Input-to-state Stability

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Outline

- Motivation for Input-to-State Stability (ISS)
- ISS Lyapunov function.
- Stability theorems.

Recall: $\mathscr{L}_\infty$ Space

- $\mathscr{L}_\infty$ Space: Space of all piecewise continuous functions $u: \mathcal{R}^+ \to \mathcal{R}^q$ satisfying

$$\|\boldsymbol{u}\|_{\mathscr{L}_\infty} = \sup_{t \in \mathcal{R}^+} \|\boldsymbol{u}(t)\|_\infty < \infty$$

$$\|\boldsymbol{u}(t)\|_\infty = \max_i |u_i(t)|$$

$$\|\boldsymbol{u}\|_{\mathscr{L}_\infty} = \sup_{t \in \mathcal{R}^+} [\max_i |u_i(t)|] < \infty$$

Nonlinear Realization

- Nonlinear realization

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{f}: D \times D_u \to \mathcal{R}^n$$

- Locally Lipschitz in $\boldsymbol{x}, \boldsymbol{u}$

$$D = \{\boldsymbol{x} \in \mathcal{R}^n : \|\boldsymbol{x}\| < r_x\}$$

$$D_u = \{\boldsymbol{u} \in \mathcal{R}^m : \|\boldsymbol{u}\| < r_u\}$$

- Assumptions guarantee local existence and uniqueness of a solution.

Questions

- Given unforced system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0})$, $\boldsymbol{f}(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}$
- $\boldsymbol{x}_e = \boldsymbol{0}$ asymptotically stable equilibrium

Q 1: $\lim_{t\to\infty} \boldsymbol{u}(t) = \boldsymbol{0} \Rightarrow \lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{0}$?

Q2: Is the system state bounded for any bounded input (BIBS)?

$$\|\boldsymbol{u}_T(t)\|_{\mathcal{L}_\infty} < \delta, \forall T \in [0, t] \Rightarrow \sup_t \|\boldsymbol{x}(t)\| < \epsilon$$

LTI Realizations

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \boldsymbol{x}(0) = \boldsymbol{x}_0$$

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)} B\boldsymbol{u}(\tau) d\tau$$

For $\boldsymbol{x}_e = \boldsymbol{0}$ asymptotically stable,

$$Re\{\lambda_i(A)\} < 0, i = 1, \ldots, n$$

$$\|e^{At}\| \le ke^{\lambda t}, \lambda < 0$$

$$\|\boldsymbol{x}(t)\| \le ke^{\lambda t}\|\boldsymbol{x}_0\| + \int_0^t ke^{\lambda(t-\tau)} \|B\| \|\boldsymbol{u}(\tau)\| d\tau$$

Answers for LTI Realizations

$$\|\boldsymbol{x}(t)\| \le ke^{\lambda t}\|\boldsymbol{x}_0\| + \int_0^t ke^{\lambda(t-\tau)} \|B\| \|\boldsymbol{u}(\tau)\| d\tau$$

$$\le ke^{\lambda t}\left\{\|\boldsymbol{x}_0\| + \|B\| \|\boldsymbol{u}_T(t)\|_{\mathcal{L}_\infty} \int_0^t e^{-\lambda\tau} d\tau\right\}$$

$$\forall T \in [0, t]$$

1. $\lim_{t\to\infty} \boldsymbol{u}(t) = \boldsymbol{0} \Rightarrow \lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{0}$
2. BIBS system

$$\|\boldsymbol{u}_T(t)\|_{\mathcal{L}_\infty} < \delta, \forall T \in [0, t] \Rightarrow \sup_t \|\boldsymbol{x}(t)\| < \epsilon$$

- Not true for nonlinear systems, in general.

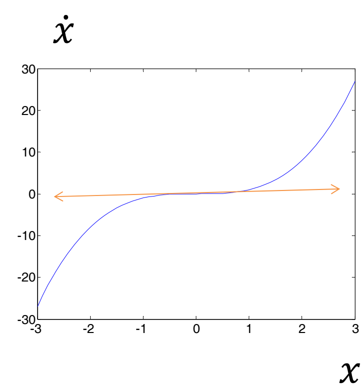
Example

- $\dot{x} = -x + (x + x^3)u$
- $u = 0 \Rightarrow \dot{x} = -x$

  (LTI system)
- $u = 1 \Rightarrow \dot{x} = x^3, x_e = 0$

  Unstable equilibrium

  State diverges for a bounded input.



Functions of Class K

Class K: continuous function $\alpha: [0, a] \rightarrow \mathcal{R}^+$ with

I. $\alpha(0) = 0$
II. $\alpha(.)$ strictly increasing.

Class K_∞: continuous function $\alpha: \mathcal{R}^+ \rightarrow \mathcal{R}^+$ with

I. $\alpha(0) = 0$
II. $\alpha(.)$ strictly increasing.
III. $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$

Class KL

Continuous function $\beta: [0, a] \times \mathcal{R}^+ \rightarrow \mathcal{R}^+$ s.t.

I. For fixed s, $\beta(r, s)$ is in class K w.r.t. r.
II. For fixed r, $\beta(r, s)$ is strictly decreasing w.r.t. s.
III. $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$

Local Input-to-State stability (ISS)

$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is locally ISS if $\exists$ a KL function β, a class K function γ, and constants $k_x, k_u \in \mathcal{R}^+$ s.t.

$\|\boldsymbol{x}(t)\| \leq \beta(\|\boldsymbol{x}_0\|, t) + \gamma\left(\|\boldsymbol{u}_T(t)\|_{\mathcal{L}_\infty}\right)$,
$\forall t \geq 0, T \in [0, t]$,
$\forall \boldsymbol{x}_0 \in D, \|\boldsymbol{x}_0\| < k_x$
$\forall \boldsymbol{u} \in D_u$,
$\|\boldsymbol{u}_T(t)\|_{\mathcal{L}_\infty} = \sup_{t \in [0,T]} [\max_i |u_i(t)|] < k_u$

Global Input-to-State stability (ISS)

$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is globally ISS if $\exists$ a KL function β, a class K_∞ function γ, and constants $k_x, k_u \in \mathcal{R}^+$ s.t.

$\|\boldsymbol{x}(t)\| \leq \beta(\|\boldsymbol{x}_0\|, t) + \gamma\left(\|\boldsymbol{u}_T(t)\|_{\mathcal{L}_\infty}\right)$,
$\forall t \geq 0, T \in [0, t]$,
$\forall \boldsymbol{x}_0 \in \mathcal{R}^n, \|\boldsymbol{x}_0\| < k_x$
$\forall \boldsymbol{u} \in \mathcal{R}^m, \|\boldsymbol{u}_T(t)\|_{\mathcal{L}_\infty} = \sup_{t \in [0,T]} [\max_i |u_i(t)|] < k_u$

Implications of ISS: Unforced System

$$\|\boldsymbol{x}(t)\| \le \beta(\|\boldsymbol{x}_0\|, t) + \gamma\left(\|\boldsymbol{u}_T(t)\|_{\mathscr{L}_\infty}\right)$$

Unforced system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0}), \boldsymbol{x}_e = \boldsymbol{0}$

- $\gamma(0) = 0$ (class K)
- $\|\boldsymbol{x}(t)\| \le \beta(\|\boldsymbol{x}_0\|, t)$
  $\forall t \ge 0, \forall \boldsymbol{x}_0 \in D, \|\boldsymbol{x}_0\| < k_x$
- $\beta(\|\boldsymbol{x}_0\|, t) \to 0$ as $t \to \infty$ (class KL)
- $\boldsymbol{x}_e = \boldsymbol{0}$ is asymptotically stable.

Interpretation of ISS

Bounded input $\|\boldsymbol{u}_T(t)\|_{\mathscr{L}_\infty} < \delta, \gamma$ (class K)

$$\|\boldsymbol{x}(t)\| \le \beta(\|\boldsymbol{x}_0\|, t) + \gamma\left(\|\boldsymbol{u}_T(t)\|_{\mathscr{L}_\infty}\right)$$
$$\le \beta(\|\boldsymbol{x}_0\|, t) + \gamma(\delta)$$
$$\forall t \ge 0, \forall \boldsymbol{x}_0 \in \mathcal{R}^n, \|\boldsymbol{x}_0\| < k_x$$

- $\beta(\|\boldsymbol{x}_0\|, t) \to 0$ as $t \to \infty$ (class KL)and
  $$\|\boldsymbol{x}(t)\| \le \gamma(\delta)$$

$\gamma(\delta)$= ultimate bound of the system

- System is ultimately bounded (or globally ultimately bounded)

Alternative ISS Definition

$$\|\boldsymbol{x}(t)\| \le \max\left\{\beta(\|\boldsymbol{x}_0\|, t), \gamma\left(\|\boldsymbol{u}_T(t)\|_{\mathscr{L}_\infty}\right)\right\}$$

- For $(a, b) > 0$,
  $$\max\{a, b\} \le a + b \le \max\{2a, 2b\}$$

Useful definition for some proofs.

ISS Lyapunov Function

- A continuously differentiable function is an ISS Lypunov function on D if $V: D \to \mathcal{R}$ if $\exists$ class K functions $\alpha_i, i = 1,2,3, \alpha_x$, s.t.
  $$\alpha_1(\|\boldsymbol{x}\|) \le V(\boldsymbol{x}) \le \alpha_2(\|\boldsymbol{x}\|), \forall \boldsymbol{x} \in D$$
  $$\dot{V}(\boldsymbol{x}) \le -\alpha_3(\|\boldsymbol{x}\|), \forall \boldsymbol{x} \in D, \forall \boldsymbol{u} \in D_u$$
  $$\|\boldsymbol{x}\| \ge \alpha_x(\|\boldsymbol{u}\|)$$
- If $D = \mathcal{R}^n, D_u = \mathcal{R}^m, \alpha_i \in K_\infty, i = 1,2,3$, then it is an ISS Lypunov function

Properties of ISS Lyapunov Function

- Lemma 3.1, p. 80, $V(\boldsymbol{x})$ pos. def. in D **if & only if**

$$\alpha_1(\|\boldsymbol{x}\|) \leq V(\boldsymbol{x}) \leq \alpha_2(\|\boldsymbol{x}\|), \forall \boldsymbol{x} \in D$$

- Outside of the ball $\{\boldsymbol{x} \in \mathcal{R}^n : \|\boldsymbol{x}\| < \alpha_x(\|\boldsymbol{u}\|)\}$, $\dot{V}(\boldsymbol{x})$ is negative definite along the trajectories of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$

Local ISS **Theorem 7.1**

- If $\exists$ an ISS Lypunov function on D, $V: D \to \mathcal{R}$ for $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ then it is ISS with

$$\|\boldsymbol{x}(t)\| \leq \beta(\|\boldsymbol{x}_0\|, t) + \gamma\left(\|\boldsymbol{u}_T(t)\|_{\mathscr{L}_\infty}\right)$$

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_x$$

$$k_x = \alpha_2^{-1}(\alpha_1(r_x)),$$

$$k_u = \alpha_x^{-1} \min\{k_x, \alpha_x(r_u)\}$$

$$D = \{\boldsymbol{x} \in \mathcal{R}^n : \|\boldsymbol{x}\| < r_x\}$$

$$D_u = \{\boldsymbol{u} \in \mathcal{R}^m : \|\boldsymbol{u}\| < r_u\}$$

Global ISS **Theorem 7.2**

- If $\exists$ an ISS Lyapunov function on D if $V: D \to \mathcal{R}$ for $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ then it is ISS with

$$\|\boldsymbol{x}(t)\| \leq \beta(\|\boldsymbol{x}_0\|, t) + \gamma\left(\|\boldsymbol{u}_T(t)\|_{\mathscr{L}_\infty}\right)$$

- $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_x$

$$k_x = \alpha_2^{-1}(\alpha_1(r)),$$

$$k_u = \alpha_x^{-1} \min\{k_x, \alpha_x(r_u)\}$$

$$D = \mathcal{R}^n, \qquad D_u = \mathcal{R}^m$$

$$\alpha_1, \alpha_2, \alpha_3 \in K_\infty$$

Example

- Check ISS for the system $\dot{x} = -ax^3 + u, a > 0$
- ISS Lyapunov function candidate $V(x) = \frac{1}{2}x^2$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \forall x \in \mathcal{R},$$

$$\alpha_i = V, i = 1,2$$

$$\dot{V}(x) = x\dot{x} = -a(1-\theta)x^4 - (a\theta x^4 - xu), \theta \in (0,1)$$

$$\leq -a(1-\theta)x^4 \text{ for } (a\theta x^4 - xu) > 0$$

$$|x| \geq \alpha_x(|u|) = \left[\frac{|u|}{a\theta}\right]^{1/3}$$

$$\dot{V}(x) \leq -\alpha_3(|x|), \forall x \in \mathcal{R}, \forall u \in \mathcal{R}, |x| \geq \alpha_x(|u|)$$

- $\alpha_i \in K_\infty, i = 1,2,3$, then the system is globally ISS

Example

- Check ISS for the system $\dot{x} = -ax^3 + x^2 u, a > 0$
- ISS Lyapunov function candidate $V(x) = \frac{1}{2}x^2$

$$\dot{V}(x) = x\dot{x} = -a(1-\theta)x^4 - (a\theta x^4 - x^3 u),$$
$$\theta \in (0,1)$$
$$\dot{V}(x) \le -a(1-\theta)x^4 \text{ for } (a\theta x^4 - x^3 u) > 0$$
$$|x| \ge \alpha_x(|u|) = \frac{|u|}{a\theta}$$
$$\dot{V}(x) \le -\alpha_3(|x|), \forall x \in \mathcal{R}, \forall u \in \mathcal{R}, |x| \ge \alpha_x(|u|)$$

- $\alpha_i \in K_\infty, i = 1,2,3$, then the system is globally ISS

Example

- Check ISS for the system $\dot{x} = -ax^3 + x(1+x^2)\, u, a > 0$
- ISS Lyapunov function candidate $V(x) = \frac{1}{2}x^2$

$$\dot{V}(x) = x\dot{x} = -a(1-\theta)x^4 - (a\theta x^4 - x^2(1+x^2)u)$$
$$\theta \in (0,1)$$
$$\dot{V}(x) \le -a(1-\theta)x^4 \text{ for } (a\theta x^4 - x^2(1+x^2)u) > 0$$
$$|x| \ge \alpha_x(|u|) = \sqrt{(1+r^2)|u|/(a\theta)}, |x| < r = k_x$$
$$\dot{V}(x) \le -\alpha_3(|x|), \forall x \in D = \{x \in \mathcal{R}: |x| < r\}$$
$$\forall u \in \mathcal{R}, |x| \ge \alpha_x(|u|), \gamma(u) = \sqrt{(1+r^2)|u|/(a\theta)}$$
$$k_u = \alpha_x^{-1}[\min\{k_x, \alpha_x(r_u)\}] = \alpha_x^{-1}(r) = (a\theta r)/(1+r^2)$$

- $\alpha_i \in K, i = 1,2,3$, then the system is locally ISS

ISS Theorems 7.3, 7.4

Theorem 7.3: The system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is locally (globally) ISS if and only if ∃ an ISS Lyapunov function satisfying the conditions of Theorem 7.1 (Theorem 7.2).

Theorem 7.4: The system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is locally ISS if (i) $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is continuously differentiable, and (ii) the autonomous system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0})$ has an asymptotically stable equilibrium $\boldsymbol{x}_e = \boldsymbol{0}$

Recall: continuously differential implies **locally** Lipschitz.

Theorem 7.5

The system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is locally ISS if

(i) $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is continuously differentiable and globally Lipschitz, and

(ii) the autonomous system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0})$ has an exponentially stable equilibrium $\boldsymbol{x}_e = \boldsymbol{0}$

Theorem 7.6: ISS Lyapunov

- A continuous function $V: D \to \mathcal{R}$ is an ISS Lypunov function on D **iff** $\exists$ class K functions $\alpha_i, i = 1,2,3, \alpha_u$, s.t.

$$\alpha_1(\|\boldsymbol{x}\|) \le V(\boldsymbol{x}) \le \alpha_2(\|\boldsymbol{x}\|), \forall \boldsymbol{x} \in D$$
$$\dot{V}(\boldsymbol{x}) \le -\alpha_3(\|\boldsymbol{x}\|) + \alpha_u(\|\boldsymbol{u}\|), \forall \boldsymbol{x} \in D, \forall \boldsymbol{u} \in D_u$$

(differential dissipation inequality, storage function V)

- If $D = \mathcal{R}^n, D_u = \mathcal{R}^m, \alpha_i \in K_\infty, i = 1,2,3$, and $\alpha_u \in K_\infty$ then it is an ISS Lyapunov function
- **Definition 7.2 :** Same condition for $V(x)$ but alternative Condition $\dot{V}(\boldsymbol{x}) \le -\alpha_3(\|\boldsymbol{x}\|)$

Proof: Theorem 7.6 ⇒ Definition 7.2

$$\dot{V}(\boldsymbol{x}) \le -\alpha_3(\|\boldsymbol{x}\|) + \alpha_u(\|\boldsymbol{u}\|), \forall \boldsymbol{x} \in D, \forall \boldsymbol{u} \in D_u$$
$$\dot{V}(\boldsymbol{x}) \le -(1-\theta)\alpha_3(\|\boldsymbol{x}\|) - \theta\alpha_3(\|\boldsymbol{x}\|) + \alpha_u(\|\boldsymbol{u}\|)$$
$$\theta \in (0,1)$$
$$\dot{V}(\boldsymbol{x}) \le -(1-\theta)\alpha_3(\|\boldsymbol{x}\|)$$
$$(\text{i.e.}\ (1-\theta)\alpha_3(\|\boldsymbol{x}\|)\ \text{class}\ K)$$
$$\forall \|\boldsymbol{x}\| \ge \alpha_3^{-1}\left[\frac{\alpha_u(\|u\|)}{\theta}\right]$$

As in Definition 7.2

Proof: Definition 7.2 ⇒ Theorem 7.6

$\dot{V}(\boldsymbol{x}) \le -\alpha_3(\|\boldsymbol{x}\|), \forall \boldsymbol{x} \in D, \forall \boldsymbol{u} \in D_u, \|\boldsymbol{x}\| \ge \alpha_x(\|\boldsymbol{u}\|)$

Case 1: $\|\boldsymbol{x}\| \ge \alpha_x(\|\boldsymbol{u}\|) \Rightarrow \dot{V}(\boldsymbol{x}) \le -\alpha_3(\|\boldsymbol{x}\|)$

$$\Rightarrow \dot{V}(\boldsymbol{x}) \le -\alpha_3(\|\boldsymbol{x}\|) + \alpha_u(\|\boldsymbol{u}\|), \forall \alpha_u \in K$$

Case 2: $\|\boldsymbol{x}\| < \alpha_x(\|\boldsymbol{u}\|)$, define

$$\phi(r) = \max_{\substack{\|\boldsymbol{u}\|=r \\ \|\boldsymbol{x}\|<\alpha_x(r)}} \left\{\dot{V}(\boldsymbol{x}) + \alpha_3[\alpha_x(\|\boldsymbol{u}\|)]\right\}, \qquad \phi(0) = 0$$
$$\bar{\alpha}_u(r) = \max(0, \phi(r))$$
$$\dot{V}(\boldsymbol{x}) \le -\alpha_3(\|\boldsymbol{x}\|) + \phi(r) \le -\alpha_3(\|\boldsymbol{x}\|) + \bar{\alpha}_u(r)$$

$\bar{\alpha}_u(r)$ satisfies (i) $\bar{\alpha}_u(0) = 0$, (ii) $\bar{\alpha}_u(0) \ge 0$, not monotone

$$\bar{\alpha}_u(r) \le \alpha_u(r), \alpha_u \text{class } K$$

$\dot{V}(\boldsymbol{x}) \le -\alpha_3(\|\boldsymbol{x}\|) + \alpha_u(r)$ (as in Theorem 7.6)

ISS and ISS Lyapunov function

- Dissipation inequality: shows how ISS and ISS Lyapunov function are related.

Given $D_u = \{\boldsymbol{u} \in \mathcal{R}^m: \|\boldsymbol{u}\| < r_u\}, \exists \boldsymbol{x} \in \mathcal{R}^n$ s.t.

$\alpha_3(\|\boldsymbol{x}\|) = \alpha_u(r_u), \exists d \in \mathcal{R}^+$ s.t. $\alpha_3(d) = \alpha_u(r_u)$

$\Rightarrow d = \alpha_3^{-1}(\alpha_u(r_u)), B_d = \{\boldsymbol{x} \in \mathcal{R}^n: \|\boldsymbol{x}\| \le d\}$

$$\dot{V}(\boldsymbol{x}) \le -\alpha_3(\|\boldsymbol{x}\|) + \alpha_u(\|\boldsymbol{u}\|) \le -\alpha_3(d) + \alpha_u(r_u)$$
$$\forall \|\boldsymbol{x}\| > d, \forall \boldsymbol{u} \in D_u$$

Ω_d = region bounded by the contour $V(\boldsymbol{x}) = c$,

$$c = \max_{\boldsymbol{x} \in B_d} V(\boldsymbol{x})$$

$\forall \boldsymbol{u} \in D_u$ all trajectories that enter Ω_d never leave it.

ISS Pair

- Ω_d depends on the composition $\alpha_3^{-1} \circ \alpha_u$
- ISS pair for the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$: $[\alpha_3, \alpha_u]$ determines the relationship between the bound r_x on $\boldsymbol{x}$ and the bound r_u on $\boldsymbol{u}$
- ISS pair is not unique.
- Also called a supply pair.

Big O Notation

- Given the functions $x, y: \mathcal{R} \to \mathcal{R}$

$$x(s) = O\big(y(s)\big)\ as\ s \to \infty^+ \text{ if}$$

$$\lim_{s\to\infty} \left|\frac{x(s)}{y(s)}\right| < \infty$$

$$x(s) = O\big(y(s)\big)\ as\ s \to 0^+ \text{ if}$$

$$\lim_{s\to 0} \left|\frac{x(s)}{y(s)}\right| < \infty$$

Theorems: Supply Pair

Theorem 7.7: If $[\alpha_3, \alpha_u]$ is a supply pair for the globally ISS system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$, then $\exists$ a supply pair $[\tilde{\alpha}_3, \tilde{\alpha}_u]$ for the system with

$$\alpha_u(r) = O(\tilde{\alpha}_u(r)) as\ r \to \infty^+ \text{ and } \tilde{\alpha}_3 \in K_\infty$$

Theorem 7.8: If $[\alpha_3, \alpha_u]$ a supply pair for the globally ISS system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$, then $\exists$ a supply pair $[\tilde{\alpha}_3, \tilde{\alpha}_u]$ for the system with

$$\alpha_3(r) = O(\tilde{\alpha}_3(r)) as\ r \to 0^+ \text{ and } \tilde{\alpha}_u \in K_\infty$$

Proof shows how to construct new ISS pairs using α_1 and α_2 (bounds on V and not V itself)

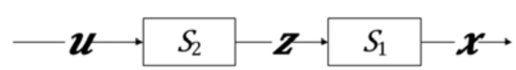
Cascade Connection

$$S_1: \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}), \qquad S_2: \dot{\boldsymbol{z}} = \boldsymbol{g}(\boldsymbol{z}, \boldsymbol{u})$$

- Show that two the ISS systems in cascade form an ISS system

$$\dot{V}^{S_1}(\boldsymbol{x}) \le -\alpha_3^{S_1}(\|\boldsymbol{x}\|) + \alpha_u^{S_1}(\|\boldsymbol{z}\|)$$

$$\dot{V}^{S_2}(\boldsymbol{z}) \le -\alpha_3^{S_2}(\|\boldsymbol{z}\|) + \alpha_u^{S_2}(\|\boldsymbol{u}\|)$$



Lemma 7.1

- Given the ISS systems S_1 and S_2 with ISS pairs $\left[\alpha_3^{S_1}, \alpha_u^{S_1}\right]$ and $\left[\alpha_3^{S_2}, \alpha_u^{S_2}\right]$, respectively.

(i) Define $\tilde{\alpha}_3^{S_2} = \begin{cases} \alpha_3^{S_2}(s), s \text{ "small"} \\ \alpha_u^{S_2}(s), s \text{ "large"} \end{cases}$

$\exists \tilde{\alpha}_u^{S_2}\ s.t. \left[\tilde{\alpha}_3^{S_2}, \tilde{\alpha}_u^{S_2}\right]$ is an ISS pair of S_2

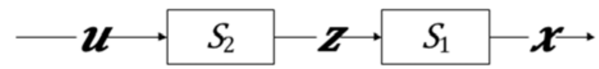
(i) Define $\tilde{\alpha}_u^{S_1} = \alpha_3^{S_2}/2$

$\exists \tilde{\alpha}_3^{S_1}\ s.t. \left[\tilde{\alpha}_3^{S_1}, \tilde{\alpha}_u^{S_1}\right]$ is an ISS pair of S_1

ISS of Cascade

Theorem 7.9: The cascade interconnection of two ISS systems $S_2: \boldsymbol{u} \rightarrow \boldsymbol{z},\ S_1: \boldsymbol{z} \rightarrow \boldsymbol{x}$
is the ISS system $S: \boldsymbol{u} \rightarrow \overline{\boldsymbol{x}},\ \overline{\boldsymbol{x}} = col\{\boldsymbol{x}, \boldsymbol{z}\}$

Theorem 7.10: The cascade interconnection of two locally ISS systems $S_2: \boldsymbol{u} \rightarrow \boldsymbol{z},\ S_1: \boldsymbol{z} \rightarrow \boldsymbol{x}$
is the locally ISS system $S: \boldsymbol{u} \rightarrow \overline{\boldsymbol{x}},\ \overline{\boldsymbol{x}} = col\{\boldsymbol{x}, \boldsymbol{z}\}$



Proof

- Using Lemma 7.1:$\tilde{\alpha}_u^{S_1} = \tilde{\alpha}_3^{S_2}/2$

$$\dot{V}^{S_1}(\boldsymbol{x}) \leq -\tilde{\alpha}_3^{S_1}(\|\boldsymbol{x}\|) + \tilde{\alpha}_3^{S_2}(\|\boldsymbol{z}\|)/2$$

$$\dot{V}^{S_2}(\boldsymbol{z}) \leq -\tilde{\alpha}_3^{S_2}(\|\boldsymbol{z}\|) + \tilde{\alpha}_u^{S_2}(\|\boldsymbol{u}\|)$$

- Define the ISS Lyapunov function for $S: \boldsymbol{u} \rightarrow \overline{\boldsymbol{x}}$

$\overline{\boldsymbol{x}} = col\{\boldsymbol{x}, \boldsymbol{z}\}$ as $V(\overline{\boldsymbol{x}}) = V^{S_1}(\boldsymbol{x}) + V^{S_2}(\boldsymbol{z})$

$$\dot{V}(\overline{\boldsymbol{x}}) = \dot{V}^{S_1}(\boldsymbol{x}) + \dot{V}^{S_2}(\boldsymbol{z})$$
$$\leq \underbrace{-\tilde{\alpha}_3^{S_1}(\|\boldsymbol{x}\|) - \tilde{\alpha}_3^{S_2}(\|\boldsymbol{z}\|)/2} + \tilde{\alpha}_u^{S_2}(\|\boldsymbol{u}\|)$$

Therefore, S is ISS

Asymptotic Stability

Consider $S_1: \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}),\ S_2: \dot{\boldsymbol{z}} = \boldsymbol{g}(\boldsymbol{z}, \boldsymbol{0})$

Corollary 7.1: If $S_1: z \rightarrow x$ is locally ISS and the equilibrium $z_e = 0$ of S_2 is asymptotically stable, then the equilibrium $\bar{x}_e = 0$ of their cascade $S: u \rightarrow \bar{x}, \bar{x} = col\{x, z\}$, is locally asymptotically stable.

Corollary 7.2: If $S_1: z \rightarrow x$ is locally ISS and the equilibrium $z_e = 0$ of S_2 is globally asymptotically stable, then the equilibrium $\bar{x}_e = 0$ of their cascade $S: u \rightarrow \bar{x}, \bar{x} = col\{x, z\}$, is globally asymptotically stable.