Lecture notes for Feb 8, 2022
Discharging method and face coloring

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Theorem 1 *Let G be a planar graph with no cycle with length k, for every $4 \leq k \leq 9$. Then $\chi(G) \leq 3$.*

Proof. Let G be a minimum counterexample. Fix a drawing of G to make it a plane graph. So G has minimum degree at least 3, and G is 2-connected (so every face is bounded by a cycle).

Define $\text{ch}(v) = 2\deg(v) - 6$ for every $v \in V(G)$. Define $\text{ch}(f) = \deg(f) - 6$ for every $f \in F(G)$. So $\sum_{x \in V(G) \cup F(G)} \text{ch}(x) = -12$. Note that $\text{ch}(v) \geq 0$ for all vertices v, and $\text{ch}(f) \geq 0$ unless f is a 3-face, since G has no cycle with length 4 or 5.

Do the following:

(R1) For each face with length at least 4, send 1 charge to each its adjacent 3-face.

(R2) For each vertex v of degree at least 4, send 1 charge to each face f with length at least 10 such that some edge is incident with v, f and a 3-face.

Let ch' be the function indicating the final number of charges.

For each vertex v of degree 3, $\text{ch}'(v) = \text{ch}(v) = 0$. For each vertex v of degree at least 4, if it is incident with t_v 3-faces and sends charge to k_v faces, then $k_v \leq 2t_v$ and $t_v + k_v \leq \deg(v)$, so $k_v \leq \frac{2}{3}\deg(v)$ and $\text{ch}'(v) = \text{ch}(v) - k_v \geq \text{ch}(v) - \frac{2}{3}\deg(v) = \frac{4}{3}\deg(v) - 6$. Hence if $\deg(v) \geq 5$, then $\text{ch}'(v) \geq 0$. And if $\deg(v) = 4$, then it sends 1 charge to at most 2 faces, so $\text{ch}'(v) \geq \text{ch}(v) - 2 \geq 0$. So $\text{ch}'(v) \geq 0$ for every $v \in V(G)$.

Now we show $\mathrm{ch}'(f) \geq 0$ for every face f. For each 3-face f, since there exists no 4-cycle, all faces adjacent to f are not 3-faces, so $\mathrm{ch}'(f) = \mathrm{ch}(f)+3 = 0$. Let f be a face with length at least 4. Note that f is bounded by a cycle C. Let P be a maximal subpath of C such that each edge of P is contained in a 3-face. Hence each internal vertex of P has degree at least 4. So f sends 1 charge to each 3-face containing an edge of P and receives 1 charge from each internal vertex of P. Hence f loses 1 charge involving with P. Note that there are at most $\lfloor \frac{1}{2}\deg(f) \rfloor$ such maximal subpaths. So $\mathrm{ch}'(f) \geq \mathrm{ch}(f) - \lfloor \frac{1}{2}\deg(f) \rfloor = \lceil \frac{1}{2}\deg(f) \rceil - 6$. Hence $\mathrm{ch}'(f) \geq 0$ if f has length at least 11.

Now suppose that $\mathrm{ch}'(f) < 0$. So f is a face with length at most 10. Since there exists no cycle of length at most 9, f has length 10. Hence there are exactly 5 maximal paths mentioned in the previous paragraph. So every vertex incident with f is incident with an edge shared by f and a 3-face. Hence if some vertex incident with f has degree at least 4, then f receives 1 charge by (R1), so $\mathrm{ch}'(f) \geq \mathrm{ch}(f) - \lfloor \frac{1}{2}\deg(f) \rfloor + 1 \geq 0$. Hence every vertex incident with f has degree at most 3. Since G has minimum degree at least 3, every vertex incident with f has degree exactly 3. That is, every vertex on C has degree 3. Let $C = v_1v_2...v_{10}v_1$. Since G is 2-connected and has no cycle of length between 4 and 9, C is induced. So each v_i has a neighbor u_i in G but not in C. By the minimality of $|V(G)|$, there exists a proper 3-coloring ϕ of $G - V(C)$. If $\phi(u_i)$ is identical for all i, say $\phi(u_i) = 1$ for $i \in [10]$, then we can extend ϕ to be a proper 3-coloring of G by assigning colors 2 and 3 to vertices in C, a contradiction. So we may assume that $\phi(u_1) \neq \phi(u_{10})$ by symmetry. Then we can extend ϕ to a proper 3-coloring of G by defining $\phi(v_1) = \phi(u_{10})$, and then for each $2 \leq i \leq 9$, defining $\phi(v_i)$ to be a color in $[3] - \{\phi(v_{i-1}), \phi(u_i)\}$, and then defining $\phi(v_{10}) \in [3] - \{\phi(v_1), \phi(v_9), \phi(u_{10})\}$ (note that it is possible since $\phi(u_{10}) = \phi(v_1)$), a contradiction.

Therefore, $\mathrm{ch}'(x) \geq 0$ for every $x \in V(G) \cup F(G)$. So $-12 = \sum_{x \in V(G) \cup F(G)} \mathrm{ch}(x) = \sum_{x \in V(G) \cup F(G)} \mathrm{ch}'(x) \geq 0$, a contradiction. ■

1 Equivalent formulation of 4CT

The Four Color Theorem (4CT) can be proved by discharging method, but it is too complicated to give details in this class, so we will not provide a proof here. Instead of proving 4CT, we will show the deepness of 4CT by giving several equivalent formulations of 4CT. There are more than 30 such reformulations in the literature. Even though many of them can be derived straightforwardly, some of them are surprising and stated in ways that are completely unrelated to graph theory. However, the only known proof is the purely graph theoretical one that uses the discharging method.

1.1 Face coloring

We first show a simple reformulation in terms of face coloring. It is actually the form that is closest to Guthrie's original question.

It is convenient to consider the dual graph. The *dual graph* of a plane multigraph G, denoted by G^*, is the plane multigraph such that

- $V(G^*) = F(G)$, and each vertex of G^* is drawn at a point in the corresponding face of G,
- there exists a bijection ι from $E(G)$ to $E(G^*)$ such that if x, y are the faces of G at the left-hand-side and the right-hand-side of an edge e of G, respectively, then $\iota(e) = \{x, y\}$, and
- edges of G^* are drawn in the natural way.

Note that G^* is always connected. So if G is not connected, then G and $(G^*)^*$ are not isomorphic. The converse is true.

Proposition 2 *If G is a connected plane multigraph, then $(G^*)^*$ and G are isomorphic.*

For a plane multigraph G, a *proper k-face-coloring* is a function $f : F(G) \to [k]$ such that for every edge e of G, if x, y are the faces of G at the left-hand-side and right-hand-side of e respectively, then $f(x) \neq f(y)$. Hence a proper k-face-coloring of G is a proper k-coloring of G^*. Note that if G has a cut-edge, then G^* has a loop, so there exists no proper face-coloring of G.

Proposition 3 *The following are equivalent.*

1. *Every planar loopless multigraph is 4-colorable.*
2. *Every triangulation is 4-colorable.*

3. *Every plane graph whose dual graph is a cubic loopless multigraph is 4-colorable.*

4. *Every 2-edge-connected cubic plane graph is 4-face-colorable.*

Proof. ($1 \Rightarrow 2$) It is obvious since every triangulation is planar.

($2 \Rightarrow 1$) It suffices to show (simple) planar graphs are 4-colorable since adding or deleting parallel edges do not change the chromatic number. And we can add edges to make a plane graph a triangulation without decreasing the chromatic number.

($2 \Leftrightarrow 3$) Every plane graph whose dual graph is a cubic loopless multigraph is a triangulation. And the dual graph of a triangulation is a cubic loopless multigraph.

($3 \Rightarrow 4$) Let G be a 2-edge-connected cubic plane graph. Since G is connected, $G = (G^*)^*$. Since G is 2-edge-connected and cubic, G^* is simple. Hence G^* is a plane graph whose dual is a cubic loopless multigraph, so G^* is 4-colorable by Statement 3. So G is 4-face-colorable.

($4 \Rightarrow 3$) Let G be a plane graph whose dual graph is a cubic loopless multigraph. Then G is connected. So $G = (G^*)^*$. If G^* has a cut-edge, then $G = (G^*)^*$ has a loop, a contradiction. So G^* is 2-edge-connected and cubic. Hence G^* is 4-face-colorable by Statement 4. So $G = (G^*)^*$ is 4-colorable. ∎