ' and DE; consequently, $PQ'ED$ is an isosceles trapezoid. In particular, it is cyclic, and so $Q' = Q$. In the same way R is the second intersection of ray AP with ω.

Now, because of the two isosceles trapezoids we have found, we conclude

$$EQ = PD = FR$$

as desired. □

Discussion. https://artofproblemsolving.com/community/c5h1629606p10226149

Problem 13. Let $ABCD$ be a circumscribed quadrilateral, and let I be its incircle. Points P and Q lies on segments AI and CI, respectively such that $\measuredangle PBQ = \frac{1}{2}\measuredangle ABC$. Prove that $\measuredangle QDP = \frac{1}{2}\measuredangle CDA$.

Proof. TBA □

Discussion. https://kvant.ras.ru/pdf/2019/2019-02.pdf

Problem 14 (USA TSTST 2019)**.** Let ABC be an acute triangle with orthocentre H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcentre of triangle AEF, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to BC meet on Γ.

Proof. Let T be the point on Γ such that $DT \perp BC$, and let $Q = TH \cap \Gamma$.

Lemma 6. *Quadrilaterals $BEHQ$ and $CFHQ$ cyclic.*

Proof. Angle chasing:

$$\measuredangle BQH = \measuredangle BDT = 90^\circ - \measuredangle DBC = 90^\circ - \measuredangle KAF = \measuredangle AEF.$$

A similar angle chase proves $CFHQ$ cyclic. □

Now QHT is the **radical axis** of $(BEHQ)$ and $(CFHQ)$. To finish, observe that

$$\measuredangle KEH = 90^\circ - \measuredangle A = \measuredangle EBH,$$

which implies KE tangent to $(BEHQ)$; similarly, KF is tangent to $(CFHQ)$. Thus, since $KE = KF$, K lies on the **radical axis**, as desired. □

Another solution. Define H_B, H_C as the reflections of H over AC, AB, respectively and let D_A be the reflection of D over BC.

Lemma 7. *Lines H_BE and H_CF concur at D.*

Proof. By **Pascal theorem,** they concur at a point $D' \in \Gamma$. Define $A' = AD' \cap \odot(AEF)$. Observe that

$$\begin{aligned}\sphericalangle AFA' &= \sphericalangle EFA' + \sphericalangle AFE\\ &= \sphericalangle EAA' + (90^\circ - \sphericalangle FHH_C)\\ &= 90^\circ + \sphericalangle D'AC - \sphericalangle D'H_CC\\ &= 90^\circ,\end{aligned}$$

so AA' is a diameter and $D = D'$, as desired. □

Lemma 8. $D_A \in EF$.

Proof. Let D_B be the reflection of D over AC. Then

$$\sphericalangle CED_B = \sphericalangle DEC = \sphericalangle AEH_B = \sphericalangle HEA,$$

so $D_B \in EF$. By **Steiner theorem** H, D_A, D_B are collinear. Hence the claim follows. □

Define $Y = BC \cap DD_A$ and $X = DD' \cap \Gamma$, and then X' as the reflection of X over BC. Note that AXD_AH is a parallelogram, so $AH = XD_A = DX'$ and $AHX'D$ is also a parallelogram. Now we have equality of **crossratios**:

$$1 = (A, A'; D, EF \cap AA') \stackrel{H}{=} (\infty, HA' \cap XD; D, D_A)$$

but since Y is the midpoint of DD_A we must have H, A', Y collinear. Finally, we have

$$1 = (X, X'; Y, \infty) \stackrel{H}{=} (HX \cap AA', \infty; A', A)$$

so HX bisects AA', which is what we needed to prove. □

Discussion. https://artofproblemsolving.com/community/c6h1863135p12608496

Problem 15 (USA TSTST 2019)**.** Let ABC be a triangle with incentre I. Points K and L are chosen on segment BC such that the incircles of ABK and ABL are tangent at P, and the incircles of ACK and ACL are tangent at Q. Prove that $IP = IQ$.

Proof. Let us denote by ω_1, ω_2, γ_1, γ_2 incircles of triangles ABK, ACL, ABL and ACK, respectively. We first prove the following general lemma:

Lemma 9. *For any points K and L on BC of triangle ABC, if I_B, J_B, I_C, and J_C denote the incentres of ω_1, γ_1, ω_2 and γ_2, respectively. Then I_BI_C, J_BJ_C, and BC concur at a point T (possibly at infinity).*

Proof. We can easily determine the equality of **crossratios**:

$$A(B, I_B; I, J_B) = \frac{\sin \frac{1}{2}\sphericalangle BAC \cdot \sin \frac{1}{2}\sphericalangle KAL}{\sin \frac{1}{2}\sphericalangle KAC \cdot \sin \frac{1}{2}\sphericalangle BAL} = A(C, I_C; I, J_C),$$

and the result follows. □

Another solution. Z Kvanta □

Now, let P and Q be a points from the problem. Using the above lemma, let us denote by $R \in BC$ the common exsimilarity centre of pairs of circles (ω_1, ω_2) and (γ_1, γ_2).

We are now ready to complete the proof. Since point R is the exsimilicentre of the incircles of ABK and ACL, then

$$\frac{PI_B}{RI_B} = \frac{QJ_C}{RJ_C}.$$

Now by **Menelaus theorem**,

$$\frac{I_B P}{PI} \cdot \frac{IQ}{QJ_C} \cdot \frac{J_C R}{RI_B} = 1 \implies IP = IQ.$$

□

Another solution. We start with the following lemmas:

Lemma 10. *There exists an* ***inversion*** *ι at R swapping $\{\omega_1, \omega_2\}$ and $\{\gamma_1, \gamma_2\}$.*

Proof. Consider the **inversion** at R swapping ω_1 and ω_2. Since ω_1 and γ_1 are tangent, the image of γ_1 is tangent to ω_2 and is also tangent to BC. The circle γ_2 is on the correct side of γ_1 to be this image. □

Lemma 11. *Circles ω_1, ω_2, γ_1, γ_2 share a common* ***radical centre.***

Proof. Let Ω be the circle with centre R fixed under ι, and let k be the circle through P centred at the radical centre of Ω, ω_1, γ_1. Then k is actually orthogonal to Ω, ω_1, γ_1, so k is fixed under ι and k is also orthogonal to ω_2 and γ_2. Thus the centre of k is the desired radical centre. □

The desired statement immediately follows. Indeed, letting S be the radical centre, it follows that SP and SQ are the common internal tangents to $\{\omega_1, \gamma_1\}$ and $\{\omega_2, \gamma_2\}$. Since S is the radical centre, $SP = SQ$. In light of $\measuredangle SPI = \measuredangle SQI = 90°$, it follows that $IP = IQ$, as desired. □

Discussion. https://artofproblemsolving.com/community/c6h1863131p12608472, https://kvant.ras.ru/pdf/2012/2012-01-b.pdf

Problem 16 (USAMO 2018)**.** In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of ABE intersects line CB at B and P, and the circumcircle of ADE intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\measuredangle MAC = 90°$.

Proof. We start with the following lemma:

Lemma 12. *The self-intersecting quadrilateral $PQDB$ is cyclic.*

Proof. By **power of a point** from C:

$$CQ \cdot CD = CA \cdot CE = CB \cdot CP.$$

□

Lemma 13. *Point E lies on line PQ.*

Proof. We have

$$\measuredangle AEP = \measuredangle ABP = \measuredangle ABC = \measuredangle ADC = \measuredangle ADQ = \measuredangle AEQ.$$

□

Therefore A is the **Miquel point** of cyclic quadrilateral PBQD. To finish, let $H := PD \cap BQ$. By properties of the **Miquel point**, we have A is the foot from H to CE. But also, points M, A, H are collinear by **Pappus theorem** on BPG and DQF, as desired. □

Discussion. https://artofproblemsolving.com/community/c5h1630185p10232392

Problem 17 (Oral Moscow Geometry Olympiad 2013)**.** The trapezoid $ABCD$ is inscribed in the circle ω ($AD \parallel BC$). The circles inscribed in the triangles ABC and ABD touch the base of the trapezoid BC and AD at points P and Q respectively. Points X and Y are the midpoints of the arcs BC and AD of circle ω that do not contain points A and B respectively. Prove that lines XP and YQ intersect on the circle ω.

Proof. Consider circle Ω tangent to BC and AD and P' and Q' and to circle ω at Z. By the **Sawayama lemma** $I, J \in P'Q'$. Since $AD \parallel BC$ we see that $P = P'$ and $Q = Q'$. Therefore XP and YQ intersect at point $Z \in \omega$ by **shooting lemma**. □

Discussion. http://olympiads.mccme.ru/ustn/resh13ge.pdf, http://kvant.mccme.ru/pdf/2008/2008-04.pdf

Problem 18 (IMO shortlist 2018)**.** Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\measuredangle PXM = \measuredangle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

Proof. Let $S = XM \cap PY$, $T = YM \cap PX$, so that quadrilateral $XYST$ is cyclic by the given angles. We denote by O the centre of this circle.

Lemma 14. *We have $OM \perp BC$. Hence $PA \perp AOM$.*

Proof. This follows by **butterfly theorem** if one extends BC to a chord of the circle, since then M is the midpoint of that chord. □

Thus, since $M \in OA$ and $PA \perp AOM$, we conclude A coincides with the **Miquel point** of quadrilateral $STXY$ (**EXERCISE**). Therefore $PAXY$ is cyclic. □

Another solution. We **invert** about P. The new problem is as follows:

Refolmulation. Let PBC be a triangle and Γ its circumcircle. The P-**Apollonius circle** intersects Γ again at M and the tangent to Γ at P again at A. Points X and Y lie on segments PB and PC respectively so that $\measuredangle PMX = \measuredangle PMY$. Show that A, X, Y are collinear.

Proof. Let $Z = PM \cap AX$ and $Y' = AX \cap PC$. Note that $\measuredangle PMA = 90°$ by the **inversion**. Since MP bisects $\measuredangle XMY$, by the **crossratio lemma** we have

$$(MA, MZ; MX, MY) = -1 = (P, M; B, C) \overset{P}{=} (A, Z; X, Y').$$

Therefore $Y = Y'$, which solves the problem. □

□

Another solution. Let Q be the point on AM such that $\measuredangle QXB = 90°$. Notice the cylic quadrilateral $BMQX$, thus

$$\measuredangle CYM = \measuredangle BXM = \measuredangle BQM = \measuredangle CQM \implies CYQM \text{ concyclic.}$$

Thus $\measuredangle CYQ = 90°$ or A, X, Y, P, Q lies on circle with diameter PQ. □

Discussion. https://artofproblemsolving.com/community/u53544h1876755p127531 06

Problem 19 (IMO shortlist 2002)**.** Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

Proof. We use the following lemma:

Lemma 15. *Given two circles ω_1 and ω_2 and two points E, F lying on the two different common external tangents of the circles, let the second tangents from E, F to ω_1 intersect at P, and similarly Q for ω_2. Then PQ passes through the insimilicenter $-(\omega_1, \omega_2)$ of the two given circles.*

Proof. Indeed, by **Monge's Theorem,** it suffices to check that there exists a circle tangent to the four lines FP, FQ, EP, EQ, which is just a matter of segment length chasing (**EXERCISE**). □

Back to the main problem:

- First suppose that such a point F exists. Let the tangency points of AB, CD with the respective inscribed circles be R, S. Then by **Brianchon theorem** (**EXERCISE**), R and S both lie on PQ, where P, Q are defined similarly as in the lemma 15. But $-(\omega_1, \omega_2) \in PQ$, i.e. R, S are mapped to one another by the internal homothety, hence $AB \parallel CD$.
- Conversely, if $AB \parallel CD$, let S_1 and S_2 be tangency points of ω_1 and ω_2 with AB and CD, respectively. Then the homothety which sends AB to CD, sends S_1 to S_2, so $P := AC \cap BD$, S_1, S_2 are collinear. Let $K := AE \cap S_1S_2$, $L = DE \cap S_1S_2$, then by converse to the **Pappus theorem** we see that

BK and CL intersects on AD. Call this point F. From converse to the **Brianchon theorem** for pentagons DS_2CEKF and $LEBS_1AF$ we get that F satisfies problem assumptions (**EXERCISE**).

□

Discussion. https://artofproblemsolving.com/community/c6h546183p3160591

Problem 20 (IMO shortlist 2015)**.** Let ABC be a triangle with $\measuredangle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

Proof. Let $K = AD \cap CQ$. We claim that triangles ABC and DBQ are similar. Indeed, $\measuredangle BCA = \measuredangle BQD = 90°$ and since $PDQ \sim PQB$ and CH bisects AD, we have

$$\frac{DQ^2}{BQ^2} = \frac{PQ^2}{PB^2} = \frac{PD}{PB} = \frac{AH}{BH} = \tan^2 B = \frac{AC^2}{BC^2},$$

and our claim holds. Thus B is the centre of the **spiral similarity** which sends AD to CQ, so $\measuredangle KDB = \measuredangle KQB$, and K lies on ω. □

Discussion. https://artofproblemsolving.com/community/c6h1268851p6622204

Problem 21 (IMO shortlist 2017)**.** In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

Proof. Let I_A be the centre of ω, T is the second intersection of AD and ω. Let N be the midpoint of DT. Since $\measuredangle AEI_A = \measuredangle AFI_A = \measuredangle ANI_A = 90°$, we get that $I_A, N \in \odot(AEF)$.

Now we claim that $MPQT$ is cyclic. Indeed, simply observe that from **power of a point** we have

$$DP \cdot DQ = DA \cdot DN = DM \cdot DT.$$

Now let R be the point on BC such that RT is tangent to ω. Since R is the **pole** of AD w.r.t. ω, R must lie on EF by **La Hire's Theorem.** Hence, again by **power of a point** we get

$$RP \cdot RQ = RE \cdot RF = RT^2,$$

so RT also touches $\odot(MPQ)$ hence we are done. □

Another solution. Let I_A be the centre of ω and let N be the midpoint of AI_A, which is the centre of $\odot(AEF)$. Notice that MN is A-midline of triangle AI_AD so $MN \perp BC \implies MP = MQ$.

Lemma 16. *Circle $\Omega = \odot(M, MP)$ is orthogonal to ω.*

Proof. Let P_1 be the reflection of P across M. Let $X = DP_1 \cap I_AP$. Notice that

$$DP_1 \parallel AP \perp PI_A \implies \measuredangle PXP_1 = 90^\circ \implies X \in \Omega.$$

Moreover, $I_AX \cdot I_AP = I_AD^2$, implying the desired orthogonality. □

We can finish the solution with two different ways:

- Apply 16 and the converse of **Casey's Theorem** on degenerate circles M, P, Q and circle ω, (**EXERCISE**),
- **Invert** around Ω. Clearly it maps $\omega \to \omega$, $\odot(MPQ) \to PQ$, which obviously touches ω so we are done.

□

Discussion. https://artofproblemsolving.com/community/c6h1671273p10632290

Problem 22 (USAMO 2016)**.** An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in AB$, $Q \in AC$, and N, $P \in BC$. Let S be the intersection of MN and PQ. Denote by ℓ the angle bisector of $\measuredangle MSQ$. Prove that OI is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

Proof. We start with the following lemma:

Lemma 17. *The locus of the point P, whose sum of the oriented distances*

$$\operatorname{dist}(P, BC) + \operatorname{dist}(P, CA) + \operatorname{dist}(P, AB)$$

is fixed, is a line which is perpendicular to the line connecting the incenter and the circumcenter of triangle ABC.

Proof. If we take points Y, Z on rays $\overrightarrow{CA}$, $\overrightarrow{BA}$, such that $CY = BZ = BC$, then $OI \perp YZ$ (**EXERCISE**). If δ_A, δ_B, δ_C denote the oriented distances from P to BC, CA, AB, then using oriented areas, we get

$$[BZYC] = [PCB] + [PBZ] + [PZY] + [PYC] = \frac{BC \cdot (\delta_A + \delta_B + \delta_C)}{2} + [PZY].$$

Hence, if $\delta_A + \delta_B + \delta_C$ is constant, then $[PZY]$ is constant $\implies$ oriented distance from P to YZ is constant $\implies$ P moves on a line parallel to YZ, i.e. perpendicular to OI. □

Let T be a point varies on the external bisector of $\measuredangle MSQ$. From Lemma 17 it suffices to prove

$$\operatorname{dist}(T, BC) + \operatorname{dist}(T, CA) + \operatorname{dist}(T, AB) \tag{2}$$

(oriented distance) is constant. Using oriented area we have

$$[TNP] + [TQA] + [TAM] = [AMNPQ] - [TMN] - [TPQ] = [AMNPQ] = \text{constant},$$

so combining $AM = NP = QA$ we conclude that 2 is fixed. □

Discussion. https://artofproblemsolving.com/community/c5h1231009p6220306

Problem 23 (IMO shortlist 2008)**.** There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$$\begin{aligned}\sphericalangle PAB + \sphericalangle PDC = \sphericalangle PBC + \sphericalangle PAD &= \\ = \sphericalangle PCD + \sphericalangle PBA = \sphericalangle PDA + \sphericalangle PCB &= 90^\circ\end{aligned}$$

if and only if the diagonals AC and BD are perpendicular.

Proof. Suppose P exists; then, the angle condition implies $\sphericalangle APD + \sphericalangle BPC = 180^\circ$, so by **isogonal conjugate theorem for quadrilaterals**, P has an isogonal conjugate Q. However, then we have (**EXERCISE**)

$$\sphericalangle AQB = \sphericalangle BQC = \sphericalangle CQD = \sphericalangle DQA = 90^\circ,$$

which is only possible if $ABCD$ has perpendicular diagonals.

Now suppose that $ABCD$ has perpendicular diagonals, and let Q be their intersection. Then $\sphericalangle AQD + \sphericalangle BQC = 180^\circ$, so Q has an isogonal conjugate P, which is in fact the point P we want (**EXERCISE**). $\square$

Discussion. https://artofproblemsolving.com/community/c6h287868p1555924

Problem 24 (USA TST for EGMO 2019)**.** Let the excircle of a triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Denote by γ the circumcircle of triangle $A_1B_1C_1$ and assume that γ passes through vertex A.

(a) Show that AA_1 is a diameter of γ.
(b) Show that the incenter of triangle ABC lies on line B_1C_1.

Proof. Let I_A, I_B and I_C be centres of excircles of triangle ABC. Let V be the circumcenter of triangle $I_AI_BI_C$ (Bevan Point of ABC). Then

$$\sphericalangle B_1VC_1 = \sphericalangle B_1AC_1 = \sphericalangle B_1A_1C_1,$$

which means that V lies on γ such that V is the antipode of A.

Suppose $V \neq A_1$. Then, $VA_1 \perp AA_1$ along with the fact that $VA_1 \perp BC$ gives that A must lie on BC, i.e. a contradiction. Thus, $V = A_1$, proving Part (a). Part (b) follows by applying **Pappus' Theorem** on I_C, A, I_B and B, A_1, C (with symbols having their usual meanings). Hence, done. $\square$

Discussion. https://artofproblemsolving.com/community/c6h1771386p11625837

Problem 25 (USA January TST for IMO 2017)**.** Let ABC be a triangle with altitude AE. The A-excircle touches BC at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

Proof. Let $FG \cap BC = T$, $MD \cap GF = P$. Call the A-excircle ω, and (IBC) as γ. Let $\omega \cap \gamma = \{X, Y\}$, where X is closer to B.

We basically have to show that if M is the midpoint of AE, then the perpendicular bisector l of AE intersects DF, DG at two points K, L whose midpoint is M. If I is the incenter of $\triangle ABC$, then M, I, D are collinear and so

$$(K, L; M, \infty_l) \stackrel{D}{=} (G, F; P, T)$$

To show that M is the midpoint of KL, we have to show that the left ratio above is -1. So we have to show that P lies on the **polar** of T with respect to ω, so we have to show that T lies on the **polar** of I.

By **radical axis** on γ, ω, (ABC), we find that $T \in XY$. But since $\measuredangle TXI_A = \pi/2 = \measuredangle TYI_A$. hence IX, IY are tangents to ω from I, and so XY is the **polar** of I, and we are done. □

Discussion. https://artofproblemsolving.com/community/c6h1388622p7732197

Problem 26 (ELMO 2016)**.** In triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\measuredangle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\measuredangle XSY = \measuredangle XTY = 90°$. Finally, let γ be the circumcircle of triangle AST. Prove that γ is tangent to the circumcircle and incircle of triangle ABC.

Proof. First, we claim that X and Y are the incenter and excenter of AST. To see this, recall that $\measuredangle AXB = \measuredangle AYC$ are right angles (well known). Now let $K = AXY \cap BC$ and let L be the foot of the external $\measuredangle A$-bisector. Then $(KL; BC) = -1$, so projection onto AI gives $(AK; XY) = -1$. Now, since $\measuredangle YSX = 90°$, we see that SX and SY are bisectors of $\measuredangle AST$. The same statement holds for $\measuredangle ATS$, which proves the claim.

In particular, this implies that AS and AT are **isogonal** to each other, and therefore γ is tangent to circumcircle of ABC.

As for second part of a problem, denote $(XSTY)$ by ω, centered at a point M, which is midpoint of arc ST of γ. Now, it's easy to see $IXD \sim IDY$, therefore $ID^2 = IX \cdot IY$ and thus the incircle is orthogonal to ω. Therefore an **inversion** around ω fixes the incircle. Now γ is mapped to line BC, which is obviously tangent to incircle. Therefore γ was tangent too. □

Discussion. https://artofproblemsolving.com/community/c6h1262194p6556907

Problem 27 (Iranian TST 2018)**.** Let ω be the circumcircle of isosceles triangle ABC ($AB = AC$). Points P and Q lie on ω and BC respectively such that $AP = AQ$. Suppose AP and BC intersect at R. Prove that the tangents from B and C to the incircle of triangle AQR (different from BC) are concurrent on ω.

Proof. Suppose that P is closer to B. Let the bisector of $\measuredangle PAQ$ intersect BC at H. Let PH intersect AQ at G. Clearly PG is tangent to ω. Note that the segment QG is the reflection of PR over AH. Therefore

$$\measuredangle PGA = \measuredangle PGQ = \measuredangle ARQ = 180 - \measuredangle RPB - \measuredangle RBP =$$
$$= 180 - \measuredangle ACB - (180 - \measuredangle PBA - \measuredangle ABC) = \measuredangle PBA,$$

thus G lies on ω, and the result follows from **Poncelet's Porism.** □

Discussion. https://artofproblemsolving.com/community/c6h1628676p10217476

Problem 28 (Iranian TST 2017)**.** Let P be a point in the interior of quadrilateral $ABCD$ such that:

$$\measuredangle BPC = 2\measuredangle BAC \quad , \quad \measuredangle PCA = \measuredangle PAD \quad , \quad \measuredangle PDA = \measuredangle PAC$$

Prove that:

$$\measuredangle PBD = \left|\measuredangle BCA - \measuredangle PCA\right|.$$

Proof. Let T be the point such that $\triangle BPT \overset{+}{\sim} \triangle APC \overset{+}{\sim} \triangle DPA$ and S be the intersection of AC, BT. Clearly, S lies on $\odot(ABP)$, (CPT), so from $\measuredangle BPC = 2\measuredangle BAC$ we get $\measuredangle BAC = \measuredangle BTC \implies T \in \odot(ABC)$, hence

$$\underbrace{\measuredangle PBD = \measuredangle PTA}_{\because\ \triangle BPT \overset{+}{\sim} \triangle DPA} = \measuredangle PTB - \measuredangle ATB = \measuredangle PCA - \measuredangle ACB.$$

□

Discussion. https://artofproblemsolving.com/community/c6h1437519p8153735

References

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- Polish Mathematical Olympiad - `https://om.mimuw.edu.pl`
- Homepage of Dominik Burek - `http://dominik-burek.u.matinf.uj.edu.pl`