Useful Identities and Inequalities in Algebra 1.0

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Typesetting and Editing
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1 Preface

On April 27, 2011, I created a thread[1] on MathLinks to compile useful algebraic identities. My goal was to bring community members together to share results which were valuable, yet seemed to exist mostly within the repertoires of certain experts. On Friday, March 4, 2016, I remembered the project and locked the thread because it was being kept alive by only by a few enthusiasts posting obscure results.

It is important to recognize that this compilation is not an encyclopedia. I have deliberately included only those results which I believe will be repeatedly relevant for readers and/or inspire further exploration in the area. To illustrate, there are several astonishing multivariable identities attributed to Euler, Lebesgue and Ferrari which were tempting to include, but despite their beauty they are likely to be useless in practice.

I hope that this document will be found useful. It may be distributed physically or electronically, in whole or in part, but only for noncommercial purposes. Credit should be given to the contributors.

Samer Seraj
March 6, 2016

2 Identities

Two Variables

$$
\begin{aligned}
x^n - y^n &= (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})\\
x^2 - y^2 &= (x-y)(x+y)\\
x^4 - y^4 &= (x-y)(x+y)(x^2+y^2)
\end{aligned}
$$

$$
\begin{aligned}
x^n + y^n &= (x+y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}) \quad \text{for odd } n\\
x^2 + y^2 &= (x \pm y)^2 \mp 2xy
\end{aligned}
$$

$$
\begin{aligned}
x^4 + 4y^4 &= (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy) \quad \textbf{(Sophie Germain)}\\
x^4 + x^2y^2 + y^4 &= (x^2 + xy + y^2)(x^2 - xy + y^2)
\end{aligned}
$$

$$
\begin{aligned}
x^2 + y^2 + (x \pm y)^2 &= 2(x^2 \pm xy + y^2)\\
x^4 + y^4 + (x \pm y)^4 &= 2(x^2 \pm xy + y^2)^2
\end{aligned}
$$

$$
\begin{aligned}
(x+y)^3 - x^3 - y^3 &= 3xy(x+y)\\
(x+y)^5 - x^5 - y^5 &= 5xy(x+y)(x^2+xy+y^2)\\
(x+y)^7 - x^7 - y^7 &= 7xy(x+y)(x^2+xy+y^2)^2
\end{aligned}
$$

[1] The original thread: `http://artofproblemsolving.com/community/c6h404216`

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Three Variables

$$
\begin{aligned}
x^2+y^2+z^2+2(xy+yz+zx) &= (x+y+z)^2 \\
2(x^2+y^2+z^2) \pm 2(xy+yz+zx) &= (x\pm y)^2+(y\pm z)^2+(z\pm x)^2 \\
x^2+y^2+z^2-(xy+yz+zx) &= (x-y)^2+(x-z)(y-z) \\
&= (y-z)^2+(y-x)(z-x) \\
&= (z-x)^2+(z-y)(x-y) \\
6(x^2+y^2+z^2)-6(xy+yz+zx) &= (2x-y-z)^2+(2y-z-x)^2+(2z-x-y)^2
\end{aligned}
$$

$$
\begin{aligned}
(xy+yz+zx)(x+y+z) &= (x^2y+y^2z+z^2x)+(xy^2+yz^2+zx^2)+3xyz \\
&= x(y-z)^2+y(z-x)^2+z(x-y)^2+9xyz \\
&= (x+y)(y+z)(z+x)+xyz \\
(x+y+z)^3-(x^3+y^3+z^3) &= 3(x+y)(y+z)(z+x)
\end{aligned}
$$

$$
\begin{aligned}
(x+y)(y+z)+(y+z)(z+x)+(z+x)(x+y) &= x^2+y^2+z^2+3(xy+yz+zx) \\
(x-y)(y-z)+(y-z)(z-x)+(z-x)(x-y) &= xy+yz+zx-(x^2+y^2+z^2)
\end{aligned}
$$

$$
\begin{aligned}
(x-y)(y-z)(z-x) &= (xy^2+yz^2+zx^2)-(x^2y+y^2z+z^2x) \\
3(x-y)(y-z)(z-x) &= (x-y)^3+(y-z)^3+(z-x)^3
\end{aligned}
$$

$$
\begin{aligned}
(x+y+z)(x-y)(y-z)(z-x) &= (xy^3+yz^3+zx^3)-(x^3y+y^3z+z^3x) \\
(xy+yz+zx)(x-y)(y-z)(z-x) &= (x^2y^3+y^2z^3+z^2x^3)-(x^3y^2+y^3z^2+z^3x^2)
\end{aligned}
$$

$$
\begin{aligned}
x^3+y^3+z^3-3xyz &= (x+y+z)(x^2+y^2+z^2-xy-yz-zx) \\
(x+y+z)^3-24xyz &= (-x+y+z)^3+(x-y+z)^3+(x+y-z)^3 \\
x^3+y^3+z^3+(x+y)^3+(y+z)^3+(z+x)^3 &= 3(x+y+z)(x^2+y^2+z^2)
\end{aligned}
$$

$$
(x+y+z)(-x+y+z)(x-y+z)(x+y-z) = 2x^2y^2+2y^2z^2+2z^2x^2-(x^4+y^4+z^4)
$$

$$
\begin{aligned}
(xy^2+yz^2+zx^2)(x^2y+y^2z+z^2x) &= x^2y^2z^2+(x^2+yz)(y^2+zx)(z^2+xy) \\
(x^2y+y^2z+z^2x-xyz)^2+(xy^2+yz^2+zx^2-xyz)^2 &= (x^2+y^2)(y^2+z^2)(z^2+x^2)
\end{aligned}
$$

More Variables

$$
x^4+y^4+z^4+w^4-4xyzw = (x^2-w^2)^2+(y^2-z^2)^2+2(xw-yz)^2
$$

$$
2(x^4+y^4+z^4+w^4-4xyzw) = (x^2+y^2+z^2+w^2)^2-(-x+y+z+w)(x-y+z+w)(x+y-z+w)(x+y+z-w)
$$

$$
\begin{aligned}
(x^2+ny^2)(z^2+nw^2) &= (xz-nyw)^2+n(xw+yz)^2 \\
&= (xz+nyw)^2+n(xw-yz)^2 \quad \textbf{(Brahmagupta)}
\end{aligned}
$$

The $n=1$ case was discovered by **Diophantus** but is named after **Fibonacci**.

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3 Inequalities

Arithmetic Mean - Geometric Mean
For any non-negative reals $x_1, x_2, \ldots, x_n$ we have

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{\frac{1}{n}}.$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Cauchy-Schwarz
For any reals $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, we have

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \geq (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2.$$

Equality holds if and only if there exists real constants μ and λ, not both 0, such that

$$(\mu x_1, \mu x_2, \ldots, \mu x_n) = (\lambda y_1, \lambda y_2, \ldots, \lambda y_n).$$

It can be proven using **Lagrange's Identity**

$$\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) - \left(\sum_{k=1}^{n} x_k y_k\right)^2 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (x_i y_j - x_j y_i)^2.$$

Titu's Lemma
Derived immediately from Cauchy-Schwarz, we have

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n}.$$

We can replace all of the x_i with 1's, or replace all of the y_i with 1's to obtain

$$\begin{cases} \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} & \geq \frac{n^2}{x_1 + x_2 + \cdots + x_n} \\ x_1^2 + x_2^2 + \cdots + x_n^2 & \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{n} \end{cases}.$$

In both cases, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Weighted AM-GM
For any non-negative reals $x_1, x_2, \ldots, x_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, we have

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \geq x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}.$$

Equality holds if and only if all of the x_i with $\lambda_i > 0$ are equal.

Power Means Ladder
For non-zero integers $p \geq q$ and positive reals $x_1, x_2, \ldots, x_n$ we have

$$\left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \geq \left(\frac{x_1^q + x_2^q + \cdots + x_n^q}{n}\right)^{\frac{1}{q}}.$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$. Note that if some of the x_i are 0, then p and q must be positive in order to avoid division by those x_i.

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Schur

For any non-negative reals x, y, z and $r > 0$ we have

$$x^r(x-y)(x-z) + y^r(y-x)(y-z) + z^r(z-x)(z-y) \geq 0.$$

Equality holds if and only if $x = y = z$, or two of x, y, z are equal and the third is 0.

Rearrangement

Let σ be a permutation of $\{1, 2, \ldots, n\}$. For any reals $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$, we have

$$x_1y_1 + \cdots + x_ny_n \geq x_{\sigma(1)}y_1 + \cdots + x_{\sigma(n)}y_n \geq x_ny_1 + \cdots + x_1y_n.$$

Chebyshev

Derived immediately from adding n cyclic Rearrangement inequalities for any reals $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$, we have

$$n(x_1y_1 + x_2y_2 + \cdots + x_ny_n) \geq (x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n).$$

If instead the sequences are oppositely sorted, then the inequality sign is reversed.

Hölder

Let $\{x_{i_1}\}_{i=1}^n, \{x_{i_2}\}_{i=1}^n, \ldots, \{x_{i_m}\}_{i=1}^n$ be sequences of nonnegative reals, and let $\{\lambda_i\}_{i=1}^n$ be a sequence of nonnegative reals such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. Then

$$\prod_{i=1}^n (x_{i_1} + x_{i_2} + \cdots + x_{i_m})^{\lambda_i} \geq \sum_{j=1}^m (x_{1_j}^{\lambda_1} x_{2_j}^{\lambda_2} \cdots x_{n_j}^{\lambda_n}).$$

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4 Substitutions

Identity Product

$$xyz = 1 \iff x = \frac{a}{b}, \quad y = \frac{b}{c}, \quad z = \frac{c}{a}$$

Identity Sum

$$x + y + z = 0 \iff x = a - b, \quad y = b - c, \quad z = c - a$$

Ravi

A triangle has sides a, b, c if and only if there exist reals x, y, z such that

$$a = x + y, \quad b = y + z, \quad c = z + x.$$

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