First-Order Linear Differential Equations:

A First order linear differential equation is an equation of the form

$$y' + P(x)y = Q(x).$$

Where P and Q are functions of x. If the equation is written in this form it is called standard form. The equation is called *first order* because it only involves the function y and first derivatives of y. We can solve this equation in general but it is better to understand *how* to solve it than it is to just memorize the solution.

Integrating factors

We are trying to solve $y' + P(x)y = Q(x)$. We make the following observation (using the product rule):

$$\frac{d}{dx}[y \cdot e^{\int P(x)dx}] = y' \cdot e^{\int P(x)dx} + y \cdot \frac{d}{dx}[e^{\int P(x)dx}]$$

Now $\frac{d}{dx}[e^{\int P(x)dx}] = e^{\int P(x)dx}P(x)$ So we have

$$\frac{d}{dx}[y \cdot e^{\int P(x)dx}] = e^{\int P(x)dx}[y' + P(x)y]$$

We define $u(x) = e^{\int P(x)dx}$ to be the integrating factor which has the property that

$$\frac{d}{dx}[u(x)y] = u(x)[y' + P(x)y].$$

Solving $y' + P(x)y = Q(x)$:

Now we start with our equation and multiply both sides by the integrating factor:

$$\left(\frac{d}{dx}[u(x)y] = \right) u(x)[y' + P(x)y] = u(x)Q(x).$$

Now we can integrate both sides and notice that the left-hand side is already the derivative of something!

$$\int \frac{d}{dx} u(x) y dx = \int u(x)Q(x)dx.$$

$$u(x)y = \int u(x)Q(x)dx \Rightarrow y = \frac{1}{u(x)} \int u(x)Q(x)dx.$$

Solving $y' + P(x)y = Q(x)$:

The general solution to the first order linear differential equation is given by

$$y(x) = \frac{1}{u(x)} \int u(x)Q(x)dx,$$

with

$$u(x) = e^{\int P(x)dx}.$$

Now let's do an example.

Example: $y' + 4xy = x$

First we note that this is already in standard form with $P(x) = 4x$, and $Q(x) = x$.
The first step is to find the integrating factor

$$u(x) = e^{\int P(x)dx} = e^{\int 4xdx} = e^{2x^2}.$$

Note that we do not need the general form of the integral until the last step.
The next step is to find $y = \frac{1}{u(x)} \int u(x)Q(x)dx$.
Recall that $\frac{1}{e^{2x^2}} = e^{-2x^2}$.

Example: $y' + 4xy = x$ continued

We have $y = \frac{1}{u(x)} \int u(x)Q(x)dx, \text{ with } u(x) = e^{2x^2}$, so we have to solve

$$y = e^{-2x^2} \int xe^{2x^2} dx.$$

This integral can be done with the method of substitution, let $z = 2x^2$, then $dz = 4xdx$

$$\int xe^{2x^2} dx = \int \frac{1}{4} e^z dz = \frac{1}{4} e^z + C = \frac{1}{4} e^{2x^2} + C.$$

Almost done, now let

$$y = e^{-2x^2} [\frac{1}{4} e^{2x^2} + C] = \frac{1}{4} + Ce^{-2x^2}.$$

Note how important the $+C$ is in the integral, without it we lose an entire part of the solution!

Example: $y' + 4xy = x$ continued

We can have the utmost confidence in our solution if we check our work. If we did everything right then we will be able to solve the differential equation with our answer, i.e.

$$y = \frac{1}{4} + Ce^{-2x^2} \text{ should satisfy } y' + 4xy = x.$$

Now

$$y' = Ce^{-2x^2}(-4x) = -4Cxe^{-2x^2}.$$

So

$$y' + 4xy = -4Cxe^{-2x^2} + 4xCe^{-2x^2} + 4x\frac{1}{4} = x$$

and it works!

Is there another way to do that problem?

Yes, we can also solve that problem using separation of variables.

$$\frac{dy}{dx} + 4xy = x \Rightarrow \frac{dy}{dx} = x(1-4y) \Rightarrow \frac{1}{(1-4y)}dy = xdx.$$

Try solving it this way and see if you get the same answer (you should!)

Another Example:

Find the solution to

$$xy' + y = x^2 + 1$$

First we write

$$y' + \frac{1}{x}y = x + \frac{1}{x} \Rightarrow P(x) = \frac{1}{x} \text{ and } Q(x) = x + \frac{1}{x}.$$

Therefore

$$u(x) = e^{\int P(x)dx} = x,$$

and

$$y = \frac{1}{u(x)} \int Q(x)u(x)dx = \frac{1}{x} \int x^2 + 1dx = \frac{1}{3}x^2 + 1 + Cx^{-1}.$$

Solve $xy' + y = x^2 + 1$

Let's check our answer $y = \frac{1}{3}x^2 + 1 + Cx^{-1}$:

$$y' = \frac{2}{3}x - Cx^{-2}$$

$$\Rightarrow xy' + y = \frac{2}{3}x^2 - Cx^{-1} + \frac{1}{3}x^2 + 1 + Cx^{-1}$$

$$\Rightarrow xy' + x = x^2 + 1!$$