Geometry

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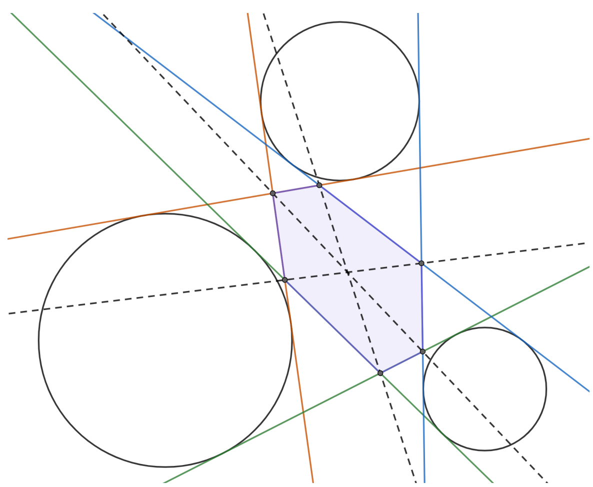
Problem. Three non-intersecting circles are drawn with their pairwise common internal tangents. The six tangents form a hexagon. Prove that this hexagon satisfies the property that it's principle diagonals intersect at one point.



Solution. Let the circles be γ_1, γ_2, and γ_3. Considering the circles intersect at the circular points, take the dual of the configuration to obtain the following problem.

Let 3 conics γ_1, γ_2, and γ_3 have 2 common tangents. Then, there exist some 2 intersections of γ_2 and γ_3, some 2 intersections of γ_3 and γ_1, and some 2 intersections of γ_3 and γ_1 such that these 6 intersections are co-conic.

Let γ_2 and γ_3 intersect at X_1, X_2, X_3, and X_4, let γ_3 and γ_1 intersect at Y_1, Y_2, Y_3, and Y_4, and let γ_1 and γ_2 intersect at Z_1, Z_2, Z_3, and Z_4, and suppose without loss of generality that we are trying to prove that X_1, X_2, Y_1, Y_2, Z_1, and Z_2 are co-conic. Send X_1 and X_2 to the circular points. Then, by converse of radical axis on conic $Y_1Y_2Z_1Z_2$ with circles γ_2 and γ_3 it suffices that $\overline{Y_1Y_2}, \overline{Z_1Z_2}$, and $\overline{X_3X_4}$ concur. Undoing the homography and the dual gives that we want that the exsimilicenter of γ_2 and γ_3, the insimilicenter of γ_3 and γ_1, and the insimilicenter of γ_1 and γ_2 are collinear, which is true by Monge's theorem. $\square$