

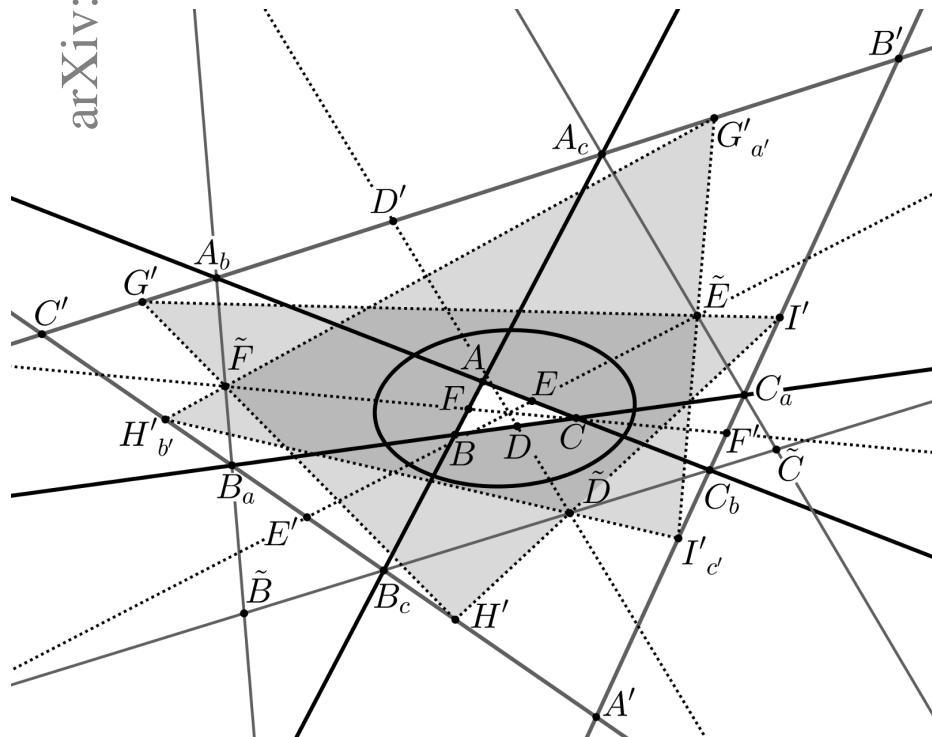
Non-Euclidean shadows of classical projective theorems

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§ 1

Introduction

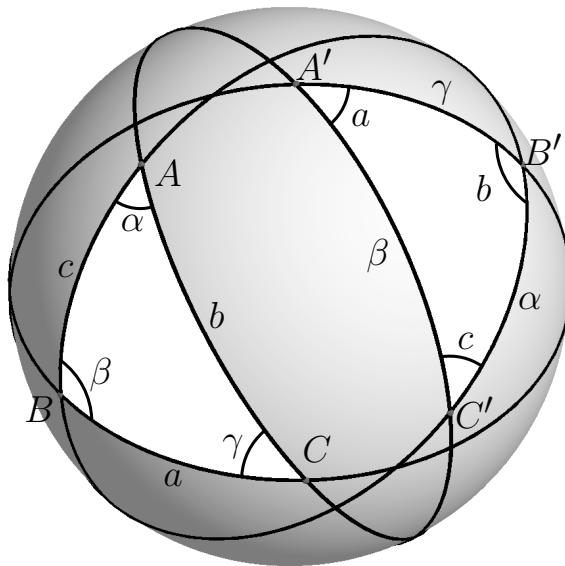
Cayley-Klein projective models for hyperbolic and elliptic (spherical) geometries have the disadvantage of being non-conformal. However, they have some interesting virtues: geodesics are represented by straight lines, allowing to easily visualize incidence properties, and they give an unified treatment of the three classic planar geometries: euclidean, spherical and hyperbolic. They can be introduced into any course on projective geometry, and so they can give to undergraduate students an early contact with non-euclidean geometry before learning more advanced topics such as riemannian geometry or calculus in a complex variable. Within a course on projective geometry, the (extremely beautiful in itself) projective theory of conics can be enhanced by introducing Cayley-Klein models.

Almost any projective theorem about conics might have multiple interpretations as different theorems of non-euclidean geometry. For example, Chasles' polar triangle Theorem (Theorem 4.3) asserts that a projective triangle and its polar triangle with respect to a conic are perspective: the lines joining the corresponding vertices are concurrent. This theorem has, at least, the following corollaries in non-euclidean geometry:

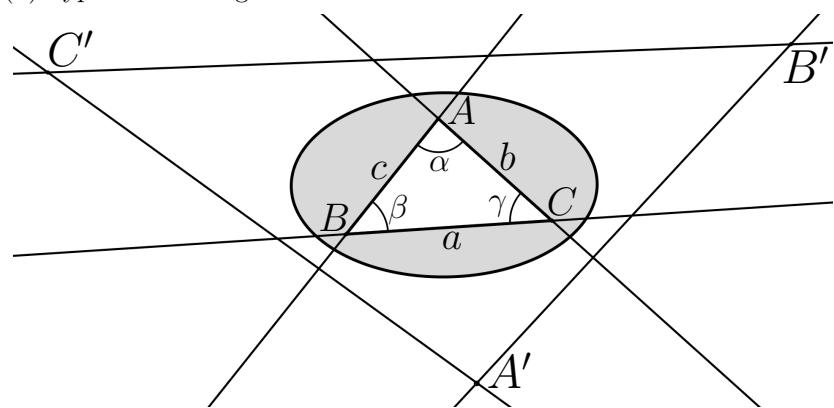
- the three altitudes of a spherical triangle are concurrent
- the three altitudes of a hyperbolic triangle are: (i) concurrent; or (ii) parallel (they are asymptotic through the same side); or (iii) ultra-parallel (they have a common perpendicular).
- the common perpendiculars to opposite sides of a hyperbolic right-angled hexagon are concurrent.

Following [18] (and [17], indeed!), we say that these three results are different non-euclidean *shadows* of Chasles' Theorem.

(a) elliptic triangle



(b) hyperbolic triangle



(c) right-angled hexagon

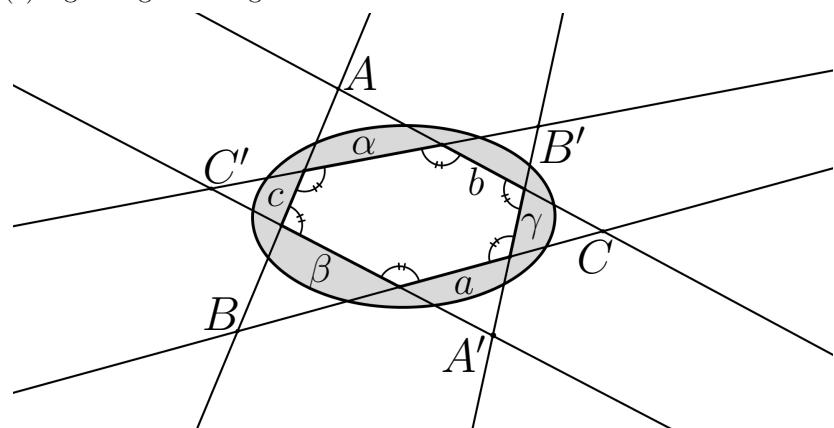


Fig. 1.1: Generalized triangles

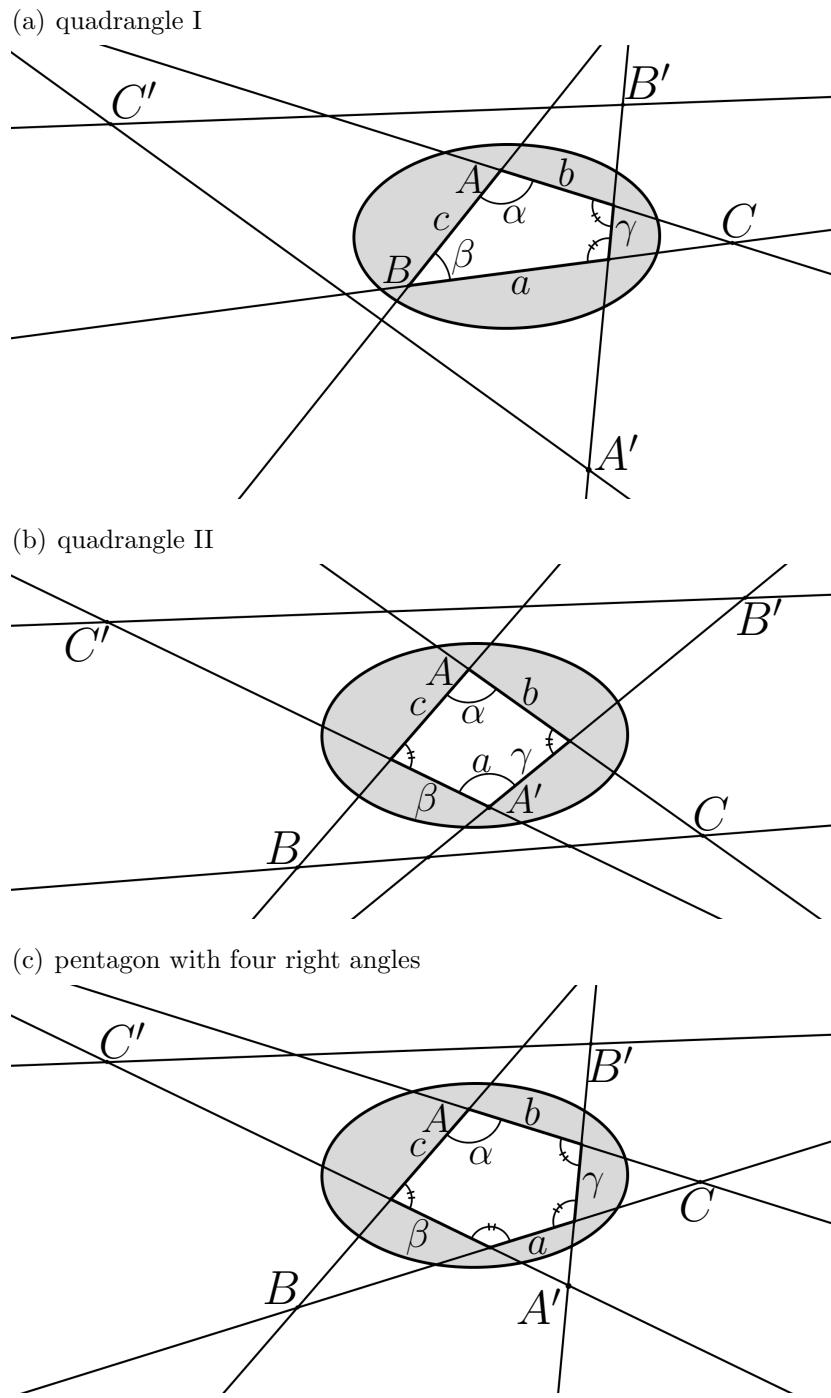


Fig. 1.2: Generalized triangles II

Aside from spherical and hyperbolic triangles, in the hyperbolic plane there are many other figures which verify similar trigonometric relations as the triangles do, such as Lambert and Saccheri quadrilaterals, right-angled

pentagons and hexagons, etc.. Following [4], we will refer as *generalized triangles* to all these elliptic and hyperbolic figures which have trigonometry.

All generalized triangles are exactly the same figure when we look at them wearing “projective glasses”: in Cayley-Klein models they are the result of intersecting a projective triangle $\mathcal{T} = \widehat{ABC}$ with its polar triangle $\mathcal{T}' = \widehat{A'B'C'}$ with respect to the absolute conic Φ of the model (see Figures 1.1–1.4, where right angles are denoted with the symbol \triangleleft). With some general position assumptions, projective theorems involving a triangle and its polar triangle with respect to a conic will not depend on the relative position of both triangles with respect to a conic, and so they will have some different shadows as non-euclidean theorems about different generalized triangles. In this work we will explore the non-euclidean shadows of some classical theorems from planar geometry. We will adopt a classic (old-fashioned?) point of view. For more modern approaches to this subject see [18] or [23, 24], for example.

Pascal’s and Chasles’ theorems and classical triangle centers In the previous example about Chasles’ Theorem and the concurrency of altitudes, we can see that the relation of a triangle with its polar one with respect to a conic is of key importance in the non-euclidean treatment of triangles. The set of midpoints of the sides of a triangle has a particular structure that is deduced essentially from Pascal’s Theorem, and this structure allows to prove the concurrency of medians, of side bisectors and of angle bisectors of a triangle.

Desargues’ Theorem and the non-euclidean Euler line For an euclidean triagle, the orthocenter, the circumcenter and the barycenter are collinear. The line passing through these three points is the *Euler line* of the triangle, and it contains also many other interesting points such as the center of the *nine-point circle*. The nine-point circle is the circle passing through the midpoints of the sides of the triangle, and it contains also the feet of the altitudes of the triangle and the midpoints of the segments joining the orthocenter with the vertices of the triangle.

The Euler line and the nine-point circle have no immediate analogues in non-euclidean gometry because, in general, for a hyperbolic or elliptic triangle, the orthocenter, the circumcenter and the barycenter are non-collinear, and the midpoints of the sides and the feet of the altitudes are not concyclic. Thus, it is usually said that in non-euclidean geometry the Euler line and the nine-point circle do not exist. Nevertheless, we claim that they exist and that they are not unique. We will propose a non-euclidean version of these two objects, and a different version of them is proposed in [1]. The

line that we propose as Euler line is the line which is called *orthoaxis* in [24]. We claim that there are enough reasons for this line to deserve the name *Euler line*. Giving credit to [24], we will call it *Euler-Wildberger line*.

Some of the classical centers of an euclidean triangle can be defined in multiple ways, all of them equivalent. However, two such definitions, equivalent under the euclidean point of view, could be non-equivalent in the non-euclidean world. We will illustrate this fact for the barycenter and the circumcenter. Using alternative definitions of these points, we will show how every non-euclidean triangle has an *alternative barycenter* and an *alternative circumcenter*, different from the standard ones. These alternative centers are collinear with the orthocenter of the triangle, and we will say that the line passing through them is the *Euler-Wildberger line* of the triangle. All these constructions have been introduced before in [23, 24] with a different notation. We give new proofs of them based on a reiterated application of Desargues' Theorem.

Beyond the Euler line, we construct a *nine-point conic* which is a non-euclidean version of the nine-point circle.

Menelaus' Theorem and non-euclidean trigonometry Hyperbolic trigonometry is a recurrent topic in most treatments on non-euclidean geometry since the early works of N. I. Lobachevsky and J. Bolyai (see [6, 8]). Due to its connection with different topics as Riemann surfaces, low-dimensional topology or special relativity, it has been treated also from different viewpoints in more recent works such as [3, 4, 10, 21]. There exists a close connection between elliptic and hyperbolic trigonometry. Trigonometric relations between sides and angles of spherical and hyperbolic triangles look much more the same, with some replacements between sines and cosines into hyperbolic sines and cosines and vice versa.

Generalized triangles are characterized by having exactly six defining magnitudes among sides and non-right angles. The value (segment length or angular measurement) of each of these magnitudes is related with a side of \mathcal{T} or \mathcal{T}' : the cross ratio of four points on this side provides the square power of a (circular or hyperbolic) trigonometric function of the corresponding magnitude.

There are four kinds of generalized triangles whose trigonometric relations are simpler than in the general case: elliptic and hyperbolic right-angled triangles (Figures 6.2(a) and 6.2(b) respectively), Lambert quadrilaterals (Figure 6.3(a)) and right-angled pentagons (Figure 6.3(b)). They appear when we force one of the non-right angles of a generalized triangle to be a right angle. For this reason, we will refer to them as *generalized right-angled triangles*. In §6 we show that all the trigonometric relations for

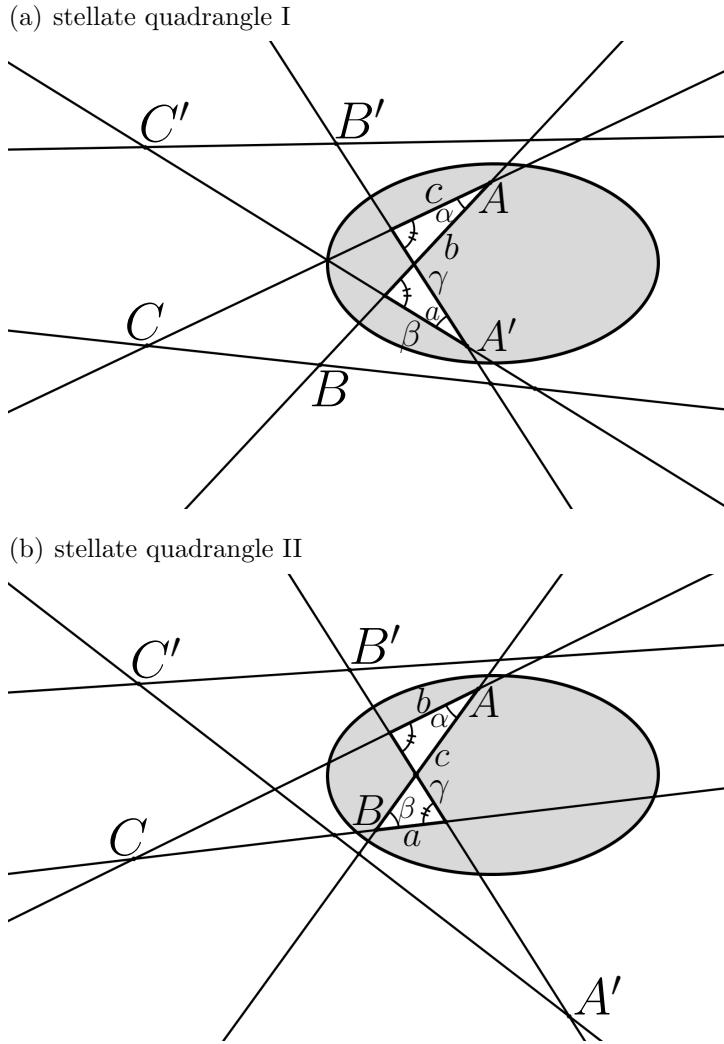


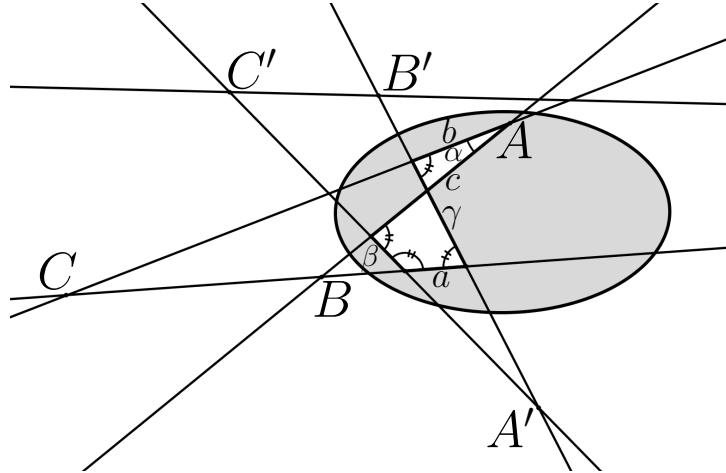
Fig. 1.3: hyperbolic stellate generalized triangles

generalized right-angled triangles are non-euclidean shadows of Menelaus' Theorem.

A generalized triangle can be constructed by pasting together two generalized right-angled triangles. This decomposition allows us to deduce the trigonometric relations of generalized triangles from the trigonometric relations of the right-angled ones. Thus, the whole non-euclidean trigonometry can be deduced from Menelaus' Theorem.

Carnot's Theorem and... Carnot's Theorem? In §7 we extend the arguments applied in §6 to Menelaus' Theorem to a theorem of Carnot on affine triangles (Theorem 7.1). We obtain that its non-euclidean shadows

(a) stellate pentagon with four right angles



(b) stellate right-angled hexagon

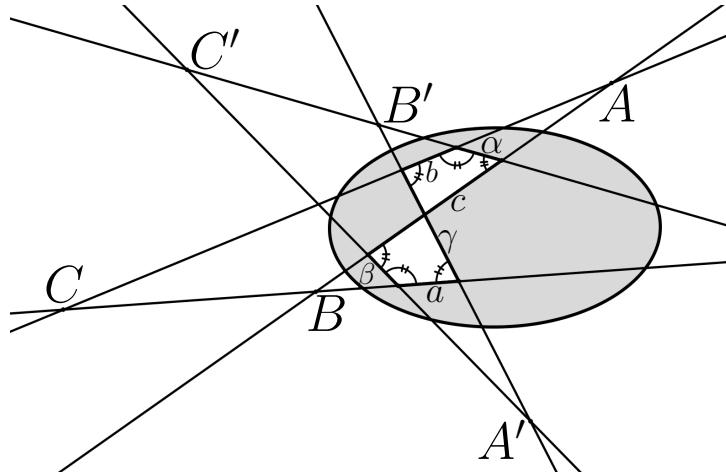


Fig. 1.4: hyperbolic stellate generalized triangles II

are the non-euclidean versions (for generalized triangles) of another classical theorem of Carnot (Theorem 7.2) for euclidean triangles!

Where do laws of cosines come from? If the projective figure behind every generalized triangle is the same (triangle and polar triangle) and the measurements of its sides and angles are given in projective terms (cross ratios), it is natural to expect that the trigonometric relations of generalized triangles have indeed a projective basis (this viewpoint has been proposed also in [18]). In §6, Menelaus' Theorem succeeds in providing all the trigonometric relations of right-angled figures and the law of sines for any, not necessarily right-angled, generalized triangle in a straightforward way.

However, Menelaus' Theorem fails with the law of cosines.

In §6, we introduce projective versions of the law of sines and the law of cosines of generalized triangles, but the non-euclidean trigonometric formulae that we obtain as translations of projective formulae are *squared*: the trigonometric functions appearing in each formula are always raised to the square power. In order to obtain the standard (*unsquared*) trigonometric formulae, we must proceed with an *unsquaring* process involving some choices of \pm signs. The unsquaring process is straightforward for all the projective trigonometric formulae with the only exception of the law of cosines. In this sense, the projective law of cosines given in §6 is unsatisfactory, and it is natural to ask if there exists a *better projective law of cosines*: a formula relating cross ratios of points of a triangle and its polar triangle with respect to Φ such that it translates in a straightforward way into the different laws of cosines for any generalized triangle.

Looking for the answer to this question, in §8 we reverse the strategy of earlier chapters. While in the rest of the book we have tried to find interesting non-euclidean theorems that can be deduced from classical projective ones, in §8 we started with some different non-euclidean results with the conviction that there must be a unique projective theorem (perhaps a non-classical one) hidden behind them. The projective formula behind all the laws of cosines is found after a detailed study of the set of midpoints of a triangle and its polar one. The midpoints allow to define a concept of “orientation” of a triangle based in purely projective techniques, and to obtain the actual, unsquared, trigonometric functions required. Although we don't recognize any classical theorem as the essence of this projective law of cosines, in its proof we will need another classical theorem of affine geometry: Van Aubel's Theorem on cevians¹.

Appendix: Laguerre's formula for rays One of the starting points of this subject, prior to the work of Cayley [5], is Laguerre's formula (3.3) expressing an euclidean angle in terms of the cross-ratio of four concurrent lines. As this formula uses a cross-ratio between *lines*, it cannot distinguish an angle from its supplement. In the appendix we propose a slight modification of Laguerre's formula in such a way that it is valid for computing angles between *rays*.

In order to make the paper more self-contained, we will introduce in §2 some basics from projective geometry, and in §3 we will present Cayley-Klein models for planar hyperbolic and elliptic geometries. Apart from the notation that we introduce in them, a reader familiar with this topic can skip §2 and §3.1.

¹The existence of this theorem was pointed out to the author by M. Avendano.

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§ 2

Basics of planar projective geometry.

We will assume that the reader is familiar with the basic concepts of real and complex planar projective geometry: the projective plane and its fundamental subsets (points, lines, pencils of lines, conics), and their projectivities (collineations, correlations). Nevertheless, we will review some concepts and results needed for understanding the rest of the paper. For rigorous definitions and proofs we refer to [7, 22], for example.

We consider the real projective plane \mathbb{RP}^2 standardly embedded in the complex projective plane \mathbb{CP}^2 . We consider the objects lying in \mathbb{CP}^2 , and they can be real or imaginary depending on how they intersect with \mathbb{RP}^2 . A point in \mathbb{CP}^2 is real if it lies in \mathbb{RP}^2 , and it is imaginary otherwise. A line l in \mathbb{CP}^2 is real if $l \cap \mathbb{RP}^2$ is a real projective line, and it is imaginary otherwise. A nondegenerate conic Φ in \mathbb{CP}^2 is real if $\Phi \cap \mathbb{RP}^2$ is a nondegenerate real projective conic, and it is imaginary otherwise.

For two points A, B in \mathbb{CP}^2 , let AB denote the line joining them. For two lines a, b in \mathbb{CP}^2 , let $a \cdot b$ denote the intersection point between them. For a line a and a conic Φ , let $a \cdot \Phi$ denote the set of intersection points between them. In \mathbb{CP}^2 , $a \cdot \Phi$ has one or two (perhaps imaginary) points, while in \mathbb{RP}^2 the intersection set $a \cdot \Phi$ can consist of 0, 1 or 2 real points. For a real line a and a real conic Φ , we say that a is *exterior*, *tangent* or *secant* to Φ if the intersection set $a \cdot \Phi$ has 0, 1 or 2 real points, respectively.

2.1 Cross ratios

Given a line $r \subset \mathbb{CP}^2$ and four different collinear points $A, B, C, D \in r$, their *cross ratio* is given by

$$(ABCD) = \frac{|AC|}{|BC|} \cdot \frac{|AD|}{|BD|}, \quad (2.1)$$

where $|XY| = Y - X$ once we have chosen a set of nonhomogeneous coordinates in r such that none of the points A, B, C, D is at infinity. This formula does not depend on the chosen set of coordinates, and so the cross ratio is well-defined.

The cross ratio $(ABCD)$ depends on the ordering of the four points A, B, C, D . By simple computations it can be checked that it fulfills the following relations:

$$(ABDC) = (BACD) = (ABCD)^{-1}, \quad (2.2a)$$

$$(ACBD) = (DBCA) = 1 - (ABCD), \quad (2.2b)$$

and for any point E collinear with A, B, C, D it is

$$(ABCD) = (ABED)(ABCE) = (EBCD)(AECD). \quad (2.2c)$$

Proposition 2.1 *The cross ratio of four different points is a number different from 0 and 1.*

Proof. If the four points A, B, C, D are different, the four numbers in the right-hand side of (2.1) are different from 0 and thus $(ABCD) \neq 0$. Applying (2.2b), we have also $(ABCD) \neq 1$. ■

Cross ratio can be defined even if some of the four points coincide, or if one of the points lies at infinity in the chosen set of coordinates. When one of the four points is the point at infinity of the line, the cross ratio coincides with a harmonic ratio of the three remaining points. For example, if D is at infinity we have

$$(ABCD) = \frac{|AC|}{|BC|}. \quad (2.3)$$

The fundamental property of cross ratio is that it is invariant under the operations of projection from a point and section with a line ¹. Let

¹ The invariance of cross-ratio under projection and section can be checked experimentally: (i) draw four collinear points on a blackboard; (ii) take a ruler and put it in front of you (with one eye closed!) making it coincide in your sight with the line containing the four points depicted; (iii) write down the numbers on the ruler that correspond to the four points in your sight; (iv) compute the cross-ratio of these four numbers. Repeat the experiment from another place of the classroom, with another ruler (cm., inches,...), etc. The resulting cross-ratio will be approximately the same.

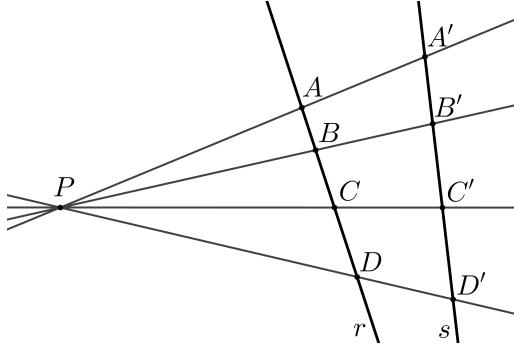


Fig. 2.1: cross ratio is invariant under projection and section

A, B, C, D be four points on a line r , let s be another line, and let P be a point not incident with r or s . Take the lines a, b, c, d joining the point P with A, B, C, D respectively, and take the points A', B', C', D' in s given by (Figure 2.1)

$$A' = a \cdot s, \quad B' = b \cdot s, \quad C' = c \cdot s, \quad D' = d \cdot s.$$

Then, the following relation holds:²

$$(ABCD) = (A'B'C'D'). \quad (2.4)$$

Property (2.4) allows us to define the cross ratio of four concurrent lines. If a, b, c, d are four concurrent lines and r is a line not concurrent with them, by taking the points A, B, C, D where r intersects a, b, c, d respectively we can define

$$(ab\bar{c}d) := (ABCD).$$

An interesting property of cross ratio for real points is the following (see [22, vol. II, Theorem 17]):

Lemma 2.2 *If A, B, C, D are four points in a real line p , then*

$$(ABCD) < 0 \iff A, B \text{ separate } C, D.$$

For the sake of simplicity, in the projective context we will consider segments and triangles just as the subsets of the set of points of the projective plane composed by their vertices, without entering deeper discussions about separation or convexity. This will not be the same in the (non-euclidean) geometric context: segments and triangles will retrieve their usual meaning when considered in the hyperbolic or elliptic plane.

²If we take one of the four lines a, b, c, d as the line at infinity and we apply (2.3), then (2.4) is Thales' theorem.

2.2 Segments

A *segment* is a pair $\{P, Q\}$ of different points of the projective plane. The segment $\{P, Q\}$ is denoted by \overline{PQ} . The points P, Q are the *endpoints* of the segment \overline{PQ} . Sometimes we will use the same name for a segment and for the line that contains it.

2.3 Triangles

A *triangle* is a set $\{P, Q, R\}$ composed by three noncollinear points (the *vertices* of the triangle). Because they are noncollinear, in particular the three vertices of a triangle are different. The triangle $\mathcal{T} = \{P, Q, R\}$ is denoted by \widehat{PQR} . A line joining two vertices of the triangle is a *side* of \mathcal{T} . The vertex P and the side q of a triangle are *opposite* to each other if $P \notin q$. If p, q, r are three nonconcurrent lines, then we denote by \widehat{pqr} the triangle having p, q, r as its sides.

2.4 Quadrangles

A *quadrangle* (see Figure 2.2) is the figure composed by four points (the *vertices* of the quadrangle), no three of which are collinear, and all the lines joining any two of them (the *sides* of the quadrangle). If the intersection point P of two sides r, s of the quadrangle is not a vertex of the quadrangle, we say that P is a *diagonal point* of the quadrangle and that r and s are *opposite sides* to each other. A quadrangle has six sides, arranged in three pairs of opposite sides, and thus it has three diagonal points. The three diagonal points of the quadrangle are noncollinear: they are the vertices of the *diagonal triangle* of the quadrangle.

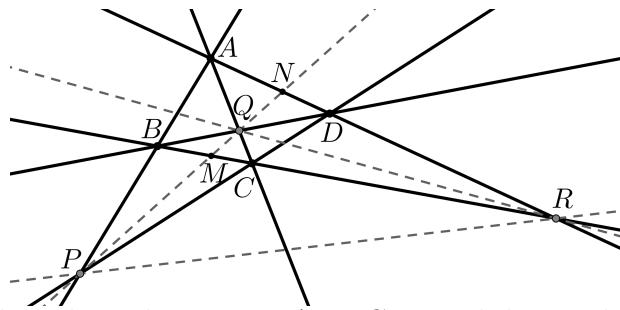


Fig. 2.2: quadrangle with vertices A, B, C, D and diagonal points P, Q, R .

2.5 Poles and polars

Let Φ be a nondegenerate conic.

The conic Φ allows us to associate to each point P of the plane a line $\rho(P)$ which is called the *polar line* of P with respect to Φ . If P does not belong to Φ , there are two (perhaps imaginary) lines u, v through P which are tangent to Φ . If U, V are the two contact points with Φ of the lines u, v respectively, then it is $\rho(P) = UV$ (see Figure 2.3(a)). If P lies on Φ , then $\rho(P)$ is the unique line tangent to Φ through P .

The map ρ is the *polarity* induced by Φ , and it is in fact a bijection: every line p of the plane has a unique point P such that $\rho(P) = p$. If p is not tangent to Φ , then $p \cdot \Phi$ is composed by two (perhaps imaginary) points U, V . In this case, if u, v are the two lines tangent to Φ at U, V respectively, the point $P = u \cdot v$ verifies that $\rho(P) = p$. If p is tangent to Φ , then $p \cdot \Phi$ contains only one point P , and it is $\rho(P) = p$. If $\rho(P) = p$, we say that P is the *pole* of p with respect to Φ and we denote also $\rho(p) = P$.

Proposition 2.3 *The polarity ρ induced by Φ is a correlation: a bijection between the set of points and the set of lines of the projective plane that preserves incidence. In particular, the poles of concurrent lines are collinear and the polars of collinear points are concurrent.*

See [22, vol. I, p.124] for a proof of this statement.

A quadrangle \mathcal{Q} is *inscribed* into Φ if the four vertices of \mathcal{Q} belong to Φ . Quadrangles are important in the theory of conics due to the following theorem (see [22, vol. I, p.123]).

Theorem 2.4 *The diagonal triangle of a quadrangle inscribed into Φ is self-polar with respect to Φ : each side is the polar of its opposite vertex with respect to Φ . Conversely, every self-polar triangle with respect to Φ is the diagonal triangle of a quadrangle inscribed into Φ .*

If Φ is a real conic and P is a real point, Theorem 2.4 provides an algorithm for drawing $\rho(P)$ even if P is interior to Φ (the tangent lines to Φ through P are imaginary in this case, so we can't draw them!): (i) draw two secant lines a, b to Φ through P and take the points A_1, A_2 , and B_1, B_2 lying in $a \cdot \Phi$ and $b \cdot \Phi$ respectively; (ii) for the quadrangle with vertices A_1, A_2, B_1, B_2 , find the two diagonal points Q, R different from P ; and (iii) the polar $\rho(P)$ of P with respect to Φ is the line QR (see Figure 2.3). In order to find the pole of a line p , take two points $A, B \in p$ and draw their polars $\rho(A), \rho(B)$: the pole of p is the intersection point $\rho(A) \cdot \rho(B)$.

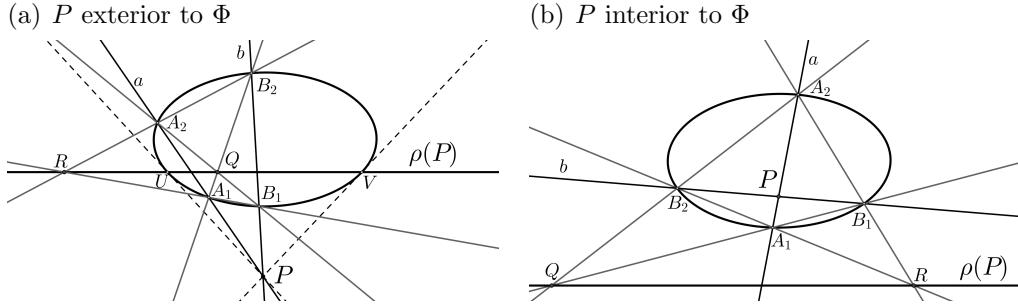


Fig. 2.3: drawing polars

2.6 Conjugate points and lines

Let p be a line not tangent to Φ . The polarity with respect to Φ defines a *conjugacy map* $\rho_p : p \rightarrow p$ given by

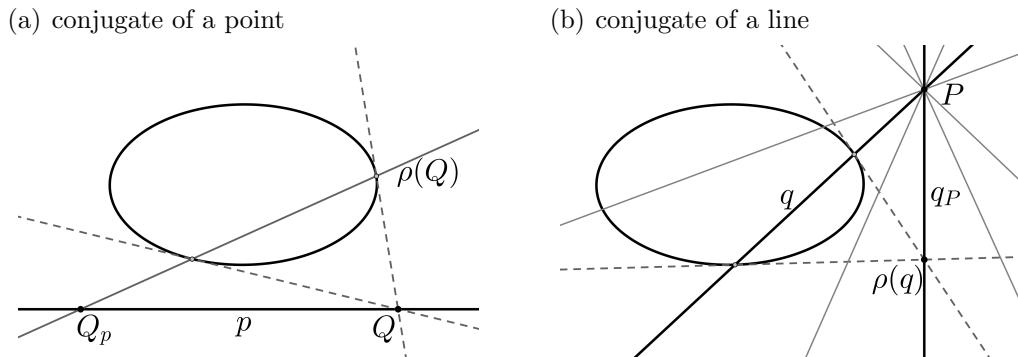
$$\rho_p(Q) = p \cdot \rho(Q) \quad \text{for all } Q \in p.$$

We say that $\rho_p(Q)$ is the *conjugate point* of Q in p with respect to Φ , and we denote it by Q_p (see Figure 2.4(a)). If P is a point not lying on Φ , we define the map ρ_P from the pencil of lines through P onto itself given by

$$\rho_P(q) = P \rho(q),$$

for any line q with $P \in q$. We say that $\rho_p(q)$ is the *conjugate line* of q with respect to Φ , and we denote it by q_P (see Figure 2.4(b)). Note that: (i) if $Q \in p$ lies on Φ (resp. q incident with P is tangent to Φ), then it is $Q_p = Q$ (resp. $q_P = q$); and (ii) the conjugacy maps ρ_p, ρ_P are involutive, that is: $(Q_p)_p = Q$ and $(q_P)_P = q$.

Polarity and conjugacy are projective transformations and therefore they preserve cross ratios: if p is a projective line and A, B, C, D are four points


 Fig. 2.4: conjugacy with respect to Φ

lying on p it is

$$(ABCD) = (\rho(A)\rho(B)\rho(C)\rho(D)) = (A_p B_p C_p D_p), \quad (2.5)$$

and if P is a point and a, b, c, d are four lines incident with P it is

$$(a b c d) = (\rho(a)\rho(b)\rho(c)\rho(d)) = (a_P b_P c_P d_P). \quad (2.6)$$

2.7 Harmonic sets and harmonic conjugacy

Two collinear segments \overline{AB} and \overline{CD} form an *harmonic set* if

$$(ABCD) = -1.$$

Quadrangles are intimately related with harmonic sets: if P, Q are two diagonal points of a quadrangle \mathcal{Q} , the two lines of the quadrangle not incident with P or Q intersect the line PQ in two points M, N verifying $(PQMN) = -1$. Indeed, if \mathcal{Q} is the quadrangle of Figure 2.2, taking the points Q_1, Q_2 given by

$$Q_1 = QR \cdot AB, \quad Q_2 = QR \cdot CD,$$

by successive projections and sections we have

$$(PQMN) = (PQ_1BA) = (PQ_2DC) = (PQNM),$$

and thus by (2.2a) it is $(PQMN) = (PQMN)^{-1}$. This only could happen if $(PQMN) = \pm 1$, but by Proposition 2.1 it cannot be $(PQMN) = 1$. Therefore it must be $(PQMN) = -1$.

Given three different collinear points A, B, C , the *harmonic conjugate of C with respect to A, B* is the unique point D , also collinear with A, B, C , such that $(ABCD) = -1$. By (2.2b), if D is the harmonic conjugate of C with respect to A, B , then D is the harmonic conjugate of C with respect to A, B . If p is the line containing all these points, the *harmonic conjugacy map* $\tau_{AB} : p \rightarrow p$ that leaves A and B fixed and maps each point of p different from A and B to its harmonic conjugate with respect to A and B is an involution of p that is, indeed, a projectivity. Every projective involution on a line has two (perhaps imaginary) fixed points, and it coincides with the harmonic conjugacy on the line with respect to its fixed points. In particular, if p is not tangent to Φ and U, V are the intersection points of p with Φ , the conjugacy ρ_p on p with respect to Φ coincides with harmonic conjugacy τ_{UV} on p with respect to U, V and for any point $A \in p$ it is

$$(UVAA_p) = -1.$$

An interesting property of harmonic sets, which will be useful later, is the following.

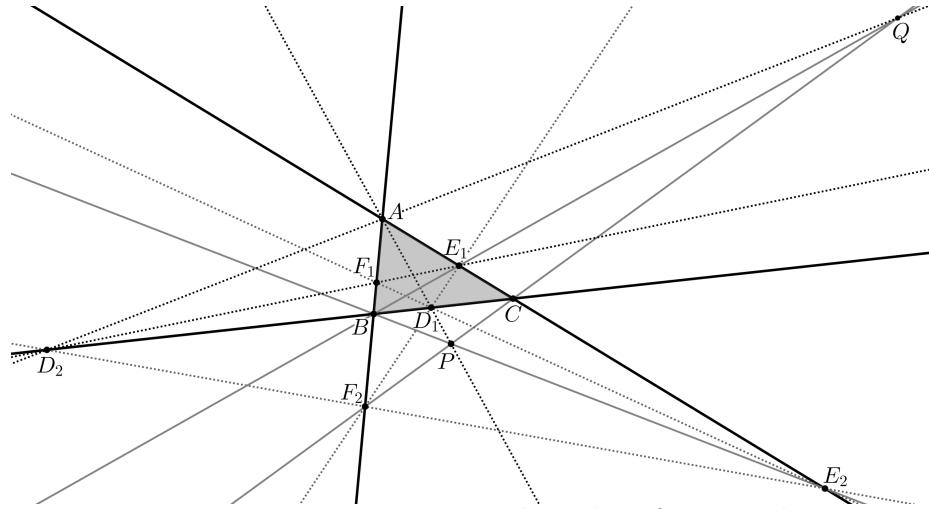


Fig. 2.5: Harmonic sets on the sides of a triangle.

Lemma 2.5 Let $\mathcal{T} = \widehat{ABC}$ be a projective triangle, and let F_1, F_2 and E_1, E_2 be two points on the side AB and on the side CA respectively such that

$$(ABF_1F_2) = (CAE_1E_2) = -1.$$

The diagonal points D_1, D_2 different from A of the quadrilateral $\mathcal{Q} = \{E_1, E_2, F_1, F_2\}$ lie on BC and $(BCD_1D_2) = -1$.

Proof. See Figure 2.5. Consider the point $P = BE_2 \cdot CF_2$, and consider also $D_1 = AP \cdot BC$. By considering the quadrangle $\{C, P, E_2, D_1\}$ we have that E_2D_1 intersects the line AB at the harmonic conjugate of F_2 with respect to A, B . That is, E_2D_1 passes through F_1 . In the same way, using the quadrangle $\{B, P, F_2, D_1\}$ for example, it is proved that E_1 is collinear with F_2 and D_1 .

Consider now the point $Q = BE_1 \cdot CF_2$, and take $D_2 = AQ \cdot BC$. By considering the quadrangle $\{C, Q, E_1, D_2\}$ we have that E_1D_2 intersects the line AB at F_1 . A similar argument can be used to conclude that D_2, E_2, F_2 are collinear.

The points D_1, D_2 so constructed are the diagonal points different from A of the quadrangle $\mathcal{Q} = \{E_1, E_2, F_1, F_2\}$ and it is $(BCD_1D_2) = -1$.

■

2.8 Conics whose polarity preserves real elements

For the rest of the paper, we will consider a nondegenerate conic Φ with respect to which the polar of a real point is always a real line³. This property is satisfied by all real conics and also by a set of imaginary conics. Moreover, two such conics are equivalent under the group of real projective collineations if and only if both are real or both are imaginary (see [22, vol. II, p. 186]). Through the whole paper, when we were talking about polar lines or points and conjugate lines or points, it must be understood that we are talking about polar lines or points, and conjugate lines or points *with respect to Φ* .

³The conics with this property are those that can be expressed by means of a 2nd degree equation with real coefficients.

§ 3

Cayley-Klein models for hyperbolic and elliptic planar geometries

In 1871, in his paper [14] Felix Klein presented his projective interpretation of the geometries of Euclid, Lobachevsky and Riemann and he introduced for them the names *parabolic*, *hyperbolic* and *elliptic*, respectively. Klein completed the study [5] just made in 1859 by Arthur Cayley by adding the geometry of Lobachevsky and studying also the three-dimensional cases (Cayley only paid attention to euclidean and spherical planar geometries). For the planar models, Klein considers the non-degenerate conic Φ in the projective plane $\mathbb{RP}^2 \subset \mathbb{CP}^2$ (*the absolute conic*). When Φ is a real conic, the interior points of Φ compose the hyperbolic plane, and when Φ is an imaginary conic the whole \mathbb{RP}^2 compose the elliptic plane. When Φ degenerates into a single line ℓ_∞ , then $\mathbb{RP}^2 \setminus \ell_\infty$ gives a model of the euclidean (parabolic) plane.

We will use the common term *the plane* \mathbb{P} either for the hyperbolic plane (when Φ is a real conic) or for the elliptic plane (when Φ is an imaginary conic). Geodesics in these models are given by the intersection with \mathbb{P} of real projective lines, and rigid motions are given by the set of real collineations that leave invariant the absolute conic.

In the hyperbolic plane, we still use expressions as “the intersection point of two lines” or “concurrent lines” in a projective sense, even if the referred point of intersection or concurrency is not interior to Φ . A common notation in hyperbolic geometry is to call “parallel” to those lines intersecting on Φ and “ultraparallel” to those lines intersecting outside Φ . If a, b are hyperbolic lines intersecting outside Φ , then $\rho(a \cdot b)$ is the common perpendicular to a and b .

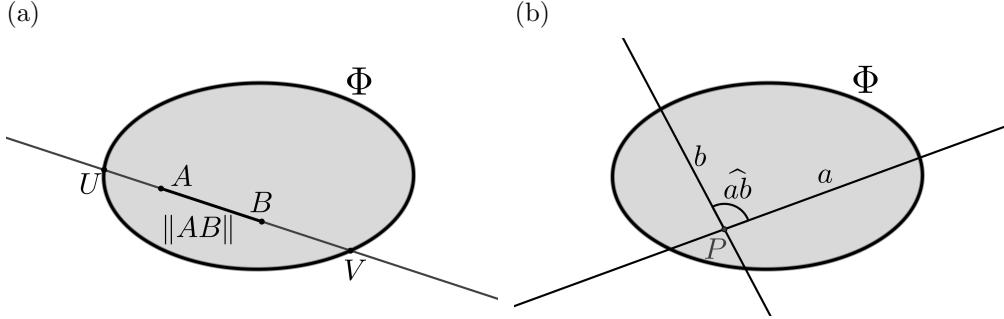


Fig. 3.1: distances and angles in the hyperbolic plane

3.1 Distances and angles

For two points $A, B \in \mathbb{P}$, let U, V be the two intersection points of the line AB with Φ . The points U, V will be two real points if the absolute conic is real (Figure 3.1(a)), and they will be imaginary if the absolute conic is imaginary. The non-euclidean distance $\|AB\|$ between them is given by

$$\|AB\| := \frac{1}{2} \log (UVAB) \quad (3.1)$$

if the absolute conic is real, and by

$$\|AB\| := \frac{1}{2i} \log (UVAB) \quad (3.2)$$

if the absolute conic is imaginary. The factors $\frac{1}{2}, \frac{1}{2i}$ before the natural logarithms of the above expressions are arbitrary, and they are taken in order to make the hyperbolic and elliptic planes' curvature equal to -1 or $+1$ respectively. In the elliptic case the total length of a line turns out to be π .

For two lines a, b such that $a \cdot b \in \mathbb{P}$, let u, v be the two tangent lines to Φ through $a \cdot b$. The lines u, v are both imaginary because of the hypothesis $a \cdot b \in \mathbb{P}$ (Figure 3.1(b)). The angle \widehat{ab} between the lines a, b at their intersection point $a \cdot b$ is given in the hyperbolic and elliptic cases by the same formula

$$\widehat{ab} := \frac{1}{2i} \log (uvab). \quad (3.3)$$

This formula is similar to the formula given in [15] for computing euclidean angles, and for this reason we will refer to it as *Laguerre's formula*. Note that \widehat{ab} is an angle between *lines*, not an angle between *rays* as usual (see [11, 13, 16]). In particular, \widehat{ab} does not distinguish an angle from its supplement and it takes values between 0 and π . We'll come back on this discussion in §8 and in the Appendix.

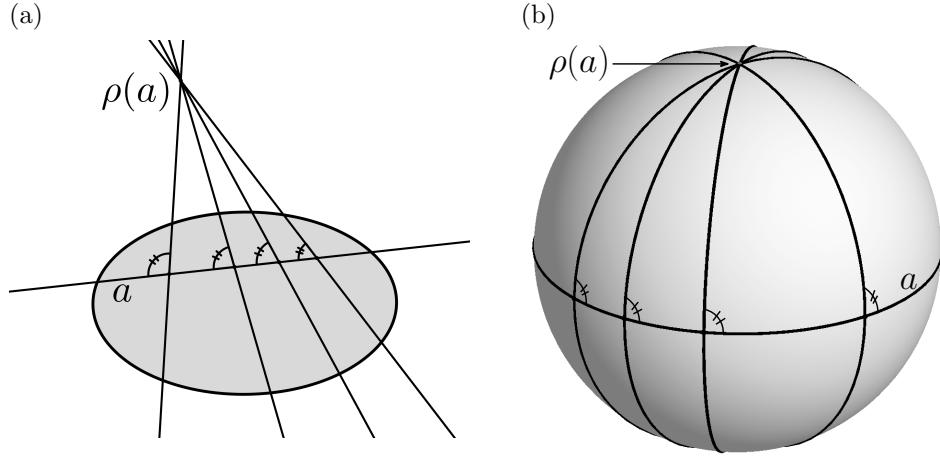


Fig. 3.2: orthogonality in Cayley-Klein models

In our pictures, we will depict the hyperbolic plane as the interior points of an ellipse in $\mathbb{R}^2 = \mathbb{RP}^2 \setminus \ell_\infty$, and the elliptic plane as a round 2-sphere of unit radius where antipodal points are identified. In this picture of the elliptic plane it is easier to visualize the pole-polar relation: lines are represented by great circles of the sphere, and the pole of a line is the pair of antipodal points representing the north and south poles when the line is chosen as the equator of the 2-sphere. This elliptic model has the virtue also of being locally isometric. In particular, angles and lengths of segments lower than π are the same in the modelling sphere as in the elliptic plane.

Conjugacy of lines with respect to the absolute characterizes perpendicularity in both models (Figure 3.2, see [19, p.36]).

Proposition 3.1 *The lines a, b such that $a \cdot b \in \mathbb{P}$ are perpendicular if and only if they are conjugate.*

3.2 Midpoints of a segment

Take a segment \overline{AB} in the projective plane whose endpoints A, B do not lie on Φ and such that the line $p = AB$ is not tangent to Φ . Let P be the pole of p , let a, b be the lines joining P with A, B respectively, and let A_1, A_2 and B_1, B_2 be the intersection points with Φ of a and b respectively (Figure 3.3(a)). The points A_1, A_2, B_1, B_2 are the vertices of a quadrangle \mathcal{Q} inscribed in Φ having P as a diagonal point. By Theorem 2.4, the other two diagonal points Q, R of \mathcal{Q} lie in p and it is $R = Q_p$. The segments \overline{AB} and $\overline{QQ_p}$ form a harmonic set and so it is $(ABQQ_p) = -1$.

Let U, V be the two intersection points of p with Φ . As

$$(UVQQ_p) = (ABQQ_p) = -1,$$

the harmonic conjugacy of p with respect to Q, Q_p sends the points U, V, A, B into the points V, U, B, A respectively. In particular, it is $(UVAQ) = (VUBQ)$, and by (2.2a) it is $(VUBQ) = (UVQB)$. By (2.2c), we have

$$(UVAB) = (UVAQ)(UVQB) = (UVAQ)^2. \quad (3.4)$$

In the same way, it can be checked that

$$(UVAB) = (UVAQ_p)^2 = (UVQB)^2 = (UVQ_pB)^2. \quad (3.5)$$

If the points A, B, Q belong to \mathbb{P} , from the distance formulae (3.1) and (3.2), we have

$$\|AB\| = 2\|AQ\| = 2\|BQ\|.$$

This property suggest the following definition:

Definition 3.2 *The points Q, Q_p are the midpoints of the segment \overline{AB} .*

If $A, B \in \mathbb{P}$, when Φ is a real conic there is exactly one of the midpoints of the projective segment \overline{AB} , say Q , lying in the hyperbolic segment bounded by A and B . In this case, Q is the geometric midpoint of the segment and Q_p is the pole of the perpendicular bisector of this segment. When Φ is an imaginary conic, then A and B divide the line AB into two elliptic segments whose respective midpoints are Q and Q_p .

It is clear that the lines PQ, PQ_p are the polars of Q_p, Q respectively. As these lines are orthogonal to p through the midpoints of \overline{AB} , they are the *segment bisectors* or simply *bisectors* of \overline{AB} . After dualizing:

Remark 3.3 *The polars of the midpoints of \overline{AB} are the angle bisectors or bisectors of $\rho(A)\rho(B)$.*

An interesting characterization of midpoints is the following:

Lemma 3.4 *Let p be a line not tangent to Φ , let $p \cdot \Phi = \{U, V\}$, and take two points $A, B \in p$ different from U, V . If C, D are two points of AB such that*

$$(ABCD) = (UVCD) = -1,$$

then C, D are the midpoints of \overline{AB} .

Proof. Let C, D be two points of p such that $(ABCD) = (UVCD) = -1$. We have seen that the midpoints Q, Q_p of \overline{AB} verify also the identity $(UVQQ_p) = (ABQQ_p) = -1$. If we take the harmonic involutions τ_{QQ_p} and τ_{CD} , of p with fixed points Q, Q_p and C, D respectively, the composition $\tau_{QQ_p} \circ \tau_{CD}$ fixes the points U, V, A, B and so it must be the identity on p . This implies that $\{Q, Q_p\} = \{C, D\}$. ■

An immediate consequence of this lemma is:

Lemma 3.5 *Let A, B be two points not in Φ such that $p = AB$ is not tangent to Φ . Then, the midpoints of \overline{AB} are also the midpoints of $\overline{A_pB_p}$.*

Proof. Let U, V be the intersection points of p with Φ , and let Q, Q_p be the midpoints of \overline{AB} . It is clear that $(UVQQ_p) = -1$. If we apply on p the conjugacy ρ_p with respect to Φ , we get

$$(ABQQ_p) = (A_pB_pQ_pQ) = -1,$$

and thus by Lemma 3.4 Q, Q_p are the midpoints of $\overline{A_pB_p}$. ■

The previous lemma gives an alternative way for constructing the midpoints of \overline{AB} using polars (Figure 3.3(b)): (i) take the polars $\rho(A), \rho(B)$; (ii) take the intersection points A'_1, A'_2 and B'_1, B'_2 of $\rho(A)$ and $\rho(B)$ respectively with Φ ; and (iii) the midpoints of \overline{AB} are the diagonal points of the quadrangle $\{A'_1, A'_2, B'_1, B'_2\}$ different from $\rho(AB)$. This is the result of applying the original construction of the midpoints to the segment $\overline{A_pB_p}$.

In the general case, we will work with segments such that the line they belong to is *in general position* with respect to Φ , that is, not tangent to Φ . Nevertheless, sometimes we will talk about the midpoints of a segment whose underlying line could be tangent to Φ . For dealing with such limit cases, we extend the concept of “midpoint” to such segments in the natural way. If in Figure 3.3(b) we move continuously the points A, B in order to make p tangent to Φ while A, B remain out of Φ , one of the points A'_1, A'_2 (say A'_1) become coincident with one of the points B'_1, B'_2 (say B'_1) and with one of the points Q, Q_p (say Q) at the contact point of p with Φ (see Figure 3.4). The other midpoint Q_p will become the point $p \cdot A'_2B'_2$ and the identity $(ABQQ_p) = -1$ remains unchanged.

Definition 3.6 *Let A, B be two points not lying on Φ such that the line AB is tangent to Φ at a point Q . We say that the midpoints of the segment \overline{AB} are Q and the harmonic conjugate of Q with respect to A, B .*

Lemma 3.4 inspires the following notation. If Q is a point not in Φ and p is a line through Q not tangent to Φ , we will say that the harmonic involution τ_{QQ_p} of p with respect to Q and Q_p is the *symmetry of p with*

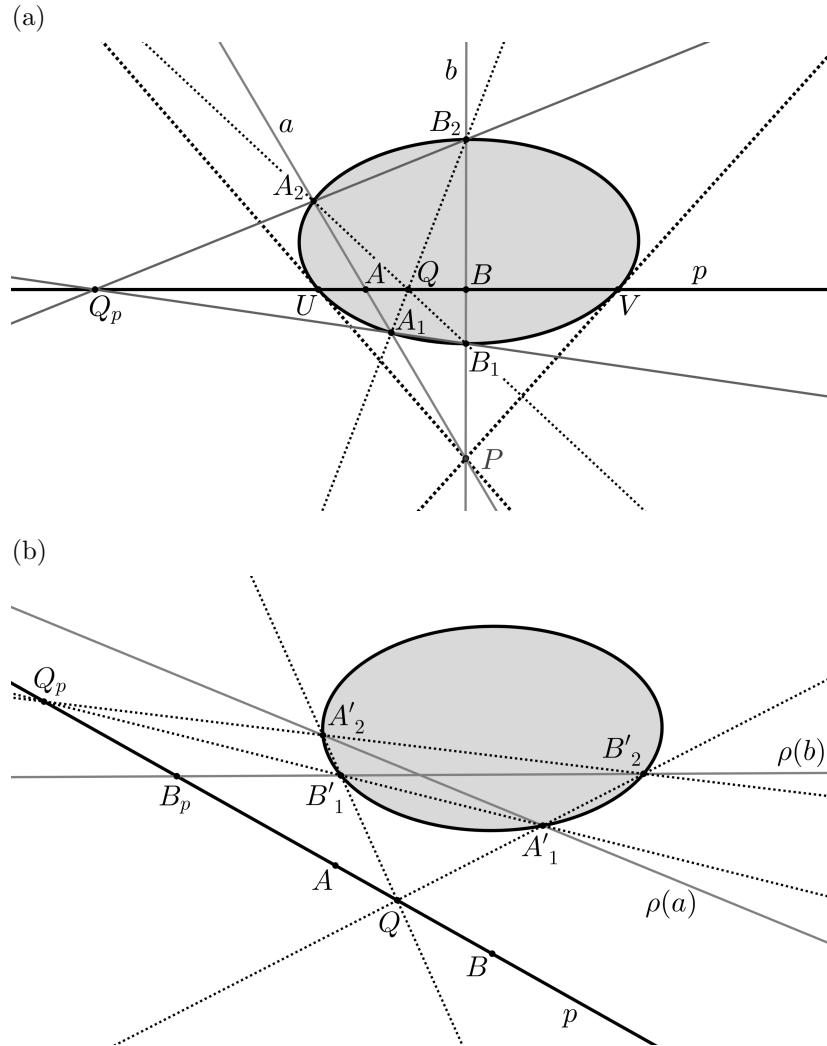


Fig. 3.3: midpoints of a segment

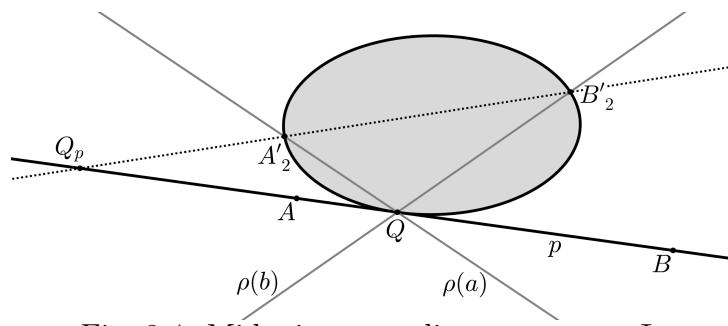


Fig. 3.4: Midpoints on a line tangent to Φ

respect to Q . If we consider the symmetry with respect to Q acting on every line through Q not tangent to Φ at once, we obtain the *symmetry of \mathbb{P} with respect to Q* , which corresponds projectively to the harmonic homology of the projective plane with center Q and axis $\rho(Q)$.

3.3 Proyective trigonometric ratios

The points U, V of (3.2) and the lines u, v of (3.3) are imaginary. This is the reason why it is more usual to use another cross ratios when working with distances and angles.

Theorem 3.7 *Let A, B be two points in \mathbb{P} , and let p be the line joining them. If \mathbb{P} is the hyperbolic plane, then*

$$(ABB_p A_p) = \cosh^2 \|AB\|; \quad (3.6)$$

and if \mathbb{P} is the elliptic plane, then

$$(ABB_p A_p) = \cos^2 \|AB\|. \quad (3.7)$$

Let a, b be two projective lines such that their intersection point P lies in \mathbb{P} . Then,

$$(abb_P a_P) = \cos^2 \widehat{ab}. \quad (3.8)$$

For proving this theorem, we need the following lemma:

Lemma 3.8 *Let p be a projective line not tangent to Φ , let U, V be the two intersection points of p with Φ , and let A, B be two points of p different from U, V . Then*

$$4(ABB_p A_p) = (UVAB) + (UVBA) + 2. \quad (3.9)$$

Identity (3.9) appears in [19, p. 24]. We give a different proof.

Proof. We will make a reiterated use of cross ratio identities (2.2). We have

$$\begin{aligned} (ABB_p A_p) &= (B_p A_p AB) = (UA_p AB)(B_p UAB) = \\ &= (VA_p AB)(UVAB)(B_p UAB). \end{aligned}$$

If Q, Q_p are the two midpoints of the segment \overline{AB} , the symmetry τ_{QQ_p} sends U, A, A_p into V, B, B_p respectively and vice versa. So it is

$$(VA_p AB) = (A_p VBA) = (B_p UAB),$$

Φ	p	A	B	$\mathbf{C}(c)$	$\mathbf{S}(c)$	$\mathbf{T}(c)$
im	—	—	—	$\cos^2 \ AB\ $ $\cos^2 \widehat{ab}$	$\sin^2 \ AB\ $ $\sin^2 \widehat{ab}$	$\tan^2 \ AB\ $ $\tan^2 \widehat{ab}$
real	sec	int	int	$\cosh^2 \ AB\ $ $-\sinh^2 \ AB_p\ $	$-\sinh^2 \ AB\ $ $\cosh^2 \ AB_p\ $	$-\tanh^2 \ AB\ $ $-\coth^2 \ AB_p\ $
			ext	$\cosh^2 \ A_pB_p\ $ $-\sinh^2 \ A_pB\ $	$-\sinh^2 \ A_pB_p\ $ $\cosh^2 \ A_pB\ $	$-\tanh^2 \ A_pB_p\ $ $-\coth^2 \ A_pB\ $
		ext	int	$\cos^2 \widehat{ab}$	$\sin^2 \widehat{ab}$	$\tan^2 \widehat{ab}$
	ext	ext	ext			

Table 3.1: non-euclidean translations of projective trigonometric ratios.

and therefore

$$(ABB_p A_p) = (UVAB)(B_p UAB)^2.$$

The cross ratio $(B_p UAB)$ can be splitted as $(B_p UVB)(B_p UAV)$, and because of the identity $(UVB_p B) = -1$, we have

$$\begin{aligned} (B_p UVB) &= \frac{1}{(UB_p VB)} = \frac{1}{1 - (UVB_p B)} = \frac{1}{2}; \\ (B_p UAV) &= (UB_p VA) = 1 - (UVB_p A) = 1 - (UVBA)(UVB_p B) = \\ &= 1 + (UVBA); \end{aligned}$$

which finally gives

$$(ABB_p A_p) = (UVAB)(B_p UAB)^2 = \frac{1}{4}(UVAB)[1 + (UVBA)]^2.$$

The proof now follows from (2.2a). ■

Proof of Theorem 3.7. We will prove only (3.6). The identities (3.7) and (3.8) can be proved in a similar way.

Let assume that \mathbb{P} is the hyperbolic plane, and let denote $\|AB\|$ simply by d . Then,

$$\cosh d = \frac{1}{2}(e^d + e^{-d}) = \frac{1}{2}[(UVAB)^{1/2} + (UVAB)^{-1/2}],$$

and so by Lemma 3.8 it is

$$\cosh^2 d = \frac{1}{4}[(UVAB) + (UVBA) + 2] = (ABB_p A_p).$$

■
Expresions (3.6–3.8) suggest the following notation. Take a segment $c = \overline{AB}$ whose endpoints $A, B \in \mathbb{RP}^2$ don't lie on Φ and such that the line

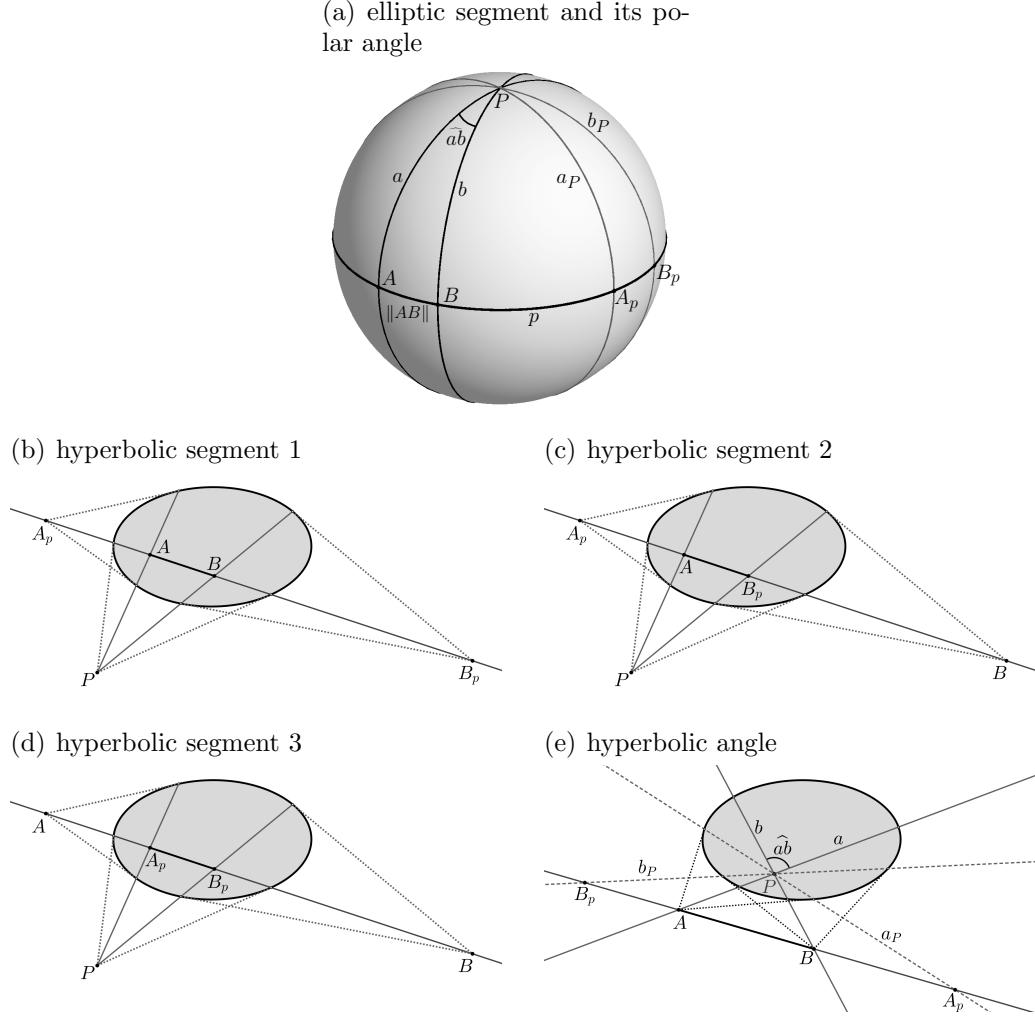


Fig. 3.5: Positions of a projective segment with respect to the conic

AB containing c is not tangent to Φ . We define the *projective trigonometric ratios* of the segment c (compare [23]) as:

$$\begin{aligned}\mathbf{C}(c) &:= (ABB_cA_c) \\ \mathbf{S}(c) &:= 1 - \mathbf{C}(c) = (AB_cBA_c) \\ \mathbf{T}(c) &:= \frac{\mathbf{S}(c)}{\mathbf{C}(c)} = \frac{1}{(ABB_cA_c)} - 1 = \\ &= (ABA_cB_c) - 1 = -(AA_cBB_c).\end{aligned}$$

In the limit case when $B = A_c$, we take

$$\mathbf{C}(c) = 0, \quad \mathbf{S}(c) = 1 \quad \text{and} \quad \mathcal{T}(c) = \infty,$$

and we say that the segment $\overline{AA_c}$ is *right*. The segments $\overline{A_cB}$ and $\overline{AB_c}$ are *complementary segments* of \overline{AB} . The identities (2.2a) and (2.2b) imply that $\mathbf{C}(c)$, $\mathbf{S}(c)$ and $\mathbf{T}(c)$ do not depend on the chosen ordering of the endpoints A, B of the segment c . Proposition 3.9 below follows directly from Theorem 3.7 and from the properties (2.2a) and (2.2b) of cross ratios (see Figure 3.5).

Proposition 3.9 *Depending on the type of absolute conic (real or imaginary) we are working with and, when Φ is a real conic, on the relative positions of the endpoints A, B of the segment c with respect to Φ (Figure 3.5), the projective trigonometric ratios \mathbf{C} , \mathbf{S} and \mathbf{T} have the non-euclidean trigonometric translations listed in Table 3.1.*

In Table 3.1, *im*, *ext*, *sec* and *int* mean *imaginary*, *exterior to Φ* , *secant to Φ* and *interior to Φ* , respectively.

§ 4

Pascal's and Chasles' theorems and classical triangle centers

We will deal with the most common triangle centers for generalized triangles: orthocenter, circumcenter, barycenter and incenter. Although they are very well-known for triangles, their projective interpretation provides equivalent centers for any generalized triangle.

4.1 Quad theorems

Let us state without proof four classical theorems of projective planar geometry.

Theorem 4.1 (Desargues' Theorem) *Let \widehat{ABC} and $\widehat{A'B'C'}$ be two projective triangles. The lines AA' , BB' , CC' are concurrent if and only if the*

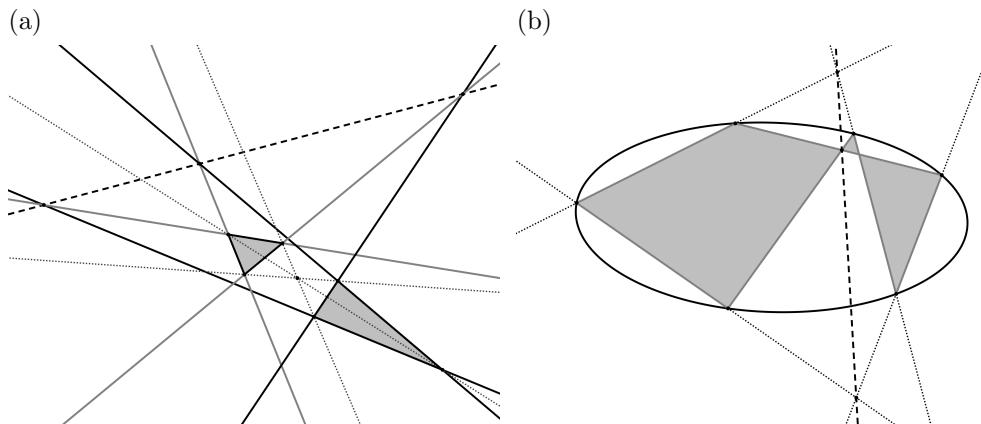


Fig. 4.1: Desargues' and Pascal's Theorems

points

$$AB \cdot A'B', \quad BC \cdot B'C', \quad CA \cdot C'A'$$

are collinear.

Theorem 4.2 (Pascal's Theorem) *If an hexagon is inscribed in a conic, the three pairs of opposite sides meet in collinear points.*

Theorem 4.3 (Chasles' polar triangle Theorem) *A triangle and its polar triangle with respect to a conic are perspective.*

Theorem 4.4 (Pappus' Involution Theorem) *The three pairs of opposite sides of a quadrangle meet any line (not through a vertex) in three pairs of an involution.*

Theorem 4.4 is a partial version of Desargues' Involution Theorem (see [7, p. 81]). Using this theorem, a given quadrangle in the projective plane determines a *quadrangular involution* on every line not through a vertex. If \mathcal{Q} is a quadrangle, we usually denote by $\sigma_{\mathcal{Q}}$ the quadrangular involution that it induces on the line considered. In the limit case where the line passes through a vertex of \mathcal{Q} , this vertex is a double point of $\sigma_{\mathcal{Q}}$. For the sake of simplicity, Chasles' polar triangle Theorem will be called "Chasles' Theorem".

4.2 Triangle notation

Along the whole text we will deal with a projective triangle \mathcal{T} with vertices A, B, C and sides

$$a = \overline{BC}, \quad b = \overline{CA}, \quad c = \overline{AB}.$$

The polar triangle of \mathcal{T} will be denoted by \mathcal{T}' , its sides a', b', c' are the polars of A, B, C respectively with respect to the absolute conic Φ , and its vertices

$$A' = b' \cdot c', \quad B' = c' \cdot a', \quad C' = a' \cdot b',$$

are the poles of a, b, c respectively. We will assume always that \mathcal{T} and \mathcal{T}' are *in general position*: the vertices and sides of \mathcal{T} and \mathcal{T}' are all different and no vertex from \mathcal{T} or \mathcal{T}' lies on Φ . This last condition implies also that no side of \mathcal{T} or \mathcal{T}' is tangent to Φ . Depending on the type of conic (real or imaginary) we are working with and, in the case of a real conic, on the relative position of the triangle \mathcal{T} with respect to the conic Φ , the

triangles \mathcal{T} and \mathcal{T}' can produce in \mathbb{P} any of the generalized triangles of Figures 1.1–1.4.

Inspired by Proposition 3.1, we say that \mathcal{T} is *right-angled* if two of its sides are conjugate to each other. In this case, only two of its sides can be conjugate because if the sides a and b are both conjugate to c , it turns out that $a \cdot b$ is the pole of c , in contradiction with the general position assumptions.

The conjugate points of the vertices of \mathcal{T} at the sides they belong to are

$$\begin{aligned} A_b &= b \cdot a' & B_c &= c \cdot b' & C_a &= a \cdot c' \\ A_c &= c \cdot a' & B_a &= a \cdot b' & C_b &= b \cdot c'. \end{aligned}$$

In the same way, the conjugate lines of the sides of \mathcal{T} are

$$\begin{aligned} a_B &= BA' & b_C &= CB' & c_A &= AC' \\ a_C &= CA' & b_A &= AB' & c_B &= BC'. \end{aligned}$$

Note that it is

$$\begin{aligned} A_b &= B'_{a'} & B_c &= C'_{b'} & C_a &= A'_{c'} & A_c &= C'_{a'} & B_a &= A'_{b'} a & C_b &= B'_{c'} \\ a_B &= b'_{A'} & b_C &= c'_{B'} & c_A &= a'_{C'} & a_C &= c'_{A'} & b_A &= a'_{B'} & c_B &= b'_{C'}. \end{aligned}$$

The midpoints of the segments \overline{BC} , \overline{CA} and \overline{AB} are D, D_a, E, E_b and F, F_c respectively. Equivalently, the midpoints of $\overline{B'C'}, \overline{C'A'}$ and $\overline{A'B'}$ are denoted $D', D'_{a'}, E', E_b'$ and $F', F'_{c'}$ respectively.

Most of all these points and lines are depicted in Figure 4.2.

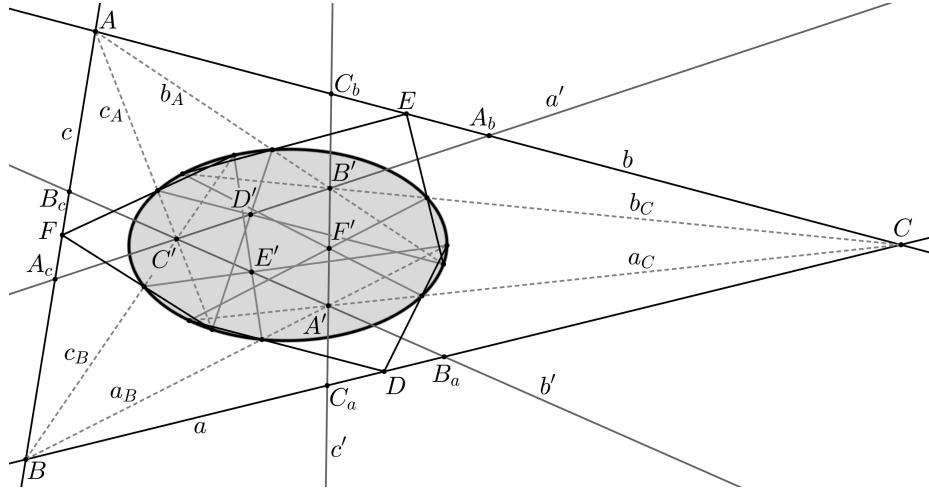


Fig. 4.2: Triangle and its polar one. Midpoints and conjugate points and lines

As the lines a, a' are different, among the points $D, D_a, D', D'_{a'}$, there must be at least three different points. In some cases, a midpoint of \overline{BC}

and a midpoint of $\overline{B'C'}$ could coincide at the intersection point A_0 of a with a' . We want to clarify what happens in this limit situation. Note that A_0 is also the pole of the line AA' .

Lemma 4.5 *Assume that D and D' are both different from A_0 . If one of the points $D_a, D'_{a'}$ coincide with A_0 , the other one coincides too, and this happens if and only if the lines AA' and DD' coincide.*

Proof. Assume that $D'_{a'} = A_0$. This implies that the polar AA' of A_0 passes through D' . The point $H_A = AA' \cdot a$ is the conjugate point of A_0 in a . Projecting the line a' onto a since A' we get

$$-1 = (B'C'D'A_0) = (C_a B_a H_A A_0).$$

By Lemmas 3.5 and 8.10 this implies that A_0, H_A are the midpoints of \overline{BC} .

On the other hand, if $AA' = DD'$, the polar of D passes through the pole of AA' , which is A_0 and so it is $D_a = A_0$, and equivalently it is $D'_{a'} = A_0$.

■

The line AA' is the line orthogonal to a through A , and therefore it is the *altitude* of \mathcal{T} through A . In the situation of previous lemma, in the limit case when D_a and $D'_{a'}$ coincide, the altitude AA' is also a *segment bisector* of \overline{BC} (as it passes through D), and an *angle bisector* of bc (as it passes through D' , cf. Remark 3.3). This is the reason why in this situation we say that the triangle \mathcal{T} is *isosceles* at A . The triangle \mathcal{T} is *equilateral* if it is isosceles at its three vertices. As we can expect, if \mathcal{T} is isosceles at A the sides incident with A have the same “length”: let B_1, B_2 and C_1, C_2 be the intersection points of b and c with the absolute conic Φ ; then

Proposition 4.6 *The triangle \mathcal{T} is isosceles at A if and only if we can label B_1, B_2, C_1, C_2 such that*

$$(B_1 B_2 A C) = (C_1 C_2 A B). \quad (4.1)$$

Proof. If \mathcal{T} is isosceles at A , the midpoint $D'_{a'}$ of $\overline{B'C'}$ coincides with A_0 . By §3.2, the midpoints of $\overline{B'C'}$ are the diagonal points of the quadrangle $\{B_1, B_2, C_1, C_2\}$ different from A , so we can label B_1, B_2, C_1, C_2 such that $B_1 C_1 \cdot B_2 C_2 = D'_{a'} = A_0$, and (4.1) follows from projection and section since A_0 .

On the other hand, if (4.1) holds, we consider the midpoint

$$D'_* = B_1 C_1 \cdot B_2 C_2$$

of $\overline{B'C'}$. If we take the point $C_* = BD'_* \cdot b$, it turns out that

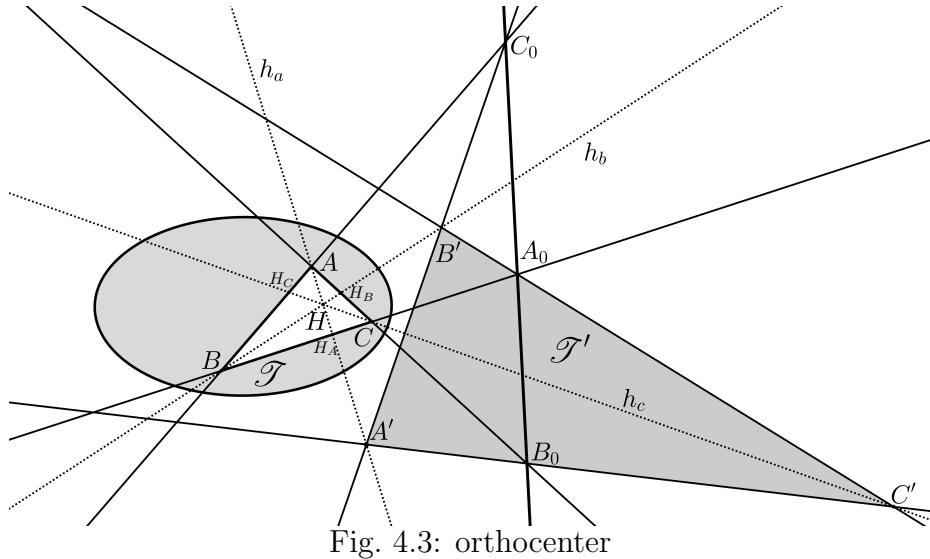
$$(B_1 B_2 A C_*) = (C_1 C_2 A B)$$

by projection and section since D'_* , and then (4.1) implies that $C_* = C$. Therefore, D'_* belongs to BC , it coincides with A_0 and the triangle \mathcal{T} is isosceles at A . ■

Proposition 4.7 *If \mathcal{T} is isosceles at two vertices, it is equilateral.* ■

As in the hypotheses of Lemma 4.5, we will always assume that D and D' are different from A_0 . In the same way, taking the points $B_0 = b \cdot b'$ and $C_0 = c \cdot c'$, we will always assume that E and E' are different from B_0 and that F and F' are different from C_0 . In particular, it must be $D \neq D'$, $E \neq E'$ and $F \neq F'$.

4.3 Chasles' Theorem and the orthocenter



The altitudes of \mathcal{T} through the vertices A, B and C are the lines

$$h_a = AA', \quad h_b = BB' \quad \text{and} \quad h_c = CC',$$

respectively. A straightforward consequence of Chasles' Theorem is:

Theorem 4.8 (Concurrency of altitudes) *The lines h_a, h_b and h_c are concurrent.*

We say that the intersection point H of h_a, h_b and h_c is the *orthocenter* of \mathcal{T} . Note that the orthocenter of \mathcal{T} is also the orthocenter of \mathcal{T}' .

The poles of h_a, h_b, h_c are the points A_0, B_0, C_0 respectively. Because h_a, h_b, h_c intersect at the point H , their poles A_0, B_0, C_0 lie on the polar h of H . In our constructions of §5.1 to §5.4 the line h will have the same relation to the triangle \mathcal{T} as the line at infinity has with any Euclidean triangle.

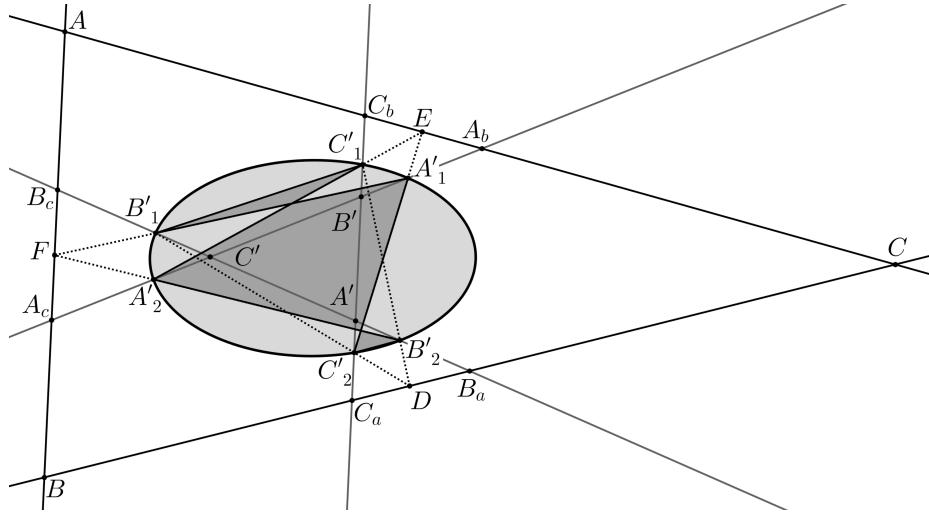


Fig. 4.4: Pascal's Theorem and the midpoints of a triangle

4.4 Pascal's Theorem and classical centers

First of all, we will introduce on the points D, D', E, E', F, F' another assumption additional to those of the last paragraph of §4.2: we will assume from now on that D, E, F are non-collinear and that D', E', F' are non-collinear too. We show how this always can be done with the points D, E, F . If \mathcal{T} is not isosceles at any vertex: (i) we take as D and E any of the midpoints of \overline{BC} and \overline{CA} respectively; (ii) as D, E cannot be collinear with both F, F_c , we choose as F a midpoint of \overline{AB} not collinear with D, E ; and (iii) this point F will be different from C_0 because \mathcal{T} is not isosceles at C . If \mathcal{T} is isosceles only at A : (i) we choose as D the midpoint of \overline{BC} different from A_0 ; (ii) we choose as E any of the midpoints of \overline{CA} ; and (iii) repeat steps (ii) and (iii) of the non-isosceles case. Finally, if \mathcal{T} is equilateral, we take as D, E, F the midpoint of $\overline{BC}, \overline{CA}, \overline{AB}$ different from A_0, B_0, C_0 respectively. In this case, as A_0, B_0, C_0 are collinear it follows from Lemma 2.5 that D, E, F cannot be collinear.

By Lemma 2.5, we know that $EF \cdot E_b F_c$ and $EF_c \cdot E_b F$ are two points on a which are harmonic conjugates with respect to B, C . In fact, these two points are D, D_a .

Lemma 4.9 *The points D_a, E, F are collinear. The points D, D_a are the diagonal points different from A of the quadrangle $\{E, E_b, F, F_c\}$.*

Proof. Let A'_1, A'_2, B'_1, B'_2 and C'_1, C'_2 be the intersection points of a', b' and c' respectively with Φ . By §3.2, the midpoints of \overline{AB} are the diagonal points different from C' of the quadrangle $\{A'_1, A'_2, B'_1, B'_2\}$. In the same way, the

midpoints of \overline{BC} are the diagonal points different from A' of the quadrangle $\{B'_1, B'_2, C'_1, C'_2\}$, and the midpoints of \overline{CA} are the diagonal points different from B' of the quadrangle $\{C'_1, C'_2, A'_1, A'_2\}$.

Imagine that we have given the names A'_1, A'_2, B'_1, B'_2 and C'_1, C'_2 to the points of $a' \cdot \Phi$, $b' \cdot \Phi$ and $c' \cdot \Phi$ respectively in such a way that

$$D = B'_1 C'_2 \cdot B'_2 C'_1 \quad \text{and} \quad E = C'_1 A'_2 \cdot C'_2 A'_1.$$

Taking the hexagon $A'_1 B'_1 C'_1 A'_2 B'_2 C'_2$ inscribed in Φ , by Pascal's Theorem the points E, F are collinear with the point $B'_1 C'_1 \cdot B'_2 C'_2$, which is a midpoint of \overline{BC} (see Figure 4.4). Because we have assumed that D, E, F are non-collinear, it must be $D_a = B'_1 C'_1 \cdot B'_2 C'_2$.

Taking the hexagon $B'_1 C'_1 A'_1 B'_2 C'_2 A'_2$, the points E_b, F_c are also collinear with D_a and therefore D_a is a diagonal point of the quadrangle $\{E, E_b, F, F_c\}$. In the same way it can be proved that D is another diagonal point of $\{E, E_b, F, F_c\}$. ■

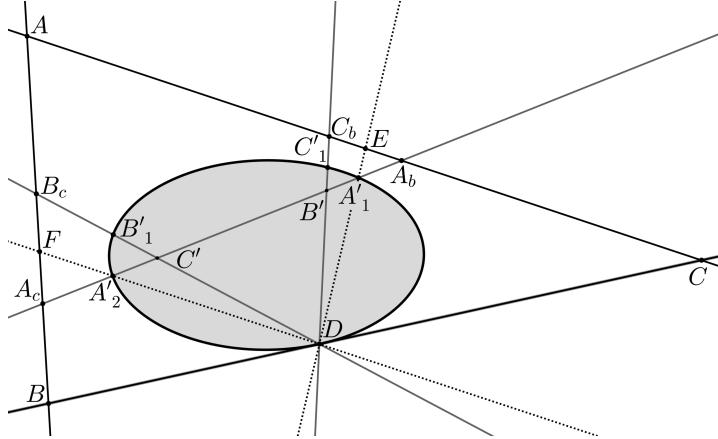
As the polar of D_a is the bisector of \overline{BC} through D , straightforward shadows of this lemma are (compare [19, Thm. 7]):

Theorem 4.10 *Let T be an elliptic or hyperbolic triangle, and let $\frac{1}{2}T$ be a medial triangle of T , i.e. a triangle whose vertices are midpoints of the sides of T . The side bisectors of T through the vertices of $\frac{1}{2}T$ are the altitudes of $\frac{1}{2}T$.*

Theorem 4.11 *Let H be a hyperbolic right-angled hexagon, and let $\frac{1}{2}H$ be a triangle whose vertices are the midpoints of alternate sides of H . The side bisectors of H through the vertices of $\frac{1}{2}H$ are the altitudes of $\frac{1}{2}H$.*

The dual figure of a quadrangle is a *quadrilateral*: the figure composed by four lines not three of which are concurrent (the *sides* of the quadrilateral), and the six points at which these four lines intersect in pairs (the *vertices* of the quadrilateral). For any vertex P of a quadrilateral \mathcal{M} there is exactly another vertex Q of \mathcal{M} such that the line PQ is not a side of \mathcal{M} . We say that P and Q are *opposite vertices* of \mathcal{M} and that PQ is a *diagonal line* of \mathcal{M} . The three diagonal lines of \mathcal{M} compose the *diagonal triangle* of \mathcal{M} . A glance at Figure 2.2 gives the following result.

Lemma 4.12 *Let \mathcal{Q} be a quadrangle, let P be a diagonal point of \mathcal{Q} and let x, y, z, w be the four sides of \mathcal{Q} not passing through P . Then, the vertices of the quadrilateral $\{x, y, z, w\}$ are the vertices of \mathcal{Q} together with the two diagonal points of \mathcal{Q} different from P . The diagonal triangle of $\{x, y, z, w\}$ has as sides the sides of \mathcal{Q} passing through P and the side of the diagonal triangle of \mathcal{Q} not passing through P .*


 Fig. 4.5: Triangle with one side tangent to Φ

The following is an interesting property of midpoints (compare [18, Theorem 22.5]). Its proof is straightforward from Lemmas 4.9 and 4.12.

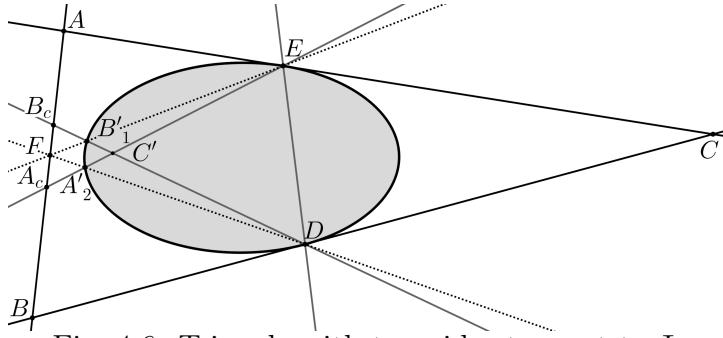
Theorem 4.13 *The midpoints of the sides of \mathcal{T} are the vertices of a quadrilateral $\mathcal{M}_{\mathcal{T}}$ whose diagonal triangle is \mathcal{T} .*

We say that $\mathcal{M}_{\mathcal{T}}$ is the *midpoint quadrilateral* of \mathcal{T} . This theorem has many important consequences as it allows to prove the concurrence of medians, the concurrence of side bisectors and (after dualizing) the concurrence of angle bisectors of generalized triangles.

Although we are interested in triangles \mathcal{T} , \mathcal{T}' in general position with respect to Φ , in some of our constructions we will need to use some accessory triangles which could not verify the general position assumptions. In particular, we wonder if Theorem 4.13 is valid for triangles with sides tangent to Φ .

Lemma 4.14 *Theorem 4.13 remains valid even if the triangle \widehat{ABC} has sides tangent to Φ .*

Proof. Let assume that the side $a = BC$ is tangent to Φ , while AB and CA are not. Let D be the point of tangency of a with Φ . The polars b', c' of B, C respectively pass through D . Let A'_1, A'_2 be the intersection points of a' with Φ , and let B'_1, C'_1 be the intersection points of b', c' with Φ respectively different from D (Figure 4.5). The midpoints of \overline{AB} are the diagonal points different from C' of the quadrangle $\{A'_1, A'_2, B'_1, D\}$, and the midpoints of \overline{CA} are the diagonal points different from B' of the quadrangle $\{A'_1, A'_2, C'_1, D\}$. Thus, the line DA'_1 pass through a midpoint of \overline{AB} and through a midpoint of \overline{CA} , and so does the line DA'_2 . In other words, D is a diagonal point of the quadrangle $\{E, E_b, F, F_c\}$. By Definition 3.6 and


 Fig. 4.6: Triangle with two sides tangent to Φ

Lemma 2.5 it follows that the other midpoint D_a of \overline{BC} is the other diagonal point of the quadrangle $\{E, E_b, F, F_c\}$ different from A .

Let assume now that the sides a, b are tangent to Φ while c is not. Let D, E be the contact points of a, b with Φ and let D_a, E_b be the midpoints of $\overline{BC}, \overline{CA}$ different from D, E respectively. The polar c' of C is the line DE . Let A'_2, B'_1 be the intersection points of a', b' with Φ different from D, E respectively (Figure 4.6). The points F, F_c are the diagonal points of the quadrangle $\{D, E, A'_2, B'_1\}$ different from C . Consider the points $D_a^* = EF \cdot a$ and $E_b^* = DF \cdot b$. By Pascal's Theorem on the degenerate hexagon $DDEEB'_1A'_2$ the points F_c, E_b^* and D_a^* are collinear. Taking the quadrangle $\{F_c, E_b^*, E, F\}$ it must be $D_a^* = D_a$ and, equivalently, $E_b^* = E_b$. Therefore, F, F_c are the diagonal points of $\{D, D_a, E, E_b\}$ different from C .

In the two previous cases, Lemma 4.12 completes the proof.

Finally, let assume that a, b, c are tangent to Φ at the points D, E, F respectively. In this case, \widehat{DEF} is the polar triangle \mathcal{T}' of \mathcal{T} . As \mathcal{T} and \mathcal{T}' are perspective by Chasles' Theorem, the points $D_0 = a \cdot EF$, $E_0 = b \cdot FD$, $F_0 = c \cdot DE$ belong to the line h . In particular, D, D_0 are the diagonal points of the quadrangle $\{E, E_0, F, F_0\}$ different from A . This implies that $(BCDD_0) = -1$, and thus D_0 is the midpoint of \overline{BC} different from D . In the same way, E_0, F_0 are the midpoints of $\overline{CA}, \overline{AB}$ different from E, F respectively. The midpoints D, D_0, E, E_0, F, F_0 are the vertices of the quadrilateral $\{a', b', c', d\}$. ■

Theorem 4.15 (Concurrence of medians) *The medians AD , BE and CF of \mathcal{T} are concurrent.*

Proof. By considering the triangles $\mathcal{T} = \widehat{ABC}$ and \widehat{DEF} , by Theorem 4.13 the intersection points of corresponding sides are

$$AB \cdot DE = F_c, \quad BC \cdot EF = D_a, \quad CA \cdot FD = E_b,$$

which are collinear. The result follows from Desargues' Theorem. ■

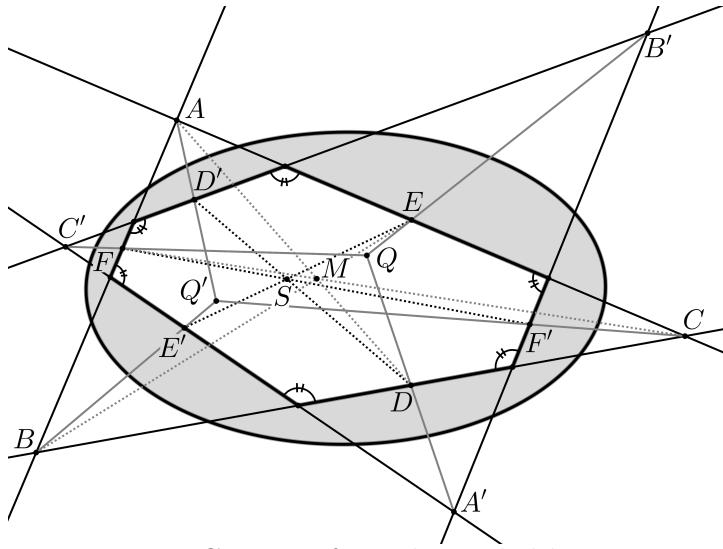


Fig. 4.7: Centers of a right-angled hexagon

Theorem 4.16 (Concurrence of side bisectors) *The side bisectors $A'D$, $B'E$ and $C'F$ of \mathcal{T} are concurrent.*

Proof. The points D_a , E_b , and F_c are, respectively, the poles of the lines $A'D$, $B'E$ and $C'F$. Because D_a , E_b , and F_c are collinear, their polars are concurrent. ■

It is interesting to note that *the side bisectors of \mathcal{T} are the angle bisectors of \mathcal{T}'* (cf. Remark 3.3). Thus, if D' , E' , F' are non-collinear midpoints of $\overline{B'C'}$, $\overline{C'A'}$, $\overline{A'B'}$, respectively, we have (compare [8, 10.21]):

Corollary 4.17 (Concurrence of angle bisectors) *The angle bisectors AD' , BE' , CF' of \mathcal{T} are concurrent.*

A point where three concurrent medians of \mathcal{T} intersect is a *barycenter* of \mathcal{T} , a point where three concurrent side bisectors of \mathcal{T} intersect is a *circumcenter* of \mathcal{T} , and a point where three angle bisectors of \mathcal{T} intersect is an *incenter* of \mathcal{T} . Each generalized triangle has four barycenters, four circumcenters and four incenters. All these centers have different geometric interpretations depending on the relative position of the figure with respect to Φ .

Another center, which as the orthocenter is shared by \mathcal{T} and \mathcal{T}' , is given by the following result.

Theorem 4.18 *The lines DD' , EE' and FF' are concurrent.*

This theorem fits perfect for a right-angled hexagon.

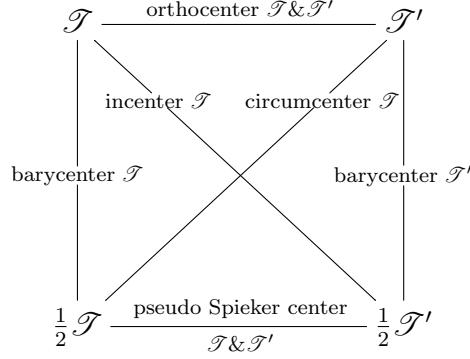


Fig. 4.8: concurrency graph of the triangles \mathcal{T} , \mathcal{T}' , $\frac{1}{2}\mathcal{T}$ and $\frac{1}{2}\mathcal{T}'$

Theorem 4.19 *In a hyperbolic right-angled hexagon the lines joining the midpoints of opposite sides are concurrent.*

In Figure 4.7 it is depicted a right-angled hexagon with the intersection point S of the lines joining opposite midpoints. In the same figure, we have depicted also: three alternate *medians* of the hexagon, lines orthogonal to a side through the midpoint of its opposite side, and the intersection point M of them; and its two circumcenters (or incenters, for this figure both concepts are the same), the points Q and Q' where the orthogonal bisectors of alternate sides intersect.

In order to interpret Theorem 4.18 for elliptic or hyperbolic triangles, note that the point D' is the pole of a bisector d of \widehat{bc} , and therefore DD' is the line through D orthogonal to d . As the points D', E', F' are non-collinear, their polars are non-concurrent. This leads us to the shadow of Theorem 4.18 for hyperbolic or elliptic triangles.

Theorem 4.20 *Let T be a hyperbolic or elliptic triangle with vertices A, B, C and opposite sides a, b, c respectively. Let D, E, F be non-collinear midpoints of a, b, c respectively, and let d, e, f be non-concurrent bisectors of T through A, B, C respectively. The lines orthogonal to d, e, f through D, E, F respectively are concurrent.*

This theorem is true also in Euclidean geometry, where the lines orthogonal to d, e, f through D, E, F are in fact angle bisectors of the medial triangle of T (the triangle whose vertices are the midpoints of the sides of T). The concurrency point of the angle bisectors of the medial triangle of T is the *Spieker center* of the triangle T . In the non-euclidean case, the lines DD' , EE' and FF' are not necessarily bisectors of \widehat{DEF} , and because

of this we say that the concurrency point of DD' , EE' and FF' given by Theorem 4.18 is the *pseudo Spieker center* of \mathcal{T} .

Surprisingly, the proof of Theorem 4.18 is harder than expected, and it is left to §8.2 (p. 89).

The pseudo Spieker center closes an interesting “concurrency graph”. Let denote by $\frac{1}{2}\mathcal{T}$ and $\frac{1}{2}\mathcal{T}'$ the medial triangles \widehat{DEF} and $\widehat{D'E'F'}$. The triangles \mathcal{T} , \mathcal{T}' , $\frac{1}{2}\mathcal{T}$ and $\frac{1}{2}\mathcal{T}'$ are perspective in pairs. If we codify each perspectivity between triangles as an edge joining the corresponding triangles, we obtain the complete graph of Figure 4.8.

§ 5

Desargues' Theorem, alternative triangle centers and the Euler-Wildberger line

In §4 we have presented the most common centers of a triangle, using their standard definitions. In euclidean geometry, these points can be defined in multiple ways, all of them equivalent, but these definitions which are equivalent in euclidean geometry could be non-equivalent in the non-euclidean context. Thus, definitions that in euclidean geometry produce the same center in non-euclidean geometry produce different centers, all of them having reminiscences of their euclidean analogue. Desargues' Theorem will allow us to construct a collection of such *pseudocenters* having some interesting properties.

5.1 Pseudobarycenter

The barycenter of an euclidean or non-euclidean triangle is the intersection point of the medians of the triangle. In euclidean geometry, the barycenter of a triangle T is also the barycenter of its medial triangle and the barycenter of its double triangle (the triangle whose medial triangle is T). This is not true in non-euclidean geometry.

We define the *double triangle* \mathcal{T}'' of \mathcal{T} as the triangle whose sides

$$a'' = A_0A, \quad b'' = B_0B, \quad c'' = C_0C,$$

are the orthogonal lines to the altitudes h_a, h_b, h_c through the vertices A, B, C of \mathcal{T} respectively. Let $A'' = b'' \cdot c'', B'' = c'' \cdot a'', C'' = a'' \cdot b''$ be the vertices of \mathcal{T}'' . In euclidean geometry, the lines a'', b'', c'' are parallel to a, b, c

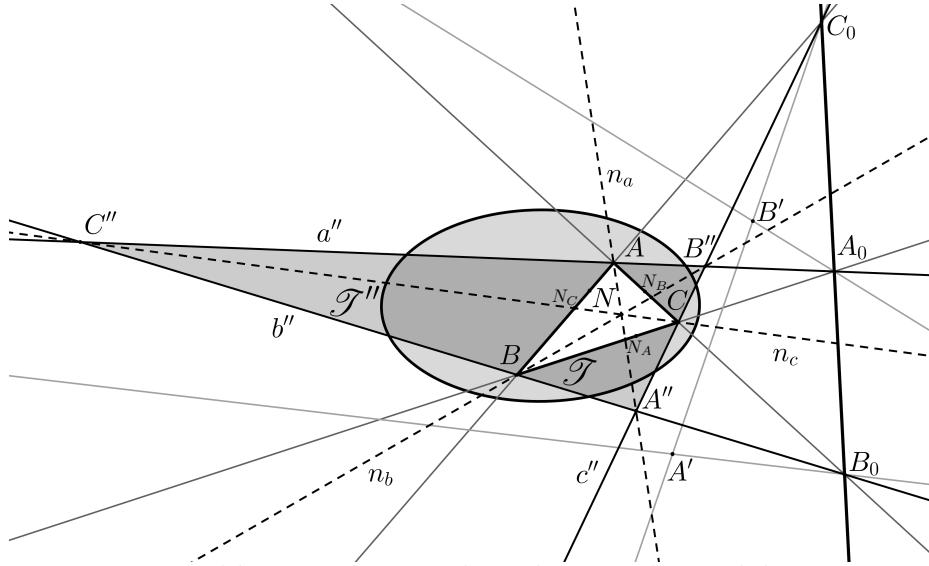


Fig. 5.1: Double triangle, pseudomedians and pseudobarycenter

respectively and the lines AA'', BB'', CC'' coincide with the medians of \mathcal{T} . This does not happen in the non-euclidean geometries, so we will call *pseudomedians* of \mathcal{T} to the lines $n_a = AA'', n_b = BB'', n_c = CC''$.

Proposition 5.1 *The pseudomedians of \mathcal{T} are concurrent.*

Proof. The sides a, b, c of the triangle \mathcal{T} intersect the sides a'', b'', c'' of \mathcal{T}'' at the collinear points A_0, B_0, C_0 respectively. The result follows from Desargues' Theorem. ■

We say that the point N of intersection of the pseudomedians of \mathcal{T} is the *pseudobarycenter* of \mathcal{T} , and that the points $N_A = n_a \cdot a$, $N_B = n_b \cdot b$, $N_C = n_c \cdot c$ where each pseudomedian of \mathcal{T} intersects its opposite side are the *pseudomidpoints* of \mathcal{T} (compare [1]).

Proposition 5.2 *The altitudes of \mathcal{T} are orthogonal to the sides of the pseudomedial triangle $\widehat{N_A N_B N_C}$.*

Proof. It suffices to prove that h_a is orthogonal to $N_B N_C$. The triangles $\widehat{N_B N_C}$ and $\widehat{B A'' C}$ are perspective with perspective center N . By Desargues' Theorem, the intersection points

$$N_B A \cdot B A'' = C A \cdot C'' A'' = B_0, \quad A N_C \cdot A'' C = A B \cdot A'' B'' = C_0, \quad N_B N_C \cdot B C$$

are collinear. This implies that $N_B N_C \cdot B C = B C \cdot h = A_0$. The point A_0 is the pole of h_a , so the lines $N_B N_C$ and h_a are conjugate. ■

The following proposition, trivial in euclidean geometry, is Theorem 16 of [24].

Proposition 5.3 Even in the non-euclidean geometries, the points A, B, C are midpoints of the sides of \mathcal{T}'' .

Proof. The quadrangle BCB_0C_0 has A and A_0 as diagonal points, and it is $AA_0 \cdot BB_0 = C''$ and $AA_0 \cdot CC_0 = B''$. Thus, it is $(AA_0B''C'') = -1$. As the points A and A_0 are conjugate with respect to Φ , by Lemma 3.4 A and A_0 are the midpoints of the segment $\overline{B''C''}$. The same holds for B, B_0 and C, C_0 , which are the midpoints of $\overline{C''A''}$ and $\overline{A''B''}$ respectively. ■

5.2 Pseudocircumcenter

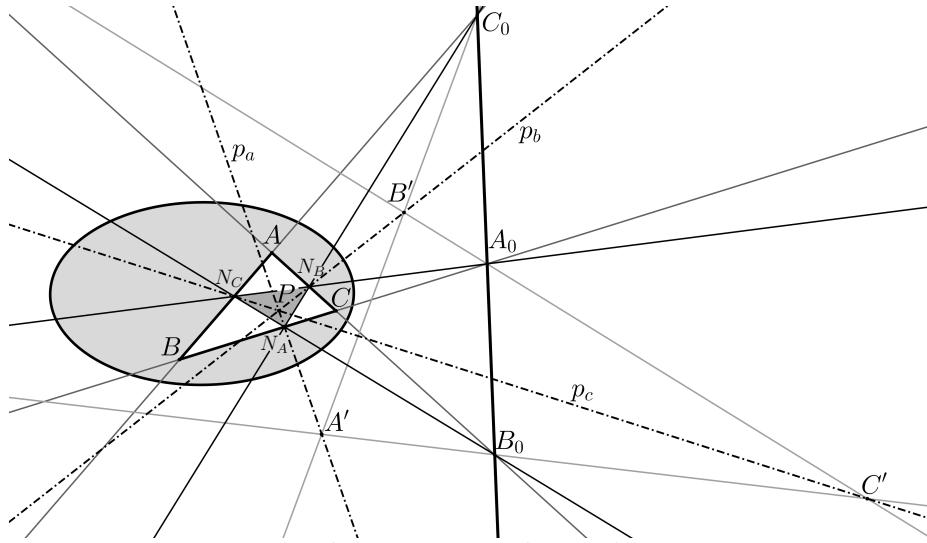


Fig. 5.2: Pseudobisectors and pseudocircumcenter

Let p_a, p_b, p_c be the orthogonal lines to a, b, c through the pseudomidpoints N_A, N_B, N_C of T , respectively. We will say that p_a, p_b, p_c are the *pseudobisectors* of the sides a, b, c of \mathcal{T} respectively.

Proposition 5.4 The pseudobisectors of \mathcal{T} are concurrent.

Proof. By Proposition 5.2, the sides of the triangle $\widehat{N_A N_B N_C}$ pass through A_0, B_0 and C_0 . The result follows by applying Desargues' Theorem to the triangles $\widehat{N_A N_B N_C}$ and \mathcal{T}' . ■

We will call *pseudocircumcenter* to the point P where the pseudobisectors intersect.

The pseudocircumcenter has other analogies with the classical circumcenter of euclidean geometry. Consider the lines

$$\begin{aligned} a_B &= BA', & b_C &= CB', & c_A &= AC', \\ a_C &= CA', & b_A &= AB', & c_B &= BC', \end{aligned}$$

and the points

$$A_1 = b_C \cdot c_B, \quad B_1 = c_A \cdot a_C, \quad C_1 = a_B \cdot b_A,$$

as in Figure 5.3. In euclidean geometry, the points A_1, B_1, C_1 lie on the circumcircle of \mathcal{T} , and they are the symmetric points of A, B, C respectively through the circumcenter. This does not happen in general in non-euclidean geometry, but nevertheless we have:

Proposition 5.5 *The lines AA_1, BB_1, CC_1 intersect at the pseudocircumcenter P .*

Proof. It suffices to show that P lies on AA_1 .

We consider the triangles $\widehat{A_1B'C'}$ and $\widehat{AN_BN_C}$. We have:

$$A_1B' \cdot AN_B = C, \quad B'C' \cdot N_B N_C = A_0, \quad C'A_1 \cdot N_C A = B.$$

Because the points B, C, A_0 are collinear, by Desargues' Theorem the triangles must be perspective. Thus, the line A_1A is concurrent with the pseudobisectors $B'N_B = p_b$ and $C'N_C = p_c$, whose intersection point is P .

■

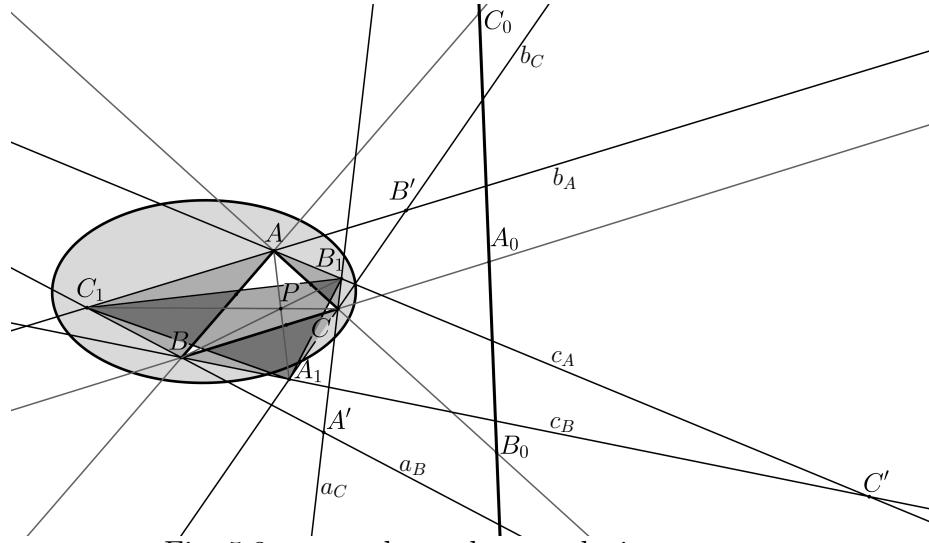


Fig. 5.3: more about the pseudocircumcenter

The definition of the points A_1, B_1, C_1 is symmetric with respect to \mathcal{T} and \mathcal{T}' , so we have also:

Proposition 5.6 *The lines $A'A_1, B'B_1, C'C_1$ intersect at the pseudocircumcenter P' of \mathcal{T}' .*

The triangle $\widehat{A_1B_1C_1}$ has a property similar to that of Proposition 5.2 for the triangle $\widehat{N_AN_BN_C}$:

Proposition 5.7 *The altitudes of \mathcal{T} are orthogonal to the sides of $\widehat{A_1B_1C_1}$.*

Proof. The proof is also identical to that of Proposition 5.2. It suffices to prove that h_a is orthogonal to B_1C_1 . The triangles $\widehat{C_1AB_1}$ and $\widehat{CA_1B}$ are perspective with perspective center P . By Desargues' Theorem, the intersection points

$$C_1A \cdot CA_1 = B', \quad AB_1 \cdot A_1B = C', \quad B_1C_1 \cdot BC,$$

are collinear. This implies that $B_1C_1 \cdot BC = BC \cdot h = A_0$, and so the lines B_1C_1 and h_a are conjugate. ■

Another interesting property is the following:

Proposition 5.8 *The points A', A'', A_1 are collinear.*

Proof. Take the triangles $\widehat{BC'B_0}$ and $\widehat{CB'C_0}$. The lines $BC = a$, $C'B' = a'$ and $B_0C_0 = h$ concur at the point A_0 . Thus, the intersection points

$$\begin{aligned} BC' \cdot CB' &= c_B \cdot b_C = A_1 \\ C'B_0 \cdot B'C_0 &= b' \cdot c' = A' \\ B_0B \cdot C_0C &= b'' \cdot c'' = A'' \end{aligned}$$

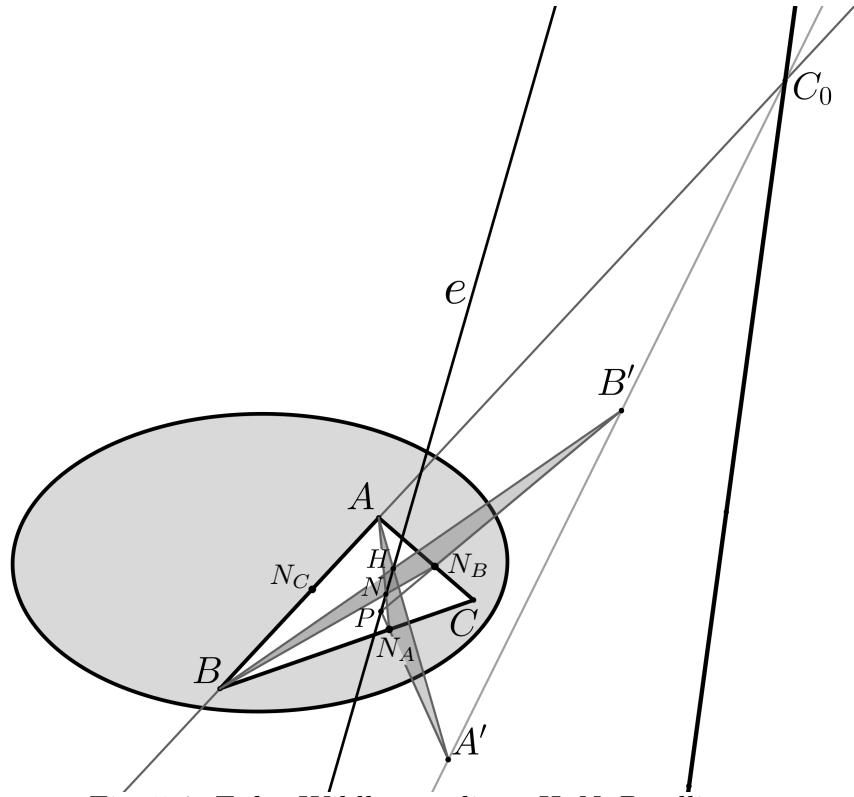
are collinear. ■

The previous proposition has the following geometric translation:

Proposition 5.9 *The lines $A''A_1, B''B_1, C''C_1$ are orthogonal to a, b, c respectively.*

5.3 The Euler-Wildberger line

Theorem 5.10 *The orthocenter H , the pseudobarycenters N and N' of \mathcal{T} and \mathcal{T}' respectively, and the pseudocircumcenters P and P' of \mathcal{T} and \mathcal{T}' respectively are collinear.*


 Fig. 5.4: Euler-Wildberger line. H, N, P collinear

Proof. Consider the triangles $\widehat{AA'N_A}$ and $\widehat{BB'N_B}$ (Figure 5.4). The lines AB , $A'B'$ and N_AN_B concur at C_0 . By Desargues' Theorem, the intersection points

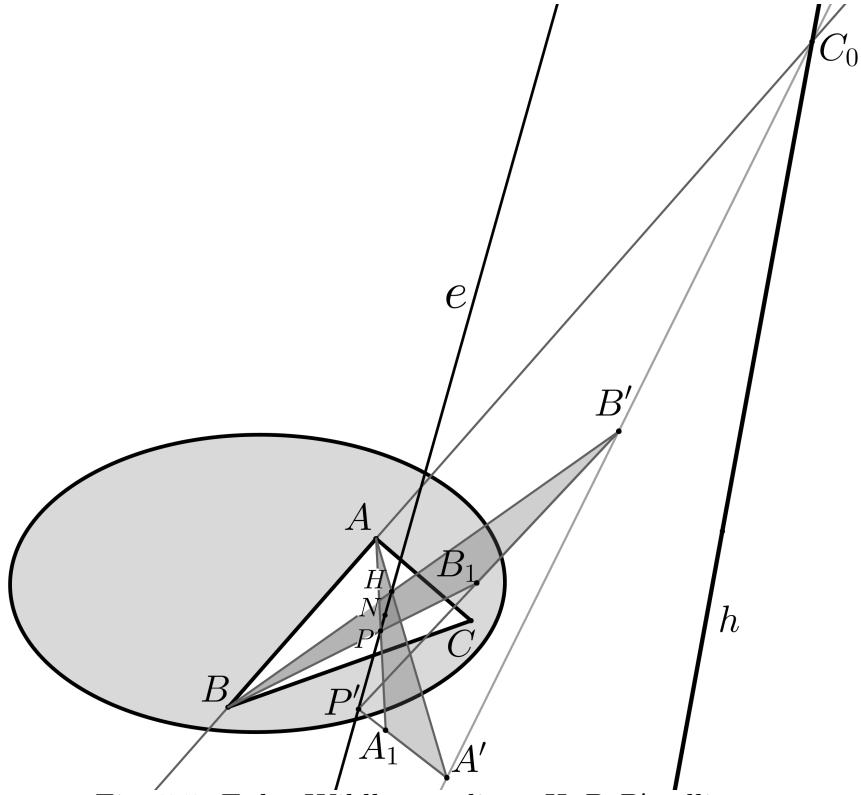
$$\begin{aligned} AA' \cdot BB' &= h_a \cdot h_b = H \\ A'N_A \cdot B'N_B &= p_a \cdot p_b = P \\ N_AA \cdot N_BB &= n_a \cdot n_b = N \end{aligned}$$

are collinear. Let e be the line through H, N, P . In the same way, it can be proved that H, P' and N' belong to a line e' .

Let consider now the triangles $\widehat{AA'A_1}$ and $\widehat{BB'B_1}$ (Figure 5.5). The lines $AB = c$, $A'B' = c'$ and A_1B_1 concur at the point C_0 . The intersection points

$$\begin{aligned} AA' \cdot BB' &= H \\ A'A_1 \cdot B'B_1 &= P' \\ AA_1 \cdot BB_1 &= P \end{aligned}$$

are collinear and so it must be $e = e'$. ■


 Fig. 5.5: Euler-Wildberger line. H, P, P' collinear

Thus, the line e joining H, N, N', P and P' is a non-euclidean version of the Euler line. This is the line which is called orthoaxis in [24], and because of this we say that e is the *Euler-Wildberger line* of \mathcal{T}' . Another interesting non-euclidean version of the Euler line is the *Euler-Akopyan line* given in [1].

Theorem 5.11 *The pseudobarycenter N' of \mathcal{T}' is the pole of the orthic axis σ of \mathcal{T} .*

The *orthic axis* σ of \mathcal{T} is the trilinear polar of the orthocenter H with respect to the triangle \mathcal{T} . If

$$H_A = a \cdot h_a, \quad H_B = b \cdot h_b, \quad H_C = c \cdot h_c,$$

are the feet of the altitudes of \mathcal{T} , by Desargues' Theorem the points

$$a \cdot H_B H_C, \quad b \cdot H_C H_A, \quad c \cdot H_A H_B$$

are collinear, and σ is the line joining them.

Proof. The polar of H_A is the line $a''' = \rho(a)\rho(h_a) = A'A_0$. Hence, the polar triangle of the orthic triangle $\widehat{H_A H_B H_C}$ of \mathcal{T} is the double triangle \mathcal{T}''' of \mathcal{T}' . If $b''' = \rho(H_B)$ and $c''' = \rho(H_C)$ are the other two sides of \mathcal{T}''' and

$$A''' = b''' \cdot c''', \quad B''' = c''' \cdot a''', \quad C''' = a''' \cdot b'''$$

are the vertices of \mathcal{T}''' , then we have also

$$A''' = \rho(H_B H_C), \quad B''' = \rho(H_C H_A), \quad C''' = \rho(H_A H_B).$$

The polar line of $a \cdot H_B H_C$ is the line joining A' and A''' . Thus, it is a pseudomedian of \mathcal{T}' , and the pole of σ is the intersection of the pseudomedians of \mathcal{T}' , which is the pseudobarycenter N' of \mathcal{T}' . ■

Because e passes through N' , we have the following corollary, which is also true in the euclidean case.

Corollary 5.12 *The Euler-Wildberger line is the line orthogonal to the orthic axis of \mathcal{T} through the orthocenter H .*

The previous corollary can be used to define the line e before introducing the pseudocenters N, P . This is the way followed in [23].

5.4 The nine-point conic

The nine-point circle of euclidean triangle is a particular version of a more general construction from projective geometry: the *eleven point conic*. Let us describe briefly this object.

Let \mathcal{Q} be a quadrangle in the projective plane with vertices A, B, C, D , and let ℓ be a line not through a vertex of \mathcal{Q} . Let q_1, q_2, \dots, q_6 be the six sides of \mathcal{Q} , and assume that q_1 and q_2 , q_3 and q_4 , q_5 and q_6 are pairs of opposite sides of \mathcal{Q} . For each $i = 1, 2, \dots, 6$, take the intersection point Q_i of q_i with ℓ , and consider also the harmonic conjugate L_i of Q_i with respect to the two vertices of \mathcal{Q} lying on q_i . Let $\sigma_{\mathcal{Q}}$ be the quadrangular involution that \mathcal{Q} induces in ℓ , and let I, J be the two fixed points of $\sigma_{\mathcal{Q}}$.

Theorem 5.13 (The eleven-point conic) *The points, I, J , the diagonal points of \mathcal{Q} and the points L_1, L_2, \dots, L_6 lie on a conic.*

Proof. See [2, vol. II, pp. 41-42] ■

We say that the conic given by this theorem is the *eleven-point conic* of the quadrangle \mathcal{Q} and the line ℓ .

When \widehat{ABC} is an euclidean triangle, D is its orthocenter and ℓ is the line at infinity, the conic given by Theorem 5.13 is the nine-point circle of \widehat{ABC} .

Let recover our projective triangle $\mathcal{T} = \widehat{ABC}$ as before. By analogy with the euclidean case, we consider the quadrangle $\mathcal{Q} = \{A, B, C, H\}$ and the line h polar of the orthocenter H with respect to Φ , and we will study the eleven-point conic Γ of \mathcal{Q} and h .

The diagonal points of \mathcal{Q} are the feet of the altitudes of \mathcal{T} , so it is $H_A, H_B, H_C \in \Gamma$.

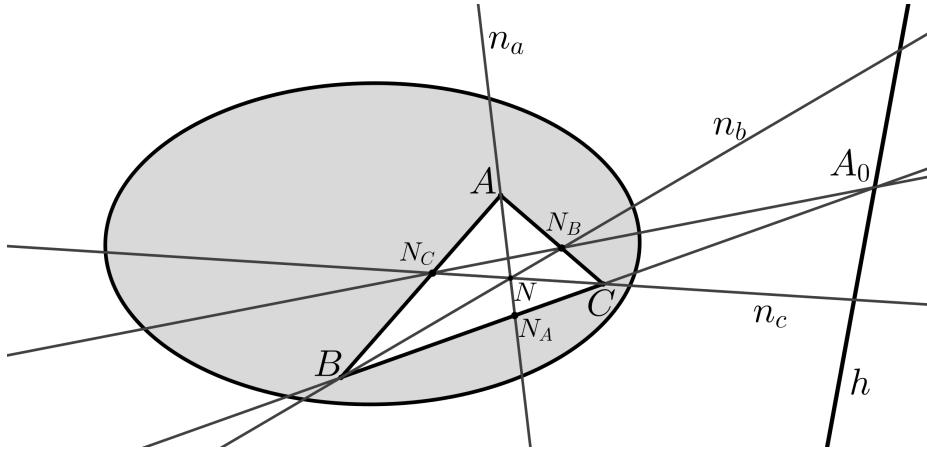


Fig. 5.6: N_A is the harmonic conjugate of A_0 with respect to B, C

Let consider now the sides $a = BC, b = CA, c = ab$ of \mathcal{Q} . The intersection points $a \cdot h, b \cdot h, c \cdot h$ are A_0, B_0, C_0 respectively.

Lemma 5.14 *The harmonic conjugate of A_0 with respect to B and C is N_A respectively.*

Proof. By considering the quadrangle $\{N, A, N_B, N_C\}$ (Figure 5.6) and using that $A_0 \in N_B N_C$, it can be seen that N_A is the harmonic conjugate of A_0 with respect to B, C . ■

Thus, we have $N_A \in \Gamma$ and in the same way $N_B, N_C \in \Gamma$.

Finally, consider the side $CH = h_c$ of the quadrangle \mathcal{Q} . Let Q_C be the intersection point of h_c with h . We need to find the harmonic conjugate L_C of Q_C with respect to C, H . Take the lines $r_a = A_0 H$ and $r_b = B_0 H$, and the points $R_{a,b} = r_a \cdot b$ and $R_{b,a} = r_b \cdot a$. If we take the quadrangle $\mathcal{R}_C = \{A_0, B_0, R_{a,b}, R_{b,a}\}$, we have that C and H are diagonal points of \mathcal{R}_C and thus $L_C = R_{a,b} R_{b,a} \cdot h_c$ (Figure 5.7). In a similar way, using also the line $r_c = C_0 H$ and the points $R_{b,c} = r_b \cdot c, R_{c,b} = r_c \cdot b, R_{c,a} = r_c \cdot a$ and $R_{a,c} = r_a \cdot c$, we can find other two points L_A, L_B of Γ as in Figures 5.8 and 5.9. The lines r_a, r_b, r_c are the perpendicular lines to the altitudes through the orthocenter, and so in euclidean geometry they are parallel to the sides of the triangle. The quadrangular involution $\sigma_{\mathcal{Q}}$ of h coincides with the

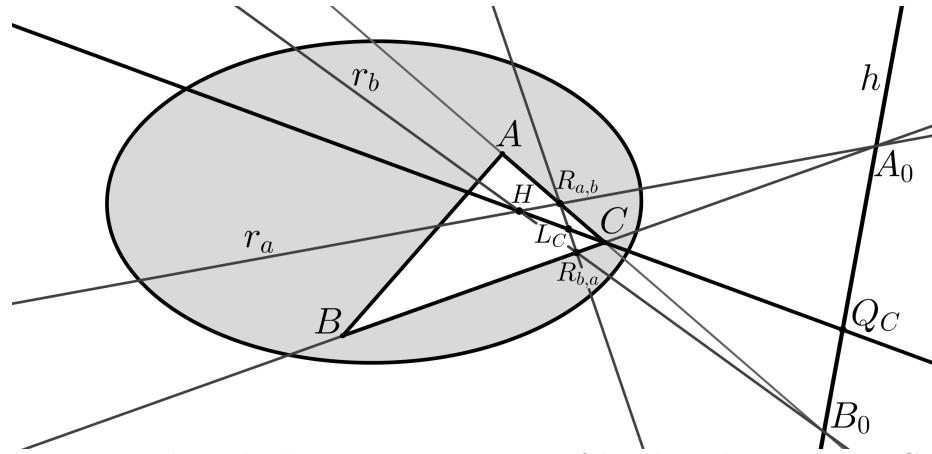


Fig. 5.7: Finding the harmonic conjugate of $h_c \cdot h$ with respect to C, H

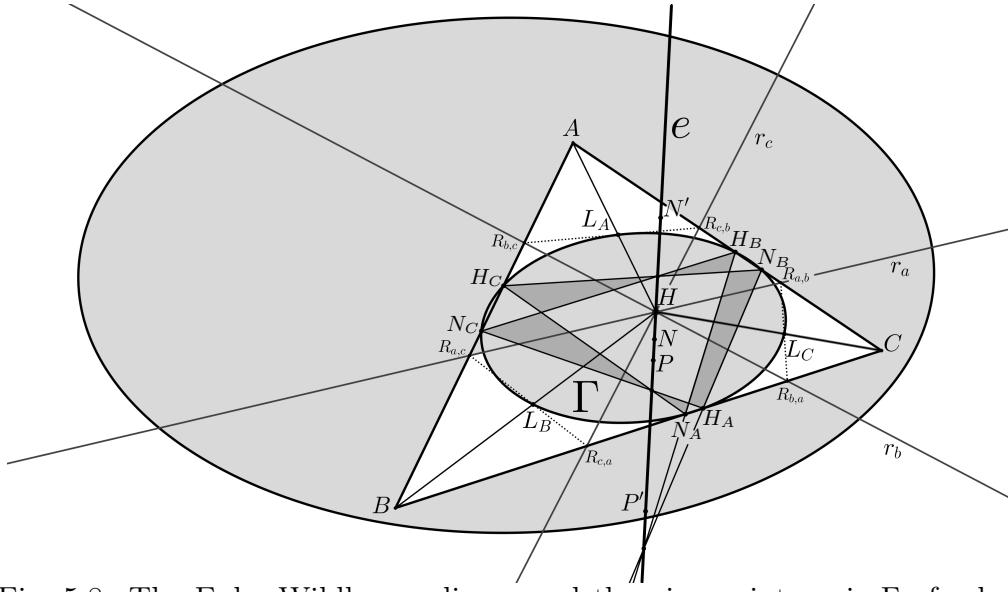


Fig. 5.8: The Euler-Wildberger line e and the nine-point conic Γ of a hyperbolic triangle

conjugacy ρ_h with respect to Φ , so if Φ is imaginary, or if Φ is real and H is interior to Φ , the fixed points I, J of $\sigma_{\mathcal{Q}}$ are imaginary.

In analogy with euclidean geometry, we have

Theorem 5.15 *The Euler-Wildberger line e is the Pascal line of the hexagon $\mathcal{H} = H_A N_B H_C N_A H_B N_C$ (see Figures 5.8 and 5.9).*

Proof. It suffices to prove that the opposite sides $H_A N_B$ and $N_A H_B$ of the hexagon \mathcal{H} inscribed in Γ intersect at e . If we consider the hexagon $A N_A H_B B N_B H_A$, with alternate vertices in the lines a, b , by Pappus' Theo-

rem the intersection points of opposite sides

$$AN_A \cdot BN_B = N, \quad N_A H_B \cdot N_B H_A, \quad H_B B \cdot H_A A = H$$

are collinear. This completes the proof. ■

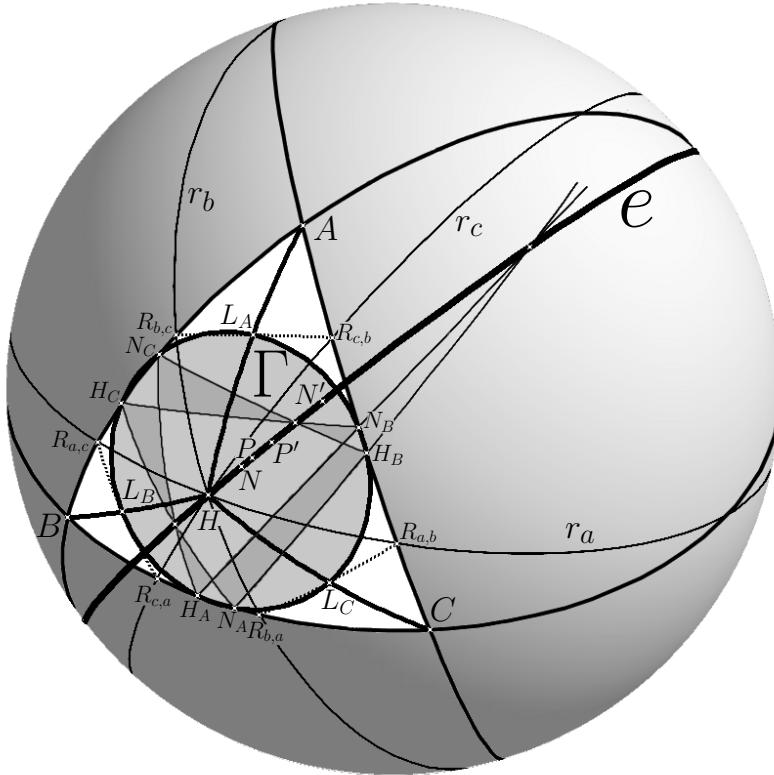


Fig. 5.9: The Euler-Wildberger line e and the nine-point conic Γ of an elliptic triangle

There are some more properties of the Euler-Wildberger line e and the nine-point conic Γ that we have checked experimentally (working with a real conic Φ in GeoGebra [12]), but for which we have no proofs. Among them:

1. The midpoints of \overline{HP} and the pole E of e with respect to Φ are the vertices of a self-polar triangle with respect to Φ and Γ .
2. The line e is a symmetry axis of Γ .
3. When Γ is an ellipse¹, its center is a midpoint of \overline{HP} and the orthogonal line to e through the center is also a symmetry axis of Γ .

¹ Γ is always an ellipse if Φ is imaginary. See [6] for a classification of conics in the hyperbolic plane.

It should be interesting to find proofs for these statements and also to find more analogies between the euclidean Euler line and nine-point circle and their noneuclidean versions proposed here or in [1].

§ 6

Menelaus' Theorem and non-euclidean trigonometry

The following theorem is classical Menelaus' Theorem as it is usually stated in affine geometry.

Theorem 6.1 (Menelaus' Theorem) *Let $\mathcal{T} = \widehat{XYZ}$ be a triangle in the affine plane. The points X_1, Y_1, Z_1 on the lines $x = YZ$, $y = ZX$, $z = XY$ respectively are collinear if and only if*

$$\frac{|XZ_1|}{|YZ_1|} \cdot \frac{|YX_1|}{|ZX_1|} \cdot \frac{|ZY_1|}{|XY_1|} = 1. \quad (6.1)$$

Identity (2.3) allows to make a projective interpretation of Menelaus' Theorem by introducing the line at infinity as part of the figure (see [22, vol. II, pp. 89-90]): if $X_\infty, Y_\infty, Z_\infty$ are the points at infinity of the lines x, y, z respectively, then (6.1) becomes

$$(XYZ_1Z_\infty)(YZX_1X_\infty)(ZXY_1Y_\infty) = 1. \quad (6.2)$$

In this projective version of (6.1), the line at infinity can be replaced with any other line of the projective plane. Assume that X_1, Y_1, Z_1 are collinear, and let s be the line they belong to. If we consider another line r and its intersection points with x, y and z :

$$X_0 = x \cdot r, \quad Y_0 = y \cdot r, \quad Z_0 = z \cdot r,$$

by Menelaus' Theorem it is

$$(XYZ_0Z_\infty)(YZX_0X_\infty)(ZXY_0Y_\infty) = 1. \quad (6.3)$$

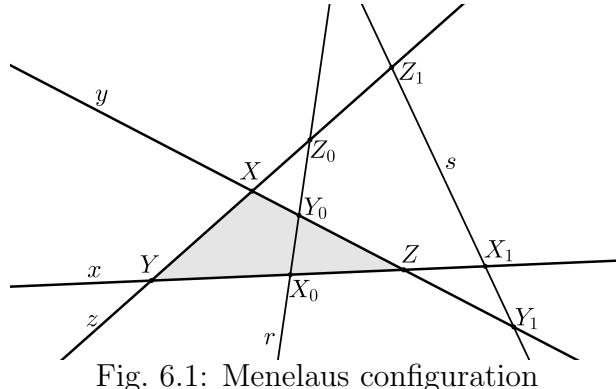


Fig. 6.1: Menelaus configuration

If we apply the identity (2.2c) to each cross ratio in (6.2), we have

$$(XYZ_1Z_0)(XYZ_0Z_\infty)(YZX_1X_0)(YZX_0X_\infty)(ZX Y_1Y_0)(ZX Y_0Y_\infty) = 1,$$

and by (6.3) the previous identity turns into

$$(XYZ_1Z_0)(YZX_1X_0)(ZX Y_1Y_0) = 1. \quad (6.4)$$

We will say that the figure composed by the triangle \widehat{xyz} and the two lines r, s as above is a *Menelaus configuration* with triangle \widehat{xyz} and transversals r, s . In order to avoid degenerate cases, we will assume that the five lines involved in a Menelaus' configuration are always in general position: they are all distinct and no three of them are concurrent.

Thus, we have proved the following corollary of Theorem 6.1, which will be enough for our purposes:

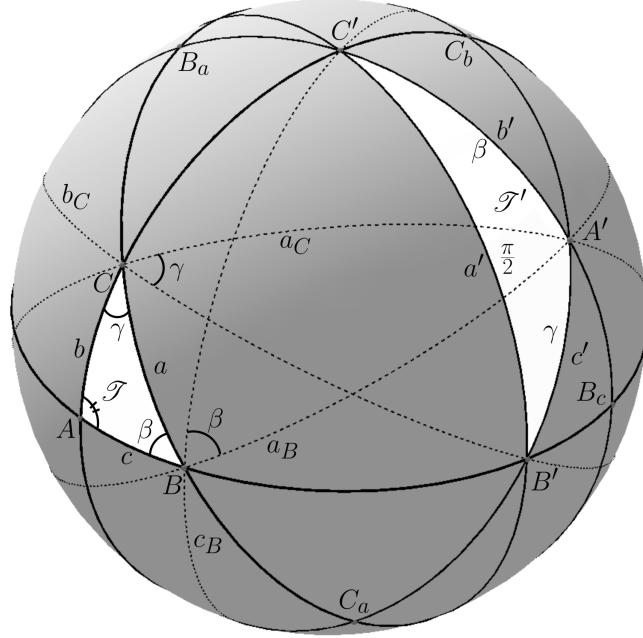
Corollary 6.2 (Menelaus' Projective Formula) *For any Menelaus' configuration with triangle $\mathcal{T} = \widehat{xyz}$ and transversals r, s , if we label the intersection points of the lines of the figure (with the only exception of $r \cdot s$) as in Figure 6.1, the identity (6.4) holds.*

6.1 Trigonometry of generalized right-angled triangles

Consider a projective triangle \mathcal{T} with vertices A, B, C and its polar triangle \mathcal{T}' as before. Assume from now on that \mathcal{T} is right-angled: b and c are conjugate to each other; and that $A \in \mathbb{P}$. Thus, $C' \in b$ and $B' \in c$. Taking the conjugate points and lines

$$\begin{aligned} B_a &= a \cdot b', & B_c &= c \cdot b', & a_B &= BA', & c_B &= BC', \\ C_a &= a \cdot c', & C_b &= b \cdot c', & a_C &= CA', & b_C &= CB', \end{aligned}$$

(a) spherical right-angled triangle



(b) hyperbolic right-angled triangle

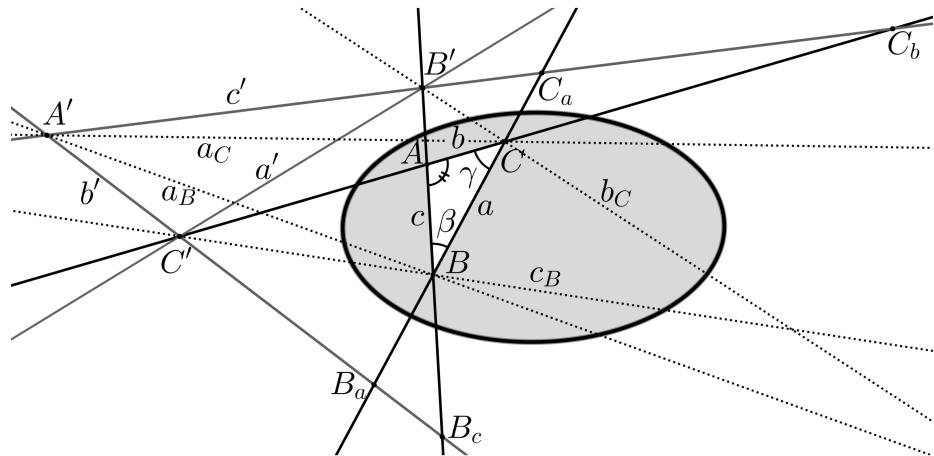


Fig. 6.2: generalized right-angled triangles I

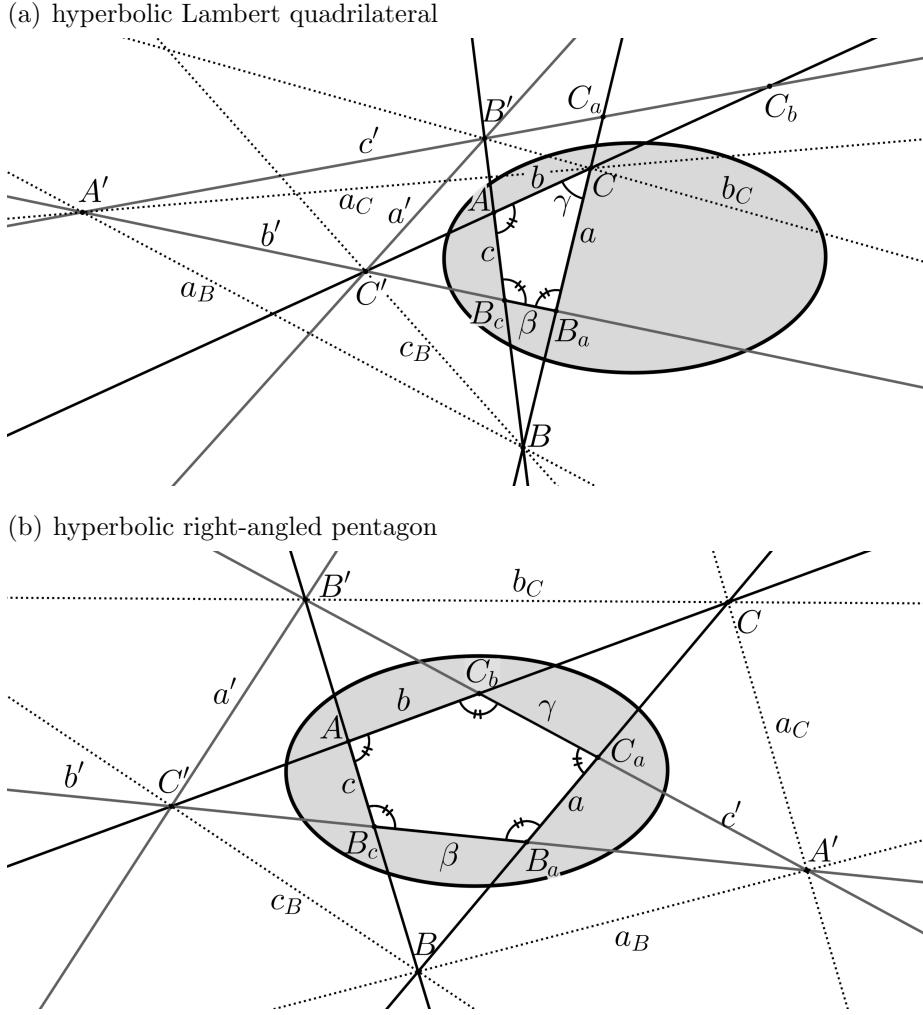


Fig. 6.3: generalized right-angled triangles II

the following relations hold

$$\begin{aligned} A_b &= b \cdot a' = C', & B_a &= A'_{b'}, & B_c &= C'_{b'}, & b_A &= AB' = c, \\ A_c &= c \cdot a' = B', & C_a &= A'_{c'}, & C_b &= B'_{c'}, & c_A &= AC' = b. \end{aligned}$$

The projective figure composed by the triangles $\mathcal{T}, \mathcal{T}'$ and the rest of points and lines considered above has four different geometric interpretations as generalized right-angled triangles. As the lines b and c are conjugate to each other, they form a right angle at the point A . When Φ is an imaginary conic (and so \mathbb{P} is the elliptic plane) \mathcal{T} is an elliptic right-angled triangle (Figure 6.2(a)). When Φ is a real conic, the triangles \mathcal{T} and \mathcal{T}' can produce three different hyperbolic polygons: (i) a right-angled hyperbolic triangle when B, C are also interior to Φ (Figure 6.2(b)); (ii) a Lambert quadrilateral,

when one of the vertices B, C turns out to be exterior to Φ (Figure 6.3(a)); and (iii) a right-angled pentagon, when both vertices B, C are considered exterior to Φ but the line BC is secant to Φ (Figure 6.3(b)). If the line BC is exterior to Φ , there appears again a Lambert quadrilateral inside \mathbb{P} and this is the same as case (ii).

Our strategy for obtaining trigonometric relations for generalized right-angled triangles will follow these steps:

1. Take a Menelaus' configuration with the five lines a, b, c, b', c' by choosing three of the lines as the sides of the triangle and the other two lines as the transversals of the configuration.
2. Apply Menelaus' Projective Formula (6.4) in order to obtain a formula relating three cross ratios of points on the figure.
3. Translate the three cross ratios of the formula just obtained into projective trigonometric ratios associated to the sides of the triangles \mathcal{T} and \mathcal{T}' . We so obtain a *projective trigonometric formula* associated with the figure.
4. For every particular generalized right-angled triangle, translate the projective trigonometric formula obtained in the previous step into a non-euclidean trigonometric formula using Table 3.1.

After the fourth step, we will obtain a squared non-euclidean trigonometric formula due to the presence of the square power in all the circular and hyperbolic trigonometric ratios of Table 3.1. As we will explain later, the subjacent unsquared trigonometric formula will be the result of removing all the square powers and choosing positive signs in the trigonometric functions of the corresponding squared formula.

By an abuse of notation, we will denote each of the segments \overline{AB} , \overline{BC} , \overline{CA} , $\overline{A'B'}$, $\overline{C'A'}$ with the name of the line that contains it.

6.1.1 Non-euclidean Pythagorean Theorems

Take the Menelaus' configuration with triangle \widehat{abc} and transversals b', c' (Figure 6.4(a)). By Menelaus' Projective Formula we have:

$$\begin{aligned}
 & (ABB_cB') (BCB_aC_a) (CAC'C_b) = 1 \stackrel{\substack{B'=A_c \\ C'=A_b}}{\Longleftrightarrow} \\
 \iff & (ABB_cA_c) (BCB_aC_a) (CAA_bC_b) = 1 \stackrel{(2.2a)}{\Longleftrightarrow} \\
 \iff & \mathbf{C}(c) \frac{1}{\mathbf{C}(a)} \mathbf{C}(b) = 1 \iff \mathbf{C}(a) = \mathbf{C}(b)\mathbf{C}(c) \quad (\mathbf{T1})
 \end{aligned}$$

If we give the names a, b, c, β, γ to the sides and angles of generalized right-angled triangles as in Figures 6.2 and 6.3, by Table 3.1 identity (**T1**) translates into:

- for an elliptic right-angled triangle

$$\cos^2 a = \cos^2 b \cos^2 c. \quad (6.5)$$

- for a hyperbolic right-angled triangle

$$\cosh^2 a = \cosh^2 b \cosh^2 c, \quad (6.6)$$

- for a Lambert quadrilateral

$$-\sinh^2 a = \cosh^2 b (-\sinh^2 c), \quad (6.7)$$

- for a right-angled pentagon

$$\cosh^2 a = (-\sinh^2 b) (-\sinh^2 c), \quad (6.8)$$

If we look for the unsquared versions of formulae (6.6 – 6.5), we must look at each figure separately.

In the three hyperbolic figures there is no discussion if we assume, as usual, that the length of a segment is always positive. In this case, because hyperbolic sines and cosines are positive for positive arguments, it must be

$$\cosh a = \cosh b \cosh c$$

for the hyperbolic right angled triangle (this formula is known as the *hyperbolic Pythagorean theorem*),

$$\sinh a = \cosh b \sinh c$$

for the Lambert quadrilateral, and

$$\cosh a = \sinh b \sinh c$$

for the right-angled pentagon.

The elliptic case is subtler than the hyperbolic ones because $\cos x$ can be positive or negative for $x \in (0, \pi)$. We need more geometric information for deciding which of the two unsquared versions of (6.5):

$$\cos a = \cos b \cos c, \quad \text{or} \quad \cos a = -\cos b \cos c;$$

is the correct one, or if both are correct but they apply to different figures. Because elliptic segments have their length in $[0, \pi]$, we will extend the usual terms for angles: *acute*, *right* and *obtuse*; to segments in the natural way. The proof of the following proposition is left to the reader.

Proposition 6.3 (elliptic triangles I) *In the triangle \mathcal{T} of Figure 6.2(a), the hypotenuse a is obtuse if and only if exactly one of the catheti b, c is obtuse. In particular, if the triangle \mathcal{T} has an obtuse side, it has exactly 2 obtuse sides.*

Proposition 6.3 implies that if one of the three ratios $\cos a, \cos b, \cos c$ is negative, then exactly another one is also negative while the remaining one is positive. Therefore, the correct formula for elliptic right-angled triangles is

$$\cos a = \cos b \cos c,$$

which is known as the *spherical Pythagorean theorem*.

6.1.2 More trigonometric relations

There are $\binom{5}{2} = 10$ different Menelaus' configurations that we can build from the five lines a, b, c, b', c' . If we exclude those relations that are equivalent after renaming the lines of the figure, there rest the six configurations depicted in Figure 6.4.

- Taking the triangle $\widehat{acc'}$ and the transversals b, b' (see Figure 6.4(b)), by Menelaus' Projective Formula we have:

$$\begin{aligned} & (C_a BCB_a) (BB' AB_c) (B'C_a C_b A') = 1 \iff \\ & \iff (BC_a B_a C) (BA_c AB_c) (B'A'_c B'_c A') = 1 \iff \\ & \iff \frac{1}{\mathbf{S}(a)} \mathbf{S}(c) \frac{1}{\mathbf{S}(c')} = 1 \iff \mathbf{S}(c) = \mathbf{S}(a) \mathbf{S}(c') \quad (\mathbf{T2}) \end{aligned}$$

- Taking the triangle $\widehat{cb'c'}$ and the transversals a, b (Figure 6.4(c)), we arrive to:

$$\begin{aligned} & (A'B'C_B C_a) (B'B_c AB) (B_c A' C' B_a) = 1 \iff \\ & \iff (A'B' B'_c A'_c) (A_c B_c AB) (C'_{b'} A' C' A'_{b'}) = 1 \iff \\ & \iff \mathbf{C}(c') \frac{1}{\mathbf{C}(c)} \frac{1}{\mathbf{S}(b')} = 1 \iff \mathbf{C}(c') = \mathbf{C}(c) \mathbf{S}(b') \quad (\mathbf{T3}) \end{aligned}$$

- Taking the triangle $\widehat{acb'}$ and transversals b, c' (Figure 6.4(d)) we obtain:

$$\mathbf{T}(c) = \mathbf{T}(a) \mathbf{C}(b'). \quad (\mathbf{T4})$$

- Taking the triangle $\widehat{ab'c'}$ and transversals b, c (Figure 6.4(e)), we arrive to:

$$\mathbf{C}(a) = \frac{1}{\mathbf{T}(b')} \frac{1}{\mathbf{T}(c')} \quad (\mathbf{T5})$$

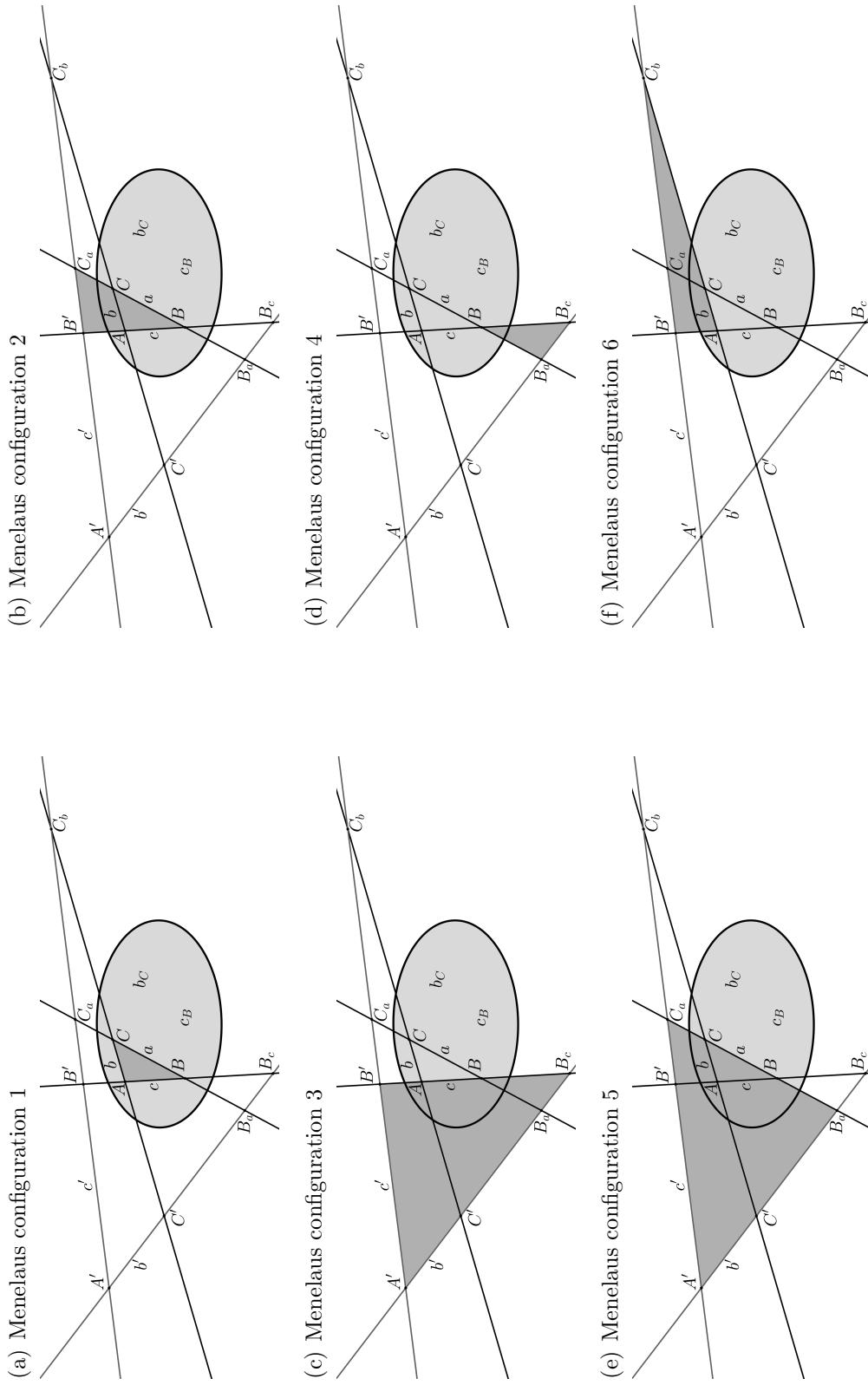


Fig. 6.4: Menelaus' configurations for a generalized right-angled triangle

6.1. Trigonometry of generalized right-angled triangles

	Elliptic right-angled triangle		Hyperbolic right-angled pentagon
	right-angled triangle	Lambert quadrilateral	right-angled pentagon
T1	$\cos a = \cos c \cos b$	$\cosh a = \cosh c \cosh b$	$\sinh a = \sinh c \cosh b$
T2	$\sin c = \sin a \sin \gamma$	$\sinh c = \sinh a \sin \gamma$	$\cosh c = \cosh a \sinh \gamma$
T3	$\cos \gamma = \cos c \sin \beta$	$\cos \gamma = \cosh c \sinh \beta$	$\cosh \gamma = \sinh c \sinh \beta$
T4	$\tan c = \tan a \cos \beta$	$\tanh c = \tanh a \cos \beta$	$\coth c = \tanh a \cosh \beta$
T5	$\cos a = \cot \beta \cot \gamma$	$\cosh a = \cot \beta \cot \gamma$	$\cosh a = \coth \beta \coth \gamma$
T6	$\tan c = \sin b \tan \gamma$	$\tanh c = \sinh b \tan \gamma$	$\coth c = \sinh b \tan \gamma$

Table 6.1: trigonometric relations for generalized right-angled triangles

- Finally, taking the triangle $\widehat{bcc'}$ and transversals a, b' (Figure 6.4(f)), we get:

$$\mathbf{T}(c) = \mathbf{S}(b)\mathbf{T}(c'). \quad (\mathbf{T}6)$$

It can be seen that the projective trigonometric formulae (T4), (T5) and (T6) can be deduced from (T1), (T2) and (T3).

If we apply Table 3.1 to expressions (T2) to (T6), for each general right-angled triangle we obtain a collection of squared non-euclidean trigonometric formulae associated with each figure. In order to decide which is the unsquared correct formula corresponding to each squared formula, we need to use the geometric properties of each figure. In the right-angled hyperbolic pentagon there will be no discussion because all the relevant magnitudes of the figure are segments. For right-angled hyperbolic triangles and Lambert quadrilaterals, it is enough to remark that:

Proposition 6.4 *A hyperbolic right-angled triangle and a Lambert quadrilateral cannot have an obtuse angle.*

Proposition 6.4 is a simple consequence of the fact that the sum of the angles of a hyperbolic triangle is lower than π . For elliptic right-angled triangles, besides Proposition 6.3 it should be necessary also the properties listed in Proposition 6.5 (whose proof is left to the reader).

Proposition 6.5 (elliptic triangles II) *The right-angled triangle \mathcal{T} of Figure 6.2(a) verifies:*

- *the angles of \mathcal{T} are equal to the sides of the polar triangle \mathcal{T}' and vice versa;*
- *one of the angles β, γ of \mathcal{T} is obtuse (right) if and only if its opposite side is also obtuse (right).*

Using Propositions 6.3, 6.4 and 6.5, it can be seen that the (unsquared) non-euclidean trigonometric translations of formulae (T2 – T6) for each generalized right-angled triangle are those listed in Table 6.1 (we include also the translations of (T1) for completeness).

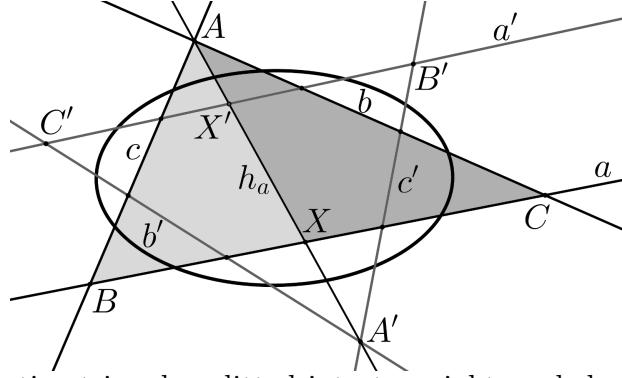


Fig. 6.5: projective triangle splitted into two right-angled projective triangles

6.2 Trigonometry of generalized, non right-angled, triangles

Let us consider again the triangle $\mathcal{T} = \widehat{ABC}$ with sides a, b, c and its polar triangle $\mathcal{T}' = \widehat{A'B'C'}$ with sides a', b', c' as in § 4.2. We assume again that \mathcal{T} is in general position with respect to Φ and \mathcal{T}' , but we will not assume now that \mathcal{T} has conjugate lines. Thus, \mathcal{T} could be an elliptic or hyperbolic triangle, or it could compose with \mathcal{T}' any of the generalized triangles depicted in Figures 1.1–1.4. We will see how all the trigonometric formulae for these figures can be deduced from the results of the previous section.

Consider the line $h_a = AA'$, and take the point $H_A = h_a \cdot a$ and its conjugate point in a , which is the point A_0 (see § 4). For simplifying the notation, we denote now X, X_a to the points H_A, A_0 respectively. Take also the point $X' = h_a \cdot a'$ (Figure 6.5). The line h_a decomposes the Figure $\mathcal{T} \cup \mathcal{T}'$ into two Menelaus' configurations with some common lines. We will use the following segments (see Figure 6.5):

$$p = \overline{AX}, \quad a_1 = \overline{BX}, \quad a_2 = \overline{CX}, \quad a'_1 = \overline{C'X'}, \quad a'_2 = \overline{B'X'}.$$

6.2.1 The general (squared) law of sines.

If we apply the identity (T2) to the projective right-angled triangles \widehat{AXB} and \widehat{AXC} we obtain

$$\mathbf{S}(p) = \mathbf{S}(c)\mathbf{S}(b') \quad \text{and} \quad \mathbf{S}(p) = \mathbf{S}(b)\mathbf{S}(c'),$$

respectively, and so the ratios $\mathbf{S}(b)/\mathbf{S}(b')$ and $\mathbf{S}(c)/\mathbf{S}(c')$ must coincide. Repeating the process, they must coincide also with the ratio $\mathbf{S}(a)/\mathbf{S}(a')$. Therefore, the following identity holds

$$\frac{\mathbf{S}(a)}{\mathbf{S}(a')} = \frac{\mathbf{S}(b)}{\mathbf{S}(b')} = \frac{\mathbf{S}(c)}{\mathbf{S}(c')}. \quad (6.9)$$

We will call to identity (6.9) the *general law of sines* (compare [4, Theorem 2.6.20] and [18, Theorem 22.6]).

If we apply Table 3.1 to (6.9) for each generalized triangle, we obtain the law of sines associated to each figure. For example, for an elliptic triangle (Figure 1.1(a)), the unsquared translation of (6.9) is

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma},$$

while for a hyperbolic triangle (Figure 1.1(b)) we have

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}.$$

For example, for the hyperbolic pentagon with four right angles (Figure 1.2(c)) we obtain

$$\frac{\sinh a}{\sin \alpha} = \frac{\cosh b}{\sinh \beta} = \frac{\cosh c}{\sinh \gamma}.$$

6.2.2 The general (squared) law of cosines

In order to obtain a projective trigonometric formula similar to the cosine rules of generalized triangles we need the following lemma.

Lemma 6.6 *The following relations hold:*

$$(BXCC_a)^2 = \frac{\mathbf{T}(a)}{\mathbf{T}(a_2)}, \quad (6.10)$$

$$\mathbf{C}(a_1) = \left(\sqrt{\mathbf{C}(a)\mathbf{C}(a_2)} + \sqrt{\mathbf{S}(a)\mathbf{S}(a_2)} \right)^2. \quad (6.11)$$

Proof. The proof relies in the properties (2.2a), (2.2b) and (2.2c) of cross ratios.

By applying (2.2c) twice and (2.2a) once we have:

$$(BXCC_a) = (X_a XCC_a)(BX_a CC_a) = \frac{1}{(XX_a CC_a)} (B_a X_a CC_a)(BB_a CC_a).$$

As cross ratios are invariant under conjugacy, it is

$$(B_a X_a C C_a) = (B X C_a C) = \frac{1}{(B X C C_a)},$$

and so we obtain

$$(B X C C_a) = \frac{1}{(B X C C_a)} \frac{(B B_a C C_a)}{(X X_a C C_a)} \Rightarrow (B X C C_a)^2 = \frac{\mathbf{T}(a)}{\mathbf{T}(a_2)}.$$

By applying (2.2c) again, we have:

$$\begin{aligned} \mathbf{C}(a_1) &= (B X X_a B_a) = (B X C_a B_a)(B X X_a C_a) = \\ &= (C X C_a B_a)(B C C_a B_a)(C X X_a C_a)(B C X_a C_a) = \\ &= (C X C_a B_a) \mathbf{C}(a) \mathbf{C}(a_2) (B C X_a C_a). \end{aligned}$$

Therefore

$$(C X C_a B_a) = (C_a X_a C B) = (B C X_a C_a).$$

This implies:

$$\mathbf{C}(a_1) = \mathbf{C}(a) \mathbf{C}(a_2) (B C X_a C_a)^2 = \mathbf{C}(a) \mathbf{C}(a_1) [1 - (B X_a C C_a)]^2.$$

Using (6.10), we obtain

$$(B X_a C C_a) = (X X_a C C_a) (B X C C_a) = -\mathbf{T}(a_2) \sqrt{\frac{\mathbf{T}(a)}{\mathbf{T}(a_2)}} = -\sqrt{\mathbf{T}(a) \mathbf{T}(a_2)}.$$

Then,

$$\begin{aligned} \mathbf{C}(a_1) &= \mathbf{C}(a) \mathbf{C}(a_2) [1 + \sqrt{\mathbf{T}(a) \mathbf{T}(a_2)}]^2 = \\ &= (\sqrt{\mathbf{C}(a) \mathbf{C}(a_2)} + \sqrt{\mathbf{S}(a) \mathbf{S}(a_2)})^2. \end{aligned}$$

■

From the proof of Lemma 6.6 it follows that (6.10) and (6.11) are valid¹ for any three points B, C, X lying on a line a not tangent to Φ , independently of the figure $\mathcal{T} \cup \mathcal{T}'$.

If we apply (T1) to the projective right-angled triangles \widehat{ABX} and \widehat{ACX} , we obtain

$$\mathbf{C}(c) = \mathbf{C}(p) \mathbf{C}(a_1) \quad \text{and} \quad \mathbf{C}(b) = \mathbf{C}(p) \mathbf{C}(a_2)$$

¹Formula (6.11) is a projective version of the trigonometric formulae for the angle sum $\cos(\alpha \pm \beta), \sin(\alpha \pm \beta), \cosh(\alpha \pm \beta)$ and $\sinh(\alpha \pm \beta)$.

respectively. It can be deduced

$$\mathbf{C}(c) = \frac{\mathbf{C}(b)}{\mathbf{C}(a_2)} \mathbf{C}(a_1),$$

and by (6.11), we arrive to

$$\begin{aligned} \mathbf{C}(c) &= \frac{\mathbf{C}(b)}{\mathbf{C}(a_2)} \left(\sqrt{\mathbf{C}(a)\mathbf{C}(a_2)} + \sqrt{\mathbf{S}(a)\mathbf{S}(a_2)} \right)^2 = \\ &= \mathbf{C}(b) \left(\sqrt{\mathbf{C}(a)} + \sqrt{\mathbf{S}(a)\mathbf{T}(a_2)} \right)^2. \end{aligned}$$

If we apply (T4) to the triangle \widehat{ACX} , we obtain that $\mathbf{T}(a_2) = \mathbf{T}(b)\mathbf{C}(c')$, and so we have

$$\mathbf{C}(c) = \left(\sqrt{\mathbf{C}(a)\mathbf{C}(b)} + \sqrt{\mathbf{S}(a)\mathbf{S}(b)\mathbf{C}(c')} \right)^2. \quad (6.12)$$

We will call to identity (6.12) the *general law of cosines* (compare [4, Theorem 2.6.20]).

For a given generalized triangle, there is a great difference between the general law of sines (6.9) and the general law of cosines (6.12) when we want to obtain their unsquared non-euclidean trigonometric translations. The unsquared trigonometric translations of (6.9) depend on a simple choice of signs that can be done in an straightforward way. On the other hand, the unsquared trigonometric translations of (6.12) depend on a multiple choice of signs due to the presence of the two square roots and the square power in the right-hand side of (6.12).

Every non-euclidean generalized triangle T have six laws of cosines, one based in each side of T and another one based in each non-right angle of T . If \mathcal{T} and \mathcal{T}' are the two projective triangles, polar to each other, that produce the non-euclidean polygon T , these six cosine laws are the unsquared trigonometric translations of (6.12), when the segment c varies along the three sides of \mathcal{T} and the three sides of \mathcal{T}' . When T is a hyperbolic generalized triangle, in [4] it is shown how to obtain all the cosine rules of T by taking an orientation on the elements (sides and non-right angles) of T and assigning some ± 1 coefficients to these elements. In our context we would like to have another *projective general law of cosines* different from (6.12) whose trigonometric translations are straightforward and give the cosine rules for each (hyperbolic and elliptic) generalized triangle. We'll explore the existence of this formula in §8. In order to obtain it, it would be desirable to express in projective terms the actual (unsquared) trigonometric functions associated to the measurements of segments and angles.

Nevertheless, every generalized triangle is the result of pasting together two generalized right-angled triangles along a side (Figures 1.1–1.4). This

6.2. Trigonometry of generalized, non right-angled, triangles

allows us to deduce the trigonometry of all generalized triangles from the trigonometry of generalized right-angled triangles. This suffices for presenting the whole non-euclidean trigonometry as a corollary of Menelaus' Theorem.

§ 7

Carnot's Theorem and... Carnot's Theorem?

There are many geometric results which are known as *Carnot's Theorem*. One of them is a generalization of Menelaus' Theorem (see Figure 7.1(a)):

Theorem 7.1 (Carnot's Theorem on affine triangles)

Let \widehat{XYZ} be a triangle in the affine plane. The six points $X_1, X_2 \in YZ$, $Y_1, Y_2 \in ZX$, $Z_1, Z_2 \in XY$ on the sides of \mathcal{T} lie on a conic if and only if

$$\frac{|X_1Y|}{|X_1Z|} \cdot \frac{|X_2Y|}{|X_2Z|} \cdot \frac{|Y_1Z|}{|Y_1X|} \cdot \frac{|Y_2Z|}{|Y_2X|} \cdot \frac{|Z_1X|}{|Z_1Y|} \cdot \frac{|Z_2X|}{|Z_2Y|} = 1.$$

Another “Carnot's Theorem” is Theorem 7.2 below.

Let T be a triangle in euclidean plane with vertices A, B, C and sides $a = BC, b = CA, c = AB$ as usual. Let A^*, B^*, C^* be three points on the lines a, b, c respectively, and let a^*, b^*, c^* be the perpendicular lines to a, b, c through the points A^*, B^*, C^* respectively. Let $a_1, a_2, b_1, b_2, c_1, c_2$ denote the euclidean lengths of the segments $\overline{BA^*}, \overline{CA^*}, \overline{CB^*}, \overline{AB^*}, \overline{AC^*}, \overline{BC^*}$, respectively (see Figure 7.1(b)).

Theorem 7.2 (Carnot's Theorem on euclidean triangles)

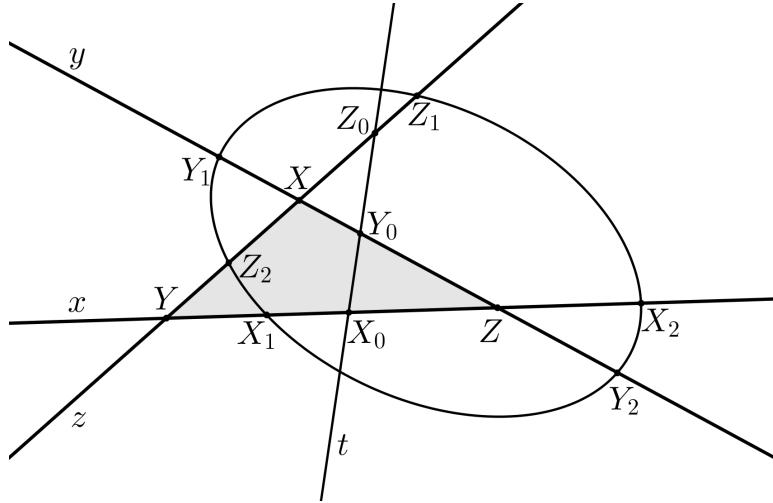
The lines a^*, b^*, c^* are concurrent if and only if

$$a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$$

This theorem can be proved by simple application of Pythagoras' Theorem. For a proof of Theorem 7.1, see See [22, vol. II, p. 90]. We say that the points A^*, B^*, C^* such that a^*, b^*, c^* are concurrent are *Carnot points* of T .

A version of Theorem 7.2 for hyperbolic triangles is stated in [9]. Another versions for different generalized triangles can be constructed using the

(a) Carnot's theorem on projective triangles



(b) Carnot's theorem on euclidean triangles

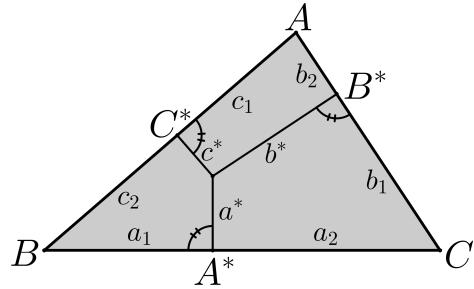


Fig. 7.1: Carnot's theorems

non-euclidean versions of Pythagoras' theorem (see §6.1.1), or its projective version (**T1**). What we will see in the following is that *the non-euclidean shadows of (the projective version of) Theorem 7.1 are the non-euclidean versions of Theorem 7.2 for generalized triangles.*

As we did with Menelaus' Theorem in §6, we can make a projective interpretation of Theorem 7.1 by introducing the line at infinity as part of the figure. We will use the same notation as in §6. Let x, y, z, t be four projective lines such that no three of them are concurrent. Consider the projective triangle $\mathcal{T} = \widehat{xyz}$ and take the points

$$\begin{aligned} X &= y \cdot z, & Y &= z \cdot x, & Z &= x \cdot y, \\ X_0 &= x \cdot t, & Y_0 &= y \cdot t, & Z_0 &= z \cdot t. \end{aligned}$$

Then, Theorem 7.1 turns into

Theorem 7.3 (Carnot's Theorem on projective triangles)

Six points $X_1, X_2 \in x$, $Y_1, Y_2 \in y$, $Z_1, Z_2 \in z$ on the sides of \mathcal{T} lie on a conic if and only if

$$(XYZ_0Z_1)(XYZ_0Z_2)(YZX_0X_1)(YZX_0X_2)(ZXY_0Y_1)(ZXY_0Y_2) = 1.$$

Exercise 7.4 Prove Lemma 4.9 and Theorem 4.13 based in Menelaus' and Carnot's Theorems instead of Pascal's Theorem.

We will use the following corollary instead of Theorem 7.3 itself.

Corollary 7.5 Let $\mathcal{T} = \widehat{XYZ}$ be a triangle in the projective plane, and let $X_1, X_2 \in YZ$, $Y_1, Y_2 \in ZX$, $Z_1, Z_2 \in XY$ be six points on the sides of \mathcal{T} different from X, Y, Z lying on a conic. Let $X_0 \in YZ$, $Y_0 \in ZX$, $Z_0 \in XY$ be another three points on the sides of \mathcal{T} . If the points X_0, Y_0, Z_0 are collinear then

$$(XYZ_0Z_1)(XYZ_0Z_2)(YZX_0X_1)(YZX_0X_2)(ZXY_0Y_1)(ZXY_0Y_2) = 1. \quad (7.1)$$

The converse of Corollary 7.5 is not true, but we have the following result instead.

Proposition 7.6 Let $\mathcal{T} = \widehat{XYZ}$ be a projective triangle, and let $X_1, X_2 \in YZ$, $Y_1, Y_2 \in ZX$, $Z_1, Z_2 \in XY$ be six points on the sides of \mathcal{T} different from X, Y, Z lying on a conic. Let $X_0 \in YZ$, $Y_0 \in ZX$, $Z_0 \in XY$ be another three points on the sides of \mathcal{T} , and consider also the point $Z'_0 = XY \cdot X_0Y_0$. If the identity (7.1) holds, then it is $Z_0 = Z'_0$ (and so X_0, Y_0, Z_0 are collinear) or Z_0 is the harmonic conjugate of Z'_0 with respect to X, Y .

Proof. Assume that (7.1) is true. By Theorem 7.3 we have

$$(XYZ'_0Z_1)(XYZ'_0Z_2)(YZX_0X_1)(YZX_0X_2)(ZXY_0Y_1)(ZXY_0Y_2) = 1.$$

This relation, together with (7.1) imply that

$$(XYZ'_0Z_1)(XYZ'_0Z_2) = (XYZ_0Z_1)(XYZ_0Z_2)$$

or equivalently by (2.2)

$$\begin{aligned} (XYZ'_0Z_1)(XYZ_1Z_0) &= (XYZ_0Z_2)(XYZ_2Z'_0) \iff \\ \iff (XYZ'_0Z_0) &= (XYZ_0Z'_0) \iff (XYZ'_0Z_0) = \pm 1. \end{aligned}$$

If $(XYZ'_0Z_0) = +1$, because X, Y are different points, it must be $Z_0 = Z'_0$, and if $(XYZ'_0Z_0) = -1$, then Z_0 is the harmonic conjugate of Z'_0 with respect to X, Y . ■

When dealing with our projective triangle \mathcal{T} and its polar triangle \mathcal{T}' with respect to Φ as in §4.2, we have the following:

Theorem 7.7 *The points $A_b, A_c, B_a, B_c, C_a, C_b$ lie on a conic.*

Proof. The lines $ac'ba'cb'$ are, in this order, the sides of the hexagon \mathcal{H} with consecutive vertices $B_a, C_a, C_b, A_b, A_c, B_c$. By Chasles' Theorem the lines AA' , BB' , CC' are concurrent, and so, by Desargues' Theorem, the points $a \cdot a'$, $b \cdot b'$ and $c \cdot c'$ are collinear. Thus, the opposite sides of the hexagon \mathcal{H} intersect at three collinear points. By the converse of Pascal's Theorem, the six vertices of the hexagon \mathcal{H} must lie on a conic. ■

It can be deduced from the previous proof that this theorem holds for any pair of perspective triangles¹. When the triangle \mathcal{T} has two conjugate sides, the conic given by Theorem 7.7 is a degenerate conic (see Figure ??). Theorem 7.7, together with Corollary 7.5, allows us to obtain geometric information about a non-euclidean generalized triangle when the collinear points X_0, Y_0, Z_0 of Corollary 7.5 are suitably chosen.

For a similar notation to that of Corollary 7.5 and Proposition 7.6, we will use now the names $A_1, A_2, B_1, B_2, C_1, C_2$ for the points $B_a, C_a, C_b, A_b, A_c, B_c$, respectively. Let A^*, B^*, C^* be three points lying on the sides a, b, c of the triangle \mathcal{T} respectively, and consider the lines

$$a^* = A^*A', \quad b^* = B^*B', \quad c^* = C^*C',$$

and the segments

$$\begin{aligned} a_1 &= \overline{BA^*}, & b_1 &= \overline{CB^*}, & c_1 &= \overline{AC^*}, \\ a_2 &= \overline{CA^*}, & b_2 &= \overline{AB^*}, & c_2 &= \overline{BC^*}. \end{aligned}$$

Theorem 7.8 *If the lines a^*, b^*, c^* are concurrent, the following relation holds:*

$$\mathbf{C}(a_1) \mathbf{C}(b_1) \mathbf{C}(c_1) = \mathbf{C}(a_2) \mathbf{C}(b_2) \mathbf{C}(c_2). \quad (7.2)$$

Before proving Theorem 7.8, we will prove the following lemma:

Lemma 7.9 *Let p be a projective line not tangent to Φ , and let X, Y, Z be three points of p not lying on Φ . Then,*

$$(XYZ_p X_p) (XYZ_p Y_p) = \frac{\mathbf{C}(\overline{XZ})}{\mathbf{C}(\overline{YZ})}.$$

Proof.

$$(ZXZ_p Y_p) \stackrel{(2.5)}{=} (Z_p X_p ZY) \stackrel{(2.2)}{=} (ZY Z_p X_p).$$

¹In fact, any pair of perspective triangles are polar to each other with respect to a conic (see [7, p. 65]).

So we have

$$(XYZ_pX_p)(XYZ_pY_p) \stackrel{(2.2)}{=} \frac{(XYZ_pX_p)}{(YXZ_pY_p)} \stackrel{(2.2c)}{=} \frac{\cancel{(YZZ_pX_p)}}{\cancel{(ZXZ_pY_p)}}(XZZ_pX_p) = \frac{\mathbf{C}(\overline{YZ})}{\mathbf{C}(\overline{XZ})}.$$

■

Proof of Theorem 7.8. Let H^* be the intersection point of the lines a^*, b^*, c^* , and let h^* be the polar line of H^* . Consider also the three points:

$$A_0^* = a \cdot h^*, \quad B_0^* = b \cdot h^*, \quad C_0^* = c \cdot h^*.$$

By Theorem 7.7 and Corollary 7.5 we have that

$$(ABC_0^*C_1)(ABC_0^*C_2)(BCA_0^*A_1)(BCA_0^*A_2)(CAB_0^*B_1)(CAB_0^*B_2) = 1. \quad (7.3)$$

Because $a^* = A^*A' = A'H^*$, we have that

$$\rho(a^*) = \rho(A') \cdot \rho(H^*) = a \cdot h^* = A_0^*.$$

Moreover, as $A^* \in a^*$ it is $\rho(A^*) \ni \rho(a^*)$ and this implies that A_0^* and A^* are conjugate to each other. In the same way, we have that B_0^*, C_0^* are the conjugate points of B^*, C^* in b, c respectively. Finally, by Lemma 7.9

$$\begin{aligned} (ABC_0^*C_1)(ABC_0^*C_2) &= \frac{\mathbf{C}(\overline{AC^*})}{\mathbf{C}(\overline{BC^*})} = \frac{\mathbf{C}(c_1)}{\mathbf{C}(c_2)}, \\ (BCA_0^*A_1)(BCA_0^*A_2) &= \frac{\mathbf{C}(\overline{BA^*})}{\mathbf{C}(\overline{CA^*})} = \frac{\mathbf{C}(a_1)}{\mathbf{C}(a_2)}, \\ (CAB_0^*B_1)(CAB_0^*B_2) &= \frac{\mathbf{C}(\overline{CB^*})}{\mathbf{C}(\overline{AB^*})} = \frac{\mathbf{C}(b_1)}{\mathbf{C}(b_2)}. \end{aligned}$$

This completes the proof. ■

Due to Proposition 7.6, identity (7.2) is a necessary but not sufficient condition for the concurrency of the lines a^*, b^*, c^* in Theorem 7.8. In order to obtain a partial converse of Theorem 7.8 we need to introduce some new notation. For any pair of points X, Y in the projective plane such the line $z = XY$ is not tangent to Φ , the *Carnot involution on z with respect to X, Y* is the composition $\zeta_{XY} = \rho_z \tau_{XY} \rho_z$, where ρ_z and τ_{XY} are the conjugacy with respect to Φ and the harmonic conjugacy with respect to X, Y respectively as introduced in §2. For any other point W on z , $\zeta_{XY}(W)$ is the *Carnot conjugate* of W with respect to X, Y . With the same notation as that introduced before Theorem 7.8, let denote now by H^* the point $b^* \cdot c^*$ and consider also the line $d^* = A'H^*$ and the point $D^* = a \cdot d^*$.

Theorem 7.10 *If the identity (7.2) holds, then it is $A^* = D^*$ or A^* is the Carnot conjugate of D^* with respect to B, C .*

Proof. Let A_0, B_0, C_0 be the conjugate points of A^*, B^*, C^* in a, b, c respectively. Let D_0 be the conjugate point of D^* in a . By the proof of Theorem 7.8, it is $D_0 = a \cdot B_0 C_0$. By Lemma 7.9, if (7.2) holds, then (7.3) also holds. By Proposition 7.6, if (7.3) holds, it is $A_0 = D_0$ or A_0 is the harmonic conjugate $\tau_{BC}(D_0)$ of D_0 with respect to B, C , and this implies that

$$A^* = \rho_a(A_0) = \rho_a(D_0) = D^*$$

or

$$A^* = \rho_a(A_0) = \rho_a(\tau_{BC}(D_0)) = \rho_a(\tau_{BC}(\rho_a(D^*))) = \zeta_{BC}(D^*) .$$

■

If the vertices A, B, C of the projective triangle \mathcal{T} and the points A^*, B^*, C^* lie in the non-euclidean plane \mathbb{P} , then \mathcal{T} is a non-euclidean triangle and a^*, b^*, c^* are the perpendicular lines to a, b, c through A^*, B^*, C^* respectively. Let denote also by $a_1, a_2, b_1, b_2, c_1, c_2$ the non-euclidean lengths in \mathbb{P} of the segments $a_1, a_2, b_1, b_2, c_1, c_2$, respectively. Theorem 7.8 implies the following (Figure 7.2(a)):

Theorem 7.11 *If \mathbb{P} is the elliptic plane and the lines a^*, b^*, c^* are concurrent, then*

$$\cos a_1 \cos b_1 \cos c_1 = \cos a_2 \cos b_2 \cos c_2 . \quad (7.4)$$

Theorem 7.10 shows how we can obtain three points A^*, B^*, C^* for which the identity (7.4) holds and such that the lines a^*, b^*, c^* are not concurrent. If this is the case, we say that the points A^*, B^*, C^* are *fake Carnot points* of \mathcal{T} . This cannot happen if \mathcal{T} is a triangle in the hyperbolic plane (Figure 7.2(b)).

Theorem 7.12 *Hyperbolic triangles have no fake Carnot points: if \mathbb{P} is the hyperbolic plane, the lines a^*, b^*, c^* are concurrent if and only if*

$$\cosh a_1 \cosh b_1 \cosh c_1 = \cosh a_2 \cosh b_2 \cosh c_2 . \quad (7.5)$$

Proof. If the lines a^*, b^*, c^* are concurrent, then (7.5) is a consequence of Theorem 7.8 and Proposition 3.9.

If (7.5) holds, then (7.2) also holds by Proposition 3.9. Thus, by Theorem 7.10, we have that A^* equals D^* or the Carnot conjugate $\zeta_{BC}(D^*)$ of D^* with respect to B, C , where D^* is, as in Theorem 7.10, the orthogonal projection of $H^* = b^* \cdot c^*$ into the line a .

The polar triangle \mathcal{T}' of \mathcal{T} divides the projective plane into four triangular regions, and one of them contains the absolute conic Φ (see Figure 7.2(b)). Let us call \mathcal{T}'_Φ to this region. The point H^* does not necessarily lie in \mathbb{P} , but because B^* and C^* are in \mathbb{P} it must lie in \mathcal{T}'_Φ . This implies that

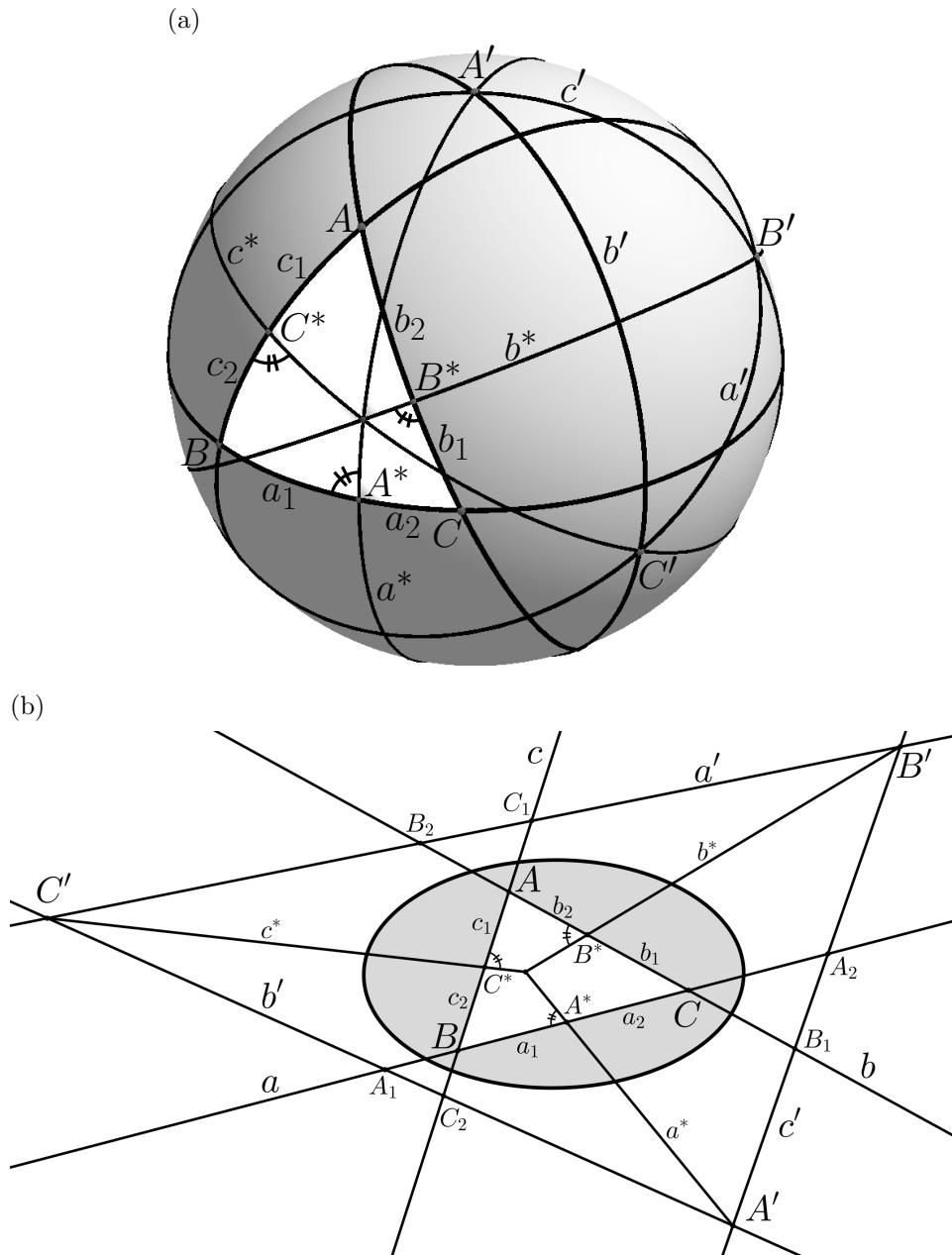


Fig. 7.2: Carnot's theorem on non-euclidean triangles

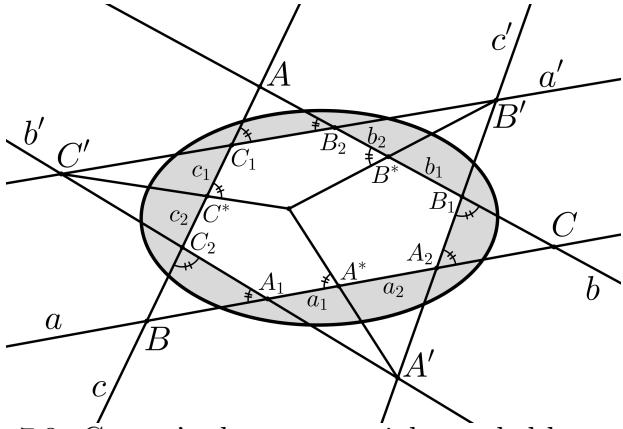


Fig. 7.3: Carnot's theorem on right-angled hexagons

D^* also lies in \mathcal{T}'_Φ , and we will see that in this case the Carnot conjugate of D^* with respect to B, C cannot be in \mathbb{P} . If U, V are the intersection points of a with Φ , as it is mentioned in §2 the conjugacy involution ρ_a of a coincides with the harmonic conjugacy τ_{UV} with respect to U, V .

For any real projective line x , any two different real points Y, Z on x divide the line into two connected subsets (segments), say x_0 and x_1 , and the harmonic conjugacy τ_{YZ} with respect to Y, Z sends x_0 onto x_1 and vice versa (cf. Lemma 2.2). Therefore,

$$\begin{aligned} D^* \in \mathbb{P} &\Rightarrow \rho_a(D^*) \notin \mathbb{P} \Rightarrow \\ &\Rightarrow \tau_{BC}(\rho_a(D^*)) \text{ belongs to the hyperbolic segment } \overline{BC} \subset \mathbb{P} \Rightarrow \\ &\Rightarrow \zeta_{BC}(D^*) = \rho_a(\tau_{BC}(\rho_a(D^*))) \notin \mathbb{P}. \end{aligned}$$

On the other hand, if D^* does not lie in \mathbb{P} , because it is in \mathcal{T}'_Φ , its conjugate $\rho_a(D^*)$ lies in \mathbb{P} but outside the hyperbolic segment \overline{BC} . Thus, $\tau_{BC}(\rho_a(D^*))$ is in the hyperbolic segment \overline{BC} , and so $\zeta_{BC}(D^*)$ lies outside \mathbb{P} . In any case, because B^* and C^* are in \mathbb{P} , it cannot be $\zeta_{BC}(D^*) \in \mathbb{P}$. Because A^* belongs to \mathbb{P} , it must be $A^* = D^*$ and in consequence the lines a^*, b^*, c^* are concurrent.

■

Finally, we will present a version of Theorem 7.2 for right-angled hyperbolic hexagons. Let \mathcal{H} be a right-angled hexagon in the hyperbolic plane with consecutive vertices $A_1, A_2, B_1, B_2, C_1, C_2$. Consider three points A^*, B^*, C^* lying on the alternate sides $a = A_1A_2, b = B_1B_2, c = C_1C_2$ of \mathcal{H} respectively, and let a^*, b^*, c^* be the perpendicular lines to a, b, c through A^*, B^*, C^* respectively. Let $a_1, a_2, b_1, b_2, c_1, c_2$ denote now the hyperbolic lengths of the segments $\overline{A_1A^*}, \overline{A_2A^*}, \overline{B_1B^*}, \overline{B_2B^*}, \overline{C_1C^*}, \overline{C_2C^*}$ (see Figure 7.3). If we construct the projective triangle \mathcal{T} with vertices $A = b \cdot c$,

$B = c \cdot a$ and $C = a \cdot b$, we can apply Theorem 7.8 to \mathcal{T} and the points A^*, B^*, C^* . After translating the projective trigonometric ratios of (7.2) to our construction using Proposition 3.9, we obtain:

Theorem 7.13 *If the lines a^*, b^*, c^* are concurrent, then*

$$\sinh a_1 \sinh b_1 \sinh c_1 = \sinh a_2 \sinh b_2 \sinh c_2.$$

§ 8

Where do laws of cosines come from?

The projective trigonometric ratios **C**, **S**, **T** have the inconvenience of the presence of the square power in all their geometric translations (Table 3.1). In §6 we saw that they are sufficient to deal with figures whose trigonometric formulae are simple, and that they are not suitable to work with the law of cosines of a generalized non-euclidean triangle, for example. We are looking for projective expressions associated with segments or angles whose non-euclidean translation give the actual, *unsquared*, trigonometric ratios of the corresponding segment or angle. This can be done using the midpoints of segments as accessory points. Another construction appears in the Appendix as Formula (A.2).

8.1 Midpoints and the unsquaring of projective trigonometric ratios

Let A, B be two points not lying on Φ and such that the line p joining them is not tangent to Φ . Let D, D_p be the two midpoints of \overline{AB} . The symmetry τ_{DD_p} of p with respect to D, D_p maps A and A_p into B and B_p respectively. This implies that

$$(ABDA_p) = (BADB_p),$$

and so it is

$$(ABB_pA_p) = (ABDA_p)(ABB_pD) = (BADB_p)(ABB_pD) = (ABB_pD)^2.$$

Moreover,

$$(ABB_pD) = (ABD_pD)(ABB_pD_p) = -(ABB_pD_p).$$

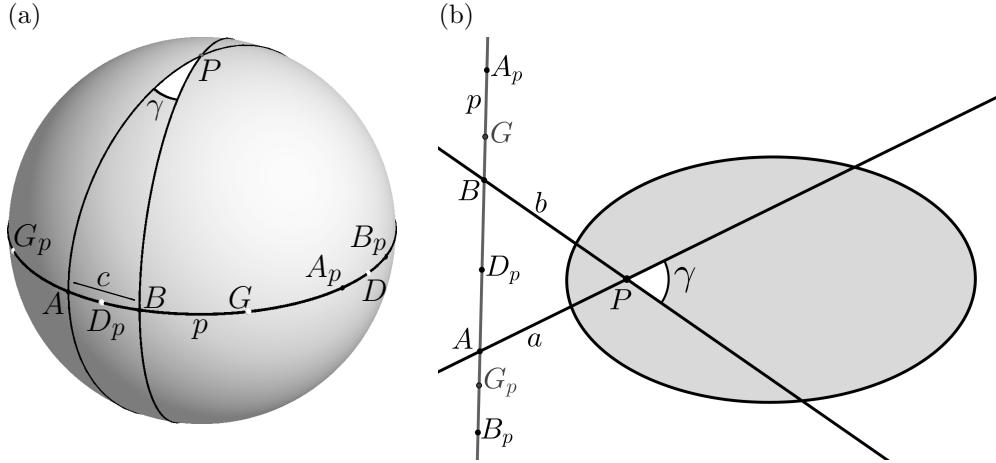


Fig. 8.1: Projective segments and their midpoints I

Remark 8.1 The previous identities imply that the two square roots of $\mathbf{C}(\overline{AB}) = (ABB_pA_p)$ are given by (ABB_pD) and (ABB_pD_p) .

By definition, it is $\mathbf{S}(\overline{AB}) = \mathbf{C}(\overline{AB}_p)$. Therefore, the two square roots of $\mathbf{S}(\overline{AB})$ are given by (AB_pBG) and (AB_pBG_p) , where G, G_p are the two midpoints of \overline{AB}_p .

If we want to compute a square root of $\mathbf{C}(\overline{AB})$ and a square root of $\mathbf{S}(\overline{AB})$, we must choose a midpoint of the segment \overline{AB} and a midpoint of the complementary segment \overline{AB}_p . By Lemma 3.5, the midpoints of \overline{AB} are also the midpoints of $\overline{A_pB_p}$ and the midpoints of \overline{AB}_p are the midpoints of $\overline{A_pB}$. We say that the midpoints of \overline{AB}_p are the *complementary midpoints* of \overline{AB} .

Definition 8.2 The segment \overline{AB} is oriented if we have chosen for it a midpoint and a complementary midpoint as preferred midpoints. If \overline{AB} is oriented, we define its two associated vectors \overrightarrow{AB} and \overrightarrow{BA} as the ordered pairs (A, B) and (B, A) , respectively, together with the preferred midpoints of \overline{AB} . We define also the projective trigonometric ratios of \overrightarrow{AB} and \overrightarrow{BA} :

$$\begin{aligned} \mathbf{c}(\overrightarrow{AB}) &= (ABB_pD), & \mathbf{c}(\overrightarrow{BA}) &= (BAA_pD), \\ \mathbf{s}(\overrightarrow{AB}) &= (AB_pBG), & \mathbf{s}(\overrightarrow{BA}) &= (BA_pAG). \end{aligned} \tag{8.1}$$

where D and G are respectively the preferred midpoint and the preferred complementary midpoint of \overline{AB} .

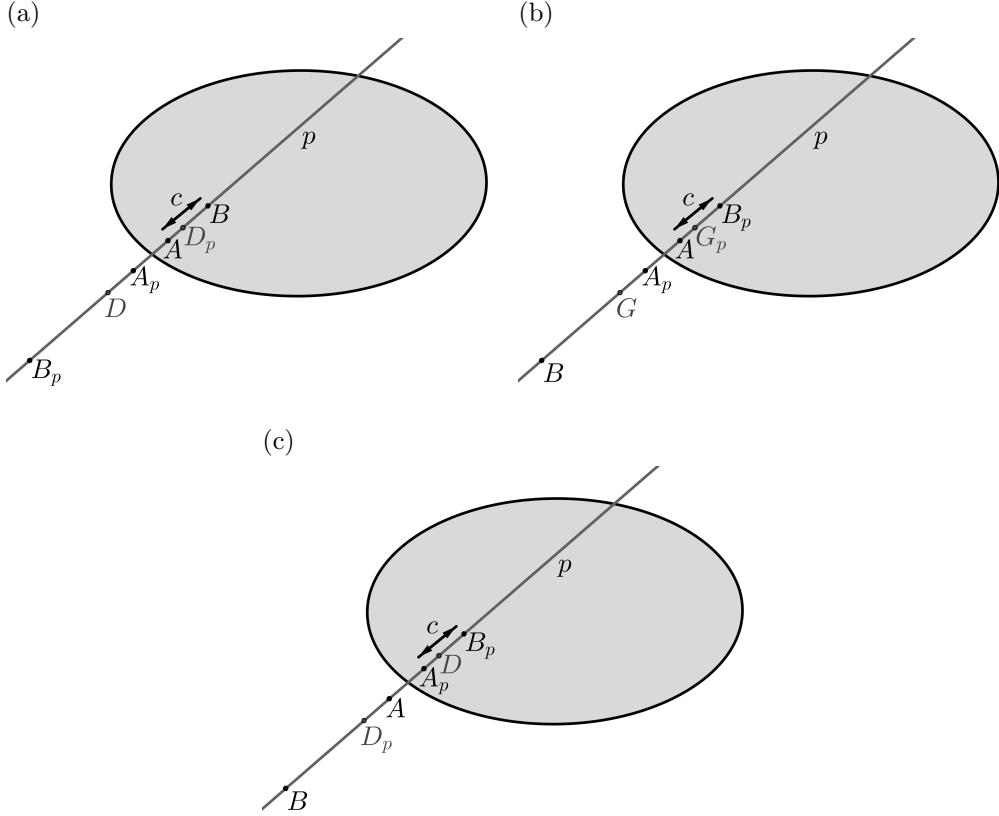


Fig. 8.2: Projective segments and their midpoints II

From now on, we will reserve the term *projective trigonometric ratios* for the functions \mathbf{c} and \mathbf{s} just defined, while the functions $\mathbf{C}, \mathbf{S}, \mathbf{T}$ defined in §3 will be called *squared projective trigonometric ratios*.

Note that the functions \mathbf{c} and \mathbf{s} for oriented segments work as it is expected in analogy with the circular and hyperbolic cosine and sine functions, because it is:

$$\begin{aligned}\mathbf{c}(\overrightarrow{AB}) &= (ABB_pD) \stackrel{\tau_{DD_p}}{=} (BAA_pD) = \mathbf{c}(\overrightarrow{BA}), \\ \mathbf{s}(\overrightarrow{AB}) &= (AB_pBG) \stackrel{\tau_{DD_p}}{=} (BA_pAG_p) = -\mathbf{s}(\overrightarrow{BA}),\end{aligned}$$

and so \mathbf{c} works like an “even” function while \mathbf{s} behaves like an “odd” function. For a simpler notation we use the expressions $\mathbf{c}(AB), \mathbf{s}(AB)$ instead of $\mathbf{c}(\overrightarrow{AB}), \mathbf{c}(\overrightarrow{AB})$.

If Φ is imaginary, the points D, D_p, G, G_p are real points. If A, B are the points depicted in Figure 8.1(a), and we choose as preferred midpoints for them the points D and G of the same figure, by Lemma 2.2 it is

$$\mathbf{c}(AB) = (ABB_pD) > 0 \quad \text{and} \quad \mathbf{s}(AB) = (AB_pBG) > 0$$

Therefore, in this case it is $\mathbf{c}(AB) = \cos c$ and $\mathbf{s}(AB) = \sin c$, where c is the elliptic distance between A and B as depicted in Figure 8.1(a). If a, b are the lines joining A, B with the pole P of p and γ is the angle between a and b depicted in Figure 8.1(a), it is also $\mathbf{c}(AB) = \cos \gamma$ and $\mathbf{s}(AB) = \sin \gamma$.

When Φ is real and p is exterior to Φ , the four points D, D_p, G, G_p are again real points. In the situation of Figure 8.1(b), if D and G are the preferred midpoints of \overline{AB} , it is $\mathbf{c}(AB) > 0$ and also $\mathbf{s}(AB) > 0$. Let consider again the lines a, b joining A, B with the pole P of p . Then, because the angle γ in this figure is an acute angle (observe that the line a and its conjugate $a_P = PA_p$ form a right angle), it must be $\mathbf{c}(AB) = \cos \gamma$ and $\mathbf{s}(AB) = \sin \gamma$.

When Φ is a real conic and p is secant to Φ , there are some possibilities depending on the relative positions of the points A, B with respect to Φ . In this case, the midpoints of \overline{AB} are real points if and only if the complementary midpoints are imaginary. This will imply that for any choice of preferred midpoints for the segment \overline{AB} , one of the projective trigonometric ratios $\mathbf{c}(AB), \mathbf{s}(AB)$ will be real while the other is pure imaginary. If we want to translate the projective trigonometric ratios $\mathbf{c}(AB), \mathbf{s}(AB)$ into geometric trigonometric ratios associated with a hyperbolic magnitude, we need to decide the *sign* (positive or negative when the number is real, *positive imaginary* or *negative imaginary*¹ when the number is pure imaginary) of $\mathbf{c}(AB)$ and $\mathbf{s}(AB)$. The sign of the real number among $\mathbf{c}(AB)$ and $\mathbf{s}(AB)$ can always be decided “visually”, without doing explicit computations, using Lemma 2.2. On the contrary, the sign of the pure imaginary number among $\mathbf{c}(AB)$ and $\mathbf{s}(AB)$ cannot be decided in a simple way.

If both points A, B are interior to Φ , the two midpoints D, D_p are real points, while the two complementary midpoints G, G_p are imaginary points. Thus, we have no visual way for choosing a preferred complementary midpoint. In the situation of Figure 8.2(a), if D is the preferred midpoint of \overline{AB} , we have $\mathbf{c}(AB) > 0$ and so it is $\mathbf{c}(AB) = \cosh c$, where c is the hyperbolic distance between A and B . On the other hand, if G is a complementary midpoint of \overline{AB} , the cross ratio $(AB_p BG)$ must be pure imaginary because by Lemma 2.2 it is

$$(AB_p BG)^2 = \mathbf{S}(AB) = (AB_p BA_p) < 0.$$

Thus, for any choice of preferred complementary midpoint it will be $\mathbf{s}(AB) = \pm i \sinh c$, and the correct sign of this equality cannot be decided without doing explicit computations.

¹ We use the terms *positive imaginary* and *negative imaginary* in the obvious way: a positive (resp. negative) imaginary number has the form $i\lambda$, with $\lambda \in \mathbb{R}$ positive (resp. negative).

In the same way, if both points A, B are exterior to Φ , for any choice of preferred midpoints it will be $\mathbf{c}(AB) = \pm \cosh c$ and $\mathbf{s}(AB) = \pm i \sinh c$, where c is the hyperbolic distance between the points A_p and B_p . For example, in Figure 8.2(c), if D is the preferred midpoint of the segment \overline{AB} it is $\mathbf{c}(AB) = \cosh c$.

If exactly one of the two points A, B is interior to Φ , then the midpoints of \overline{AB} are imaginary numbers while the complementary midpoints are real. In Figure 8.2(b), for example, if G is the preferred complementary midpoint of \overline{AB} it is $\mathbf{s}(AB) = \cosh c$ while for any choice of a preferred midpoint of \overline{AB} it will be $\mathbf{c}(AB) = \pm i \sinh c$, where c is the hyperbolic distance between A and B_p .

In our aim of obtaining a projective version of the law of cosines, we will use the functions \mathbf{c}, \mathbf{s} just defined, but we have the annoying problems of dealing with imaginary midpoints, and of deciding the sign of projective trigonometric ratios when they are pure imaginary. As we will see, we can overcome these drawbacks: we will give a construction such that no imaginary midpoint needs to be explicitly constructed and such that no sign of pure imaginary projective trigonometric ratio needs to be explicitly computed.

8.2 The magic midpoints. Oriented triangles

Consider a projective triangle \mathcal{T} and its polar triangle \mathcal{T}' as in §4.2. The points D, D_a and G, G_a are the midpoints and the complementary midpoints of \overline{BC} , respectively. In the same way, the points E, E_b and H, H_b are the midpoints and the complementary midpoints of \overline{CA} , respectively, and F, F_c and I, I_c are the midpoints and the complementary midpoints of \overline{AB} , respectively. As usual, the same notation but adding an apostrophe

$$D', D'_{a'}, G', G'_{a'}, E', E'_{b'}, H', H'_{b'}, F', F'_{c'}, I', I'_{c'}$$

holds for the midpoints and complementary midpoints of \mathcal{T}' .

In order to simplify notation, we say that the midpoints (complementary midpoints) of a segment whose endpoints are vertices of \mathcal{T} are midpoints (complementary midpoints) of the corresponding side of \mathcal{T} and also midpoints (complementary midpoints) of \mathcal{T}' .

If we want to apply our functions \mathbf{c}, \mathbf{s} to the triangles \mathcal{T} and \mathcal{T}' , we need to orient each side of both triangles in the sense of Definition 8.2. It should be interesting to do so in a coherent way. For the midpoints of \mathcal{T} and \mathcal{T}' it is not difficult to establish a choice criterion based on Theorem 4.13. We have assumed that D, E, F are non-collinear, and this implies that

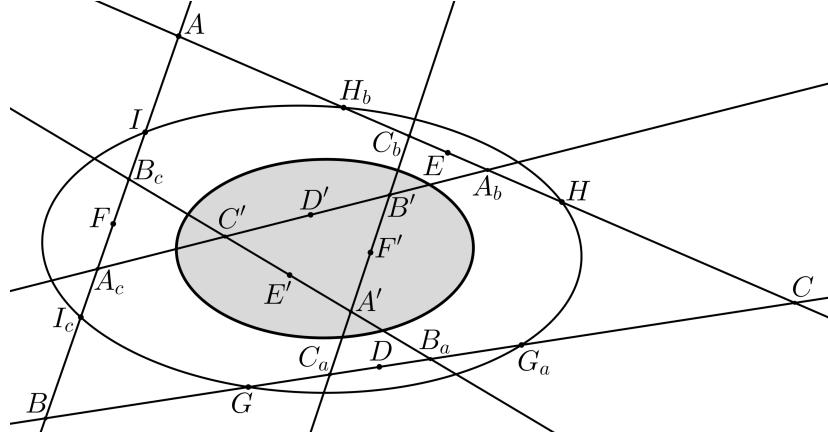


Fig. 8.3: Complementary midpoints lie on a conic

D, E, F_c are collinear. There is a big difference between choosing D, E, F or D, E, F_c as preferred midpoints. We will say that the preferred midpoints of the sides of \mathcal{T} are *coherently chosen* if they are not collinear, and the same for the preferred midpoints of \mathcal{T}' . Thus, if we choose the midpoints D, E, F and D', E', F' with the assumptions of §4.2 as preferred midpoints of their respective segments, they will be coherently chosen. The choice criterion is not as easy for the complementary midpoints of \mathcal{T} , because they have not the same structure as the midpoints.

Theorem 8.3 *The complementary midpoints of the triangle \mathcal{T} lie on a conic.*

Before proving this theorem, we need the following lemma:

Lemma 8.4 *D, D_a are the midpoints of $\overline{GG_a}$ and G, G_a are the midpoints of $\overline{DD_a}$.*

Proof. By Lemma 3.4, it suffices to prove that $(GG_aDD_a) = -1$. The composition $\tau_{GG_a} \circ \tau_{DD_a}$ sends B, B_a, C, C_a into B_a, B, C_a, C respectively. Therefore, it is $\tau_{GG_a} \circ \tau_{DD_a} = \rho_a$, and so

$$\tau_{GG_a}(D) = \tau_{GG_a}(\tau_{DD_a}(D)) = \rho(D) = D_a \Rightarrow (GG_aDD_a) = -1.$$

■

Proof of Theorem 8.3. Assume that the midpoints D, E, F of \mathcal{T} are not collinear (see Figure 8.3). Then, D_a, E_b, F_c are collinear.

By Lemma 8.4, we have

$$(BCD_aG) \stackrel{\tau_{DD_a}}{=} (CBD_aG_a) \Rightarrow (BCD_aG)(BCD_aG_a) = 1.$$

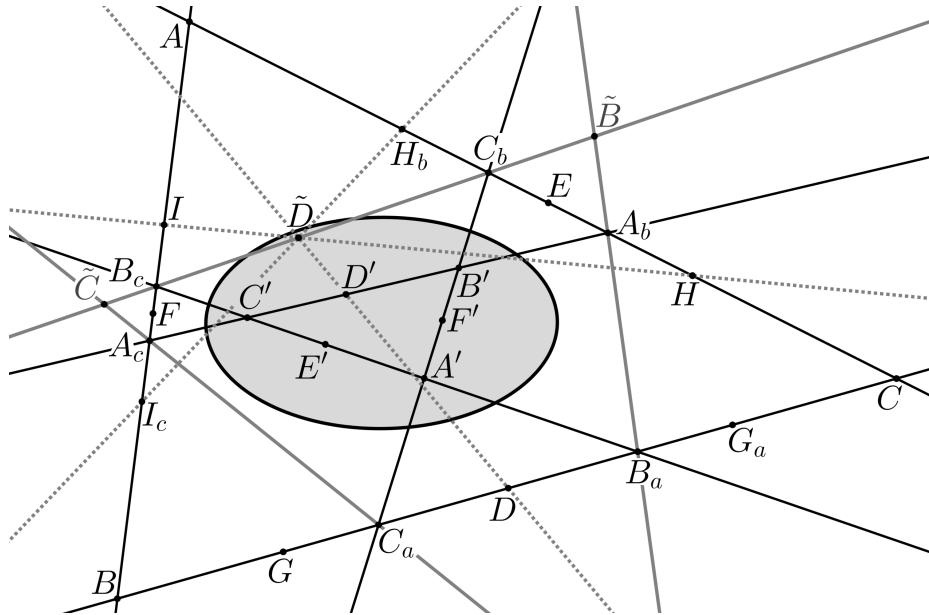


Fig. 8.4: Concurrence of lines at a magic midpoint

In the same way, it is

$$(CAE_bH)(CAE_bH_b) = 1 \quad \text{and} \quad (ABF_cI)(ABF_cI_c) = 1.$$

Therefore,

$$(BCD_aG)(BCD_aG_a)(CAE_bH)(CAE_bH_b)(ABF_cI)(ABF_cI_c) = 1.$$

The result now is a consequence of Carnot's Theorem on projective triangles (Theorem 7.3). ■

Thus, until now there is no remarkable difference between choosing G, H, I or G, H, I_c as preferred complementary midpoints of \mathcal{T} . We must make a deeper exploration for finding a choice criterion for the preferred complementary midpoints of \mathcal{T} . This will need... some magic.

Let fix our sight now in the segment $\overline{B_cC_b}$. Its midpoints, and the midpoints of $\overline{C_aA_c}$ and $\overline{A_bB_a}$ are very interesting because they are “in the middle of everywhere”. We need to introduce some new notation. Let $\tilde{a}, \tilde{b}, \tilde{c}$ be the lines B_cC_b, C_aA_c, A_bB_a respectively, let $\tilde{\mathcal{T}}$ be the triangle with sides $\tilde{a}, \tilde{b}, \tilde{c}$, and consider the vertices of $\tilde{\mathcal{T}}$:

$$\tilde{A} = \tilde{b} \cdot \tilde{c}, \quad \tilde{B} = \tilde{c} \cdot \tilde{a}, \quad \tilde{C} = \tilde{a} \cdot \tilde{b}.$$

The poles of $\tilde{a}, \tilde{b}, \tilde{c}$ are, respectively, the points

$$A_1 = b_C \cdot c_B, \quad B_1 = c_A \cdot a_C, \quad C_1 = a_B \cdot b_A,$$

of §5. By the proof of Proposition 5.7, the lines a, a' and B_1C_1 are concurrent, and this implies that their poles A', A and \tilde{A} respectively are collinear. Equivalently, \tilde{B} is collinear with B and B' and \tilde{C} is collinear with C and C' . Let $\tilde{D}, \tilde{D}_{\tilde{a}}$ be the midpoints of $\overline{B_cC_b}$.

Theorem 8.5 *The set $\{\tilde{D}, \tilde{D}_{\tilde{a}}\}$ coincides with:*

I *the set of intersection points $\{HI \cdot H_bI_c, HI_c \cdot H_bI\}$;*

II *the set of intersection points $\{H'I' \cdot H'_{b'}I'_{c'}, H'I'_{c'} \cdot H'_{b'}I'\}$; and*

III *the set of midpoints of $\overline{\tilde{B}\tilde{C}}$.*

Moreover,

IVa *if \mathcal{T} is not isosceles at A , the set $\{\tilde{D}, \tilde{D}_{\tilde{a}}\}$ coincides with the set of intersection points $\{DD' \cdot D_aD'_{a'}, DD'_{a'} \cdot D_aD'\}$;*

IVb *if \mathcal{T} is isosceles at A , one of the points $\tilde{D}, \tilde{D}_{\tilde{a}}$ coincides with A_0 while the other is collinear with D and D' .*

In particular, $\tilde{D}, \tilde{D}_{\tilde{a}}$ are: (i) the diagonal points different from A of the quadrangle whose vertices are the complementary midpoints H, H_b, I, I_c of \overline{CA} and \overline{AB} ; (ii) the diagonal points different from A' of the quadrangle whose vertices are the complementary midpoints $H', H'_{b'}, I', I'_{c'}$ of $\overline{C'A'}$ and $\overline{A'B'}$; and (iii) if \mathcal{T} is not isosceles at A , the diagonal points different from A_0 of the quadrangle whose vertices are the midpoints $D, D_a, D', D'_{a'}$ of \overline{BC} and $\overline{B'C'}$ (Figure 8.4).

Proof. We consider the triangle $\widehat{ABC_bC_b}$. Its midpoints are the vertices of a complete quadrilateral. This proves **I**. An equivalent construction proves **II**.

We prove **IV** before **III**. In order to prove **IV** we need:

Claim 8.5.1 *The triangle \mathcal{T} is isosceles at A if and only if a, a' and \tilde{a} are concurrent.*

Proof of Claim 8.5.1. If, a, a', \tilde{a} are concurrent, it is

$$(ABB_cA_c) = (ACC_bA_b).$$

Let π_{A_0} be the perspectivity from b onto c with center at A_0 . Because A, C, C_b, A_b are four different points, their cross ratio is a number different from 0, 1, and so it has two different square roots which are also different from 0, 1. By Remark 8.1, the two square roots of (ACC_bA_b) are given by (ACC_bE) and (ACC_bE_b) , and so the two square roots of (ABB_cA_c) must

be given by $(ABB_c\pi_{A_0}(E))$ and $(ABB_c\pi_{A_0}(E_b))$. This implies that the set $\{\pi_{A_0}(E), \pi_{A_0}(E_b)\}$ coincides with $\{F, F_c\}$. As we have assumed that D, E, F are non-collinear and that D is different from A_0 , this implies that EF and E_bF_c pass also through A_0 and so $A_0 = D_a$.

The previous argument can be reversed: if $D_a = A_0$, the line EF passes through A_0 . By taking again the projection π_{A_0} , we have that the cross-ratios (CAA_bE) and (BAA_cF) must be equal, and so their respective square powers (CAA_bC_b) and (BAA_cB_c) must be equal too. This implies that $\tilde{a} = B_cC_b$ passes also through A_0 and therefore a, a', \tilde{a} are concurrent. ■

Therefore, if \mathcal{T} is not isosceles at A , the triangle $\mathcal{T}_a = \widehat{aa'\tilde{a}}$ is an actual triangle because its sides are not concurrent. In this case, we will show that the midpoints of \mathcal{T}_a coincide with the midpoints of the segments \overline{BC} , $\overline{B'C'}$, $\overline{B_cC_b}$.

The vertices of the triangle \mathcal{T}_a are the points

$$A_0 = a \cdot a', \quad J = a \cdot \tilde{a}, \quad J' = a' \cdot \tilde{a}.$$

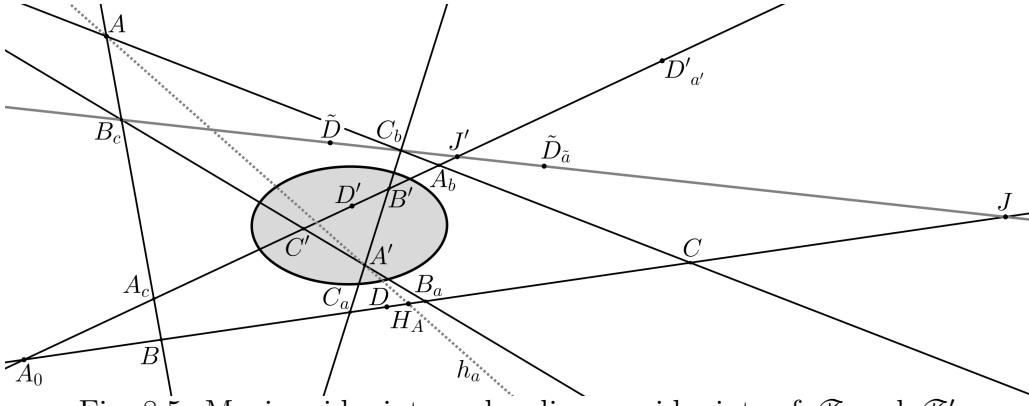
Let D, D_a be the midpoints of \overline{BC} . We must prove that they are also the midpoints of the segment $\overline{A_0J}$. The conjugate point of A_0 in a is the point H_A where the altitude h_a of \mathcal{T} intersects the side a . If we consider the quadrangle $\mathcal{Q}_2 = \{A, A', B_c, C_b\}$, the quadrangular involution $\sigma_{\mathcal{Q}_2}$ induced by \mathcal{Q}_2 on a sends the points B, C, J into the points C_a, B_a, H_A respectively and vice versa (Figure 8.5). The composition $\rho_a \circ \sigma_{\mathcal{Q}_2}$ sends the points B, C, B_a, C_a into the points C, B, C_a, B_a , and so it coincides with the symmetry τ_{DD_a} of a with respect to D . Then, it is

$$\tau_{DD_a}(J) = \rho_a(H_A) = A_0,$$

and therefore, by Lemma 3.4, the points D, D_a are also the midpoints of the segment $\overline{A_0J}$.

In the same way, it can be proved that the midpoints $D', D'_{a'}$ of $\overline{B'C'}$ are also the midpoints of $\overline{A_0J'}$, and a similar argument can be used to conclude that the midpoints $\tilde{D}, \tilde{D}_{\tilde{a}}$ of $\overline{B_cC_b}$ are also the midpoints of $\overline{JJ'}$. The polar of J is the line $A'A_1$ and the polar of J' is the line AA_1 . If we consider the quadrangular involution $\sigma_{\mathcal{Q}_3}$ on \tilde{a} induced by the quadrangle $\mathcal{Q}_3 = \{A, B, C, A_1\}$, we have that $\sigma_{\mathcal{Q}_3}$ sends B_c, C_b, J into $(C_b)_{\tilde{a}}, (B_c)_{\tilde{a}}, J'_{\tilde{a}}$ respectively and vice versa. The composition $\rho_{\tilde{a}} \circ \sigma_{\mathcal{Q}_3}$ coincides with the symmetry $\tau_{\tilde{D}\tilde{D}_{\tilde{a}}}$ of \tilde{a} with respect to \tilde{D} and sends J into J' . Thus, $\tilde{D}, \tilde{D}_{\tilde{a}}$ are the midpoints of $\overline{JJ'}$.

If \mathcal{T} is isosceles at A , the lines a, a' and \tilde{a} concur at A_0 , and the quadrangle $\mathcal{Q}_4 = \{A, A', B_0, C_0\}$ shows that A_0 and $AA' \cdot \tilde{a} = DD' \cdot \tilde{a}$ are the midpoints of $\overline{B_cC_b}$.


 Fig. 8.5: Magic midpoints and ordinary midpoints of \mathcal{T} and \mathcal{T}'

III. Consider the quadrangle $\mathcal{Q}_1 = \{B, C, B', C'\}$. The quadrangular involution $\sigma_{\mathcal{Q}_1}$ on \tilde{a} sends

$$(B_c)_{\tilde{a}} = BC' \cdot \tilde{a}, \quad \tilde{B} = BB' \cdot \tilde{a}, \quad J,$$

into

$$(C_b)_{\tilde{a}} = CB' \cdot \tilde{a}, \quad \tilde{C} = CC' \cdot \tilde{a}, \quad J',$$

and vice versa. Thus, $\sigma_{\mathcal{Q}_1}$ coincides with $\tau_{\tilde{D}\tilde{D}_a}$, and we conclude that \tilde{D}, \tilde{D}_a are the midpoints of $\overline{\tilde{B}\tilde{C}}$. ■

Remark 8.6 By Lemma 4.14, the statement of the previous theorem remains true even if \tilde{a} is tangent to Φ .

We will say that the triangle $\widetilde{\mathcal{T}}$ is the *magic triangle* of \mathcal{T} and that the midpoints of the sides of $\widetilde{\mathcal{T}}$ are the *magic midpoints* of \mathcal{T} . By their construction, the magic triangle and the magic midpoints of \mathcal{T} are also the magic triangle and the magic midpoints of \mathcal{T}' .

Theorem 8.5 gives the key for defining a coherent orientation on a triangle in a purely projective way. The triangle \mathcal{T} is *oriented* if we have oriented each side of \mathcal{T} and each side of \mathcal{T}' . Assume that \mathcal{T} is oriented. Following our previous notation, let D, E, F and G, H, I be the preferred midpoints and the preferred complementary midpoints of $\overline{BC}, \overline{CA}, \overline{AB}$, respectively, and let D', E', F' and G', H', I' be the preferred midpoints and the preferred complementary midpoints of $\overline{B'C'}, \overline{C'A'}, \overline{A'B'}$, respectively.

Definition 8.7 The triangle \mathcal{T} is *coherently oriented* if (see Figure 8.6)

- D, E, F are non-collinear and different from A_0, B_0, C_0 resp.;
- D', E', F' are non-collinear and different from A_0, B_0, C_0 resp.;

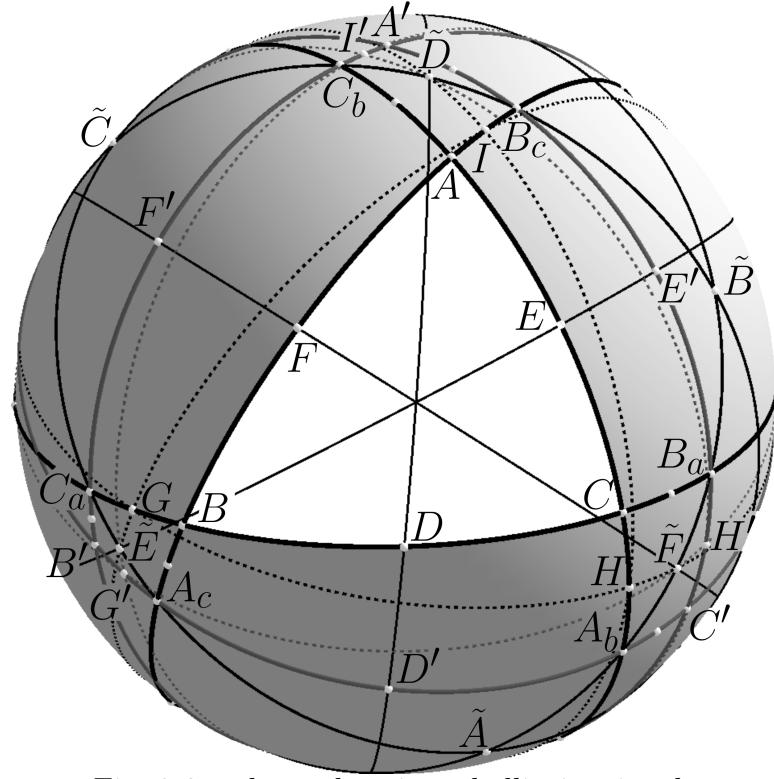


Fig. 8.6: coherently oriented elliptic triangle

and there exist non-collinear magic midpoints $\tilde{D}, \tilde{E}, \tilde{F}$ lying on $\tilde{a}, \tilde{b}, \tilde{c}$ respectively such that

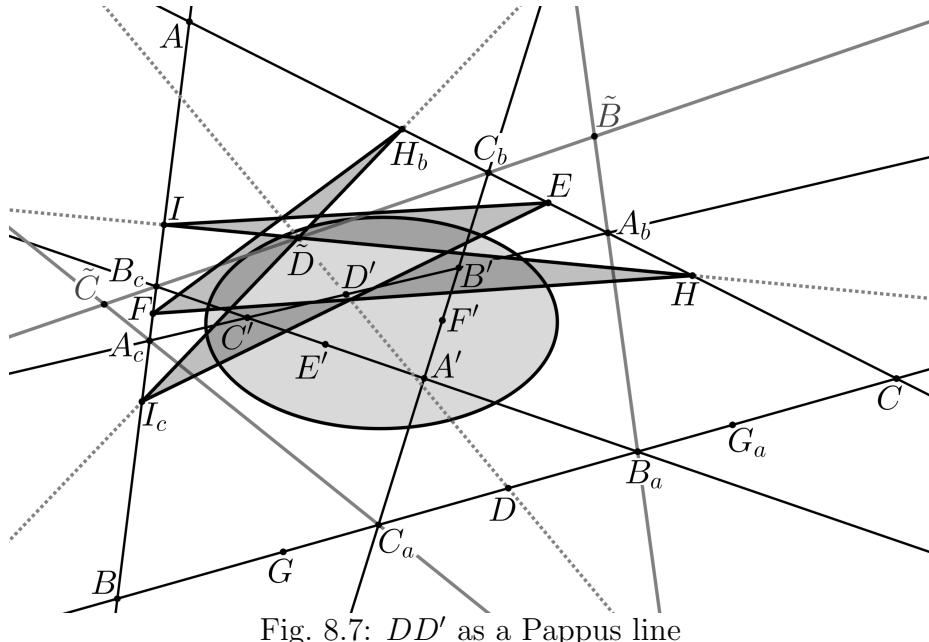
- \tilde{D} is the intersection point of DD', HI and $H'I'$;
- \tilde{E} is the intersection point of EE', IG and $I'G'$;
- \tilde{F} is the intersection point of FF', GH and $G'H'$.

In Figure 8.6 we have depicted \mathcal{T} as an elliptic triangle. This is the only situation where midpoints, complementary midpoints and magic midpoints of \mathcal{T} and \mathcal{T}' are all real points.

We will introduce some more properties about this construction. In particular, we can complete a forgotten task of our to-do list: the proof of Theorem 4.18.

Digression: the proof of Theorem 4.18 We need another classical construction from projective geometry.

Lemma 8.8 Let p, p' be two different projective lines, and let X, Y, Z, T and X', Y', Z', T' be two tetrads of points on p and on p' respectively such


 Fig. 8.7: DD' as a Pappus line

that $(XYZT) = (X'Y'Z'T')$. Then, the Pappus lines of the hexagons $XY'ZX'YZ'$, $XY'TX'YT'$, $XZ'TX'ZT'$ and $YZ'TY'ZT'$ coincide.

The proof of this Lemma is left as an exercise. A look at [7, p. 41] would help.

Remark 8.9 Assume that \mathcal{T} is coherently oriented. If E, E_b, H, H_b and F, F_c, I, I_c are taken as the tetrads of the previous Lemma, the resulting Pappus line is DD' (see Figure 8.7). Similar constructions hold for EE' and FF' .

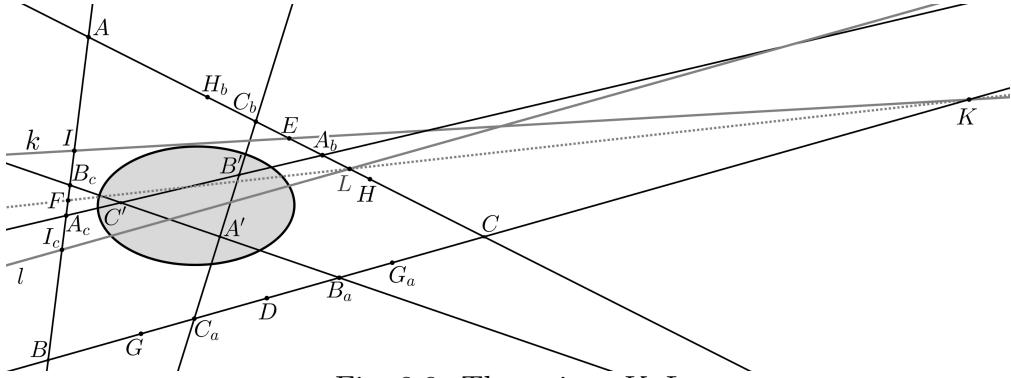
Proof. By Lemma 8.4, the two tetrads are harmonic sets, and so they fullfil the hypothesis of Lemma 8.8. It suffices to remember that by the chosen notation it is $EF_c \cdot E_bF = D$ and $HI \cdot H_bI_c = \tilde{D}$. ■

In Figure 8.7 we can see DD' as the Pappus line of the hexagon $EIHFH_bI_c$.

Proof of Theorem 4.18. We will divide the proof into some substeps, each of them presented as a claim.

Let consider the lines $k = EI$ and $l = I_cD_a$ and the points $K = a \cdot k$ and $L = b \cdot l$.

Claim 8.9.1 The points F, K, L are collinear (see Figure 8.8).


 Fig. 8.8: The points K, L

Proof of Claim 8.9.1. Let consider the quadrangle $\mathcal{Q} = \{E, D_a, K, L\}$, and the involution $\sigma_{\mathcal{Q}}$ that it induces on c . We have that

$$\begin{aligned} EK \cdot c &= I, \quad D_aL \cdot c = I_c \quad \Rightarrow \quad \sigma_{\mathcal{Q}}(I) = I_c; \\ EL \cdot c &= A; \quad D_aK \cdot c = B \quad \Rightarrow \quad \sigma_{\mathcal{Q}}(A) = B. \end{aligned}$$

This implies that $\sigma_{\mathcal{Q}}$ must coincide with the symmetry τ_{FF_c} on c with respect to F, F_c . As $ED_a \cdot c = F$, it must be also $KL \cdot c = F$. ■

Take the points $M = EI \cdot FH_b$ and $N = EF \cdot H_b I_c$.

Claim 8.9.2 *The points B, M, N are collinear (see Figure 8.9).*

Proof of Claim 8.9.2. Take the triangles $\widehat{BD_a I_c}$ and $\widehat{MEH_b}$. The intersection points of corresponding sides are:

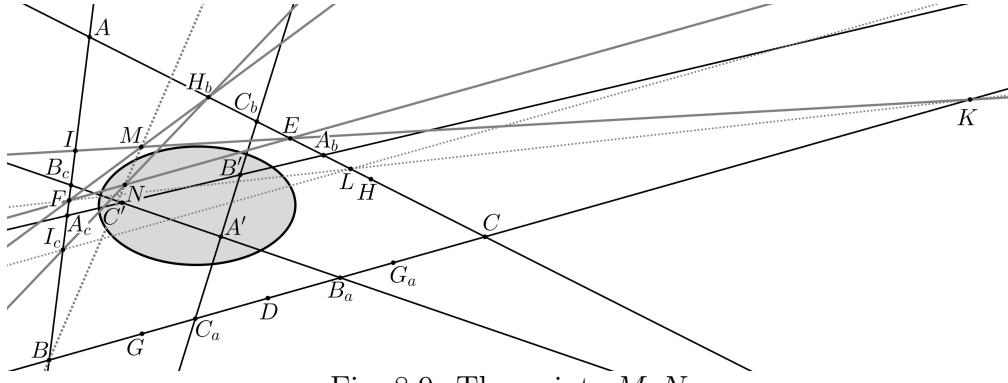
$$\begin{aligned} BD_a \cdot ME &= a \cdot k = K; \\ D_a I_c \cdot EH_b &= l \cdot b = L; \text{ and} \\ I_c B \cdot H_b M &= c \cdot FH_b = F; \end{aligned}$$

which are collinear by Claim 8.9.1. By Desargues' Theorem, both triangles are perspective: the lines $BM, D_a E, I_c H_b$ are concurrent. In other words, the line BM passes through $D_a E \cdot I_c H_b = EF \cdot I_c H_b = N$. ■

Consider the point $S = DD' \cdot EE'$.

Claim 8.9.3 *The points G_a, N, S are collinear.*

Proof of Claim 8.9.3. Consider the triangles $\widehat{BFG_a}$ and \widehat{MES} . By Remark 8.9, the point M lies on $\tilde{d} = DD'$ and the point $M' = DI \cdot FG_a$ lies on EE' . This implies that the intersection points of corresponding sides of


 Fig. 8.9: The points M, N

both triangles are:

$$\begin{aligned} BF \cdot ME &= c \cdot ME = I; \\ FG_a \cdot ES &= M'; \text{ and} \\ G_aB \cdot SM &= a \cdot \tilde{d} = D. \end{aligned}$$

As we have just said, $M' \in DI$: by Desargues' Theorem, both triangles are perspective. The lines BM, FE, G_aS are concurrent, but by Claim 8.9.2 the intersection point $BM \cdot FE$ is N . This completes the proof. ■

We can repeat step by step the proof of the previous claim but taking as starting point $S' = DD' \cdot FF'$ and considering the triangles \widehat{BEG}_a and \widehat{MFS}' in order to conclude that the points G_a, N, S' are collinear, and this implies that S and S' must coincide. This completes the proof of Theorem 4.18 ■

End of the digression

We will use some of the notation from the proof of Theorem 4.18 in the proof of this small lemma.

Lemma 8.10 *The points D, D_a are the midpoints of the segment whose endpoints are $J_1 := a \cdot HI$ and $J_2 = a \cdot H_b I_c$.*

Proof. Consider the points $N = EF \cdot H_b I_c$ and $N' = EF \cdot HI$. Assume by simplicity that N and N' are different points (the case $N = N'$ would follow as a limit case). In the proof of Theorem 4.18 we have seen that the point N is collinear with G_a and S , where S is the intersection point of the lines DD', EE' and FF' . In an exactly similar way, it can be proved that N' is collinear with G and S .

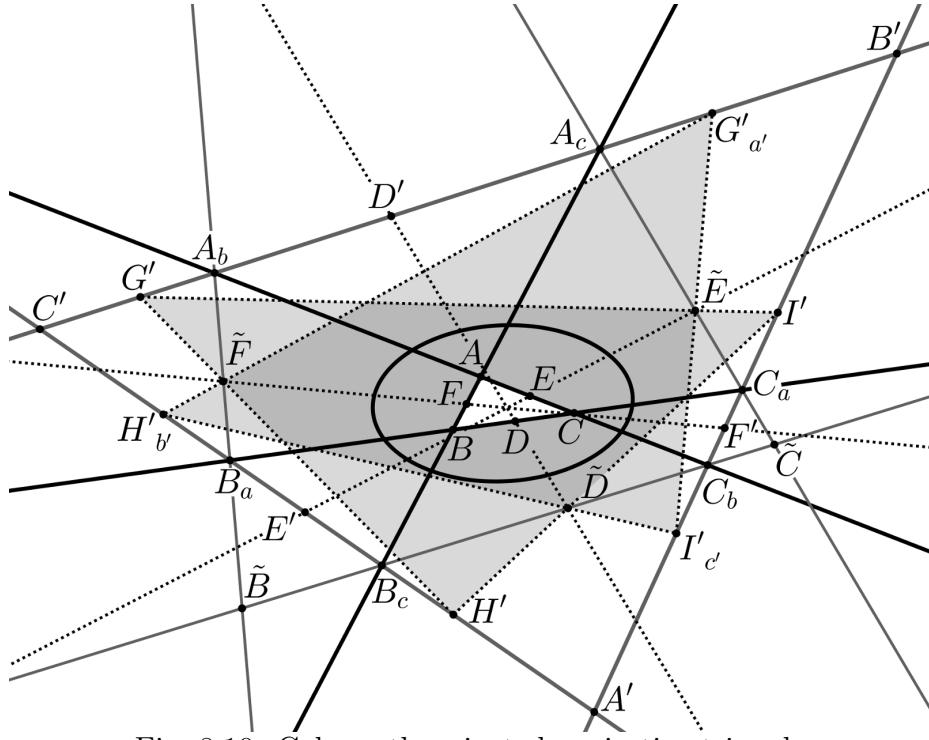


Fig. 8.10: Coherently oriented projective triangle

Let recover the magic midpoint $\tilde{D} = HI \cdot H_b I_c$, and consider the quadrangle $\mathcal{Q} = \{\tilde{D}, Q, N, N'\}$ and the involution $\sigma_{\mathcal{Q}}$ that it induces on a . We have that

$$\sigma_{\mathcal{Q}}(D) = \sigma_{\mathcal{Q}}(\tilde{A}Q \cdot a) = NN' \cdot a = EF \cdot a = D_a,$$

and that $\sigma_{\mathcal{Q}}(J_1) = G_a$ and that $\sigma_{\mathcal{Q}}(J_2) = G$. Therefore,

$$(J_1 J_2 D D_a) = (G_a G D_a D) = -1.$$

The result now follows from Lemma 3.4. ■

Remark 8.11 D, D_a are also the midpoints of the segment whose endpoints are $a \cdot HI_c$ and $a \cdot H_b I$.

Remark 8.12 None of the points D, D_a can be collinear with two of the points H, H_b, I, I_c .

Definition 8.7 allows us to develop a complete trigonometry for generalized triangles. Along the next sections, we will assume that the triangle \mathcal{T} is not right-angled and that it is coherently oriented, and we will use for the preferred midpoints and complementary midpoints of \mathcal{T} and \mathcal{T}' the same notation as in Definition 8.7. A good exercise would be to rewrite §6.1 using the functions \mathbf{c}, \mathbf{s} in these terms.

8.3 The general law of sines

Theorem 8.13 (Projective Law of sines) *If \mathcal{T} is coherently oriented, then*

$$\frac{\mathbf{s}(AB)}{\mathbf{s}(A'B')} = \frac{\mathbf{s}(BC)}{\mathbf{s}(B'C')} = \frac{\mathbf{s}(CA)}{\mathbf{s}(C'A')} \quad (8.2)$$

Proof. Assume for instance that \mathcal{T} is the triangle depicted in Figure 8.10. If we apply Menelaus' Projective Formula to the Menelaus' configuration with triangle $\widehat{AB_cC_b}$ and transversals a and HI , we obtain

$$(AB_cBI)(B_cC_bJ\tilde{D})(C_bACH) = 1,$$

where $J = a \cdot \tilde{a}$ as in §8.2. Note that $\mathbf{s}(AB) = (AB_cBI)$ and that $\mathbf{s}(AC) = (AC_bCH)$. Therefore, the previous equality is equivalent to

$$\frac{\mathbf{s}(AC)}{\mathbf{s}(AB)} = (C_bB_cJ\tilde{D}).$$

On the other hand if we apply Menelaus' Projective Formula to the Menelaus' configuration with triangle $\widehat{A'B_cC_b}$ and transversals a' and $H'I'$, we obtain

$$(A'B_cC'H')(B_cC_bJ'\tilde{D})(C_bA'B'I') = 1,$$

where $J' = a' \cdot \tilde{a}$, and this last formula is equivalent to

$$\frac{\mathbf{s}(A'C')}{\mathbf{s}(A'B')} = (C_bB_cJ'\tilde{D}).$$

By the proof of Theorem 8.5, if \mathcal{T} is not isosceles at A we know that $\tilde{D}, \tilde{D}_{\tilde{a}}$ are the midpoints of $\overline{JJ'}$. In this case, as $\tilde{D}, \tilde{D}_{\tilde{a}}$ are also the midpoints of $\overline{B_cC_b}$, the simmetry $\tau_{\tilde{D}\tilde{D}_{\tilde{a}}}$ on \tilde{a} with respect to \tilde{D} gives

$$(C_bB_cJ\tilde{D}) = (B_cC_bJ'\tilde{D}) = \frac{1}{(C_bB_cJ'\tilde{D})}.$$

Thus

$$\frac{\mathbf{s}(AC)}{\mathbf{s}(AB)} = \frac{\mathbf{s}(A'B')}{\mathbf{s}(A'C')}.$$

This argument also works If \mathcal{T} is isosceles at A , because in this case it is $(C_bB_cJ\tilde{D}) = (C_bB_cJ'\tilde{D}) = -1$.

As, $\mathbf{s}(AC) = -\mathbf{s}(CA)$ and $\mathbf{s}(A'C') = -\mathbf{s}(C'A')$, we obtain

$$\frac{\mathbf{s}(CA)}{\mathbf{s}(C'A')} = \frac{\mathbf{s}(AB)}{\mathbf{s}(A'B')}.$$

The rest of equalities if (8.2) are proved in the same way. ■

8.4 The general law of cosines

For proving our main result, we will need an extra help from some other classic theorems not mentioned before. After the previous chapters, a reader familiar with the classical theorems of affine and projective geometry would have the feeling that “someone’s missing”.

Theorem 8.14 (Ceva’s Theorem²) *Let $\mathcal{T} = \widehat{XYZ}$ be a projective triangle, and let X_1, Y_1, Z_1 be three points on the lines YZ, ZX, XY respectively. Let r be a line not incident with X, Y , or Z , and consider the points $X_0 = r \cdot YZ, Y_0 = r \cdot ZX$, and $Z_0 = r \cdot XY$ (Figure 8.11). The lines XX_1, YY_1, ZZ_1 are concurrent if and only if (compare (6.4))*

$$(XYZ_1Z_0)(YZX_1X_0)(ZXY_1Y_0) = -1 \quad (8.3)$$

We make a projective interpretation of Ceva’s Theorem exactly as we did with Menelaus’ Theorem in §6. It is usually said that Menelaus’ and Ceva’s theorems are *dual* to each other, but perhaps it should be said that they are *harmonic* to each other because harmonic conjugacy provides the equivalence between both theorems:

Lemma 8.15 *Let $\mathcal{T} = \widehat{XYZ}$ be a projective triangle, let X_1, Y_1 and Z_1 be three points on the lines YZ, ZX and XY respectively, and let X_2, Y_2 and Z_2 be their harmonic conjugates with respect to Y and Z , Z and X and X and Y , respectively. The lines XX_1, YY_1 and ZZ_1 are concurrent if and only if the points X_2, Y_2 and Z_2 are collinear.*

The proof of this lemma follows from Lemma 2.5, and it is left as an exercise.

We will use Ceva’s Theorem for proving another theorem, closely related to Menelaus’ and Ceva’s ones. As before, we propose a projective interpretation of the classical affine theorem.

Theorem 8.16 (Van Aubel’s Theorem on cevians) *Let $\mathcal{T} = \widehat{XYZ}$ be a projective triangle, and let X_1, Y_1, Z_1 be three points on the lines YZ, ZX, XY respectively. Let r be a line not incident with X, Y , or Z , and consider the points $X_2 = r \cdot XX_1, Y_0 = r \cdot XZ$, and $Z_0 = r \cdot XY$ (Figure 8.11). If the lines XX_1, YY_1, ZZ_1 are concurrent and Q is their concurrence point, then*

$$(XX_1QX_2) = (XYZ_1Z_0) + (XZY_1Y_0). \quad (8.4)$$

Proof. Consider the points

$$X_0 = r \cdot YZ, \quad Y_2 = r \cdot YY_1, \quad \text{and} \quad Z_2 = r \cdot ZZ_1.$$

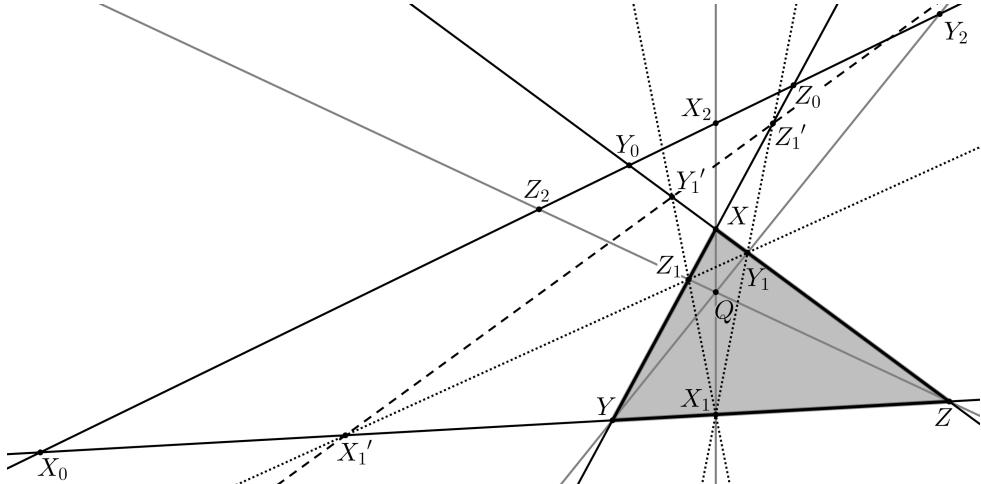


Fig. 8.11: Ceva's and Van Aubel's Theorems

By Ceva's Theorem, identity (8.3) holds. By projecting from Y , we have

$$(XX_1QX_2) = (Z_0X_0Y_2X_2),$$

and by projecting from Q we have

$$\begin{aligned} (XYZ_1Z_0) &= (X_2Y_2Z_2Z_0), \\ (YZX_1X_0) &= (Y_2Z_2X_2X_0), \\ (ZXY_1Y_0) &= (Z_2X_2Y_2Y_0). \end{aligned}$$

Ceva's identity (8.3) implies that

$$(Y_2Z_2X_2X_0) = \frac{-1}{(X_2Y_2Z_2Z_0)(Z_2X_2Y_2Y_0)}$$

By applying cross-ratio identities (2.2), we obtain

$$\begin{aligned} (Z_0X_0Y_2X_2) &= (X_2Y_2X_0Z_0) = (X_2Y_2Z_2Z_0)(X_2Y_2X_0Z_2) = \\ &= (X_2Y_2Z_2Z_0)[1 - (Z_2Y_2X_0X_2)] = \\ &= (X_2Y_2Z_2Z_0)[1 - (Y_2Z_2X_2X_0)] = \\ &= (X_2Y_2Z_2Z_0)[1 - \frac{-1}{(X_2Y_2Z_2Z_0)(Z_2X_2Y_2Y_0)}] = \\ &= (X_2Y_2Z_2Z_0) + \frac{1}{(Z_2X_2Y_2Y_0)} = (X_2Y_2Z_2Z_0) + (X_2Z_2Y_2Y_0) = \\ &= (XYZ_1Z_0) + (XZY_1Y_0). \end{aligned}$$

■

Now we are ready to state and prove our main result:

Theorem 8.17 (Projective law of cosines) *If the triangle \mathcal{T} is coherently oriented, then:*

$$\mathbf{c}(BC) = -\mathbf{s}(AB)\mathbf{s}(CA)\mathbf{c}(B'C') - \mathbf{c}(AB)\mathbf{c}(CA). \quad (8.5)$$

Proof.

Instead of (8.5), we will prove the equivalent dual formula

$$\mathbf{c}(B'C') = -\mathbf{s}(A'B')\mathbf{s}(C'A')\mathbf{c}(BC) - \mathbf{c}(A'B')\mathbf{c}(C'A'). \quad (8.6)$$

We can assume that \mathcal{T} is the triangle depicted in Figure 8.10. In that figure, \mathcal{T} is depicted as a hyperbolic triangle, and we have depicted also the polar triangle \mathcal{T}' , the magic triangle $\widetilde{\mathcal{T}}$, the preferred midpoints of \mathcal{T} and \mathcal{T}' , the complementary midpoints of \mathcal{T}' and the magic midpoints $\tilde{D}, \tilde{E}, \tilde{F}$ of \mathcal{T} . The figure misses the complementary midpoints of \mathcal{T} because they are imaginary.

We have that

$$\begin{aligned} \mathbf{c}(A'B') &= \mathbf{c}(B'A') = (B'A'C_aF') \stackrel{\rho_{c'}}{=} (C_bC_aA'F'_{c'}) = \\ &= -(C_bC_aA'F') = -(A'F'C_bC_a), \end{aligned}$$

and so

$$\mathbf{c}(A'B')\mathbf{c}(A'C') = \mathbf{c}(A'B')\mathbf{c}(C'A') = -(A'F'C_bC_a)(C'A'B_aE').$$

By applying Menelaus' Projective Formula (6.4) to the Menelaus configuration with triangle $\widehat{C'A'F'}$ and transversals B_aC_b and $E'C_a$, we get

$$(C'A'B_aE')(A'F'C_bC_a)(F'C'ZY) = 1,$$

where $Z = B_aC_b \cdot F'C'$ and $Y = E'C_a \cdot F'C'$ (Figure 8.12), and therefore

$$\mathbf{c}(A'B')\mathbf{c}(C'A') = -(C'F'ZY).$$

On the other hand, we have that

$$\mathbf{c}(BC) = (BCC_aD) = (B_aC_aCD_a) = -(B_aC_aCD),$$

and so

$$-\mathbf{s}(C'A')\mathbf{c}(BC) = (C'B_aA'H')(B_aC_aCD).$$

We apply Menelaus' Projective Formula to the Menelaus configuration with triangle $\widehat{C'B_aC_a}$ and transversals $A'C$ and $H'D$ in order to obtain

$$(C'B_aA'H')(B_aC_aCD)(C_aC'XW) = 1,$$

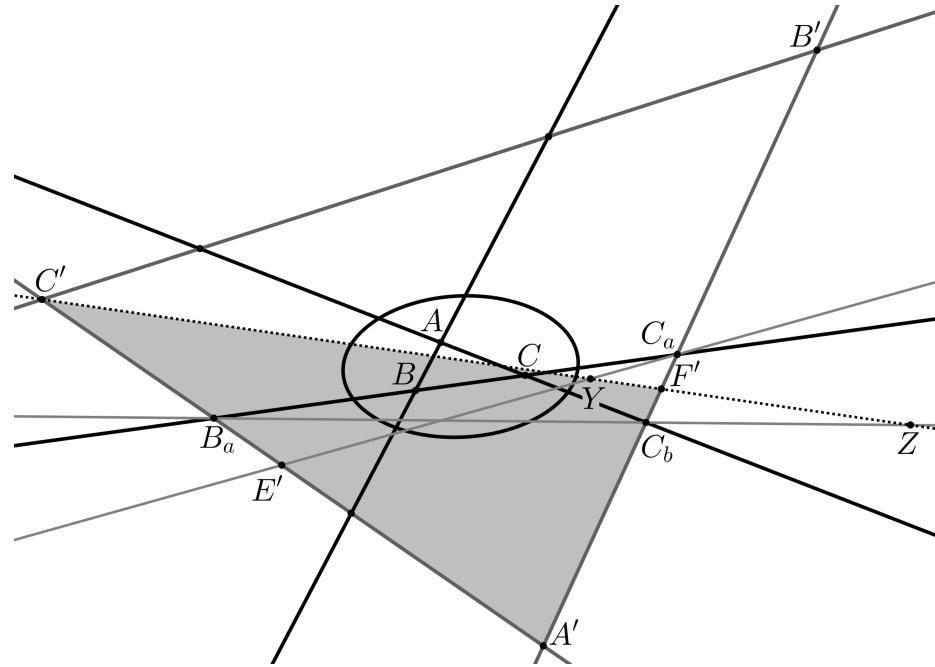


Fig. 8.12: Menelaus' Projective Formula 1, the points Y, Z

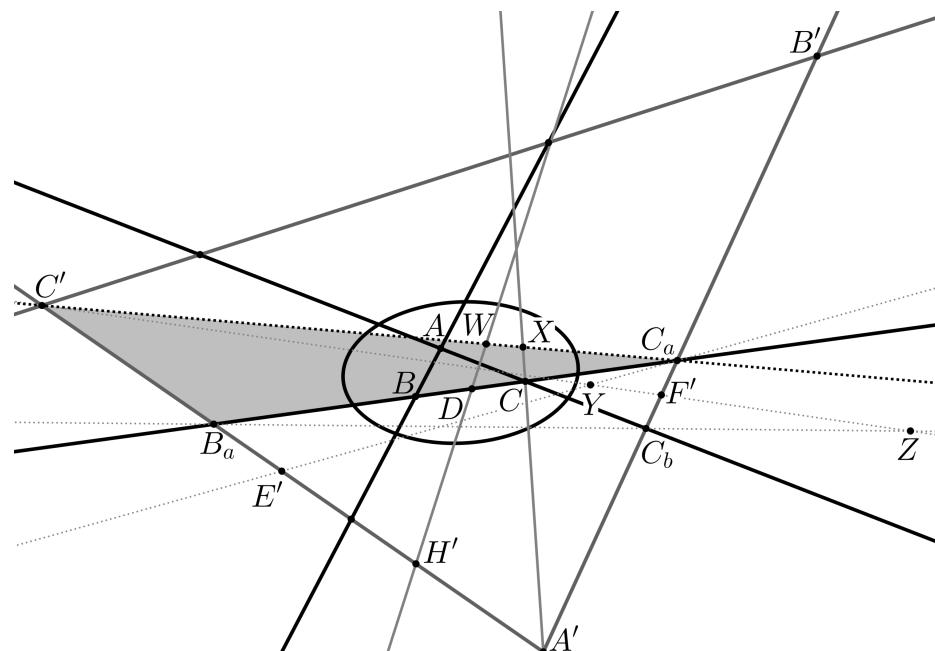
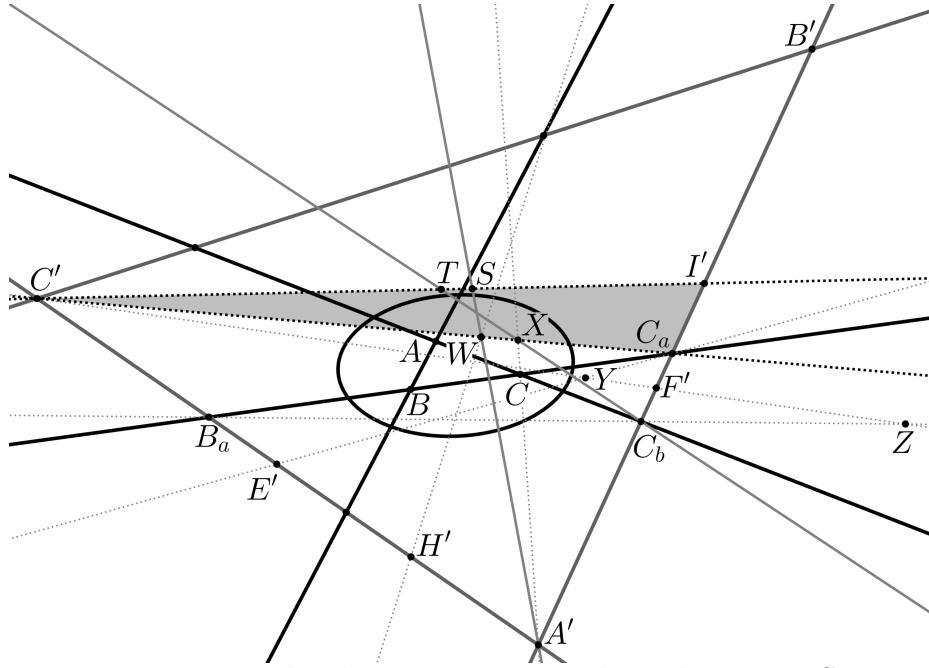


Fig. 8.13: Menelaus' Projective Formula 2, the points W, X


 Fig. 8.14: Menelaus' Projective Formula 3, the points S, T

where $X = A'C \cdot C_aC'$ and $W = H'D \cdot C_aC'$ (Figure 8.13). Therefore, it is

$$-\mathbf{s}(C'A')\mathbf{c}(BC) = (C'C_aXW).$$

Finally, we have that

$$\mathbf{s}(A'B') = (A'C_bB'I') \stackrel{\tau_{I'I'}}{=} (C_bA'C_aI') = (C_aI'C_bA'),$$

and then

$$-\mathbf{s}(A'B')\mathbf{s}(C'A')\mathbf{c}(BC) = (C_aI'C_bA')(C'C_aXW).$$

By applying Menelaus' Projective Formula to the Menelaus configuration with triangle $\widehat{C'C_aI'}$ and transversals XC_b and WA' we obtain

$$(C'C_aXW)(C_aI'C_bA')(I'C'TS) = 1,$$

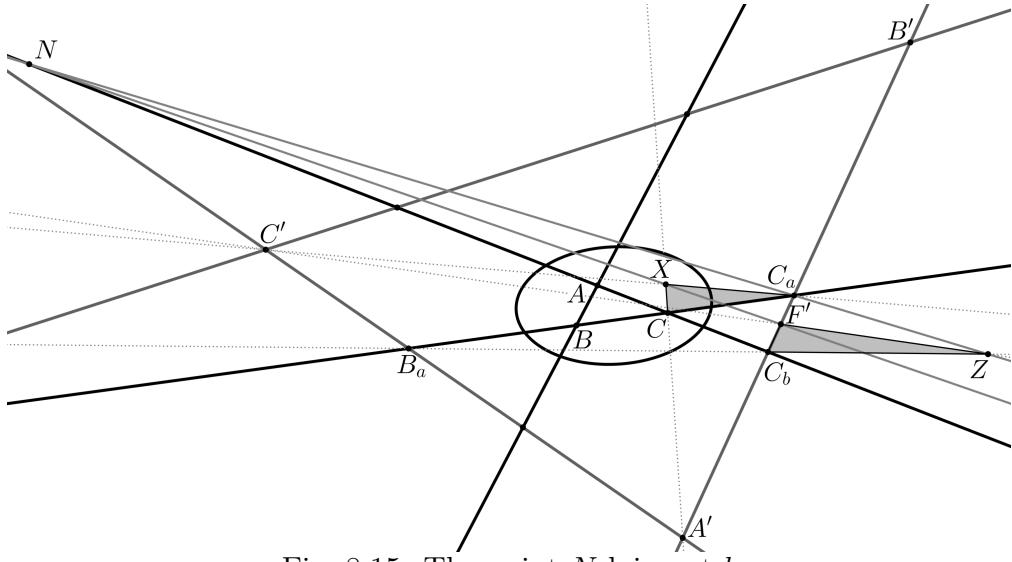
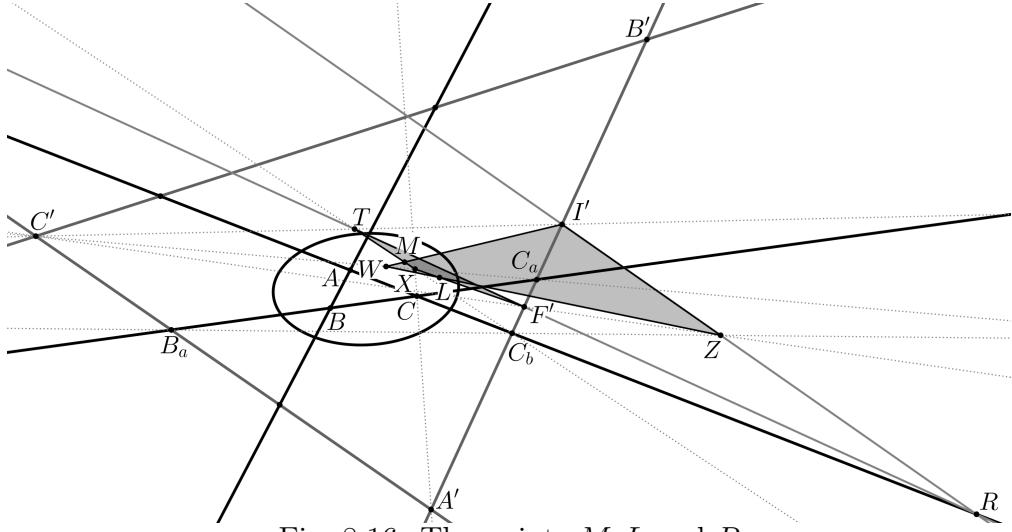
where $T = XC_b \cdot I'C'$ and $S = WA' \cdot I'C'$ (Figure 8.14). Therefore, it is

$$-\mathbf{s}(A'B')\mathbf{s}(C'A')\mathbf{c}(BC) = (C'I'TS).$$

The proof of Theorem 8.17 relies now in the proof of the identity

$$(C'B'A_bD') = (C'I'TS) + (C'F'ZY).$$

We will need to prove some small claims before doing so.


 Fig. 8.15: The point N lying at b

 Fig. 8.16: The points M , L and R

Claim 8.17.1 *The intersection point R of the lines ZI' and $F'T$ lies on b .*

Proof of Claim 8.17.1. Consider the triangles $\widehat{CC_aX}$ and $\widehat{C_bZF'}$. The intersection points of the sides of both triangles are

$$CC_a \cdot C_bZ = B_a, \quad C_aX \cdot ZF' = C', \quad XC \cdot F'C_b = A',$$

which are collinear. By Desargues' Theorem, the lines $CC_b = b$, C_aZ and XF' are concurrent: let N be their intersection point (Figure 8.15).

The triangles \widehat{WITZ} and $\widehat{XTF'}$ are perspective from C' (Figure 8.16), therefore the points $M = I'W \cdot TX$, $L = ZW \cdot XF'$ and R are collinear.

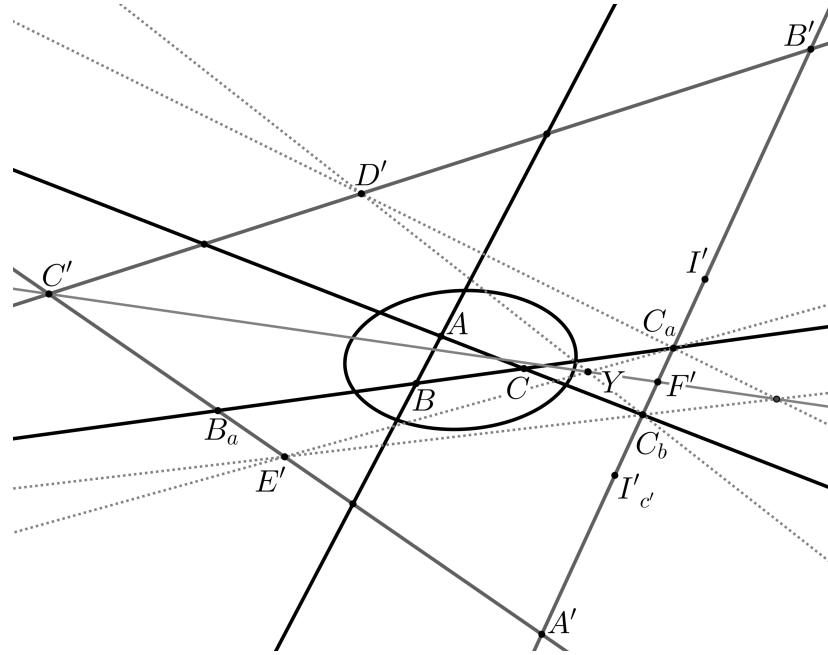


Fig. 8.17: Concurrency of lines in Claim 8.17.2

Finally, the triangles \widehat{MLX} and $\widehat{I'ZC_a}$ are perspective from W (Figure 8.16) and so the points

$$R = ML \cdot I'Z, \quad N = LX \cdot ZC_a, \quad C_b = XM \cdot C_aI'$$

are collinear. ■

Claim 8.17.2 *The lines $C'F'$, C_aD' and C_bE' are concurrent. The lines $C'F'$, C_bD' and C_aE' are also concurrent. In other words, the line $C'F'$ is a diagonal line of the quadrangle $\{C_a, C_b, E', D'\}$.*

Proof of Claim 8.17.2. Let Y^* be the intersection point of the lines C_aE' and C_bD' , and consider the quadrilateral $\mathcal{Q} = \{C', D', Y^*, E'\}$. The quadrangular involution $\sigma_{\mathcal{Q}}$ on c' sends the points A' and B' into the points C_b and C_a respectively and vice versa, and so it coincides with $\tau_{I'I'c'}$. Then,

$$C'Y^* \cdot c' = \tau_{I'I'c'}(D'E' \cdot c') = \tau_{I'I'c'}(F'_{c'}) = F'.$$

Therefore, Y^* must coincide with Y (Figure 8.17).

The rest of the Claim can be proved in a similar way. ■

It is interesting to remark that, because H', D are midpoints of the segments $\overline{C'B_a}, \overline{B_aC_a}$ respectively, the point W is a midpoint of $\overline{C'C_a}$. By considering the triangle $\overline{B'C'C_a}$, the line $D'W$ must intersect c' at a midpoint of $\overline{B'C_a}$. Thus, it must be $D'W \cdot c' = I'$ or $D'W \cdot c' = I'_{c'}$.

Claim 8.17.3 *The points D' , W , $I'_{c'}$ are collinear.*

Proof. Let d be the line joining the points D , D' and \tilde{D} , and take the points:

$$D_1^* = d \cdot c', \quad D_2^* = d \cdot b', \quad D_3^* = d \cdot C_a C'.$$

Let π_1 be the perspectivity from c' onto b' through the point D'_1 , let π_2 be the perspectivity from b' onto $C_a C'$ through the point D , and let π_3 be the perspectivity from $C_a C'$ onto c' through the point D' . Let consider also the projectivity π of c' onto itself given by the composition $\pi_3 \circ \pi_2 \circ \pi_1$.

Because the line d joins the points D , D' and \tilde{D} , it is

$$D_1^* \xrightarrow{\pi_1} D_2^* \xrightarrow{\pi_2} D_3^* \xrightarrow{\pi_1} D_1^*,$$

and so D_1^* is a fixed point of π .

On the other hand, looking at the midpoints of the triangle $\widehat{C'B_aC_a}$, it is

$$I' \xrightarrow{\pi_1} H' \xrightarrow{\pi_2} W',$$

and

$$I'_{c'} \xrightarrow{\pi_1} H'_{b'} \xrightarrow{\pi_2} W',$$

where W' is the conjugate point of W in $C_a C'$. Looking at the midpoints of the triangle $\widehat{B'C'C_a}$, it can be

$$\pi_3(W) = I'_{c'} \quad \text{and} \quad \pi_3(W') = I'$$

or

$$\pi_3(W) = I' \quad \text{and} \quad \pi_3(W') = I'_{c'}.$$

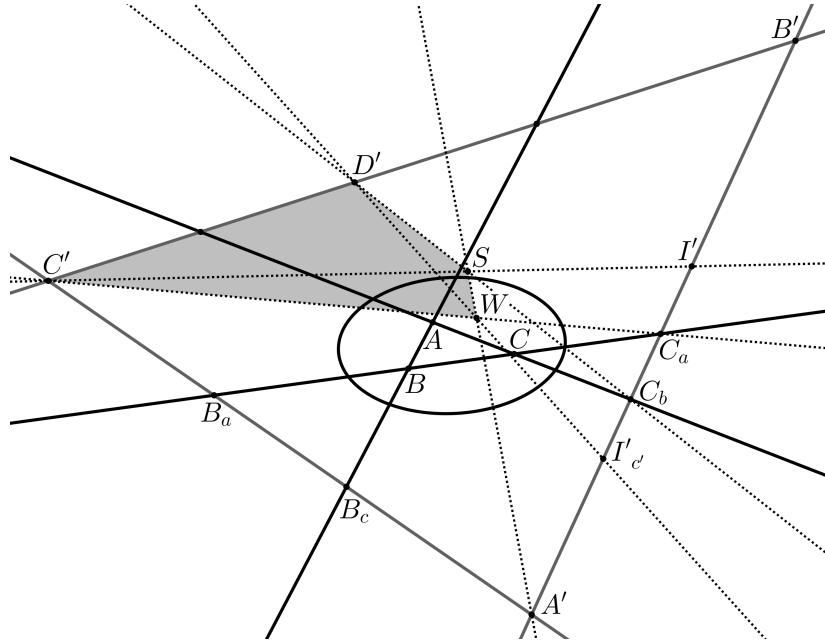
In the latter case, it turns out that I' , $I'_{c'}$ are also fixed points of π . If the point D_1^* coincides with I' or $I'_{c'}$, because $\tilde{D} \in d$ it would also be $H'_{b'} \in d$ or $H' \in d$, and this is impossible by Remark 8.12. Therefore, D_1^* is different from I' , $I'_{c'}$, and so the map π must be the identity map on c' .

If we consider the point $K = C' \tilde{D} \cdot c'$, we have

$$K \xrightarrow{\pi_1} C' \xrightarrow{\pi_2} C' \xrightarrow{\pi_3} B'.$$

If $K = B'$, it should be $\tilde{D} \in a'$ and also $D \in a'$, and this is not possible because \mathcal{T} is coherently oriented. Therefore, the points K and B' are different and π cannot be the identity map. Thus, it must be $\pi_3(W) = I'_{c'}$ and $\pi_3(W') = I'$. ■

Claim 8.17.4 *The point S lies in $C_b D'$.*


 Fig. 8.18: The point S lies in C_bD'

Proof of the Claim. By Claim 8.17.3, the line $D'W$ passes through $I'_{c'}$. If we consider the quadrangle $\mathcal{Q} = \{C', D', S, W\}$, the quadrilateral involution $\sigma_{\mathcal{Q}}$ on c' sends A', I' into $B', I'_{c'}$ respectively and vice versa. The involution $\sigma_{\mathcal{Q}}$ coincides with the symmetry $\tau_{F'F'_{c'}}$ of c' with respect to F' and it must send C_a into C_b . Therefore, it is $D'S \cdot c' = C_b$. ■

Consider the line $j_0 = C'R$ and the points $J_0, J'_0 \in j$ given by

$$J_0 = C_bD' \cdot j_0 \quad \text{and} \quad J'_0 = c' \cdot j_0.$$

By projecting since C_b the line a' onto j_0 , we have

$$(C'B'A_bD') = (C'J'_0RJ_0).$$

Let consider the triangle $\widehat{C'F'I'}$ and its cevians that intersect at R . Thus, J'_0 is the basepoint of the cevian through C' , T is the basepoint of the cevian through F' , and Z is the basepoint of the cevian through I' . Because Y, S are collinear with C_b and D' (Claims 8.17.4 and 8.17.2), the points Y, S, J_0 are collinear, and therefore, by Van Aubel's Theorem it is

$$(C'J'_0RJ_0) = (C'F'ZY) + (C'I'TS).$$

This completes the proof of Theorem 8.17 ■

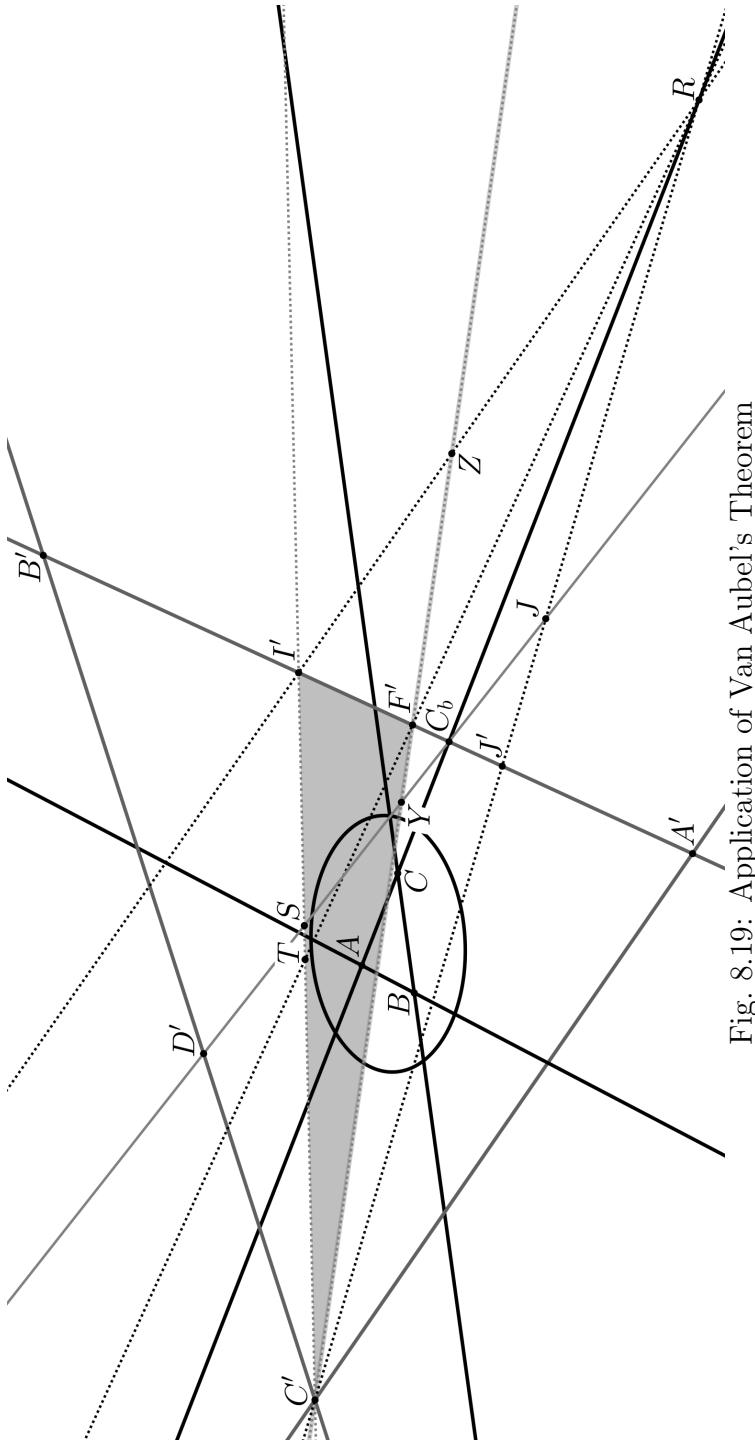


Fig. 8.19: Application of Van Aubel's Theorem

8.5 Some examples

Finally, we will see that the projective laws of sines and cosines *work* for obtaining elliptic and hyperbolic trigonometric formulae. We show how they can be used to deduce the corresponding geometric laws for any generalized triangle. The first thing that we must do for any of those figures is to draw the points among the midpoints, complementary midpoints and magic midpoints of \mathcal{T} and \mathcal{T}' that are real points, and to label them following the requirements of Definition 8.7. We will assume always that the projective triangles depicted are coherently oriented. Then, we have to translate the projective trigonometric ratios of the projective law of cosines (8.5) into circular or hyperbolic trigonometric ratios concerning the different magnitudes of the figure. Each figure will be characterized by six geometric magnitudes that we will call $a, b, c, \alpha, \beta, \gamma$. The magnitudes a, b, c denote always a segment length, while α, β or γ could denote an angular measure or a segment length. After that, we must deduce the sign to be put in the different terms of the geometric laws of cosines so obtained. When all the points on a cross-ratio are real points, its sign is deduced using Lemma 2.2. The main problem is to deduce the sign of a pure imaginary cross ratio (see the footnote on page 81), which appears when one of the midpoints or complementary midpoints is imaginary. We cannot deduce the sign of an imaginary without doing explicit computations. We didn't any explicit computation in the whole book (just some tricky manipulations with cross-ratios, indeed), and we will not start now. As we will see, using only the "visible" (real) points of each figure, we can deduce the relative sign of an imaginary cross ratio of (8.5) with respect to another one. This will be done using the Projective Law of sines or using Menelaus' Theorem together with Lemma 2.2, and it will be enough for our purposes.

8.5.1 Elliptic triangles

Let \mathcal{T} be the elliptic triangle depicted in Figure 8.20.

By applying Lemma 2.2 to the cross ratios involving the Projective Law of sines we have:

$$\begin{aligned}
 \mathbf{s}(AB) &= (AB_cBI) < 0 & \implies \mathbf{s}(AB) = -\sin c \\
 \mathbf{s}(BC) &= (BC_aCG) < 0 & \implies \mathbf{s}(BC) = -\sin a \\
 \mathbf{s}(CA) &= (CA_bAH) < 0 & \implies \mathbf{s}(CA) = -\sin b \\
 \mathbf{s}(A'B') &= (A'C_bB'I') < 0 & \implies \mathbf{s}(A'B') = -\sin \gamma \\
 \mathbf{s}(B'C') &= (B'A_cC'G') < 0 & \implies \mathbf{s}(B'C') = -\sin \alpha \\
 \mathbf{s}(C'A') &= (C'B_aA'H') < 0 & \implies \mathbf{s}(C'A') = -\sin \beta.
 \end{aligned}$$

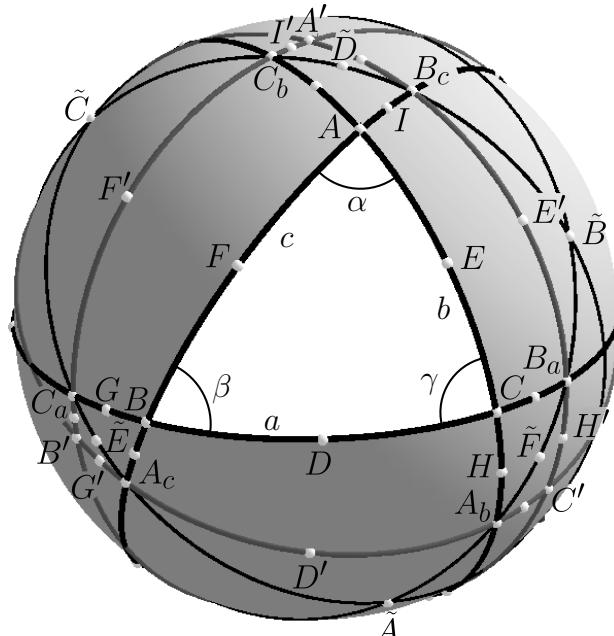


Fig. 8.20: Elliptic triangle

The law of sines for an elliptic triangle is:

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

We have also:

$$\begin{aligned} \mathbf{c}(AB) &= (ABB_cF) < 0 \implies \mathbf{c}(AB) = -\cos c \\ \mathbf{c}(BC) &= (BCC_aD) < 0 \implies \mathbf{c}(BC) = -\cos a \\ \mathbf{c}(CA) &= (CAA_bE) < 0 \implies \mathbf{c}(CA) = -\cos b \\ \mathbf{c}(A'B') &= (A'B'C_bF') > 0 \implies \mathbf{c}(A'B') = \cos \gamma \\ \mathbf{c}(B'C') &= (B'C'A_cD') > 0 \implies \mathbf{c}(B'C') = \cos \alpha \\ \mathbf{c}(C'A') &= (C'A'B_aE') > 0 \implies \mathbf{c}(C'A') = \cos \beta. \end{aligned}$$

Therefore, from the Projective Law of cosines (8.5), we get

$$\cos a = \sin c \sin b \cos \alpha + \cos c \cos b,$$

and from its dual (8.6):

$$\cos \alpha = \sin \gamma \sin \beta \cos a - \cos \gamma \cos \beta.$$

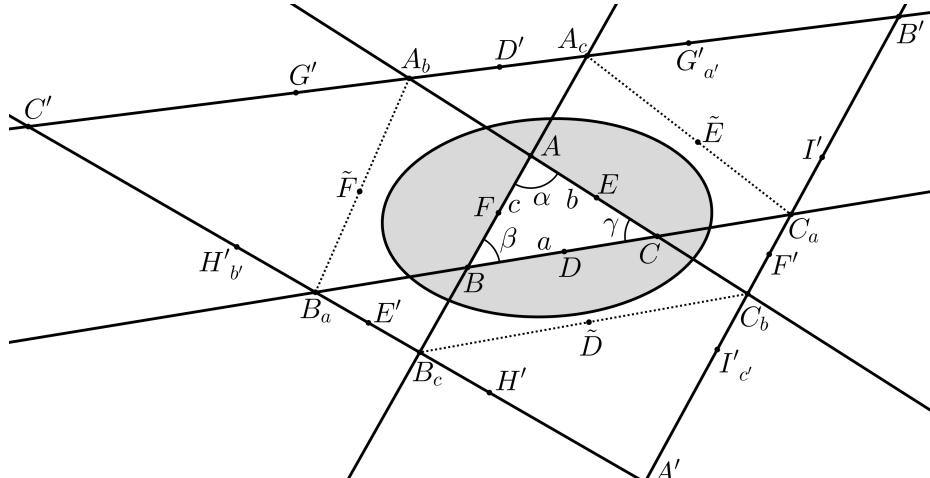


Fig. 8.21: Hyperbolic triangle

8.5.2 Hyperbolic triangles

Consider now the hyperbolic triangle given by \mathcal{T} in Figure 8.21. The sides and angles of \mathcal{T} have the same names as in the elliptic case. We have

$$\begin{aligned}
 \mathbf{s}(AB) &= (AB_c BI) = \text{imaginary} & \mathbf{s}(AB) &= \pm i \sinh c \\
 \mathbf{s}(BC) &= (BC_a CG) = \text{imaginary} & \mathbf{s}(BC) &= \pm i \sinh a \\
 \mathbf{s}(CA) &= (CA_b AH) = \text{imaginary} & \mathbf{s}(CA) &= \pm i \sinh b \\
 \mathbf{s}(A'B') &= (A'C_b B'I') > 0 & \mathbf{s}(A'B') &= \sin \gamma \\
 \mathbf{s}(B'C') &= (B'A_c C'G') > 0 & \mathbf{s}(B'C') &= \sin \alpha \\
 \mathbf{s}(C'A') &= (C'B_a A'H') > 0 & \mathbf{s}(C'A') &= \sin \beta .
 \end{aligned}$$

From the Projective Law of sines (8.2) we deduce

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma} .$$

In particular, $\mathbf{s}(AB)$, $\mathbf{s}(BC)$ and $\mathbf{s}(CA)$ have all the same imaginary sign: all of them are positive imaginary or all of them are negative imaginary; and the product of any two of these ratios is a negative real number.

We have, also

$$\begin{aligned}
 \mathbf{c}(AB) &= (ABB_c F) < 0 & \mathbf{c}(AB) &= -\cosh c \\
 \mathbf{c}(BC) &= (BCC_a D) < 0 & \mathbf{c}(BC) &= -\cosh a \\
 \mathbf{c}(CA) &= (CAA_b E) < 0 & \mathbf{c}(CA) &= -\cosh b \\
 \mathbf{c}(A'B') &= (A'B'C_b F') > 0 & \mathbf{c}(A'B') &= \cos \gamma \\
 \mathbf{c}(B'C') &= (B'C'A_c D') > 0 & \mathbf{c}(B'C') &= \cos \alpha \\
 \mathbf{c}(C'A') &= (C'A'B_a E') > 0 & \mathbf{c}(C'A') &= \cos \beta .
 \end{aligned}$$

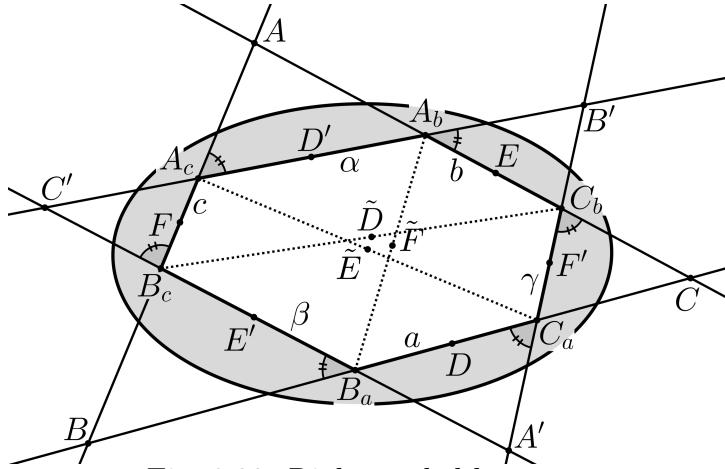


Fig. 8.22: Right-angled hexagon

Thus, the Projective Law of cosines (8.5) gives

$$\cosh a = -\sinh c \sinh b \cos \alpha + \cosh c \cosh b,$$

while its dual (8.6)

$$\cos \alpha = \sin \gamma \sin \beta \cosh a - \cos \gamma \cos \beta.$$

8.5.3 Right-angled hexagons

Consider now a right-angled hexagon with sides $a, \gamma, b, \alpha, c, \beta$ generated by \mathcal{T} and \mathcal{T}' in the hyperbolic plane as in Figure 8.22. In this case, all the complementary midpoints of \mathcal{T} and \mathcal{T}' are imaginary.

As the hyperbolic sines are always positive, the law of sines for this figure must be

$$\frac{\sinh a}{\sinh \alpha} = \frac{\sinh b}{\sinh \beta} = \frac{\sinh c}{\sinh \gamma}.$$

Let study now the Projective Law of cosines (8.5). By Lemma 2.2, we know that the four real cross ratios of this formula: $\mathbf{c}(BC)$, $\mathbf{c}(B'C')$, $\mathbf{c}(AB)$ and $\mathbf{c}(CA)$ are positive. This implies that the product $\mathbf{s}(AB)\mathbf{s}(CA)$ must be negative. Otherwise, the two members of (8.5) would have different sign, which is impossible. Therefore, the law of cosines for a right-angled hexagon is

$$\cosh a = \sinh c \sinh b \cosh \alpha - \cosh c \cosh b.$$

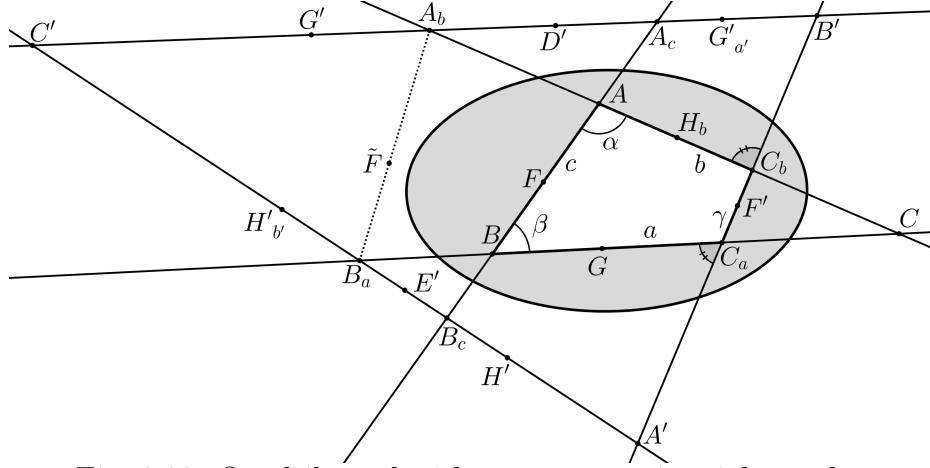


Fig. 8.23: Quadrilateral with two consecutive right angles

8.5.4 Quadrilateral with consecutive right angles

We study now the quadrilateral with consecutive right angles depicted in Figure 8.23. We know that

$$\begin{aligned}
 \mathbf{s}(AB) &= (AB_c BI) = \text{imaginary} & \mathbf{s}(AB) &= \pm i \sinh c \\
 \mathbf{s}(BC) &= (BC_a CG) < 0 & \mathbf{s}(BC) &= -\cosh a \\
 \mathbf{s}(CA) &= (CA_b AH) < 0 & \mathbf{s}(CA) &= -\cosh b \\
 \mathbf{s}(A'B') &= (A'C_b B'I') = \text{imaginary} & \mathbf{s}(A'B') &= \pm i \sinh \gamma \\
 \mathbf{s}(B'C') &= (B'A_c C'G') > 0 & \mathbf{s}(B'C') &= \sin \alpha \\
 \mathbf{s}(C'A') &= (C'B_a A'H') > 0 & \mathbf{s}(C'A') &= \sin \beta.
 \end{aligned}$$

The Projective Law of sines (8.2) tells us that $\mathbf{s}(AB)$ and $\mathbf{s}(A'B')$ have different imaginary sign, as their quotient is a negative real number. The law of sines for this figure is

$$\frac{\cosh a}{\sin \alpha} = \frac{\cosh b}{\sinh \beta} = \frac{\sinh c}{\sinh \gamma}.$$

For the projective cosines, we have:

$$\begin{aligned}
 \mathbf{c}(AB) &= (ABB_c F) < 0 & \mathbf{c}(AB) &= -\cosh c \\
 \mathbf{c}(BC) &= (BCC_a D) = \text{imaginary} & \mathbf{c}(BC) &= \pm \sinh a \\
 \mathbf{c}(CA) &= (CAA_b E) = \text{imaginary} & \mathbf{c}(CA) &= \pm \sinh b \\
 \mathbf{c}(A'B') &= (A'B'C_b F') > 0 & \mathbf{c}(A'B') &= \cosh \gamma \\
 \mathbf{c}(B'C') &= (B'C'A_c D') > 0 & \mathbf{c}(B'C') &= \cos \alpha \\
 \mathbf{c}(C'A') &= (C'A'B_a E') > 0 & \mathbf{c}(C'A') &= \cos \beta.
 \end{aligned}$$

Let us start now with the dual Projective Law of cosines (8.6). The four real cross ratios of this formula: $\mathbf{c}(B'C')$, $\mathbf{s}(C'A')$, $\mathbf{c}(A'B')$ and $\mathbf{c}(C'A')$

are positive. This implies that the product $\mathbf{s}(A'B')\mathbf{c}(BC)$ is a negative real number, and then $\mathbf{s}(A'B')$ and $\mathbf{c}(BC)$ have the same imaginary sign. The formula that we obtain from (8.6) is

$$\cos \alpha = \sinh \gamma \sin \beta \sinh a - \cosh \gamma \cos \beta.$$

We return now to the Projective Law of cosines (8.5). In this formula we have three unknown signs: those of $\mathbf{c}(BC)$, $\mathbf{s}(AB)$ and $\mathbf{c}(CA)$.

We have seen by the Projective law of sines that $\mathbf{s}(AB)$ and $\mathbf{s}(A'B')$ have different imaginary sign, and now we know also that $\mathbf{s}(A'B')$ and $\mathbf{c}(BC)$ have the same imaginary sign. Thus, $\mathbf{c}(BC)$ and $\mathbf{s}(AB)$ have different imaginary sign.

On the other hand, if we apply Menelaus Projective formula to the triangle \widehat{ABC} with transversals c' and DE , we get

$$(ABC_0F_c)(BCC_aD)(CAC_bE) = 1.$$

As $(ABC_0F_c) > 0$ by Lemma 2.2, we conclude that

$$\mathbf{c}(BC) = (BCC_aD) \quad \text{and} \quad \mathbf{c}(CA) = (CAA_bE) = (ACC_bE)$$

have the same imaginary sign. Therefore (8.5) gives

$$\sinh a = -\sinh c \cosh b \cos \alpha + \cosh c \sinh b.$$

There are other two laws of cosines for this figure which are not equivalent to those given above. These are the laws of cosines given by the segments \overline{AB} and $\overline{A'B'}$. The first of them is

$$\mathbf{c}(AB) = -\mathbf{s}(BC)\mathbf{s}(CA)\mathbf{c}(A'B') - \mathbf{c}(BC)\mathbf{c}(CA).$$

As $\mathbf{c}(BC)$ and $\mathbf{c}(CA)$ have the same imaginary sign, their product is a negative real number. So,

$$\cosh c = \cosh a \cosh b \cosh \gamma - \sinh a \sinh b.$$

The last law of cosines we are interested in is the dual of the previous one:

$$\mathbf{c}(A'B') = -\mathbf{s}(B'C')\mathbf{s}(C'A')\mathbf{c}(AB) - \mathbf{c}(B'C')\mathbf{c}(C'A').$$

All the cross-ratios involved here are real, and they give

$$\cosh \gamma = \sin \alpha \sin \beta \cosh c - \cos \alpha \cos \beta.$$

Appendix A

Laguerre's formula for rays

As it is mentioned in §3, Laguerre's formula (3.3) for computing angles in Cayley-Klein models is valid for angles between lines, not for angles between rays, and it cannot distinguish an angle from its supplement. We propose a modification of Laguerre's formula that allows us to compute angles between rays. Although our formula fits perfectly in the projective model for the hyperbolic plane, it can be adapted also to the elliptic and euclidean models.

A projective conic as a 1-dimensional object is equivalent to a projective line or a pencil of lines through a point. The key for this similarity is given by Steiner's theorem.

Theorem A.1 (Steiner's Theorem) *Let X, Y be two points on the non-degenerate conic Θ . The map from the pencil of lines through X to the pencil of lines through Y that maps the line XP into the line YP for all $P \in \Theta$ is a projectivity.*

As a corollary of Steiner's Theorem, we obtain the following result (which is also known as “Chasles' Theorem”):

Corollary A.2 *Let A, B, C, D be four points on the conic Θ , and let X, Y be another two points on the same conic. If a, b, c, d are the four lines joining the point X with A, B, C, D respectively, and a', b', c', d' are the four lines joining Y with A, B, C, D respectively, then*

$$(abcd) = (a'b'c'd').$$

This corollary allows us to define the cross ratio of four points on a conic as the cross ratio of the lines joining them with any other point on the conic.

Definition A.3 *Given four points A, B, C, D on the nondegenerate conic Θ , we define their cross ratio over Θ , and we denote it by $(ABCD)_\Theta$, by the equality*

$$(ABCD)_\Theta := (abcd),$$

where a, b, c, d are the lines joining A, B, C, D , respectively, with any other point X of Θ .

Assume now that \mathbb{P} is the hyperbolic plane and take a point $P \in \mathbb{P}$. Consider two lines a, b through P , and take the intersection points A_1, A_2 and B_1, B_2 of a and b with Φ respectively. The hyperbolic line, that we also denote by a , determined by the projective line a is divided by P into two rays a_1, a_2 starting from P and ending at A_1, A_2 respectively. In the same way, the line b is divided by P into the two rays b_1, b_2 starting at P and ending at B_1, B_2 respectively. Let u, v be the two tangent lines to Φ through P , and let U, V be their contact points with Φ , respectively. The line $p = UV$ is the polar line of P . If A, B are the intersection points with p of the lines a, b respectively, the two diagonal points Q, Q_p of the quadrangle $\mathcal{Q} = \{A_1, A_2, B_1, B_2\}$ different from P are the midpoints of the segment \overline{AB} . If, for example, it is $Q = p \cdot A_1 B_2$, by projecting from B_2 we have

$$(UVQB) = (UVA_1B_1)_\Phi,$$

and so by (3.5) it is

$$(uvab) = (UVA_1B_1) = (UVQB)^2 = (UVA_1B_1)_\Phi^2.$$

This suggest the following definition:

Definition A.4 *The angle between the rays a_1, b_1 is given by the formula*

$$\widehat{a_1b_1} = \frac{1}{i} \log(UVA_1B_1)_\Phi. \quad (\text{A.1})$$

With this definition, angles between rays take values between 0 and 2π as expected. If we denote by α the angle $\widehat{a_1b_1}$, by applying (3.9), it can be seen that

$$\cos \alpha = 2(A_1B_1B_2A_2)_\Phi - 1.$$

Moreover, if we draw the conjugate lines a_P, b_P of a, b at P and we label their intersection points with Φ as in Figure A.1 ⁽¹⁾, it can be proved that

$$\cos \alpha = (A_1B_1B'_1A'_1)_\Phi. \quad (\text{A.2})$$

Formula (A.2) shows that (A.1) works, because it gives a positive cosine for acute angles and a negative cosine for obtuse angles (cf. Lemma 2.2 for conics). If the angle α between the rays a_1, b_1 is acute as in Figure A.1, the points A_1, B_1 do not separate the points A'_1, B'_1 . In the same figure, the

¹We label the points A'_1, A'_2, B'_1, B'_2 in such a way that the lines $A_1B_2, B_1A_2, A'_1B'_1, A'_2B'_2$ are concurrent.

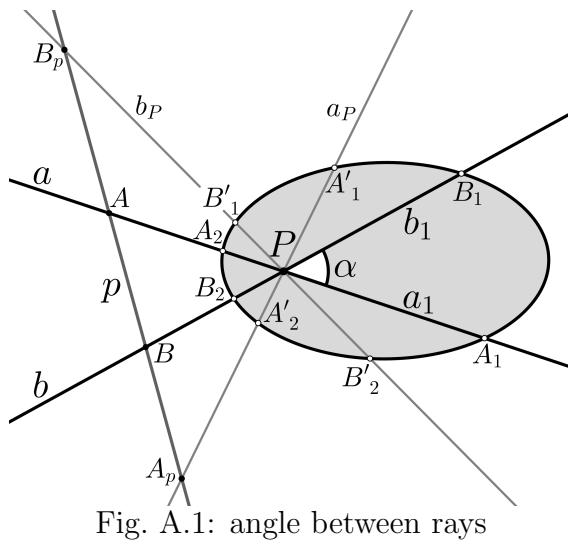


Fig. A.1: angle between rays

angle between the rays a_1 and b_2 is obtuse because A'_1, A'_2 separate A_1 from B_2 , and in this case (A.2) will give a negative cosine for $a_1 b_2$.

Formula (A.1) is valid for euclidean, hyperbolic and elliptic geometries if we replace the absolute conic Φ with any circle centered at P . In euclidean geometry, U, V are Poncelet's circular points at infinity.

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