

Functional Equations Problems

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Dedicated to pco.

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1 Definitions

- \mathbb{N} is the set of positive integers.
- $\mathbb{N} \cup \{0\} = \mathbb{N}^*$ is the set of non-negative integers.
- \mathbb{Z} is the set of integers.
- \mathbb{Q} is the set of rational numbers.
- \mathbb{R}^+ is the set of positive real numbers.
- \mathbb{R} is the set of real numbers.
- If a function f is defined on the set A to the set B , we write $f : A \rightarrow B$ and read “ f is a function from the set A to the set B .”

2 Problems

1. Find all surjective functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(n) \geq n + (-1)^n, \forall n \in \mathbb{N}$.
2. Find all functions $g : \mathbb{R} \rightarrow \mathbb{R}$ such that for any real numbers x and y

$$g(x+y) + g(x)g(y) = g(xy) + g(x) + g(y).$$

3. Find all real valued functions defined on the reals such that for every real x, y

$$f(x^2 - y^2) = xf(x) - yf(y).$$

4. Find all real valued functions defined on the reals such that for every real x, y :

$$f(xf(x) + f(y)) = f(x)^2 + y.$$

5. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(f(n)) + (f(n))^2 = n^2 + 3n + 3$ for all positive integers n .

6. Let n be a positive integer. Find all strictly increasing functions $f : \mathbb{N}^* \rightarrow \mathbb{N}^*$ such that the equation

$$\frac{f(x)}{k^n} = k - x$$

has an integral solution x for all $k \in \mathbb{N}$.

7. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$f\left(\frac{x+y}{2}\right) = \frac{2f(x)f(y)}{f(x) + f(y)} \quad \forall x, y \in \mathbb{R}^+.$$

8. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(1-x) = 1 - f(f(x)) \quad \forall x \in \mathbb{R}.$$

9. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$f(1 + xf(y)) = yf(x+y) \quad \forall x, y \in \mathbb{R}^+.$$

10. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$f(xf(y)) = f(x+y) \quad \forall x, y \in \mathbb{R}^+.$$

11. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4yf(x) \quad \forall x, y \in \mathbb{R}.$$

12. Find all functions $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y) + g(x-y) = 2h(x) + 2h(y) \quad \forall x, y \in \mathbb{R}.$$

13. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y+z) = f(x) \cdot f(1-y) + f(y) \cdot f(1-z) + f(z) \cdot f(1-x) \quad \forall x, y, z \in \mathbb{R}.$$

14. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(f(x) - f(y)) = (x - y)^2 f(x + y) \quad \forall x, y \in \mathbb{R}.$$

15. Find all functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ such that

- If $x < y$, then $f(x) < f(y)$;
- for all $x, y \in \mathbb{R}$, we have $f(xy) = g(y)f(x) + f(y)$.

16. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f((x+z)(y+z)) = (f(x) + f(z))(f(y) + f(z)) \quad \forall x, y, z \in \mathbb{R}.$$

17. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that satisfy

$$f(x^3 + y^3) = x^2 f(x) + y f(y^2)$$

for all $x, y \in \mathbb{R}$.

18. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that satisfy

$$f(m + n f(m)) = f(m) + m f(n)$$

for all m and n .

19. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x)f(y) = f(x+y) + xy$ for all $x, y \in \mathbb{R}$.

20. Find all functions $f : \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$ such that $x \cdot 3^{f(y)}$ divides $f(x) \cdot 3^y$ for all $x, y \in \mathbb{N} \cup \{0\}$.

21. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y)f(x-y) = (f(x)f(y))^2 \quad \forall x, y \in \mathbb{R}.$$

22. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(x+y)(f(x) - f(y)) = (x-y)f(x+y) \quad \forall x, y \in \mathbb{R}.$$

23. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f((f(x) + y)) = f(x^2 - y) + 4f(x)y \quad \forall x, y \in \mathbb{R}.$$

24. Find all the functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ such that

$$f(m + n - mn) = f(m) + f(n) - f(mn) \quad \forall m, n \in \mathbb{Z}$$

25. Find all functions $f : (0, 1) \rightarrow (0, 1)$ such that $f(\frac{1}{2}) = \frac{1}{2}$ and

$$(f(ab))^2 = (af(b) + f(a))(bf(a) + f(b)) \quad \forall a, b \in (0, 1).$$

26. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{Q}$ such that

$$f(x + y + f(x)) = x + f(x) + f(y) \quad \forall x, y \in \mathbb{Q}.$$

27. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 + f(y)) = (x - y)^2 f(x + y) \quad \forall x, y \in \mathbb{R}.$$

28. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

- $f(x + y) = f(x) + f(y) \quad \forall x, y \in \mathbb{R},$
- $f(x) = x^2 f(\frac{1}{x}) \quad \forall x \in \mathbb{R} \setminus \{0\}.$

29. Let $a > \frac{3}{4}$ be a real number. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(f(x)) + a = x^2 \quad \forall x \in \mathbb{R}.$$

30. Find all injective functions $f : \mathbb{N} \rightarrow \mathbb{N}$ which satisfy

$$f(f(n)) \leq \frac{n + f(n)}{2} \quad \forall n \in \mathbb{N}.$$

31. Find all continuous functions $f(x), g(x), q(x) : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2) + f(y^2) = [q(x) - q(y)]g(x + y) \quad \forall x, y \in \mathbb{R}.$$

32. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ so that

$$f(x + y) + f(x - y) = 2f(x) \cos y \quad \forall x, y \in \mathbb{R}.$$

33. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x - f(y)) = f(x) + x \cdot f(y) + f(f(y)) \quad \forall x, y \in \mathbb{R}.$$

34. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$f(f(x)) = 6x - f(x) \quad \forall x \in \mathbb{R}^+.$$

35. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x + y) + f(xy) + 1 = f(x) + f(y) + f(xy + 1) \quad \forall x, y \in \mathbb{R}.$$

36. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x)f(y \cdot f(x) - 1) = x^2 f(y) - f(x) \quad \forall x, y \in \mathbb{R}.$$

37. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^3 + y^3) = x \cdot f(x^2) + y \cdot f(y^2) \quad \forall x, y \in \mathbb{R}.$$

38. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{R}$ such that

$$|f(x) - f(y)| \leq (x - y)^2 \quad \forall x, y \in \mathbb{Q}.$$

39. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}^+$ such that

$$f(x + y) = f(x^2 + y^2) \quad \forall x \in \mathbb{R}^+.$$

40. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$x^2 y^2 (f(x + y) - f(x) - f(y)) = 3(x + y)f(x)f(y) \quad \forall x, y \in \mathbb{R}.$$

41. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(f(x) + f(y) + f(z)) = f(f(x) - f(y)) + f(2xy + f(z)) + 2f(xz - yz)$$

for all reals x, y .

42. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $m^2 + f(n)|(f(m))^2 + n$ for all positive integers m, n .

43. Let n be a positive integer. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y^n \quad \forall x, y \in \mathbb{R}.$$

44. Find all the functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$3f(f(f(n))) + 2f(f(n)) + f(n) = 6n \quad \forall n \in \mathbb{N}.$$

45. Find all functions $f : \mathbb{N}^* \rightarrow \mathbb{N}^*$ satisfying

$$(f^2(m) + f(n)) \mid (m^2 + n)^2$$

for any two positive integers m and n .

46. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$f\left(\frac{2xy}{x+y}\right) = \frac{2f(x)f(y)}{f(x) + f(y)} \quad \forall x, y \in \mathbb{R}^+.$$

47. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(xy) = \max\{f(x), y\} + \min\{f(y), x\} \quad \forall x, y \in \mathbb{R}.$$

48. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\bullet \quad f(x + f(y)) = y + f(x) \quad \forall x, y \in \mathbb{R}, \text{ and}$$

- The set $A = \frac{f(x)}{x}$ is finite.

49. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(f(x) + f(y)) + f(f(x)) = 2f(x) + y \quad \forall x, y \in \mathbb{R}.$$

50. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2(z^2+1)+f(y)(z+1)) = 1-f(z)(x^2+f(y))-z((1+z)x^2+2f(y)) \quad \forall x, y, z \in \mathbb{R}.$$

51. Prove that there is no bijective function $f : \{1, 2, 3, \dots\} \rightarrow \{0, 1, 2, 3, \dots\}$ such that

$$f(mn) = f(m) + f(n) + 3f(m)f(n).$$

52. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1 \quad \forall x, y \in \mathbb{R}.$$

53. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(xf(x+y)) = f(yf(x)) + x^2 \quad \forall x, y \in \mathbb{R}.$$

54. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = f\left(x^2 + \frac{x}{3} + \frac{1}{9}\right) \quad \forall x \in \mathbb{R}.$$

55. Given $0 < p < 2$, find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(f(x)) = f(x) + px \quad \forall x \in \mathbb{R}.$$

56. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x + xy + f(y)) = \left(f(x) + \frac{1}{2}\right) \left(f(y) + \frac{1}{2}\right) \quad \forall x, y \in \mathbb{R}.$$

57. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(f(x) + y) = f(x + y) + xf(y) - xy - x + 1 \quad \forall x, y \in \mathbb{R}.$$

58. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$x(f(x) + f(-x) + 2) + 2f(-x) = 0 \quad \forall x \in \mathbb{R}.$$

59. Find all non-decreasing functions $f : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$ such that for each $x, y \in \mathbb{R}^+ \cup \{0\}$

$$f\left(\frac{x + f(x)}{2} + y\right) = 2x - f(x) + f(f(y)).$$

60. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$(1 + f(x)f(y))f(x + y) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}.$$

61. For function $f : \mathbb{R} \rightarrow \mathbb{R}$ given that $f(x^2 + x + 3) + 2 \cdot f(x^2 - 3x + 5) = 6x^2 - 10x + 17$. Calculate $f(2009)$.

62. Find all the functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

63. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(1) = 1$ and

$$f\left(f(x)y + \frac{x}{y}\right) = xyf(x^2 + y^2)$$

for all real numbers x and y with $y \neq 0$.

64. Find all functions f , defined on the positive real numbers and taking real numbers such that

$$f(x) + f(y) \leq \frac{f(x+y)}{2}, \quad \frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y}$$

for all $x, y > 0$.

65. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(xf(y-x)) = f(yf(x)) - x^2 \quad \forall x, y \in \mathbb{R}.$$

66. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that :

$$f(x + f(y + f(z))) = f(x) + f(f(y)) + f(f(f(z))) \quad \forall x, y, z \in \mathbb{R}.$$

67. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying the identity

$$f(x)f(y) = y^\alpha f\left(\frac{x}{2}\right) + x^\beta f\left(\frac{y}{2}\right) \quad \forall x, y \in \mathbb{R}^+$$

Where α, β are given real numbers.

68. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, we have

$$f(x+y) + f(x)f(y) = f(xy) + (y+1)f(x) + (x+1)f(y).$$

69. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y) = 2f(x)f(y) + 3f(x) + 3f(y) + 3 \quad \forall x, y \in \mathbb{R}.$$

70. Find all the continuous bounded functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(f(x))^2 - (f(y))^2 = f(x+y)f(x-y) \text{ for all } x, y \in \mathbb{R}.$$

71. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x+y) + f(x)f(y) = f(xy) + 2xy + 1$ for all real numbers x and y .

72. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 + y^2) = f(f(x)) + f(xy) + f(f(y)) \quad \forall x, y \in \mathbb{R}.$$

73. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$(x+y)f(f(x)y) = x^2f(f(x) + f(y)) \quad \forall x, y \in \mathbb{R}^+.$$

74. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y^2) \geq (y+1)f(x) \quad \forall x, y \in \mathbb{R}.$$

75. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x)f(y) \leq f(xy) \text{ and } f(x) + f(y) \leq f(x+y) \quad \forall x, y \in \mathbb{R}.$$

76. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{R}^+$ such that

- $f(x) \geq 0 \quad \forall x \in \mathbb{Q}, \quad f(x) = 0 \iff x = 0,$
- $f(xy) = f(x) \cdot f(y),$
- $f(x+y) \leq \max\{f(x), f(y)\}$

77. Determine all function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$xf(y) - yf(x) = f\left(\frac{y}{x}\right)$$

for all $x, y \in \mathbb{R}$ with $x \neq 0$.

78. Determine all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\sum_{k=1}^n \frac{1}{f(k) \cdot f(k+1)} = \frac{f(f(n))}{f(n+1)} \quad \forall n \in \mathbb{N}.$$

79. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $m, n \in \mathbb{N}$,

$$(2^m + 1)f(n)f(2^m n) = 2^m f(n)^2 + f(2^m n)^2 + (2^m - 1)^2 n.$$

80. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) - 2xf(y) + f(x) \quad \forall x, y \in \mathbb{R}.$$

81. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(f(x) - y^2) = f(x)^2 - 2f(x)y^2 + f(f(y)) \quad \forall x, y \in \mathbb{R}.$$

82. Find all functions $f : [0, +\infty) \rightarrow [0, +\infty)$ such that:

$$f(x + f(x) + 2y) = 2x + f(2f(y)) \quad \forall x, y \in [0, +\infty).$$

83. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2) + f(xy) = f(x)f(y) + yf(x) + xf(x+y)$$

for all $x, y \in \mathbb{R}$.

84. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{Q}$ such that

$$f(x + f(x) + 2y) = 2x + 2f(f(y)) \quad \forall x, y \in \mathbb{Q}.$$

85. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

- $f(\frac{x+f(x)}{2} + y + f(2z)) = 2x - f(x) + f(f(f(y))) + 2f(f(z)) \quad \forall x, y, z \in \mathbb{R},$
- $f(f(0)) = f(0).$

86. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which satisfy the following conditions:

- $f(x + f(y)) = f(x)f(y)$ for all $x, y > 0$;
- there are at most finitely many x with $f(x) = 1$.

87. Find all functions $f : \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$ such that for all $m, n \in \mathbb{N} \cup \{0\}$,

$$mf(n) + nf(m) = (m+n)f(m^2 + n^2).$$

88. Find all functions $f : (0, 1) \rightarrow \mathbb{R}$ such that

$$f(xyz) = xf(x) + yf(y) + zf(z)$$

for all real numbers $x, y, z \in (0, 1)$.

89. Find all functions $f : \mathbb{Z} \mapsto \mathbb{Z}$ satisfying the condition: $f(x^3 + y^3 + z^3) = f(x)^3 + f(y)^3 + f(z)^3$.

90. Determine all real functions $f(x)$ that are defined and continuous on the interval $(-1, 1)$ and that satisfy the functional equation

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)} \quad (x, y, x+y \in (-1, 1)).$$

91. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^n + 2f(y)) = (f(x))^n + y + f(y) \quad \forall x, y \in \mathbb{R}, \quad n \in \mathbb{Z}_{\geq 2}.$$

92. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 + y^2) = f(x^2) + f(y^2) + 2f(x)f(y) \quad \forall x, y \in \mathbb{R}.$$

93. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y)f(x-y) = (f(x) + f(y))^2 - 4x^2f(y) \quad \forall x, y \in \mathbb{R}.$$

94. Find all injective functions $f : \mathbb{N} \rightarrow \mathbb{R}$ such that

$$f(1) = 2, f(2) = 4, \text{ and } f(f(m) + f(n)) = f(f(m)) + f(n) \quad \forall m, n \in \mathbb{N}.$$

95. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for any real numbers $a, b, c, d > 0$ satisfying $abcd = 1$, we have

$$(f(a) + f(b))(f(c) + f(d)) = (a + b)(c + d).$$

96. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2) \left(f(x)^2 + f\left(\frac{1}{y^2}\right) \right) = 1 + f\left(\frac{1}{xy}\right) \quad \forall x, y \in \mathbb{R} \setminus \{0\}.$$

97. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(f(x) - f(y)) = f(f(x)) - 2x^2f(y) + f(y^2) \quad \forall x, y \in \mathbb{R}.$$

98. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$f(x+1) = f(x) + 1 \text{ and } f\left(\frac{1}{f(x)}\right) = \frac{1}{x} \quad \forall x, y \in \mathbb{R}^+.$$

99. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x + f(x)f(y)) = f(x) + xf(y) \quad \forall x, y \in \mathbb{R}.$$

100. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) + f(y) - f(x+y) = xy \quad \forall x, y \in \mathbb{R}.$$

3 Link to Solutions

1. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=410614>
2. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=320016>
3. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=320299>
4. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=320331>
5. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=319947>
6. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=321004>
7. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=321965>
8. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=321967>
9. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=323174>
10. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=323169>
11. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=328125>
12. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=324002>
13. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=324666>
14. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=325068>
15. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=325439>
16. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=325471>
17. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=150112>
18. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=326890>
19. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=329072>
20. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=328849>
21. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=329548>
22. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=329724>
23. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=329761>
24. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=331073>
25. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=331738>
26. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=271589>
27. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=334067>

28. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=334070>
29. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=334624>
30. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=334842>
31. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=334873>
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33. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=335522>
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35. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=336764>
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39. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=339865>
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41. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=337225>
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59. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=352088>
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82. <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=378715>
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