<u>Sunandan Adhikary</u>

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1 (a) $f(x) = \frac{k}{x^2}$; $x \in \{1, 2, \dots\}$ $k \in \mathbb{R}$

$= 0$; otherwise

$f(x)$ to be PMF (i) $k > 0$ since $f(x) > 0$ for $x \in \{1, 2..\}$ i.e. in Range(x)

(ii) $f(x) \leq 1 \Rightarrow k \leq x^2$ $\forall x \in \{1, 2..\}$

(iii) $\sum_{x=1}^{\infty} f(x) = 1 \Rightarrow k \sum_{x=1}^{\infty} \frac{1}{x^2} = 1 \Rightarrow \sum_{1}^{\infty} \frac{1}{x^2} = 1/k$

$\because \sum_{x=1}^{\infty} \frac{1}{x^2} > 1 \Rightarrow k < 1$ $[\because k > 0]$

$\therefore$

(b) $f(x) = \frac{k}{|x|}$; $x \in \mathbb{Z} \setminus \{0\}$ $k \in \mathbb{R}$

$= 0$; else

for $f(x)$ to be PDF

(i) $k > 0$ $\forall x \in \mathbb{Z} \setminus \{0\}$

(ii) $f(x) \leq 1$ $\forall x \in \mathbb{Z} \setminus \{0\}$

$\Rightarrow$ $k \leq x$ $\forall x \geq 0$

$k \geq x$ $\forall x < 0$

(iii) $\int_{-\infty}^{\infty} f(x) dx = 1$

$\Rightarrow \int_{-\infty}^{0-} f(x) dx + f(x=0) + \int_{0-}^{\infty} f(x) dx = 1 \Rightarrow \int_{-\infty}^{0-} \frac{k}{-x} dx + \int_{0+}^{\infty} \frac{k}{x} dx = 1$

$\Rightarrow 2 \int_{0}^{\infty} \frac{k}{x} dx = 1 \Rightarrow 2k \log x \Big|_{0}^{\infty} = 1$ ---- this can not be true for any value of $k \in \mathbb{R}$ since the series sum $= \infty$

$\therefore$ $\boxed{\exists \text{ no such } k (\in \mathbb{R}) \text{ for which } f(x) \text{ is a pdf}}$

2) A random variable X over the probability space (Ω, f, P) has the following properties

(i) $X : \Omega \to \mathbb{R}$ (ii) $X^{-1}(x) = \{\omega \mid X(\omega) = x\} \in f$ when $x \in \mathbb{R}$

Now if there exists a function $F_X(x)$ that follows the following properties ~~ie~~ $\forall x \in \mathbb{R}$

(i) $0 \leq F_X(x) \leq 1$

(ii) $\lim_{x \to -\infty} F_X(x) = 0$

(iii) $\lim_{x \to \infty} F_X(x) = 1$

(iv) $x_1 \geq x_2 \iff F_X(x_1) \geq F_X(x_2)$

(v) $\lim_{x \to a+} F_X(x) = \lim_{x \to a} F_X(x)$ i.e. $F_X(x)$ is right continuous

[or $\lim_{\delta \to 0} [F_X(x+\delta) - F(x)] = 0 \quad \forall \delta > 0$]

then Lebasgue Decomposition theorem states

$F_X(x) = G_X(x) + H_X(x)$ s.t. $\lim_{x \to a-} G_X(x) = \lim_{x \to a} G_X(x) = \lim_{x \to a+} G_X(x)$

$\Rightarrow G_X(x)$ is continuous

and $\lim_{x \to a+} H_X(x) = \lim_{x \to a} H_X(x)$

$\Rightarrow H_X(x)$ is right-continuous

If $F_X(X) = P(X \leq x)$ i.e. F_X is Cumulative Density Function ~~of~~ the random variable X then we can characterize X following the charcteristics of G_X and H_X eg.

(i) If $G_X(x) = 0$ then X is a discrete random variable and the discontinuties denotes the probabilities of the corresponding points i.e. $\lim_{x \to a-} F_X(x) - F(a) = P(X=a)$

(ii) If $H_X(x) = 0$ then X is a continuous random variable and $\left.\frac{d}{dx} F_X(x)\right|_{x=a} = P(X=a)$ at all continuous points of G_X

(iii) if $G_X(x) \neq 0$ and $H_X(x) \neq 0$ the X is a Hybrid random variable that has characteristics of both of the above.

③ Geometric Distribution: Let, X be a ~~discrete~~ random variable that denotes the number of independent consecutive bernoulli trials required to win once and 'p' is the probability of winning in each bernoulli trial, then X is distributed following a geometric distribution with parameter 'p'.

i.e $f(x) = (1-p)^{x-1} p \quad x = \{0,1,2..\}$
$\quad = 0 \quad$ else

is considered to be the PMF of X

CDF $F(x) = \sum_{t=0}^{x} f(t) = \sum_{t=0}^{x} p(1-p)^{t-1} = \cancel{p} \frac{1-(1-p)^t}{1-1+\cancel{p}} =$
$= 1-(1-p)^t$

$\therefore P(X > x+y) = 1 - F(x+y) = (1-p)^{x+y} = (1-p)^x (1-p)^y$
$= (1-F(x))(1-F(y))$
$= P(X>x)\,P(X>y)$
$\Rightarrow$ Memoryless

So, Geometric Distribution follows the memoryless property

Exponential Distribution: Let ~~rand~~ X be a random variable that denotes the time passed between 2 consecutive events that occurs in an average const rate in a poisson process of λ, then X follows the following ~~distr~~ PDF and is following an exponential distribution with parameter 'λ'.

i.e $f(x) = \lambda e^{-\lambda x} \quad \forall x \in \mathbb{R}_+$ i.e $x \in [0, \infty)$

CDF $F(x) = \int_{0}^{x} \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}$

$\therefore P(X > x+y) = 1 - F(x+y) = e^{-\lambda(x+y)} = e^{-\lambda x} \cdot e^{-\lambda y}$
$= (1-F(x))(1-F(y))$
$= P(X>x)\,P(X>y)$
$\Rightarrow$ Memoryless

Therefore the exponentially ~~random~~ distributed random variable is memoryless

(4) To have duplicate Roots ~~ea~~ of the quadratic equ.

$ax^2+bx+c=0$ (i) $b^2 = 4ac$ must be true

(ii) $a \neq 0$ for the equ. to remain quadratic

Since $a, b, c \in \{-3,-2,-1,0,1,2,3\}^3 = D^3$ and all possibilities ~~are~~ are equally likely then a takes 6 possible values [leaving '0']

~~both~~ and b and c both takes 7 possible values resulting to $\underline{6 \times 7 \times 7}$ = total possible versions of ~~eq~~ quadratic equ. We consider this our sample sp $S^3 \subset D^3$.

There are following combinations of (a, b, c) for which condition (i) and (ii) are true i.e.

$a, b, c \in \{(1,2,1), (1,-2,1), (-1,2,-1), (-1,-2,-1), (x,0,0)\}$ where x ~~does~~ $\in \{-3,-2,-1,1,2,3\}$

$\Rightarrow$ total 10 possibilities

$\therefore$ Probability of Duplicate roots $= \dfrac{10}{6 \times 7 \times 7} = \dfrac{5}{147}$

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(5) Let S_{dept} be the event of student s ~~is~~ Department "dept".
is from

and d_{dept} be the event ~~of~~ of a student from Department "dept" dropping the course. [dept ∈ Department]

$P(S_{AGFE}) = 15/50$ $\quad$ $P(d_{AGFE}) = \cancel{\tfrac{15}{50}} \times 0.1$

$P(S_{CS}) = 20/50$ $\quad$ $P(d_{CS}) = \cancel{\tfrac{20}{50}} \times 0.07$

$P(S_{EE}) = 15/50$ $\quad$ $P(d_{EE}) = \cancel{\tfrac{15}{50}} \times 0.05$

where Department = {AGFE, CS, EE}

∴ Probability that a student who dropped course is from AGFE

$= P(S_{AGFE} | d)$ where d is the event that a student dropped the course

$$P(S_{AGFE} | d) = \frac{P(S_{AGFE}) \cdot P(d_{AGFE})}{\sum_{\forall\, dept \in Department} P(S_{dept}) \cdot P(d_{dept})} = \frac{1.5/50}{(1.5 + 1.4 + 0.75)/50}$$

$$= \frac{150}{365} = \frac{30}{73}$$

[∵ S_{dept} and d_{dept} are independent

$\Rightarrow P(S_{dept}, d_{dept}) = P(S_{dept})\, P(d_{dept})$

and $P(d) = \sum P(S, d)$]

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⑥ Let X_i be the r.v. that denotes the number appeared on the top face of ith dice ~~when it is rolled ith time~~. after throwing them independently.

$\therefore P(X_1=6) = 1/6$, $P(X_2=6) = 1/6$ since ~~all possibilities are~~ dice is fair

$\therefore P(X_1=6, X_2=6) = P(X_1=6)\,P(X_2=6) = 1/36$ [$\because X_1, X_2$ are independent event outcomes].

(i) $P(X_1=x_1, X_2=x_2) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$ [$\because$ Each throws are independent and $x_1, x_2 \in \{1\ldots 6\}$]

$\therefore P(X_1=x_1, X_2=x_2, \{X_1=6) = \dfrac{1/36}{1/6} = \dfrac{1}{6}$

$P(X_1=x_1, X_2=x_2, \{X_2=6) = \dfrac{1/36}{1/6} = \dfrac{1}{6}$

~~$P(X_1=x_1, X_2=x_2 \mid X_1=6, X_2=6) = 1/36$~~

~~$P(X_1=6, X_2=6) =$~~ $1/36 = P(X_1=x_1, X_2=x_2 \{x_1=6, x_2=6)$

$\therefore$ Probability of having one six already been appeared

$= P(X_1=x_1, X_2=x_2 \{x_1=6 \text{ or } x_2=6) = \frac{1}{6}+\frac{1}{6}-\frac{1}{36}$

$= \frac{11}{36}$

$\therefore$ ~~$P(X_1=6, X_2=6)$~~ $P(X_1=x_1, X_2=x_2, x_1=x_2=6 \mid x_1=6 \text{ or } x_2=6)$

$$= \frac{P(X_1=x_1, X_2=x_2, x_1=x_2=6, x_1=6 \text{ or } x_2=6)}{P(X_1=x_1, X_2=x_2, x_1=6 \text{ or } x_2=6)}$$

$$= \frac{1/36}{11/36} = \underline{\underline{\frac{1}{11}}}$$

(ii) $P(X_1=x_1, X_2=x_2, x_1=6, x_2=6 \mid x_1+x_2>6) = \dfrac{P(X_1=x_1, X_2=x_2, x_1=x_2=6, \ldots)}{P(X_1=x_1, X_2=x_2, x_1+x_2>6)}$

$= \dfrac{1/36}{1-P(x_1+x_2\le 6)} = \dfrac{1/36}{1-P(x_1+x_2=1)-P(x_1+x_2=2) - P(x_1+x_2=3)-P(x_1+x_2=4) - P(x_1+x_2=5)-P(x_1+x_2=6)}$

$= \dfrac{1/36}{1-\frac{15}{36}} = \dfrac{1}{21}$

7) (i) $f(x) = \frac{1}{2} e^{-|x-2|}$ $x \in \mathbb{R}$ is a PDF because

(a) $f(x) > 0$ $\forall x \in \mathbb{R}$ i.e range of x

(b) $f(x) \leq 1$ $\forall x \in \mathbb{R}$

(c) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{2} \frac{1}{2} e^{-(-(x-2))} dx + \int_{2}^{\infty} \frac{1}{2} e^{-(x-2)} dx$

2nd Part: $x-2 = z$, $dx = dz$

1st Part: $x-2 = -z'$, $dx = -dz'$

$$= \int_{+\infty}^{0} \frac{1}{2} e^{-z'}(-dz') + \int_{0}^{\infty} \frac{1}{2} e^{-z} dz$$

$$= \int_{0}^{\infty} e^{-z} dz' = -e^{-z}\Big|_{0}^{\infty} = \underline{\underline{1}}$$

(ii) Mean: $\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{2} \frac{x}{2} e^{-(-x+2)} dx + \int_{2}^{\infty} \frac{x}{2} e^{-(x-2)} dx$

1st: $x-2 = z$, $dx = dz$

$$= \int_{-\infty}^{0} \frac{(z+2)}{2} e^{z} dz + \int_{0}^{\infty} \frac{z+2}{2} e^{-z} dz$$

$-z = z'$, $-dz = dz'$

$$= \int_{\infty}^{0} \frac{-z'+2}{2} e^{-z'}(-dz') + \int_{0}^{\infty} \frac{z+2}{2} e^{-z} dz$$

$$= \int_{0}^{\infty} \frac{-z}{2} e^{-z} dz + \int_{0}^{\infty} e^{-z} dz + \int_{0}^{\infty} \frac{z}{2} e^{-z} dz + \int_{0}^{\infty} e^{-z} dz$$

$\underline{\underline{\mathbb{E}(X) = 2}}$

$$= -2e^{-z}\Big|_{0}^{\infty} = 2\left[-e^{-\infty} + e^{0}\right] = \underline{\underline{2}}$$

Mode: $= \arg\max_{x} f(x)$, $\frac{d}{dx} f(x) = -\frac{1}{2} e^{-|x-2|}$ at $x > 2$

$= +\frac{1}{2} e^{-|x-2|}$ at $x \leq 2$

$= 0$ at $x = 2$

$\therefore$ $\underline{\underline{\text{mode}(X) = 2}}$

Median: $f(x)$ is a function i.e symmetric w.r.t $x = 2$

because $f(x) = \frac{1}{2} e^{-(x-2)}$ for $x > 2$

$= \frac{1}{2} e^{-(-x+2)}$ $x < 2$

$= 1/2$ $x = 2$

$\therefore f(-z) = f(z)$ for $z = x-2$

$\Rightarrow f(z)$ is symmetric about 0

$\Rightarrow f(x)$ ... x 2

$\therefore F(2) = 1/2 \Rightarrow$ $\underline{\underline{\text{Median} = 2}}$

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(8) $(i+1)$th Bernoulli trial probability $= P_{i+1} = P_i/2$ for $i = 1,2,3$

and $P_1 = 2/3$

$\therefore P_2 = \frac{1}{3}$ $P_3 = \frac{1}{6}$ $P_4 = \frac{1}{12}$

Since they are independent their joint probability is product of their marginals.

If random variable X denotes the number of success among 4 trials $\Rightarrow P(X \geq 2) = 1 - P(X \leq 1)$ [$\because X$ only takes values in $\{0, 1, 2, 3, 4\}$]

$$= 1 - [P_1 \bar{P}_2 \bar{P}_3 \bar{P}_4 + \bar{P}_1 P_2 \bar{P}_3 \bar{P}_4 + \bar{P}_1 \bar{P}_2 P_3 \bar{P}_4 + \bar{P}_1 \bar{P}_2 \bar{P}_3 P_4 + \bar{P}_1 \bar{P}_2 \bar{P}_3 \bar{P}_4]$$

[$\bar{P}_i = (1 - P_i)$ for $i \in \{1,2,3,4\}$]

$$= 0.3564$$

(5) $f(x) = \begin{cases} (1-p)\,p^i & \text{for } x \in [i, i+1) \\ 0 & \text{otherwise} \end{cases}$ $\Big|$ $i = 0, 1, 2, \ldots$; $p \in (0, 1)$

(b) $CDF = \sum_{i=0}^{x} f(x) = \sum_{i=0}^{x} (1-p)\,p^i = (1-p)\dfrac{1-p^{x+1}}{1-p} = 1-p^{x+1} = F(x)$

$\therefore$ Median $= 2 \Rightarrow F(2) = 1/2 \Rightarrow 1-p^{2+1} = 1/2 \Rightarrow \boxed{p = \dfrac{1}{\sqrt[3]{2}}}$

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(10) $f(x) = \frac{k}{2^x}$ $x = 0, 1, 2 \ldots,\ k \in \mathbb{R}$

$= 0$ else

For $f(x)$ to be a valid PMF

(i) $f(x) > 0$ for $x = 0, 1, 2 \ldots$

$\Rightarrow k > 0$

(ii) $f(x) \le 1 \Rightarrow k \le 2^x \ \forall x = 0, 1, \ldots$

(iii) $F(x) = \sum_{x=0}^{\infty} f(x) = 1 \Rightarrow \sum_{x=0}^{\infty} \frac{k}{2^x} = k \frac{1}{1 - 1/2} = 1 \Rightarrow \underline{k = 1/2}$

~~Now for x = 0, 2, 4, ... any x that is multiple by 2^k can be chosen with~~ $\cancel{\sum_{x \in \{0,2,4 \ldots\}} \frac{1}{2^{x+1}}}$ ~~Probability = 1/2~~

Now probability to choose any x i.e multiple of 2 is given by

$$\sum_{x \in \{0,2,4 \ldots\}} \frac{1}{2^{x+1}} = \frac{1/2}{1 - 1/2^2} = 2/3 = P(x \text{ div } 2) \quad [\text{div} = \text{divisible by}]$$

Similarly probability of $P(X = x \text{ div } 3) = \frac{1/2}{1 - 1/2^3} = \frac{4}{7}$ and

$P(X = x \text{ div } 6) = \frac{1/2}{1 - 1/2^6} = \frac{32}{63}$

$\therefore P(X = x \text{ div } 2 \text{ or } 3) = \frac{2}{3} + \frac{4}{7} - \frac{32}{63} = \frac{46}{63}$