Computer Aided Control System Design

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Information and Questions

- Lectures and General Comments :

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- Please consult

  $\boxed{http://toledo.kuleuven.ac.be}$

  for

  - e–mail addresses of teaching assistants
  - downloads of course material
  - up–to–date information + FAQ on
    - exercise sessions
    - practical sessions
    - course notes

Another (not regularly updated ☹) webpage, which contains general information in English about the course, can be found at

http://www.esat.kuleuven.ac.be/~eneman/cacsd/cacsd.html

Exercise and practical sessions

Exercise sessions :

- session 1 : State space modeling.
- session 2 : Control design.
- session 3 & 4 : Two practical control design problems.

Practical sessions (choose 1 out of 5) :

1. Linear inverted pendulum.
2. Rotary inverted pendulum.
3. Seesaw.
4. Linear flexible joint.
5. Rotary flexible joint.

Examination Requirements

	Contents	Points
1.	Exercise sessions • 4 exercise sessions. • Input=prepare them *a priori* ! • Output=report + simulink schemes. • Deadline: try to reach an agreement with Prof. De Moor	30%
2.	Practical session • Choose 1 out of 5 practical sessions. • Overall time required: $\leq$ 5 afternoons. • Output=report. • Deadline: try to reach an agreement with Prof. De Moor	20%
3.	Final examination : oral defense • Input: 2 reports + simulink schemes. • Transfer Function: Jury(BDM,OB,KE,IG,DVD, ...). • Output: Jury's Evaluation.	50%
	Total	100%

What you should be able to do when you have *successfully* completed this course . . .

- Plant modeling in state space.
- Defining control specifications.
- Control system analysis.
  - Controllability/observability.
  - Stability.
  - Performance analysis.
  - Elementary sensitivity and robustness analysis.
- Controller design.
  - State feedback control design.
    Pole-placement, LQR.
  - Observer design.
    Pole-placement, Kalman filter.
  - Compensator design.
    Separation principle, LQG.
  - Reference introduction and tracking.
  - Basics of MPC.

List of book-references

- Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, *Feedback Control of Dynamic Systems*, Third Edition, Addison-Wesley, 1994.
- Gene F. Franklin, J. David Powell and M.L. Workman, *Digital Control of Dynamic Systems*, 2nd ed., Addison-Wesley, 1990.
- R. Dorf, Modern Control Systems, 7th ed., Addison-Wesley, 1995.
- M. Green and D.J.N. Limebeer, *Linear Robust Control*, Prentice Hall, 1995.
- K.J. Åström and B. Wittenmark, *Adaptive Control*, Second Edition, Addison-Wesley, 1995
- A. Saberi, B.M. Chen and P. Sannuti, *Loop Transfer Recovery: Analysis and Design*, Springer-Verlag, 1993
- T. Chen and B. Francis, *Optimal Sampled-Data Control Systems*, Springer, 1995
- R.J. Vaccaro, *Digital Control, A State Space Approach*, McGraw-Hill, 1995
- L. Ljung and T. Glad, *Modelling of Dynamic Systems*, Prentice-Hall, 1994
- R.N. Bateson, *Introduction to Control System Technology*, Fourth Edition, Merril MacMillan, 1993
- N.S. Nise, *Control System Engineering*, Second Edition, Benjamin-Cummings, 1995
- K. Zhou, J.C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, 1995
- . . . don't hesitate to consult the library (LIBIS) . . .

http://www.libis.kuleuven.ac.be/

List of journals

- IEEE Transactions on Automatic Control
- International Journal on Control
- SIAM Journal on Control and Optimization
- Automatica
- European Control Journal
- Journal A
- IEEE Transactions on Control Systems Technology
- ...

List of Conferences

- Benelux Meeting on Systems and Control
- European Control Conference
- American Control Conference
- MTNS: Mathematical Theory of Networks and Systems
- IFAC: World Congress
- IEEE Conference on Decision and Control
- ...

List of organisations

- IEEE (Institute of Electrical and Electronics Engineers), Control Systems Society
- IFAC (International Federation of Automatic Control)
- BIRA (Belgisch Instituut voor Regeltechniek en Automatisatie)
- Belgian Graduate School on Systems and Control
- ISA (Instrumentation Society of America)
- SIAM (Society for Industrial and Applied Mathematics)
- ...

Software references

- Matlab
  - Matlab
  - Simulink
  - Toolboxes :
    * Control System
    * System Identification (some of the algorithms were developed at ESAT, e.g. N4SID)
    * MMLE3 Identification
    * Mu-Analysis and Synthesis
    * Neural Network
    * Optimization
    * Robust Control
    * Signal Processing
    * Statistics
    * Symbolic Math
    * Extended Symbolic Math
- RaPID (developed at ESAT)

- Xmath
  - Control Design Module
  - Interactive Control Design Module
  - Interactive System Identification Module (developed at ESAT)
  - Optimization Module
  - Robust Control Module
  - Model Reduction Module
  - Signal Analysis Module
  - X_μ Module
  - GUI Module
  - SystemBuild
    * BlockLibrary
    * AutoCode (automatic generation of C-code)
    * DocumentIt (automatic documentation of block diagrams)

Control Companies

- Fisher-Rozemount
- Honeywell
- ABB
- Setpoint-IPCOS
- Eurotherm
- Cambridge-Control
- ISI
- Yokogawa
- Siemens
- ISMC (KUL spin-off)
- …and many others …

Applications

Literally: everywhere, all sectors, all industries, economy, process industry, mechatronic design, ….

Chapter 1

Introduction

Control system engineering is concerned with modifying the behavior of dynamical systems to achieve certain pre-specified goals.

Control design loop:

- Modeling:

  Physical plants $\Rightarrow$ Mathematical models.

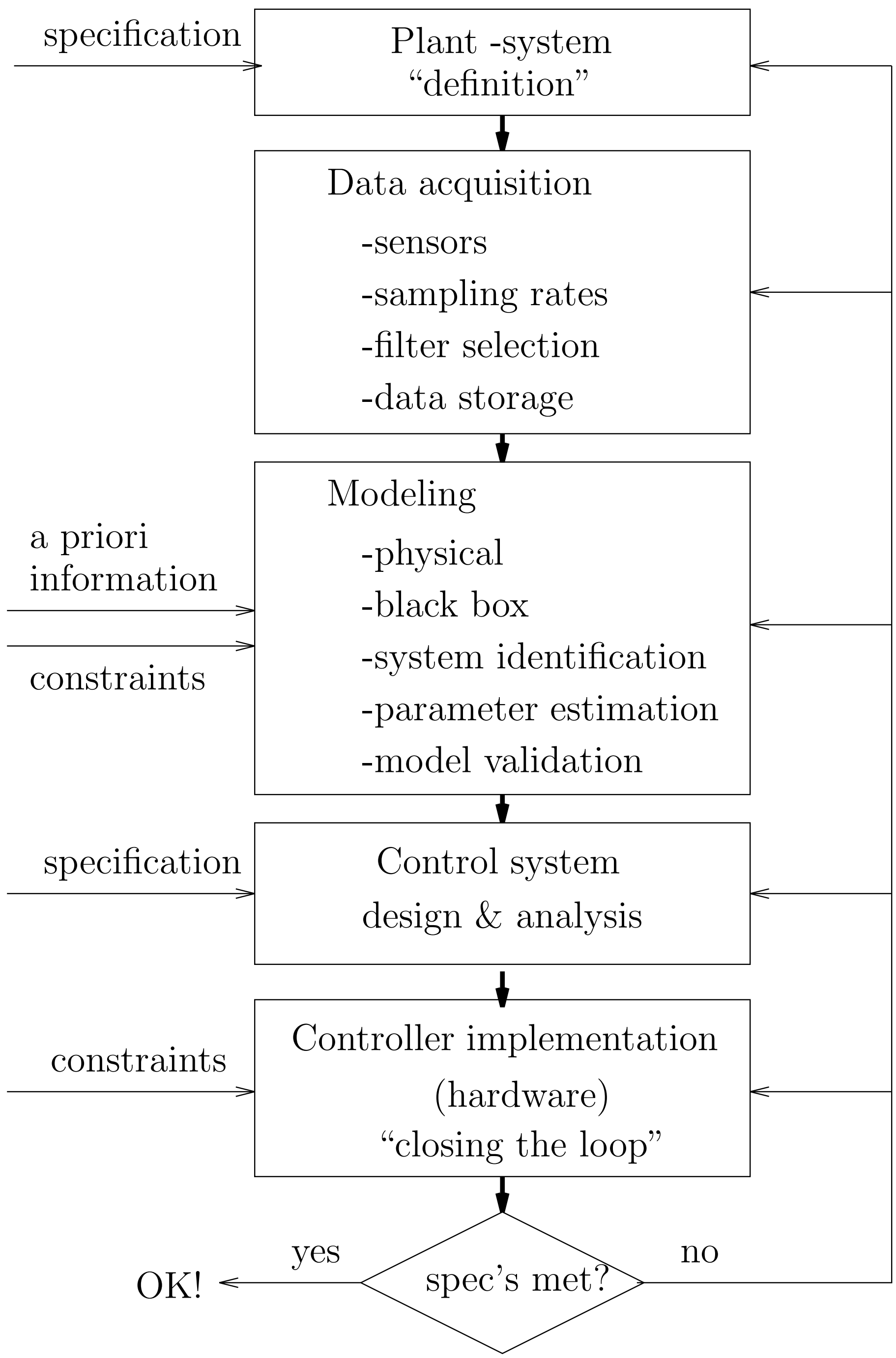
- Analysis:

  Given performance specifications, check whether the specifications are satisfied; Robustness issues.

- Design:

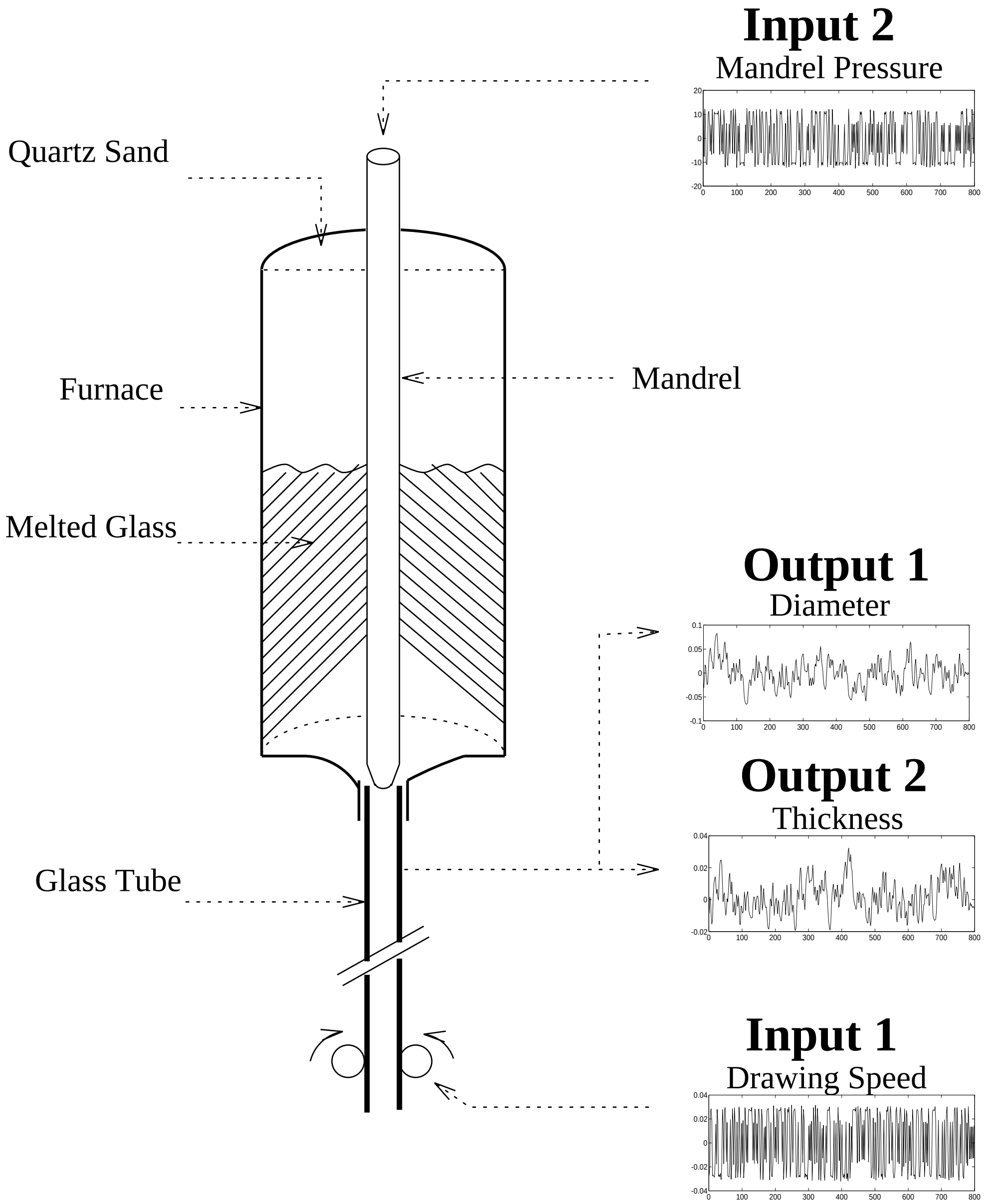
  Design controllers such that the closed-loop system satisfies the specifications.

Control system design



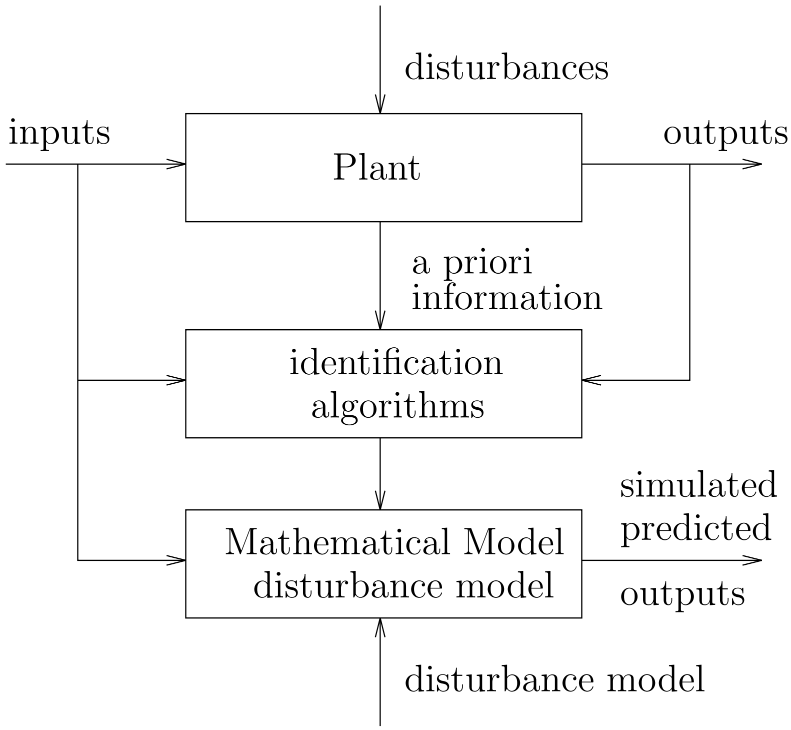
3 Motivating Examples

Philips Glass Tube Manufacturing Process

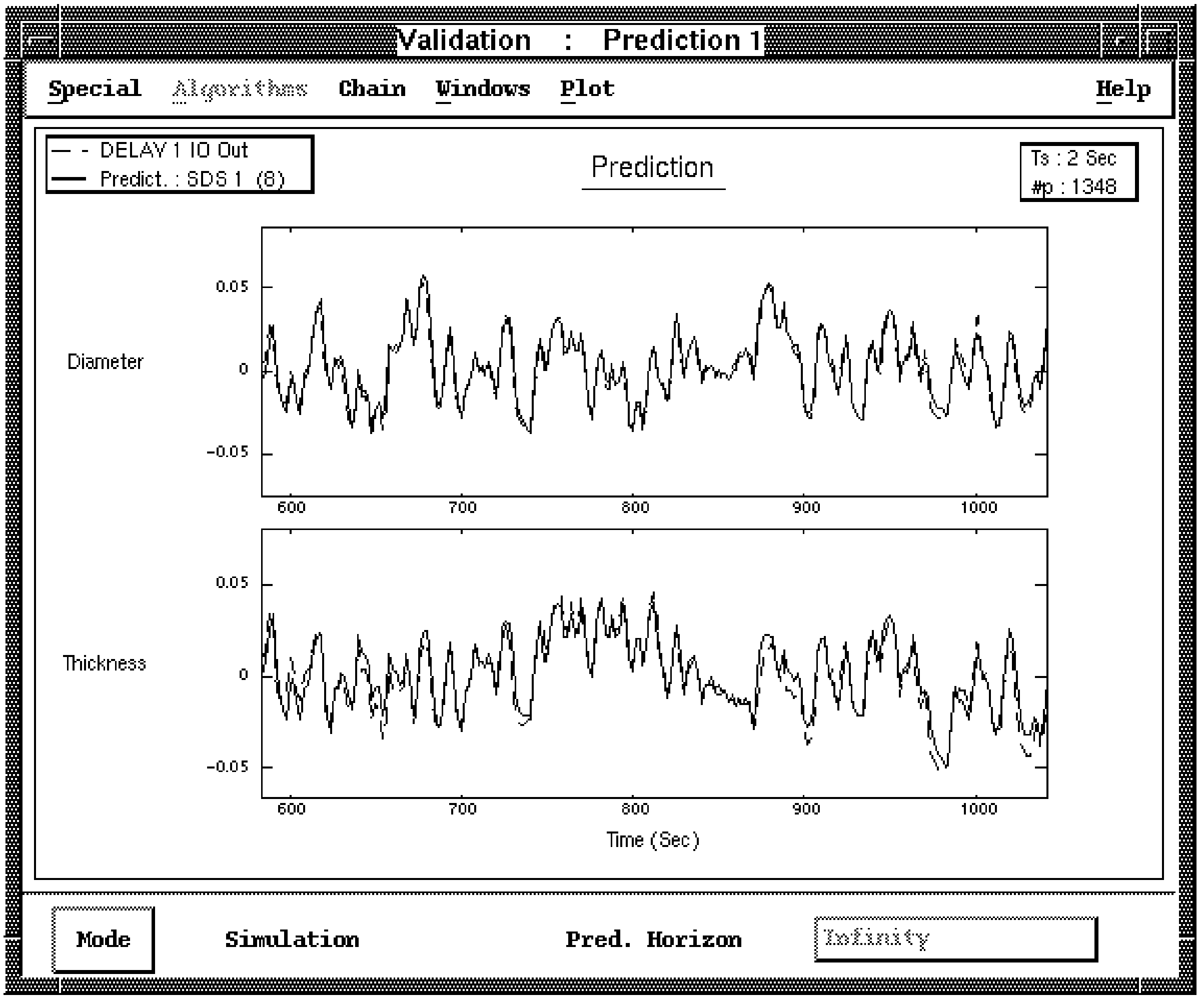


Control design specifications: design a controller such that the wall thickness and diameter are as constant as possible.

Modeling via system identification:

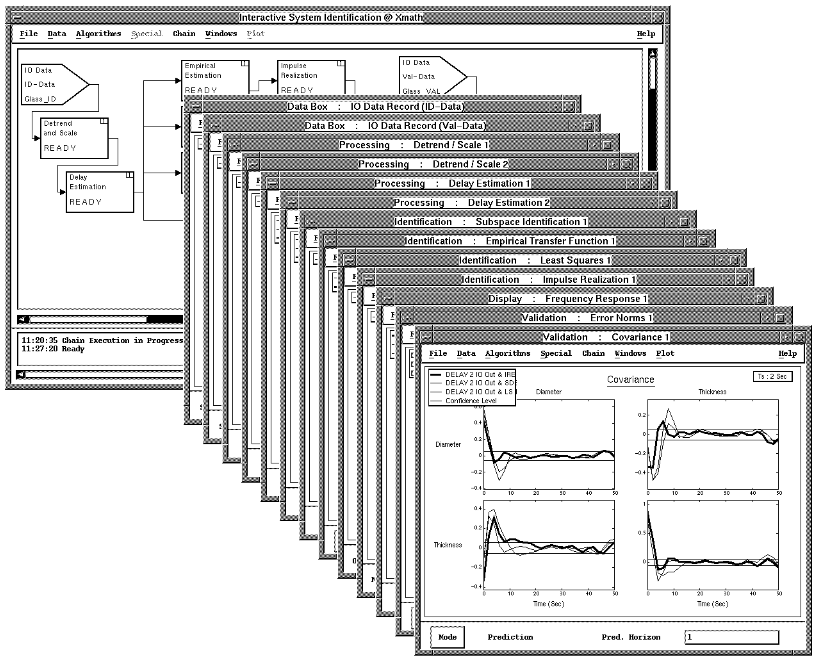


Plot with simulation and validation results:



System identification toolbox

- Xmath GUI identification toolbox.



- Matlab Toolbox of system identification.
- RaPID (some algorithms developed at ESAT/SISTA)
- ······

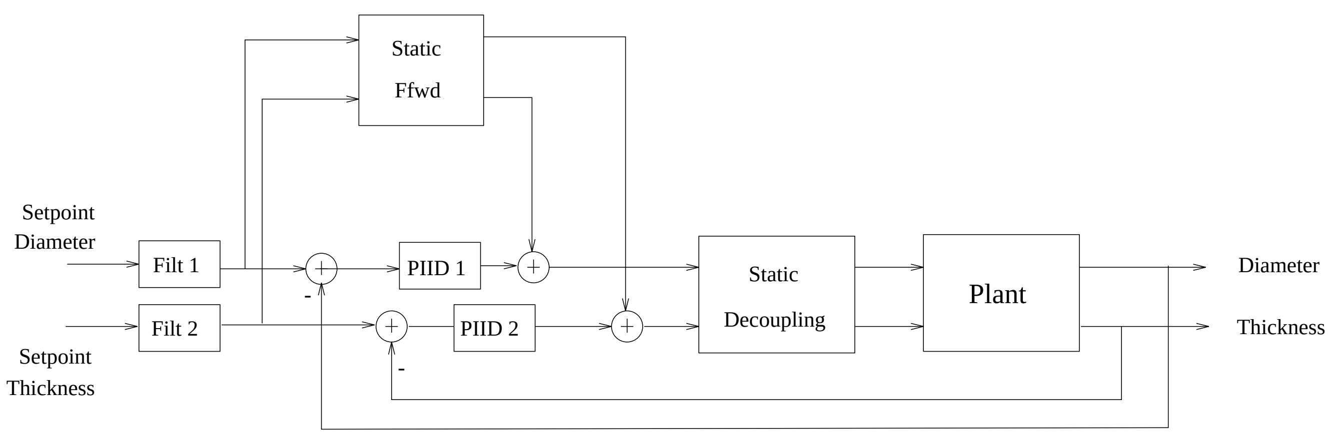
After system identification, a 9th order linear discrete-time model with two inputs and two outputs was obtained:

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + w_k, \\
y_k &= Cx_k + Du_k + v_k.
\end{aligned}
$$

$$
\mathcal{E}\left\{ \begin{bmatrix} w_k \\ v_k \end{bmatrix} \begin{bmatrix} w_l^T & v_l^T \end{bmatrix} \right\} = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta_{kl}
$$

where A, B, C, D are matrices, u_k is the control input and w_k and v_k are process noise and measurement noise respectively.

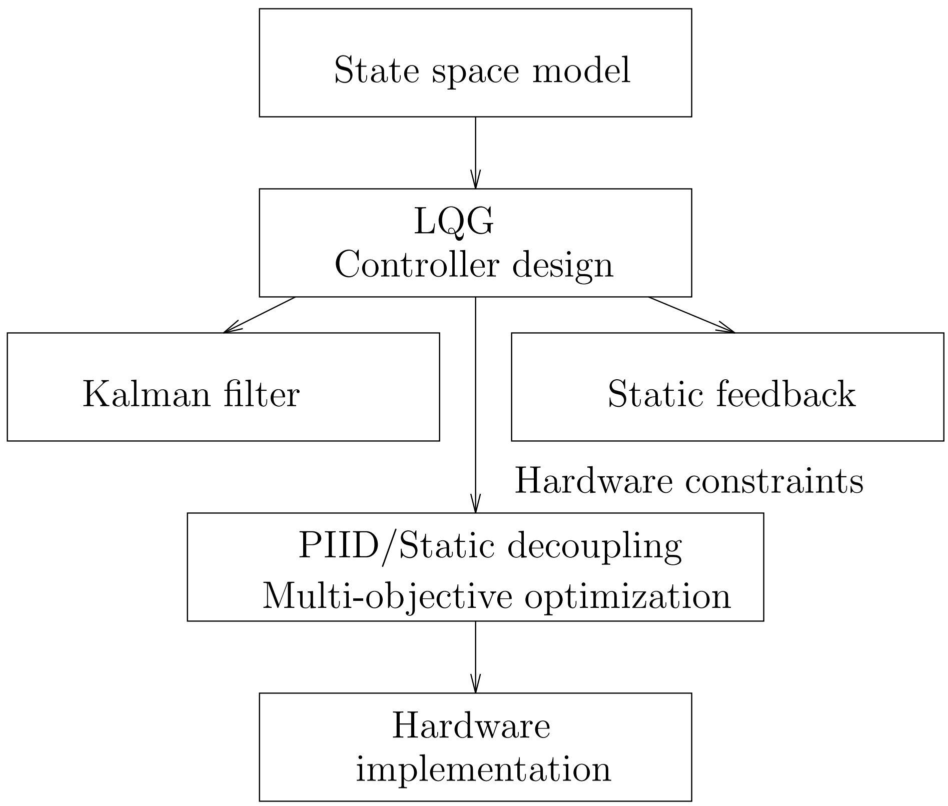
Control design results:



$$
\text{Controller} = \begin{cases} \text{two feedforward filters} \\ \text{a static feedforward controller} \\ \text{a static decoupling controller} \\ \text{two PIID controllers} \end{cases}
$$

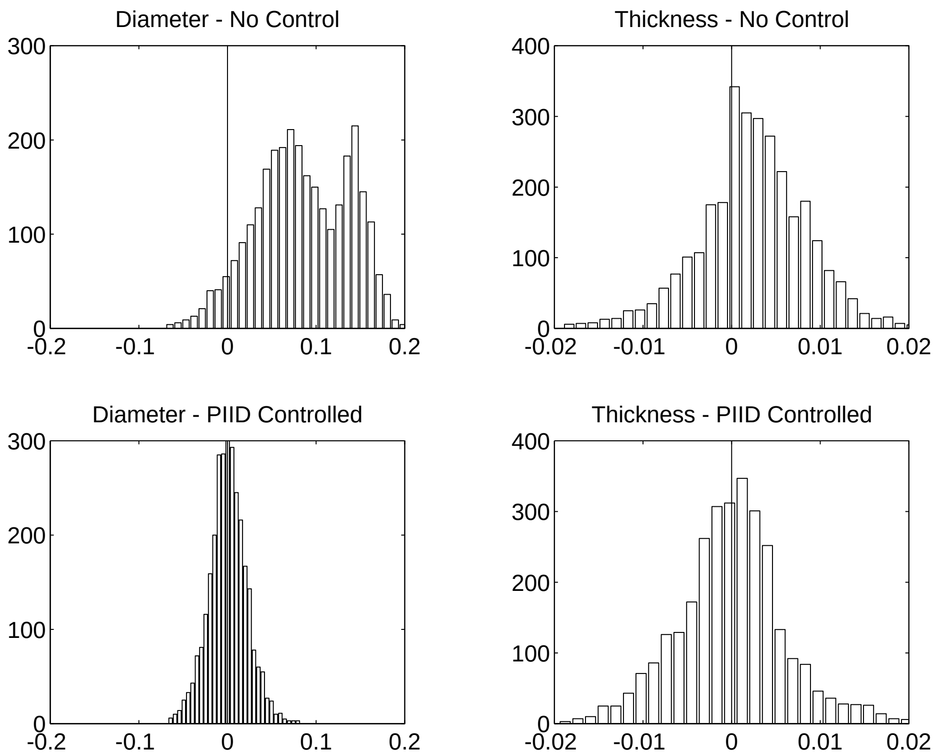
The two PIIDs control the decoupled loops. Parameter tuning follows from a multi-objective optimization algorithm.

The control design loop for this application :

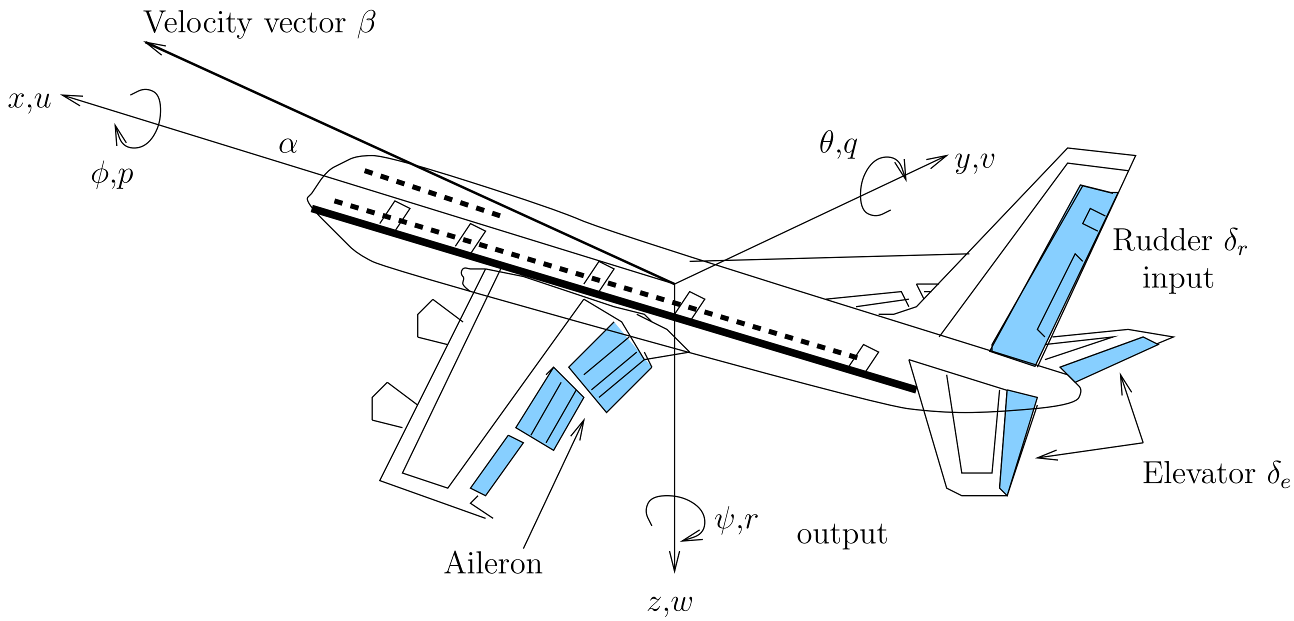


Quality improvement via control:

Histogram of the measured diameter and thickness



Boeing 747 aircraft control:



x, y, z	$=$	position coordinates	ϕ	$=$	roll angle
u, v, w	$=$	velocity coordinates	θ	$=$	pitch angle
p	$=$	roll rate	ψ	$=$	yaw angle
q	$=$	pitch rate	β	$=$	slide-slip angle
r	$=$	yaw rate	α	$=$	angle of attack

Control specifications for lateral control:

- Stability
- Damping ratio $\simeq 0.5$

Modeling:

A lateral perturbation model in horizontal flight for a nominal forward speed of 774 ft/s at 40.000 ft can be derived based on physical laws. A 4th order, linear, single-input/single-output system was obtained :

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned}$$

where

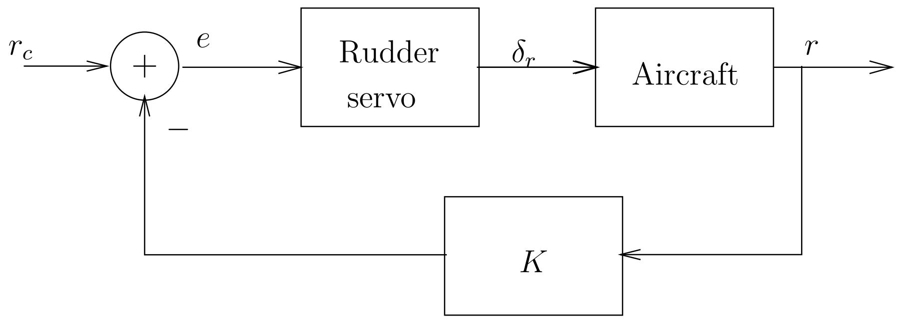
$$A = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0 \\ -3.0500 & 0.3880 & -0.4650 & 0 \\ 0 & 0.0805 & 1.0000 & 0 \end{bmatrix},$$

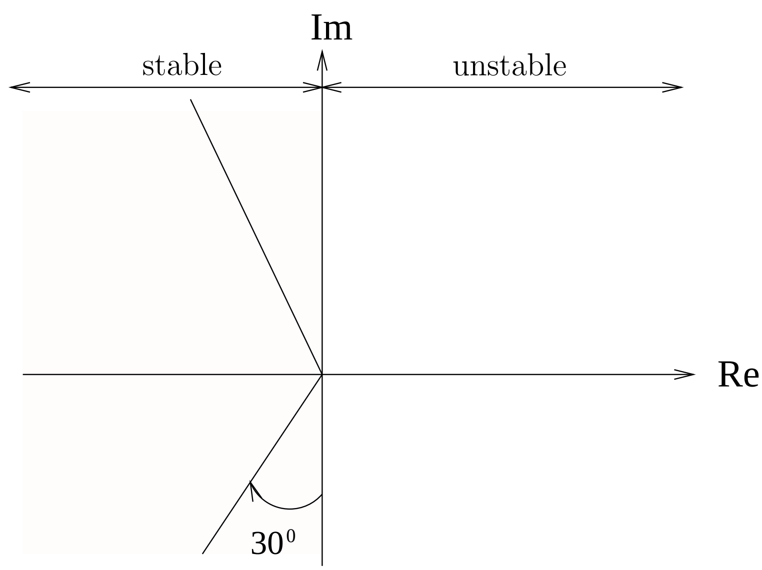
$$B = \begin{bmatrix} 0.0073 \\ -0.4750 \\ 0.1530 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \; D = 0,$$

and $x = \begin{bmatrix} \beta & r & p & \phi \end{bmatrix}^T, \; u = \delta_r, \; y = r.$

Control design:

The controller is a proportional feedback from yaw rate to rudder, designed based on the root-locus method.



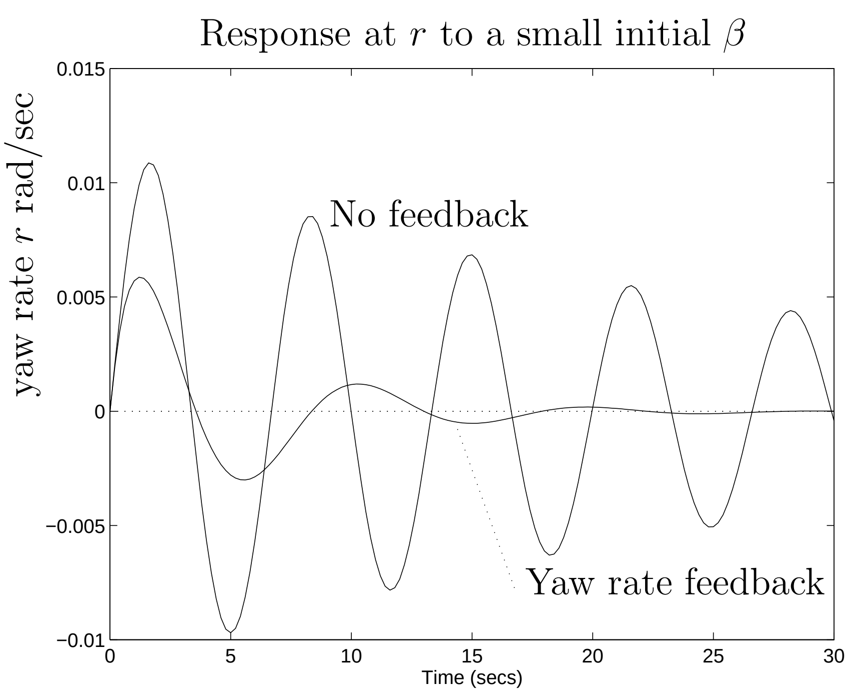


damping ratio = 0.5

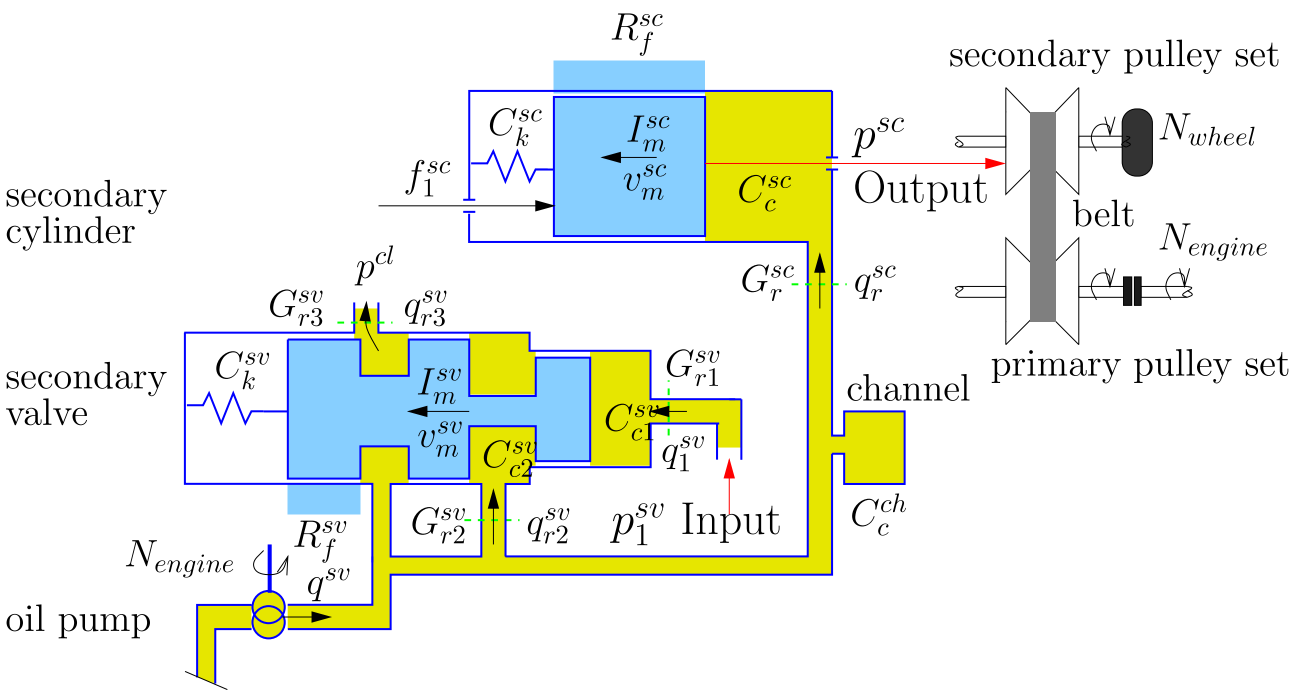
Analysis:

- For good pilot handling, the damping ratio of the system should be around 0.5. The open loop system, without control, has a damping ratio of 0.03, far less than 0.5. With the controller, the damping ratio is 0.35, near to 0.5 : a big improvement !

- Consider the initial-condition response for initial slide-slip angle $\beta_0 = 1°$:



Automobile control: CVT control



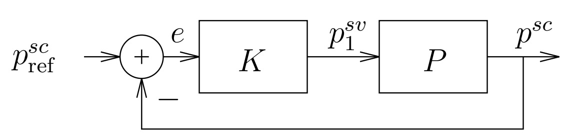
$$
\begin{aligned}
C_k &= \text{spring constant} \\
I_m &= \text{inertia} \\
R &= \text{friction resistor} \\
G_r &= \text{hydraulic impedance} \\
p &= \text{pressure} \\
q &= \text{oil flow} \\
v_m &= \text{velocity of a mass} \\
f &= \text{force}
\end{aligned}
$$

Control specifications (tracking problem):

- stability,
- no overshoot,
- steady state error on the step response $\leq 2\%$,
- rise time of the step response is $\leq$ 50ms.

Modeling:
A 6th order single-input/single-output nonlinear system was obtained by physical (hydraulic and mechanical) modeling.

Controller design:
The control problem is basically a tracking problem:



A PID controller is designed using optimal and robust control methods.

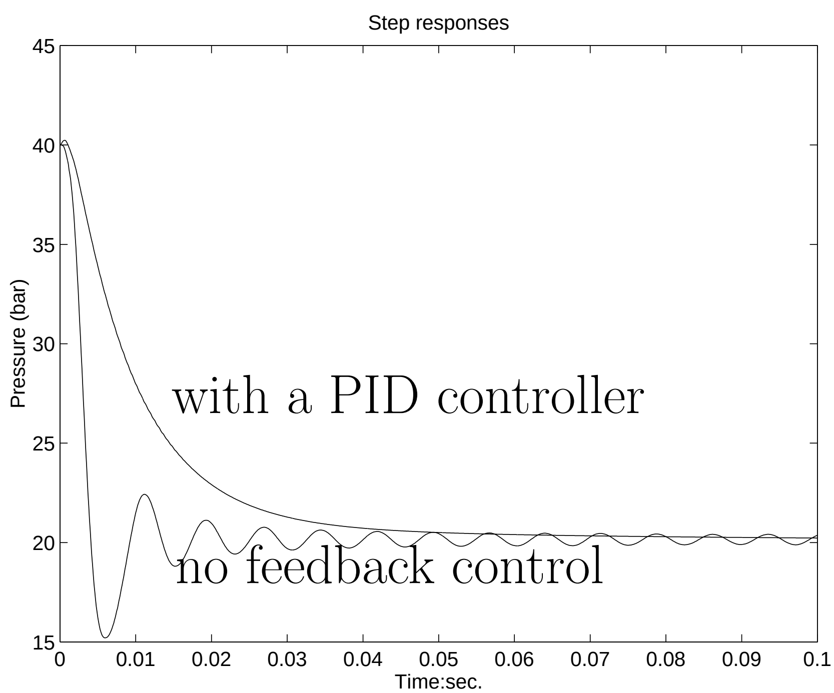
Analysis:

It is required that there is no overshoot and that the rise time and steady state error of the step time response should be less than 50 ms and 2% respectively.

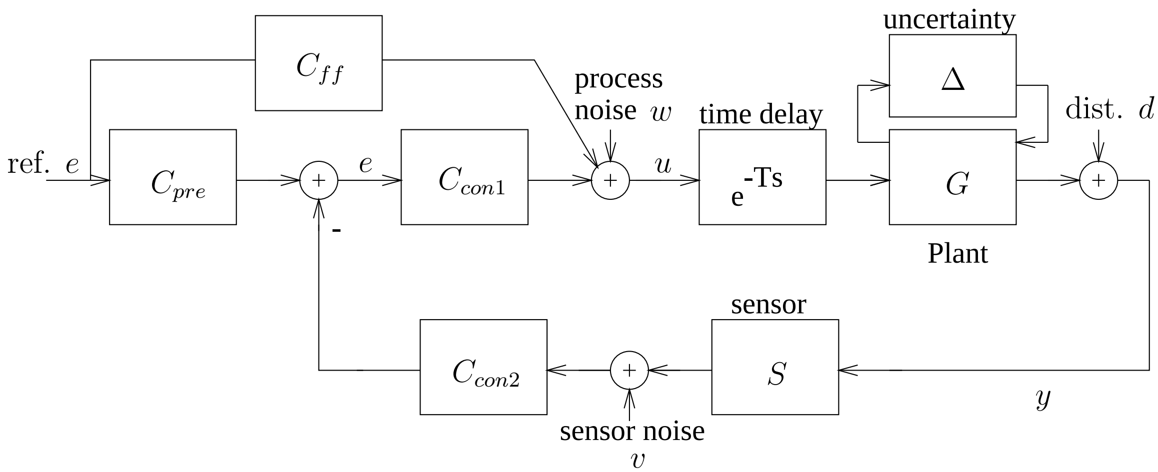
Without feedback control: overshoot 20% with oscillation!

With a PID controller: no overshoot, rise time is less than 30ms, no steady-state error.

Step (from 40 bar to 20 bar) responses:



General Control Configuration



Systems and Models

- Linear - Nonlinear systems

  A system L is linear

  $$\Leftrightarrow$$

  if input u_1 yields output $L(u_1)$ and input u_2 yields output $L(u_2)$, then :

  $$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2),\ c_1, c_2 \in \mathbb{R}$$

  In this course, we only consider linear systems.

- Lumped - distributed parameter systems

  Many physical phenomena are described mathematically by partial differential equations (PDEs). Such systems are called distributed parameter systems. Lumped parameter systems are systems which can be described by ordinary differential equations (ODEs).
  Example :

  - Diffusion equation $\rightarrow$ discretize in space
  - Heat equation

  In this course, we only consider lumped parameter systems.

- Time invariant - time varying systems

  A system is time varying if one or more of the parameters of the system may vary as a function of time, otherwise, it is time invariant.

  Example: Consider a system

  $$M\frac{d^2y}{dt} + F\frac{dy}{dt} + Ky = u(t)$$

  If all parameters (M, F, K) are constant, it is time invariant. Otherwise, if any of the parameters is a function of time, it is time varying.

  In this course, we only consider time-invariant systems.

- Continuous time - discrete time systems

  A continuous time system is a system which describes the relationship between time continuous signals, and can be described by differential equations. A discrete system is a system which describes the relationship between discrete signals, and can be described by difference equations.

  In this course, we'll consider systems both in continuous and discrete time.

- Causal - a-causal systems

  A system is called causal, if the output to time T depends only on the input up to time T, for every T, otherwise it is called a-causal.

  In this course, we only consider causal systems.

<u>This course</u>:

Linear		
Lumped parameters	systems in	discrete time
Time-invariant		continuous time
Causal		

Realistic? In many real cases, YES!

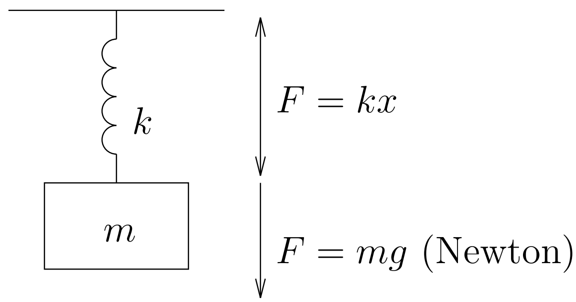
- Industrial processes around an equilibrium point: glass oven, aircraft and CVT.
- Linearization of a nonlinear system $\rightarrow$ linear system.
- ...

System Modeling

System models are developed in two ways mainly:

- Physical modeling consists of applying various laws of physics, chemistry, thermodynamics, ... , to derive ODE or PDE models. It is modeling from "First Principles".

  Example:

$$m\frac{d^2x}{dt^2} = mg - kx.$$

- Empirical modeling or identification consists of developing models from observed or collected data.

  Experiments:
  - System identification, *e.g.*: glass oven
  - Parameter estimation

Chapter 2

State Space Models

General format :

(valid for any nonlinear causal system)

$$\text{CT}: \begin{array}{l} \dot{x} = f(x, u), \\ y = h(x, u). \end{array} \qquad \text{DT}: \begin{array}{l} x_{k+1} = f(x_k, u_k), \\ y_k = h(x_k, u_k). \end{array}$$

where

$$\begin{aligned} x &= \text{ the state of the system, } n \times 1\text{-vector} \\ u &= \text{ the input of the system, } m \times 1\text{-vector} \\ y &= \text{ the output of the system, } p \times 1\text{-vector} \\ f &= \text{ state equation vector function} \\ h &= \text{ output equation vector function} \\ n &= \text{ number of states} \Rightarrow n\text{-th order system} \\ m &= \text{ number of inputs} \\ p &= \text{ number of outputs} \end{aligned}$$

Single-Input Single-Output system (SISO) : $m = p = 1$,
Multi-Input Multi-Output system (MIMO) : $m, p > 1$,
Multi-Input Single-Output system (MISO) : $m > 1,\ p = 1$,
Single-Input Multi-Output system (SIMO) : $m = 1,\ p > 1$.

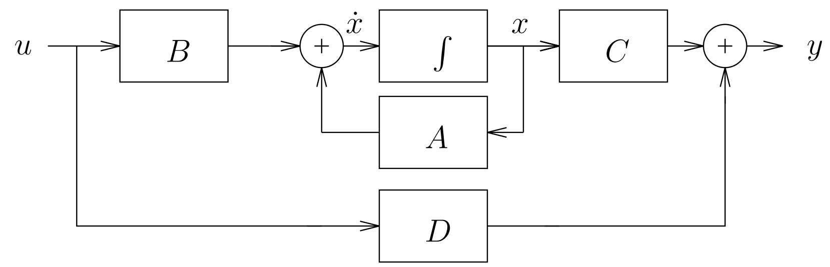
Linear Time Invariant (LTI) systems

(only valid for linear causal systems)

Continuous-time system:

$$\dot{x} = Ax + Bu$$

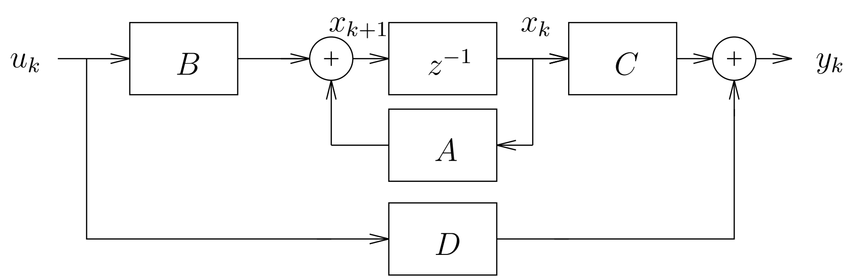
$$y = Cx + Du$$



Discrete-time system:

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$



A: $n \times n$ system matrix

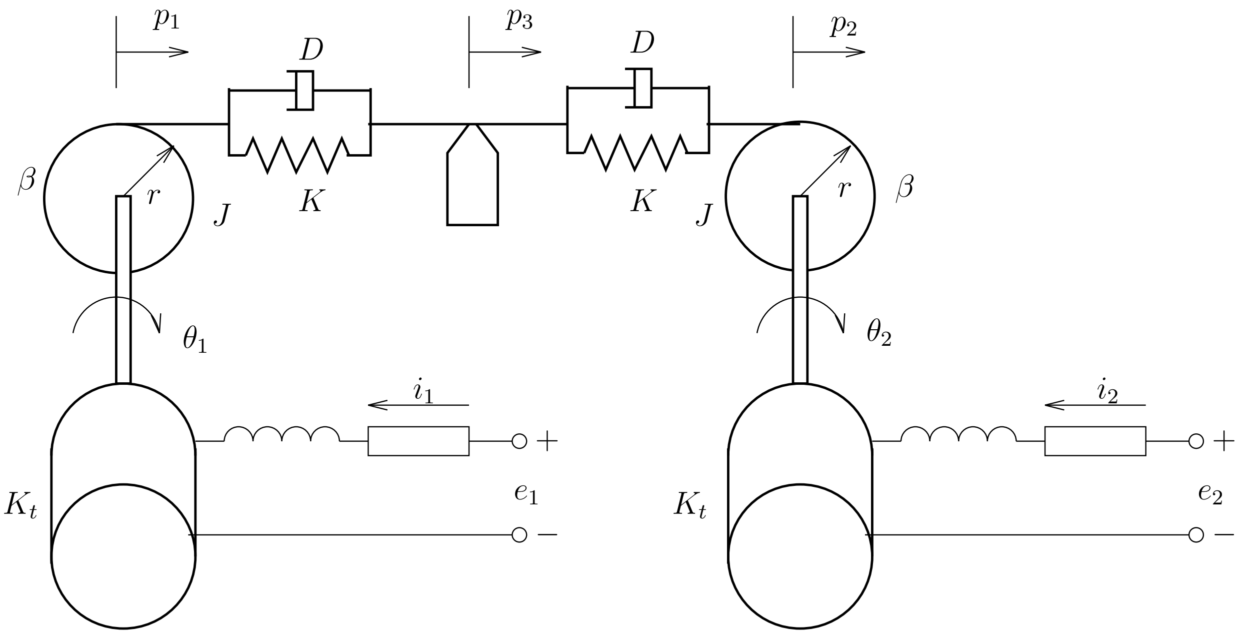
B: $n \times m$ input matrix

C: $p \times n$ output matrix

D: $p \times m$ direct transmission matrix

An example of linear state space modeling:

Example: Tape drive control - state space modeling
Process description:



The drive motor on each end of the tape is independently controllable by voltage resources e_1 and e_2. The tape is modeled as a linear spring with a small amount of viscous damping near to the static equilibrium with tape tension 6N. The variables are defined as deviations from this equilibrium point.

The equations of motion of the system are:

$$
\begin{aligned}
\text{capstan 1:} \quad & J\frac{d\omega_1}{dt} = Tr - \beta\omega_1 + K_t i_1, \\
\text{speed of } x_1\text{:} \quad & \dot{p}_1 = r\omega_1, \\
\text{motor 1:} \quad & L\frac{di_1}{dt} = -Ri_1 - K_e\omega_1 + e_1, \\
\text{capstan 2:} \quad & J\frac{d\omega_2}{dt} = -Tr - \beta\omega + K_t i_2, \\
\text{speed of } x_1\text{:} \quad & \dot{p}_2 = r\omega_2, \\
\text{motor 2:} \quad & L\frac{di_2}{dt} = -Ri_2 - K_e\omega_2 + e_2, \\
\text{Tension of tape:} \quad & T = \frac{K}{2}(p_2 - p_1) + \frac{D}{2}(\dot{p}_2 - \dot{p}_1), \\
\text{Position of the head:} \quad & p_3 = \frac{p_1 + p_2}{2}.
\end{aligned}
$$

Description of variables and constants:

$$
\begin{aligned}
D &= \text{ damping in the tape-stretch motion} \\
&= 20 \text{ N/m·sec}, \\
e_{1,2} &= \text{ applied voltage, V}, \\
i_{1,2} &= \text{ current into the capstan motor}, \\
J &= \text{ inertia of the wheel and the motor} \\
&= 4\times10^{-5}\text{kg·m}^2, \\
\beta &= \text{capstan rotational friction, } 400\text{kg·m}^2/\text{sec}, \\
K &= \text{ spring constant of the tape, } 4\times10^4\text{N/m}, \\
K_e &= \text{ electrical constant of the motors} = 0.03\text{V·sec}, \\
K_t &= \text{ torque constant of the motors} = 0.03\text{N·m/A}, \\
L &= \text{ armature inductance} = 10^{-3}\text{ H}, \\
R &= \text{ armature resistance} = 1\Omega, \\
r &= \text{ radius of the take-up wheels, 0.02m}, \\
T &= \text{ tape tension at the read/write head, } N, \\
p_{1,2,3} &= \text{ tape position at capstan 1,2 and the head}, \\
\dot{p}_{1,2,3} &= \text{ tape velocity at capstan 1,2 and the head}, \\
\theta_{1,2} &= \text{ angular displacement of capstan 1,2}, \\
\omega_{1,2} &= \text{ speed of drive wheels} = \dot{\theta}_{1,2}.
\end{aligned}
$$

With a time scaling factor of 10^3 and a position scaling factor 10^{-5} for numerical reasons, the state equations become:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned}$$

where

$$x = \begin{bmatrix} p_1 \\ \omega_1 \\ p_2 \\ \omega_2 \\ i_1 \\ i_2 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ -0.1 & -0.35 & 0.1 & 0.1 & 0.75 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0.1 & 0.1 & -0.1 & -0.35 & 0 & 0.75 \\ 0 & -0.03 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -0.03 & 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ -0.2 & -0.2 & 0.2 & 0.2 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; y = \begin{bmatrix} p_3 \\ T \end{bmatrix}, \; u = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

State-space model, transfer matrix and impulse response

Continuous-time system:

$$\begin{array}{l} \text{state-space} \\ \text{equations} \end{array} \left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array} \right. \quad \xrightarrow[x(0)=0]{\text{Laplace}} \quad \underbrace{G(s) = \frac{Y(s)}{U(s)} = D + C(sI - A)^{-1}B}_{\text{transfer matrix}}$$

$$\text{in practice : } D = 0$$

$$\Downarrow$$

$$\underbrace{G(t) = Ce^{At}B}_{\text{impulse response matrix}} \quad \overset{\text{Laplace}}{\longleftrightarrow} \quad \underbrace{G(s) = \sum_{i=1}^{\infty} CA^{i-1}Bs^{-i}}_{\text{transfer matrix}}$$

Discrete-time system:

$$\begin{array}{l} \text{state-space} \\ \text{equations} \end{array} \left\{ \begin{array}{r} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k + Du_k \end{array} \right. \quad \overset{z-\text{transf}}{\underset{x_0=0}{\longleftrightarrow}} \quad \underbrace{G(z) = \frac{Y(z)}{U(z)} = D + C(zI - A)^{-1}B}_{\text{transfer matrix}}$$

$$\Downarrow$$

$$\underbrace{G(k) = \left\{ \begin{array}{cl} D & : k = 0 \\ CA^{k-1}B & : k \geq 1 \end{array} \right.}_{\text{impulse response matrix}} \quad \overset{z-\text{transf}}{\longleftrightarrow} \quad \underbrace{G(z) = \sum_{i=1}^{\infty} CA^{i-1}Bz^{-i} + D}_{\text{transfer matrix}}$$

In case of a SISO system, $G(t)$ or $G(k)$ is the impulse response.
For MIMO $G(t)$, $G(k)$ are matrices containing the $m \times p$ impulse responses (one for every input-output pair).

Linearization of a nonlinear system about an equilibrium point

Consider a general nonlinear system in continuous time :

$$\frac{dx}{dt} = f(x, u)$$

$$y = h(x, u)$$

For small deviations about an equilibrium point (x_e, u_e, y_e) for which

$$f(x_e, u_e) = 0$$

$$y_e = h(x_e, u_e)$$

we define

$$x = x_e + \Delta x, \quad u = u_e + \Delta u, \quad y = y_e + \Delta y,$$

and obtain

$$\frac{dx}{dt} = \frac{d\Delta x}{dt} = f(x, u) = f(x_e + \Delta x, u_e + \Delta u)$$

and

$$y_e + \Delta y = h(x, u) = h(x_e + \Delta x, u_e + \Delta u).$$

By first order approximation we obtain a linear state space model from Δu to Δy :

$$\frac{d\Delta x}{dt} = f(x_e + \Delta x, u_e + \Delta u)$$

$$\Downarrow \; f(x_e, u_e) = 0$$

$$\frac{d\Delta x}{dt} \overset{(1)}{=} \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x_e,u_e}}_{n \times n} \Delta x + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{x_e,u_e}}_{n \times m} \Delta u$$

$$\begin{array}{cc} \Downarrow & \Downarrow \\ A & B \end{array}$$

and

$$y_e + \Delta y = h(x_e + \Delta x, u_e + \Delta u)$$

$$\Downarrow \; y_e = h(x_e, u_e)$$

$$\Delta y \overset{(1)}{=} \underbrace{\left.\frac{\partial h}{\partial x}\right|_{x_e,u_e}}_{p \times n} \Delta x + \underbrace{\left.\frac{\partial h}{\partial u}\right|_{x_e,u_e}}_{p \times m} \Delta u$$

$$\begin{array}{cc} \Downarrow & \Downarrow \\ C & D \end{array}$$

Conclusion : use a linear approximation about equilibrium.

Example :

Consider a decalcification plant which is used to reduce the concentration of calcium hydroxide in water by forming a calcium carbonate precipitate.

The following equations hold (simplified model) :

- chemical reaction : $\mathrm{Ca(OH)_2} + \mathrm{CO_2} \rightarrow \mathrm{CaCO_3} + \mathrm{H_2O}$
- reaction speed : $r = c[\mathrm{Ca(OH)_2}][\mathrm{CO_2}]$
- rate of change of concentration :

$$\frac{d[\mathrm{Ca(OH)_2}]}{dt} = \frac{k}{V} - \frac{r}{V}$$
$$\frac{d[\mathrm{CO_2}]}{dt} = \frac{u}{V} - \frac{r}{V}$$

k and u are the inflow rates in moles/second of calcium hydroxide and carbon dioxide respectively. V is the tank volume in liters.

Let the inflow rate of carbon dioxide be the input and the concentration of calcium hydroxide be the output.

A nonlinear state space model for this reactor is :

$$\begin{aligned}\frac{d[\mathrm{Ca(OH)_2}]}{dt} &= \frac{k}{V} - \frac{c}{V}[\mathrm{Ca(OH)_2}][\mathrm{CO_2}] \\ \frac{d[\mathrm{CO_2}]}{dt} &= \frac{u}{V} - \frac{c}{V}[\mathrm{Ca(OH)_2}][\mathrm{CO_2}] \\ y &= [\mathrm{Ca(OH)_2}]\end{aligned}$$

The state variables are $x_1 = [\mathrm{Ca(OH)_2}]$ and $x_2 = [\mathrm{CO_2}]$.

In equilibrium we have :

$$\frac{k}{V} - \frac{c}{V}[\mathrm{Ca(OH)_2}]_{\mathrm{eq}}[\mathrm{CO_2}]_{\mathrm{eq}} = \frac{k}{V} - \frac{c}{V}X_1X_2 = 0$$

$$\frac{u_{\mathrm{eq}}}{V} - \frac{c}{V}[\mathrm{Ca(OH)_2}]_{\mathrm{eq}}[\mathrm{CO_2}]_{\mathrm{eq}} = \frac{U}{V} - \frac{c}{V}X_1X_2 = 0$$

$$Y = [\mathrm{Ca(OH)_2}]_{\mathrm{eq}} = X_1$$

For small deviations about the equilibrium :

$$\begin{aligned}\frac{d\Delta x_1}{dt} &= -\frac{c}{V}X_2\Delta x_1 - \frac{c}{V}X_1\Delta x_2 \\ \frac{d\Delta x_2}{dt} &= -\frac{c}{V}X_2\Delta x_1 - \frac{c}{V}X_1\Delta x_2 + \frac{1}{V}\Delta u \\ \Delta y &= \Delta x_1\end{aligned}$$

so,

$$A = \begin{bmatrix} -\frac{cX_2}{V} & -\frac{cX_1}{V} \\ -\frac{cX_2}{V} & -\frac{cX_1}{V} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{V} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

If

$$k = 0.1\ \frac{\text{mole}}{\text{sec}}, \qquad c = 0.5\ \frac{l^2}{\text{sec}\cdot\ \text{mole}}, \qquad U = 0.1\ \frac{\text{mole}}{\text{sec}},$$

$$X_1 = 0.25\ \frac{\text{mole}}{\text{l}}, \quad X_2 = 0.8\ \frac{\text{mole}}{\text{l}}, \qquad V = 5\ \text{l}$$

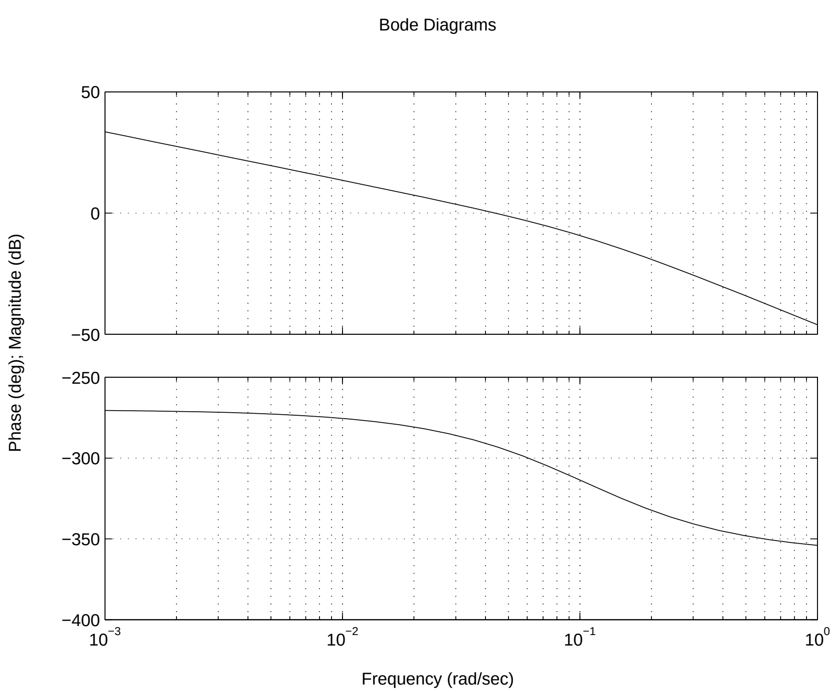
then

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} -0.08 & -0.025 & 0 \\ -0.08 & -0.025 & 0.2 \\ \hline 1 & 0 & 0 \end{array}\right]$$

The corresponding transfer function is

$$\frac{\Delta y(s)}{\Delta u(s)} = \frac{-0.005}{s^2 + 0.105s}$$

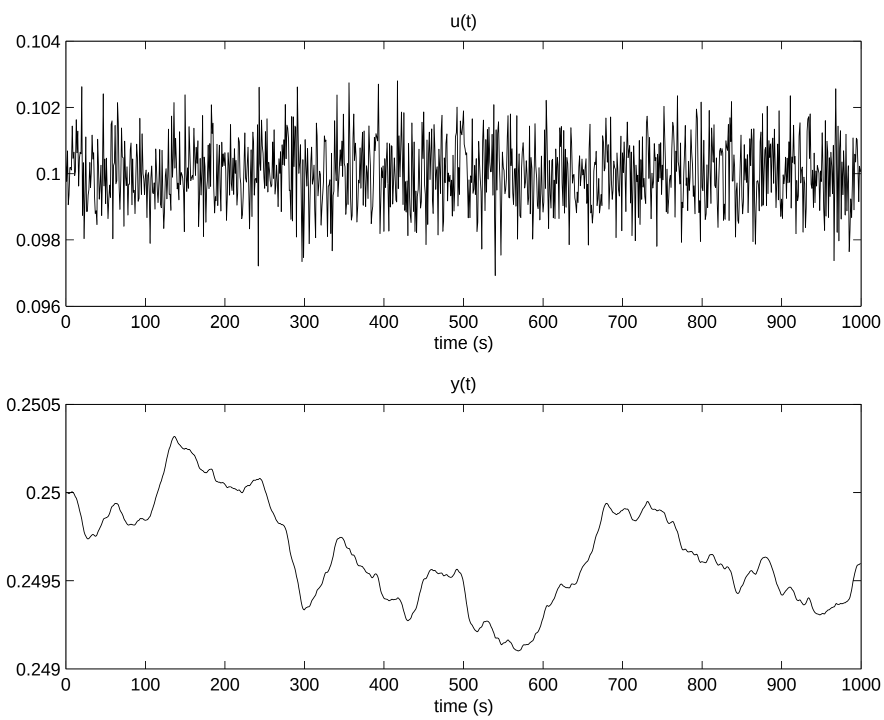
and its bode plot



One can also obtain a linear state space model for this chemical plant from linear system identification.

For small deviations about the equilibrium point the dynamics can be described fairly well by a linear (A, B, C, D)-model. Hence, a small white noise disturbance Δu was generated and was added to the equilibrium value U. We applied this signal to the input of a nonlinear model of the chemical reactor (Simulink model for instance), i.e. $u(t) = U + \Delta u$.

The following input-output set was obtained :



By applying a linear system identification algorithm (N4SID), the following 2nd-order model was obtained :

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} 0.0015 & -0.0589 & -0.0867 \\ 0.0027 & -0.1066 & 0.1713 \\ \hline 0.2546 & 0.129 & 0 \end{array}\right]$$

The corresponding transfer function is

$$\frac{y(s)}{u(s)} = \frac{-1.299.10^{-7}s - 0.005}{s^2 + 0.1051s + 1.346.10^{-8}}$$

It has 2 poles at $-1.281.10^{-7}$ and -1.051.

As $-1.281.10^{-7}$ lies close to 0, and taking in account the properties of the manually derived linear model of page 45, we conclude that the plant has one integrator pole. Hence, it might be better to fix one pole at $s = 0$. In this way it is guaranteed that the linear model obtained by system identification is stable.
The transfer function which was obtained using this modified identification procedure is

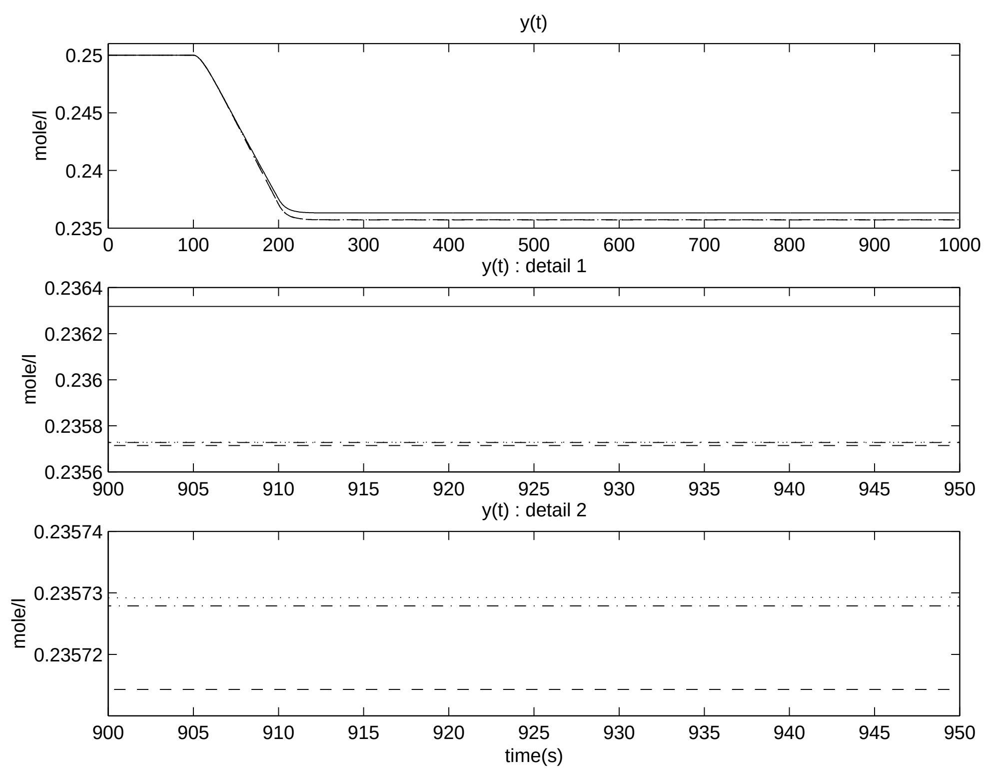
$$\frac{y(s)}{u(s)} = \frac{-1.68.10^{-8}s - 0.005}{s^2 + 0.1051s}$$

The different models are validated by comparing their response to the following input :

$$\begin{aligned}\Delta u(t) &= 0.003 \quad \text{if } 100 < t < 200 \\ &= 0 \qquad\quad \text{elsewhere}\end{aligned}$$

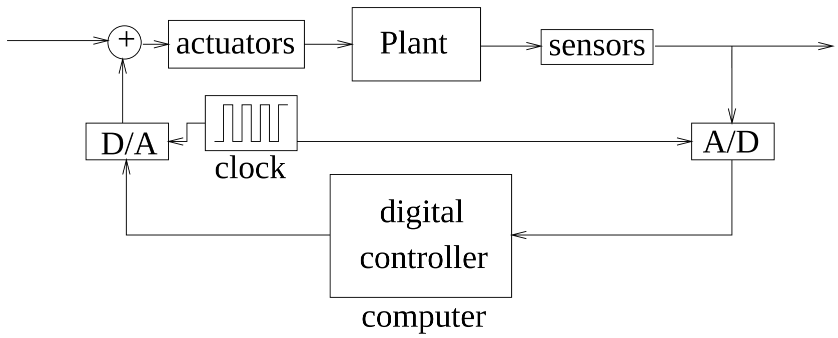
Four responses are shown :

- the nonlinear system (—)
- the linearised model obtained by hand (page 45) (- -)
- the 2nd-order linear model obtained from N4SID ($\cdots$)
- the linear model obtained from N4SID having a fixed integrator pole by construction (-.)

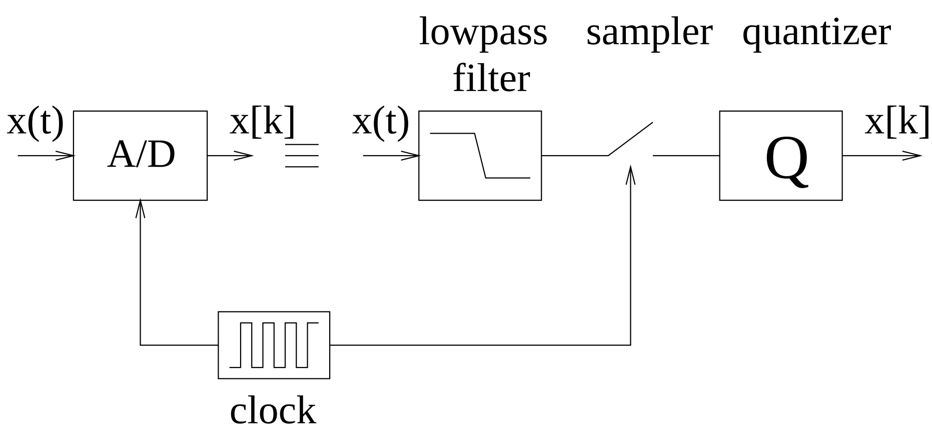


Digitization

in this course we want to control physical plants using a digital computer :

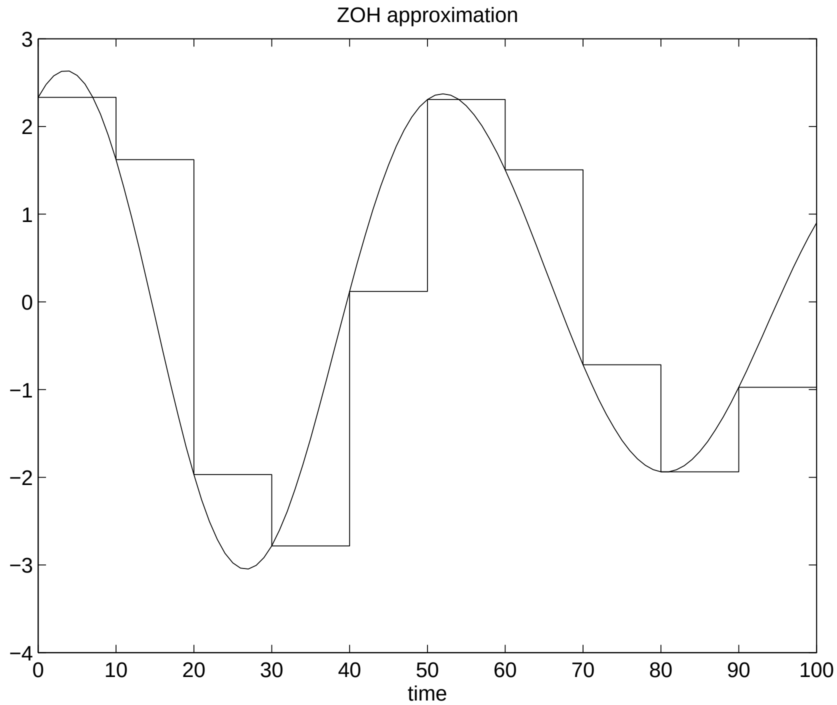


- analog-to-digital converter :



  - the analog anti-aliasing filter filters out high frequent components (> Nyquist frequency)
  - the filtered analog signal is sampled and quantized in order to obtain a digital signal. For quantization 10 to 12 bits are common.

- digital-to-analog converter :
  - zero-order hold



    - introduces a delay
    - frequency spectrum deformation
  - first-order hold
  - n-th order polynomial (more general case) : fit a n-th order polynomial through the n+1 most recent samples and extrapolate to the next time instance

Digitization of state space models

1. discretization by applying numerical integration rules :

   a continuous-time integrator

$$\dot{x}(t) = e(t)$$

$$\Leftrightarrow$$

$$sX(s) = E(s)$$

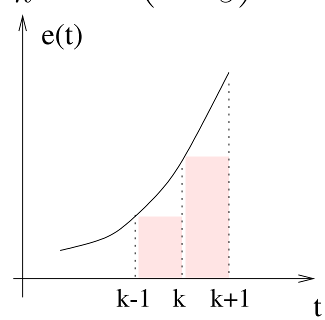
   can be approximated using

   - the forward rectangular rule or Euler's method

$$\frac{x_{k+1} - x_k}{T_s} = e_k \quad \text{with } x_k = x(kT_s)$$

$$\Leftrightarrow$$

$$\frac{z-1}{T_s} X(z) = E(z)$$

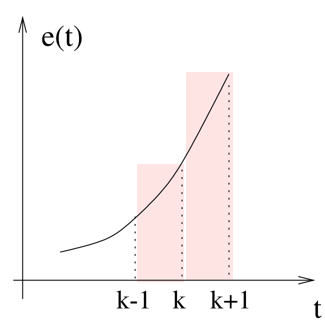


   - the backward rectangular rule

$$\frac{x_{k+1} - x_k}{T_s} = e_{k+1}$$

$$\Leftrightarrow$$

$$\frac{z-1}{zT_s} X(z) = E(z)$$

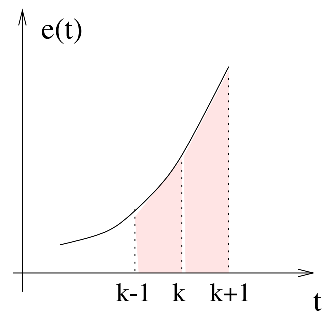


- the trapezoid rule or bilinear transformation

$$\frac{x_{k+1} - x_k}{T_s} = \frac{e_{k+1} + e_k}{2}$$

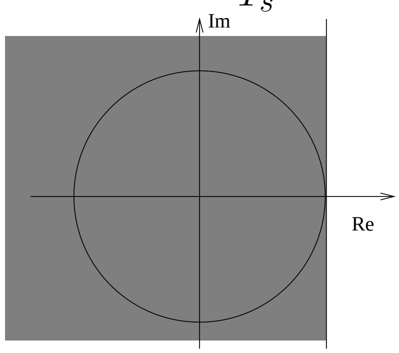
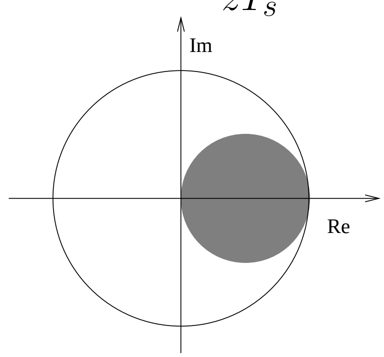
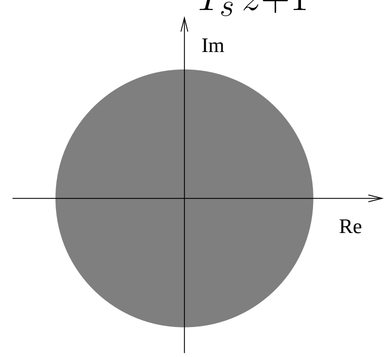
$$\Leftrightarrow$$

$$\frac{2}{T_s}\frac{z-1}{z+1}X(z) = E(z)$$



Change a continuous model $G(s)$ into a discrete model $G_d(z)$ by replacing all integrators with their discrete equivalents :

■ = projection of the left half-plane

Euler	backward rect.	bilinear transf.
$s \longrightarrow \frac{z-1}{T_s}$ 	$s \longrightarrow \frac{z-1}{zT_s}$ 	$s \longrightarrow \frac{2}{T_s}\frac{z-1}{z+1}$ 

Except for the forward rectangular rule stable continuous poles (grey zone) are guaranteed to be placed in stable discrete areas, i.e. within the unit circle.

The bilinear transformation maps stable poles $\longrightarrow$ stable poles and unstable poles $\longrightarrow$ unstable poles.

The following continuous-time model is given :

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} sX &= AX + BU \\ Y &= CX + DU \end{aligned}$$

following Euler's method s is replaced by $\frac{z-1}{T_s}$, so

$$\frac{z-1}{T_s} X = AX + BU$$

or

$$zX = (I + AT_s)X + BT_sU$$

the output equation $Y = CX + DU$ remains. A similar calculation can be done for the backward rectangular rule and the bilinear transformation resulting in the following table :

	Euler	backward rect.	bilinear transf.
A_d	$I + AT_s$	$(I - AT_s)^{-1}$	$(I - \frac{AT_s}{2})^{-1}(I + \frac{AT_s}{2})$
B_d	BT_s	$(I - AT_s)^{-1}BT_s$	$(I - \frac{AT_s}{2})^{-1}BT_s$
C_d	C	$C(I - AT_s)^{-1}$	$C(I - \frac{AT_s}{2})^{-1}$
D_d	D	$D + C(I - AT_s)^{-1}BT_s$	$D + C(I - \frac{AT_s}{2})^{-1}\frac{BT_s}{2}$

2. discretization by assuming zero-order hold ...

$$\dot{x} = Ax + Bu$$

$\Rightarrow$

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{convolution integral}}$$

Let the sampling time be T_s, then

$$\begin{aligned} x(t+T_s) &= e^{A(t+T_s)}x(0) + \int_0^{t+T_s} e^{A(t+T_s-\tau)}Bu(\tau)d\tau \\ &= e^{AT_s}x(t) + e^{AT_s}\underbrace{\int_t^{t+T_s} e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{approximated}} \end{aligned}$$

$$\begin{aligned} \overset{\text{ZOH}}{\Rightarrow} x_{k+1} &\overset{t=kT_s}{=} e^{AT_s}x_k + e^{AT_s}A^{-1}(I - e^{-AT_s})Bu_k \\ &= \underbrace{e^{AT_s}}_{A_d}x_k + \underbrace{A^{-1}(e^{AT_s} - I)B}_{B_d}u_k \end{aligned}$$

One can prove that $G_d(z)$ can be expressed as

$$G_d(z) \overset{\text{ZOH}}{=} (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right\}$$

For this reason this discretization method is sometimes called step invariance mapping.

3. discretization by zero-pole mapping :
   following the previous method the poles of $G_d(z)$ are related to the poles of $G(s)$ according to $z = e^{sT_s}$. If we assume by simplicity that also the zeros undergo this transformation, the following heuristic may be applied:
   (a) map poles and zeros according to $z = e^{sT_s}$
   (b) if the numerator is of lower order than the denominator, add discrete zeros at -1 until the order of the numerator is one less than the order of the denominator. A lower numerator order corresponds to zeros at ∞ in continuous time. By discretization they are put at -1.
   (c) adjust the DC gain such that

$$\lim_{s\to 0} G(s) = \lim_{z\to 1} G_d(z)$$

Example :

given the following SISO system

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} -0.2 & -0.5 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 0.4 & 0 \end{array}\right]$$

$$\Rightarrow G(s) = C(sI - A)^{-1}B + D = \frac{s + 0.4}{s^2 + 0.2s + 0.5}.$$

Now compare the following discretization methods ($T_s = 1$ sec.) :

1. bilinear transformation :
   (a) A_d, B_d, C_d and D_d are calculated using the conversion table
   (b) $G_{\text{bilinear}}(z) = C_d(zI - A_d)^{-1}B_d + D_d$.
   $$= \frac{0.4898 + 0.1633z^{-1} - 0.3265z^{-2}}{1 - 1.4286z^{-1} + 0.8367z^{-2}}$$
2. ZOH :
   (a) $A_d = e^{AT_s}$, $B_d = A^{-1}(e^{AT_s} - I)B$, $C_d = C$ and $D_d = D$
   (b) $G_{\text{ZOH}}(z) = C_d(zI - A_d)^{-1}B_d + D_d$.
   $$= \frac{1.0125z^{-1} - 0.6648z^{-2}}{1 - 1.3841z^{-1} + 0.8187z^{-2}}$$

3. pole-zero mapping :

   (a) the poles of $G(s)$ are $-0.1 + j0.7$ and $-0.1 - j0.7$

   (b) there is one zero at -0.4

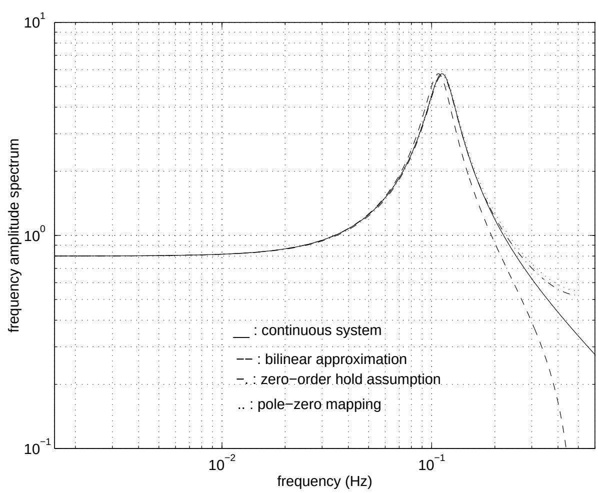
   (c)

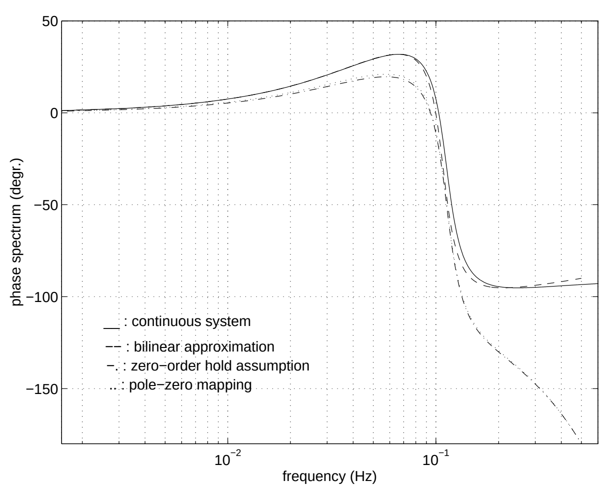
$$G_{\mathrm{pz}} = K \frac{z - e^{-0.4}}{\left(z - e^{-0.1+j0.7}\right)\left(z - e^{-0.1-j0.7}\right)}$$

$$= K \frac{z^{-1} - 0.6703 z^{-2}}{1 - 1.3841 z^{-1} + 0.8187 z^{-2}}$$

$$\text{hence } K = \frac{\lim_{s\to 0} G(s)}{\lim_{z \to 1} G_{\mathrm{pz}}(z)} = 1.0546$$

Compare the bode plots :





Sampling rate selection :

- the lower the sampling rate, the lower the implementation cost (cheap microcontroller & A/D converters), the rougher the response and the larger the delay between command changes and system response.

- an absolute lower bound is set by the specification to track a command input with a certain frequency :

$$f_s \geq 2\mathrm{BW}_{\mathrm{cl}}$$

  f_s is the sampling rate and $\mathrm{BW}_{\mathrm{cl}}$ is the closed-loop system bandwidth.

- if the controller is designed for disturbance rejection,

$$f_s > 20\mathrm{BW}_{\mathrm{cl}}$$

- when the sampling rate is too high, finite-word effects show up in small word-size microcontrollers (<10 bits).

- for systems where the controller adds damping to a lightly damped mode with resonant frequency f_r,

$$f_s > 2f_r$$

In practice, the sampling rate is a factor 20 to 40 higher than $\mathrm{BW}_{\mathrm{cl}}$.

Advantages of state space models

- More general models: LTI and Nonlinear Time Varying (NTV).
- Geometric concepts: more mathematical tools (linear algebra).
- Internal and external descriptions: "divide and conquer" strategy.
- Unified framework: the same for SISO and MIMO.

Geometric properties of linear state space models

Canonical forms

Control canonical form (SISO):

$$\dot{x} = A_c x + B_c u,\ \ y = C_c x.$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},\ \ B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$C_c = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$$

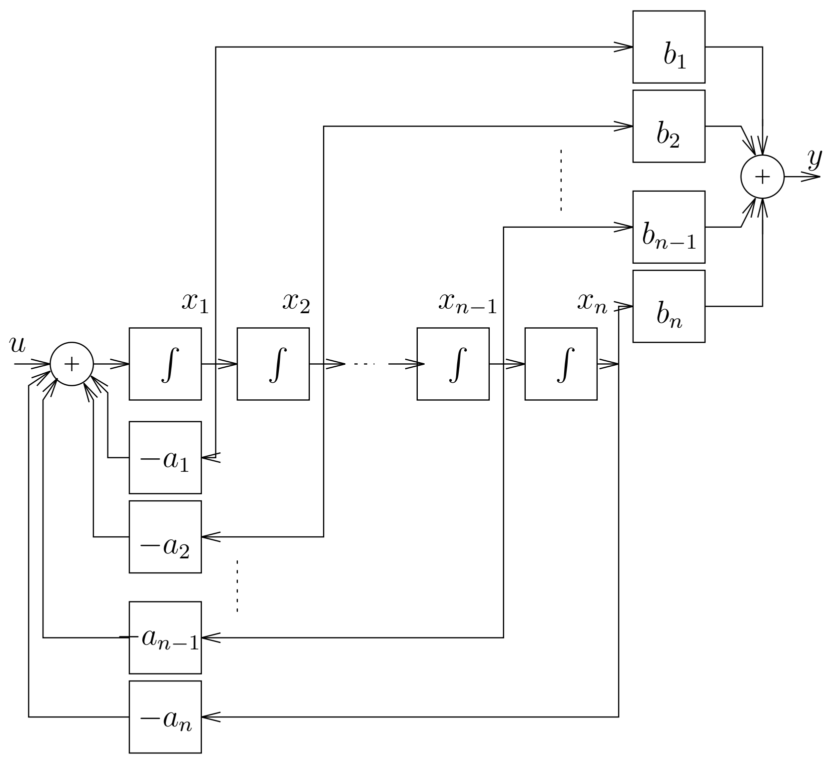
Corresponding transfer function:

$$G(s) = \frac{b(s)}{a(s)}$$

$$b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n$$

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$$

Block diagram for the control canonical form



Modal canonical form:

$$\dot{x} = A_m x + B_m u,\ y = C_m x.$$

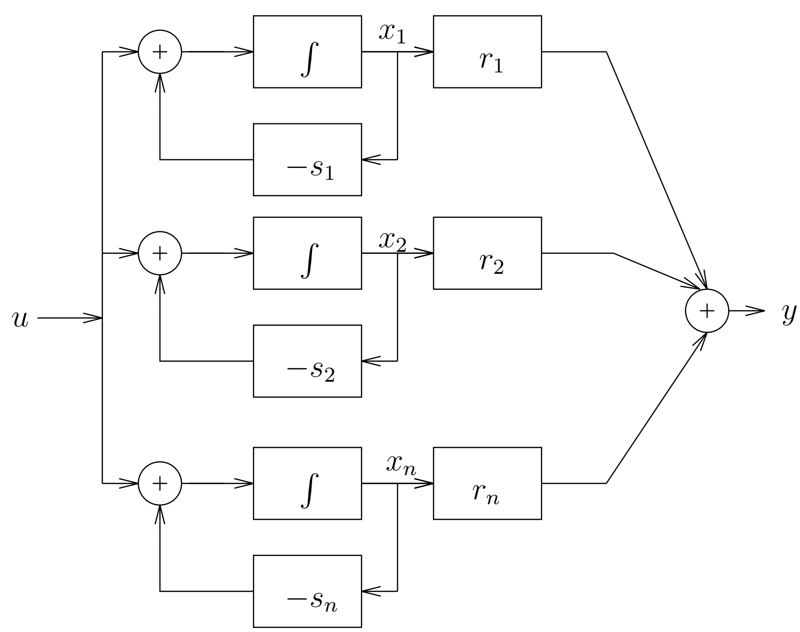
$$A_m = \begin{bmatrix} -s_1 & & & \\ & -s_2 & & \\ & & \ddots & \\ & & & -s_n \end{bmatrix},\ B_m = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C_m = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$$

Corresponding transfer function:

$$G(s) = \sum_{i=1}^{n} \frac{r_i}{s + s_i}$$

Block diagram for the modal canonical form:



Similarity transformation

A state space model (A, B, C, D) is not unique for a physical system. Let

$$x = T\bar{x}, \det(T) \neq 0,$$

$$\bar{A} = T^{-1}AT,\ \ \bar{B} = T^{-1}B, \bar{C} = CT, \bar{D} = D$$

$\Rightarrow$

$$\dot{\bar{x}} = T^{-1}AT\bar{x} + T^{-1}Bu = \bar{A}\bar{x} + \bar{B}u,$$

$$y = CT\bar{x} + Du = \bar{C}\bar{x} + Du$$

T is chosen to give the most convenient state-space description for a given problem (*e.g.*control or modal canonical forms).

Example: Eigenvalue decomposition of A:

$$A = X\Lambda X^{-1},\ \ \Lambda = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ -\beta_1 & \alpha_1 & & & \\ & & \ddots & & \\ & & & \lambda_1 & \\ & & & & \ddots \end{bmatrix}$$

$$\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array} \quad \overset{X^{-1}x = z}{\Longrightarrow} \quad \begin{array}{l} \dot{z} = \Lambda z + X^{-1}Bu \\ y = CXz + Du \end{array}$$

Controllability

An n-th order system is called controllable if one can reach any state x from any given initial state x_0 in a finite time.

Controllability matrix:

$$\mathcal{C} = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array} \right]$$

$$\text{rank}(\mathcal{C}) = n \Leftrightarrow (A, B) \text{ controllable}$$

Explanation:
Consider a linear (discrete) system of the form

$$x_{k+1} = Ax_k + Bu_k$$

Then

$$x_{k+1} = A^{k+1}x_0 + A^kBu_0 + \cdots + Bu_k$$

such that

$$x_{k+1} - A^{k+1}x_0 = \left[\begin{array}{cccc} B & AB & \cdots & A^kB \end{array} \right] \left[\begin{array}{c} u_k \\ u_{k-1} \\ \vdots \\ u_0 \end{array} \right]$$

Note that

$$\operatorname{rank}\left[\begin{array}{cccc} B & AB & \cdots & A^k B \end{array}\right] = \operatorname{rank}\left[\begin{array}{cccc} B & AB & \cdots & A^{n-1} B \end{array}\right]$$

for $k \geq n-1$ (Cayley-Hamilton Theorem).

There exists always a vector $\begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$ if $\operatorname{rank}(\mathcal{C}) = n$.

Remarks: Controllability matrices for a continuous time system and a discrete time system are of the same form.

Observability

An n-th order system is called observable if knowledge of the input u and the output y over a finite time interval is sufficient to determine the state x.

Observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$\text{rank}(\mathcal{O}) = n \Leftrightarrow (A, C) \text{ observable}$$

Explanation:

Consider an autonomous system

$$x_{k+1} = Ax_k$$

$$y_k = Cx_k$$

Then we find easily

$$y_k = CA^k x_0$$

and

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \end{bmatrix} x_0$$

Note that

$$\text{rank} \begin{bmatrix} C \\ CA \\ \cdots \\ CA^k \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix}$$

for $k \geq n - 1$.

One can always determine x_0 if $\text{rank}(\mathcal{O}) = n$.

Remarks: Observability matrices for a continuous time system and a discrete time system are of the same form.

The Popov-Belevitch-Hautus tests (PBH)

PBH controllability test:
(A, B) is not controllable if and only if there exists a left eigenvector $q \neq 0$ of A such that

$$A^T q = q\lambda$$

$$B^T q = 0$$

In other words, (A, B) is controllable if and only if there is no left eigenvector q of A that is orthogonal to B.
If there is a λ and q satisfying the PBH test, then we say that the mode corresponding to eigenvalue λ is uncontrollable (uncontrollable mode), otherwise it is controllable (controllable mode).
How to understand (for SISO) ? Put A in the modal canonical form:

$$A = \begin{bmatrix} * & & \\ & \lambda_i & \\ & & * \end{bmatrix}, \; B = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}, \Rightarrow q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Uncontrollable mode: $\dot{x}_i = \lambda_i x_i$.

PBH observability test:

(A, C) is not observable if and only if there exists a right eigenvector $p \neq 0$ of A such that

$$Ap = p\lambda$$

$$Cp = 0$$

If there is a λ and p satisfy the PBH test, then we say that the mode corresponding to eigenvalue λ is unobservable (unobservable mode), otherwise it is observable (observable mode).

How to understand for SISO ? Put A in a modal canonical form.

Stability/Stabilizability/Detectability

Stability:
A system with system matrix A is unstable if A has an eigenvalue λ with real$(\lambda) \geq 0$ for a continuous time system and $|\lambda| \geq 1$ for a discrete time system.

Stabilizability:
(A, B) is stabilizable if all unstable modes are controllable.

Detectability:
(A, C) is detectable if all unstable modes are observable.

Example

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k + Du_k
\end{aligned}
$$

$$
\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{rrrr|r} 1.1 & 0 & 0 & 0 & 1 \\ 0 & -0.5 & 0.6 & 0 & 2 \\ 0 & -0.6 & -0.5 & 0 & -1 \\ 0 & 0 & 0 & 2 & 2 \\ \hline 0 & 1 & 2 & 3 & 1 \end{array}\right]
$$

Modes	Stable?	Stabilizable?	Detectable?
1.1	no	yes	no
$-0.5 \pm 0.6i$	yes	yes	yes
2	no	yes	yes

Kalman decomposition and minimal realization

Kalman decomposition

Given a state space system $[A, B, C, D]$. Then we can always find an invertible similarity transformation T such that the transformed matrices have the structure

$$TAT^{-1} = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} r_1 & r_2 & r_3 & r_4 \end{array} \\ \begin{pmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\overline{co}} \end{pmatrix} \end{array}$$

$$TB = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \!\!\begin{array}{c} m \\ \begin{pmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{pmatrix} \end{array}, \quad CT^{-1} = \begin{array}{c} \\ p \end{array} \!\!\begin{array}{c} \begin{array}{cccc} r_1 & r_2 & r_3 & r_4 \end{array} \\ \begin{pmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{pmatrix} \end{array}$$

where

$$r_1 = \text{rank}\,(\mathcal{OC}),\ r_2 = \text{rank}\,(\mathcal{C}) - r_1,$$

$$r_3 = \text{rank}\,(\mathcal{O}) - r_1,\ r_4 = n - r_1 - r_2 - r_3.$$

The subsystem

$$[\ A_{co}, B_{co}, C_{co}, D\ ]$$

is controllable and observable. The subsystem

$$[\ \begin{pmatrix} A_{co} & 0 \\ A_{21} & A_{c\bar{o}} \end{pmatrix}, \begin{pmatrix} B_{co} \\ B_{c\bar{o}} \end{pmatrix}, \begin{pmatrix} C_{co} & 0 \end{pmatrix}, D\ ]$$

is controllable. The subsystem

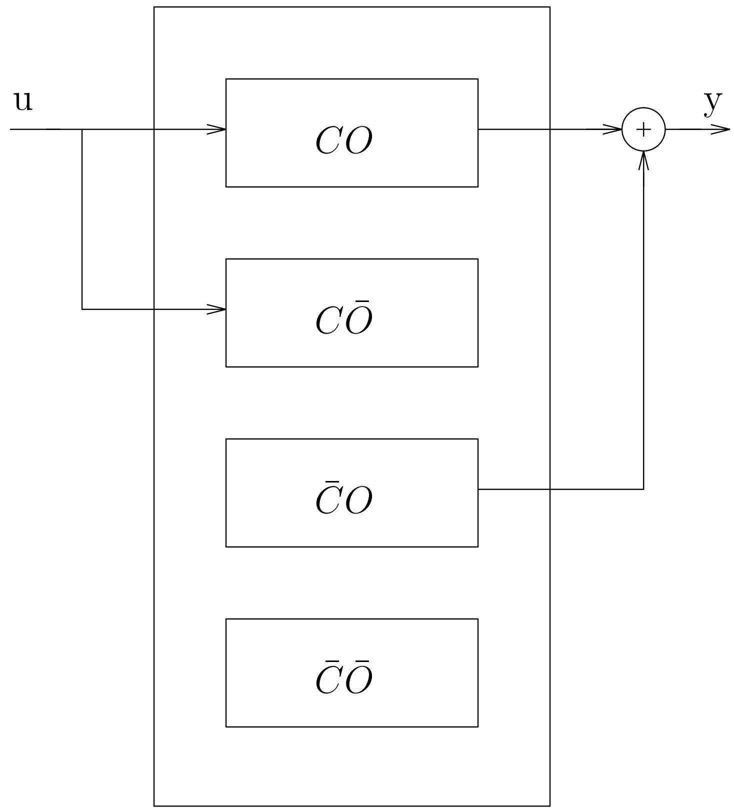
$$[\ \begin{pmatrix} A_{co} & A_{13} \\ 0 & A_{\bar{c}o} \end{pmatrix}, \begin{pmatrix} B_{co} \\ 0 \end{pmatrix}, (\ C_{co} \ \ C_{\bar{c}o}\ ), D\ ]$$

is observable. The subsystem

$$[\ A_{\overline{co}}\ , 0, 0, D\ ]$$

is neither controllable nor observable.

Kalman decomposition



Minimal realization:

A minimal realization is one that has the smallest-size A matrix for all triplets $[A, B, C]$ satisfying

$$G(s) = D + C(sI - A)^{-1}B$$

where $G(s)$ is a given transfer matrix.

$\left[\begin{array}{cccc} A & B & C & D \end{array} \right]$ is minimal $\Leftrightarrow$ controllable and observable.

Let $\left[\begin{array}{cccc} A_i & B_i & C_i & D \end{array} \right]$, $i = 1, 2$, be two minimal realizations of a transfer matrix, then

$$\left[\begin{array}{cccc} A_1 & B_1 & C_1 & D \end{array} \right]$$
$$T \Downarrow \Uparrow T^{-1}$$
$$\left[\begin{array}{cccc} A_2 & B_2 & C_2 & D \end{array} \right]$$

with

$$T = \mathcal{C}_1\mathcal{C}_2^T(\mathcal{C}_2\mathcal{C}_2^T)^{-1}$$

Input/output properties of state-space models

Transfer matrix

General transfer matrix for a state space model :

$$G(\xi) = D + C(\xi I - A)^{-1}B$$

ξ can be s (CT) or z (DT).

Poles

Characteristic polynomial of matrix A:

$$\begin{aligned}\alpha(\xi) &= \det(\xi I - A) \\ &= \alpha_n \xi^n + \alpha_{n-1}\xi^{n-1} + \cdots + \alpha_1 \xi + \alpha_0\end{aligned}$$

Characteristic equation:

$$\alpha(\xi) = 0$$

The eigenvalues $\lambda_i, i = 1, \cdots, n$ of the system matrix A are called the poles of the system.
The pole polynomial is defined as

$$\pi(\xi) = \Pi_{i=1}^{n}(\xi - \lambda_i)$$

(= characteristic equation up to within a scalar)

Physical interpretation of a pole :

Consider the following second order system :

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} \alpha & \beta & b_1 \\ -\beta & \alpha & b_2 \\ \hline c_1 & c_2 & 0 \end{array}\right]$$

The transfer matrix is given by (try to verify it) :

$$\begin{aligned} G(\xi) &= C(\xi I - A)^{-1}B + D \\ &= \frac{\xi(b_1c_1 + b_2c_2) + \beta(b_2c_1 - b_1c_2) - \alpha(b_1c_1 + b_2c_2)}{\xi^2 - 2\alpha\xi + \alpha^2 + \beta^2} \end{aligned}$$

There are 2 poles at $\alpha \pm j\beta$.

There is a zero at

$$\frac{\alpha(b_1c_1 + b_2c_2) - \beta(b_2c_1 - b_1c_2)}{b_1c_1 + b_2c_2}$$

In continuous time the impulse response takes on the form:

$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s(b_1c_1+b_2c_2)+\beta(b_2c_1-b_1c_2)-\alpha(b_1c_1+b_2c_2)}{s^2-2\alpha s+\alpha^2+\beta^2}\right\}$$

$$= (A_m\cos(\beta t)+B_m\sin(\beta t))e^{\alpha t} = C_m e^{\alpha t}\cos(\beta t+\gamma)$$

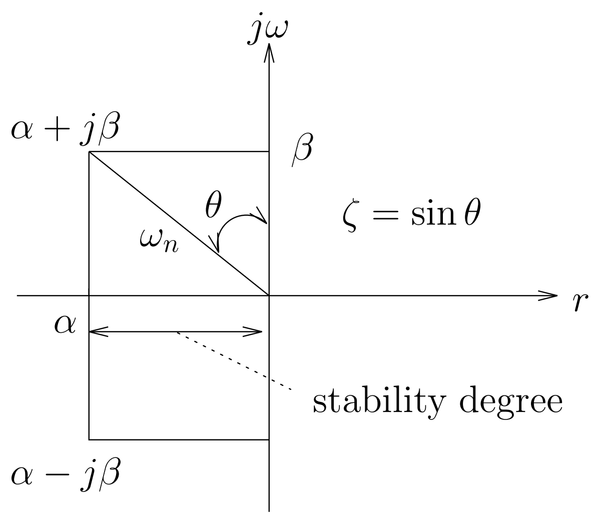
with $A_m = b_1c_1+b_2c_2$ and $B_m = b_2c_1-b_1c_2$

$$\Rightarrow\ C_m = \sqrt{(b_1^2+b_2^2)(c_1^2+c_2^2)}$$

$$\Rightarrow\ \gamma = \text{atan}\left(\frac{-B_m}{A_m}\right)$$

Define the damping ratio ζ and natural frequency ω_n:

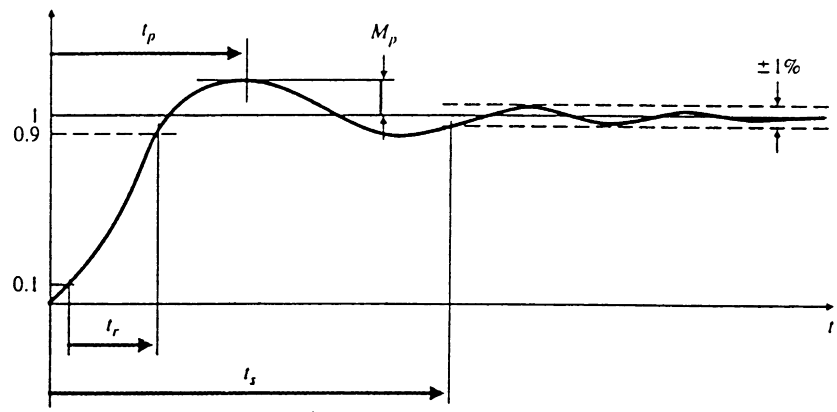
$$\alpha = -\zeta\omega_n,\ \ \beta = \omega_n\sqrt{1-\zeta^2}$$



For a second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

the time response looks like :



$$\begin{aligned} \text{rise time} \;\; & t_r \simeq \frac{1.8}{\omega_n} \\ \text{settling time} \;\; & t_s = \frac{4.6}{\zeta\omega_n} \\ \text{peak time} \;\; & t_p = \frac{\pi}{\omega_d},\ \omega_d = \omega_n\sqrt{1-\zeta^2} \\ \text{overshoot} \;\; & M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}},\ 0 \le \zeta < 1 \end{aligned}$$

In general : if a continuous-time system $(A, B, C, 0)$ has poles at $\alpha \pm j\beta \Rightarrow$ the impulse response will have time modes of the form

$$A_m e^{\alpha t} \cos(\beta t + \gamma)$$

$$\left.\begin{array}{l} A_m\text{: amplitude} \\ \gamma\text{: phase} \end{array}\right\} \text{determined by } B \text{ and } C.$$

In discrete time the impulse response matrix $G_d(k)$ takes on the form :

$$G_d(k) = C_d A_d^{k-1} B_d, \qquad k \geq 1$$

which can be proven to be a sum of terms of the form :

$$C_m b^{k-1} \cos(\omega(k-1) + \gamma)$$

each of which satisfies a second order linear system

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} \alpha & \beta & b_1 \\ -\beta & \alpha & b_2 \\ \hline c_1 & c_2 & 0 \end{array}\right]$$

Parameterize A as

$$A = \begin{bmatrix} b\cos\omega & b\sin\omega \\ -b\sin\omega & b\cos\omega \end{bmatrix}$$

then

$$A^k = b^k \begin{bmatrix} \cos\omega k & \sin\omega k \\ -\sin\omega k & \cos\omega k \end{bmatrix}$$

Now,

$$
\begin{aligned}
CA^kB &= (A_m \cos \omega k + B_m \sin \omega k) b^k \\
&= C_m b^k \cos(\omega k + \gamma)
\end{aligned}
$$

$$\text{with } A_m = C_m \cos \gamma = b_1 c_1 + b_2 c_2$$

$$\text{and } B_m = -C_m \sin \gamma = b_2 c_1 - b_1 c_2$$

$$
\begin{aligned}
&\Rightarrow \; C_m = \sqrt{(b_1^2 + b_2^2)(c_1^2 + c_2^2)} \\
&\Rightarrow \; b = \sqrt{\alpha^2 + \beta^2} \\
&\Rightarrow \; \omega = \operatorname{atan}\left(\frac{\beta}{\alpha}\right) \\
&\Rightarrow \; \gamma = \operatorname{atan}\left(\frac{-B_m}{A_m}\right)
\end{aligned}
$$

Transmission zeros

Definition: The zeros of a LTI system are defined as those values $\zeta \in \mathbb{C}$ for which the rank of the transfer matrix $G(\xi)$ is lower than its normal rank (=rank of $G(\xi)$ for almost all values of ξ):

$$\mathrm{rank}(G(\zeta)) < \text{normal rank}$$

Property: Let ζ be a zero of $G(\xi)$ $(p \times m)$, then

$$\mathrm{rank}\,(G(\zeta)) < \text{normal rank},$$

$$\Downarrow \text{ if } \zeta \text{ is not a pole of } G(\xi)$$

$$\exists v \neq 0 \; s.t. \left[D + C(\zeta I - A)^{-1}B\right] v = 0,$$

$$\Downarrow \text{ define } w \stackrel{\Delta}{=} (\zeta I - A)^{-1}Bv$$

$$Dv + Cw = 0, \; (\zeta I - A)w - Bv = 0,$$

$$\Downarrow$$

$$\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = 0$$

How to find zeros for square MIMO systems ($p = m$) with invertible D ?

$$(\zeta I - A)w - Bv = 0,\ Cw + Dv = 0$$

$$\Downarrow$$

$$Aw + Bv = w\zeta,\ v = -D^{-1}Cw$$

$$\Downarrow$$

$$(A - BD^{-1}C)w = w\zeta$$

$$\begin{aligned} \zeta &= \text{eigenvalue of } A - BD^{-1}C, \\ w &= \text{corresponding eigenvector !} \end{aligned}$$

For other cases :

- Generalized eigenvalue problem.
- Use Matlab function "tzero".

Minimum and non-minimum phase systems:
If a system has an unstable zero (in the right half–plane (RHP) or outside the unit circle), then it is a non-minimum phase (NMP) system, otherwise it is a minimum phase (MP) system.

Physical explanation (continuous time) :

Let ζ be a real zero, then there will exist vectors x_0 and u_0 such that:

$$\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

This means if we have an input $u_0 e^{\zeta t}$, there exists an initial state x_0 such that the response is

$$y(t) = 0.$$

For complex zeros :

if ζ is a complex zero, then also its complex conjugate ζ^* is a zero. Try to prove that if

$$\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

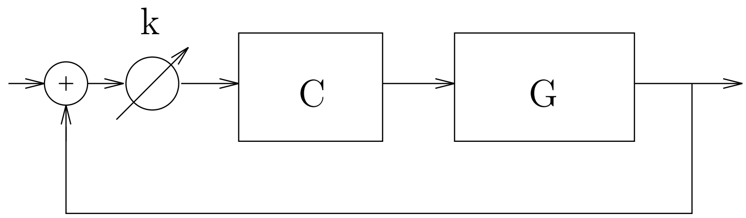
the output $y(t)$ will be exactly zero if the input is

$$u(t) = |u_0| e^{\mathcal{R}e\{\zeta\}t} \cdot \cos(\mathcal{I}m\{\zeta\}t + \phi_{u_0})$$

and the initial state is chosen to be $\mathcal{R}e\{x_0\}$. "$\cdot$" is the elementwise multiplication, both "cos()" and "$e^{()}$" are assumed to be elementwise operators, $\mathcal{R}e\{x_0\}$ is the real part of x_0, $\mathcal{I}m\{\zeta\}$ is the imaginary part of ζ and $u_0 = |u_0| \cdot e^{j\phi_{u_0}}$. Try to derive equivalent formulas for the discrete-time case.

Why zeros are important :

- Limited control system performance



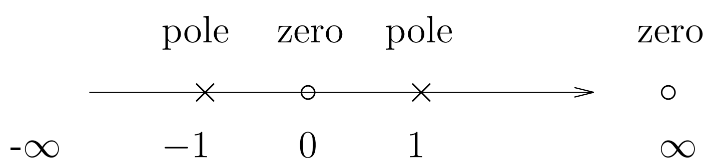
If G is NMP, k can not go to ∞, since unstable zeros (which become unstable poles) put a limit on high gains.

- Stability of a stabilizing controller, Parity Interlacing Property (PIP) :

Let G be unstable. Then, G can be stabilized by a controller C which itself is stable $\Leftrightarrow$ between every pair of real RHP zeros of G (including at ∞), there is an even number of poles.

Example:

$$G(s) = \frac{s}{s^2 - 1}$$



$G(s)$ can not be stabilized by a stable controller.

RHP zeros can cause undershoot !

Let

$$G(s) = \frac{\Pi_{i=1}^{n}(1 - \frac{s}{\zeta_i})}{\Pi_{i=1}^{n+r}(1 - \frac{s}{\lambda_i})},$$

where r = relative degree of $G(s)$. Let $y(t)$ be the step response.

$$\text{Fact : } \begin{cases} y(0) \text{ and its first } r-1 \text{ derivative are } 0; \\ y^{(r)} \text{ is the first non-zero derivative;} \\ y(\infty) = G(0). \end{cases}$$

The system has undershoot $\Leftrightarrow$ the steady state value $y(\infty)$ has a sign opposite to $y^{(r)}(0)$, i.e.

$$y^{(r)}(0)y(\infty) < 0.$$

It can be proven that the system has undershoot $\Leftrightarrow$ there is an odd number of RHP zeros.

An example of state-space analysis

Example: Tape drive control - state space analysis

Controllability/observability

Direct rank test using SVD:
The singular values of $\mathcal{C}$ are

$$4.5762,\ 4.3861,\ 1.2940,\ 1.2758,\ 0.5737,\ 0.4787,$$

so the system is controllable (rank=6).
The singular values of $\mathcal{O}$ are

$$2.5695,\ 1.4504,\ 0.7145,\ 0.7071,\ 0.5171,\ 0.2356,$$

so the system is observable (rank=6).

PBH test:

There are 6 (independent) left eigenvectors associated with the 6 eigenvalues of A :

$$\begin{bmatrix} -0.1480 \pm 0.0826i \\ 0.0703 \pm 0.5380i \\ 0.1480 \mp 0.0826i \\ -0.0703 \mp 0.5380i \\ 0.2994 \pm 0.2954i \\ -0.2994 \mp 0.2954i \end{bmatrix}, \begin{bmatrix} 0.0766 \\ 0.5624 \\ 0.0766 \\ 0.5624 \\ 0.4218 \\ 0.4218 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0.4892 \\ 0.0000 \\ 0.4892 \\ 0.5105 \\ 0.5105 \end{bmatrix}, \begin{bmatrix} -0.0046 \\ -0.0227 \\ 0.0046 \\ 0.0227 \\ -0.7067 \\ 0.7067 \end{bmatrix}, \begin{bmatrix} 0.0000 \\ -0.0295 \\ 0.0000 \\ -0.0295 \\ -0.7065 \\ -0.7065 \end{bmatrix}$$

It can be easily checked that none of them is orthogonal to B. The result is the same as the rank test above.

The same can be done to investigate the observability.

Poles and zeros:

Poles: the eigenvalues of A are

$$-0.2370 \pm 0.5947i, 0,\ -0.2813,\ -0.9760,\ -0.9687$$

The system is not stable since there is a pole at zero.

The first two oscillatory poles are from the spring-mass system consisting of the tape and the motor/capstan inertias. The zero pole reflects the pure integrative effect of the tape on the capstans.

Zeros: it can be checked in Matlab using function “tzero” that there is only one zero at -2. The eigenvector of the matrix

$$\begin{bmatrix} -2I-A & -B \\ C & 0 \end{bmatrix}$$

associated with the eigenvalue 0 is

$$\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \text{ with } x_0 = \begin{bmatrix} -0.1959 \\ 0.1959 \\ 0.1959 \\ -0.1959 \\ -0.4571 \\ 0.4571 \end{bmatrix},\ u_0 = \begin{bmatrix} 0.4630 \\ -0.4630 \end{bmatrix}.$$

Thus, let

$$x(0) = x_0,\ u(t) = u_0 e^{-2t},$$

then the output $y(t)$ will be always zero, $\forall t \geq 0$.

Physical explanation of the zero:

Since the initial values $i_1(0) = -i_2(0)$ and the inputs $e_1(t) = -e_2(t)$, and the motor drive systems in both sides are symmetric, the two capstans will rotate at the same speed, but in opposite directions. So the tape position over the read/write head will remain at 0 if the initial value is at 0.

The tension in the tape is the sum of the tensions in the tape spring and the tape damping. The input voltages vary such a way that the tensions in the tape spring and the tape damping cancel each other out.

Matlab Functions

c2d

c2dm

canon

expm

tf2ss

ss2tf

(d)impulse

(d)step

minreal

tzero

ctrb

obsv

ctrbf

obsvf

(d)lsim

ss

ssdata

Chapter 3

State Feedback - Pole Placement

Motivation

Whereas classical control theory is based on output feedback, this course mainly deals with control system design by state feedback. This model-based control strategy consists of

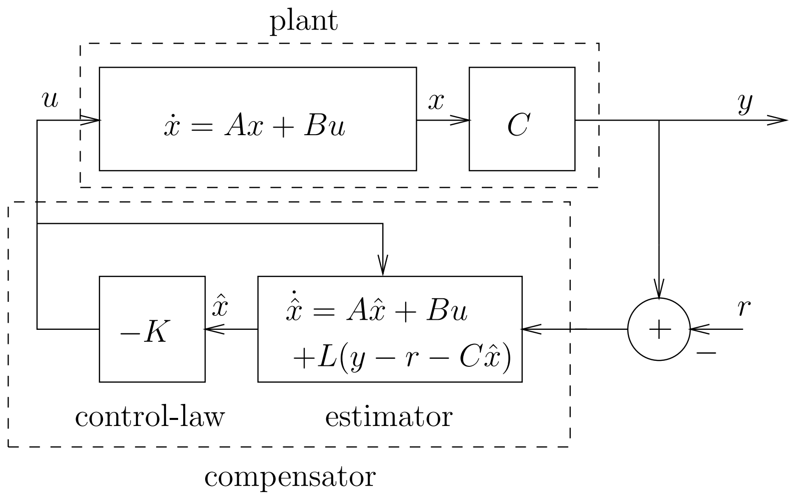
Step 1. State feedback control-law design.

Step 2. Estimator design to estimate the state vector.

Step 3. Compensation design by combining the control law and the estimator.

Step 4. Reference input design to determine the zeros.

Schematic diagram of a state-space design example



Control-law design by state feedback : a motivation

Example: Boeing 747 aircraft control

The complete lateral model of a Boeing 747 (see also page 22), including the rudder actuator (an hydraulic servo) and washout circuit (a lead compensator), is

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx + Du.\end{aligned}$$

where

$$A = \begin{bmatrix} -10 & 0 & 0 & 0 & 0 & 0 \\ 0.0729 & -0.0558 & -0.997 & 0.0802 & 0.0415 & 0 \\ -4.75 & 0.598 & -0.115 & -0.0318 & 0 & 0 \\ 1.53 & -3.05 & 0.388 & -0.465 & 0 & 0 \\ 0 & 0 & 0.0805 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.3333 \end{bmatrix}$$

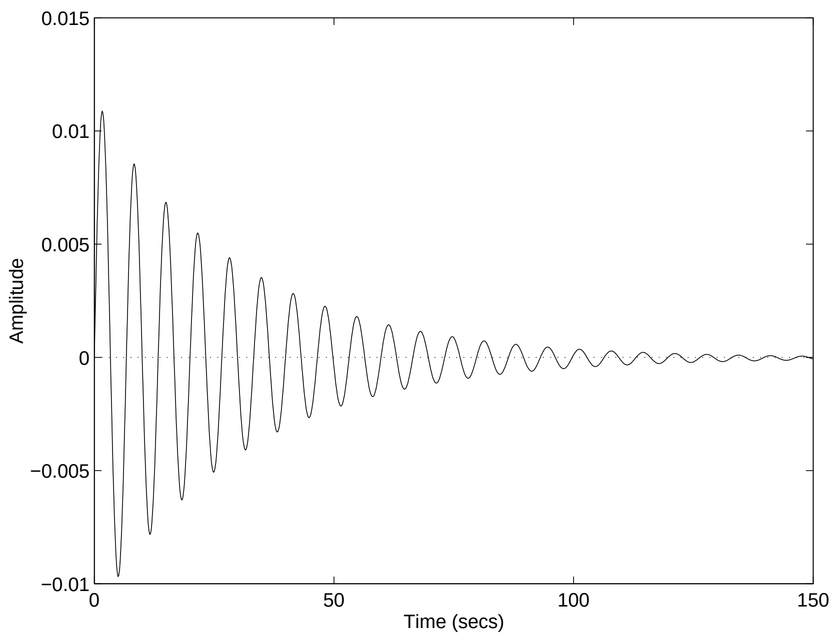
$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -0.3333 \end{bmatrix}, D = 0.$$

The system poles are

$$-0.0329 \pm 0.9467i, \; -0.5627, \; -0.0073, \; -0.3333, \; -10.$$

The poles at $-0.0329 \pm 0.9467i$ have a damping ratio $\zeta = 0.03$ which is far from the desired value $\zeta = 0.5$. The following figure illustrates the consequences of this small damping ratio.

Initial condition response with $\beta = 1°$.



To improve this behavior, we want to design a control law such that the closed loop system has a pair of poles with a damping ratio close to 0.5.

General Format of State Feedback

Control law

$$u = -Kx, \quad K: \quad \text{constant matrix.}$$

For single input systems (SI):

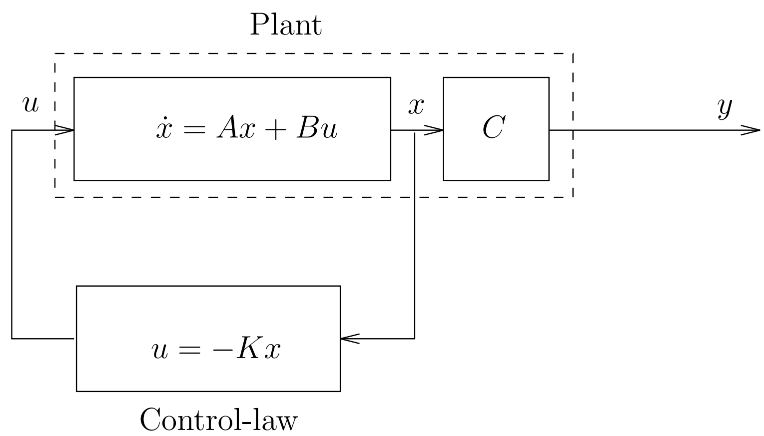
$$K = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}$$

For multi input systems (MI):

$$K = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{p1} & K_{p2} & \cdots & K_{pn} \end{bmatrix}$$

Note: one sensor is needed for each state $\Rightarrow$ disadvantage. We'll see later how to deal with this problem (estimator design).

Structure of state feedback control



Pole Placement

Closed-loop system:

$$\dot{x} = Ax + Bu, u = -Kx. \quad \Rightarrow \quad \dot{x} = (A - BK)x$$

poles of the closed loop system

$$\Updownarrow$$

roots of $\det(sI - (A - BK))$

Pole-placement:

> Choose the gain K such that the poles of the closed loop systems are in specified positions.

More precisely, suppose that the desired locations are given by

$$s = s_1, s_2, \cdots, s_n$$

where $s_i, i = 1, \cdots, n$ are either real or complex conjugated pairs, choose K such that the characteristic equation

$$\alpha_c(s) \stackrel{\Delta}{=} \det\left(sI - (A - BK)\right)$$

equals

$$(s - s_1)(s - s_2)\ldots(s - s_n).$$

Pole-placement - direct method

Find K by directly solving

$$\det(sI - (A - BK)) = (s - s_1)(s - s_2)\cdots(s - s_n)$$

and matching coefficients in both sides.

Disadvantage:

- Solve nonlinear algebraic equations, difficult for n>3.

  Example Let $n = 3$, $m = 1$. Then the following 3rd order equations have to be solved to find $K = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$:

$$\sum_{1\le i\le 3}(a_{ii} - b_iK_i) = \sum_{1\le i\le 3} s_i,$$

$$\sum_{1\le i<j\le 3}\begin{vmatrix} a_{ii} - b_iK_i & a_{ij} - b_iK_j \\ a_{ji} - b_jK_i & a_{jj} - b_jK_j \end{vmatrix} = \sum_{1\le i<j\le 3} s_is_j,$$

$$\begin{vmatrix} a_{11} - b_1K_1 & a_{12} - b_1K_2 & a_{13} - b_1K_3 \\ a_{21} - b_2K_1 & a_{22} - b_2K_2 & a_{23} - b_2K_3 \\ a_{31} - b_3K_1 & a_{32} - b_3K_2 & a_{33} - b_3K_3 \end{vmatrix} = s_1s_2s_3.$$

- You never know whether there IS a solution K. (But THERE IS one if (A, B) is controllable!)

Pole-placement for SI: Ackermann's method

Let A_c, B_c be in a control canonical form, then

$$A_c - B_c K_c = \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & \cdots & \cdots & -a_n - K_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

and

$$\det\left(sI - (A_c - B_c K_c)\right) =$$

$$s^n + (a_1 + K_1)s^{n-1} + (a_2 + K_2)s^{n-2} + \cdots + (a_n + K_n)$$

If

$$(s - s_1)(s - s_2)\cdots(s - s_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

then

$$K_1 = -a_1 + \alpha_1, K_2 = -a_2 + \alpha_2, \cdots, K_n = -a_n + \alpha_n.$$

Design procedure (when A,B are not in control canonical form):

- Transform (A, B) to a control canonical form (A_c, B_c) with a similarity transformation T.
- Find control law K_c with the procedure on the previous page.
- Transform K_c: $K = K_c T^{-1}$.

Note that the system (A, B) must be controllable.

Property:
For SI systems, control law K is unique!

The design procedure can be expressed in a more compact form :

$$K = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \alpha_c(A),$$

where $\mathcal{C}$ is the controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

and

$$\alpha_c(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_n I.$$

Pole-placement for MI - SI generalization

Fact: If (A, B) is controllable, then for almost any $K_r \in \mathbb{R}^{m\times n}$ and almost any $v \in \mathbb{R}^m$, $(A - BK_r, Bv)$ is controllable.

$$\Downarrow$$

From the pole placement results for SI, there is a $K_s \in \mathbb{R}^{1\times n}$ so that the eigenvalues of $A - BK_r - (Bv)K_s$ can be assigned to desired values.

$$\Downarrow$$

Also the eigenvalues of $A - BK$ can be assigned to desired values by choosing a state feedback in the form of

$$u = -Kx = -(K_r + vK_s)x.$$

Design procedure:

- Arbitrarily choose K_r and v such that $(A - BK_r, Bv)$ is controllable.
- Use Ackermann's formula to find K_s for $(A - BK_r, Bv)$.
- Find state feedback gain $K = K_r + vK_s$.

Pole-placement for MIMO - Sylvester equation

Let Λ be a real matrix such that the desired closed-loop system poles are the eigenvalues of Λ. A typical choice of such a matrix is:

$$\Lambda = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ -\beta_1 & \alpha_1 & & & \\ & & \ddots & & \\ & & & \lambda_1 & \\ & & & & \ddots \end{bmatrix},$$

which has eigenvalues: $\alpha_1 \pm j\beta_1, \ldots, \lambda_1, \ldots$. which are the desired poles of the closed-loop system. For controllable systems (A, B) with static state feedback,

$$A - BK \sim \Lambda.$$

$\Rightarrow$ There exists a similarity transformation X such that:

$$X^{-1}(A - BK)X = \Lambda,$$

or

$$AX - X\Lambda = BKX.$$

The trick to solve this equation: split up the equation by introducing an arbitrary auxiliary matrix G:

$$\begin{aligned} AX - X\Lambda &= BG, \quad \text{(Sylvester equation in } X\text{)} \\ KX &= G. \end{aligned}$$

The Sylvester equation is a matrix equation that is linear in X. If X is solved for a known G, then

$$K = GX^{-1}.$$

Design procedure:

- Pick an arbitrary matrix G.
- Solve the Sylvester equation for X.
- Obtain the static feedback gain $K = GX^{-1}$.

Properties:

- There is always a solution for X if A and Λ have no common eigenvalues.
- For SI, K is unique, hence independent of the choice of G.
- For certain special choices of G this method may fail ($e.g. X$ not invertible or ill conditioned). Then just try another G.

Examples for pole placement

Example Boeing 747 aircraft control - control law design with pole placement (Ackermann's method)

Desired poles:

$$-0.0051,\ -0.468,\ -1.106,\ -9.89,\ -0.279 \pm 0.628i$$

which have a maximum damping ratio 0.4.

From the desired poles

$$\alpha_c(s) = s^6 + 12.03s^5 + 23.01s^4 + 19.62s^3 + 10.55s^2 + 2.471s + 0.0123,$$

$$\alpha_c(A) = \begin{bmatrix} 8844.70 & 0 & 0 & 0 & 0 & 0 \\ 368.20 & 7.88 & 2.43 & -0.61 & -0.16 & 0 \\ 4220.20 & -2.47 & 5.93 & -0.46 & -0.23 & 0 \\ -1459.94 & -2.75 & -25.73 & 2.78 & 1.09 & 0 \\ 115.27 & 25.80 & -6.71 & -0.82 & 0.08 & 0 \\ -436.60 & -5.95 & 0.06 & 0.28 & 0.06 & 0.13 \end{bmatrix}$$

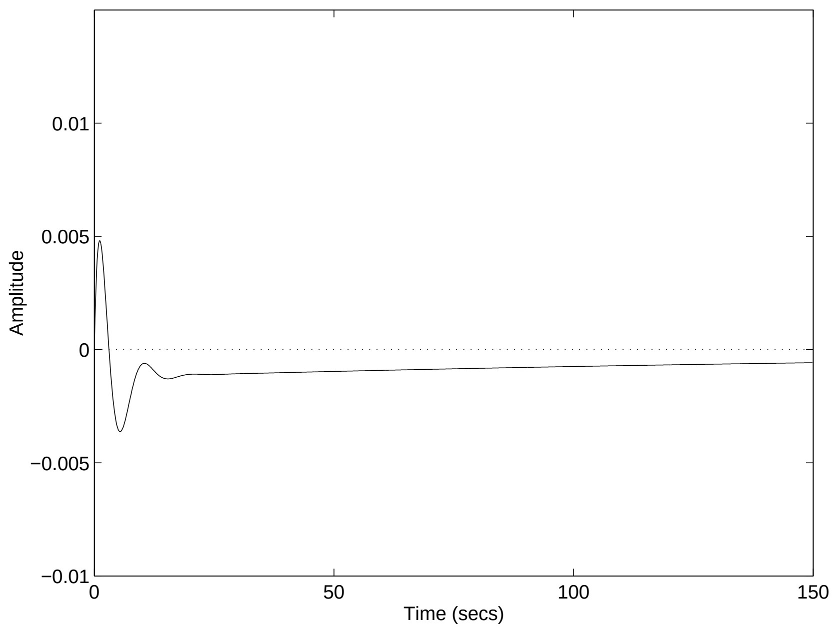
is obtained and hence the controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} 1.00 & -10.00 & 100.00 & -1000.00 & 10000.00 & -100000 \\ 0 & 0.07 & 4.12 & -42.22 & 418.15 & -4180.52 \\ 0 & -4.75 & 48.04 & -477.48 & 4774.34 & -47746.07 \\ 0 & 1.53 & -18.08 & 167.47 & -1664.37 & 16651.01 \\ 0 & 0 & 1.15 & -14.21 & 129.03 & -1280.03 \\ 0 & 0 & -4.75 & 49.62 & -494.03 & 4939.01 \end{bmatrix}.$$

Then the control law is

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \alpha_c(A)$$
$$= \begin{bmatrix} 1.06 & -0.19 & -2.32 & 0.10 & 0.04 & 0.49 \end{bmatrix}.$$

Plot of the initial condition response with $\beta_0 = 1^\circ$:



Much better!

Example Tape drive control - control law design with pole placement (Sylvester equations)
Desired poles:

$$-0.451 \pm 0.937i,\ -0.947 \pm 0.581i,\ -1.16,\ -1.16.$$

Take an arbitrary matrix G:

$$G = \begin{bmatrix} 1.17 & 0.08 & -0.70 & 0.06 & 0.26 & -1.45 \\ 0.63 & 0.35 & 1.70 & 1.80 & 0.87 & -0.70 \end{bmatrix}.$$

Solve the Sylvester equation for X:

$$AX - X\Lambda_c = BG,$$

where

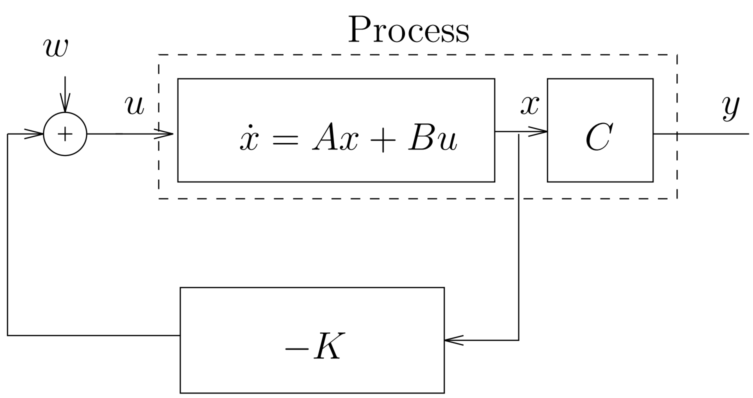
$$\Lambda_c = \begin{bmatrix} -0.451 & 0.937 & & & & \\ -0.937 & -0.451 & & & & \\ & & -0.947 & 0.581 & & \\ & & -0.581 & -0.947 & & \\ & & & & -1.16 & \\ & & & & & -1.16 \end{bmatrix}.$$

$$X = \begin{bmatrix} -0.27 & -1.83 & 1.00 & 0.86 & 2.31 & -10.78 \\ 0.92 & 0.29 & -0.23 & -0.70 & -1.34 & 6.25 \\ 0.56 & -0.76 & -2.26 & -6.25 & 6.42 & -5.74 \\ 0.23 & 0.43 & -0.75 & 3.62 & -3.72 & 3.33 \\ -0.62 & 0.91 & 0.17 & 1.19 & 1.40 & -7.87 \\ -0.58 & 0.33 & 2.99 & -3.15 & 4.75 & -3.76 \end{bmatrix}.$$

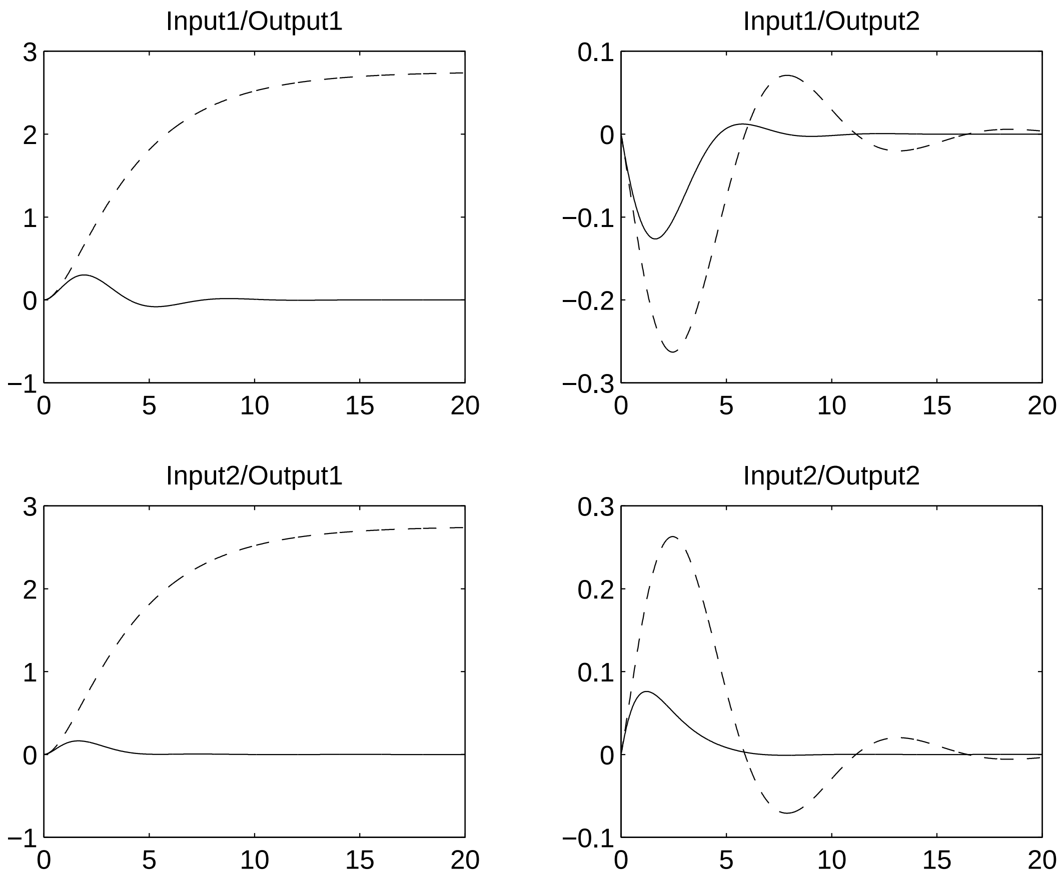
Obtain the static feedback gain $K = GX^{-1}$:

$$K = \begin{bmatrix} 0.55 & 1.58 & 0.32 & 0.56 & 0.67 & 0.05 \\ 0.60 & 0.60 & 0.68 & 3.24 & -0.21 & 1.74 \end{bmatrix}.$$

The closed loop system looks like :



Impulse response (to w):



Dashed line: without feedback, sensitive to process noise.

Solid line: with state feedback, much better!

Property:

Static state feedback does not change the transmission zeros of a system:

$$\text{zeros}(A, B, C, D) = \text{zeros}(A - BK, B, C - DK, D)$$

Proof :

If ζ is a zero of (A, B, C, D), then (when ζ is not a pole), there are u and v such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \zeta$$

Now let $\bar{u} = u$, $\bar{v} = Ku + v$, then

$$\begin{bmatrix} A - BK & B \\ C - DK & D \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} \zeta$$

$\Rightarrow \zeta$ is a zero of $(A - BK, B, C - DK, D)$.

Pole location selection

Dominant second-order poles selection

Use the relations between the time specifications (rise time, overshoot and settling time) and the second-order transfer function with complex poles at radius ω_n and damping ratio ζ.

1. Choose the closed-loop poles for a high-order system as a desired pair of dominant second-order poles.
2. Select the rest of the poles to have real parts corresponding to sufficiently damped modes, so that the system will mimic a second-order response with reasonable control effort.
3. Make sure that the zeros are far enough into the left half-plane to avoid having any appreciable effect on the second-order behavior.

Prototype design

An alternative for higher-order systems is to select prototype responses with desirable dynamics.

- ITAE transfer function poles : a prototype set of transient responses obtained by minimizing a certain criterion of the form

  $$J = \int_0^\infty t|e|dt.$$

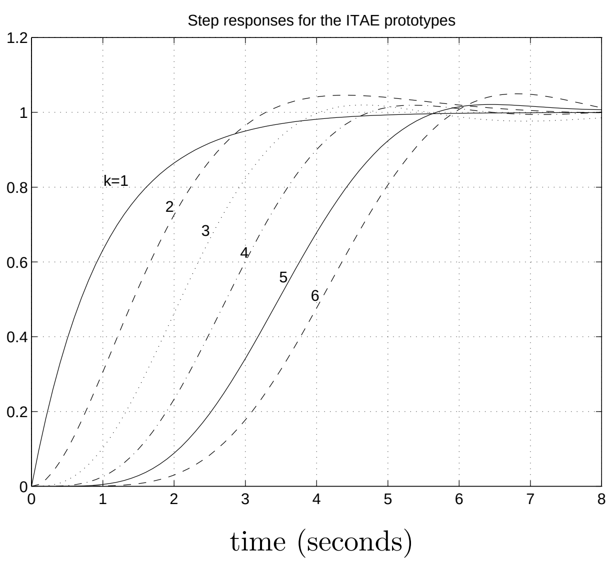
  Property: fast but with overshoot.

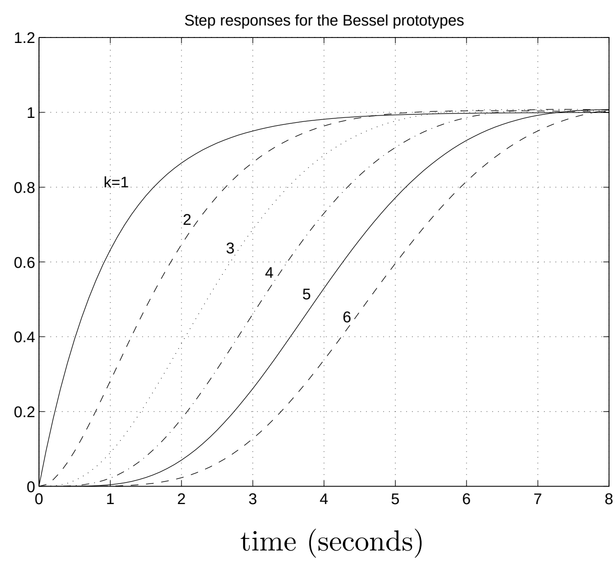
- Bessel transfer function poles : a prototype set of transfer functions of $1/B_n(s)$ where $B_n(s)$ is the nth-degree Bessel polynomial.

  Property: slow without overshoot.

Prototype Response Poles:

	k	Pole Location
(a) ITAE	1	-1
T.F. poles	2	$-0.7071 \pm 0.7071j$
	3	$-0.7081,\ -0.5210 \pm 1.068j$
	4	$-0.4240 \pm 1.2630j,\ 0.6260 \pm 0.4141j$
(b) Bessel	1	-1
T.F. poles	2	$-0.8660 \pm 0.5000j$
	3	$-0.9420, -0.7455 \pm 0.7112j$
	4	$-0.6573 \pm 0.8302j,\ -0.9047 \pm 0.2711j$





Pole locations should be adjusted for faster/slower response. A time scaling with factor α can be applied by replacing the Laplace variable s in the transfer function by s/α.

Examples

Example Tape drive control - Selection of poles. The poles selection methods above are basically for SI systems. Thus consider only the one-motor (the left one) of the tape drive system and set the inertia J of the right wheel 3 times larger than the left one. Then

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx + Du,\end{aligned}$$

where

$$x = \begin{bmatrix} p_1 \\ \omega_1 \\ p_2 \\ \omega_2 \\ i_1 \end{bmatrix},\ A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -0.1 & -0.35 & 0.1 & 0.1 & 0.75 \\ 0 & 0 & 0 & 2 & 0 \\ 0.1 & 0.1 & -0.1 & -0.35 & 0 \\ 0 & -0.03 & 0 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T,\ C = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 \\ -0.2 & -0.2 & 0.2 & 0.2 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ y = \begin{bmatrix} p_3 \\ T \end{bmatrix},\ u = e_1.$$

Specification: the position p_3 has no more than 5% overshoot and a rise time of no more than 4 sec. Keep the peak tension as low as possible.

Pole placement as a dominant second-order system:

Using the formulas on page 79, we find

$$\text{Overshoot } M_p < 5\% \Rightarrow \text{damping ratio } \zeta = 0.6901.$$
$$\text{Rise time } t_r < 4 \text{ sec.} \Rightarrow \text{natural frequency } \omega_n = 0.45$$

The formulas on page 79 are only approximative, therefore we take some safety margin and choose for instance $\zeta = 0.7$ and $\omega_n = 1/1.5$.

$$\Downarrow$$

$$\text{Poles}: \frac{-0.707 \pm 0.707j}{1.5}$$

Other poles far to the left: $-4/1.5$, $-4/1.5$, $-4/1.5$. $\Rightarrow$

$$K = \begin{bmatrix} 8.5123 & 20.3457 & -1.4911 & -7.8821 & 6.1927 \end{bmatrix}.$$

Control law:

$$u = -Kx + 7.0212r,$$

where r is the reference input such that y follows r (in steady state $y = r$).

Pole placement using an ITAE prototype:

Check the step responses of the ITAE prototypes and observe that the rise time for the 5th order system is about 5 sec. So let $\alpha = 5/4 = 1.25$. From the ITAE poles table, the following poles are selected:

$(-0.8955, -0.3764 \pm 1.2920j, -0.5758 \pm 0.5339j) \times 1.25.$

$\Rightarrow K = \begin{bmatrix} 1.9563 & 4.3700 & 0.5866 & 0.8336 & 0.7499 \end{bmatrix}.$

Control law:

$$u = -Kx + 2.5430r.$$

Pole placement using a Bessel prototype:

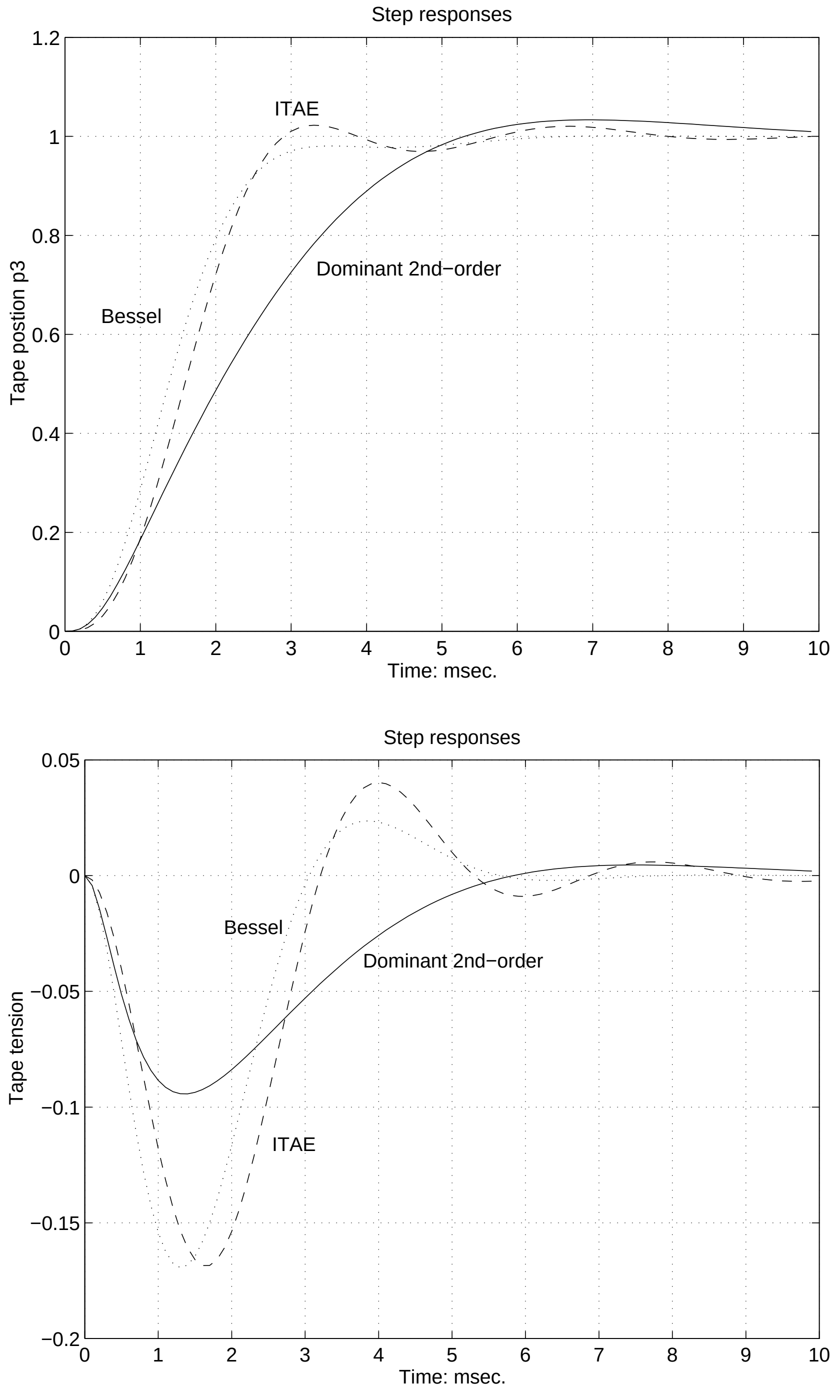
Check the step responses of the Bessel prototypes, it appears that the rise time for the 5th order system is about 6 sec. So let $\alpha = 6/4 = 1.5$. From the Bessel poles table, the following poles are selected:

$(-1.3896, -0.8859 \pm 1.3608j, -1.2774 \pm 0.6641j) \times 1.5.$

$\Rightarrow K = \begin{bmatrix} 3.9492 & 9.1131 & 2.3792 & 5.2256 & 2.9662 \end{bmatrix}.$

$$u = -Kx + 6.3284r.$$

Step responses:



Matlab Functions

poly

real

polyvalm

acker

lyap

place

Chapter 4

State Feedback – LQR

Motivation

Quadratic minimization : least squares

Consider the state space model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k. \end{aligned}$$

Find the control inputs u_k, $k = 0, \ldots, N-1$ such that

$$J_N = \sum_{k=0}^{N-1} (w_k - y_k)^T (w_k - y_k)$$

is minimized, where w_k, $k = 0, \ldots, N-1$ is a given sequence. N is a predefined horizon.
The smaller J_N, the closer the sequence y_k is to the sequence w_k.

The solution can be derived in terms of the following input-output equations :

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}}_{\mathcal{O}_N} x_0 + \underbrace{\begin{bmatrix} H_0 & 0 & \cdots & \cdots & 0 \\ H_1 & H_0 & \cdots & \cdots & 0 \\ H_2 & H_1 & H_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{N-1} & H_{N-2} & \cdots & \cdots & H_0 \end{bmatrix}}_{\mathcal{H}_N} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{u}$$

where H_k are called the Markov parameters (impulse response matrices) defined as :

$$H_0 = D,\ H_k = CA^{k-1}B,\ k = 1, 2, \cdots, N-1.$$

Let vector w be defined in a similar way as y, then

$$\begin{aligned} J_N \ = \ & w^T w + x_0^T \mathcal{O}_N^T \mathcal{O}_N x_0 + u^T \mathcal{H}_N^T \mathcal{H}_N u \\ & -2w^T \mathcal{O}_N x_0 - 2w^T \mathcal{H}_N u + 2x_0^T \mathcal{O}_N^T \mathcal{H}_N u \end{aligned}$$

is the quadratic criterion to be minimized.
Now find the optimal u as the solution to $\min_u J_N$. In order to do so we set the derivatives of J_N with respect to the vector u to zero, resulting in :

$$\mathcal{H}_N^T \mathcal{H}_N u = \mathcal{H}_N^T w - \mathcal{H}_N^T \mathcal{O}_N x_0.$$

When $\mathcal{H}_N^T\mathcal{H}_N$ is invertible, then

$$u_{\text{opt}} = (\mathcal{H}_N^T\mathcal{H}_N)^{-1}\mathcal{H}_N^T(w - \mathcal{O}_N x_0),$$

which can be written as

$$\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{N-1} \end{bmatrix} x_0 + \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}.$$

Constrained minimization : basics of MPC

The least squares solution can be complemented with linear constraints, for instance on the magnitude of the input sequence or on its derivative. This then leads to a quadratic programming problem (see Matlab *quadprog.m*). One could require for instance that all components of the input u are smaller in absolute value than a certain prespecified threshold or that they have to be nonnegative (to avoid that too large or too small inputs are applied to the actuators), that the first derivative of the input u stays within certain bounds (to avoid abrupt changes in the input), etc ... .

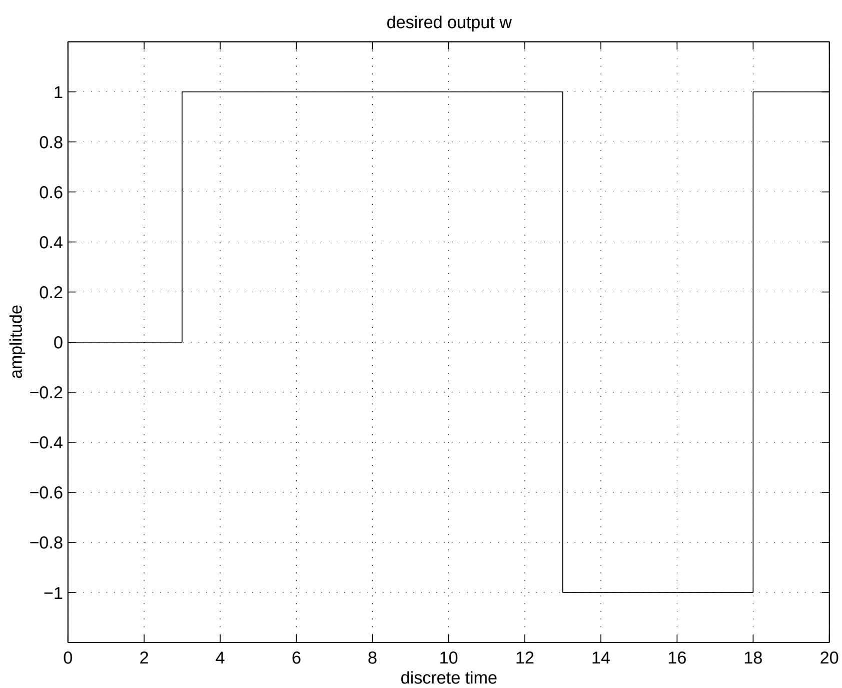
Example :

Consider the following SISO system

$$x_{k+1} = \begin{bmatrix} 0.7 & 0.5 & 0 \\ -0.5 & 0.7 & 0 \\ 0 & 0 & 0.9 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u_k, \quad x_0 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$

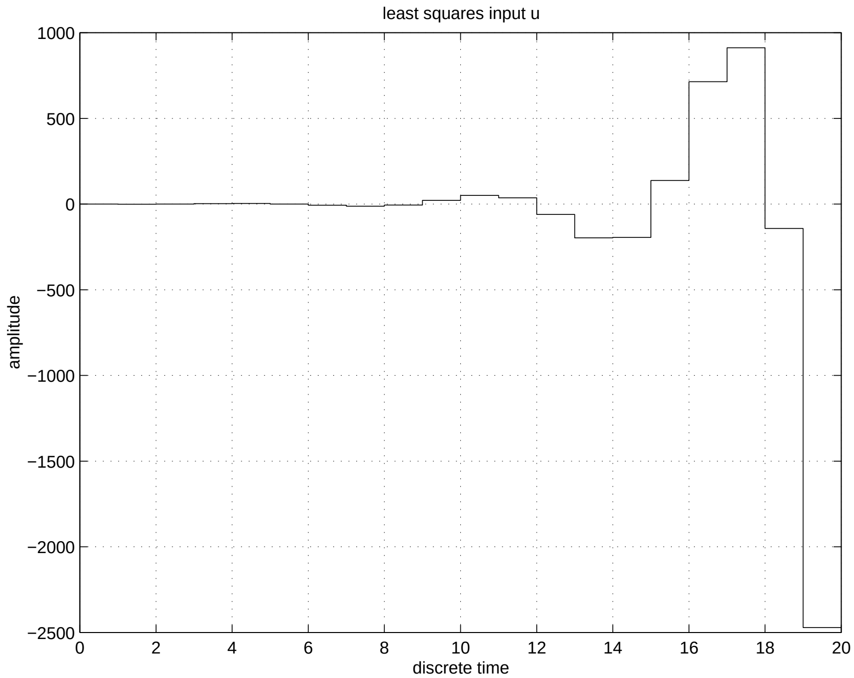
$$y_k = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} x_k + 0.5 u_k.$$

We want the output to be as close as possible to a predefined w by minimizing $(w-y)^T(w-y)$, where w looks like (N=20) :



Construct $\mathcal{H}_N$ and $\mathcal{O}_N$. Now compute u as the least squares solution to $\min_u (w-y)^T(w-y)$ (formula on top of page 127).

The input u is shown in the figure below.



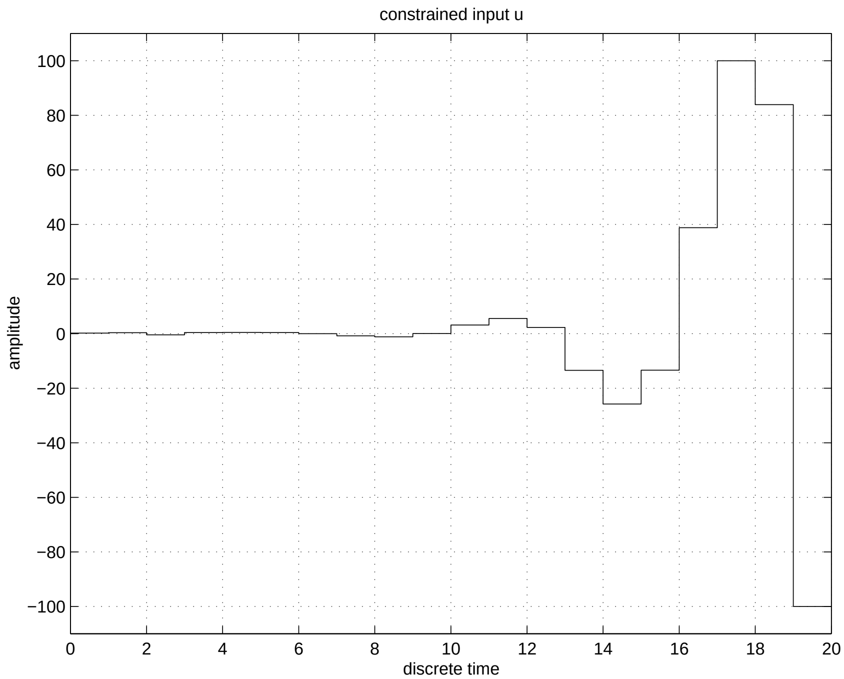
It appears that if this input is applied to the system the output is exactly the desired block wave we wanted.

The input however may be too large. Now let's assume that for safety reasons the input should stay between -100 and 100. Then we have to solve a constrained optimization problem minimizing $(w-y)^T(w-y)$ with respect to u and subject to $-100 \leq u_k \leq 100$. This comes down to a quadratic programming problem which basically solves

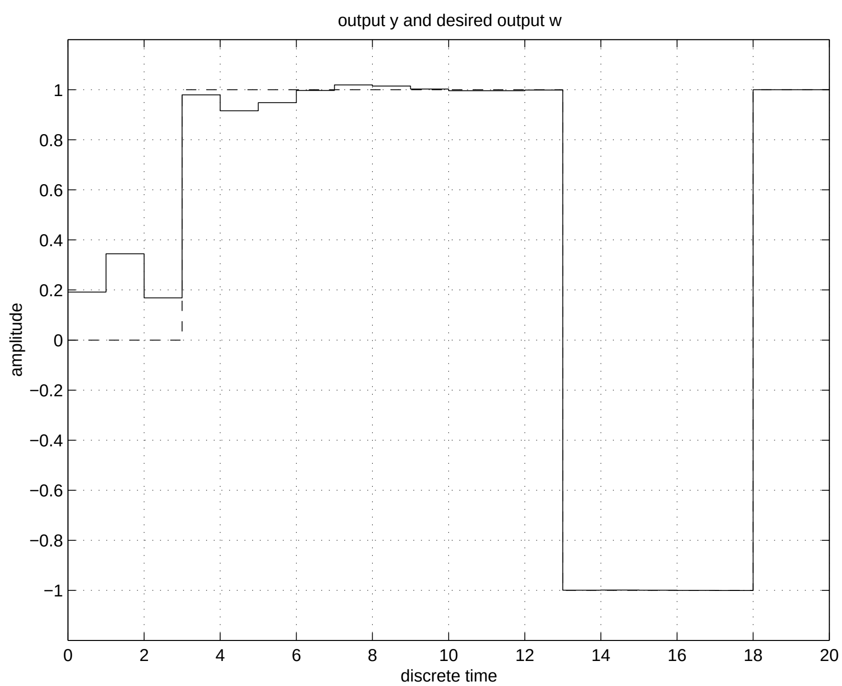
$$\min_u \frac{1}{2} u^T H u + f^T u \qquad \text{subject to } Au \leq b.$$

Matlab program *quadprog.m* can be used to solve this equation.

H and f follow from the expression for J_N on page 126 and A and b depend on the constraints which are being applied. The input is shown in the next figure



and clearly remains between the predefined bounds. The output follows the block wave quite well :



A more general quadratic criterion is one in which the inputs and the outputs are weighted with two weighting matrices $R \in \mathbb{R}^{mN \times mN}$ and $Q \in \mathbb{R}^{pN \times pN}$ such that

$$J_N = u^T R u + (y - w)^T Q (y - w).$$

In this way the controller can be better adjusted to the designer's wishes. Furthermore, it appears that this approach is numerically more robust.
Try to express J_N as a function of w, u, x_0, ... as on page 126. The optimal solution is now given by

$$u_{\text{opt}} = \min_u J_N = (R + \mathcal{H}_N^T Q \mathcal{H}_N)^{-1} (\mathcal{H}_N^T Q w - \mathcal{H}_N^T Q \mathcal{O}_N x_0).$$

Also in this case the minimization criterion can be complemented with linear constraints.

The problem with the controller so far is that the input sequences are not really generated by state feedback. If the system model differs from the real system due to inaccurate modeling or nonlinearities, the previous method might fail. In practice therefore feedback is inserted. Further, only the first element of vector u_{opt} is applied to the system and u_{opt} is recomputed each sample instance. This basically leads to the so–called Model–based Predictive Controller (MPC).

Linear Quadratic Regulator (LQR)

Solving a quadratic programming problem is a computationally intensive operation. For this purpose we propose to use a simpler quadratic cost function of the form

$$\sum_{k=0}^{N}(x_k^T Q x_k + u_k^T R u_k)$$

where both Q and R are nonnegative definite. We also constrain the optimization problem in this case, but now requiring that $u_k = -Kx_k$. By introducing state feedback the regulator is armed against model uncertainties, plant changes, ... Remark that using this simpler approach we cannot force the input to stay within predefined bounds, as with the MPC.

The central question is whether we can find a stabilizing feedback matrix K in an easy way without having to solve a complex optimization problem. In this chapter it will become clear how to choose a feedback matrix K that is optimal w.r.t. a quadratic criterion such as the one defined on the previous page. This will lead to the Linear Quadratic Regulator (LQR). Afterwards we'll show how to design a state estimator.

This overcomes that all state variables need to be measured. Measuring all state variables is too expensive or can be even practically impossible. The chapter on reference introduction and integral design shows how to extend the feedback equation to $u_k = -Kx_k + f(\text{ref})$ such that tracking of an external signal is feasible.

First example of LQR :
Consider the system

$$x_{k+1} = x_k + u_k$$

and apply static feedback $u_k = -\kappa x_k$. Suppose that the cost function is given by

$$\sum_{k=0}^{\infty}(x_k^2 + \rho u_k^2).$$

If ρ is small, we don't care about the magnitude of the control inputs (cheap control) and we are mainly interested in a fast response, requiring that the terms x_k^2 are small. If on the other hand ρ is large, we weigh the control input heavily in the quadratic cost function (high cost).
We find easily that

$$x_k = (1-\kappa)^k x_0$$

Now,

$$\begin{aligned}\sum_{k=0}^{\infty}(x_k^2+\rho u_k^2) &= x_0^2\frac{(1+\rho\kappa^2)}{1-(1-\kappa)^2} \qquad \text{if } 0<\kappa<2 \\ &= \infty \qquad \text{if } \kappa\leq 0 \text{ or } \kappa>2\end{aligned}$$

Observe that the cost is quadratic in x_0. In case that $0<\kappa<2$ we find the optimal value of κ by setting the derivative with respect to κ equal to 0.

$$\kappa_{\text{opt}} = \frac{\sqrt{1+4\rho}-1}{2\rho}$$

For this choice of κ, the optimal cost is

$$\frac{2\rho x_0^2}{\sqrt{1+4\rho}-1} = \frac{x_0^2}{\kappa_{\text{opt}}}$$

For low cost, cheap control, we have :

$$\lim_{\rho\rightarrow 0}\kappa_{\text{opt}} = 1$$

The corresponding cost is $x_0^2(1+\rho)$. For high cost control, we find that $\kappa_{\text{opt}}\approx\frac{1}{\sqrt{\rho}}$ if ρ is large. κ_{opt} is small. It barely stabilizes the system (closed-loop eigenvalue is $1-\frac{1}{\sqrt{\rho}}$) but the plant input is small.

LQR

Quadratic minimization : state feedback

Given the linear time invariant discrete time system

$$x_{k+1} = Ax_k + Bu_k,\ x_0 \text{ is known.}$$

Find a control sequence $u_k, k = 0, \ldots, N-1$ such that

$$J_N = \frac{1}{2}\sum_{k=0}^{N-1}[x_k^T Q x_k + u_k^T R u_k] + \frac{1}{2}x_N^T Q_N x_N$$

is minimized. R, Q and Q_N are nonnegative definite weighting matrices.

Lagrange multipliers method :

Take the state equations for $k = 0, \ldots, N-1$ as N constraints. There will be one Lagrange multiplier vector, l_{k+1}, for each value of k. Write the Lagrangean J_N as

$$\sum_{k=0}^{N-1} \left[\frac{1}{2}x_k^T Q x_k + \frac{1}{2}u_k^T R u_k + l_{k+1}^T(-x_{k+1} + Ax_k + Bu_k)\right] + \frac{1}{2}x_N^T Q_N x_N.$$

Setting the derivatives with respect to u_k, l_{k+1} and the x_k to zero results in a set of

$$\begin{aligned} \text{control equations :} \quad & u_k^T R + l_{k+1}^T B = 0, \\ \text{state equations :} \quad & -x_{k+1} + Ax_k + Bu_k = 0, \\ \text{adjoint equations :} \quad & x_k^T Q - l_k^T + l_{k+1}^T A = 0, \\ \text{final state equations :} \quad & Q_N x_N - l_N = 0. \end{aligned}$$

$\Rightarrow$ Two-point boundary value problem.

The adjoint equations may also be written as

$$l_k = A^T l_{k+1} + Qx_k$$

This equation runs backwards in time. In order to solve this set of coupled difference equations we need to know the initial condition on x_k and the final condition on l_k. x_0 is assumed to be given.

Sweep method to solve the two-point boundary value problem

Assume

$$l_k = S_k x_k, \quad \text{for a certain matrix } S_k.$$

The control equations become :

$$Ru_k = -B^T S_{k+1}(Ax_k + Bu_k)$$

Solving for u_k, we obtain

$$u_k = -P_{k+1}^{-1} B^T S_{k+1} A x_k, \quad \text{with } P_{k+1} = R + B^T S_{k+1} B.$$

From the adjoint equations and the state equations :

$$S_k x_k = A^T S_{k+1}(Ax_k + Bu_k) + Qx_k.$$

Combine these two equations :

$$\left[S_k - A^T \left(S_{k+1} - S_{k+1} B P_{k+1}^{-1} B^T S_{k+1}\right) A - Q\right] x_k \overset{\forall k}{=} 0$$

$\Rightarrow$ Riccati difference equation :

$$S_k = A^T \left[S_{k+1} - S_{k+1} B P_{k+1}^{-1} B^T S_{k+1}\right] A + Q.$$

This equation runs backwards in time. The boundary condition on the recursion is obtained from the final state equations :

$$l_N = Q_N x_N = S_N x_N \to S_N = Q_N.$$

The final solution for finite horizon can be found by solving for u_k :

$$u_k = -K_k x_k,$$

where

$$K_k = (R + B^T S_{k+1} B)^{-1} B^T S_{k+1} A.$$

This matrix K_k is the desired optimal time-varying feedback matrix. Note that the optimal gain sequence is independent of x_0. It can be pre-computed if the system matrices A and B are known and R, Q, Q_N and N are specified.
The optimal cost is

$$J_N^{\min} = \frac{1}{2} x_0^T S_0 x_0.$$

Example :

Consider the following discrete-time system

$$x_{k+1} = 0.3679 x_k + 0.6321 u_k, \qquad x_0 = 1$$

Determine the control sequence that minimizes the following cost function :

$$J = \frac{1}{2} \sum_{k=0}^{9} [x_k^2 + u_k^2] + \frac{1}{2} x_{10}^2$$

Note that in this example $Q = 1$, $R = 1$ and $Q_N = 1$.

The Riccati difference equation we have to solve is

$$S_k = 0.3679 [S_{k+1} - S_{k+1} 0.6321 (1 + 0.6321 S_{k+1} 0.6321)^{-1} 0.6321 S_{k+1}] 0.3679 + 1$$

which can be simplified to

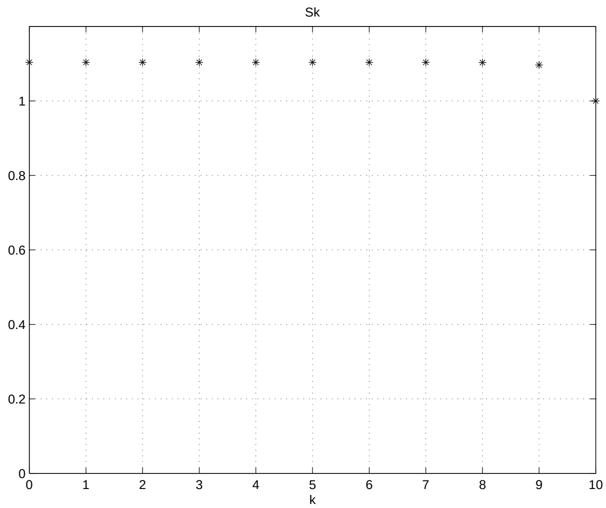
$$S_k = 1 + 0.1354 S_{k+1} [1 + 0.3996 S_{k+1}]^{-1}$$

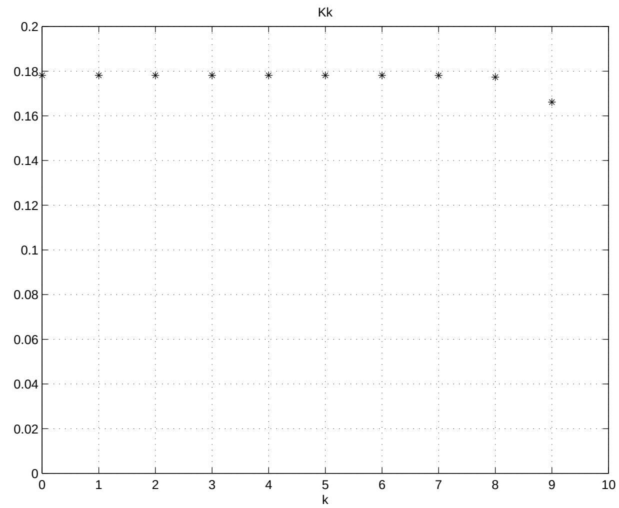
The boundary condition is $S_N = S_{10} = Q_N = 1$.

S_k can be computed backwards from $k = 9$ to $k = 0$. Notice that the values of S_k rapidly approach a steady-state value S_{ss}.

The feedback gain can be computed from

$$K_k = [1 + 0.6321 S_{k+1} 0.6321]^{-1} 0.6321 S_{k+1} 0.3679$$





The important conclusion from this example is that the control gains K_k keep their constant value until at the very end of the time horizon. This suggests that, when the horizon N is large, a 'steady-state' controller with its gain matrix fixed and equal to the initial gain will not differ much from the optimal one.

One can show that the set S_k converges to a steady-state value S. S can be found from the so-called discrete algebraic Riccati equation :

$$S = A^T \left[S - SB(R + B^T SB)^{-1} B^T S \right] A + Q.$$

The steady-state optimal control law is then time-invariant:

$$K = (R + B^T SB)^{-1} B^T SA,$$

The feedback solution for infinite horizon is

$$u_k = -Kx_k.$$

The closed-loop system matrix $A - BK$ is stable if the system (A, B) is controllable and $(A, R^{1/2})$ is observable.
The optimal cost is :

$$J_N^{\min} = \frac{1}{2} x_0^T S x_0.$$

Solving the algebraic Riccati equation

Combine the state equations and the control equations :

$$Ax_k = x_{k+1} + BR^{-1}B^T l_{k+1}$$

From the adjoint equations we have :

$$A^T l_{k+1} = l_k - Qx_k$$

We can combine these equations as

$$\begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} \begin{bmatrix} x_k \\ l_k \end{bmatrix} = \begin{bmatrix} I & BR^{-1}B^T \\ 0 & A^T \end{bmatrix} \begin{bmatrix} x_{k+1} \\ l_{k+1} \end{bmatrix}.$$

If we take the z-transform

$$\begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} \begin{bmatrix} X(z) \\ L(z) \end{bmatrix} = \begin{bmatrix} I & BR^{-1}B^T \\ 0 & A^T \end{bmatrix} \begin{bmatrix} X(z) \\ L(z) \end{bmatrix} z,$$

and call

$$y = \begin{bmatrix} X(z) \\ L(z) \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}$$

and $z = \lambda$

we get

$$\begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} y = \begin{bmatrix} I & BR^{-1}B^T \\ 0 & A^T \end{bmatrix} y\lambda,$$

which is a generalized eigenvalue problem.

If λ is an eigenvalue, the also $1/\lambda$ will be one. So, the eigenvalue spectrum is symmetric with respect to the unit circle.

We assume that the eigenvalues are distinct and $|\lambda_i| < 1$. Then $1/\lambda_i$ are all lying outside the unit circle. Also assume that the vectors v_i are linearly independent.

Now suppose w_i can be written as Sv_i :

$$\begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} \begin{bmatrix} v_i \\ Sv_i \end{bmatrix} = \begin{bmatrix} I & BR^{-1}B^T \\ 0 & A^T \end{bmatrix} \begin{bmatrix} v_i \\ Sv_i \end{bmatrix} \lambda_i.$$

This can be rewritten as

$$\begin{aligned} Av_i &= \lambda_i(I + BR^{-1}B^T S)v_i \\ \lambda_i A^T S v_i &= -Qv_i + Sv_i. \end{aligned}$$

If we pre-multiply the first equation with $A^T S$, we find

$$\lambda_i A^T S v_i = A^T S(I + BR^{-1}B^T S)^{-1} A v_i.$$

Now we can eliminate λ_i to find

$$(S - Q - A^T S(I + BR^{-1}B^T S)^{-1}A)v_i = 0.$$

Since the v_i are independent, we have

$$S - Q - A^T S(I + BR^{-1}B^T S)^{-1}A = 0$$

and it can be proven, using the matrix inversion lemma[1], that this is the algebraic Riccati equation.

Hence we have found an algorithm to solve the algebraic Riccati equation :

1. solve the generalized eigenvalue problem
$$\begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} I & BR^{-1}B^T \\ 0 & A^T \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} \lambda_i,$$
2. let v_i and w_i correspond to the eigenvalues within the unit circle,
3. define $V = [v_1 ... v_N]$ and $W = [w_1 ... w_N]$,
4. then
$$S = WV^{-1}.$$

[1] $(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$

LQR for continuous-time systems

Up till now we restricted ourselves to LQR design for discrete-time system. A similar procedure exists in continuous time. We will discuss the results briefly.

The finite-horizon problem

Be given a continuous-time system :

$$\dot{x} = Ax + Bu, \qquad u = -K(t)x$$

Consider the problem of minimizing the cost function

$$J = \int_t^T (x^T Q x + u^T R u) d\tau + x^T(T) Q_T x(T)$$

with Q, Q_T and R nonnegative definite weighting matrices and $K(t)$ the linear system over which the cost function is going to be minimized.

One can prove that the optimal $K(t) = R^{-1}B^TP(t)$. $P(t)$ is the solution to the following Riccati differential equation:

$$-\dot{P} = PA + A^TP - PBR^{-1}B^TP + Q, \qquad P(T) = Q_T$$

The infinite-horizon problem

Now let $t = 0$ and $T \to \infty$. The optimal controller is $K = R^{-1}B^TP$ where P is the solution to the continuous algebraic Riccati equation :

$$PA + A^TP - PBR^{-1}B^TP + Q = 0.$$

Note that P and K are constant matrices now.

Algebraic Riccati Equations

Consider the following Riccati equation:

$$A^T X + XA - XBB^T X + C^T C = 0$$

Existence, Uniqueness and Stabilizability

Suppose that (A, B) is stabilizable and (A, C) is detectable. Then the Riccati equation has a unique positive semidefinite solution. Moreover, this solution is stabilizing, that is,

$$A - BB^T X$$

has all its eigenvalues in the left half-plane.
Recall that $B^T X$ is the state feedback gain K for LQR, thus the closed loop system with LQR is stable.

Elimination of cross terms

So far we used LQR cost functions that made a trade-off between weighted states x and weighted inputs u. Quite often however, a trade-off between inputs u and outputs y is more appropriate.

For continuous-time systems the direct transmission matrix D is equal to 0 in practice, so $y = Cx$. A cost function, making a trade-off between inputs and outputs, can be defined

$$J = \int_0^\infty (y^T Q y + u^T R u) d\tau = \int_0^\infty (x^T C^T Q C x + u^T R u) d\tau,$$

and apparently falls down to equations we already discussed. Here, Q and R are typically chosen to be diagonal matrices.

However, in discrete-time the direct transmission matrix D typically does exist :

$$y_k = C x_k + D u_k.$$

So

$$y_k^T Q y_k = x_k^T C^T Q C x_k + u_k^T D^T Q D u_k + 2 x_k^T C^T Q D u_k,$$

such that $J = \sum y_k^T Q y_k + u_k^T R u_k$ equals

$$\sum_{k=0}^{\infty} x_k^T C^T Q C x_k + u_k^T (D^T Q D + R) u_k + 2 x_k^T C^T Q D u_k.$$

Hence, we have to solve the quadratic function

$$J = \sum_{k=0}^{\infty} x_k^T P_{xx} x_k + u_k^T P_{uu} u_k + 2 x_k^T P_{xu} u_k.$$

in which some cross terms do appear. They need to be eliminated in order to obtain a Riccati equation. Now define a new input vector v_k as (assuming P_{uu} is symmetric and invertible)

$$v_k = u_k + P_{uu}^{-1} P_{xu}^T x_k.$$

Then we find for the state space equations :

$$\begin{aligned} x_{k+1} &= (A - B P_{uu}^{-1} P_{xu}^T) x_k + B v_k \\ y_k &= (C - D P_{uu}^{-1} P_{xu}^T) x_k + D v_k \end{aligned}$$

and the cost function is transformed into

$$J = \sum_{k=0}^{\infty} x_k^T (P_{xx} - P_{xu} P_{uu}^{-1} P_{xu}^T) x_k + v_k^T P_{uu} v_k$$

without cross terms. Previously discussed techniques can now be applied.

Examples of LQR design

Example Boeing 747 aircraft control - LQR design
Minimize the sum of the energy of the output y and the energy of the control u. The main effort is to minimize the energy of y which is supposed to be zero in a steady state condition. So we put a weight $q = 9.527 > 1$ on the energy of y. The problem now is as follows.

Solve the optimization problem:

$$\min_K \left(\int_0^\infty (q y^T y + u^T u) dt \right), \; u = -Kx, x(0) = x_0$$

where y and u satisfy the lateral model of a Boeing 747.
This corresponds to

$$\min_K \int_0^\infty (x^T C^T Q C x + u^T R u) dt, \quad u = -R^{-1} B^T P x$$

if the weighting matrix for the states is $Q = q$. The weight for the control input u is unity, so $R = 1$.

The Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + C^T QC = 0$$

has a solution :

$$P = \begin{bmatrix} 1.06 & -0.19 & -2.32 & 0.10 & 0.04 & 0.49 \\ -0.19 & 3.12 & 0.13 & -0.04 & 0.02 & -0.34 \\ -2.32 & 0.13 & 5.55 & -0.20 & -0.08 & -1.22 \\ 0.10 & -0.04 & -0.20 & 0.06 & 0.03 & -0.17 \\ 0.04 & 0.02 & -0.08 & 0.03 & 0.02 & -0.10 \\ 0.49 & -0.34 & -1.22 & -0.17 & -0.10 & 1.23 \end{bmatrix}$$

and the feedback control gain

$$K = R^{-1}B^T P = \begin{bmatrix} 1.06 & -0.19 & -2.32 & 0.10 & 0.04 & 0.49 \end{bmatrix}.$$

Example Tape drive control - LQR design

The control specification is :

minimize the energy of the output (the position of the read/write head and the tension at that position) and the control output. The main purpose is mainly to minimize the energy of the output which is required to be zero. However, the control output must not be too large because of the saturation of the actuator (the drive motor here). Thus a certain amount of trial and error is usually required to choose the weighting matrix R and Q.

Let $Q = 2I$ and $R = I$ and solve the Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + C^T QC = 0.$$

The solution for the Riccati equation is

$$P = \begin{bmatrix} 1.02 & 1.73 & 0.84 & 1.36 & 0.55 & 0.45 \\ 1.73 & 4.35 & 1.36 & 3.02 & 1.78 & 1.14 \\ 0.84 & 1.36 & 1.02 & 1.73 & 0.45 & 0.55 \\ 1.36 & 3.02 & 1.73 & 4.35 & 1.14 & 1.78 \\ 0.55 & 1.78 & 0.45 & 1.14 & 0.86 & 0.46 \\ 0.45 & 1.14 & 0.55 & 1.79 & 0.46 & 0.86 \end{bmatrix}$$

and thus

$$K = R^{-1}B^T P = \begin{bmatrix} 0.55 & 1.78 & 0.45 & 1.14 & 0.86 & 0.46 \\ 0.45 & 1.14 & 0.55 & 1.78 & 0.46 & 0.86 \end{bmatrix}.$$

The closed loop poles are at

$$-0.627 \pm 0.865, \; -0.406 \pm 0.727i, \; -1.315, \; -1.041.$$

Now let $Q = 9I$ and $R = I$, then the Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + C^T QC = 0$$

has a solution:

$$P = \begin{bmatrix} 3.52 & 4.83 & 2.84 & 3.41 & 1.26 & 0.86 \\ 4.83 & 9.74 & 3.41 & 5.74 & 3.31 & 1.68 \\ 2.84 & 3.41 & 3.52 & 4.83 & 0.86 & 1.26 \\ 3.41 & 5.74 & 4.83 & 9.74 & 1.68 & 3.31 \\ 1.26 & 3.31 & 0.86 & 1.68 & 1.39 & 0.53 \\ 0.86 & 1.68 & 1.26 & 3.31 & 0.53 & 1.39 \end{bmatrix}$$

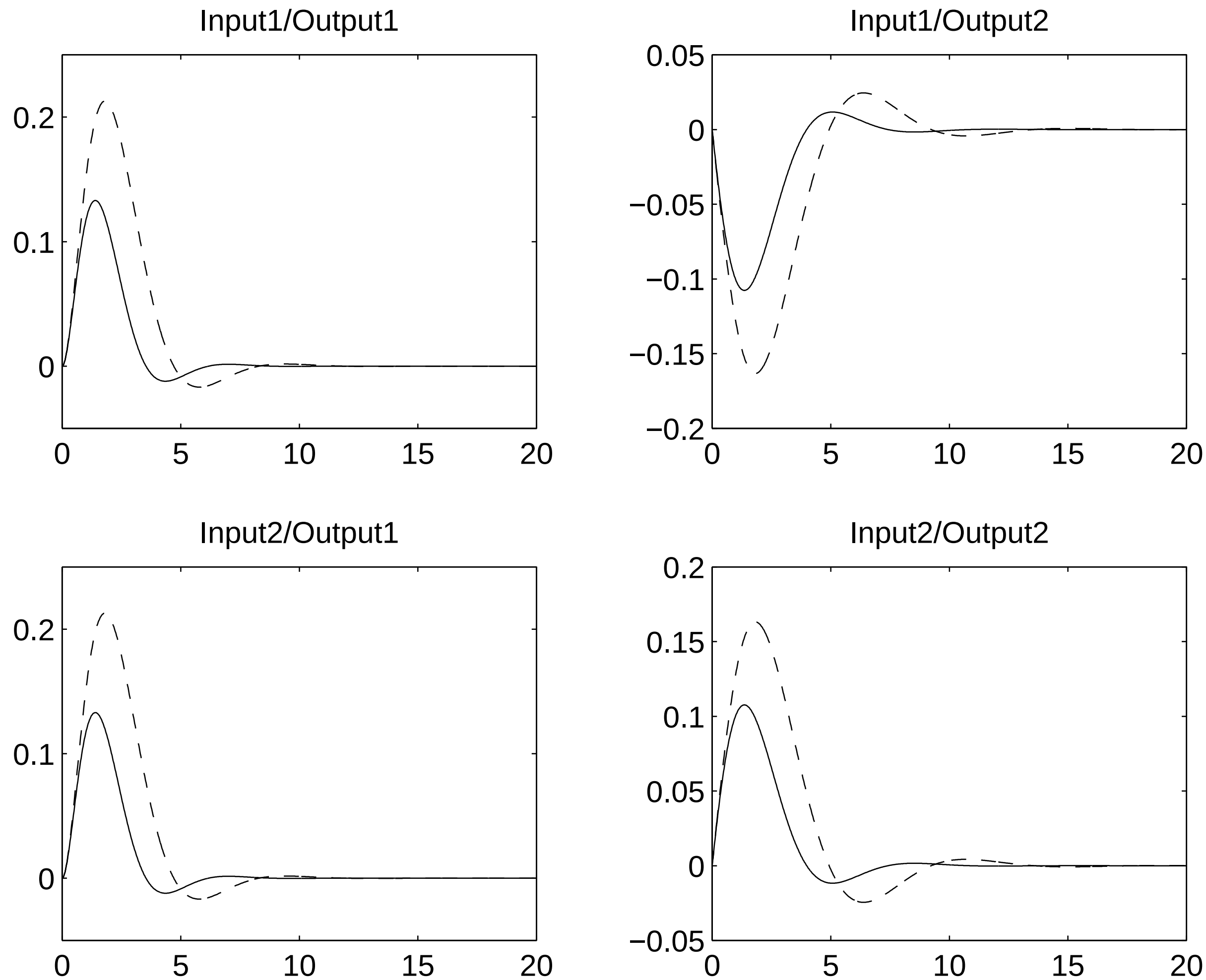
so the state feedback gain:

$$K = R^{-1}B^T P = \begin{bmatrix} 1.26 & 3.31 & 0.86 & 1.68 & 1.39 & 0.53 \\ 0.86 & 1.68 & 1.26 & 3.31 & 0.53 & 1.39 \end{bmatrix}.$$

The closed loop poles are:

$$-0.784 \pm 1.174, \; -0.574 \pm 0.907, \; -1.596, \; -1.158.$$

The output impulse responses:



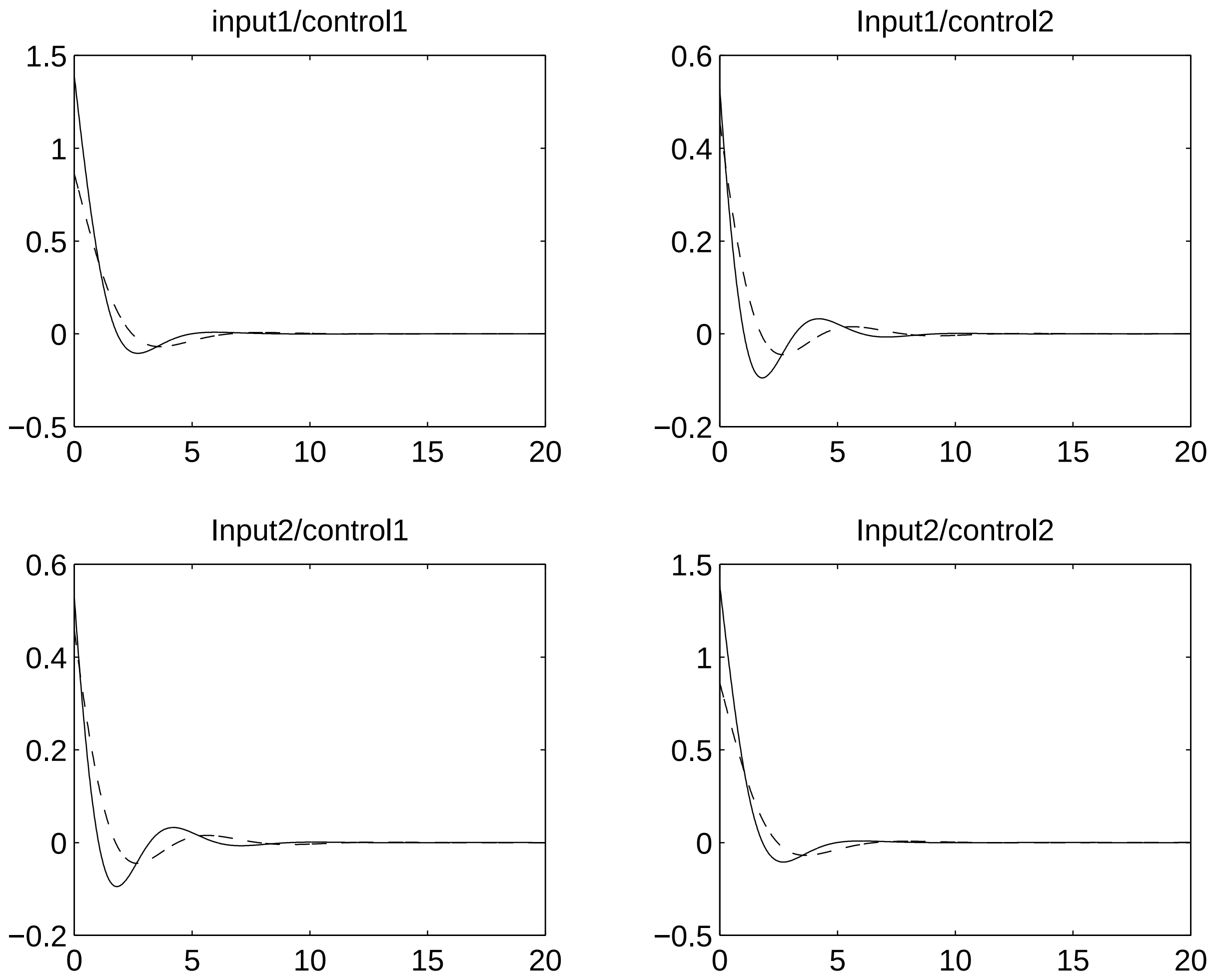
Dashed line: $Q = 2I$

Solid line: $Q = 9I$

The larger Q, the faster and smaller the transient.

(What is the price for a larger Q?)

The control output impulse responses:



Dashed line: $Q = 2I$

Solid line: $Q = 9I$

The larger Q, the larger the control output u.

Matlab Functions

are

dare

dlqr

dlqry

lqr

lqry

quadprog

rlocus

Chapter 5

Estimator Design - Pole placement

Motivation

- Control law design with state feedback needs all the state variables.
- Not all state variables are available since the cost of the required sensors may be prohibitive, or it may be physically impossible to measure all the state variables.

Estimator: reconstruct all the state variables x of a system from measurements. The reconstructed state variables $\hat{x}$ are then used for control.

Motivation example

Example Boeing 747 aircraft control

In the Boeing 747 aircraft model, there are 4 states : the side-slip angle β, the yaw rate r, the roll rate p and the roll angle ϕ. Only the yaw rate r is measured with a gyroscope, while all other states are not measured.

Challenge : Can we estimate the states β, p and ϕ from the measurement of r such that the state feedback control law can be implemented? What are the effects of the estimated states on the final closed loop system?

Full Order Estimators (Observers)

Open loop estimator

Estimator:

Reconstruct a full order model of the plant dynamics:

$$\dot{\hat{x}} = A\hat{x} + Bu.$$

$\hat{x}$ is the estimate of the state x.

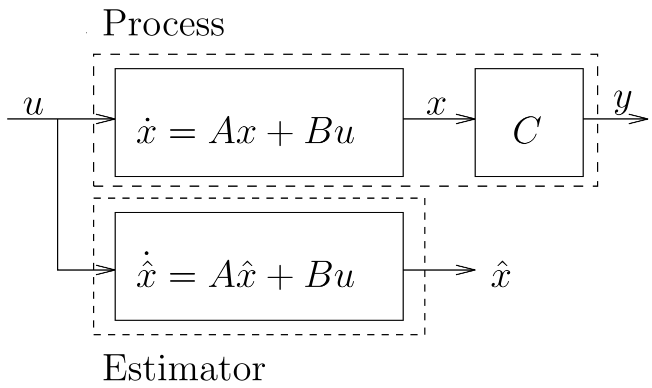
The dynamics of this estimator:

Define the error in the estimate:

$$\tilde{x} \triangleq x - \hat{x}$$

Then the dynamics of this error system are given by (subtracting the estimate from the state):

$$\dot{\tilde{x}} = A\tilde{x}, \quad \tilde{x}(0) = x(0) - \hat{x}(0).$$



Disadvantage of the open loop estimator:

- The error is converging to zero at the same rate as the natural dynamics of A (if A is stable). There is no way to influence the rate at which the state estimate converges to the true state.
- The error will NOT converge if A is unstable.
- If the model dynamics (A, B, C, D) are different from the system dynamics (due to inaccurate modeling for instance), in general, $\tilde{x}$ will not converge to 0.

Estimator with feedback

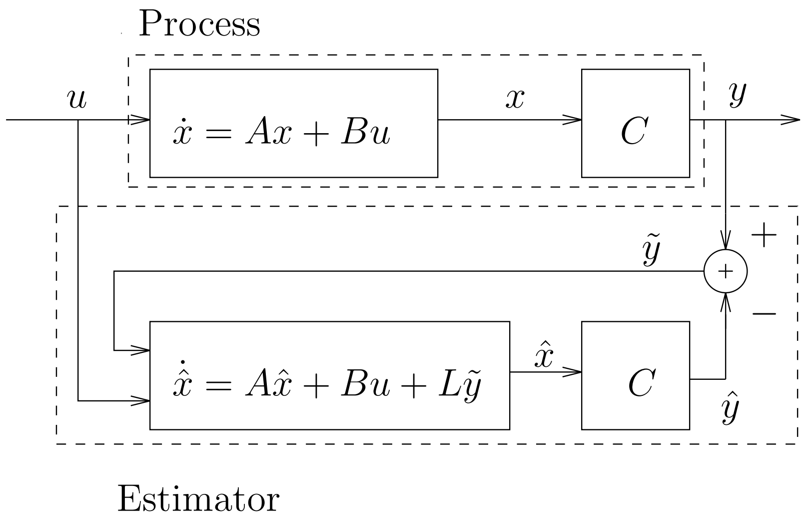
Estimator:
Insert the feedback signal of the difference between the measured and estimated outputs $(y - C\hat{x})$ to correct the model:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}).$$

where L is a constant matrix in $\mathbb{R}^{n\times p}$.

Dynamics of the estimator:
The error dynamics are given as follows (by subtracting the estimate from the state):

$$\dot{\tilde{x}} = (A - LC)\tilde{x}, \quad \tilde{x}(0) = x(0) - \hat{x}(0).$$



Properties:

- By choosing the gain matrix L, the error dynamics of $A - LC$ can be stable and much faster than the open loop dynamics A.
- In case of inaccurate A and/or C, the error dynamics can still be stabilized and be faster $\Rightarrow$ Robustness.

Estimator Design - Pole Placement

Choose the gain matrix L such that the poles of the estimator are in desired positions.

Let the desired locations be given by

$$s = s_1, s_2, \ldots, s_n$$

then the desired estimator characteristic equation is

$$\alpha_e(s) \stackrel{\Delta}{=} (s - s_1)(s - s_2) \ldots (s - s_n)$$

Direct method

Determine L by comparing coefficients of the two polynomials on both sides of the following equation :

$$\det (sI - (A - LC)) = (s - s_1)(s - s_2) \ldots (s - s_n)$$

Duality of estimation and control designs

The problem to find L such that

$$\det\left(sI - (A - LC)\right) = (s - s_1)(s - s_2)\ldots(s - s_n)$$

is equivalent to the problem to find $K = L^T$ such that

$$\det\left(sI - (A^T - C^T K)\right) = (s - s_1)(s - s_2)\ldots(s - s_n)$$

which is the problem for state feedback control design.

Further, the controllability matrix $\mathcal{C}$ of (A, B) is the transpose of the observability matrix $\mathcal{O}$ of (A^T, B^T). Thus we have the following duality relations:

Control	Estimation
A	A^T
B	C^T
K	L^T
$\mathcal{C}$	$\mathcal{O}^T$

Ackermann's formula for SISO

$$L = \alpha_e(A)\mathcal{O}^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

where $\mathcal{O}$ is the observability matrix.

Sylvester equation for MIMO:

Design procedure:

- Pick an arbitrary matrix $G \in \mathbb{R}^{p\times n}$.
- Solve the Sylvester equation for X:

$$A^T X - X\Lambda = C^T G$$

$$\Lambda = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ -\beta_1 & \alpha_1 & & & \\ & & \ddots & & \\ & & & \lambda_1 & \\ & & & & \ddots \end{bmatrix},$$

  which has eigenvalues: $\alpha_1 \pm j\beta_1, \ldots, \lambda_1, \ldots$. which are the desired poles of the estimator.

- Obtain the static feedback gain $L = (GX^{-1})^T$.

Estimator Pole Selection

The dynamics of the state estimation error are governed by the eigenvalues of $A - LC$. Obviously, all eigenvalues are chosen to be stable. Concerning the estimator pole selection there is a fundamental trade-off between sensitivity of the estimation error to sensor noise and speed of convergence. The smaller $\mathcal{R}\mathrm{e}\{\mathrm{eig}(A - LC)\}$, the faster the estimation error goes to zero. However, for small eigenvalues rapid changes in the signals will also introduce rapid changes in the estimation error. The estimated state will change nervously. In the next chapter we will describe an optimal solution for this dilemma which is the Kalman filter.

Consider a control system with process noise w and sensor noise v:

$$\begin{aligned} \dot{x} &= Ax + Bu + B_1 w, \\ y &= Cx + v \end{aligned}$$

then the estimator error will be :

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + B_1 w - Lv$$

Rules of thumb:

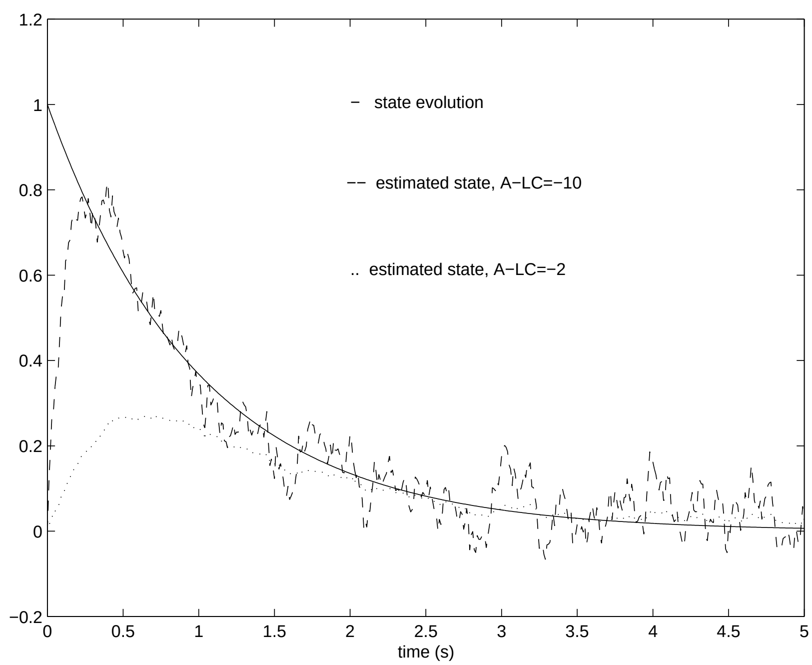
- Trade-off between a fast dynamic response (of the estimator) and a good sensor noise reduction. ⇒ "large" L or "small " L.
- Trade-off between process noise reduction and sensor noise reduction. ⇒ "large" L or "small " L.

Consider the following first order system

$$\dot{x} = -x + u$$
$$y = x + v$$

where v is sensor noise.

Now compare 2 estimators : $A-LC = -10$ and $A-LC = -2$. There is no input, but the initial state is 1. The dynamics for $A - LC = -10$ are faster but more subject to noise :



Examples of estimator design

Example Boeing 747 aircraft control - Estimator design with pole placement (Ackermann's method)
We have designed a state feedback control law, and the desired closed-loop poles are at

$$-0.0051,\ -0.468,\ -1.106,\ -9.89,\ -0.279 \pm 0.628i.$$

For a fast response of the estimator, we make the poles of the estimator 5 times faster than the desired closed-loop poles :

$$-0.0255,\ -2.34\ , -5.53,\ -49.45,\ -1.395 \pm 3.14i.$$

The characteristic polynomial of the estimator

$$\alpha_e(s) = s^6 + 60.14s^5 + 575.4s^4 + 2453s^3 + 6595s^2 + 7721s + 192.6$$

$$\alpha_e(A) =$$

$$\begin{bmatrix} -1.13e+6 & 0 & 0 & 0 & 0 & 0 \\ -4.48e+4 & -5.15e+3 & -4.49e+3 & 6.01e+2 & 2.24e+2 & 0 \\ -5.38e+5 & 3.27e+3 & -4.06e+3 & 1.91e+2 & 1.50e+2 & 0 \\ 1.81e+5 & -8.17e+3 & 1.55e+4 & -3.62e+3 & -6.12e+2 & 0 \\ -1.72e+4 & -1.45e+4 & 8.58e+3 & 4.54e+3 & -6.54e+1 & 0 \\ 5.38e+4 & 3.17e+3 & 3.84e+3 & -3.14e+1 & 5.48e+1 & -1.73e+3 \end{bmatrix}$$

The observability matrix equals

$$\mathcal{O} =$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -3.33e-1 \\ -4.75e+0 & 5.98e-1 & -4.48e-1 & -3.18e-2 & 0 & 1.11e-1 \\ 4.96e+1 & -2.04e-1 & -4.46e-1 & 7.70e-2 & 2.48e-2 & -3.70e-2 \\ -4.94e+2 & -4.90e-1 & 2.50e-1 & -1.32e-2 & -8.49e-3 & 1.23e-2 \\ 4.94e+3 & 2.17e-1 & 4.66e-1 & -4.96e-2 & -2.03e-2 & -4.12e-3 \\ -4.94e+4 & 4.18e-1 & -2.95e-1 & 5.31e-3 & 9.01e-3 & 1.37e-3 \end{bmatrix}$$

Then the feedback gain for the estimator is

$$L = \begin{bmatrix} 2.5047e+01 \\ -2.0517e+03 \\ -5.1935e+03 \\ -2.4851e+04 \\ -4.0914e+04 \\ -1.5728e+04 \end{bmatrix}$$

Example Tape drive control - Estimator design with pole placement (Sylvester equation)

The desired estimator poles are again set to be 5 times faster than the desired closed-loop poles:

$$-2.2550 \pm 4.6850i,\ -4.7350 \pm 2.9050i,\ -5.8000,\ -5.8000.$$

Take an arbitrary matrix G:

$$G = \begin{bmatrix} -0.508 & -0.248 & -0.445 & -0.209 & -1.064 & 1.133 \\ 0.885 & -0.726 & -0.613 & 0.562 & 0.352 & 0.150 \end{bmatrix}$$

Solve the Sylvester equation for X:

$$A^T X - X\Lambda_e = C^T G$$

Λ_e is a block diagonal matrix with the eigenvalues equal to the desired estimator poles:

$$\Lambda_c = \begin{bmatrix} -2.225 & 4.685 & & & & \\ -4.685 & -2.225 & & & & \\ & & -4.735 & 2.905 & & \\ & & -2.905 & -4.735 & & \\ & & & & -5.8 & \\ & & & & & -5.8 \end{bmatrix}.$$

$X =$

$$\begin{bmatrix} -4.0150e-02 & -7.4418e-02 & -3.5794e-02 & -2.4713e-02 & -1.0413e-01 & 9.2375e-02 \\ -6.1041e-02 & 3.7229e-03 & 2.4053e-02 & -3.0478e-02 & 2.4528e-02 & -3.8791e-02 \\ 4.0795e-02 & -3.4260e-02 & -5.2180e-02 & 3.4518e-02 & -7.9299e-02 & 1.0297e-01 \\ 2.1710e-02 & 1.2860e-02 & 1.5984e-03 & 9.5250e-03 & 4.1517e-02 & -3.1544e-02 \\ 2.9984e-03 & 8.9685e-03 & -4.3541e-05 & 6.1539e-03 & -3.8326e-03 & 6.0611e-03 \\ 1.0521e-03 & -3.7573e-03 & -1.1269e-03 & -1.0362e-03 & -6.4870e-03 & 4.9287e-03 \end{bmatrix}$$

Now obtain the estimator feedback gain $L = X^{-T}G^T$:

$$L = \begin{bmatrix} 5.38e+01 & -4.21e+01 \\ 2.04e+02 & -1.56e+02 \\ 3.90e+01 & -7.92e+00 \\ 4.00e+02 & -3.08e+02 \\ 6.51e+02 & -5.82e+02 \\ 1.77e+03 & -1.50e+03 \end{bmatrix}$$

Matlab Functions

obsv

poly

real

polyvalm

acker

lyap

place

Chapter 6

Optimal Estimator - Kalman Filter

Formulation

We want to control the following discrete-time system by state feedback

$$\begin{aligned} x_{k+1} &= Ax_k + B_2 u_k + B_1 w_k \\ y_k &= C_2 x_k + D_2 u_k + v_k \end{aligned}$$

So far, we know how to design a gain matrix K by pole placement or LQR. The states are not measured, but estimated by a state observer. Its dynamics are controlled by a matrix L.

A fast observer reacts more nervously to the measurement noise v, whereas a slow state estimator may be less sensitive to v, but amplifies the process noise w.

If the characteristics of the noise sources w and v are known in advance an optimal state estimator can be designed. This optimal state estimator or Kalman filter will be studied in this chapter.

Discrete-time Kalman filtering

Least squares formulation

Consider a discrete-time system

$$\begin{aligned} x_{k+1} &= Ax_k + B_2 u_k + B_1 w_k \\ y_k &= C_2 x_k + D_2 u_k + v_k. \end{aligned}$$

At a certain time k, the state space equations can be assembled in a large over-determined set of linear equations, with a vector of unknowns θ containing all previous state vectors, up to x_{k+1} :

$$\underbrace{\begin{bmatrix} -B_2 u_0 \\ y_0 - D_2 u_0 \\ -B_2 u_1 \\ y_1 - D_2 u_1 \\ \vdots \\ -B_2 u_k \\ y_k - D_2 u_k \end{bmatrix}}_{\mathbf{b}} = \underbrace{\begin{bmatrix} A & -I & & & & \\ C_2 & 0 & & & & \\ & A & -I & & & \\ & C_2 & 0 & & & \\ & & & \ddots & & \\ & & & & A & -I \\ & & & & C_2 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_k \\ x_{k+1} \end{bmatrix}}_{\theta} + \underbrace{\begin{bmatrix} B_1 w_0 \\ v_0 \\ B_1 w_1 \\ v_1 \\ \vdots \\ B_1 w_k \\ v_k \end{bmatrix}}_{\mathbf{e}}$$

We have to find an optimal state vector estimate $\hat{\theta}$ that is the 'best' solution for the set of equations

$$\mathbf{b} = \mathbf{A}\theta + \mathbf{e}.$$

Vector $\mathbf{e}$ contains stacked and weighted noise samples w_k and v_k. We assume that the characteristics of w and v are known and are such that

$$\begin{aligned} \mathcal{E}(\mathbf{e}) &= \mathbf{0} \qquad : \text{zero-mean} \\ \mathcal{E}(\mathbf{e}\mathbf{e}^{\mathbf{T}}) &= W \end{aligned}$$

W is the so-called noise covariance matrix, which in this case gives some information about the amplitude of the noise components, their mutual correlation and spectral shaping (correlation in time domain).
If the components of both w and v are stationary, mutually uncorrelated and white, having the same variance σ_w^2 and σ_v^2 respectively,

$$W = \begin{bmatrix} \sigma_w^2 B_1 B_1^T & 0 & 0 & \dots & 0 \\ 0 & \sigma_v^2 I & 0 & \dots & 0 \\ 0 & 0 & \sigma_w^2 B_1 B_1^T & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_v^2 I \end{bmatrix}$$

The optimal estimate for θ can be found as

$$\hat{\theta} = (\mathbf{A}^T W^{-1} \mathbf{A})^{-1} \mathbf{A}^T W^{-1} \mathbf{b}$$

$\hat{\theta}$ is the least squares solution to

$$\min_{\theta} \{ (\mathbf{b} - \mathbf{A}\theta)^T W^{-1} (\mathbf{b} - \mathbf{A}\theta) \}$$

based on input-output data up to time k.

Disadvantage: $k \to \infty \Rightarrow$ dimension of $\mathbf{A} \to \infty$.

Recursive least squares problem
Finite horizon case :

How can we find an optimal estimate $\hat{x}_{k+1|k}$, in the least squares sense, for the state x_{k+1}, by making use of observations of inputs u and outputs y up to time k, given $\hat{x}_{k|k-1}$, the estimate for x_k at time $k-1$?

Consider the following state observer

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + B_2 u_k + L_k(y_k - C_2\hat{x}_{k|k-1} - D_2 u_k).$$

Combining it with the system equations leads to the estimator error equation

$$\tilde{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k} = (A - L_k C_2)\tilde{x}_{k|k-1} + B_1 w_k - L_k v_k$$

We will assume that w_k and v_k have zero-mean white noise characteristics with covariance

$$\begin{aligned} \mathcal{E}(w_j w_k^T) &= Q\delta_{jk}, \\ \mathcal{E}(v_j v_k^T) &= R\delta_{jk}, \\ \mathcal{E}(v_j w_k^T) &= 0 \end{aligned}$$

and Q and R nonnegative definite weighting matrices.

Our aim now is to find an optimal estimator gain L_k that minimizes

$$\alpha^T P_{k+1} \alpha = \alpha^T \mathcal{E}((\tilde{x}_{k+1} - \mathcal{E}(\tilde{x}_{k+1}))(\tilde{x}_{k+1} - \mathcal{E}(\tilde{x}_{k+1}))^T)\alpha.$$

α is an arbitrary column vector. P_k is called the estimation error covariance matrix.

From the estimator error equation and taking in account the properties of the noise :

$$\mathcal{E}(\tilde{x}_{k+1}) = (A - L_k C_2)\mathcal{E}(\tilde{x}_k).$$

Now if we take $\mathcal{E}(\tilde{x}_0) = 0$, the mean value of the estimation error is zero for all k. Then,

$$\begin{aligned} P_{k+1} &= \mathcal{E}((\tilde{x}_{k+1})(\tilde{x}_{k+1})^T) \\ &= (A - L_k C_2) P_k (A - L_k C_2)^T + B_1 Q B_1^T + L_k R L_k^T \end{aligned}$$

So, if P_0 is properly set, P_k will be symmetric positive semidefinite for all k .

The optimal L_k or Kalman filter gain can be obtained by minimizing

$$L_k = \underset{L_k}{\operatorname{argmin}} \, \alpha^T P_{k+1} \alpha.$$

Setting the derivative with respect to L_k equal to zero,

$$\alpha^T(-2AP_kC_2^T + 2L_k(R + C_2P_kC_2^T))\alpha = 0.$$

The equality has to be fulfilled for all vectors α, so

$$L_k = AP_kC_2^T(R + C_2P_kC_2^T)^{-1}.$$

and the error covariance update equation becomes

$$P_{k+1} = AP_kA^T + B_1QB_1^T - AP_kC_2^T(R + C_2P_kC_2^T)^{-1}C_2P_kA^T$$

which is a Riccati difference equation.

Infinite horizon case :

It can be shown that if the system is controllable and observable and $k \to \infty$, matrix P_k converges to a steady-state positive semidefinite matrix P and L_k approaches a constant matrix L with

$$L = APC_2^T(R + C_2PC_2^T)^{-1}.$$

P satisfies the Discrete Algebraic Riccati Equation :

$$P = B_1QB_1^T + APA^T - APC_2^T(R + C_2PC_2^T)^{-1}C_2PA^T$$

Both for the finite horizon and the infinite horizon case, the Kalman filter equations are similar to the equations we obtained for LQR design.

Verify the duality relation between Kalman filter design and LQR by using the following conversion table and plugging in the correct matrices in the LQR equations.

Conversion table

LQR $\leftrightarrow$ Kalman filter design

LQR	Kalman filter
A	A^T
B	C_2^T
Q	$B_1 Q B_1^T$
K	L^T

Kalman filters may be designed based on LQR formulas using this table, as is done in Matlab for instance. The Kalman filter is sometimes referred to as LQE, i.e. a Linear Quadratic Estimator.

Continuous-time Kalman filtering

Consider a system:

$$\begin{aligned}\dot{x} &= Ax + B_2u + B_1w,\\ y &= C_2x + D_2u + v\end{aligned}$$

and the estimator:

$$\dot{\hat{x}} = A\hat{x} + L(y - C_2\hat{x} - D_2u) + B_2u$$

where the input disturbance noise w and the sensor noise v are zero mean white noise with covariance Q and R respectively.

Find an optimal L such that the following stochastic cost function

$$\mathcal{E}\left\{\frac{1}{T}\int_0^T \big((x-\hat{x})^T(x-\hat{x})dt\big)\right\}$$

is minimized.

Finite-horizon case :

This leads to the Riccati differential equation :

$$-\dot{P} = PA^T + AP - PC_2^T R^{-1} C_2 P + B_1 Q B_1^T, \quad P(0) = 0.$$

The optimal estimator or Kalman gain :

$$L = P(t) C_2^T R^{-1}$$

Infinite-horizon case :

This leads to the continuous Algebraic Riccati equation :

$$PA^T + AP - PC_2^T R^{-1} C_2 P + B_1 Q B_1^T = 0$$

and the corresponding Kalman gain :

$$L = PC_2^T R^{-1}$$

Duality with LQR design :

Also for the continuous time case the conversion rules on page 182 hold.

Examples of Kalman Filter Design

Example Boeing 747 aircraft control - Kalman Filter

Suppose the plant noise w enters the system in the same way as the control input and suppose the measurement noise is v. Then

$$\begin{aligned} \dot{x} &= Ax + Bu + B_1 w, \\ y &= Cx + Du + v. \end{aligned}$$

where (A, B, C, D) is the nominal aircraft model given in the previous examples. $B_1 = B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. Suppose further that w and v are white noises with the covariance of w, $R_w = 0.7$ and the covariance of v $R_v = 1$.

The Riccati equation

$$QA^T + AQ - QC^T R_v^{-1} CQ + B_1 R_w B_1^T = 0$$

has a solution:

$$Q = \begin{bmatrix} 3.50e-2 & 1.94e-3 & -1.60e-2 & 2.88e-3 & -3.08e-4 & -1.55e-3 \\ 1.94e-3 & 5.20e-1 & 3.68e-2 & -9.45e-1 & 3.43e+0 & -3.88e-2 \\ -1.60e-2 & 3.68e-2 & 8.18e-1 & -6.79e-1 & 1.34e+1 & 1.78e+0 \\ 2.88e-3 & -9.45e-1 & -6.79e-1 & 5.06e+0 & -1.03e+0 & 1.59e-1 \\ -3.08e-4 & 3.43e+0 & 1.34e+1 & -1.03e+0 & 3.51e+2 & 4.12e+1 \\ -1.55e-3 & -3.88e-2 & 1.78e+0 & 1.59e-1 & 4.12e+1 & 5.33e+0 \end{bmatrix}$$

and the Kalman filter gain is

$$L = QC^T R_v^{-T} = \begin{bmatrix} -1.5465e-02 \\ 4.9686e-02 \\ 2.2539e-01 \\ -7.3199e-01 \\ -3.0200e-01 \\ -8.2157e-15 \end{bmatrix}.$$

Now we can compare this result with the result from pole placement discussed in the previous chapter.
The Kalman Filter has poles at

$$-9.992,\ -0.1348 \pm 0.9207i,\ -0.5738,\ -0.0070,\ -0.352$$

They are slower than those from pole placement, but the Kalman filter is less (actually least) sensitive to the noises.

Example Tape drive control - Kalman filter design

Suppose again that the plant noise w enters the system in the same way as the control input and suppose that the measurement noise is v. Then

$$\begin{aligned}\dot{x} &= Ax + Bu + B_1 w,\\ y &= Cx + Du + v.\end{aligned}$$

where (A, B, C, D) is the nominal model given in the previous examples and

$$B_1 = B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T.$$

Suppose that the RMS accuracy of the tape position measurement is $2 \cdot 10^{-5}$ m and the RMS accuracy of tension measurement is 0.01 N. This means the covariance matrix of v is

$$R_v = \begin{bmatrix} 4 \cdot 10^{-10} & 0 \\ 0 & 0.0001 \end{bmatrix}.$$

The covariance matrix of w is given as

$$R_w = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}.$$

The following Riccati equation

$$QA^T + AQ - QC^T R_v^{-1} CQ + B_1 R_w B_1^T = 0$$

has as solution :

$$Q=\begin{bmatrix} 2.23e-1 & 3.16e-3 & 2.22e-1 & 3.02e-3 & 2.39e-4 & 1.07e-4 \\ 3.16e-3 & 4.81e-4 & 3.02e-3 & 3.35e-4 & 2.22e-4 & 5.33e-5 \\ 2.22e-1 & 3.02e-3 & 2.23e-1 & 3.16e-3 & 1.07e-4 & 2.39e-4 \\ 3.02e-3 & 3.35e-4 & 3.16e-3 & 4.81e-4 & 5.33e-5 & 2.22e-4 \\ 2.39e-4 & 2.22e-4 & 1.07e-4 & 5.33e-5 & 4.75e-4 & 1.65e-5 \\ 1.07e-4 & 5.33e-5 & 2.39e-4 & 2.22e-4 & 1.65e-5 & 4.75e-4 \end{bmatrix}$$

and the Kalman filter gain is

$$L = QC_2^T R_v^{-T} = \begin{bmatrix} 5.56e-02 & -1.67e+00 \\ 7.74e-04 & -5.71e-01 \\ 5.56e-02 & 1.67e+00 \\ 7.74e-04 & 5.71e-01 \\ 4.32e-05 & -6.02e-01 \\ 4.32e-05 & 6.02e-01 \end{bmatrix}.$$

The Kalman filter poles are at

$-0.5882 \pm 0.9226i,\ -1.1695,\ -0.0633\ , -0.2735\ , -0.969.$

Again these poles are slower, but less sensitive to the noise than those used in the pole placement design example from a previous chapter.

Matlab Functions

are

dkalman

dlqe

kalman

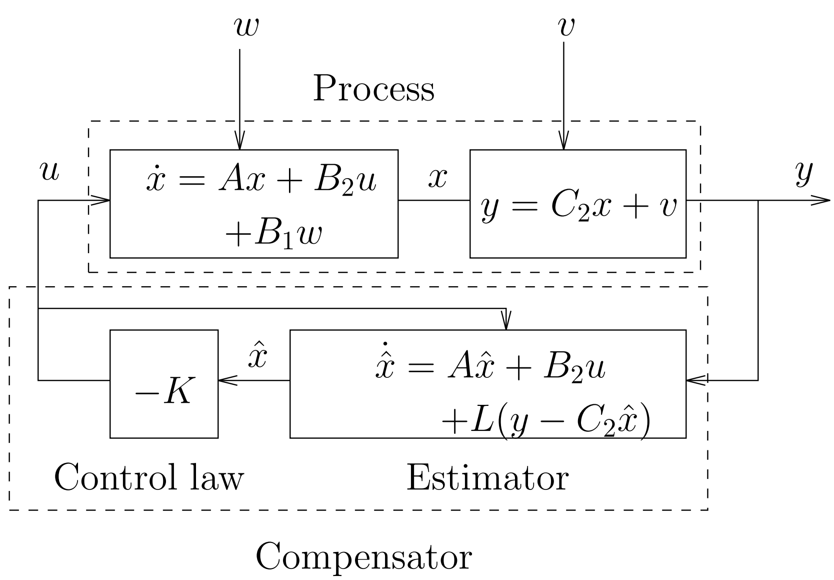
lqe

Chapter 7

Compensator Design

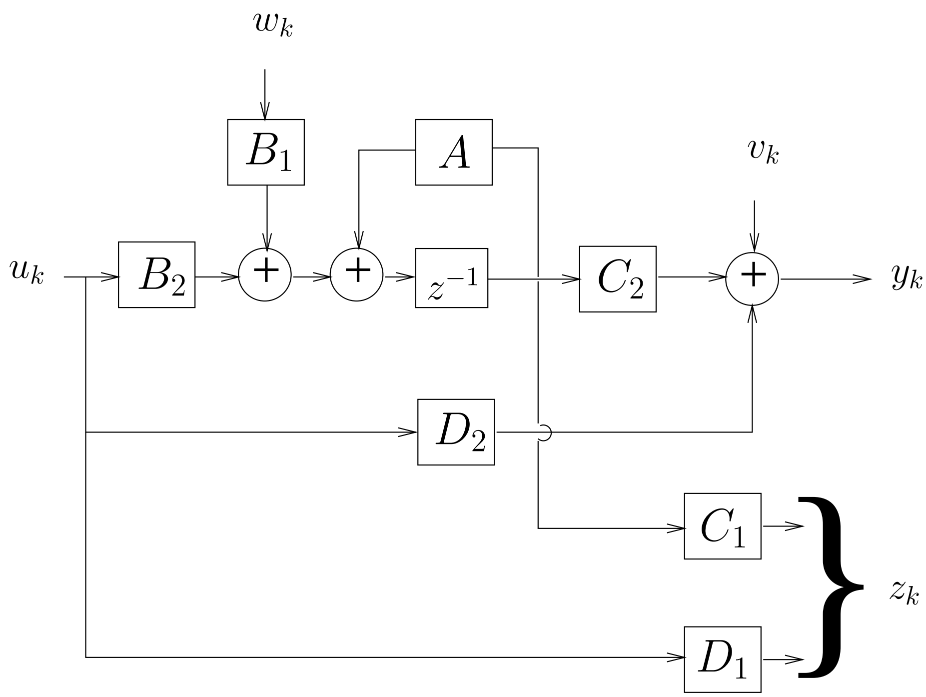
Compensator Structure

Compensator = Control Law + Estimator



Standard Plant Formulation

In control theory the following general system description, called standard plant, is often used



$$
\begin{aligned}
x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k \\
y_k &= C_2 x_k + D_2 u_k + v_k \\
z_k &= \begin{bmatrix} C_1 x_k \\ D_1 u_k \end{bmatrix}
\end{aligned}
$$

where x_k is the state vector, z_k is the regulated output vector, y_k is the measurement vector, u_k is the control input, w_k and v_k are process noise and measurement noise respectively.

Clearly there are 2 types of inputs :

- the actuator or control input u_k : inputs manipulated by the compensator
- the exogeneous input w_k : all other input signals

and 2 types of outputs :

- measured output y_k : inputs to the compensator
- regulated output z_k : all outputs that are of interest for control. These outputs may be virtual and not measured. They typically define the state feedback cost function
$$J_N = \sum_{k=0}^{N} z_k^T z_k = \sum_{k=0}^{N} (x_k^T C_1^T C_1 x_k + u_k^T D_1^T D_1 u_k)$$
so, using the LQR formalism, by definition of z_k,
$Q = C_1^T C_1$ and $R = D_1^T D_1$.

State-space Description

Plant :

$$
\begin{aligned}
x_{k+1} &= Ax_k + B_2 u_k + B_1 w_k, \\
z_k &= \begin{bmatrix} C_1 x_k \\ D_1 u_k \end{bmatrix}, \\
y_k &= C_2 x_k + D_2 u_k + v_k.
\end{aligned}
$$

Estimator :

$$
\hat{x}_{k+1} = A\hat{x}_k + L(y_k - C_2\hat{x}_k - D_2 u_k) + B_2 u_k.
$$

Control law :

$$
u_k = -K\hat{x}_k.
$$

Compensator :

$$
\begin{aligned}
\hat{x}_{k+1} &= (A - B_2 K - LC_2 + LD_2 K)\hat{x}_k + Ly_k, \\
u_k &= -K\hat{x}_k.
\end{aligned}
$$

Closed-Loop System

A state-space model for plant + compensator is

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A & -B_2K \\ LC_2 & A - B_2K - LC_2 \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} w_k \\ v_k \end{bmatrix},$$

$$y_k = \begin{bmatrix} C_2 & -D_2K \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} + \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} w_k \\ v_k \end{bmatrix}.$$

Using the error system (estimator) dynamics

$$\tilde{x}_{k+1} = (A - LC_2)\tilde{x}_k + B_1 w_k - L v_k,$$

we obtain

$$\begin{bmatrix} x_{k+1} \\ \tilde{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A - B_2K & B_2K \\ 0 & A - LC_2 \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ B_1 & -L \end{bmatrix} \begin{bmatrix} w_k \\ v_k \end{bmatrix},$$

$$y_k = \begin{bmatrix} C_2 - D_2K & D_2K \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix} + \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} w_k \\ v_k \end{bmatrix}.$$

Separation Principle: Pole Placement

The characteristic equation of the closed-loop system

$$\det \begin{bmatrix} sI - A + B_2K & -B_2K \\ 0 & sI - A + LC_2 \end{bmatrix} = 0$$

The poles of the closed-loop system can be obtained from

$$\det(sI - A + BK)\det(sI - A + LC) = \alpha_c(s)\alpha_e(s) = 0$$

$\Rightarrow$ Separation Principle (for Pole Placement):

> The set of poles of the closed-loop system consists of the union of the control poles and estimator poles.

Design procedure :

- Choose the desired poles for the state feedback control loop. Determine the control law K via pole placement.
- Choose the desired poles for the estimator. Determine the estimator feedback gain L via pole placement.
- Combine the estimator and the control law to get the final compensator.

Linear Quadratic Gaussian Control

Measurement feedback control

Let w and v be Gaussian zero mean white noises.

Our aim is to find an output feedback controller $u = \mathbf{K}y$ such that

$$\lim_{T\to\infty} \left\{ \frac{1}{T} \int_0^T z^T z dt \right\} \quad \text{or} \quad \lim_{K\to\infty} \left\{ \frac{1}{K} \sum_{k=0}^{K} z_k^T z_k \right\}$$

is minimized.

This problem is called the Linear Quadratic Gaussian (LQG) control problem.
The difference between LQG and LQR is that LQR uses state feedback and hence $\mathbf{K}$ is just a constant matrix while LQG uses measurement or output feedback and hence $\mathbf{K}$ is a dynamic system.

Stochastic Separation Principle

It can be proven that the solution to the LQG optimal control problem consists in using the optimal state estimator (Kalman filter) together with the same feedback as would have been applied had the states been measured (LQR). This principle relies on the assumption of linearity, the Gaussian character of the noise and the use of a quadratic optimization criterion.

Design procedure (summary) :

- LQR design: find an optimal state feedback control law $u_k = -Kx_k$ for the system

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k, \\ z_k &= \begin{bmatrix} C_1 x_k \\ D_1 u_k \end{bmatrix}. \end{aligned}$$

where $K = (D_1^T D_1 + B_2^T S B_2)^{-1} B_2^T S A$, in which S is a solution for the Riccati equation :

$$S = A^T \left[S - SB_2(D_1^T D_1 + B_2^T S B_2)^{-1} B_2^T S\right] A + C_1^T C_1.$$

- Kalman filter design : find an optimal estimator

$$\hat{x}_{k+1} = A\hat{x}_k + L(y_k - C_2\hat{x}_k - D_2 u_k) + B_2 u_k$$

for the system

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k, \\ y_k &= C_2 x_k + D_2 u_k + v_k, \end{aligned}$$

where $L = APC_2^T(R + C_2 P C_2^T)^{-1}$, in which P is a solution for the Riccati equation :

$$P = B_1 Q B_1^T + APA^T - APC_2^T(R + C_2 P C_2^T)^{-1} C_2 P A^T.$$

with $R = \mathrm{cov}(v)$ and $Q = \mathrm{cov}(w)$.

- Combination of the LQR control law and the Kalman filter : the final controller **K** is given as follows :

$$\begin{aligned}\hat{x}_{k+1} &= (A - B_2K - LC_2 + LD_2K)\hat{x}_k + Ly_k,\\ u_k &= -K\hat{x}_k.\end{aligned}$$

Design strategy for LQG :

Trade-off between speed of time responses and attenuation of noises by choosing the covariance matrices of w and v.

Similarly,

Trade-off between the energy of the states and energy of the control effort by choosing the weighting matrices $C_1^TC_1$ and $D_1^TD_1$.

Compensator Dynamics

Compensator :

$$\begin{aligned} \hat{x}_{k+1} &= (A - B_2K - LC_2 + LD_2K)\hat{x}_k + Ly_k, \\ u_k &= -K\hat{x}_k. \end{aligned}$$

Characteristic equation of the compensator :

$$\det(zI - A + B_2K + LC_2 - LD_2K) = 0$$

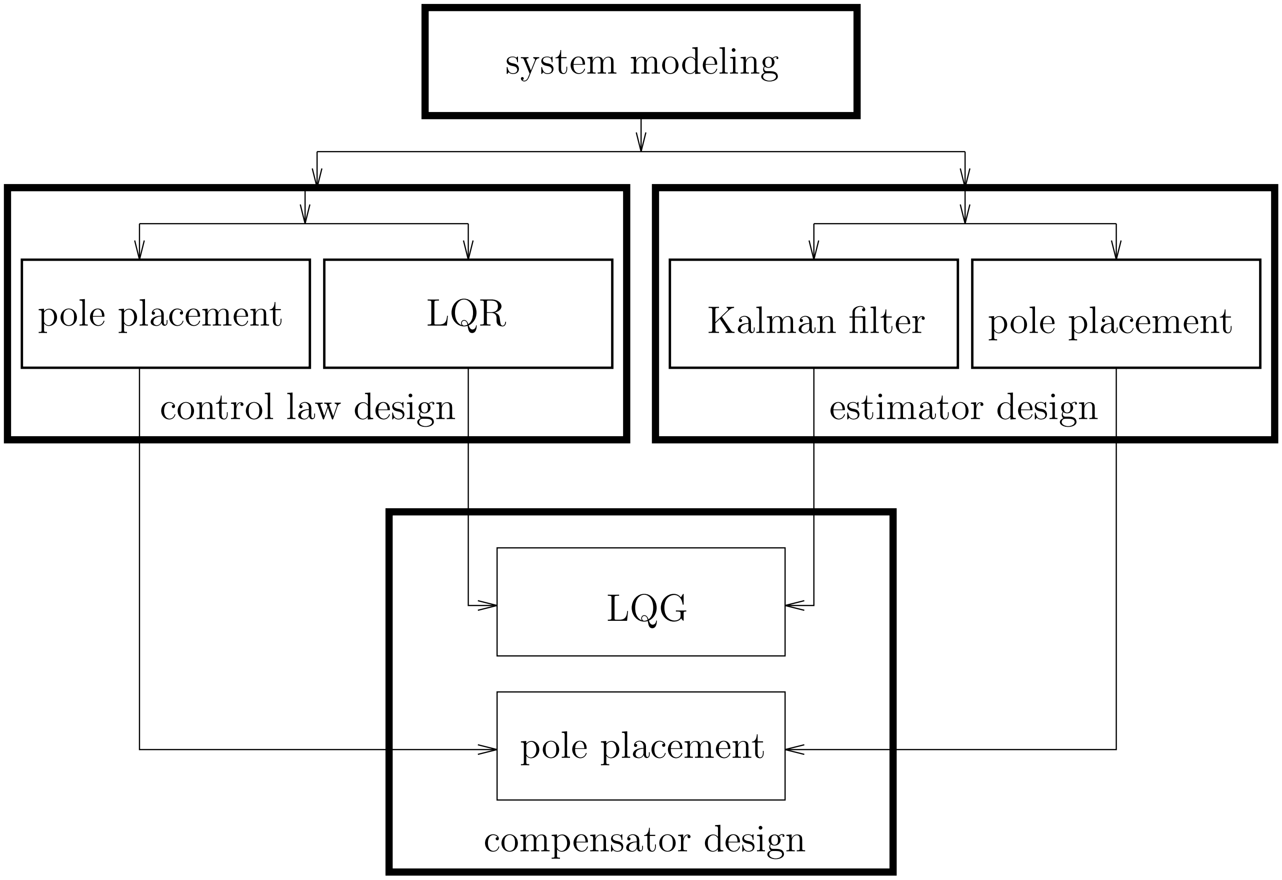
$\Rightarrow$

The poles of the compensator are not necessary in the unit circle, the compensator might be unstable.

Transfer function of the compensator :

$$\mathbf{K}(z) = -K(zI - A + B_2K + LC_2 - LD_2K)^{-1}L.$$

Summary of State Space Control Design



Examples of Compensator Design

Example Boeing 747 Aircraft control - Compensator design

The complete model of the aircraft is

$$\begin{aligned}\dot{x} &= Ax + Bu + Bw, \\ z &= \begin{bmatrix} q^{1/2}Cx \\ u \end{bmatrix}, \\ y &= Cx + v.\end{aligned}$$

where q is the weighting factor and w and v are zero mean white noises with covariance R_w and R_v respectively. (A, B, C, D) is the nominal aircraft model ($D = 0$).

First, we design a compensator via pole placement. From results of pole placement design examples in chapter 3 and chapter 5, we know that the state feedback gain

$$K = \begin{bmatrix} 1.06 & -0.19 & -2.32 & 0.10 & 0.04 & 0.49 \end{bmatrix}$$

places the poles of the control loop at

$$-0.0051,\ -0.468,\ -1.106,\ -9.89,\ -0.279 \pm 0.628i$$

and the estimator gain

$$L = \begin{bmatrix} 2.5047e+01 \\ -2.0517e+03 \\ -5.1935e+03 \\ -2.4851e+04 \\ -4.0914e+04 \\ -1.5728e+04 \end{bmatrix}$$

places the estimator poles at

$$-0.0255,\ -2.34\ , -5.53,\ -49.45,\ -1.395 \pm 3.14i.$$

The compensator then is

$$\begin{aligned} \dot{\hat{x}} &= (A - BK - LC)\hat{x} + Ly, \\ u &= -K\hat{x}. \end{aligned}$$

The poles of the compensator are then at

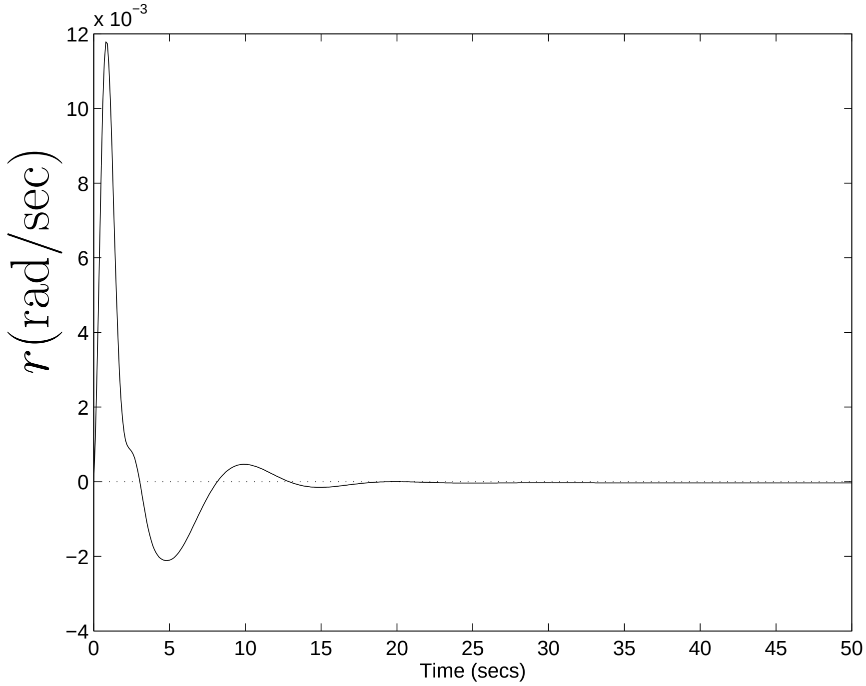
$$-51.35,\ -4.074 \pm 10.12i,\ -0.8372 \pm 0.6711i,\ -0.027.$$

So the compensator itself is stable, which is preferred since otherwise an accident resulting in a breakdown of the control loop will cause the control output to go to infinity.

The transfer function of the compensator $K(s)$ is

$$\begin{aligned} K(s) &= -K(sI - A + BK + LC)^{-1}L \\ &= 10^2 \frac{-8.42s^5 - 92.3s^4 - 72.1s^3 + 74.9s^2 + 113s + 2.55}{s^6 + 61.2s^5 + 640s^4 + 7086s^3 + 11029s^2 + 7316s + 194}. \end{aligned}$$

The initial condition response with $\beta_0 = 1^\circ$:



The response is almost the same as with state feedback control but with a slightly larger peak value and with the slowly decaying error (see the plot on page 97) filtered out here.

Now consider LQG design. Let $q = 9.527$, $R_w = 0.7$ and $R_v = 1$. From the results of LQR and Kalman filter design examples in chapter 4 and 6, we obtained a state feedback gain K

$$K = \begin{bmatrix} 1.06 & -0.19 & -2.32 & 0.10 & 0.04 & 0.49 \end{bmatrix}$$

and an estimator gain

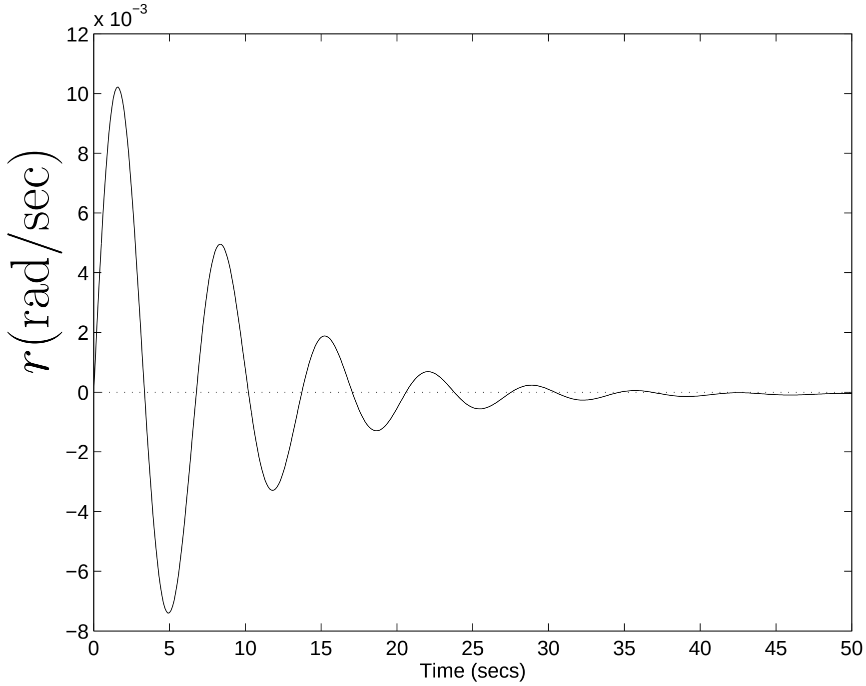
$$L = \begin{bmatrix} -1.5465e-02 \\ 4.9686e-02 \\ 2.2539e-01 \\ -7.3199e-01 \\ -3.0200e-01 \\ -8.2157e-15 \end{bmatrix}.$$

The poles of the compensator with the K and L above are at:

$-9.845,\ -1.405,\ -0.2616 \pm 0.5571i,\ -0.4756,\ -0.0049.$

Again the compensator is stable.

The initial condition response with $\beta_0 = 1°$:



This initial condition response shows that the Kalman filter is too slow compared with the state feedback control loop.

The reason for this result is that the measurement noise v is large compared with the process noise w, so that the Kalman filter has to reduce the measurement noise influence using a small estimator L, which slows down the Kalman filter. Now increase the process noise covariance to $R_w = 700$. The Kalman filter gain is now

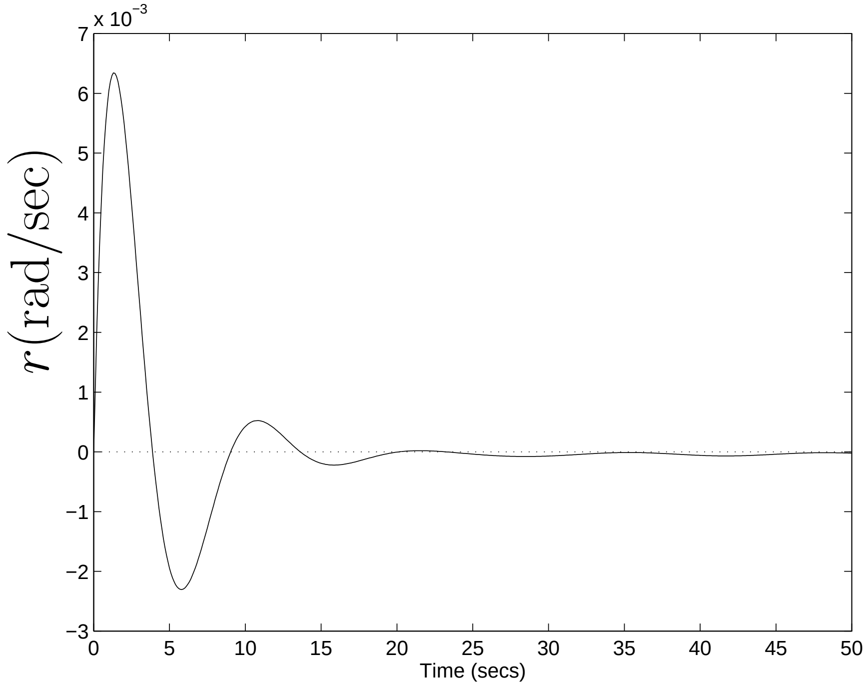
$$L = \begin{bmatrix} -8.0289e+00 \\ -3.3434e-01 \\ 8.2913e+00 \\ -4.8082e+00 \\ -3.4682e+00 \\ 3.5982e-11 \end{bmatrix}$$

The poles of the compensator are at

$$-9.845,\ -1.405,\ -0.2616 \pm 0.5571i,\ -0.4756,\ -0.0049.$$

It's not clear whether or not the Kalman filter is speeding up from these poles location. But the following initial condition response shows that the result is much better.

The initial condition response with $\beta_0 = 1^\circ$:



This response is even better than the one from pole placement and with a better noise attenuation.

Example Tape drive control - LQG design

The complete model of the tape drive is

$$\begin{aligned}\dot{x} &= Ax + Bu + Bw,\\ z &= \begin{bmatrix} Q^{1/2}Cx \\ u \end{bmatrix},\\ y &= Cx + v.\end{aligned}$$

where Q is a weighting matrix, w and v are zero mean white noises with covariance R_w and R_v respectively.

Pole placement design:

From the results of the pole placement design examples for control law and estimator design in chapter 3 and 5, we know that the state feedback gain K

$$K = \begin{bmatrix} 0.55 & 1.58 & 0.32 & 0.56 & 0.67 & 0.05 \\ 0.60 & 0.60 & 0.68 & 3.24 & -0.21 & 1.74 \end{bmatrix}$$

places the poles of the control loop at

$$-0.451 \pm 0.937i,\ -0.947 \pm 0.581i,\ -1.16,\ -1.16$$

and the estimator gain

$$L = \begin{bmatrix} 5.3833e+01 & -4.2084e+01 \\ 2.0377e+02 & -1.5615e+02 \\ 3.9038e+01 & -7.9226e+00 \\ 4.0009e+02 & -3.0809e+02 \\ 6.5103e+02 & -5.8228e+02 \\ 1.7691e+03 & -1.5000e+03 \end{bmatrix}$$

places the estimator poles at

$$-2.255 \pm 4.685i, \; -4.735 \pm 2.905i, \; -5.800, \; -5.800.$$

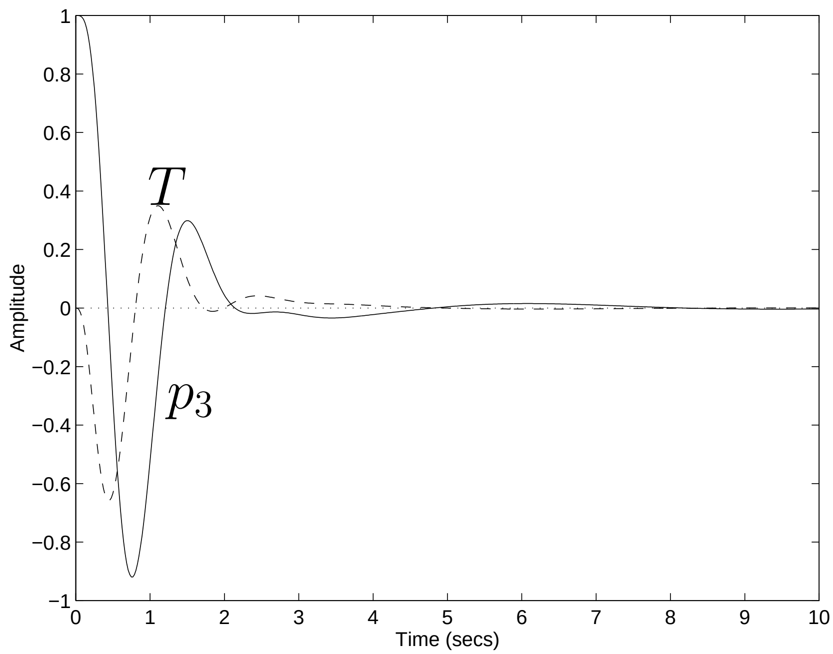
The compensator then is

$$\begin{aligned} \dot{\hat{x}} &= (A - BK - LC)\hat{x} + Ly, \\ u &= -K\hat{x}. \end{aligned}$$

The compensator has poles at

$$-1.696 \pm 8.531,\ -12.09,\ -5.847,\ -3.335 \pm 0.2469i.$$

The initial condition response with $x_1(0) = x_3(0) = 1$ looks like



Now consider LQG design.

Let

$$Q = 2I_2, R_w = 0.001I_4, R_v = \begin{bmatrix} 4 & 0 \\ 0 & 0.0001 \end{bmatrix}.$$

From the results of the LQR and Kalman filter design examples in chapter 4 and 6, we obtain the optimal state feedback gain K

$$K = \begin{bmatrix} 0.55 & 1.78 & 0.45 & 1.14 & 0.86 & 0.46 \\ 0.45 & 1.14 & 0.55 & 1.78 & 0.46 & 0.86 \end{bmatrix}$$

and the optimal estimator gain

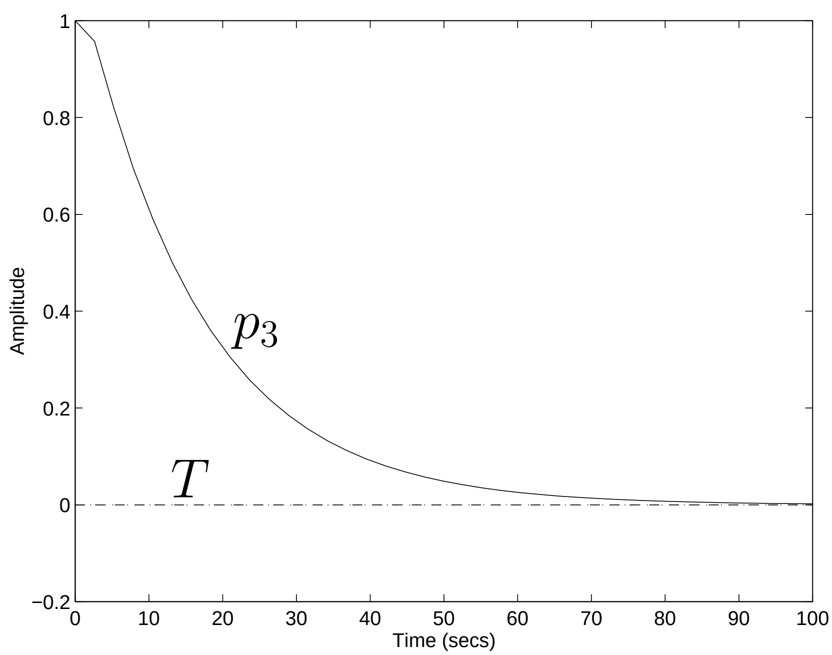
$$L = \begin{bmatrix} 5.56e-02 & -1.67e+00 \\ 7.74e-04 & -5.71e-01 \\ 5.56e-02 & 1.67e+00 \\ 7.74e-04 & 5.71e-01 \\ 4.32e-05 & -6.02e-01 \\ 4.32e-05 & 6.02e-01 \end{bmatrix}.$$

The poles of the compensator with the K and L above are at

$$-0.7204 \pm 1.093i,\ -0.6294 \pm 0.90502i,\ -1.308,\ -1.365.$$

These poles are obviously slower than those for pole placement estimator design. As a consequence, the time response is slower.

The initial condition response with $x_1(0) = x_3(0) = 1$:



Matlab Functions

initial

kalman

lqg

lqgreg

lqr

reg

Chapter 8

Reference Introduction – Integral Control

Reference Input – Zero Design

Motivation

A controller obtained by combining a control law with an estimator is essentially a regulator design : the characteristic equations of the controller and the estimator are basically chosen for good disturbance rejection. However, it does not lead to tracking, which is evidenced by a good transient response of the combined system to command changes. A good tracking performance is obtained by properly introducing the reference input into the system. This is equivalent to design proper zeros from the reference input to the output.

Reference input – full state feedback

Discrete-time :

The reference signal r_k is typically the signal that the output y_k is supposed to follow. To ensure zero steady-state error to a step input r_k, the feedback control law has to be modified.

Modification of the control law :

- Calculate the steady-state values x_{ss} and u_{ss} of the state x_k and the output y_k for the step reference r_{ss} (=the steady-state of step reference r_k) :

$$\begin{aligned} x_{ss} &= Ax_{ss} + Bu_{ss} \\ r_{ss} &= Cx_{ss} + Du_{ss} \end{aligned}$$

  Let $x_{ss} = N_x r_{ss}$ and $u_{ss} = N_u r_{ss}$, then

$$\begin{bmatrix} A - I & B \\ C & D \end{bmatrix} \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

  $\Rightarrow$

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A - I & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

- Modify the control law:

$$u_k = N_u r_k - K(\hat{x}_k - N_x r_k) = -K\hat{x}_k + \underbrace{(N_u + KN_x)}_{\bar{N}} r_k$$

In this way the steady-state error to a step input will be 0.

Proof :

1. Verify that the closed-loop system from r_k to y_k is given by

$$\begin{aligned} \begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} &= \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} + \begin{bmatrix} B\bar{N} \\ B\bar{N} \end{bmatrix} r_k, \\ y_k &= \begin{bmatrix} C & -DK \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} + D\bar{N} r_k. \end{aligned}$$

2. If $|\mathrm{eig}(A - BK)| < 1$ and $|\mathrm{eig}(A - LC)| < 1$ we obtain the following steady-state equations :

$$\begin{aligned} x_{ss} &= Ax_{ss} - BK\hat{x}_{ss} + B\bar{N} r_{ss} \\ \hat{x}_{ss} &= x_{ss} \\ y_{ss} &= (C - DK)x_{ss} + D\bar{N} r_{ss} \\ u_{ss} &= -Kx_{ss} + \bar{N} r_{ss} \end{aligned}$$

$$\Downarrow$$

$$y_{ss} = Cx_{ss} + D(-Kx_{ss} + \bar{N} r_{ss}) = Cx_{ss} + Du_{ss} = r_{ss}$$

So the transfer matrix relating y and r is a unity matrix at DC $\Rightarrow$ zero steady–state tracking error, steady–state decoupling.

Note that:

- r_k is an exogenous signal, the reference introduction will NOT affect the poles of the closed-loop system.
- $\begin{bmatrix} A-I & B \\ C & D \end{bmatrix}^{-1}$ must exist, and thus for MIMO

  number of references = number of outputs

- also for MIMO, reference introduction implies a steady-state decoupling between different reference and output pairs. This means that $y_{ss} = r_{ss}$.
- some properties of this controller are discussed on page 227.

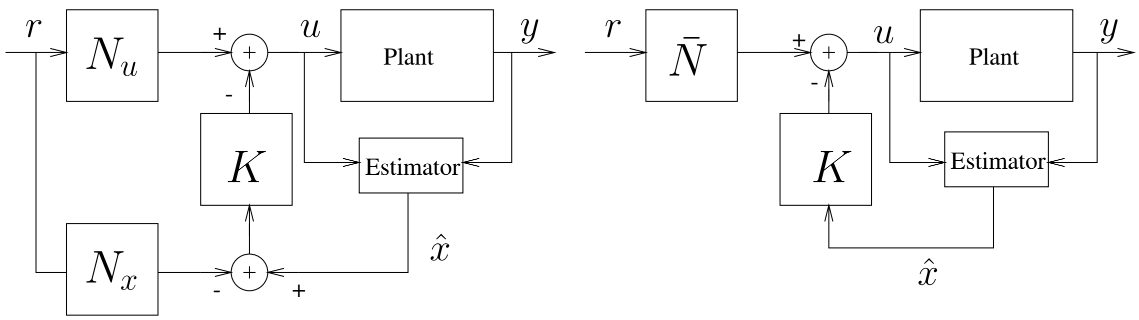
Continuous-time :

Try to verify that in this case

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

There are two types of interconnections for reference input introduction with full state-feedback :

$$\begin{aligned} \text{Type I:} \quad & u_k = N_u r_k - K(\hat{x}_k - N_x r_k) \\ \text{Type II:} \quad & u_k = -K\hat{x}_k + \underbrace{(N_u + KN_x)}_{\bar{N}} r_k \end{aligned}$$



For a type II interconnection, the control law K used in the feedback ($u_k = -K\hat{x}_k$) and in the reference feedforward ($\bar{N} = N_u + KN_x$) should be exactly the same, otherwise there is a steady-state error. There is no such problem in type I.

$\Rightarrow$

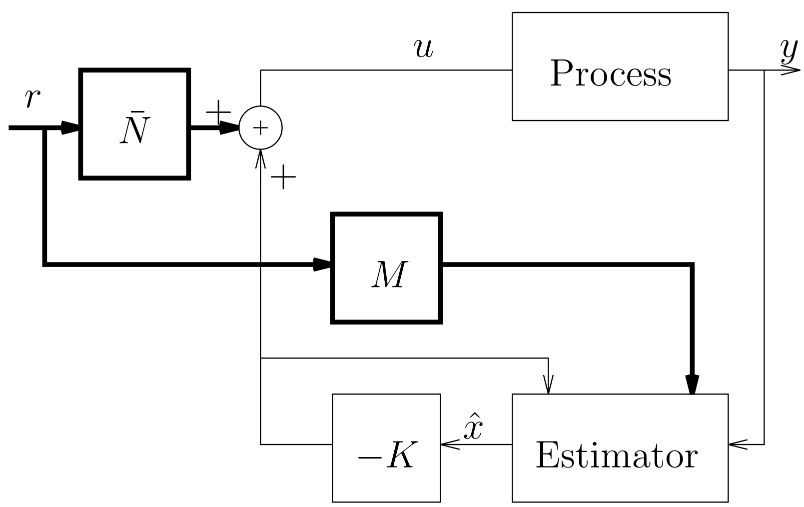
Type I is more ROBUST to parameter errors than Type II.

Reference Input - General Compensator

Plant and compensator model :

$$\begin{aligned} \text{Plant :} \quad & x_{k+1} = Ax_k + Bu_k, \\ & y_k = Cx_k + Du_k; \\ \text{Compensator :} \quad & \hat{x}_{k+1} = (A - BK - LC + LDK)\hat{x}_k \\ & \qquad + Ly_k, \\ & u_k = -K\hat{x}_k \end{aligned}$$

The structure of a general compensator with reference input r :



The general compensator is defined by the following closed-loop equations from r_k to y_k :

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} + \begin{bmatrix} B\bar{N} \\ M \end{bmatrix} r_k,$$

$$y_k = \begin{bmatrix} C & -DK \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} + D\bar{N}r_k.$$

Hence, the equations defining the compensator are

$$\begin{aligned} \hat{x}_{k+1} &= (A - BK - LC + LDK)\hat{x}_k + Ly_k \\ &\quad +(M - LD\bar{N})r_k, \\ u_k &= -K\hat{x}_k + \bar{N}r_k \end{aligned}$$

where $M \in \mathbb{R}^{n\times m}$ and $\bar{N} \in \mathbb{R}^{p\times m}$.

The estimator error dynamics are

$$\tilde{x}_{k+1} = (A - LC)\tilde{x}_k + B\bar{N}r_k - Mr_k.$$

Poles:

Characteristic equation:

$$\det\left(zI - \begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix}\right) = 0.$$

This is the same characteristic equation as without reference introduction. So introducing references will NOT change the poles.

Zeros :

The equations for a transmission zero are (see page 82)

$$\det \begin{bmatrix} \zeta I - A & BK & -B\bar{N} \\ -LC & \zeta I - A + BK + LC & -M \\ C & -DK & D\bar{N} \end{bmatrix} = 0$$

$$\Leftrightarrow$$

$$\begin{bmatrix} \zeta I - A & BK & -B\bar{N} \\ -LC & \zeta I - A + BK + LC & -M \\ C & -DK & D\bar{N} \end{bmatrix} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\neq 0} = 0$$

$$\Leftrightarrow$$

$$\det\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \det\begin{bmatrix} \zeta I - A + BK + LC & -M \\ -K & \bar{N} \end{bmatrix} = 0$$

The first term determines the transmission zeros of the open loop system while the second term corresponds to the transmission zeros of the compensator from r_k to u_k:

$$\begin{aligned} \hat{x}_{k+1} &= (A - BK - LC + LDK)\hat{x}_k + (M - LD\bar{N})r_k, \\ u_k &= -K\hat{x}_k + \bar{N}r_k \end{aligned}$$

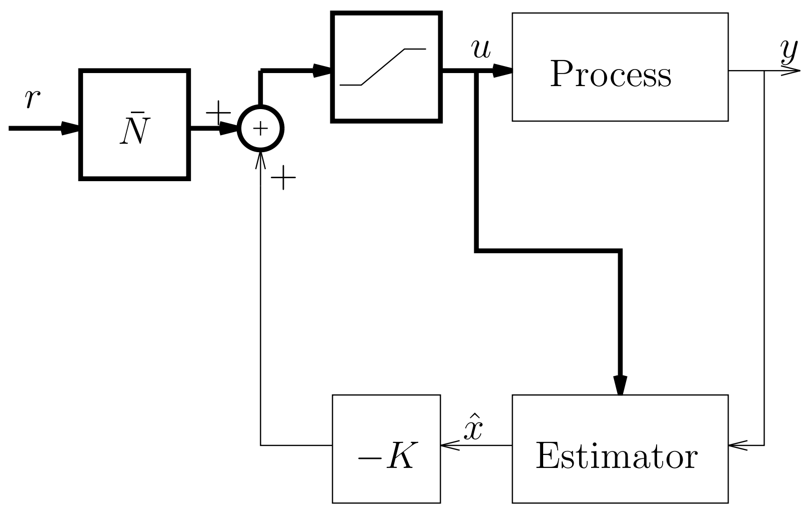
These transmission zeros are designed via reference introduction.

Autonomous estimator (cfr. pag. 218-220) :

Select M and $\bar{N}$ such that the state estimator error equation is independent of $r \Rightarrow$

$$M = B\bar{N}$$

where $\bar{N}$ is determined by the method for introducing the reference input with full state feedback.



Zeros :

The transmission zeros from r_k to u_k in this case are determined by

$$\det(\zeta I - A + LC) = 0$$

which is the characteristic equation for the estimator, hence the transmission zeros from r_k to u_k cancel out the poles of the state estimator.

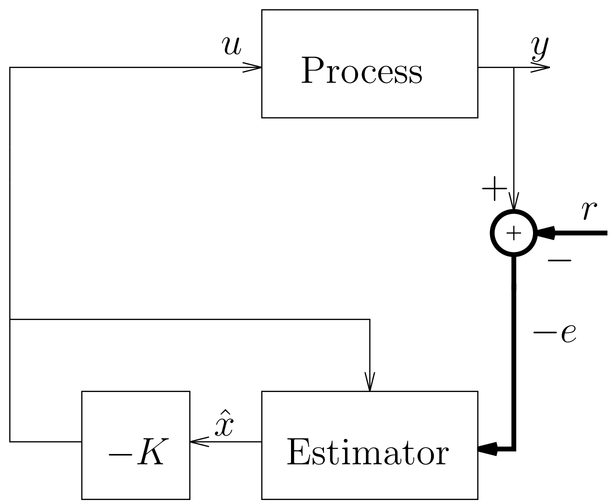
Properties :

- The compensator is in the feedback path. The reference signal r_k goes directly into both the plant and the estimator.
- Because of the pole-zero cancelation which causes "uncontrollability" of the estimator modes, the poles of the transfer function from r_k to y_k consist only of the state feedback controller poles (the roots of $\det(sI - A + BK) = 0$).
- The nonlinearity in the input (saturation) cancels out in the estimator since in this case the state estimator error equation is independent of u ($\tilde{x}_{k+1} = (A - LC)\tilde{x}_k$)

Tracking–error estimator

Select M and $\bar{N}$ such that only the tracking error, $e_k = (r_k - y_k)$, is used in the controller.

$\Rightarrow$
$$\bar{N} = 0, \; M = -L$$



The control designer is sometimes forced to use a tracking–error estimator, for instance when the sensor measures only the output error. For example, some radar tracking systems have a reading that is proportional to the pointing error, and this error signal alone must be used for feedback control.

Zeros :

The transmission zeros from r_k to y_k are determined by

$$\det\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \det\begin{bmatrix} \zeta I - A + BK + LC & L \\ -K & 0 \end{bmatrix} = 0$$

$$\Leftrightarrow$$

$$\det\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \det\begin{bmatrix} \zeta I - A & L \\ -K & 0 \end{bmatrix} = 0.$$

Once K and L are fixed by the control and estimator design, so are the zeros. So there is no way to choose the zeros.

Properties :

- The compensator is in the feedforward path. The reference signal r enters the estimator directly only. The closed-loop poles corresponding to the response from r_k to y_k are the control poles AND the estimator poles (the roots of $\det(sI - A + BK)\det(sI - A + LC) = 0$).
- In general for a step response there will be a steady-state error and there will exist a static coupling between the input-output pairs.
- Used when only the output error e_k is available.

Zero-assignment estimator (SISO) :

Select M and $\bar{N}$ such that n of the zeros of the overall transfer function are placed at desired positions. This method provides the designer with the maximum flexibility in satisfying transient-response and steady-state gain constraints. The previous two methods are special cases of this method.

Zeros of the system from r_k to u_k:

$$\det \begin{bmatrix} \zeta I - A + BK + LC & -M \\ -K & \bar{N} \end{bmatrix} = 0$$

$$\Downarrow \; \bar{M} \stackrel{\Delta}{=} M\bar{N}^{-1}$$

$$\lambda(\zeta) \stackrel{\Delta}{=} \det(\zeta I - A + BK + LC - \bar{M}K) = 0$$

Solution :

Determine $\bar{M}$ using a estimator pole-placement strategy for "system" (A_z, C_z), with

$$A_z = A - BK - LC,\ C_z = K,$$

$\bar{N}$ is determined such that the DC gain from r_k to y_k is unity.

For instance, in the case of a SISO system in continuous time, for which $D = 0$

$$\bar{N} = -\frac{1}{C(A-BK)^{-1}B[1 - K(A-LC)^{-1}(B-\bar{M})]}$$

and finally $M = \bar{M}\bar{N}$.

Example Tape drive control - reference introduction

Autonomous Estimator :

Consider the model of the tape drive on page 39. From the pole-placement design example on page 114, K is known.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2.5 & 0 & 2.5 & 0 & -0.67 & 0.67 & -0.67 & 0.67 \end{bmatrix}^T .$$

Thus,

$$N_x = \begin{bmatrix} 1 & -2.5 \\ 0 & 0 \\ 1 & 2.5 \\ 0 & 0 \\ 0 & -0.67 \\ 0 & 0.67 \end{bmatrix}, \; N_u = \begin{bmatrix} 0 & -0.67 \\ 0 & 0.67 \end{bmatrix}.$$

Let $M = B\bar{N}$. The control law is

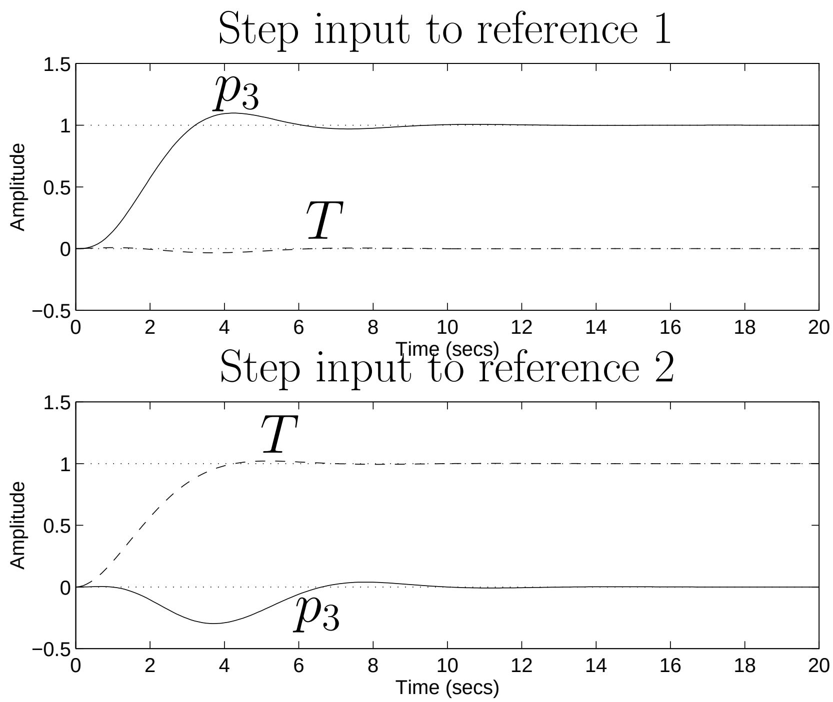
$$\begin{aligned} u &= -Kx + (N_u + KN_x)r \\ &= -Kx + \begin{bmatrix} 0.8666 & -1.6514 \\ 1.2779 & 2.1706 \end{bmatrix} r. \end{aligned}$$

Let L be the matrix from the pole placement estimator design example on page 172. Then the closed-loop system from r to y is

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B\bar{N} \\ B\bar{N} \end{bmatrix} r, \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}. \end{aligned}$$

This system is NOT controllable, as was expected (Try to prove it!).

Step responses from the reference r to the output y.



Output 1 (p_3) follows a step input to reference 1 while output 2 (T) is zero in steady state.

Output 2 (T) follows a step input to reference 2 while output 1 (p_3) is zero in steady state.

$\Rightarrow$ steady-state decoupling.

Integral Control and Robust Tracking

Motivation:

The choice of $\bar{N}$ will result in a step response with a zero steady-state error (see page 218). But the result is not robust because any change in the parameters will cause the error to be nonzero. Integral control is needed to obtain robust tracking of step inputs.

A more general method for robust tracking, called the error space approach (see page 238), can solve a broader class of tracking problems, i.e. tracking signals that do not go to zero in steady-state (a step, ramp, or sinusoidal signal).

Integral control

Augment the plant

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k \end{aligned}$$

with extra states integrating the output error $e_k = y_k - r_k$

$$x_{\mathrm{I}_{k+1}} = x_{\mathrm{I}_k} + \underbrace{Cx_k + Du_k - r_k}_{e_k}.$$

The augmented state equations become

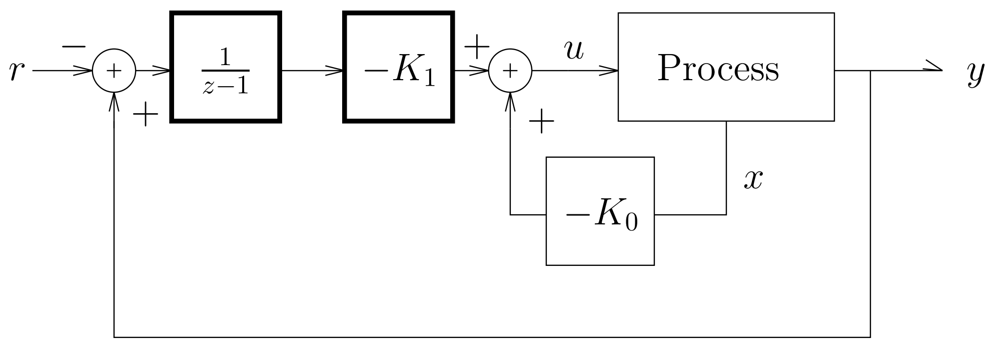
$$\begin{bmatrix} x_{\mathrm{I}_{k+1}} \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} I & C \\ 0 & A \end{bmatrix} \begin{bmatrix} x_{\mathrm{I}_k} \\ x_k \end{bmatrix} + \begin{bmatrix} D \\ B \end{bmatrix} u_k - \begin{bmatrix} I \\ 0 \end{bmatrix} r_k.$$

What are the equivalent equations in continuous-time ?

We now close the loop to stabilize the system. The feedback law is

$$u_k = -\underbrace{\begin{bmatrix} K_1 & K_0 \end{bmatrix}}_{K} \begin{bmatrix} x_{\mathrm{I}_k} \\ x_k \end{bmatrix}.$$

Use pole placement or LQR methods to design the control feedback gain K. Once the closed-loop is stable, the tracking error e goes to zero even if some parameters change.



The states of the plant x_k are estimated using a state estimator. The estimator gain L is determined using pole placement or Kalman filtering techniques. The integrator states x_{I_k} need not to be estimated as they are being computed explicitly.

What will be the closed-loop response from r_k to y_k ? Try to derive a state-space model.

Note that pole placement or LQR might not work since the augmented system is NOT always stabilizable and in this case integral control can not be used.

Tracking control - the error-space approach

Integral control is limited to step response tracking. A more general approach, the error-space approach, gives a control system the ability to track a non-decaying or even a growing input such as a step, a ramp, or a sinusoid.

Suppose the external signal, the reference, is generated by a certain dynamic system. By including the dynamic system as a part of the formulation and solving the control problem in an error space, the error approaches zero.

Given the plant dynamics

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k \end{aligned}$$

and the reference dynamics

$$r_{k+2} + \alpha_1 r_{k+1} + \alpha_2 r_k = 0,$$

the tracking error is defined as

$$e_k = y_k - r_k.$$

Define the error-space state:

$$\xi_k \stackrel{\Delta}{=} x_{k+2} + \alpha_1 x_{k+1} + \alpha_2 x_k,$$

and the error-space control:

$$\mu_k = u_{k+2} + \alpha_1 u_{k+1} + \alpha_2 u_k.$$

Then

$$e_{k+2} + \alpha_1 e_{k+1} + \alpha_2 e_k = C\xi_k + D\mu_k,$$

and the state equation for ξ_k becomes

$$\xi_{k+1} = A\xi_k + B\mu_k$$

Combining these two equations, the final error system is

$$z_{k+1} = A_e z_k + B_e \mu_k$$

where

$$z_k = \begin{bmatrix} e_k \\ e_{k+1} \\ \xi_k \end{bmatrix}, \; A_e = \begin{bmatrix} 0 & I & 0 \\ -\alpha_2 I & -\alpha_1 I & C \\ 0 & 0 & A \end{bmatrix}, \; B_e = \begin{bmatrix} 0 \\ D \\ B \end{bmatrix}.$$

Controllability of the error system:
If (A, B) is controllable and has no zero at the roots of

$$\alpha_e(z) = z^2 + \alpha_1 z + \alpha_2,$$

then (A_e, B_e) is controllable.

Control design: Pole-placement or LQR

$$\mu_k = -\begin{bmatrix} K_2 & K_1 & K_0 \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \\ \xi_k \end{bmatrix} = -K z_k$$

The actual control u_k is determined by the following internal model:

$$(u + K_0 x)_{k+2} + \sum_{i=1}^{2} \alpha_i (u + K_0 x)_{k+2-i} = -\sum_{i=1}^{2} K_i e_{k+2-i}.$$

Once the closed-loop is stable, e_k and e_{k+1} go to zero even if some parameters change.

Disturbance rejection by disturbance estimation

Motivation

If the state is not available then $-Kx$ can be replaced by the estimate $-K\hat{x}$ where $\hat{x}$ comes from the state estimator. The disturbance rejection problem consists in designing an estimator such that the error $\tilde{x} = x - \hat{x}$ goes to zero even when there is a disturbance signal with known dynamics.

Suppose that the disturbance is generated by a certain known dynamic system. The method consists in augmenting the estimator with the disturbance system in a way to cancel out the disturbance effects in the estimator output.

Augmenting the disturbance system to the plant

Given a plant with a disturbance input:

$$\begin{aligned} x_{k+1} &= Ax_k + B(u_k + w_k), \\ y_k &= Cx_k + Du_k \end{aligned}$$

and the disturbance dynamics (suppose 2nd order):

$$w_{k+2} + \alpha_1 w_{k+1} + \alpha_2 w_k = 0.$$

The final error system is

$$z_{k+1} = A_d z_k + B_d u_k$$

where

$$z = \begin{bmatrix} w_k \\ w_{k+1} \\ x_k \end{bmatrix}, \; A_d = \begin{bmatrix} 0 & I & 0 \\ -\alpha_2 I & -\alpha_1 I & 0 \\ B & 0 & A \end{bmatrix}, \; B_d = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix},$$

$$C_d = \begin{bmatrix} D & 0 & C \end{bmatrix}, D_d = D.$$

Observability:

If the plant (A, C) is observable and has no zero at any roots of

$$\alpha_d(z) = z^2 + \alpha_1 z + \alpha_2,$$

then (A_d, C_d) is observable.

Estimator for the error system:

$$\hat{z}_{k+1} = A_d \hat{z}_k + B_d u_k + L(y_k - C_d \hat{z}_k - D_d u_k).$$

The output u_k:

$$u_k = -K\hat{x}_k + \underbrace{\bar{N} r_k}_{\text{introduce reference}} - \underbrace{\hat{w}_k}_{\text{cancel disturbance}}.$$

Final closed-loop system:

$$x_{k+1} = (A - BK)x_k + B\bar{N}r_k + BK\tilde{x}_k + B\tilde{w}_k.$$

where $\tilde{x}_k = x_k - \hat{x}_k$ and $\tilde{w}_k = w_k - \hat{w}_k$.

Stable estimator $\Rightarrow \tilde{x}_k \to 0$ and $\tilde{w}_k \to 0$. The final state is NOT affected by the disturbance.

Chapter 9

Sensitivity - Robustness

Why robustness?

- The plant characteristics may be variable or time-varying. System modeling techniques identify the plant within a certain model class and with a certain amount of inaccuracy. So there always exists a plant uncertainty, which can not be described exactly by the mathematical models.
- Control systems need to be made robust against this plant variability and uncertainty.

Why frequency response analysis?

- There is a clear connection between frequency response plots and experimentally obtained data.
- Easy to learn for trained engineers familiar with classical frequency response methods.

Sensitivity

Definition

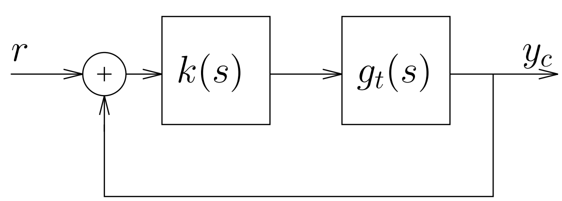
Sensitivity of a quantity α to changes in a quantity β:

$$S_\beta^\alpha \triangleq \frac{\partial \alpha}{\partial \beta} \cdot \frac{\beta}{\alpha} = \frac{\partial \alpha / \partial \beta}{\alpha / \beta}.$$

$\Rightarrow$

A measure of the relative change in α due to a relative change in β.

Sensitivity for a SISO system:



$g_t(s)$: plant, $k(s)$: controller.

The closed-loop transfer function from r to y_c:

$$h(s) = \frac{g_t(s)k(s)}{1 - g_t(s)k(s)}.$$

The sensitivity of $h(s)$ to changes in $g_t(s)$:

$$S_g^h(s) = \frac{\partial h(s)}{\partial g_t(s)} \frac{g_t(s)}{h(s)} = \frac{1}{1 - g_t(s)k(s)}$$

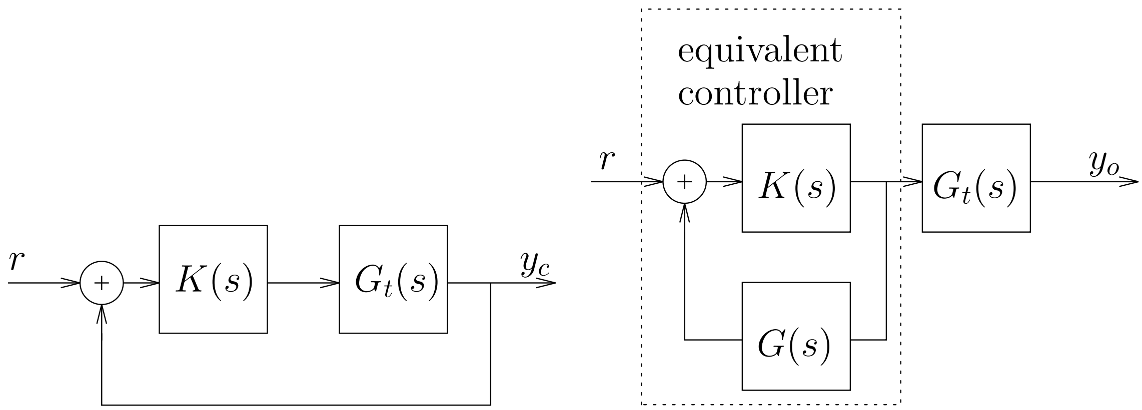
Plant variations will have a small or a large effect on the closed-loop transfer function according to the size of the sensitivity function $(1 - g_t(s)k(s))^{-1}$.

$\Rightarrow$

The robustness of the closed-loop system to plant variations is improved by making the sensitivity function small, said otherwise, by making the “loop gain” $g_t(s)k(s)$ large.

Sensitivity for a MIMO system:

Consider the following two control systems :



The equivalent open loop, $G(s)$: model,
$G_t(s)$: plant with uncertainties, $K(s)$: controller.

Since

$$
\begin{aligned}
y_c &= G_t(s)K(s)(I - G_t(s)K(s))^{-1}r, \\
y_o &= G_t(s)K(s)(I - G(s)K(s))^{-1}r.
\end{aligned}
$$

it follows that $y_c = y_o$ for all r if the system $G_t(s)$ and the model $G(s)$ are identical.

Suppose the plant $G_t(s)$ depends on a parameter δ. Then,

$$\begin{aligned}\frac{\partial y_c}{\partial \delta} &= (I - G_t(s)K(s))^{-1} \\ &\quad \times \frac{\partial G_t(s)}{\partial \delta} K(s)(I - G_t(s)K(s))^{-1} r, \\ \frac{\partial y_o}{\partial \delta} &= \frac{\partial G_t(s)}{\partial \delta} K(s)(I - G(s)K(s))^{-1} r.\end{aligned}$$

Assume $G(s) = G_t(s)|_{\delta=\delta_{\text{nom}}}$, then

$$\left.\frac{\partial y_c}{\partial \delta}\right|_{\delta=\delta_{\text{nom}}} = (I - G_t(s)K(s))^{-1} \left.\frac{\partial y_o}{\partial \delta}\right|_{\delta=\delta_{\text{nom}}}.$$

$\Rightarrow$

The sensitivity operator

$$S(s) \stackrel{\Delta}{=} (I - G_t(s)K(s))^{-1}$$

determines how changes in the plant affect the output of the closed-loop scheme given changes in the nominally equivalent open-loop scheme.

For matrices :

$$\begin{aligned}\text{if } y &= Su \\ &\Downarrow \\ ||y|| &\leq \sigma_{max}(S)||u||\end{aligned}$$

Again, the closed-loop scheme will be more or less sensitive to changes in the output depending on the "size" of $S(s)$:

$$\left\|\frac{\partial y_c(j\omega)}{\partial \delta}\right\|_{\delta=\delta_{\text{nom}}} \leq \bar{\sigma}(S(j\omega)) \left\|\frac{\partial y_o(j\omega)}{\partial \delta}\right\|_{\delta=\delta_{\text{nom}}}.$$

where $\bar{\sigma}(S(j\omega))$ is the largest singular value of the matrix $S(j\omega)$, a measure for the matrix size, and $\|\cdot\|$ is the 2-norm of a vector.

A feedback or controller design objective might be:

$$\bar{\sigma}(S(j\omega)) < 1, \ \forall\omega$$

which ensure that the closed-loop scheme is uniformally less sensitive to changes in system parameters than the open-loop scheme.

Complementary sensitivity:

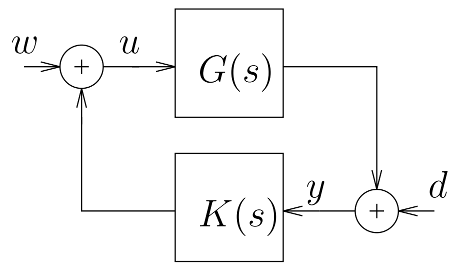
$$T(s) \stackrel{\Delta}{=} G(s)K(s)S(s) = G(s)K(s)(I - G(s)K(s))^{-1}$$

Note that $S + T = I$. The complementary sensitivity will be used later.

Robust Stability Analysis

Nyquist stability

Consider the following feedback system



Definition for internal stability :
Suppose $G(s)$ and $K(s)$ are proper[a] rational transfer function matrices and let $H(s)$ denote the closed-loop transfer function matrix from $\begin{bmatrix} w \\ d \end{bmatrix}$ to $\begin{bmatrix} u \\ y \end{bmatrix}$:

$$H(s) = \begin{bmatrix} I & -K(s) \\ -G(s) & I \end{bmatrix}^{-1}.$$

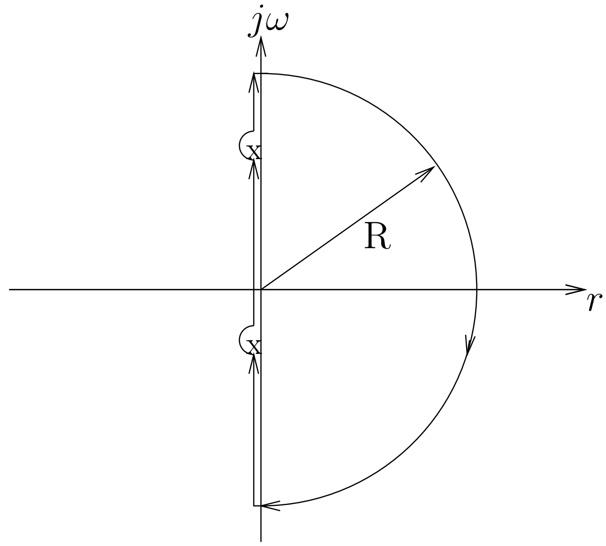
Then

- The feedback loop is called well posed if $H(s)$ is proper.
- The feedback loop is internally stable if H is stable.

[a] $H(s)$ is proper iff $\lim_{s\to\infty} H(s) = D$ and D is finite. If $D = 0$, $H(s)$ is called strictly proper.

Nyquist stability criterion for MIMO systems:

Let $G(s)$ and $K(s)$ be given proper rational transfer functions that form a well-posed closed loop, and let $G(s)$ and $K(s)$ have n_G and n_K poles (counting multiplicities) respectively in the closed-right half plane. The feedback loop is internally stable if and only if the Nyquist diagram $\Gamma = \det(I - G(s)K(s))$ makes $n_G + n_K$ anti-clockwise encirclements around the origin (without crossing it) for $s \in D_R$, where D_R is the Nyquist contour:



x: poles of $G(s)$ or $K(s)$ on the imaginary axis.

R is large enough to contain all the closed right half plane poles of $G(s)$, $K(s)$ and $H(s)$.

Small Gain Theorem

Gain of a system H

$$||H||_{gn} = \sup_{||w|| \neq 0} \frac{||Hw||}{||w||}$$

here $||.||$ is your favorite, well-chosen norm.
Not all norms will do : not every system norm is a gain.
Some gains for instance have the property that they are finite if and only if the system is stable :

$$||H||_{gn} < \infty \Leftrightarrow H \text{ is stable.}$$

The H_∞-norm is an example.

Sub-multiplicativity

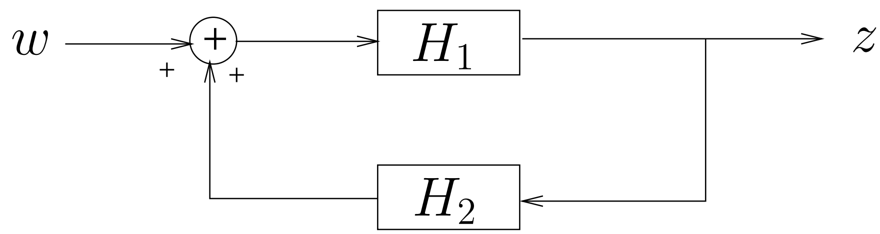
For all gains, the gain of a cascade is not larger than the product of the gains

$$||H_1 H_2||_{gn} \leq ||H_2||_{gn} ||H_1||_{gn}.$$

This is called sub-multiplicativity.

Small Gain Theorem

Consider the following system :



Assume the problem is well-posed i.e. $\det(I - H_1 H_2) \neq 0$. Then the transfer matrix from w to z is

$$G_{zw} = (I - H_1 H_2)^{-1} H_1$$

Now the small gain theorem states :

$$\boxed{\begin{array}{l} \text{If } ||H_1||_{gn}||H_2||_{gn} < 1, \text{ then} \\ \qquad ||G_{zw}||_{gn} \leq \dfrac{||H_1||_{gn}}{1 - ||H_1||_{gn}||H_2||_{gn}} \end{array}}$$

If $||.||_{gn}$ is such that finiteness means stability, then the small gain condition $||H_1||_{gn}||H_2||_{gn} < 1$ implies stability of G_{zw}.

Proof :

we have

$$z = H_1(w + H_2 z) = H_1 w + H_1 H_2 z$$

The triangle inequality delivers

$$||z|| \leq ||H_1 w|| + ||H_1 H_2 z||$$

Using the gain definitions & sub-multiplicativity

$$\begin{aligned} ||z|| &\leq ||H_1||_{gn}||w|| + ||H_1 H_2||_{gn}||z|| \\ &\leq ||H_1||_{gn}||w|| + ||H_1||_{gn}||H_2||_{gn}||z|| \end{aligned}$$

so that

$$||z||(1 - ||H_1||_{gn}||H_2||_{gn}) \leq ||H_1||_{gn}||w||.$$

From the small gain condition

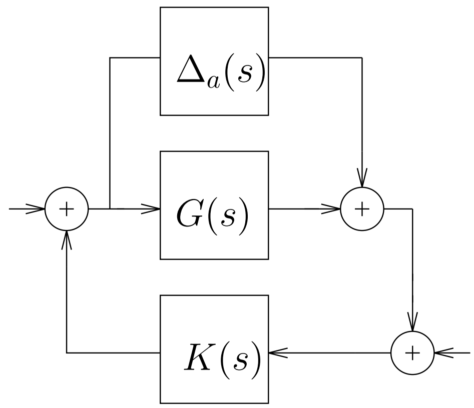
$$||H_1||_{gn}||H_2||_{gn} < 1$$

we find

$$||z|| \leq \frac{||H_1||_{gn}}{1 - ||H_1||_{gn}||H_2||_{gn}}||w||.$$

Additive model error

Consider the control system:



Feedback loop with additive model error

$G(s)$: Nominal system transfer function,

$\Delta_a(s)$: Additive stable perturbation transfer function,

$K(s)$: Controller transfer function designed to ensure the internal stability of the nominal closed-loop.

How large can $\bar{\sigma}(\Delta_a(j\omega))$ become before the closed-loop becomes unstable?

Necessary and sufficient condition for the closed-loop system to be internally stable:

$$\bar{\sigma}(\Delta_a(j\omega))\bar{\sigma}(K(j\omega)(I - G(j\omega)K(j\omega))^{-1}) < 1, \forall\omega.$$

$\Rightarrow$

- Given the gain of the perturbation:

$$\gamma_a \triangleq \max_{\omega} \bar{\sigma}(\Delta_a(j\omega)),$$

  then the stability condition is

$$\bar{\sigma}(K(j\omega)(I - G(j\omega)K(j\omega))^{-1}) < \frac{1}{\gamma_a}, \forall\omega,$$

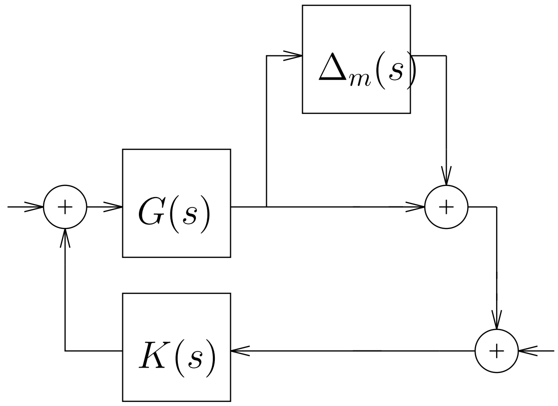
  which can be used as a design objective. And the design problem can be formulated as a minimization problem:

$$\min_{K(s)} \left(\max_{\omega} \bar{\sigma}(K(j\omega)(I - G(j\omega)K(j\omega))^{-1}) \right).$$

- The exact structure of $\Delta_a(s)$ is not needed.

Multiplicative model error

Consider the control system:



Feedback loop with multiplicative model error

$G(s)$: Nominal system transfer function,

$\Delta_m(s)$: Multiplicative stable perturbation transfer function,

$K(s)$: Controller transfer function designed to ensure the internal stability of the nominal closed-loop.

How large can $\bar{\sigma}(\Delta_m(j\omega))$ become before the closed-loop becomes unstable?

Necessary and sufficient condition for the closed-loop system to be internally stable:

$$\bar{\sigma}(\Delta_m(j\omega))\bar{\sigma}(\underbrace{G(j\omega)K(j\omega)(I - G(j\omega)K(j\omega))^{-1}}_{T(j\omega)}) < 1, \forall\omega.$$

$\Rightarrow$

- Given the gain of the perturbation:

  $$\gamma_m \stackrel{\Delta}{=} \max_{\omega} \bar{\sigma}(\Delta_m(j\omega)),$$

  then the stability condition is

  $$\bar{\sigma}(T(j\omega)) < \frac{1}{\gamma_m}, \forall\omega$$

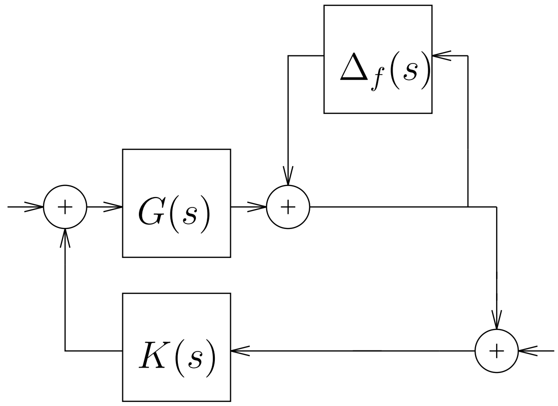
  which can be used as a design objective. And the design problem can be formulated as a minimization problem:

  $$\min_{K(s)} \left(\max_{\omega} \bar{\sigma}(T(j\omega)) \right).$$

- The exact structure of $\Delta_m(s)$ is again not needed.

Feedback multiplicative model error

Consider the control system:



Feedback loop with feedback multiplicative model error

$G(s)$: Nominal system transfer function,

$\Delta_f(s)$: Feedback multiplicative perturbation transfer function such that both $\Delta_f(s)$, and $(I - \Delta_f(s))^{-1}$ are well-posed and stable.

$K(s)$: Controller transfer function designed to ensure the internal stability of the nominal closed-loop.

How large can $\bar{\sigma}(\Delta_f(j\omega))$ become before the closed-loop becomes unstable?

Necessary and sufficient condition for the closed-loop system to be internally stable:

$$\bar{\sigma}(\Delta_f(j\omega))\bar{\sigma}(\underbrace{(I - G(j\omega)K(j\omega))^{-1}}_{S(j\omega)}) < 1, \omega \in \mathbb{R}.$$

$\Rightarrow$

- Given the gain of the perturbation

$$\gamma_f \triangleq \max_{\omega} \bar{\sigma}(\Delta_f(j\omega)),$$

  then the stability condition is

$$\bar{\sigma}(S(j\omega)) < \frac{1}{\gamma_f}, \forall\omega,$$

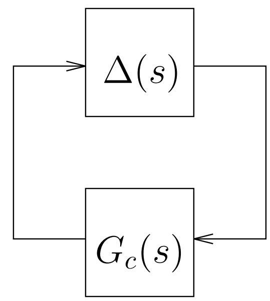
  which can be used as an design objective, and the design problem can be formulated as

$$\min_{K(s)} \left(\max_{\omega} \bar{\sigma}(S(j\omega)) \right).$$

- The exact structure of $\Delta_f(s)$ is not needed.

General case - small gain condition

Consider the following feedback loop:



General uncertainty feedback loop

$\Delta(s)$: Perturbation transfer function, stable. $G_c(s)$: Plant transfer function "seen" from the perturbation $\Delta(s)$, stable.

Model Error	$\Delta(s)$	$G_c(s)$
Additive	$\Delta_a(s)$	$K(s)(I - G(s)K(s)^{-1})$
Multiplicative	$\Delta_m(s)$	$T(s)$
Feedback Multplic.	$\Delta_f(s)$	$S(s)$

How large can $\bar{\sigma}(\Delta(j\omega))$ become before the closed-loop becomes unstable?

Small Gain Condition:
Necessary and sufficient condition for the closed-loop system to be internally stable (page 254) :

$$\bar{\sigma}(\Delta(j\omega))\bar{\sigma}(G_c(j\omega)) < 1, \forall\omega.$$

$\Rightarrow$

- Essentially the small gain condition states that if a feedback loop consists of stable systems and the loop-gain product is less than unity, then the feedback loop is internally stable. Otherwise there always exists a $\Delta(s)$ such that the closed-loop system is unstable.

- Given the gain of the perturbation

$$\gamma \triangleq \max_{\omega} \bar{\sigma}(\Delta(j\omega)),$$

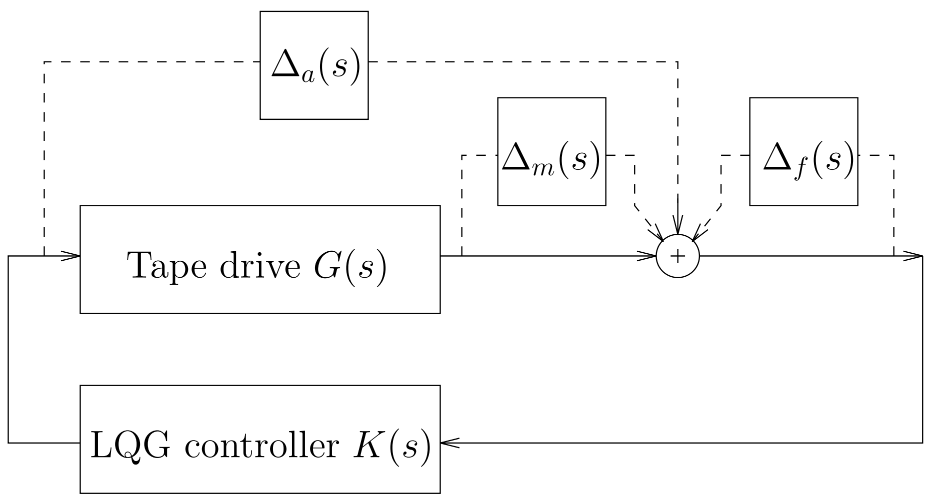
  then the stability condition is

$$\bar{\sigma}(G_c(j\omega)) < \frac{1}{\gamma}, \forall\omega$$

- There is no need to know the structure of the perturbation $\Delta(s)$.

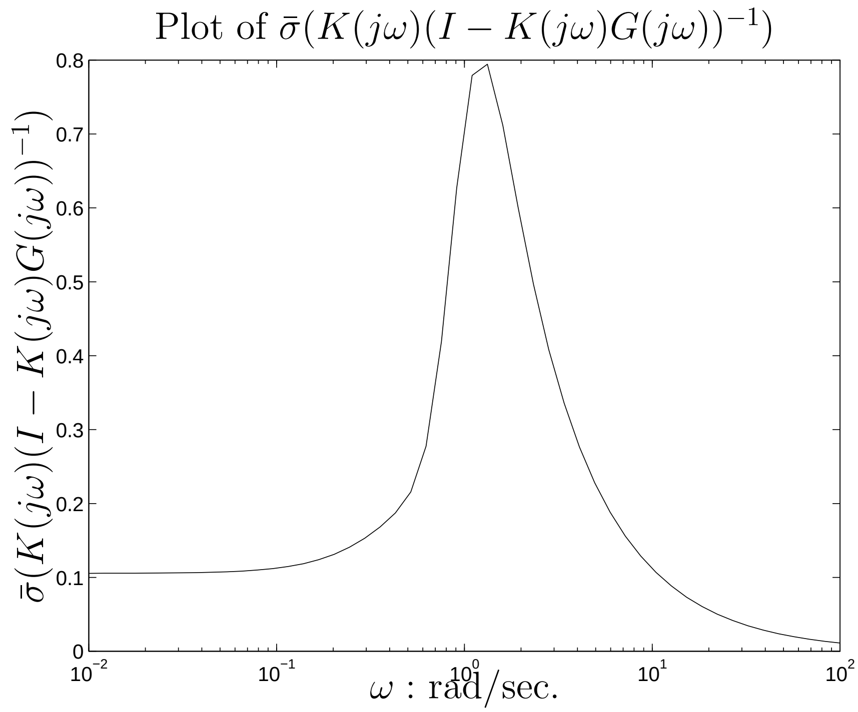
Example Tape drive control - Robustness analysis

Since only robust stability will be investigated here, all the noise inputs, the reference input and the regulated output are neglected and only the main loop (LQG design here) will be considered:



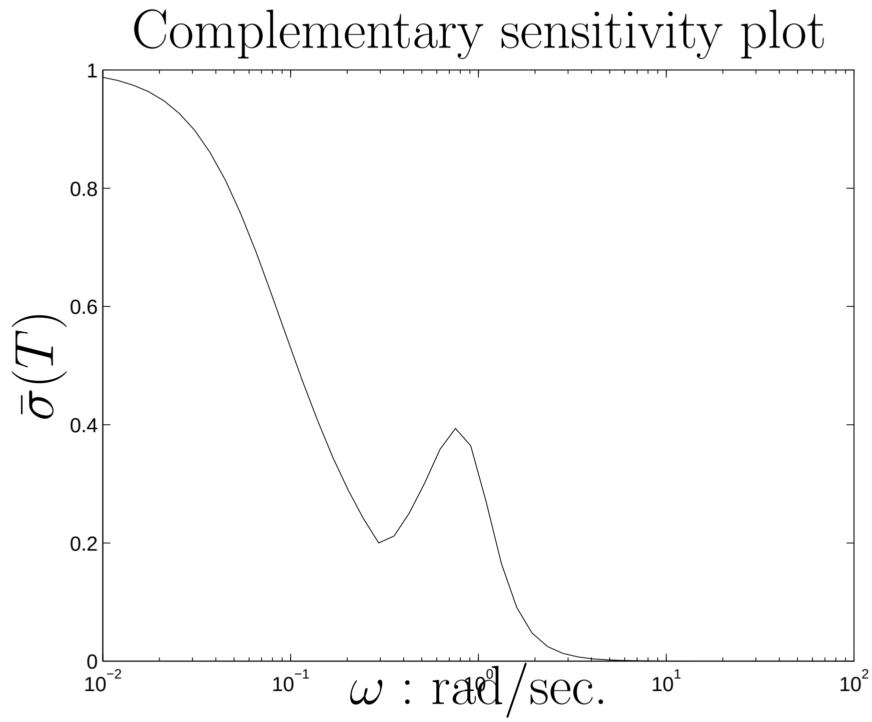
We analyze how large the additive, multiplicative and feedback multiplication perturbations $(\Delta_a(s), \Delta_m(s), \Delta_f(s))$ would be before the system becomes unstable when they are independently acting in the closed-loop.

Additive error.



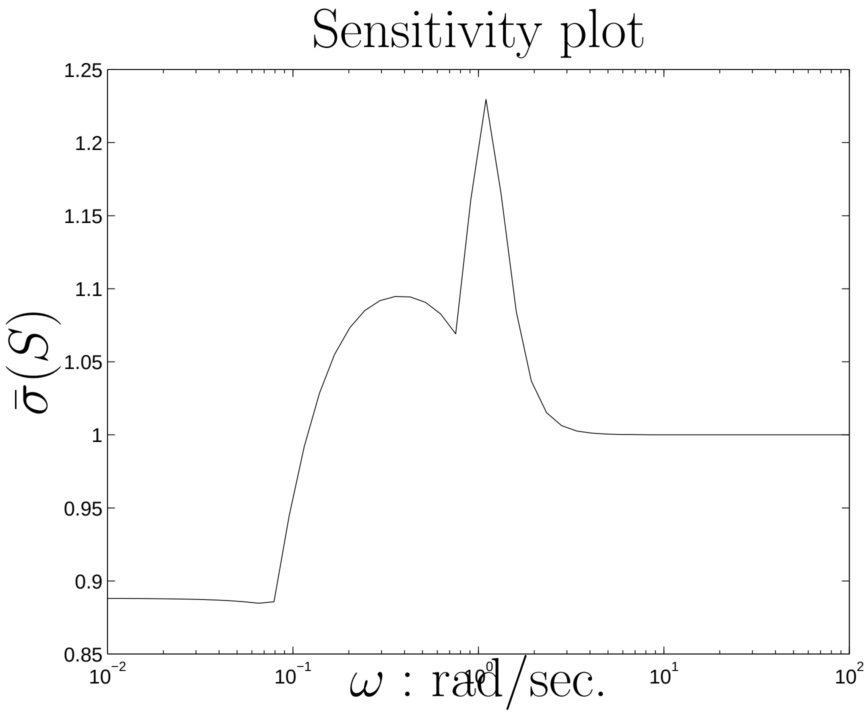
This plot shows that the the peak value of $\bar{\sigma}(K(j\omega)(I - G(j\omega)K(j\omega))^{-1})$ is less than 0.8. Thus if the gain of the additive error, $\gamma_a = \max_\omega(\Delta_a(j\omega))$, is less than $1/0.8 = 1.25$, the closed-loop system will not be destabilized by this error.

Multiplicative error.



This plot shows that the peak value of the maximum singular value of the complementary sensitivity is less than 1. Thus once the gain of the multiplicative error, $\gamma_m = \max_\omega(\Delta_m(j\omega))$, is less than 1, the closed-loop system will not be destabilized by this error.

Feedback multiplicative error.



This plot shows that the peak value of the maximum singular value of the sensitivity is less than 1.24. Thus once the gain of the feedback multiplicative error, $\gamma_f = \max_\omega(\Delta_f(j\omega))$, is less than $1/1.24 = 0.8065$ (< 1) than the closed-loop system will not be destabilized by this error. Note also that the peak value of the maximum singular value of the sensitivity is about 1.24. This means that a change in the open loop system will cause a change in the closed-loop which is 1.24 times larger (in percentage).

Summary

- Using SVD as a vehicle to *analyze* the robustness of systems with perturbations gives a simple way to generalize the frequency analysis methods from SISO to MIMO systems. However, since the structures of the perturbations are not considered, the results are often conservative. Thus it is possible that some classes of perturbations with specific structure (diagonal for instance) might never destabilize the closed-loop systems even if the gain conditions are not satisfied.
- So far we have not discussed the subject on robust control *design*, which is out of the scope of this course. Robust control design for systems with unstructured perturbations, namely H_∞ control synthesis, has been solved. The structure of the solution is similar to LQG control: combine the optimal control law and optimal estimator based on a similar separation principle as in LQG design. However, the Riccati equations are different.

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