INTRODUCTION to SWITCHED SYSTEMS;

STABILITY under ARBITRARY SWITCHING

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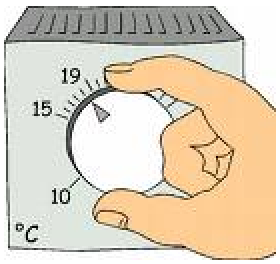
SWITCHED and HYBRID SYSTEMS

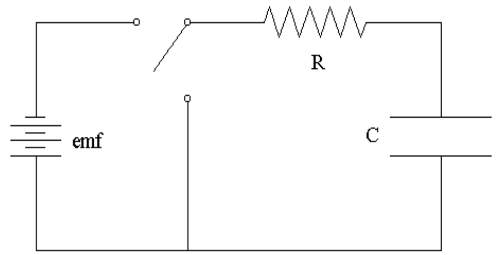
Hybrid systems combine continuous and discrete dynamics

Which practical systems are hybrid?

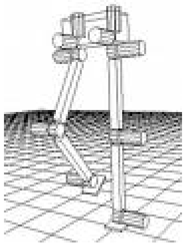


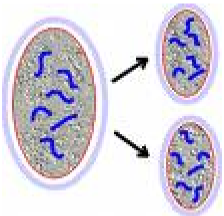
thermostat


stick shift



electric circuit

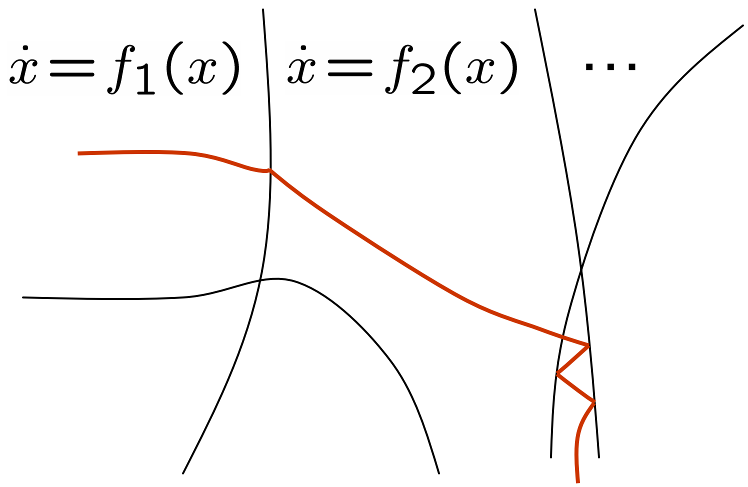

walking


cell division

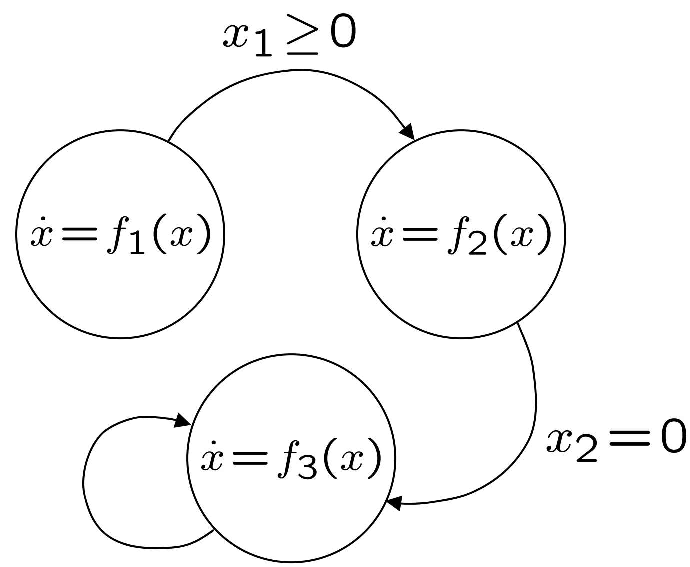
Which practical systems are not hybrid?

More tractable models of continuous phenomena

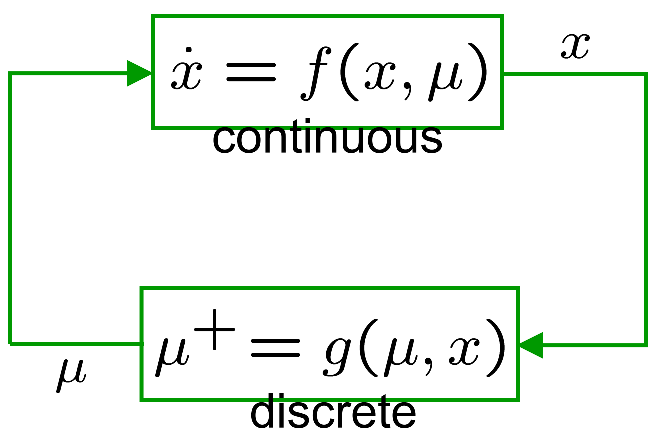
MODELS of HYBRID SYSTEMS



[Van der Schaft–Schumacher '00]



[Proceedings of HSCC]



[Nešić–L '05]

Flow: $\dot{x} \in F(x), \quad x \in C$

Jumps: $x^+ \in G(x), \quad x \in D$

[Goebel-Sanfelice-Teel]

SWITCHED vs. HYBRID SYSTEMS

Switched system:

$$\boxed{\dot{x} = f_\sigma(x)}$$

- $\dot{x} = f_p(x), \;\; p \in \mathcal{P}$ is a family of systems
- $\sigma : [0, \infty) \to \mathcal{P}$ is a switching signal

Switching can be:

- State-dependent or time-dependent
- Autonomous or controlled

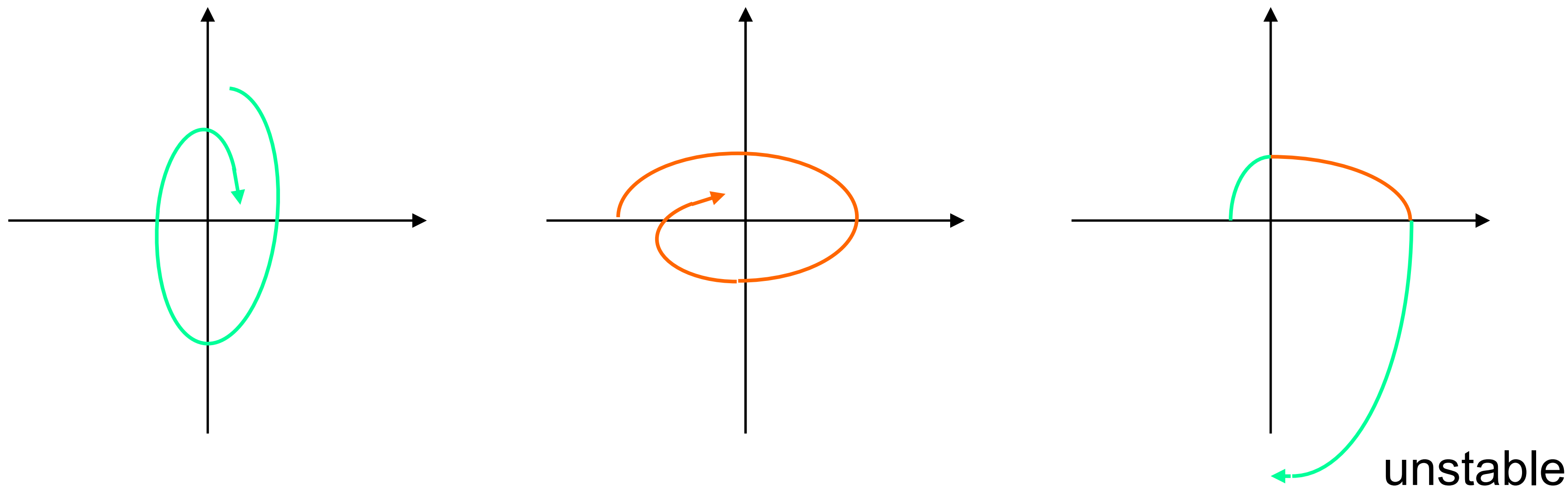
Details of discrete behavior are "abstracted away"

Discrete dynamics $\longrightarrow$ classes of switching signals

Properties of the continuous state x: stability and beyond

STABILITY ISSUE

$\dot{x} = f_1(x)$ $\qquad$ $\dot{x} = f_2(x)$ $\qquad$ $\dot{x} = f_\sigma(x)$



Asymptotic stability of each subsystem is

not sufficient for stability

TWO BASIC PROBLEMS

- Stability for arbitrary switching

- Stability for constrained switching

TWO BASIC PROBLEMS

- Stability for arbitrary switching

- Stability for constrained switching

GLOBAL UNIFORM ASYMPTOTIC STABILITY

GUAS is: Lyapunov stability

$$\forall \varepsilon \ \exists \delta \ \ |x(0)| \le \delta \Rightarrow |x(t)| \le \varepsilon \ \ \forall t \ge 0, \forall \sigma$$

plus asymptotic convergence

$$\forall \varepsilon, \delta \ \exists T \ \ |x(0)| \le \delta \Rightarrow |x(t)| \le \varepsilon \ \ \forall t \ge T, \forall \sigma$$

GUES: $\quad |x(t)| \le c e^{-\lambda t} |x(0)| \quad \forall t \ge 0, \ \forall \sigma$

COMMON LYAPUNOV FUNCTION

$\dot{x} = f_\sigma(x)$ is GUAS if (and only if) $\exists\, V$ s.t.

$$\boxed{\frac{\partial V}{\partial x} f_p(x) \le -W(x) \quad \forall x,\ \forall p}$$

where W is positive definite

V, W quadratic $\Rightarrow$ $\dot{x} = f_\sigma(x)$ is GUES

OUTLINE

Stability criteria to be discussed:

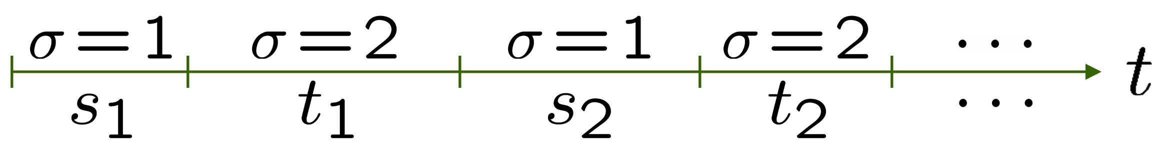
- Commutation relations (Lie algebras)
- Feedback systems (absolute stability)
- Observability and LaSalle-like theorems

Common Lyapunov functions will play a central role

COMMUTING STABLE MATRICES => GUES

$$\mathcal{P} = \{1, 2\}, \ \ A_1 A_2 = A_2 A_1$$

(commuting Hurwitz matrices)



$$x(t) = e^{A_2 t_k} e^{A_1 s_k} \cdots e^{A_2 t_1} e^{A_1 s_1} x(0)$$

$$= e^{A_2(t_k + \ldots + t_1)} e^{A_1(s_k + \ldots + s_1)} x(0) \longrightarrow 0$$

For > 2 subsystems – similarly

COMMUTING STABLE MATRICES => GUES

Alternative proof:

$\exists$ quadratic common Lyapunov function

[Narendra–Balakrishnan ’94]

$$P_1 A_1 + A_1^T P_1 = -I$$

$$P_2 A_2 + A_2^T P_2 = -P_1$$

$$\vdots$$

$$P_m A_m + A_m^T P_m = -P_{m-1}$$

$x^T P_m x$ is a common Lyapunov function

NILPOTENT LIE ALGEBRA => GUES

Lie algebra: $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{\mathsf{LA}}$

Lie bracket: $[A_1, A_2] = A_1 A_2 - A_2 A_1$

Nilpotent means sufficiently high-order Lie brackets are 0

For example: $[A_1,[A_1,A_2]] = [A_2,[A_1,A_2]] = 0$

(2nd-order nilpotent)

Recall: in commuting case $x(t) = e^{A_2 t_2}\, e^{A_1 t_1}\, x(0)$

In 2nd-order nilpotent case

$$x(t) = e^{A_1 t_5}\, e^{A_2 t_4}\, e^{A_1 t_3}\, e^{A_2 t_2}\, e^{A_1 t_1}\, x(0)$$

Hence still GUES [Gurvits '95]

SOLVABLE LIE ALGEBRA => GUES

Larger class containing all nilpotent Lie algebras

Suff. high-order brackets with certain structure are 0

Lie's Theorem: $\mathfrak{g}$ is solvable $\Leftrightarrow$ triangular form

$$A_p = \begin{pmatrix} \lambda_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

Example: $A_1 = \begin{pmatrix} -a_1 & b_1 \\ 0 & -c_1 \end{pmatrix}, \; A_2 = \begin{pmatrix} -a_2 & b_2 \\ 0 & -c_2 \end{pmatrix}$

$\dot{x}_2 = -c_\sigma x_2 \Rightarrow x_2 \to 0$ exponentially fast

$\dot{x}_1 = -a_\sigma x_1 + b_\sigma \overset{\nearrow 0}{x_2} \Rightarrow x_1 \to 0$ exp fast

$\exists$ quadratic common Lyap fcn $x^T D x, \; D$ diagonal

[Kutepov '82, L–Hespanha–Morse '99]

SUMMARY: LINEAR CASE

Lie algebra $\{A_p, p \in \mathcal{P}\}_{\mathsf{LA}}$ w.r.t. $[A_1, A_2] = A_1 A_2 - A_2 A_1$

Assuming GES of all modes, GUES is guaranteed for:

- commuting subsystems: $[A_p, A_q] = 0 \ \forall p, q \in \mathcal{P}$

$$\bigcap$$

- nilpotent Lie algebras (suff. high-order Lie brackets are 0)

$$\bigcap \qquad \text{e.g. } [A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0$$

- solvable Lie algebras (triangular up to coord. transf.)

$$\bigcap$$

- solvable + compact (purely imaginary eigenvalues)

Quadratic common Lyapunov function exists in all these cases

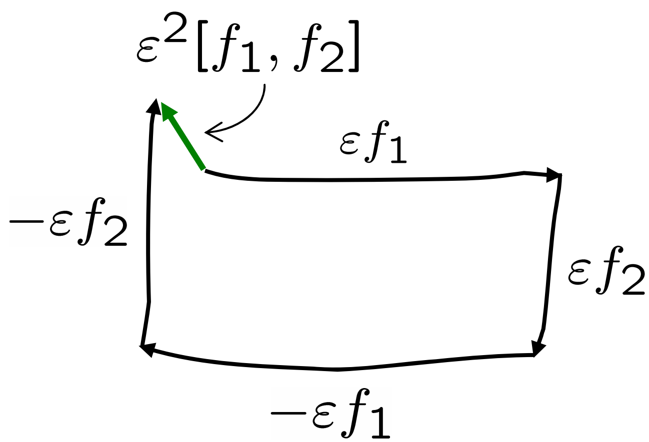
No further extension based on Lie algebra only [Agrachev–L '01]

SWITCHED NONLINEAR SYSTEMS

Lie bracket of nonlinear vector fields:

$$[f_1, f_2] := \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2$$

Reduces to earlier notion for linear vector fields (modulo the sign)



SWITCHED NONLINEAR SYSTEMS

- Commuting systems

$$[f_p, f_q] = 0 \;\Rightarrow\; \text{GUAS}$$

Can prove by trajectory analysis [Mancilla-Aguilar '00]
or common Lyapunov function [Shim et al. '98, Vu–L '05]

- Linearization (Lyapunov's indirect method)

$$A_p = \frac{\partial f_p}{\partial x}(0), \;\; p \in \mathcal{P}$$

- Global results beyond commuting case – ?

[Unsolved Problems in Math. Systems & Control Theory '04]

SPECIAL CASE

f_1, f_2 globally asymptotically stable

$$\boxed{[f_1,[f_1,f_2]] = [f_2,[f_1,f_2]] = 0}$$

Want to show: $\dot{x} = f_\sigma(x), \ \sigma \in \{1,2\}$ is GUAS

Will show: differential inclusion

$$\boxed{\dot{x} \in \mathrm{co}\{f_1(x), f_2(x)\}}$$

is GAS

OPTIMAL CONTROL APPROACH

Associated control system:

$$\boxed{\dot{x} = f(x) + g(x)u}$$

where $f := f_1, \quad g := f_2 - f_1, \quad u \in [0,1]$

(original switched system $\leftrightarrow u \in \{0,1\}$)

Worst-case control law [Pyatnitskiy, Rapoport, Boscain, Margaliot]:

fix x_0 and small enough t_f

$$\boxed{|x(t_f)|^2 \to \max_u}$$

MAXIMUM PRINCIPLE

$$H(x,u,\lambda) = \lambda^T f(x) + \underbrace{\lambda^T g(x)}_{\varphi \text{ (along optimal trajectory)}} u$$

Optimal control:

$$u(t) = 0 \text{ if } \varphi(t) < 0, \;\; u(t) = 1 \text{ if } \varphi(t) > 0$$

$$\dot{\varphi} = \lambda^T [f,g], \quad \ddot{\varphi} = \lambda^T [f,[f,g]] + \lambda^T [g,[f,g]]u = 0$$

$$\Downarrow$$

φ is linear in t

$\Downarrow$ (unless $\varphi \equiv 0$)

at most 1 switch

$$\Downarrow$$

GAS

GENERAL CASE

$$\dot{x} = f(x) + \sum_{k=1}^{m} g_k(x) u_k$$

$$\varphi_{ij} := \lambda^T (g_i(x) - g_j(x))$$

Want: φ_{ij} polynomial of degree $< r$

$\Downarrow$ (proof – by induction on m)

bang-bang with $(r+1)^m - 1$ switches

$\Downarrow$

GAS

[Margaliot–L '06, Sharon–Margaliot '07]

REMARKS on LIE-ALGEBRAIC CRITERIA

- Checkable conditions

- In terms of the original data

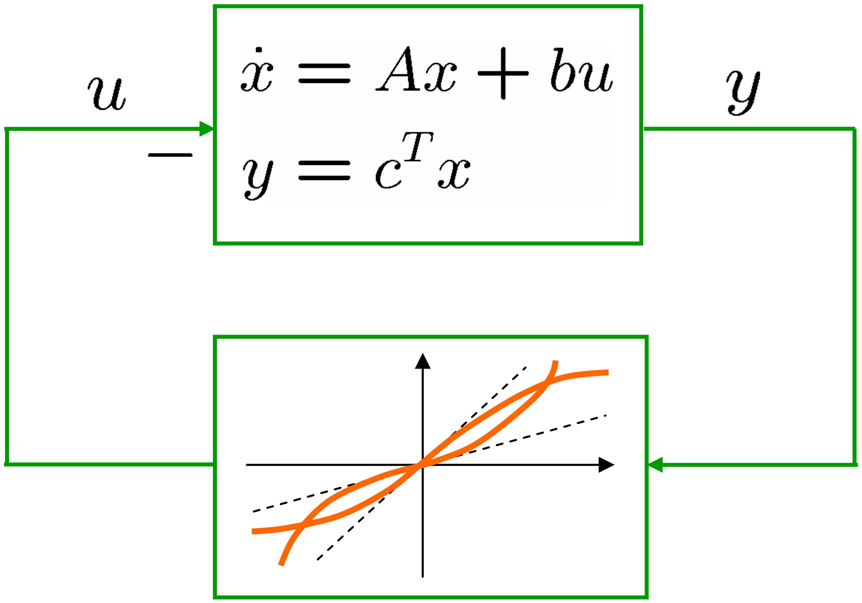
- Independent of representation

- Not robust to small perturbations

In any neighborhood of any pair of $n \times n$ matrices there exists a pair of matrices generating the entire Lie algebra $gl(n, \mathbb{R})$ [Agrachev–L '01]

How to measure closeness to a "nice" Lie algebra?

FEEDBACK SYSTEMS: ABSOLUTE STABILITY



(A, b) controllable

$$g(s) = c^T(sI - A)^{-1}b$$

$$u = -\varphi_p(y)$$

$$k_1 y^2 \le y\varphi_p(y) \le k_2 y^2 \ \forall p$$

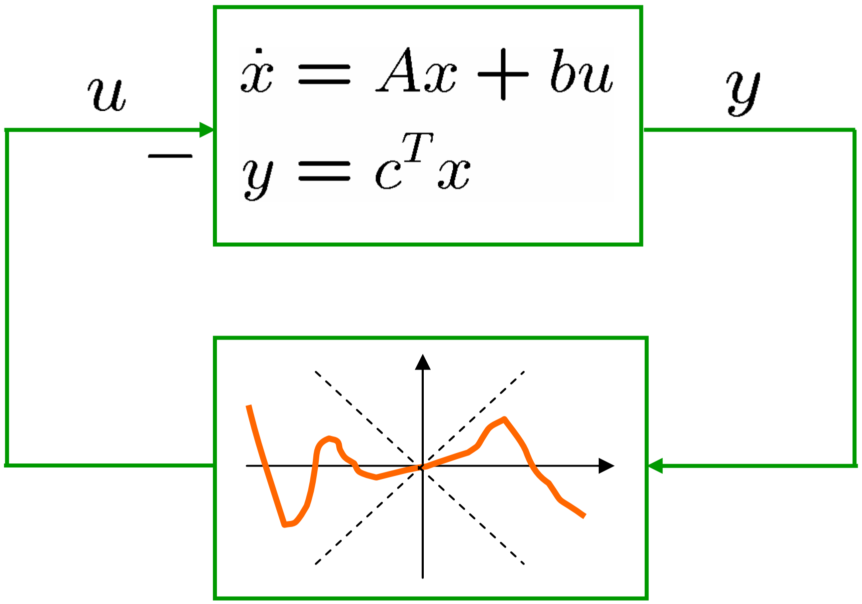
Circle criterion: $\exists$ quadratic common Lyapunov function $\Leftrightarrow$

$h(s) = \frac{1+k_2 g(s)}{1+k_1 g(s)}$ is strictly positive real (SPR): $Re\, h(i\omega) > 0$

For $k_1 = 0, k_2 = \infty$ this reduces to $g(s)$ SPR (passivity)

Popov criterion not suitable: V depends on φ_p

FEEDBACK SYSTEMS: SMALL-GAIN THEOREM



(A, b) controllable

$$g(s) = c^T(sI - A)^{-1}b$$

$$u = -\varphi_p(y)$$

$$|\varphi_p(y)| \le |y| \ \forall p$$

$$(k_1 = -1, k_2 = 1)$$

Small-gain theorem:

$\exists$ quadratic common Lyapunov function

$$\Updownarrow$$

$$\|g\|_\infty = \max_\omega |g(i\omega)| < 1$$

OBSERVABILITY and ASYMPTOTIC STABILITY

Barbashin-Krasovskii-LaSalle theorem:

$\dot{x} = f(x)$ is GAS if $\exists\, V$ s.t.

- $\dot{V} := \frac{\partial V}{\partial x} f(x) \le 0 \quad \forall x$ (weak Lyapunov function)
- $\dot{V}$ is not identically zero along any nonzero solution (observability with respect to $\dot{V}$)

Example:

$$\dot{x} = Ax, \quad V(x) = x^T P x$$

$$\left.\begin{array}{l} A^T P + PA \le -C^T C \\ (A, C) \text{ observable} \end{array}\right\} \Rightarrow \text{GAS}$$

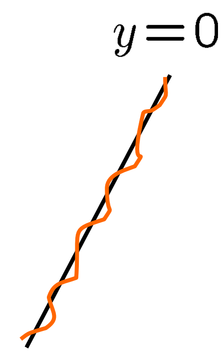
SWITCHED LINEAR SYSTEMS [Hespanha '04]

$$\boxed{\dot{x} = A_\sigma x}$$

Theorem (common weak Lyapunov function):

Switched linear system is GAS if

- $\exists\, P > 0$ s.t. $A_p^T P + P A_p \le -C_p^T C_p \quad \forall p$
- (A_p, C_p) observable for each p
- $\exists$ infinitely many switching intervals of length $\ge \tau$



To handle nonlinear switched systems and non-quadratic weak Lyapunov functions, need a suitable nonlinear observability notion

SWITCHED NONLINEAR SYSTEMS

$$\boxed{\dot{x} = f_\sigma(x)}$$

Theorem (common weak Lyapunov function):

Switched system is GAS if

- $\exists\, V$ s.t. $\frac{\partial V}{\partial x} f_p(x) \le -W_p(x) \le 0 \quad \forall\, x, \ \forall\, p$
- $\exists$ infinitely many switching intervals of length $\ge \tau$
- Each system $\dot{x} = f_p(x), \quad y = W_p(x)$

  is norm-observable:

$$\exists\, \gamma(\cdot) : \ |x(0)| \le \gamma\left(\|y\|_{[0,\tau]}\right)$$

[Hespanha–L–Sontag–Angeli '05]