Mathematics for Control Theory

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ system norms

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Reading materials

We will use:

- Michael Green and David Limebeer, *Linear Robust Control*, Prentice-Hall, 1995 (mostly Chapter 3, some material from other sections and appendices)
- Kemin Zhou, John Doyle and Keith Glover, *Robust and Optimal Control*, Prentice-Hall, 1995 (for more information specific to linear robust control).

Input-Output System

An input-output system or simply *system* is a mapping $G : \mathcal{S}_1 \mapsto \mathcal{S}_2$. We use the operator notation $z = Gw$ with $w \in \mathcal{S}_1$ and $z \in \mathcal{S}_2$.

Systems form a linear vector space (even if the systems themselves are not linear!) under the following operations:

$$(G_1 + G_2)w = G_1 w + G_2 w$$

$$(\alpha G)w = \alpha G w$$

Linear systems satisfy:

$$G(\alpha w_1 + \beta w_2) = \alpha G w_1 + \beta G w_2$$

for all $w_1, w_2 \in \mathcal{S}_1$ and all scalars α, β (taken from an appropriate field, for instance $\mathbb{R}$).

Causal Systems

As engineers, we don't like systems which react to an input before the input has been applied or that continue to "feel" the effects of the input after it is no longer applied.

Systems whose outputs up to time T depend on the input only up to time T, for any T, are called causal. Mathematically, causal systems G obey

$$P_T G P_T = P_T G$$

where P_T is the truncation operator:

$$(P_T(w))(t) = \begin{cases} w(t), & t \leq T \\ 0, & t > T \end{cases}$$

$\mathcal{L}_2$ Stability

A system is $\mathcal{L}_2$ stable if the following implication is true:

$$w \in \mathcal{L}_2[0, \infty] \Longrightarrow Gw \in \mathcal{L}_2[0, \infty]$$

.

For continuous-time, LTI systems, the well-known condition that the transfer function must not have poles in Re$(s) > 0$ is equivalent to $\mathcal{L}_2$ stability.

A measure of $\mathcal{L}_2$ norm amplification/reduction (gain) from input to output is of interest.

Review: Singular Value Decomposition

Refer to MCE/EEC 647/747 material on SVD:
`http://academic.csuohio.edu/richter_h/courses/mce647/mce647_4p5.pdf`
(slides 2,3,4)

Induced norms

Let $G : \mathcal{S}_1 \mapsto \mathcal{S}_2$. The induced norm of G is defined as

$$||G|| = \sup_{||w||-||w_1||\neq 0} \frac{||Gw - Gw_1||_{\mathcal{S}_2}}{||w - w_1||_{\mathcal{S}_1}}$$

For linear systems, we take $w_1 = 0$, $||G||$ doesn't change (HW problem).

For matrices (A a static linear system, just a transformation):

$$||A|| = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2} = \bar{\sigma}(A)$$

That is, the Euclidean norm induces the maximum singular value as an operator norm.

Note that induced norms are actually norms and they satisfy the submultiplicative property:

$$||GH|| \leq ||G||||H||$$

System 2-norm

We saw that $\mathcal{H}_2$ contained analytic complex matrix-valued functions G for which

$$||G||_2^2 \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(jw)G(jw)dw < \infty$$

When $G(jw)$ is the frequency response of an LTI system, the above defines a valid system norm.

However, the above is not an induced norm, and it does not satisfy the submultiplicative property.

Because of this, the $\mathcal{H}_2$ norm is not conducive to robustness analysis. Linear quadratic Gaussian control (LQG) is equivalent to a 2-norm minimization problem.

$\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ spaces

Space $\mathcal{L}_\infty(j\mathbb{R})$ is formed by all essentially bounded complex-valued matrix functions F such that

$$\operatorname*{ess\,sup}_{w\in\mathbb{R}} \bar{\sigma}[F(jw)] < \infty$$

Note: F is "essentially bounded" if it is bounded except in a set of measure zero. Similarly, the essential supremum is the smallest upper bound excluding such set.

$\mathcal{H}_\infty$ is a subspace of $\mathcal{L}_\infty(j\mathbb{R})$ restricted to $F(s)$ analytic and bounded in $\text{Re}(s) > 0$ (note this excludes "rhp" poles). A norm in $\mathcal{H}_\infty$ is defined by:

$$||F||_\infty = \sup_{w\in\mathbb{R}} \bar{\sigma}[F(jw)]$$

Notes

- The notations $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$ are used when the respective subspaces are restricted to stable, real-rational functions (ratios of polynomials with real coefficients) which are strictly proper (degree of the denominator $>$ degree of the numerator).
- The 2-system norm can be finite only if the transfer matrix is strictly proper ($D = 0$).
- The 2-system norm can be calculated in Matlab with `h2norm`
- The 2-system norm of an LTI system is the expected RMS value of the output when the input is a white noise $w(t)$ with unit variance:

$$||G||_2^2 = \mathcal{E}\{\frac{1}{T}\int_0^\infty z^T(t)z(t)dt\}$$

where $Z = GW$

$\mathcal{H}_\infty$ system norm

Let $G(s)$ be a transfer matrix such that $Y(s) = G(s)U(s)$. Assume $G(s) \in \mathcal{L}_\infty$. Define an induced operator norm by:

$$||G|| = \sup_{u \in \mathcal{L}_2, ||u|| \geq 1} \frac{||y||_2}{||u||_2}$$

It is shown (see Zhou and Doyle) that:

$$||G|| = ||G||_\infty = \sup_{w \in \mathbb{R}} \bar{\sigma}[G(jw)]$$

In the SISO case, $||G||_\infty$ is just the peak frequency response.

$\mathcal{H}_\infty$ system norm

The norm of an LTI system (A, B, C, D) induced by the signal norm $\mathcal{L}_2$ is the $\mathcal{H}_\infty$ norm of its transfer function $G(s) = C(sI - A)^{-1}B + D$, the maximum singular value of $G(jw)$ for $w \geq 0$.

The $\mathcal{H}_\infty$ system norm can be calculated in Matlab with a singular value plot (`sigma`) or with the `hinfnorm` command.

Time-domain interpretation of the $\mathcal{H}_\infty$ system norm

Let $G(s)$ be a $p \times m$ transfer matrix. Apply input vector $u(t)$ with sinusoidal components $u_j(t)$, $j = 1, 2...m$ having the same frequency but different amplitudes and phases:

$$u_j(t) = U_j \sin(wt + \phi_j)$$

If G is stable, the output $y(t)$ will reach a stationary regime of the form $y_i(t) = Y_i \sin(wt + \theta_i)$, $i = 1, 2...p$.

Define an amplification ratio by $||Y||/||U||$, with $Y = [Y_1,\ Y_2,\ ...Y_p]$ and $U = [U_1,\ U_2,\ ...U_m]$.

The norm $||G||_\infty$ will be the supremum of the amplifications as the frequencies, amplitudes and phases of u_j are varied over all possible combinations so that $0 < ||U|| < \infty$.

Example

Obtain a singular value plot and calculate norms $\mathcal{H}_2$ and $\mathcal{H}_\infty$ for the following transfer matrices:

$$G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+1} \\ \frac{1}{(s+1)^2} & -\frac{1}{(s+1)^2} \end{bmatrix}$$

$$G(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{s+1} \\ \frac{1}{s^2+0.2s+1} & -\frac{1}{s^2+0.2s+1} \end{bmatrix}$$