A PROBLEM ON FEUERBACH

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Contents

1 Preliminaries

The notation I will use throughout the paper for the points related to a given triangle $\triangle ABC$:

- The circumcircle of $\triangle ABC$ is denoted by ω.
- The circumcenter, orthocenter, centroid and incenter are denoted by O, H, G and I respectively.
- Midpoints of segments BC, CA and AB are given by M_a, M_b and M_c.
- The nine-point-circle of the triangle is γ.
- The incircle of the triangle is Ω.
- The Feuerbach point is denoted by F_e, and by "Feuerbach point" I mean the tangency point of $Omega$ and γ.

2 Introduction

In this paper I discuss a problem I came up with, I will try to go through the process of creating this problem (as it's probably the hardest I've came up with so far), and write my solution. But first, a lemma:

Mini Problem. In triangle $\triangle ABC$, O and I are the circumcenter and the incenter, F_e is the Feuerbach point. show that F_e is the anti-steiner point of OI wrt the median triangle $M_aM_bM_c$.

Solution. It's a special case of Fontene's Theorem 3. □

3 Coming up with the problem

The Feuerbach point is a point that exists by accident in my opinion as the nine-point-circle and the incircle barely touch and then you get a point nearly related to everything, it's very unexpected how this point is related to some geometric configurations, and in how many ways we can construct it.

3.1 Inspiration

I was trying to create a problem revolving around the Feuerbach point F_e, I remembered something about the Ponclet point of a quadrilateral, it was the following problem from our Saudi TST, to be precise:

Saudi TST. In triangle $\triangle ABC$, O and H are the circumcenter and the orthocenter respectively. Prove that the nine-point-circles of triangles AOH, BOH, COH share some common point.

The solution involved proving that the nine-point-circles of triangles BOC, COA, AOB intersected in one point, which is proved by applying Ponclet's Theorem on the concave quadrilateral

$ABCO$.

Then I thought about making the Ponclet's point of some quadrilateral having the three vertices of the triangle as it's vertices and the fourth namely T be F_e. I got this idea since the Ponclet point had to be on γ if A, B, C were vertices since γ is the nine-point-circle of ABC. Turns out the locus of such points T is the unique hyperbola passing through points A, B, C, H, I (I found some info about this locus on this place). WOW, this sounds weird, but it is quite easy to prove using the moving points method, and this added to most work in this paper is generalizable, but F_e makes everything weirdly familiar. And as I was toying around with this configuration, I found it humane to get F_e on the circumcircle ω using the homothety centered at H with ratio 2, which maps γ to ω as we know. Another very reasonable thing I did was look at the Steiner line of this new point K. And the result was *tres chic* to say the least..

3.2 The problem statement

The Problem. Points O, I and H are the circum-, in-, and ortho-centers of triangle ABC. Let F_e be the Feuerbach point (i.e the tangency point of the nine-point-circle and the incircle), let K be the reflection of H in F_e. Show that reflections of K around the sides of the triangle form a line perpendicular to line OI.

I am pretty sure the statement could be made more "mysterious" meaning "not containing words like Feuerbach and stuff" but we can work on that later, in my solution I translated stuff to the Simson line of K as you will see in the next section, but after the solution I'll also write another result (related to the Steiner line) that could be proven using a similar approach.

4 The Solution

Shall we begin?

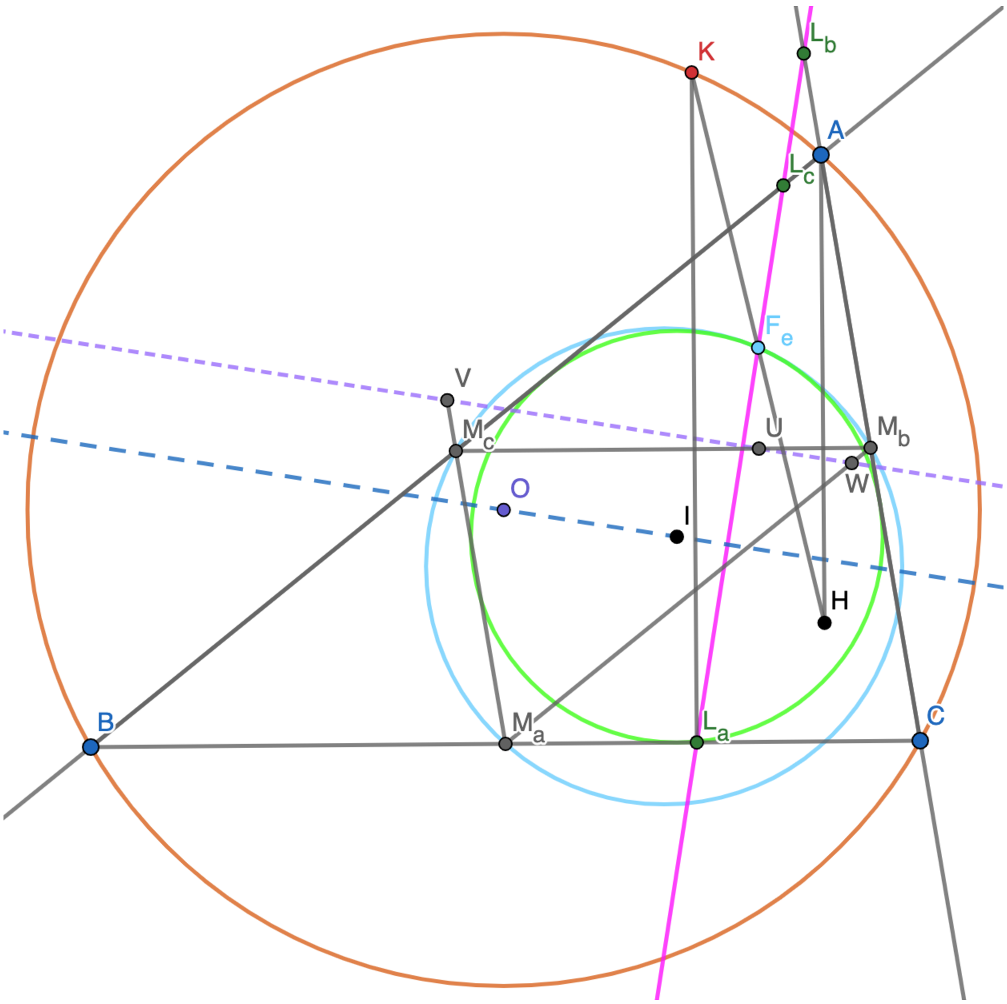

Figure 1: Very pretty Figure, like really nice coloring

Solution. So we basically need to show that the Simson or Steiner line of K say l or p is perpendicular to line OI, so we suppose l intersects the sides BC, CA, AB at L_a, L_b, L_c respectively. By definition of the Simson line, these are just projections of K onto the sides of the triangle, implying that K lies on the circles $AL_bL_c, BL_cL_a, CL_bL_a, ABC$. Thus K is the Miquel point of quadrilateral (AB , BC , CA , l). Notice now that p is the Steiner line of the quadrilateral. Now we think about a line perpendicular to that, the Gauss line. We want to prove that this Gauss line is parallel to OI and we are done. Let us then introduce the midpoints of AL_a, BL_b, CL_c as U, V, W respectively which form our Gauss line. we can conclude that U is on the midline M_bM_c, and analogously V is on M_cM_a and W on M_aM_b. We know that AH and KL_a are both perpendicular to BC hence are parallel, and since F_e and U are midpoints of KH and AL_a, then F_eU is the median line of the pair of parallel lines AH and KL_a, which means that F_eU is perpendicular to M_bM_c. Cyclically we see that U, V, W are projections of F_e on the sides of the median triangle, thus they form the Simson line of F_e wrt the median triangle. Our mini problem takes care of the rest. $\square$

5 The Similar Problem

The Similar Problem. Points O, I and H are the circum-, in-, and ortho-centers of triangle ABC. Let F_e be the Feuerbach point and let K be the reflection of H in F_e. The reflections of K around the sides of the triangle form a line l which intersect the sides BC, CA, AB in U, V, W, take M as the Miquel point of quadrilateral (AB, BC, CA, l). Also let p be the Simson line of K wrt ABC. Show that l and p meet on γ the nine-point-circle of ABC.

This problem can be solved using similar methods, I really hope you enjoyed this paper.