Geometry

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Problem. A triangle $\triangle\ ABC$ is given. M is the midpoint of segment BC. Let circle with diameter BC (ω) intersect AB, AC at F, E other than B, C respectively. The median AM meets ω at K, L where K is closer to A than L. P, Q are intersections of line EF with tangents to ω at L, K respectively, AQ, AP intersect BC at X, Y respectively. If the perpendicular line passing through B to BC met the median at Z, show that $\angle XZY = 90$.

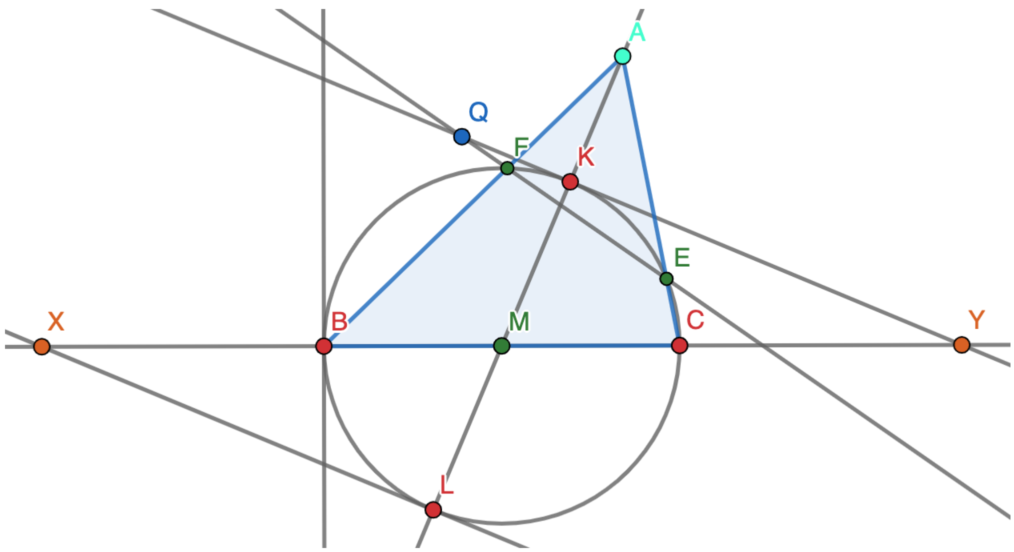

Figure 1: A is moving on the fixed median WOW!

My Solution. Fix segment BC. The circle $\odot(M, MC)$ is fixed, so let us also fix the whole median AM, thus fixing points K and L. Now we suppose that A is moving along the median with degree $degA = 1$, and projecting A through C gives us E and since C is fixed and $degA = 1$ then $degE$ is at most $2 \times degA = 2$. Similarly with F, then $degEF$ is at most $degE + degF - 1 = 3$, thus $degQ$ is at most $degEF + degl_k - 1 = 2$ (where l_k is the tangent from K which is fixed). We will prove that X, Q and A are collinear (supposing that X is the fixed intersection of tangent at L with BC), it suffices to check $degX + degQ + degA + 1$ which is at most 4 placements of A which is the easiest when:

- $A = K$
- $A = L$
- $A = M$
- $A =$ infinity point of direction KL

Left to check by the reader.
Now we have a better definition of X and Y; they are not other than the intersections of tangents from K and L to our circle centered at M. By the butterfly theorem on chords KL, KL and BC of midpoint M, M is also the midpoint of XY, and now it is trivial to show that $\triangle\, MLX$ and $\triangle\, MBZ$ are congruent and thus $MX = MY = MZ$ and then the result of $\angle XZY = 90$ is obtained. $\square$