Lemmas in Olympiad Geometry

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Introduction

Here are some lemmas which can be useful in Olympiad Geometry along with some references to other lemmas in geometry. Most are well known and some are due to the author himself, so have fun proving them and using them to the fullest advantage in your Olympiad journey.

A word of warning

You can use these lemmas on the actual Olympiad only after you prove them on the test, because you can't quote lemmas on an Olympiad.

The Lemmas

1. If H is the orthocentre of a triangle ABC and M is the midpoint of BC then the circle with AH as diameter, circumcircle of BHC and AM are concurrent.

2. If P is any point on the circumcircle of ABC and L is the nine-point centre of PBC and J is the reflection of P over L, then J is the reflection of the circumcentre over BC.

3. If the circle through vertex A and the midpoint A' of the arc BAC of the circumcircle of ABC cuts AB and AC at B' and C' respectively then BB' = CC'.

4. If O is the circumcentre and I is the incentre of a triangle then OI is the Euler line of the contact triangle.

5. Given a complete cyclic quadrilateral if any line cuts 2 opposite sides at equal distances from the centre of the circle, then it does so for each other pair too.

6. Given any quadrilateral ABCD and the midpoints X, Y, Z, W, U, V of AB, BC, CD, DA, AC, BD, and the centroids G_1, G_2, G_3, G_4 of the triangles BCD, CDA, DAB and ABC then XZ, YW, UV, AG_1, BG_2, CG_3, DG_4 are all concurrent at a point P which bisects the first three segments and divides the last four in a ratio 3:1.

7. If in the above lemma the quadrilateral is cyclic and the orthocentres of the triangles BCD, CDA, DAB and ABC are H_1, H_2, H_3, H_4 respectively then AH_1, BH_2, CH_3, DH_4 are concurrent at the reflection of the centre of the circle in P, say Q. Also Q is the midpoint of each of these segments.

8. Define a triangle k-centre Xk to be the point on the Euler line such that OXk : OH = k. Then if the quadrilateral ABCD is cyclic, and the k-centres of the triangles BCD, CDA, DAB and ABC are Xk_1, Xk_2, Xk_3, Xk_4 respectively then the quadrilateral $Xk_1Xk_2Xk_3Xk_4$ is similar and homothetic to ABCD with ratio of similitude –k i.e. they are inversely similar.

9. The sixteen incenters and excenters of the 4 triangles formed by a cyclic quadrilateral are the intersections of 2 sets of 4 parallel lines which are mutually perpendicular.

10. In a complete quadrilateral the bisectors of the angles are concurrent at 16 points. These points are intersections of 2 sets of 4 circles each, which are members of conjugate coaxial systems. The axes of these systems pass through the Miquel point of the quadrilateral.

11. The Apollonius circles are orthogonal to the circumcircle, the Brocard circle, the Lemoine line. The circumcircle, the Brocard

circle, the Lemoine line and the isodynamic points belong to a coaxial system of circles.

12. The cevian triangles of isotomic conjugates have the same area.

13. If a line makes equal angles with the opposite sides of a cyclic quadrilateral, then circles can be drawn tangent to each pair, where this line meets them, and these circles are coaxial with the original circle.

14. The medial triangle and the triangle homothetic to the original triangle at the Nagel point share a common incircle.

15. Given an angle and a circle through the vertex of the angle, cutting its bisector at a fixed point. Then the sum of the intercepts of the circle on the sides of the angle is invariant.

16. The triangle formed by the reflections of a point with respect to sides of a triangle has its centre as the isogonal conjugate of that point. Further the circumcircles of the triangle associated with the point and its isogonal conjugate are congruent.(Can be used to give an alternative proof of the existence of the isogonal conjugate)

17. The centre of a composition of 2 homotheties lies on the line joining the centres of both. (Very useful)

18. The centre of inversion swapping 2 circles is collinear with their centres. (Very useful)

19. Let C_3 be a circle coaxial with 2 circles C_1 and C_2. Then it is the locus of points such that the ratio of the powers of the point with respect to C_1 and C_2 is constant.

20. Let I_a, I_b, I_c be the excenters and M_1, M_2, M_3 be the midpoints of the arcs BAC, ABC and ACB of the circumcircle. Then I, the incentre, is the orthocentre of the excentral triangle, $M_1M_2M_3$ is the medial triangle of the excentral triangle and I_bI_cBC etc are cyclic with diameters as I_bI_c etc and IBI_aC etc are cyclic with diameters II_a etc respectively. The circumcircle of ABC is the nine point circle and ABC is the orthic triangle of the excentral triangle. Also, the contact triangle is homothetic with the excentral triangle.
21. $ABCD$ is a quadrilateral such that $BA + BC = DA + DC$ if $BA \cap DC = E, BC \cap DA = F$ then $EA + EC = FA + FC$
22. Let DEF be the orthic triangle and let PQR be the medial triangle. QR intersects PE and PF at X and Y respectively. Then PXY and DFE are similar. Also, YRF, XQE, YPQ and XRP are all isosceles.
23. The Nagel point of a triangle is the incentre of the antimedial triangle (due to the Nagel line).
24. The circle with diameter AH, the circumcircle and HM are concurrent where H is the orthocentre of ABC and M is the midpoint of BC.
25. Let M be the midpoint of BC and E, F be the feet of perpendiculars from B and C to the opposite sides respectively. Then ME and MF are tangents to the circumcircle of AEF at M and hence AM is a symmedian of AEF. Note that the second part also follows directly from the fact that EF and BC are antiparallel wrt ABC.

Some incircle lemmas

Given an acute triangle ABC inscribed in (O), incircle (I). The tangent points of (I) on BC, CA, AB are respectively D, E, F. Let (O_a) be the A-excircle, and it is tangent to BC, CA, AB at D', E', F' respectively.

1/ AD, BE, CF are concurrent; AD', BE', CE' are concurrent at I_0.

2/ $D'C = DB$

3/ $ID \cap EF = D_1$. AD_1 passes through the midpoint M of BC.

4/ ID cuts (I) at $\{D, D_2\}$. AD_2 passes through D'.

5/ $BI \cap EF = B'$, $CI \cap EF = C'$. (B, I, F, C'), (C, I, E, B') and (B, C, B', C') are the sets of concyclic points.

6/ AI cuts (BIC) at $\{A, A_0\}$. I_0 is symmetric to I wrt BC. Let K' be the foot of the attitude from A of triangle ABC. $K'I_0$ passes through A_0.

7/ $E'D' \cap DF = K$. A, K, K' are collinear (Paul Yiu's theorem).

8/ $A*$ is symmetric to A wrt O. $A * I \cap EF = W$. DW is perpendicular to EF

9/ $AI \cap BC = A_1$. $A_1D \cap A * I \in (O)$

10/ AI cuts (O) at $\{A, A_2\}$. A_2 is the circumcenter of triangle BIC.

11/ Let R be an arbitrary point lying on minor arc BC. R_1R_2 is the polar of R wrt (I). BC cuts RR_1 and RR_2 at R'_1 and R'_2 respectively. The A-mixtilinear incircle is tangent to (O) at Z. $Z \in (DA_1A_2) \cap (RR'_1R'_2)$. (Cosmin Pohoata)

12/ $IC \cap AK = C_0, IB \cap AK = B_0$. K_1 is the midpoint of AK'. K_1 lies on the radical axis of (C_0EC) and (B_0FB).

13/ (O) and (IAZ) are orthogonal.

14/ AZ cuts (DA_1A_2) at Z and Z'. $Z' \in (O_a)$.

15/ OO_a is perpendicular to EF.

16/ (OI) cuts (IAB) at I, J. IJ is parallel to DE.

17/(OI)cuts(OAB)at$\{O, J_1\}$. IJ_1passes through the midpoint $M_1$of DE

18/Let I_1be the projection of Ionto AD. $I_1M_1 \cap AC = N$. $ND$is parallel to EF.

19/The line passing through $A$and parallel to BCcuts EFat T.Let T_1be the midpoint of AT. $T_1M$is tangent to(I).

20/Let $H_b$be the orthocenter of triangle IAC. $H_bD$is perpendicular to IM .(India MO 2014).

21/Draw the diameters EE_2, $FF_2$of(I). E_2F_2cuts BCat P. $\widehat{MIP} = 90$.

22/IB, ICcut E_2F_2at E_3, F_3respectively. the perpendicular bisector of E_3F_3passes through the symmetric point of M wrt A_1.

23/$I$is the incenter of triangle $D_2E_3F_3$.

24/DK'cuts(I)at$\{D, K_1\}$. $K_1D$is the angle bisector of triangle $\widehat{BK_1C}$.

25/$(BK_1C)$is tangent to(I).

26/Let Vbe the midpoint of ID. $A * V$cuts(O)at$\{A*, V\}$. $BC$is tangent to(VAD).

27/EFcuts(O)at E_4, F_4.Let A_e, A_fbe respectively the projections of Aonto IE_4, IF_4. $DA_e, DA_f$are isogonal conjugate wrt $\widehat{EDF}$.

28/ the center of(DE_4F_4)lies on the perpendicular bisector of IA.
29/If $AB + AC = 3BC$then $(IB'C')$is tangent to(IBC).

30/$(O_aBF) \cap (O_aCE) = O_a, W_a$. $(BCW_a)$is tangent to(I)

31/Similarly define W_b, W_c,then W_aD, W_bE, W_cF, OI are concurrent.

32/The angle bisector of $\widehat{AW_cB}$ cuts (AW_cB) at W'_c,the angle bisector of $\widehat{AW_bC}$ cuts (AW_bC) at W'_b. W_b, W_c, W'_b, W'_c are concyclic.

33/Construct E_5, F_5 such that vector EE_5 =vector FF_5 =vector BC.Let I' be the intersection of BE, CF. $I'E_5 = I'F_5$.

34/Construct B_1, C_1 on the rays BA, CA respectively such that $BB_1 = CC_1 = BC$.Let I_1, O_1 be respectively the incenter and circumcenter of triangle AB_1C_1. :
34.1/ IO is perpendicular to B_1C_1.
34.2/ IO' is perpendicular to BC.
34.3/ II_1 is parallel to OO_1.
34.4/ $IO \cap I_1O1 \in (O)$.

35/Let $B*, C*$ be respectively the midpoint of arc ABC, ACB. $B*, C*$ are respectively the centers of $(O_aBA), (O_aCA)$.

36/ $B * C*$ passes through I iff $AB + AC = 3BC$.

37/ $D_2A = I_0D'$.

Further Reading and Suggestions

For more lemmas refer to Yufei Zhao's handout on "Lemmas in Euclidean Geometry" and "The Big Picture".

Also you can read extensively about Gauss-Bodenmiller's theorem, Simson lines, Miquel point of a complete quadrilateral, inversion, Morley's theorem (especially proofs), the Shooting Lemma, Utkarsh's

Isogonality lemma, Curvilinear and Mixtilinear incircles (especially Evan Chen's article), Sawayama-Thebault theorem, Monge's theorem, Monge-d'Alembert's theorem, Pascal's theorem, Desargue's theorem, Brianchon's theorem, Pappus's theorem, and some projective geometry.

For problems, see Evan Chen's book "Euclidean Geometry in Mathematical Olympiads", Sharygin's "Problems In Plane Geometry", and the AoPS forum.

For other geometrical topics, refer to Darij Grinberg's notes, which are very useful.