STABILITY of SWITCHED SYSTEMS under ARBITRARY SWITCHING

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SYSTEMS with SPECIAL STRUCTURE

- Triangular systems

- Feedback systems

    - passivity conditions

    - small-gain conditions

- 2-D systems

TRIANGULAR SYSTEMS

For linear systems, triangular form $\Rightarrow$ GUES

$$A_1 = \begin{pmatrix} -a_1 & b_1 \\ 0 & -c_1 \end{pmatrix}, \; A_2 = \begin{pmatrix} -a_2 & b_2 \\ 0 & -c_2 \end{pmatrix}$$

$\dot{x}_2 = -c_\sigma x_2 \Rightarrow x_2 \to 0$ exponentially fast

$\dot{x}_1 = -a_\sigma x_1 + b_\sigma \underset{\substack{\downarrow \\ 0}}{x_2} \Rightarrow x_1 \to 0$ exp fast

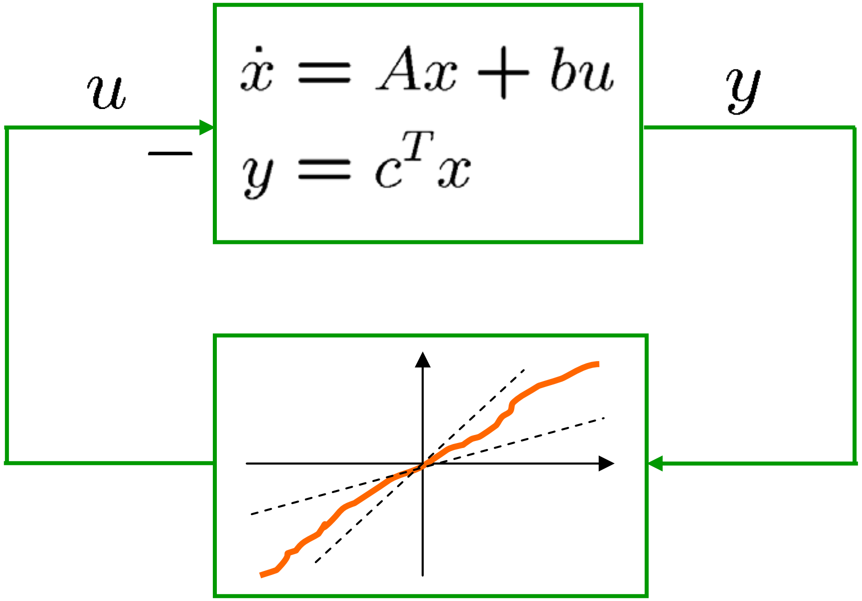
$\exists$ quadratic common Lyap fcn $x^T D x,\ D$ diagonal

For nonlinear systems, not true in general

Need to know $x_2 \to 0 \Rightarrow x_1 \to 0$ (ISS)

[Angeli & L '00]

FEEDBACK SYSTEMS: ABSOLUTE STABILITY



(A, b) controllable

$$g(s) = c^T(sI - A)^{-1}b$$

$$u = -\varphi_p(y)$$

$$k_1 y^2 \leq y\varphi_p(y) \leq k_2 y^2 \ \forall p$$

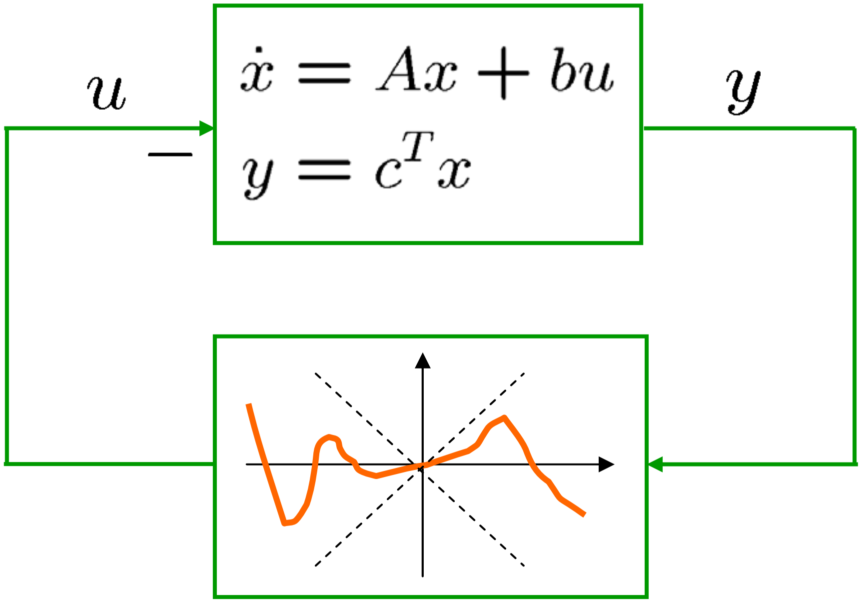
Circle criterion: $\exists$ quadratic common Lyapunov function $\Leftrightarrow$

$h(s) = \frac{1+k_2 g(s)}{1+k_1 g(s)}$ is strictly positive real (SPR): $Re\, h(i\omega) > 0$

For $k_1 = 0, k_2 = \infty$ this reduces to $g(s)$ SPR (passivity)

Popov criterion not suitable: V depends on φ_p

FEEDBACK SYSTEMS: SMALL-GAIN THEOREM



(A, b) controllable

$$g(s) = c^T(sI - A)^{-1}b$$

$$u = -\varphi_p(y)$$

$$|\varphi_p(y)| \leq |y| \;\; \forall p$$

$$(k_1 = -1, k_2 = 1)$$

Small-gain theorem:

$\exists$ quadratic common Lyapunov function

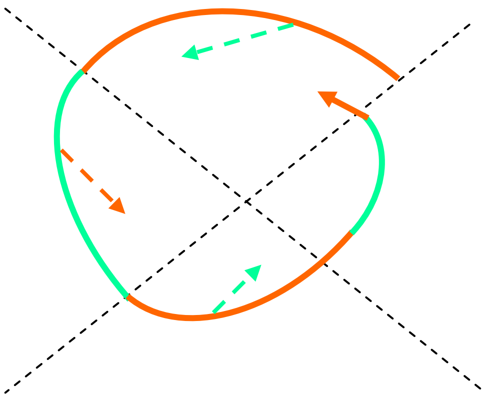
$$\Updownarrow$$

$$\|g\|_\infty = \max_\omega |g(i\omega)| < 1$$

TWO-DIMENSIONAL SYSTEMS

Necessary and sufficient conditions for GUES

known since 1970s



worst-case switching

$$\boxed{\dot{x} = A_1 x, \ \ \dot{x} = A_2 x, \ \ x \in \mathcal{R}^2}$$

$\exists$ quadratic common Lyap fcn <=>

convex combinations of $A_1, A_2, A_1^{-1}, A_2^{-1}$ Hurwitz

OBSERVABILITY and ASYMPTOTIC STABILITY

Barbashin-Krasovskii-LaSalle theorem:

$\dot{x} = f(x)$ is glob. asymp. stable (GAS) if $\exists\, V$ s.t.

- $\dot{V} := \frac{\partial V}{\partial x} f(x) \leq 0 \;\; \forall x$ (weak Lyapunov function)
- $\dot{V}$ is not identically zero along any nonzero solution (observability with respect to $\dot{V}$)

Example:

$$\dot{x} = Ax, \quad V(x) = x^T P x$$

$$\left.\begin{array}{l} A^T P + PA \leq -C^T C \\ (A, C) \text{ observable} \end{array}\right\} \Rightarrow \text{GAS}$$

SWITCHED LINEAR SYSTEMS [Hespanha '04]

$$\boxed{\dot{x} = A_\sigma x}$$

Theorem (common weak Lyapunov function):

Switched linear system is GAS if

- $\exists\, P > 0$ s.t. $A_p^T P + P A_p \le -C_p^T C_p \quad \forall p$
- (A_p, C_p) observable for each p
- $\exists$ infinitely many switching intervals of length $\ge \tau$

Want to handle nonlinear switched systems and nonquadratic weak Lyapunov functions

Need a suitable nonlinear observability notion

OBSERVABILITY: MOTIVATING REMARKS

Several ways to define observability

(equivalent for linear systems)

Benchmarks:

- observer design or state norm estimation
- detectability vs. observability
- LaSalle's stability theorem for switched systems

Joint work with Hespanha, Sontag, and Angeli

No inputs here, but can extend to systems with inputs

STATE NORM ESTIMATION

$$\dot{x} = Ax, \qquad y = Cx$$

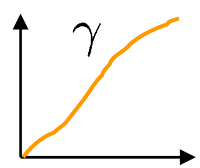
$$x(0) = W^{-1} \int_0^\tau e^{A^T t} C^T y(t) dt$$ where

$$W = \int_0^\tau e^{A^T t} C^T C e^{At} dt$$ (observability Gramian)

$$\dot{x} = f(x), \qquad y = h(x)$$

Observability definition #1:

$$\boxed{|x(0)| \le \gamma\left(\|y\|_{[0,\tau]}\right)}$$ where $\gamma \in \mathcal{K}_\infty$



This is a robustified version of 0-distinguishability

OBSERVABILITY DEFINITION #1: A CLOSER LOOK

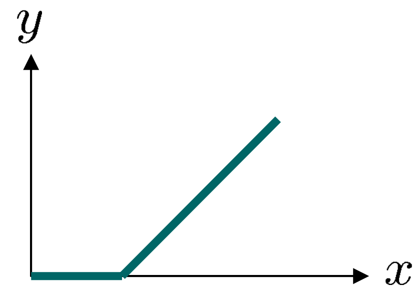
Small-time observability:

$\not\Updownarrow$ $$\forall\, \tau > 0\ \exists\, \gamma \in \mathcal{K}_\infty : \quad |x(0)| \le \gamma\left(\|y\|_{[0,\tau]}\right)$$

Large-time observability:

$$\exists\, \tau > 0,\ \ \gamma \in \mathcal{K}_\infty : \quad |x(0)| \le \gamma\left(\|y\|_{[0,\tau]}\right)$$

Counterexample: $\dot{x} = 1$



Initial-state observability:

$\Updownarrow$ $$\forall\, \tau > 0\ \exists\, \gamma \in \mathcal{K}_\infty : \quad |x(0)| \le \gamma\left(\|y\|_{[0,\tau]}\right)$$

Final-state observability:

$$\forall\, \tau > 0\ \exists\, \gamma \in \mathcal{K}_\infty : \quad |x(\tau)| \le \gamma\left(\|y\|_{[0,\tau]}\right)$$

DETECTABILITY vs. OBSERVABILITY

$$\dot{x} = Ax, \quad y = Cx$$

Detectability $\Leftrightarrow \exists\, L : \; A - LC$ is Hurwitz

$$\dot{x} = (A - LC)x + Ly, \quad |x(t)| \le ce^{-\lambda t}|x(0)| + d\|y\|_{[0,t]}$$

Observability $\Leftrightarrow$ $A - LC$ can have arbitrary eigenvalues

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$$\dot{x} = f(x), \quad y = h(x)$$

A natural detectability notion is output-to-state stability (OSS):

$$\boxed{|x(t)| \le \beta(|x(0)|, t) + \gamma\left(\|y\|_{[0,t]}\right)}$$

where $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ [Sontag-Wang]

Observability def’n #2: OSS, and β can decay arbitrarily fast

OBSERVABILITY DEFINITION #2: A CLOSER LOOK

Definition: $\forall\, \varepsilon > 0,\ \nu \in \mathcal{K}_\infty \quad \exists\, \beta \in \mathcal{KL},\ \gamma \in \mathcal{K}_\infty$:

$$|x(t)| \le \beta(|x(0)|, t) + \gamma\left(\|y\|_{[0,t]}\right) \quad \forall\, t \ge 0$$

and

$$\beta(r, \varepsilon) \le \nu(r) \quad \forall\, r \ge 0$$

Theorem: This is equivalent to definition #1 (small-time obs.)

OSS admits equivalent Lyapunov characterization:

$$|x| \ge \rho(|y|) \ \Rightarrow\ \dot{V} \le -\alpha(|x|), \qquad \alpha, \rho \in \mathcal{K}_\infty$$

For observability, α should have arbitrarily rapid growth

STABILITY of SWITCHED SYSTEMS

$$\boxed{\dot{x} = f_\sigma(x)}$$

Theorem (common weak Lyapunov function):

Switched system is GAS if

- $\exists\, V$ s.t. $\frac{\partial V}{\partial x} f_p(x) \le -W_p(x) \le 0 \quad \forall\, x,\ \forall\, p$
- $\exists$ infinitely many switching intervals of length $\ge \tau$
- Each system

$$\dot{x} = f_p(x), \qquad y = W_p(x)$$

is observable:

$$\exists\, \gamma \in \mathcal{K}_\infty : \ |x(0)| \le \gamma\left(\|y\|_{[0,\tau]}\right)$$

$y = 0$