SOPHISTICATED LIMITS WITH FOCUS ON MATHEMATICAL OLYMPIAD PROBLEMS

By

TETYANA DARIAN

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THESIS ABSTRACT

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by TETYANA DARIAN

Thesis Director:

Gabor Toth

The purpose of this thesis is to derive a variety of sophisticated limits, some in Mathematical Olympiad Problems, without using mathematically advanced concepts. In fact, this thesis requires only basic calculus and the Stolz-Cesàro theorems. The latter will be presented without proofs but with all the necessary ingredients and formulas needed in this work. The thesis is written for a mathematically mature student with a certain level of preparation to tackle these beautiful limits.

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Chapter 1

Introduction

This chapter is a brief look at what this thesis is all about. To aid us in our discussion, we will be including some definitions and results which will be stated without proof.

Important: All definitions, propositions, and theorems presented in this chapter will be reintroduced, and elaborated upon throughout the course of the thesis. The results of this chapter will be used without any explicit references. To further emphasize this, we will not number the definitions, propositions, and theorems of this chapter.

1.1 Completeness of the Real Numbers

Definition. A nonempty set S of real numbers is said to be *bounded above* if there exists some real number M such that $x \leq M$ for every x in S. In this case M is said to be an *upper bound* for S. Likewise, S is said to be *bounded below* if there exists some number m such that $m \leq x$ for all x in S; m is called a *lower bound* for S. We call S *bounded* if it is bounded both above and below.

Definition. Let S be a nonempty set of real numbers that is bounded above. A *supremum* or *least upper bound* of S, denoted $\sup S$, is a real number μ such that

1. $x \le \mu$, for all x in S;

2. If M is an upper bound for S, then $\mu \le M$.

Definition. Let S be a nonempty set of real numbers that is bounded below. An *infimum* or *greatest lower bound* of S, denoted $\inf S$, is a real number ν such that

1. $\nu \le x$, for all x in S;

2. If m is any lower bound for S, then $\nu \ge m$.

Axiom (The Completeness Axiom for $\mathbb{R}$)**.** If S is a nonempty set of real numbers that is bounded above, then $\sup S$ exists in $\mathbb{R}$.

1.2 Neighborhoods and Limit Points

Definition. Let x be any real number. The *neighborhood* of x with positive radius r is the set

$$N(x;r) = \{y \; in \; \mathbb{R} : |y - x| < r\}$$

Definition. A *deleted neighborhood* of x in $\mathbb{R}$, denoted $N'(x;r)$, is the neighborhood $N(x;r)$ with the point x itself removed.

Definition. Given a nonempty set S in $\mathbb{R}$, a point x in $\mathbb{R}$ is said to be a *limit point* of S if, for each $\varepsilon > 0$, the deleted neighborhood $N(x;\varepsilon)$ contains at least one point of S.

Theorem (Bolzano-Weierstrass Theorem)**.** If S is a bounded, infinite subset of $\mathbb{R}$, then S has a limit point in $\mathbb{R}$.

1.3 The Limit of a Sequence

Definition. The point c is a *cluster point* of the sequence $\{x_k\}$ if, for every $\varepsilon > 0$ and every k in $\mathbb{N}$, there is a $k_1 > k$ such that x_{k_1} is in $N(c;\varepsilon)$.

Definition. The sequence $\{x_k\}$ converges to x_0 and we say x_0 is the limit of $\{x_k\}$ if, for each neighborhood $N(x_0; \varepsilon)$, there exists an index k_0 such that, whenever $k > k_0$, x_k is in $N(x_0; \varepsilon)$. We write $\lim_{k \to \infty} x_k = x_0$. If a sequence $\{x_k\}$ fails to converge, for whatever reason, then we say that the sequence *diverges*.

Theorem. The limit of a convergent sequence in $\mathbb{R}$ is unique.

Theorem. A convergent sequence is bounded.

Theorem. If a sequence is unbounded, then it must diverge.

Theorem. A bounded, monotonic sequence of real numbers converges.

Definition. Let $\{x_k\}$ be any sequence. Choose any strictly monotonic increasing sequence $k_1 < k_2 < k_3 < ...$ of natural numbers. For each j in $\mathbb{N}$, let $y_j = x_{k_j}$. The sequence $\{y_j\} = \{x_{k_j}\}$ is called a *subsequence* of $\{x_k\}$.

Theorem. The point c is a cluster point of $\{x_k\}$ if and only if there exists a subsequence x_{k_j} that converges to c.

Theorem. Any bounded sequence has a cluster point.

Theorem. A sequence x_{k_j} converges to x_0 if and only if every subsequence of x_{k_j} converges to x_0.

Theorem. A bounded sequence converges if and only if it has exactly one cluster point.

Chapter 2

Stolz-Cesàro Theorems and the Stirling Formula

In this chapter, we discuss powerful criteria for convergence of sequences, all due to Otto Stolz and Ernesto Cesàro.

2.1 The Stolz-Cesàro Theorem

Let $(a_n)_{n\in\mathbb{N}_0}$ *and* $(b_n)_{n\in\mathbb{N}_0}$ *be real sequences such that* $(b_n)_{n\in\mathbb{N}_0}$ *is strictly increasing with* $\lim_{n\to\infty} b_n = \infty$. *Then, we have*

$$\lim_{n\to\infty}\inf \frac{a_n - a_{n-1}}{b_n - b_{n-1}} \leq \lim_{n\to\infty}\inf \frac{a_n}{b_n} \leq \lim_{n\to\infty}\sup \frac{a_n}{b_n} \leq \lim_{n\to\infty}\sup \frac{a_n - a_{n-1}}{b_n - b_{n-1}}.$$

In particular

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty} \frac{a_n}{b_n},$$

provided that the limit on the left-hand side exists.

2.2 The Stolz-Cesàro Theorem

(Equivalent Formulation)

Let $(a_n)_{n\in\mathbb{N}_0}$ *and* $(b_n)_{n\in\mathbb{N}_0}$ *be real sequences such that* $b_n > 0, n \in \mathbb{N}_0$ *and* $\lim_{n\to\infty} b_n = \infty$*. Then, we have*

$$\lim_{n\to\infty}\inf \frac{a_n}{b_n} \le \lim_{n\to\infty}\inf \frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \le \lim_{n\to\infty}\sup \frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \le \lim_{n\to\infty}\sup \frac{a_n}{b_n}.$$

In particular

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_1+\cdots+a_n}{b_1+\cdots+b_n},$$

provided that the limit on the left-hand side exists.

Proof. This follows directly from the previous by the substitution

$a_n \mapsto a_0 + \cdots + a_n$ and $b_n \mapsto b_0 + \cdots + b_n$, $n \in \mathbb{N}_0$.

Letting $b_n = n, n \in \mathbb{N}$, we obtain the following special cases valid for any real sequence $(a_n)_{n\in\mathbb{N}}$:

$$\lim_{n\to\infty}\inf a_n \le \lim_{n\to\infty}\inf \frac{a_1+\cdots+a_n}{n} \le \lim_{n\to\infty}\sup \frac{a_1+\cdots+a_n}{n} \le \lim_{n\to\infty}\sup a_n,$$

and

$$\lim_{n\to\infty}\inf (a_n - a_{n-1}) \le \lim_{n\to\infty}\inf \frac{a_n}{n} \le \lim_{n\to\infty}\sup \frac{a_n}{n} \le \lim_{n\to\infty}\sup (a_n - a_{n-1}).$$

In particular,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_1+\cdots+a_n}{n},$$

and

$$\lim_{n\to\infty} (a_n - a_{n-1}) = \lim_{n\to\infty} \frac{a_n}{n},$$

provided that the limits on the left-hand sides exist.

We call these *the additive Stolz-Cesàro formulas.*

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence with positive members. For $n \in \mathbb{N}$, let $b_n = \log_2(a_n)$, or equivalently, $a_n = 2^{b_n}$. Applying the exponential identities, we obtain

$$2^{\frac{b_1+\cdots+b_n}{n}} = \sqrt[n]{a_1 \ldots a_n}.$$

Using the monotonicity of the exponentiation, and the Stolz-Cesàro limit formulas above for the sequence $(b_n)_{n\in\mathbb{N}}$, we obtain

$$\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \inf \sqrt[n]{a_1 \ldots a_n} \leq \lim_{n\to\infty} \sup \sqrt[n]{a_1 \ldots a_n} \leq \lim_{n\to\infty} \sup a_n,$$

and

$$\lim_{n\to\infty} \inf \frac{a_n}{a_{n-1}} \leq \lim_{n\to\infty} \inf \sqrt[n]{a_n} \leq \lim_{n\to\infty} \sup \sqrt[n]{a_n} \leq \lim_{n\to\infty} \inf \frac{a_n}{a_{n-1}}.$$

In particular

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{a_1 \ldots a_n},$$

and

$$\lim \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} \sqrt[n]{a_n},$$

provided that the limits on the left-hand sides exist.

We call these *the multiplicative Stolz-Cesàro formulas.*

2.3 The Stirling Formula

An alternative approach for deriving some of the limits that follow in Chapter 3 is to first derive the Stirling Formula. However, the Stirling Formula is much more involved, compared to the more elementary Stolz-Cesàro formulas, and would require elaborate proofs that are not within the scope of this work.

Chapter 3

Examples of Sophisticated Limits

3.1 Preliminaries

We begin with a preliminary example:

(1)
$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e.$$

Indeed, letting $a_n = n!/n^n$, $n \in \mathbb{N}$, we calculate

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e},$$

where we use the Euler's limit relation. Now, the multiplicative Stolz-Cezàro theorem gives

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

(1) follows.

As a direct application, we have the following example (Lalescu Sequence):

$$\lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) = \frac{1}{e}.$$

Once again, an easy application of the Stolz-Cesàro Theorem gives

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}{(n+1)-n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e},$$

where we used (1).

3.2 Example 1

We are now ready to state our first limit.

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence with positive terms: $0<a_n\in\mathbb{R}, n\in\mathbb{N}$. We have the following implication

$$(2)\qquad \boxed{\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n}=L>0 \quad\Rightarrow\quad \lim_{n\to\infty}\left(\frac{a_{n+1}}{\sqrt[n+1]{(n+1)!}}-\frac{a_n}{\sqrt[n]{n!}}\right)=e\cdot\frac{L}{2}}$$

An important special case is $a_n=n^2$ as follows.

Bătineţu-Giurgiu Sequence:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)=e.$$

This follows from (2), since $\lim_{n\to\infty}\frac{(n+1)^2-n^2}{n}=\lim_{n\to\infty}\frac{2n+1}{n}=2$.

We now turn to the proof of (2). Let $(a_n)_{n\in\mathbb{N}}$ as above. We first use the Stolz-Cesàro Theorem to the effect that

$$(3)\qquad \lim_{n\to\infty}\frac{a_n}{n^2}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)^2-n^2}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n}\cdot\frac{n}{2n+1}=\frac{L}{2},$$

where we used the assumption in (2).

We take the common denominator inside the second limit in (2), and use (1) and (3) to calculate

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{\sqrt[n+1]{(n+1)!}}-\frac{a_n}{\sqrt[n]{n!}}\right)=\lim_{n\to\infty}\frac{a_n}{\sqrt[n]{n!}}\left(\frac{a_{n+1}}{a_n}\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}-1\right)$$

$$=\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\cdot\frac{a_n}{n^2}\cdot n\left(\frac{a_{n+1}}{a_n}\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}-1\right)$$

$$=e\cdot\frac{L}{2}\cdot\lim_{n\to\infty}n\left(\frac{a_{n+1}}{a_n}\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}-1\right).$$

It remains to calculate the last limit, which we write as

$$\lim_{n\to\infty}n\left(b_n-1\right)=1,$$

where

$$b_n=\frac{a_{n+1}}{a_n}\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}.$$

First, note that

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{n+1}{n}$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^2}\cdot\frac{n^2}{a_n}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{n+1}{n}=1\cdot\frac{1}{e}\cdot e=1,$$

where we used (3) again.

By continuity of the natural logarithm, we obtain

$$\lim_{n\to\infty}\ln b_n=0.$$

Returning to the main line, we have

$$\lim_{n\to\infty} n\,(b_n - 1) = \lim_{n\to\infty} n\left(e^{\ln b_n} - 1\right) = \lim_{n\to\infty} n\ln b_n \cdot \frac{e^{\ln b_n} - 1}{\ln b_n} = \lim_{n\to\infty} n\ln b_n,$$

since

$$\lim_{n\to\infty} \frac{e^{\ln b_n} - 1}{\ln b_n} = \lim_{n\to\infty} \frac{e^x - 1}{x} = \lim_{n\to\infty} \frac{e^x}{1} = 1.$$

Finally, we use the explicit formula above for $b_n, n \in \mathbb{N}$, and obtain

$$\lim_{n\to\infty} n\ln b_n = \lim_{n\to\infty} n\ln\left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\right)$$

$$= \lim_{n\to\infty} n\left(\ln\frac{a_{n+1}}{a_n} + \frac{\ln n!}{n} - \frac{(n+1)!}{n+1}\right).$$

For the first term in the last parentheses, we calculate

$$\lim_{n\to\infty} n\ln\frac{a_{n+1}}{a_n} = \lim_{n\to\infty} n\ln\left(\left(\frac{a_{n+1}}{a_n} - 1\right) + 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{a_{n+1}}{a_n} - 1\right)$$

$$= \lim_{n\to\infty} \frac{n^2}{a_n} \cdot \frac{a_{n+1} - a_n}{n} = \frac{2}{L} \cdot L = 2.$$

where we used $\lim_{n\to\infty} a_{n+1}/a_n = 1$ along with L'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{\ln\left(\left(\frac{a_{n+1}}{a_n} - 1\right) + 1\right)}{\frac{a_{n+1}}{a_n - 1}} = \lim_{x\to 0} \frac{\ln(x+1)}{x} = 1.$$

For the last two terms in the parentheses above, we use the Stolz-Cezàro Theorem again, and calculate

$$\lim_{n\to\infty} n\left(\frac{\ln n!}{n} - \frac{\ln (n+1)!}{n+1}\right)$$

$$= \lim_{n\to\infty} \frac{(n+1)\ln n! - n\ln (n+1)!}{n+1}$$

$$= \lim_{n\to\infty} \frac{(n+1)(\ln 1 + \ln 2 + \cdots + \ln n) - n(\ln 1 + \ln 2 + \cdots + \ln n + \ln (n+1))}{n+1}$$

$$= \lim_{n\to\infty} \frac{\ln n! - n\ln (n+1)}{n+1}$$

$$= \lim_{n\to\infty} \frac{(\ln (n+1)! - (n+1)\ln (n+2)) - (\ln n! - n\ln (n+1))}{(n+2)-(n+1)}$$

$$= \lim_{n\to\infty} (n+1)\ln \frac{n+1}{n+2}$$

$$= -\ln\left(\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{n+1}\right)$$

$$= -\ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}\right) = -\ln e = -1.$$

Finally, putting everything together, we obtain

$$\lim_{n\to\infty} n(b_n - 1) = 2 - 1 = 1$$

Now (2) follows.

3.3 Example 2

We have

$$\boxed{\lim_{n\to\infty} n^{\alpha_n} = e, \qquad \alpha_n = \frac{\ln \cdot n!}{n \cdot H_n^2}}, \tag{1}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, is the n^{th} partial sum of the harmonic series. [1]

To derive (1), we first prove the following two lemmas.

Lemma 1.

$$\left(\frac{n}{e}\right)^n < n! < n \cdot \left(\frac{n}{e}\right)^n, \quad n \geq 2. \tag{2}$$

Proof. First, we show

$$\left(1 + \frac{1}{n}\right)^n < e < \left(\frac{1}{n}\right)^{n+1}. \tag{3}$$

Indeed, by the trivial estimate of the integral, since $y = \frac{1}{x}$ is strictly decreasing, we have

$$\frac{1}{1+n} < \int_n^{n+1} \frac{dx}{x} < \frac{1}{n}.$$

This gives

$$\frac{1}{n+1} < \ln(n+1) - \ln n = \ln\left(\frac{n+1}{n}\right) < \frac{1}{n}.$$

Hence

$$e < \left(\frac{n+1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right)^{n+1},$$

and

$$\left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n < e.$$

Thus, (3) follows.

[1] It is well-known that $\lim_{n\to\infty} H_n = \infty$.

We now proceed to prove (2) by induction, with respect to $n \geq 2$.

Initial Case:

Let $n = 2$, then

$$e \cdot \left(\frac{2}{e}\right)^2 < 2! < 2 \cdot e \left(\frac{2}{e}\right)^2$$

$$e \cdot \frac{4}{e^2} < 2 < 2 \cdot e \cdot \frac{4}{e^2}$$

$$\frac{4}{e} < 2 < \frac{8}{e}$$

$$2 < e < 4.$$

For the general induction step $n \Rightarrow n+1$, we assume that (2) holds for n.

Therefore, we have

$$(n+1)\left(\frac{n+1}{e}\right)^{n+1} > (n+1) \cdot \frac{1}{e} \frac{(n+1)^{n+1}}{n^{n+1}} \cdot n \cdot \left(\frac{n}{e}\right)^n$$

$$> (n+1) \cdot \frac{1}{e} \cdot \left(1 + \frac{1}{n}\right)^{n+1} \cdot n! = \frac{1}{e} \cdot \left(1 + \frac{1}{n}\right)^{n+1} \cdot (n+1) > (n+1)!$$

where

$$\frac{1}{e} \cdot \left(1 + \frac{1}{n}\right)^{n+1} > 1 \quad \text{and} \quad \left(1 + \frac{1}{n}\right)^{n+1} > e.$$

Lemma 2.

$$\ln n < H_n < 1 + \ln(n+1), \quad 1 \leq n \in \mathbb{N}.$$

Proof. Once again, the trivial estimate of the integral gives

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} > \int_n^{n+1} \frac{dx}{x} = \ln(n+1)$$

and

$$\frac{1}{2} + \cdots + \frac{1}{n} < \int_1^n \frac{dx}{x} = \ln n.$$

Combined together, these give

$$\ln(n+1) < H_n < 1 + \ln n, \quad n > 1. \tag{4}$$

We are now ready to show (1). Taking the natural logarithm of (2), we obtain

$$n \cdot (\ln n - 1) < \ln n! < n \cdot (\ln n - 1) + \ln n.$$

Multiplying through the chain of inequalities by $\ln n / (n \cdot H_n^2)$, we get

$$\frac{(\ln n - 1) \cdot \ln n}{H_n^2} < \frac{\ln n \cdot \ln n!}{n \cdot H_n^2} < \frac{[n \cdot (\ln n - 1) + \ln n] \cdot \ln n}{n \cdot H_n^2}, \quad n \geq 2.$$

We replace the lower and upper bounds in (4) with H_n^2, and obtain

$$\frac{(\ln n - 1) \cdot \ln n}{[1 + \ln(n+1)]^2} < \frac{\ln \cdot \ln n!}{n \cdot H_n^2} < \frac{n \cdot (\ln n - 1) + \ln n}{n \cdot \ln n}, \quad n \geq 2.$$

For the limit of the lower bound, as $n \to \infty$, we calculate

$$\lim_{n\to\infty} \frac{(\ln n - 1) \cdot \ln n}{[1 + \ln(n+1)]^2} = \lim_{n\to\infty} \frac{1 - \frac{\ln n}{n}}{\left[\frac{1+\ln(n+1)}{\ln n}\right]^2} = 1,$$

where we used the elementary limits

$$\lim_{n\to\infty} \ln n / n = 0 \quad \text{and} \quad \lim_{n\to\infty} \ln(n+1) / \ln(n) = 1,$$

where both limit relations follows from L'Hôpital's Rule.

For the upper bound, we have

$$\lim_{n\to\infty} \frac{n \cdot (\ln n - 1) + \ln n}{n \cdot \ln n} = \lim_{n\to\infty} \left[1 - \frac{1}{\ln} + \frac{1}{n}\right] = 1.$$

Finally, by the Squeeze Theorem, we obtain

$$\lim_{n\to\infty} \frac{\ln n \cdot \ln n!}{n \cdot H_n^2} = 1.$$

Therefore,

$$\lim_{n\to\infty} n^{\alpha_n} = \lim_{n\to\infty} e^{\alpha_n \ln n} = \lim_{n\to\infty} e^{\frac{\ln n \cdot \ln n!}{n \cdot H_n^2}} = e.$$

Example 2 follows.

3.4 Example 3

Let $\quad 0 < x_0 < 1 \quad$ and $\quad x_{n+1} = x_n - x_n^2 \ , \quad n \in \mathbb{N}_0.$

Show that

$$(1) \qquad \lim_{n\to\infty} x_n = 0;$$

$$(2) \qquad \lim_{n\to\infty} nx_n = 1;$$

and find

$$(3) \qquad \boxed{\lim_{n\to\infty} \frac{n\,(1-nx_n)}{\ln n}}$$

The first two limit relations are the contents of the next two lemmas.

Lemma 3. The sequence $(x_n)_{n\in\mathbb{N}_0}$ is decreasing and we have (1).

Proof. First, we use induction to show that $x_n \in (0,1)$ for all $n \in \mathbb{N}_0$. This holds for $n = 0$ by assumption. For the general induction step $n \Rightarrow n+1$, if $x_n \in (0,1)$, then $x_{n+1} = x_n - x_n^2 = x_n\,(1-x_n) \in (0,1)$ clearly holds.

Note that this immediately implies that the sequence $(x_n)_{n\in\mathbb{N}_0}$ is decreasing, since $x_{n+1} = x_n - x_n^2 < x_n \quad$ for all $n \in \mathbb{N}_0$.

Finally, since $(x_n)_{n\in\mathbb{N}_0}$ is decreasing and bounded from below, the Bolzano-Weierstrass Theorem applies (see Sec. 1.2), yielding the existence of the limit $\lim_{n\to\infty} x_n = l$. The recurrence relation defining the sequence now gives $l = l - l^2$. Hence $l = 0$, and the lemma follows.

Lemma 4. We have the limit relation (2):

$$\lim_{n\to\infty} nx_n = 1.$$

Proof. We write the limit in (2) as

$$\lim_{n\to\infty} nx_n = \lim_{n\to\infty} \frac{n}{\frac{1}{x_n}} = \lim_{n\to\infty} \frac{(n+1)-n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}},$$

where, in the last equality, we used the additive Stolz-Cesàro Theorem with $a_n = n$ and $\ln = \frac{1}{x_n}$. Note that, by Lemma 3, we have $\lim_{n\to\infty} \frac{1}{x_n} = \infty$, so that the additive Stolz-Cesàro theorem holds.[2]

We calculate

$$\lim_{n\to\infty} \frac{(n+1)-n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \lim_{n\to\infty} \frac{x_n \cdot x_{n+1}}{x_n - x_{n+1}}$$

$$= \lim_{n\to\infty} \frac{x_n \cdot (x_n - x_n^2)}{x_n^2}$$

$$= \lim_{n\to\infty} (1 - x_n) = 1.$$

Lemma 4 follows.

Finally, we turn to the proof of limit relation (3) as follows.

$$\lim_{n\to\infty} \frac{n\,(1 - nx_n)}{\ln n} = \lim_{n\to\infty} nx_n \cdot \frac{\frac{1}{x_n} - n}{\ln n} = \lim_{n\to\infty} \frac{\frac{1}{x_n} - n}{\ln n},$$

where we used Lemma 4. We rewrite the last limit using the additive Stolz-Cesàro Theorem, and calculate

[2] In what follows, we will use the Stolz-Cesàro Theorem frequently without explicit reference to Sec.2.1.

$$\lim_{n\to\infty} \frac{[\frac{1}{x_{n+1}} - (n+1)] - [\frac{1}{x_n} - n]}{\ln(n+1) - \ln n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{x_{n+1}} - \frac{1}{x_n} - 1}{\ln\left(1 + \frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} n \cdot \left[\frac{1}{x_n \cdot (1 - x_n)} - \frac{1}{x_n} - 1\right]$$

$$= \lim_{n\to\infty} n \cdot \frac{1 - (1 - x_n) - x_n \cdot (1 - x_n)}{x_n \cdot (1 - x_n)}$$

$$= \lim_{n\to\infty} \frac{n \cdot x_n}{1 - x_n} = 1.$$

We obtain limit relation in (3).

3.5 Example 4

In the previous examples, we have seen that the Stolz-Cesàro Theorems serve as a powerful tool for calculating difficult limits. However, in some instances, L'Hôpital's Rule provides a short and more efficient approach for deriving certain limits, as we will see in the example below.

Show that

$$\boxed{\lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n}\right)^n}{e}\right]^n = \frac{1}{\sqrt{e}}} \tag{1}$$

We will derive a more general continuous limit:

$$\lim_{x\to 0^+}\left[\frac{\left(1+\frac{1}{x}\right)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}} = \frac{1}{\sqrt{e}}. \tag{2}$$

To do this, we take the natural logarithm of the limit and calculate

$$\begin{aligned}
\ln\left(\lim_{x\to 0^+}\left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}\right) &= \lim_{x\to 0^+}\frac{1}{x}\cdot\ln\frac{(1+x)^{\frac{1}{x}}}{e} \\
&= \lim_{x\to 0^+}\frac{\frac{1}{x}\ln(1+x)-1}{x} = \lim_{x\to 0^+}\frac{\ln(1+x)-x}{x^2} \\
&= \lim_{x\to 0^+}\frac{\frac{1}{1+x}-1}{2x} = \lim_{x\to 0^+}\frac{1-(1+x)}{2x(1+x)} = -\frac{1}{2}.
\end{aligned}$$

where, in the last but one equality, we used L'Hôpital's Rule.

Exponentiating (2), and hence (1) follows.

3.6 Example 5

Show that

(1) $$\boxed{\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{x_{n+1}}{y_{n+1}}}-\sqrt[n]{\frac{x_n}{y_n}}\right)=\frac{a}{b\cdot e}},$$

where

$$(x_n)_{n\geq 1},\quad x_n>0;\qquad \lim_{n\to\infty}\frac{x_{n+1}}{n^2x_n}=a>0$$

$$(y_n)_{n\geq 1},\quad y_n>0;\qquad \lim_{n\to\infty}\frac{y_{n+1}}{ny_n}=b>0.$$

Let $a_n=\sqrt[n]{\frac{x_n}{y_n}}$. We start by deriving $\lim_{n\to\infty}\frac{a_n}{n}=\frac{a}{b\cdot e}$. Applying the Stolz-Cesàro Theorem, we obtain

$$\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n\sqrt[n]{y_n}}=\lim_{n\to\infty}\frac{n\sqrt[n]{x_n}}{n^2\sqrt[n]{y_n}}\quad=\lim_{n\to\infty}\frac{\frac{1}{n^2}\sqrt[n]{x_n}}{\frac{1}{n}\sqrt[n]{y_n}}=\lim_{n\to\infty}\frac{\sqrt[n]{\frac{x_n}{n^{2n}}}}{\sqrt[n]{\frac{y_n}{n^n}}}$$

$$=\lim_{n\to\infty}\frac{\frac{x_{n+1}}{(n+1)^{2(n+1)}}\div\frac{y_{n+1}}{(n+1)^{n+1}}}{\frac{x_n}{n^{2n}}\div\frac{y_n}{n^n}}=\lim_{n\to\infty}\frac{\frac{n^{2n}}{(n+1)^{2(n+1)}}\cdot\frac{x_{n+1}}{x_n}}{\frac{n^n}{(n+1)^{n+1}}\cdot\frac{y_{n+1}}{y_n}}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{n}{n+1}\right)^{2n}\cdot\frac{x_{n+1}}{(n+1)^2\cdot x_n}}{\left(\frac{n}{n+1}\right)^{n}\cdot\frac{y_{n+1}}{(n+1)\cdot y_n}}=\frac{\frac{1}{e^2}}{\frac{1}{e}}\cdot\lim_{n\to\infty}\frac{\frac{n^2}{(n+1)^2}\cdot\frac{x_{n+1}}{n^2\cdot x_n}}{\frac{n}{n+1}\cdot\frac{y_{n+1}}{n\cdot y_n}}=\frac{1}{e}\cdot\frac{a}{b}.$$

This gives

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\left[\frac{a_{n+1}}{n+1}\cdot\frac{n}{a_n}\cdot\frac{n+1}{n}\right]=\frac{a}{be}\cdot\frac{be}{a}\cdot 1=1$$

and

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n=\lim_{n\to\infty}\left(\frac{x_{n+1}}{n^2x_n}\cdot\frac{ny_n}{y_{n+1}}\cdot n\sqrt[n]{\frac{y_n}{x_n}}\right)^{\frac{n}{n+1}}=\frac{a}{b}\cdot\frac{be}{a}=e.$$

Putting everything together, we obtain:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{x_{n+1}}{y_{n+1}}}-\sqrt[n]{\frac{x_n}{y_n}}\right) = \lim_{n\to\infty}(a_{n+1}-a_n) = \lim_{n\to\infty}\left[a_n\cdot\left(\frac{a_{n+1}}{a_n}-1\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{a_n}{n}\cdot\frac{e^{\ln(\frac{a_{n+1}}{a_n}-1)}}{\ln\left(\frac{a_{n+1}}{a_n}\right)}\cdot\ln\left(\frac{a_{n+1}}{a_n}\right)^n\right] = \frac{a}{b\cdot e}\cdot 1\cdot\ln e = \frac{a}{b\cdot e}.$$

Thus, (1) follows.

3.7 Example 6

Show that

(1)
$$\boxed{\lim_{n\to\infty}(1+e-F_n)^{n!}=1}$$

and

(2)
$$\boxed{\lim_{n\to\infty}(1+e-F_n)^{(n+1)!}=e},$$

where $F_n=\sum_{k=0}^{n}\frac{1}{k!}$ with $\lim_{n\to\infty}F_n=e$.

We first take the natural logarithm of (1) and calculate

$$\begin{aligned}\ln\left(\lim_{n\to\infty}(1+e-F_n)^{n!}\right) &= \lim_{n\to\infty} n!\cdot\ln(1+e-F_n)\\ &= \lim_{n\to\infty}\frac{\ln(1+e-F_n)}{e-F_n}n!\cdot(e-F_n)\\ &= \lim_{n\to\infty}\frac{\ln(1+e-F_n)}{e-F_n}\cdot\lim_{n\to\infty}n!\cdot(e-F_n).\end{aligned}$$

We claim that the first limit on the right-hand side is equal to 1. Indeed, we let $e-F_n=u\to 0$, as $n\to\infty$ and apply L'Hôpital's Rule (to the continuous limit), and obtain

$$\lim_{n\to\infty}\frac{\ln(1+e-F_n)}{e-F_n}=\lim_{u\to 0^+}\frac{\ln(1+u)}{u}=\lim_{u\to 0^+}\frac{\frac{1}{1+u}}{1}=1.$$

For the remaining (second) limit, we use the Stolz-Cesàro Theorem, and calculate

$$\lim_{n\to\infty} n!\cdot(e-F_n) \quad = \lim_{n\to\infty}\frac{e-F_n}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{(e-F_n-(e-F_{n-1}))}{\frac{1}{n!}-\frac{1}{(n-1)!}}$$

$$= \lim_{n\to\infty}\frac{F_{n-1}-F_n}{\frac{1}{(n-1)!}\cdot\left(\frac{1}{n}-1\right)} = \lim_{n\to\infty}\frac{-\frac{1}{n!}}{\frac{1}{(n-1)!}\cdot\left(\frac{1}{n}-1\right)} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1-\frac{1}{n}} = 0.$$

Hence, (1) follows.

The computations leading to (2) are almost verbatim to that of (1). To derive (2), the only change is to replace $n!$ by $(n+1)!$. For the last limit above, we obtain

$$\lim_{n\to\infty}(n+1)!(e-F_n) = \lim_{n\to\infty}\frac{e-F_n}{\frac{1}{(n+1)!}} = \lim_{n\to\infty}\frac{-\frac{1}{n!}}{\frac{1}{n!}\cdot\left(\frac{1}{n+1}-1\right)} = 1.$$

Thus, (2) follows.

Bibliography

[1] Steven A. Douglass. *Introduction to Mathematical Analysis*, Addison-Wesley, Reading, Massachusetts, 1996.

[2] Virgil Nicula. *Very Nice and Difficult Limits*. The Art of Problem Solving, October 16, 2010. <https://artofproblemsolving.com/community/c1090h1013815_158_very_nice_and_difficult_limits>