Lyapunov functions and stability problems

Gunnar Söderbacka, Workshop Ghana, 29.5-10.5, 2013

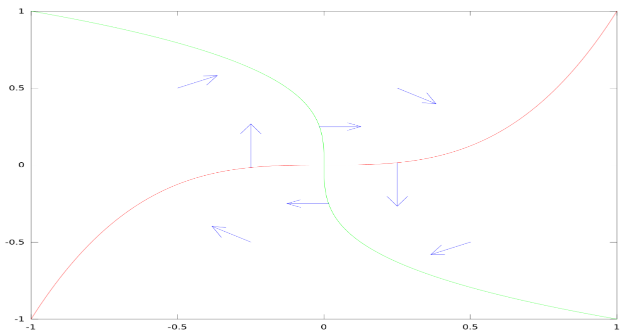
1 Introduction

In these notes we explain the power of Lyapunov functions in determining stability of equilibria and estimating basins of attraction. We concentrate on two dimensional functions. Lyapunov functions are also basis for many other methods in analysis of dynamical system, like frequency criteria and the method of comparing with other systems. The theory of Lyapunov function is nice and easy to learn, but finding a good Lyapunov function can often be a big scientific problem. Detecting new effective families of Lyapunov functions can be seen as a serious advance.

Example of stability problem

We consider the system $x' = y - x^3, y' = -x - y^3$. The only equlilibrium of this system can be calculated to be (0,0). The linearization of this system around origin is a center why the linearization cannot be used to determine the type of the equilibrium. The zero-isoclines are given by $y = x^3$ when $x' = 0$ and $x = -y^3$ for $y' = 0$. The sign analyis shown in figure 1 shows only that solutions are rotating around the equilibrium but does not tell anything about whether equilibrium is stable or not.

Figure 1: Sign analysis of system $x' = y - x^3, y' = -x - y^3$, included zero-isoclines ($x' = 0$ by red and $y' = 0$ by green). Arrows indicate the direction of the solution. These directions are determined by the sign of x' and y'.

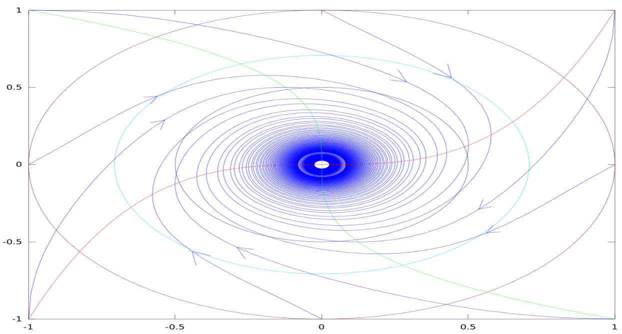


Let us now look at level curves of the function given by $V = x^2 + y^2$. They are circles around the equilibrium. Calculating the derivative of V with respect to the system we get

$$V' = V'_x x' + V'_y y' = 2xx' + 2yy' = -x^4 - y^4$$

It is clear that V' is negative everywhere except at the equilibrium so the value of V is decreasing along the solutions meaning that solutions on any level curve continue into the region bounded by the level curve and thus they tend to the equilibrium which thereby is not only locally stable but also globally stable with whole plane R^2 as basin of attraction. The phaseportrait, zero-isoclines and some level curves of V are found in figure 2. The attraction near to equilibrium is very slow why the trajectory seems to fill a whole disc. You can see that solutions pass through level curves all going inside.

Figure 2: The phase portrait of system $x' = y - x^3, y' = -x - y^3$, included zero-isoclines ($x' = 0$ by red and $y' = 0$ by green) and level curves of the Lyapunov function $V = x^2 + y^2$.



2 Definitions and main theorems

Stability definitions

Even if you have some idea now about what is stability it is good to have the exact definitions we introduce here.

Definition 1. An equilibrium is *stable* if for any neighbourhood N of the equilibrium there is a neighbourhood N' contained in N such that all solutions starting in N' remain in N.

Definition 2. An equilibrium is *asymptotically stable* if it is stable and there is a neighbourhood of the equilibrium such that any solution starting in it tends to the equilibrium for $t \to \infty$.

Definition 3. The *basin of attraction* of an equilibrium consists of all points such that a solution starting at them tends to the equilibrium.

To understand these concepts it is good to try to see them on some concrete examples, so we give an exercise. Answers are given at the end of these notes.

Exercise 1. Is the origin stable or asymptotically stable for the following systems:

1. $x' = x$, $y' = y$
2. $x' = -x$, $y' = 2y$
3. $x' = -2x$, $y' = -y$
4. $x' = y$, $y' = -x$
5. $x' = -x - x^3$, $y' = -y$
6. $x' = -x$, $y' = -y - y^2$
7. $x' = -2x - 16x^4, y' = -3y + 12y^3$

If it is asymptotically stable find its basin of attraction.

Positive and negative definite?

In order to find out about stability we use Lyapunov functions. The most difficult problem in finding proper Lyapunov functions is to prove that the function and its derivative is positive or negative in some region. We give the following definitions in order to understand the text of the theorems to be used for finding Lyapunov functions and determining stability and estimates of basin of attraction.

Definition 4. A function $V : R^n \to R$ is *positive (negative) definite* in a neighbourhood N of origo if $V(0, .., 0) = 0$ and $V(x) > (<)0$ for $x \neq (0, ..., 0)$ in N.

Definition 5. A function $V : R^n \to R$ is *positive (negative) semi-definite* in a neighbourhood N of origo if $V(0,..,0) = 0$ and $V(x) \geq (\leq) 0$ for $x \neq (0,...,0)$ in N.

To understand how to work in practice it is good to have some exercise. The answers are given at the end of these notes.

Exercise 2 a) Which of the following functions have some of these properties in R^2 (Def 4 and 5).

1. $x_1^2 + 6x_1x_2 + x_2^2$
2. $5x_1^2 - 7x_1x_2 + 3x_2^2$
3. $7x_1^2 + x_1x_2$
4. $-x_1^2 + 2x_1x_2 - x_2^2$
5. $x_1^2 + x_2^2 - x_1^3x_2$
6. $x_1 + 5x_2$
7. $x_1 + x_1^2 + x_2^2$
8. $x_1^3 + x_1^4 + x_1^2x_2^2 + x_2^4$
9. $x_1^2 - x_2^2 + x_1^4$
10. $x_1^6 + x_1^3x_2^3 + x_2^6$
11. $-x_1^2 - x_2^2 + x_2^4$
12. $4x_1^3x_2 + 4x_1x_2^3 - x_1^4 - 2x_2^4 - 8x_1^2x_2^2$
13. $x_1x_2 - x_1^2 - x_2^2$
14. $x_1^2 - 2x_1x_2 + x_2^4$

b) Do they have some of the properties in a neighbourhood of the origin?
c) Construct some three dimensional functions with each of the properties.

We remember you that a quadratic form $Ax^2 + Bxy + Cy^2$ is positive (negative) defnite if $4AC - B^2 > 0$ and $A > 0$ $(C > 0)$.

We now give the important theorems to be used for concluding stability and estimating basin of attraction.

Lyapunov stability theorems

Definition. The derivative V' along the solutions $x(t)$ is

$$V'(x) = \frac{\partial V(x)}{\partial x_1} x_1' + ... \frac{\partial V(x)}{\partial x_n} x_n'$$

We formulate the theorem for the case when the equilibirum is at origin, if this is not the case the equilibrium has to be transformed to origin (or we can use an analogous definition of positive og negative definite).

Theorem 1. Suppose $x' = X(x)$ has equilibrium at origin and there exist a bounded neighbourhood N of the origin and a function V defined in $\bar{N}$ such that
1) the first partial derivatives are continuous
2) V is positive definite
3) V' is negative semi-definite Then the origin is stable
If moreover
3) V' is negative definite then the origin is asymptotically stable and if $\bar{N}$ is given by the inequality $V(x) \leq C$ for some C then $\bar{N}$ is in the basin of attraction of the origin.

Proof. We first prove the stability statement. For any neighbourhood N' of the origin there exists a k such that the set V_k defined by $V(x) \leq k$ is a subset of N'.

If $V' \leq 0$ then V is decreasing along solutions and thus these solutions remain in V_k and we conclude stability.

To prove the asymptotical stability statement we choose a k such that the set V_k defined by $V(x) \leq k$ is a subset of $\bar{N}$. Then $V' < 0$ in V_k. Suppose a solution $x(t)$ starting in V_k does not tend to the origin. Then $V(x(t)) > k'$ and $V'(x(t)) < -m$, for some $0 < k' < k$ and $m > 0$ for all t. This means $V(x(t)) < V(x(0)) - mt$ for all t which is a contradiction to $V \geq 0$. Thus solutions in V_k must tend to the origin and we have asymptotical stability. If $\bar{N}$ is given by $V(x) \leq C$ then above we can use $k = C$ and analogously prove that solutions in $\bar{N}$ tend to the origin and thus the basin of attraction contains the region defined by $V(x) \leq C$.

Theorem 2. Suppose $x' = X(x)$ has equilibrium in the origin and there exists a function V in a bounded neighbourhood N of the origin such that in $\bar{N}$

1) the first partial derivatives are continuous
2) V is positive somewhere in any neighbourhood of the origin
3) V' is positive definite
4) $V(0) = 0$

then origin is unstable.

Proof. Unstability means that there exists a neighbourhood M of the origin such that in any neighbourhood of the origin there exists a solution starting there and leaving M. We show that we can choose $M = \bar{N}$. There exists a k such that $V(x) < k$ in N. There exists a k such that for any x inside N $V(x) \leq k$. Choose an arbitrary neighbourhood N' of the origin. In this neighbourhood there is a point x_0 where $V(x_0) > 0$. Suppose now that $x(t)$ with $x(0) = x_0$ remains in $V(x) < k$ for all $t > 0$. Then $V(x(t)) > V(x_0)$ for $t > 0$ because $V'(x) > 0$ and thus there is an $m > 0$ such that $V'(x(t)) > m$. But then $V(x(t)) = V(x(0)) + mt \to \infty$ for $t \to \infty$ contradicting $V(x) < k$. Thus in any neighbourhood of the origin there are solutions escaping $V(x) < k$ and N.

3 Examples

We provide you by some two dimensional examples of how to use Lyapunov functions to estimate basins of attraction.

From the theorems we have to find three things

1. Find where V is positive
2. Find where V' is negative
3. Find the largest level curve $V = c$ of V so that $V < c$ is inside the region where V' is negative.

Notice that different Lyapunov functions give different estimates for basin of attraction for the same system and the union of all estimates with different Lyapunov functions can also be used as a better estimate.

We can often make use of the following fact.

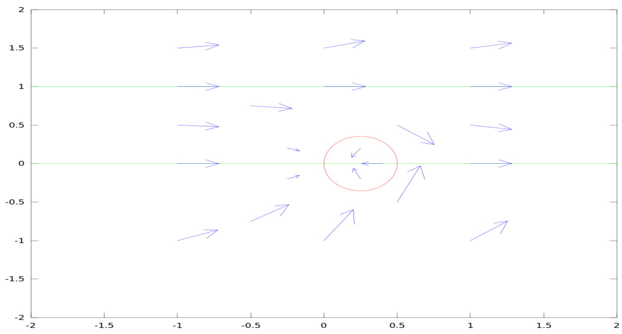
If at a point $4AC - B^2 > 0$, $A < 0$ then the value of the function given by $Ax^2 + Bxy + Cy^2$ is negative. Here we consider A, B and C as functions of x and y.

Example. Let us consider the system $x' = -x + 2x^2 + y^2$, $y' = -y + y^2$. Let us examine the positive definite function $V(x, y) = x^2 + y^2$. We calculate the derivative. $V' = 2(xx' + yy')$ which gives

$$V' = 2(x(-x + 2x^2 + y^2) + y(-y + y^2)) = 2(x^2(-1 + 2x) + y^2(x - 1 + y)).$$

If $x < 1/2$ and $y < 1 - x$ then $V' < 0$ for $(x, y) \neq (0, 0)$. The level curves of V are circles and these are inside region $V' < 0$ if the radius is less than $1/2$. Thus $V' < 0$ in the region defined by $V(x) \leq k < 1/4$ but not in $V(x) \leq 1/4$. We conclude that the region where $V(x) < 1/4$ is in the basin of attraction.

Figure 3: Zero-isoclines and sign analysis for $x' = -x + 2x^2 + y^2,\ y' = -y + y^2$.

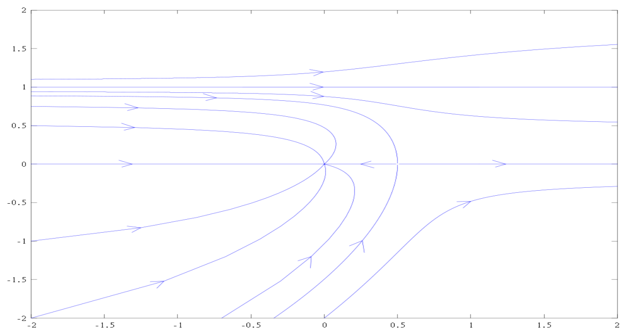


Let us compare with the phase portrait of the system. Zero isoclines are $2(x - 1/4)^2 + y^2 = 1/8$ ($x' = 0$) and $y = 0$ or 1 ($y' = 0$). There are two equilbria $(0, 0)$ and $(1/2, 0)$, a sink and a saddle resp. The sign analysis can be seen in figure 3.

The basin of attraction seems to be the region to the left of the saddle. Notice that points where $1 < y < 1 - x$ are not in the basin of attraction even if $V' < 0$ there.

In figure 4 we can see the basin of attraction of origin as the region to the left from the stable sets of a saddle.

Figure 4: Phase portrait for $x' = -x + 2x^2 + y^2,\ y' = -y + y^2$.



In figure 5 we have added to the phase portrait level curves of the Lyapunov function and boundary for the region where we surely knew $V' = 0$. We see that one level curve is inside the region, another touches the boundary $V = 1/4$ and the third goes outside the region and cannot be used.

We see that a quadratic function cannot catch the parts of the basin of attraction far to the left because of the symmetry of the function. For such estimates we need to use other Lyapunov functions or methods.

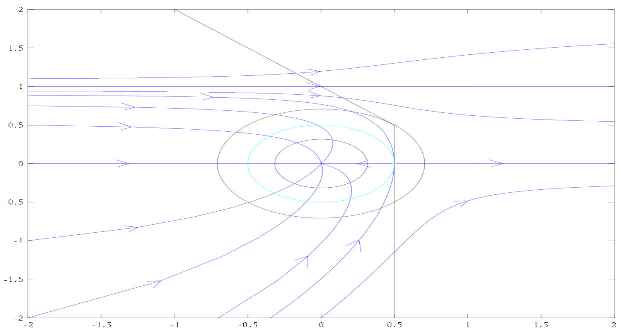
Example. Let us look at the system $x' = -2x - 3y + x^2,\ y' = x + y$. We wish to prove using Lyapunov functions that the origin is asymptotically stable and to estimate its basin of attraction. Indeed from the linearization we already know the origin is a stable focus (look at determinant and trace of Jacobian matrix).

We try with a function $V(x, y) = x^2 + Bxy + Cy^2$. For the function to be positive definite we require $4C > B^2$. Calculating the derivative of V gives:

$$V' = 2x(-2x - 3y + x^2) + B(x(x + y) + y(-2x - 3y + x^2)) + 2Cy(x + y).$$

Expanding and collecting terms gives:

Figure 5: Phase portrait for $x' = -x + 2x^2 + y^2,\ y' = -y + y^2$, level curves of $x^2 + y^2$ and boundary for region $x < 1/2$, $y < 1 - x$.



$$V' = (B-4)x^2 + (2C-6-B)xy + (2C-3B)y^2 + 2x^3 + Byx^2.$$

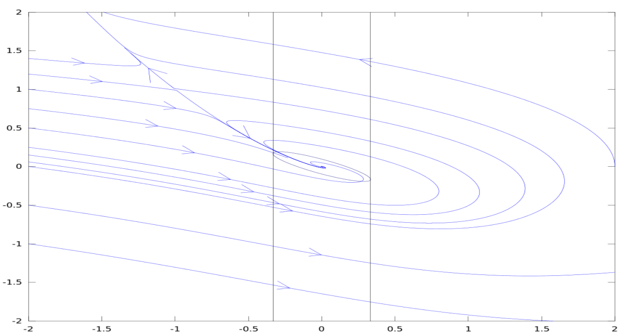
For V' to be negative definite we require $B - 4 < 0$ and $kr = 4(B-4)(2C-3B) - (2C-6-B)^2 > 0$. Trying with $C = 1$ we get $kr = 48B - 48 - 13B^2$ which is never positive. We then try with $C = 3$ and get $kr = 72B - 96 - 13B^2$ which is positive, for example, for $B = 3$. We see that that $4C > B^2$ because $4 \cdot 3 > 3^2$ so we have found a desired Lyapunov function.

Calculations give $V' = -x^2 - 3xy + 2x^3 - 3y^2 + 3yx^2$ which also can be written in the form $V' = (2x-1)x^2 + (3x-3)xy - 3y^2$

For $|x| < 1/3$ clearly $4(2x-1)(-3) - (3x-3)^2 > 0$ and thus we can conclude that V' is negative definite in the region defined by $V = x^2 + 3xy + 3y^2 < k$ for any k such that this region is inside $|x| < 1/3$ Level curves of V have vertical tangent at $y = -x/2$, the level curve is tangent to the lines $|x| = 1/3$ when $V = 1/36$ that means the V-value for $x = 1/3,\ y = -1/6$. So clearly the region $V < 1/36$ is in the basin of attraction.

From figure 6 we can see how the real basin of attraction looks like.

Figure 6: Phase portrait for $x' = -2x - 3y + x^2,\ y' = x + y$ and level curve for the Lyapunov function $V(x, y) = x^2 + 3xy + 3y^2$ for $V = \frac{1}{36}$.



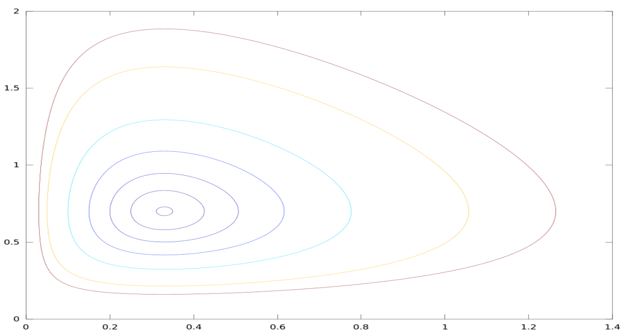
Example. Let us consider the system $x' = (s - \lambda)x,\ s' = (h(s) - x)s$, where $h(s) = (1-s)(s+a)$ and suppose $a > 0,\ (1-a)/2 < \lambda < 1$. The system then has an equilibrium at the point $(h(\lambda), \lambda)$ which can be proved to be asymptotically stable. Let us now estimate the basin of attraction using the Lyapunov function

$$V(x, s) = x - h(\lambda)\ln x + s - \lambda \ln s + h(\lambda)\ln(h(\lambda)) - h(\lambda) + \lambda \ln \lambda - \lambda$$

We show that the function is positive when $x, s > 0$, except at the equilibrium. Really $x - h(\lambda)\ln x$ takes it minimum only once for $x = h(\lambda)$ and it is equal to $-h(\lambda)\ln(h(\lambda)) + h(\lambda)$. Analogously $s - \lambda \ln s$ has minimum at $s = \lambda$.

Thus $V(h(\lambda), \lambda) = 0$, but $V(x, s) > 0$ for all other positive x and s. Because there are only two points on the intersection of the level curves with vertical and horizontal lines, the level curves look like depicted in figure 7.

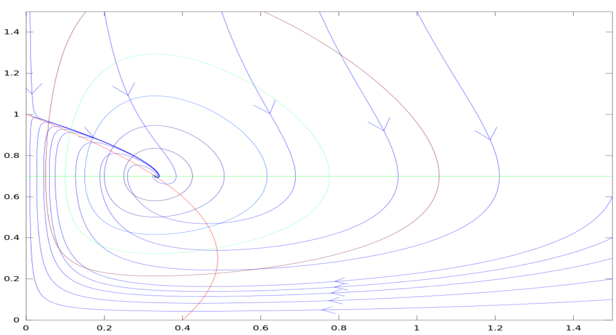
Figure 7: Level curves (for $a = 0.4,\ \lambda = 0.7$) of Lyapunov function $V(x, s) = x - h(\lambda)\ln x + s - \lambda \ln s + h(\lambda)\ln(h(\lambda)) - h(\lambda) + \lambda \ln \lambda - \lambda$.



Derivating with respect to time gives $V' = (\lambda - s)(h(\lambda) - h(s))$. If $s > \lambda$ then $h(\lambda) > h(s)$ and thus $V' < 0$ for $s > \lambda$. If $s < \lambda$ then $V' < 0$ if $h(s) > h(\lambda)$. Let $\bar{s} < \lambda$ be a number such that $h(\bar{s}) = h(\lambda)$ if such a number is existing (means $a > h(\lambda)$). Then we conclude that the region defined by $V(x, s) < V(h(\lambda), \bar{s})$ will be in the basin of attraction of the equilibrium. If such a number $\bar{s}$ is not existing we conclude that the equilibrium attracts all point with positive x and s. The reader gets confused here, indeed, this is only a weak Laypunov function which is zero for $s = \lambda$. Anyhow there exists a theorem telling that if the set where $V' = 0$ contain no whole solution (like equilibria) then the theorem for estimating basin of attraction still works. But this does not follow directly from the theorems we gave. We also here notice that we have used Theorem 1 for the equilibrium transformed to origin.

From figure 8 we can see how solutions intersect the level curves going inside to the globally attracting equilibrium.

Figure 8: Phase portrait and zero-isoclines for $x' = (s - \lambda)x,\ s' = (h(s) - x)s$, where $h(s) = (1 - s)(s + a)$ and $a = 0.4,\ \lambda = 0.7.$ and level curves for a Lyapunov function.



In general very often a quadratic functions give some estimate of the basin of attraction of an equilibrium as the linear terms dominate the behavour.

We here look at some possibilities to find Lyapunov functions in form of quadratic expressions $Ax^2+Bxy+Cy^2$ for estimating basin of attraction of some asymptotically stable equilibrium. For this we use the linearization around the equilibirum. We will such expressions which are locally Lyapunov functions and prove theorems for existence of local Lyapunov functions.

Lemma Q1. Suppose the linearization around an equilibrium of a two dimensional system of differential equations is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{3.1}$$

and suppose $\det M > 0$ and $a, d < 0$ and $b, c \neq 0$. Then $x^2 + ky^2$ is a local Lyapunov function in a neighbourhood of the equilibrium if $k = b/c$ when b and c are of the same sign and $k = -b/c$ when b and c are of opposite signs.

Proof. Calculating the derivative of V where $V(x, y) = x^2 + ky^2$ with respect to the linearized system gives

$$V' = 2a\,x^2 + 2(b + ck)\,x\,y + 2d\,k\,y^2$$

Choosing $k = b/c$ in the case b and c are of the same sign will give

$$4 \cdot 2a \cdot 2dk - \{2(b + ck)\}^2 = 16\frac{adb}{c} - 16b^2 = 16\frac{b}{c}\det M > 0$$

Thus V' is negative definite because $2a < 0$.

Choosing $k = -b/c$ in the case b and c are of the opposite sign will give $V' = 2ax^2 + 2dky^2$ which is negative definite as $a, d < 0$ and $k > 0$.

As higher order terms cannot change the sign of V and V' the function V is a local Lyapunov function for these choices of k.

Lemma Q2. Suppose the linearization around an equilibrium of a two dimensional system of differential equations is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{3.2}$$

and suppose $\det M > 0$ and $tr(M) = a + d < 0$. Then $x^2 + Bxy + Cy^2$ is a local Lyapunov function in a neighbourhood of the equilibrium if

for $c \neq 0$

$$B = B_0 = \frac{d - a}{c},\ C = C_0 = \frac{(a + d)^2 - 2bc}{2c^2}$$

and for $c = 0$ we have $B = 0$ and $C > \frac{b^2}{4ad}$.

Proof. Calculating V' we get

$$V' = (2a + Bc)x^2 + (2cC + dB + aB + 2b)xy + (2dC + bB)y^2$$

Calculating $4(2a + Bc)(2dC + bB) - (2cC + dB + aB + 2b)^2$ we get

$$-4\tilde{C}^2 + 4c(d - a)\tilde{B}\tilde{C} - ((d - a)^2 + 4\det M)\tilde{B}^2 + \frac{4\,tr(M)^2 detM}{c^2}$$

where $\tilde{B} = B - B_0, \tilde{C} = C - C_0$ in the case $c \neq 0$.

This is clearly greater than zero and as $2a + B_0c = a + d < 0$ the function V' is negative definite.

In the case $c = 0$ the expression $4(2a + Bc)(2dC + bB) - (2cC + dB + aB + 2b)^2$ takes the form

$$-(a + d)^2B^2 - 4b(a - d)B + 16adC - 4b^2 > 0$$

if we choose $B = 0$ and $C > \frac{b^2}{4ad}$.

Thus also in this case V' is negative definite.

As higher order terms cannot change the sign of V and V' the function V is a local Lyapunov function for these choices of B and C.

Example Q1a. If we choose $a = -5, b = 4, c = 1, d = -1$ then we can use Lemma Q1 to conlcude that $V = x^2 + 4y^2$ is a Lyapunov function for the linearized system. The function $V = x^2 + y^2$ will not be a Lyapunov function.

Example Q1b. If we choose $a = -3, b = -5, c = 1, d = -1$ then we can use Lemma Q1 to conlcude that $V = x^2 + 5y^2$ is a Lyapunov function for the linearized system. We will have $V' = -6x^2 - 10y^2$. The function $V = x^2 + y^2$ will not be a Lyapunov function. We will then have $V' = -6x^2 - 8xy - 2y^2$.

Example Q1c. If we choose $a = -2, b = -5, c = 1, d = -2$ then we can use Lemma Q1 to conlcude that $V = x^2 + 5y^2$ is a Lyapunov function for the linearized system. We will have $V' = -4x^2 - 20y^2$. The function $V = x^2 + y^2$ will not be a Lyapunov function. We will then have $V' = -4x^2 - 8xy - 4y^2 = -4(x + y)^2$.

Example Q2a. If we choose $a = -2, b = -4, c = 1, d = 1$ then we can use Lemma Q2 to conlcude that $V = x^2 + 3xy + 4.5y^2$ is a Lyapunov function for the linearized system. The equilibrium is here a stable focus.

Example Q2b. If we choose $a = -2, b = 1, c = -4, d = 1$ then we can use Lemma Q2 to conlcude that $V = x^2 - \frac{3}{4}xy + \frac{9}{32}y^2$ is a Lyapunov function for the linearized system. The equilibrium is here a stable focus.

Example Q2c. If we choose $a = -4, b = -5, c = 1, d = 1$ then we can use Lemma Q2 to conlcude that $V = x^2 + 5xy + 9.5y^2$ is a Lyapunov function for the linearized system. The equilibrium is here a stable node.

Example Q2d. If we choose $a = -2, b = 0, c = 1, d = -1$ then we can use Lemma Q2 to conlcude that $V = x^2 + xy + 4.5y^2$ is a Lyapunov function for the linearized system. The equilibrium is here a stable node.

Example Q2e. If we choose $a = -1, b = 0, c = 1, d = -1$ then we can use Lemma Q2 to conlcude that $V = x^2 + 2y^2$ is a Lyapunov function for the linearized system. The equilibrium is here a stable focus.

Example Q2f. If we choose $a = -2, b = 4, c = 0, d = -1$ then we can use Lemma Q2 to conlcude that $V = x^2 + 3y^2$ is a Lyapunov function for the linearized system. The equilibrium is here a stable focus.

We now introduce an example of using theorem 2.

Example. Let us consider the system $x' = x^2 - y^2$, $y' = -2xy$. We consider the function given by $V = \frac{x^3}{3} - xy^2$. Calculating the derivative we get

$$V' = (x^2 - y^2)(x^2 - y^2) - 2xy(-2xy) = (x^2 + y^2)^2$$

V' is clearly positive definite and V is positive somewhere in any neighbourhood of origin. Thus origin is unstable. The unstability of origin is anyhow easy to establish also by looking at the behaviour of solution on positive x-axis.

Lyapunov functions can also be used to find attracting regions and, for example, the location of a limit cycle. We give one example.

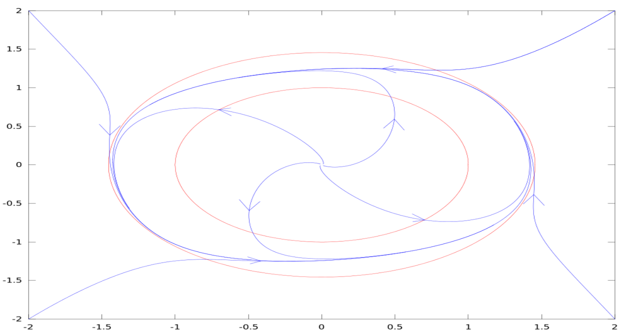
Example. We consider the system $x' = x(2-x^2-y^2)-y,\ y' = y(3-x^2-2y^2)+x$. We consider the function given by $V = x^2 + y^2$. Calculating the derivative gives $V' = R(2 - R) + y^2 - y^4$, where $R = x^2 + y^2$.

As $V' < R(2 - R) + 1/4 < 0$ for $R > R_0 = \frac{\sqrt{5}+2}{2}$ all solutions starting at distance greater than $\sqrt{R_0} < 1.46$ from origin will decrease in this distance and after some time enter region $x^2 + y^2 \leq R_0$. The system is clearly dissipative.

As $V' > R(2-R) > 0$ for $x^2+y^2 \leq 1$ all solutions, except origin, starting inside the unit circle will diverge out from it and enter region $x^2+y^2 > 1$.

This is illlustrated in figure 9 where we see a stable limit cycle inside the region given by $1 < x^2+y^2 < R_0$.

Figure 9: Phase portrait for $x' = x(2-x^2-y^2) - y,\ y' = y(3-x^2-2y^2) + x$ and the estimates for the limit cycle from Lyapunov functions



At the end we give you a small exercise for an epidemic model.

Exercise E. Consider the system $S' = -\beta IS + \mu - \mu S,\ I' = \beta IS - \gamma I - \mu I$ defined in $0 \leq S, I \leq 1$. For $\beta > \gamma + \mu$ there is an endemic equilibrium ($I > 0$). Find the endemic equilibrium and show that

$$V = \beta S - (\gamma+\mu)\ln S + \beta I - \mu(\frac{\beta}{\gamma+\mu} - 1)\ln I$$

is a weak Lyapunov function and the endemic equlibrium is globally attracting.

Can you find a quadratic Lyapunov function estimating the basin of attraction? May be for some concrete values of the parameters?

4 Answers to exercises

Exercise 1.

1. This is a linear system where the origin is an unstable node and clearly thus unstable.

2. This is a linear system where the origin is a saddle and thus only the solutions on the stable set remains in neighbourhoods of origin. The origin is unstable.

3. This is a linear system where the origin is a stable node. The origin is asymptotically stable with the whole plane R^2 as basin of attraction. It is a global attractor.

4. The origin is saddle and unstable.

5. The origin is a stable node with basin of attraction as whole plan. It is global attractor.

6. The origin is a stable node and the basin of attraction is the region where $y > -1$.

7. The origin is a stable node and the basin of attraction is the region where $x > 1/2$ and $|y| < 1/2$.

Exercise 2.

1. This is a quadratic form which can change sign in any neighbourhood of origin and thus it is neither positive nor negative definite or semi-definite.
2. This is a quadratic form which can change sign in any neighbourhood of origin and thus it is neither positive nor negative definite or semi-definite.

3. This is a quadratic form which can change sign in any neighbourhood of origin and thus it is neither positive nor negative definite or semi-definite.
4. Quadratic form, negative semi-definite. Zero for $x_1 = x_2$.
5. Positive definite near origin. It can be written in the form $(1 - x - 1x_2)x_1^2 + x_2^2$ surely positive for $x_1 x_2 < 1$.
6. Changes sign in any neighbourhood of origin and thus it is neither positive nor negative definite or semi-definite.
7. Changes sign in any neighbourhood of origin and thus it is neither positive nor negative definite or semi-definite.
8. Changes sign in any neighbourhood of origin and thus it is neither positive nor negative definite or semi-definite.
9. Changes sign in any neighbourhood of origin and thus it is neither positive nor negative definite or semi-definite.
10. Positively definite. An easy substitution makes it a quadratic form.
11. Negative definite in a neighbourhood of origin.
12. Negative definite. Can be written in form $-(x_1 - x_2)^4 - x_2^4$
13. Changes sign in any neighbourhood of origin and thus it is neither positive nor negative definite or semi-definite.
14. Changes sign in any neighbourhood of origin and thus it is neither positive nor negative definite or semi-definite.

Exericise E. $V' = -\frac{\mu}{(\gamma+\mu)S}(\beta S - \gamma - \mu)^2$ For endemic equilibrium $S = \frac{\gamma+\mu}{\beta},\ I = \mu(\frac{\beta}{\gamma+\mu} - 1)$.