LQ Optimal Control

- Stability and the Lyapunov equation
- Linear Quadratic Optimal Control
- Solution with completion of squares
- The algebraic Riccati equation
- Cheap control and asymptotic properties
- Robustness properties

Related Reading

[KK]: 9.1-9.2, 10 and [AM]: 4.4, 6.4

Lyapunov Functions for Linear Systems

We have analyzed asymptotic stability of the linear system

$$\dot{x} = Ax = f(x), \quad A \in \mathbb{R}^{n \times n}$$

by a direct consideration of e^{At}. It sheds an new light on linear stability analysis and prepares for later if we use Lyapunov theory.

Since the system is linear, let's try to use a (homogenous) **quadratic** Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$. Such functions are described by

$$V(x) = x^T P x \;\text{ with a symmetric matrix }\; P \in \mathbb{R}^{n \times n}.$$

For applying the Lyapunov theorem (Lecture 2) we need to consider

$$\partial V(x) f(x) = 2x^T PAx = x^T [A^T P + PA] x.$$

Remark. If some formulas for derivatives are not familiar to you, you should verify them by arguing on the basis of those rules that you know.

Recap: Simple Facts from Linear Algebra

Let $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ be symmetric ($Q = Q^T$ and $R = R^T$).

1. Q is positive semi-definite ($Q \succcurlyeq 0$) iff we have either:
   - $x^TQx \geq 0$ for all $x \in \mathbb{R}^n$
   - all eigenvalues of Q are non-negative
   - Q can be written as C^TC (with C of full row rank)

2. R is positive definite ($R \succ 0$) iff we have either:
   - $u^TRu > 0$ for all $u \in \mathbb{R}^m$ that are not zero
   - all eigenvalues of R are positive
   - R can be written as U^TU with a square and invertible U.

3. If a positive semi-definite matrix has a zero on the diagonal, then the corresponding row and column must be zero.

For any vector $x \in \mathbb{R}^n$ the Euclidean norm $\sqrt{x^Tx}$ is denoted by $\|x\|$.

Lyapunov Conditions for Asymptotic Stability

Theorem 2-13 requires to make sure that

$$x^T P x > 0 \;\text{ and }\; x^T[A^T P + PA]x < 0 \;\text{ for all }\; x \neq 0.$$

We hence arrive at the following result.

Theorem 1 *If there exists a $P \succ 0$ such that $A^T P + PA \prec 0$ then $\dot{x} = Ax$ is (globally) asymptotically stable.*

This result follows from general Lyapunov theory. On the next slide we provide a direct proof. In practice, the following recipe is often applied.

Theorem 2 *For any $Q = Q^T \prec 0$ (such as for example $Q = -I$) consider the following linear equation in P:*

$$A^T P + PA = Q.$$

If it has a unique positive definite solution then A is Hurwitz. Otherwise A is not Hurwitz.

Proof of Theorem 1

For some small $\alpha > 0$ the matrix $A^T P + PA + \alpha P$ is still negative definite. Therefore $x^T[A^T P + PA + \alpha P]x \le 0$ for all $x \in \mathbb{R}^n$ and hence

$$x^T[A^T P + PA]x \le -\alpha\, x^T Px. \qquad (\star)$$

For $\xi \in \mathbb{R}^n$ we need to show that $x(t) = e^{At}\xi \to 0$ for $t \to \infty$. Define

$$v(t) = x(t)^T Px(t) \ge 0.$$

We then infer with the help of (2) that

$$\dot{v}(t) = \frac{d}{dt}x(t)^T Px(t) = x(t)^T[A^T P + PA]x(t) \le -\alpha x(t)^T Px(t) = -\alpha v(t).$$

Hence $r(t) = \dot{v}(t) + \alpha v(t) \le 0$. By the variation-of-constants formula $0 \le v(t) = v(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-\tau)} r(\tau)\, d\tau \le v(0)e^{-\alpha t} \to 0$ for $t \to \infty$.

Therefore $\lim_{t\to\infty} v(t) = 0$. Since P is positive definite, it can be written as $V^T V$, V invertible. Then $v(t) = x(t)^T V^T V x(t) = \|Vx(t)\|^2 \to 0$; hence $Vx(t) \to 0$ and thus $V^{-1}Vx(t) = x(t) \to 0$ for $t \to \infty$.

Proof of Theorem 2

In what follows we present the algebraic analogue of the trajectory-oriented proof given the previous slide.

Suppose that $P \succ 0$ and $A^T P + PA = Q \prec 0$. Let $Ax = \lambda x$ with $x \in \mathbb{C}^n \setminus \{0\}$. We infer

$$0 > x^*(A^T P + PA)x = \bar{\lambda} x^* P x + \lambda x^* P x = 2\text{Re}(\lambda) x^* P x.$$

Since $x^* P x > 0$, this implies $\text{Re}(\lambda) < 0$.

The converse follows from Theorem 3.

Lyapunov Equation

Theorem 3 *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz.*

- *For every symmetric matrix $Q \in \mathbb{R}^{n\times n}$ the* **Lyapunov equation**

$$A^T P + PA = Q$$

  does have a unique symmetric solution $P \in \mathbb{R}^{n\times n}$.

- *If Q is negative semi-definite then P is positive semi-definite.*
- *If $Q \preccurlyeq 0$ and (A, Q) is observable then P is positive definite.*

The equation is well-studied also in the case that A is **not** Hurwitz. Then, for any symmetric and positive definite Q:

- either the Lyapunov equation has no solution;
- or there exists a solution but it is not unique;
- or there exists a unique solution but it is not positive definite.

Proof

Since e^{At} decays exponentially to zero for $t \to \infty$ the matrix

$$\boxed{P = -\int_0^\infty e^{A^T t} Q e^{At}\, dt}$$

is well-defined. Moreover we have

$$A^T P + PA = -\int_0^\infty A^T \left[e^{A^T t} Q e^{At}\right] + \left[e^{A^T t} Q e^{At}\right] A\, dt =$$

$$= -\int_0^\infty \frac{d}{dt}\left[e^{A^T t} Q e^{At}\right]\, dt = -\left. e^{A^T t} Q e^{At}\right|_{t=0}^{t=\infty} = Q.$$

Hence P solves the Lyapunov equation.

If $\tilde{P}$ is another solution we infer for $\Delta = \tilde{P} - P$ that $A^T\Delta + \Delta A = 0$. If we define $M(t) = e^{A^T t}\Delta e^{At}$ we have $M(\infty) := \lim_{t\to\infty} M(t) = 0$ and

$$\dot{M}(t) = e^{A^T t} A^T \Delta e^{At} + e^{A^T t}\Delta A e^{At} = e^{A^T t}[A^T\Delta + \Delta A]e^{At} = 0.$$

Hence $M(\cdot)$ is constant; thus $\Delta = M(0) = M(\infty) = 0$; hence $P = \tilde{P}$.

Proof

If $Q \preccurlyeq 0$ we infer $e^{A^T t} Q e^{At} \preccurlyeq 0$ for all $t \geq 0$ and thus

$$P = -\int_0^\infty e^{A^T t} Q e^{At}\, dt \succcurlyeq 0.$$

Now suppose that, in addition, (A, Q) is observable. If $x \in N(P)$ then

$$0 = x^T (A^T P + PA - Q)x = -x^T Q x$$

which implies $Qx = 0$. Hence

$$0 = (A^T P + PA - Q)x = PAx$$

which leads to $Ax \in N(P)$.

In summary, $AN(P) \subset N(P)$ and $N(P) \subset N(Q)$. By Theorem 4-7 we infer $N(P) = \{0\}$. This implies $P \succ 0$.

Example

The command `lyap(A,R)` solves the equation $AX + XA^T + R = 0$:

```
A=[-2 3;1 1];P=lyap(A',eye(2));eig(P)=[-0.8090;0.3090]
%%
As=[-2 3;1 1]-1.8*eye(2);P=lyap(As',eye(2))
eig(P)=[0.1089;68.2607]
%%
ev=eig(A);
As=A-ev(1)*eye(2);
P=lyap(As',eye(2))

??? Error using ==> lyap
Solution does not exist or is not unique.
```

LQ Optimal Control

There are many ways to find controls $u(.)$ such that the state of

$$\dot{x} = Ax + Bu, \quad x(0) = \xi \in \mathbb{R}^n$$

converges to zero for $t \to \infty$. Designing feedback gains by pole-placement is not simple since it is difficult to balance the speed of convergence of $x(.)$ and the "size" of the corresponding control action $u(.)$.

This motivates to **quantify** the average distance of the state-trajectory from 0 and the effort involved in the control action as

$$\int_0^\infty x(t)^T Q x(t)\, dt \quad \text{and} \quad \int_0^\infty u(t)^T R u(t)\, dt$$

respectively, where Q and R are symmetric **weighting matrices** that are positive semi-definite and positive definite respectively.

The weighting matrices allow to put individual emphasis on the different components of the state- and control-trajectories.

LQ Optimal Control

Achieving fast state-convergence to zero with the least possible effort then amounts to minimizing the **cost function**

$$\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt \tag{C}$$

over all trajectories satisfying

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = \xi \;\text{ and }\; \lim_{t\to\infty} x(t) = 0. \tag{S}$$

Definition 4 *Solving the optimal control problem of minimizing the quadratic cost function (C) over all controls* $u(.)$ *that satisfy (S) is the* **linear quadratic** *(LQ) optimal control problem (with stability).*

We stress that other cost criteria might better reflect the desired objectives; these would result in more general problems of **optimal control**.

The choice of a quadratic cost for linear systems is motivated by a beautiful problem solution and fast solution algorithms.

Choice of Weighting Matrices

Often $Q = \operatorname{diag}(q_1, \ldots, q_n)$ and $R = \operatorname{diag}(r_1, \ldots, r_m)$ are taken to be diagonal and the cost then reads as

$$\sum_{k=1}^{n} \int_0^\infty q_k x_k(t)^2 \, dt + \sum_{k=1}^{m} \int_0^\infty r_k u_k(t)^2 \, dt.$$

The scalars $q_k \geq 0$ and $r_k > 0$ allow us to balance the emphasis put on the state- and input-components:

- Large values of q_k or r_k penalize the component $x_k(t)$ or $u_k(t)$ heavier. Therefore these components are expected to be pushed to smaller values by optimal controllers.
- Small values of q_k or r_k allow for larger deviations of $x_k(t)$ from zero or for a larger action of $u_k(t)$.
- With $q_k = 0$ no emphasis is put on $x_k(t)$. For technical reasons $r_k = 0$ is not allowed: **All control components have to be penalized.**

Completion of Squares

For any symmetric matrix P and any state-trajectory with (S) we have

$$\frac{d}{dt}x(t)^T Px(t) = \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) =$$

$$= (Ax(t) + Bu(t))^T Px(t) + x(t)^T P(Ax(t) + Bu(t)) =$$

$$= x(t)^T (A^T P + PA)x(t) + x(t)^T PBu(t) + u(t)B^T Px(t).$$

Let us add on both sides $x(t)^T Qx(t)$ and $u(t)^T Ru(t)$. Now suppose $R = U^T U$ with some square invertible U. We infer

$$x(t)^T PBu(t) + u(t)B^T Px(t) + u(t)^T Ru(t) = -x(t)^T PBR^{-1}B^T Px(t) +$$

$$+ x(t)^T PBR^{-1}B^T Px(t) + x(t)^T PBu(t) + u(t)B^T Px(t) + u(t)^T Ru(t) =$$

$$= -x(t)^T PBR^{-1}B^T Px(t) + \|Uu(t) + U^{-T}B^T Px(t)\|^2.$$

This latter step is called **completion of the squares**. Purpose?

Completion of Squares

We have derived the following key relation along any system trajectory:

$$\frac{d}{dt}x(t)^T Px(t) + x(t)^T Qx(t) + u(t)^T Ru(t) =$$
$$= x(t)^T [A^T P + PA - PBR^{-1}B^T P + Q]x(t) +$$
$$+ \|Uu(t) + U^{-T}B^T Px(t)\|^2.$$

This motivates to choose $P = P^T$ as a solution of the following so-called **algebraic Riccati equation (ARE)**

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

If that was possible we could infer

$$\frac{d}{dt}x(t)^T Px(t) + x(t)^T Qx(t) + u(t)^T Ru(t) =$$
$$= \|Uu(t) + U^{-T}B^T Px(t)\|^2.$$

Completion of Squares

If we integrate over $[0, T]$ for $T > 0$ we finally arrive at

$$x(T)^T P x(T) + \int_0^T x(t)^T Q x(t) + u(t)^T R u(t)\, dt =$$
$$= \xi^T P \xi + \underbrace{\int_0^T \|U u(t) + U^{-T} B^T P x(t)\|^2\, dt}_{\geq 0}.$$

- For any trajectory of (S) we have $x(T) \to 0$ for $T \to \infty$ and thus
$$\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt \geq \xi^T P \xi.$$
The cost is **not smaller** than $\xi^T P \xi$, no matter which stabilizing control function is chosen.
- **Equality** is achieved exactly when $U u(t) + U^{-T} B^T P x(t) = 0$ or
$$u(t) = -R^{-1} B^T P x(t) \;\text{ for all }\; t \geq 0.$$

Insights

- Any solution $P = P^T$ of the ARE gives us a **lower bound** $\xi^T P \xi$ on the cost function for all admissible control functions.
- The lower bound **is attained** if we can choose the control function to satisfy $u(t) = -R^{-1}B^T P x(t)$. This could be assured as follows:

  1. Solve $\dot{x} = [A - BR^{-1}B^T P]x$ with $x(0) = \xi$ for $x(.)$.
  2. Then define the control function by $u_*(t) = -R^{-1}B^T P x(t)$.

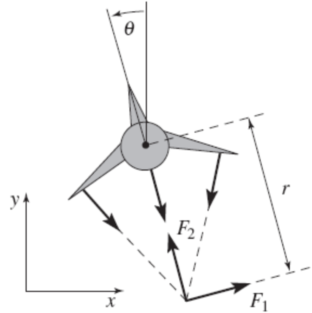
  But we need to make sure that $\lim_{t\to\infty} x(t) = 0$ which requires that

  $$A - BR^{-1}B^T P \text{ is Hurwitz.}$$

  If there exists a P as indicated then the constructed input $u_*(\cdot)$ is indeed a **unique optimal open-loop control function**.
- Moreover, the optimal control function can actually be implemented by a **feedback strategy** $u = -Fx$ with gain $F = R^{-1}B^T P$.

Example

(a) Harrier "jump jet"



(b) Simplified model

Consider Harrier at vertical take-off ([AM] pp.53,141,191) modeled as

$$\begin{aligned} m\ddot{x} &= F_1\cos(\theta) - F_2\sin(\theta) - c\dot{x}, \\ m\ddot{y} &= F_1\sin(\theta) + F_2\cos(\theta) - mg - c\dot{y}, \\ J\ddot{\theta} &= rF_1. \end{aligned}$$

Example

With state $z = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$ and input $u = (F_1, F_2)$ put the system into a first-order description and linearize at the equilibrium $u_e = (0, mg)$ and $z_e = (x_e, y_e, 0, 0, 0, 0)$. This leads to

$$\left(A \,\middle|\, B \right) = \left(\begin{array}{cccccc|cc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -g & -c/m & 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 0 & -c/m & 0 & 0 & 1/m \end{array}\right).$$

For a scale model choose the parameters

$$m = 4;\ J = 0.0475;\ r = 0.25;\ g = 9.81;\ c = 0.05.$$

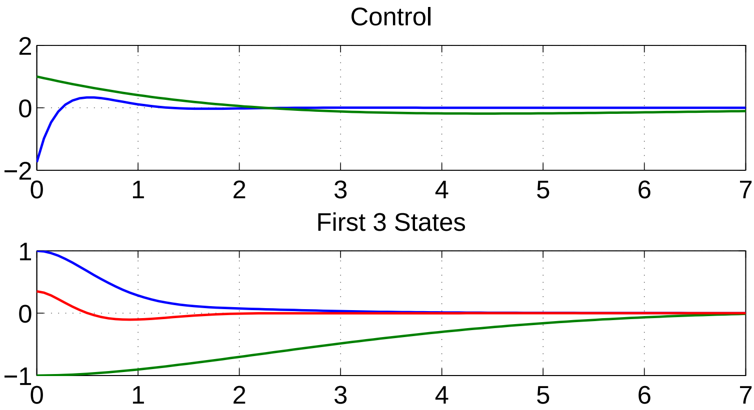
For Q and R we compute with

$$[\mathtt{F}, \mathtt{P}, \mathtt{E}] = \mathtt{lqr}(\mathtt{A}, \mathtt{B}, \mathtt{Q}, \mathtt{R})$$

the LQ-gain F, the stabilizing ARE solution P and the closed-loop eigenvalues $E = \text{eig}(A - BF)$.

Example

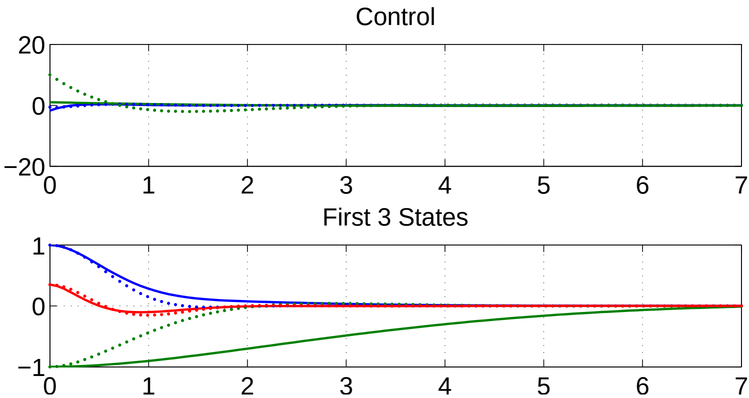
For $Q = I$, $R = I$, $\xi = (1, -1, 0.35, 0, 0, 0)$ get closed-loop responses



The second state is very slow. Also the first should be somewhat faster. This motivates to increase the penalty (weight) on these states e.g. to $Q = \mathrm{diag}(10, 100, 1, 1, 1, 1)$.

Example

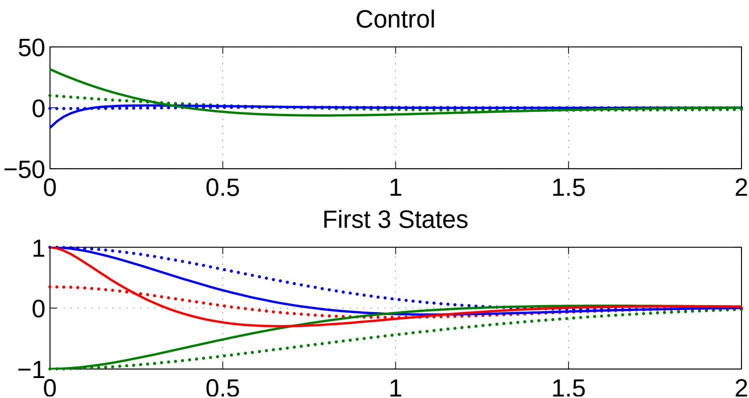
The responses are faster, at the expense of a larger control action:



Let's now allow for an even larger control action by reducing the input weight to $R = 0.1I$.

Example

This speeds up the responses further, but again at the expense of larger control actions:



By reducing $\rho > 0$ in $R = \rho I$ we put less weight on the control input. This typically comes along with high gains in the state-feedback matrix.

Riccati Theory

Definition 5 *Let* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ *satisfy* $Q = Q^T$ *and* $R \succ 0$. *The quadratic matrix equation*

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \qquad \text{(ARE)}$$

in the unknown $P \in \mathbb{R}^{n\times n}$ *is called* **algebraic Riccati equation (ARE)** *(for the linear system described by* (A, B) *and the quadratic cost function defined with* (Q, R)*).*

Any solution P *of the ARE which also satisfies*

$$\text{eig}(A - BR^{-1}B^T P) \subset \mathbb{C}^-$$

it is said to be a **stabilizing solution**.

We are typically only interested in **symmetric** solutions P of the ARE.

Can we characterize the existence of (stabilizing) solutions of the ARE?
Can we compute them (efficiently) if they exist? Yes we can ...

Riccati

Jacopo Francesco Riccati (1676-1754)

The Hamiltonian Matrix

Definition 6 *The* **Hamiltonian matrix** *of the ARE is defined as*

$$H = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \in \mathbb{R}^{2n\times 2n}.$$

Why is H relevant? If P solves the ARE we have

$$-Q - A^T P = P[A - BR^{-1}B^T P].$$

This leads to the relation

$$H \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} A - BR^{-1}B^T P & -BR^{-1}B^T \\ 0 & -[A - BR^{-1}B^T P]^T \end{pmatrix}$$

and thus

$$\begin{pmatrix} I & 0 \\ P & I \end{pmatrix}^{-1} H \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} = \begin{pmatrix} A - BR^{-1}B^T P & -BR^{-1}B^T \\ 0 & -[A - BR^{-1}B^T P]^T \end{pmatrix}$$

Hence a solution P of the ARE allows to transform H by similarity into a block-triangular form. Many insights can be extracted from here.

Riccati Theory: Main Result I

Let P be a stabilizing solution of the ARE. We then infer from

$$\text{eig}(H) = \text{eig}(A - BR^{-1}B^TP) \cup \text{eig}(-[A - BR^{-1}B^TP]^T)$$

and since $A - BR^{-1}B^TP$ is Hurwitz that

$$H \quad \text{has no eigenvalues on the imaginary axis.}$$

Of course, we also conclude that (A, B) is stabilizable. This proves one direction of the following key result.

Theorem 7 *(ARE) has a stabilizing solution iff (A, B) is stabilizable and the corresponding Hamiltonian matrix H has no eigenvalues on the imaginary axis.*

The proof is constructive and allows to determine “the” stabilizing solution of the ARE by solving an eigenvalue problem.

Let’s first prove uniqueness.

Uniqueness

Let P be a stabilizing solution of the ARE. We then infer that

$$R\begin{pmatrix} I \\ P \end{pmatrix} = \sum_{\lambda\in\mathbb{C}^-} \mathrm{ker}[(H-\lambda I)^{2n}],$$

which is the complex generalized eigenspace of H related to its eigenvalues in $\mathbb{C}^-$ (and also often just called the stable subspace of H).

If P_1 and P_2 are two stabilizing solutions we hence conclude

$$R\begin{pmatrix} I \\ P_1 \end{pmatrix} = R\begin{pmatrix} I \\ P_2 \end{pmatrix}$$

and thus $P_1 = P_2$. This proves the following result.

Lemma 8 *(ARE) has at most one stabilizing solution.*

Remark. This holds for so-called indefinite AREs as well, which are defined with some non-singular and merely symmetric matrix R.

A Property of the Hamiltonian Matrix

The eigenvalues of the real matrix H are clearly located symmetrically with respect to the real axis in the complex plane. Due to the particular structure of H the same holds with respect to the imaginary axis.

Lemma 9 *If λ is an eigenvalue of H with algebraic multiplicity k then $-\bar{\lambda}$ is as well an eigenvalue of H with the same algebraic multiplicity.*

Proof. Define the skew-symmetric (and orthogonal) matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

One easily checks that JH is symmetric. This implies $JH = (JH)^T = H^T J^T = -H^T J$ and thus

$$JHJ^{-1} = -H^T.$$

Similarity of H and $-H^T$ proves the statement.

Proof of 'if' in Theorem 7

Since H has no eigenvalue on the axis and by Lemma 9, H has n eigenvalues in the open left- and in the open right half-plane respectively.

Therefore there exists an invertible $T \in \mathbb{C}^{2n\times 2n}$ such that

$$T^{-1}HT = \begin{pmatrix} M & M_{12} \\ 0 & M_{22} \end{pmatrix} \quad \text{with } M \in \mathbb{C}^{n\times n} \text{ being Hurwitz.}$$

Just choose $T_1 \in \mathbb{C}^{2n\times n}$ as a basis of the stable subspace of H:

$$R(T_1) = \sum_{\lambda\in\mathbb{C}^-} N[(H-\lambda I)^{2n}];$$

then extend with T_2 to a non-singular matrix $T = (T_1\ T_2)$. Since $R(T_1)$ is H-invariant, the above structure follows.

Now partition T as H into four $n\times n$-blocks as

$$T = \begin{pmatrix} U & T_{12} \\ V & T_{22} \end{pmatrix} \quad \text{implying} \quad HZ = ZM \quad \text{for} \quad Z := \begin{pmatrix} U \\ V \end{pmatrix}.$$

Proof of ‘if’ in Theorem 7

The key step is to show: U is invertible and VU^{-1} is real symmetric (even though T is computed over $\mathbb{C}$ to block-triangularize H).

Step 1. $V^*U = U^*V$.

$HZ = ZM$ implies $Z^*JHZ = Z^*JZM$. Since the l.h.s. is a Hermitian matrix, so is the right-hand side. This implies $(Z^*JZ)M = M^*(Z^*J^*Z) = -M^*(Z^*JZ)$ by $J^* = -J$. Since M is Hurwitz, we infer from $M^*(Z^*JZ) + (Z^*JZ)M = 0$ that $Z^*JZ = 0$ (Theorem 3) which is indeed nothing but $V^*U - U^*V = 0$.

Step 2. $Ux = 0 \Rightarrow B^TVx = 0 \Rightarrow UMx = 0$.

$Ux = 0$ and the first row of $ZMx = HZx$ imply $UMx = (AU - BR^{-1}B^TV)x = -BR^{-1}B^TVx$ and thus $x^*V^*UMx = -x^*V^*BR^{-1}B^TVx$. By Step 1 we get $-x^*V^*BR^{-1}B^TVx = x^*U^*VMx = 0$ and thus $B^TVx = 0$. From $UMx = -BR^{-1}B^TVx$ we infer $UMx = 0$.

Proof of ‘if’ in Theorem 7

Step 3. U is invertible.

Suppose $N(U) \neq \{0\}$. Since $Ux = 0$ implies $UMx = 0$ (Step 2), $N(U)$ is M-invariant. Since non-trivial, there exists an eigenvector of M in $N(U)$, i.e., an $x \neq 0$ with $Mx = \lambda x$ and $Ux = 0$. Now the second row of $HZ = ZM$ yields $(-QU - A^T V)x = VMx$ and thus $A^T Vx = -\lambda Vx$. Since $Ux = 0$, we have $B^T Vx = 0$ (Step 2). Because (A, B) is stabilizable and Re$(-\lambda) > 0$, we infer $Vx = 0$. Since $Ux = 0$, this implies $Zx = 0$ and hence $x = 0$ because Z has full column rank. Contradiction!

Step 4. $P := VU^{-1}$ is Hermitian.

$V^*U = U^*V$ implies $U^{-*}V^* = VU^{-1}$ and hence $(VU^{-1})^* = VU^{-1}$.

Proof of 'if' in Theorem 7

Step 5. P is a (and hence the) stabilizing solution of the ARE.

From $HZ = ZM$ we infer $HZU^{-1} = ZU^{-1}(UMU^{-1})$ and hence

$$\begin{pmatrix} A - BR^{-1}B^TP \\ -Q - A^TP \end{pmatrix} = H \begin{pmatrix} I \\ P \end{pmatrix} = \begin{pmatrix} I \\ P \end{pmatrix} (UMU^{-1}).$$

The first row implies $A - BR^{-1}B^TP = UMU^{-1}$ such that $A - BR^{-1}B^TP$ is Hurwitz. The second row reads as $-Q - A^TP = P(A - BR^{-1}B^TP)$, which just means that P satisfies the ARE.

Step 6. P is real.

Since the data matrices are real, we have

$$\overline{A^TP + PA - PBR^{-1}B^TP + Q} = A^T\bar{P} + \bar{P}A - \bar{P}BR^{-1}B^T\bar{P} + Q$$

and $\overline{A - BR^{-1}B^TP} = A - BR^{-1}B^T\bar{P}$. Hence P and $\bar{P}$ are both stabilizing solutions of the ARE, which implies $P = \bar{P}$ (Lemma 8).

How to Block-Triangularize the Hamiltonian?

Let us mention three possibilities to block-triangularize the Hamiltonian:

- Choose T which block-diagonalizes H.

  One can e.g. transform H into the (suitably ordered) real or complex Jordan canonical form and extract the first n columns of T.

  In practice H is often diagonizable. Then these first n columns of T can be taken equal to n linearly independent eigenvectors of H that correspond to the eigenvalues of H in the open left half-plane.

- A numerically much more favorable way is to use the **ordered Schur decomposition**: Recall that one can always compute a **unitary** T which achieves the required block-triangular form of H.
- Modern algorithms (for large matrices) construct T with symplectic transformations on H that preserve the Hamiltonian structure.

Example

Here is some (very naive) Matlab code that computes the stabilizing ARE solution (instead of using `are` or `care`):

```
% Check stabilizability of (A,B)
% Check whether or not H has eigenvalues on imaginary axis
H=[A -B*inv(R)*B';-Q -A'];eig(H)
% Determine Z if H is diagonizable
[n,n]=size(A);[T,D]=eig(H);Z=[];
for j=1:2*n;
    if real(D(j,j))<0;Z=[Z T(:,j)];end;
end;
% Compute P
if size(Z,2)==n;
    U=Z(1:n,:);V=Z(n+1:2*n,:);P=V*inv(U);
end;
```

Riccati Theory: Additional Fact I

Typically, the ARE has infinitely many solutions. In the solution set of the ARE the stabilizing solution (if existing) has a particularly nice location.

Theorem 10 *The stabilizing solution of (ARE) is largest among all other solutions.*

Remark. Here the partial ordering among symmetric matrices P_1, P_2 is defined through $P_1 \preccurlyeq P_2 \Leftrightarrow 0 \preccurlyeq P_2 - P_1$.

Proof. Let P be the stabilizing and X any other solution of the ARE. With $\hat{A} = A - BR^{-1}B^TP$ and $\Delta = X - P$ one easily checks by subtracting the two AREs that

$$\hat{A}^T\Delta + \Delta\hat{A} = \Delta BR^{-1}B^T\Delta.$$

Since $\hat{A}$ is Hurwitz, Theorem 3 implies $\Delta \preccurlyeq 0$ thus $X \preccurlyeq P$. This shows that P is largest among all solutions.

A Second Property of the Hamiltonian Matrix

The case $Q \succcurlyeq 0$ is particularly nice. Then the eigenvalues of H on the imaginary axis

$$\mathbb{C}^0 := \{\lambda \in \mathbb{C} \mid \mathsf{Im}(\lambda) = 0\} = \{i\omega \mid \omega \in \mathbb{R}\}$$

are determined by the uncontrollable modes of (A, B), denoted as

$$\mathsf{eig}(A - sI \;\; B) = \{\lambda \in \mathbb{C} \mid \mathsf{rk}(A - \lambda I \;\; B) < n\}$$

and the unobservable modes of (A, Q), denotes as

$$\mathsf{eig}\begin{pmatrix} A - sI \\ Q \end{pmatrix} = \{\lambda \in \mathbb{C} \mid \mathsf{rk}\begin{pmatrix} A - \lambda I \\ Q \end{pmatrix} < n\},$$

on the imaginary axis.

Theorem 11 *If $Q \succcurlyeq 0$ then*

$$\mathit{eig}(H) \cap \mathbb{C}^0 = \left(\mathit{eig}(A - sI \;\; B) \cup \mathit{eig}\begin{pmatrix} A - sI \\ C \end{pmatrix}\right) \cap \mathbb{C}^0.$$

Proof

If H has the eigenvalue $\lambda \in \mathbb{C}^0$ we infer

$$\begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \lambda \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad \text{for some} \quad \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \neq 0.$$

With $R = U^T U$ and $Q = C^T C$ we get

$$Ae_1 - BU^{-1}[BU^{-1}]^T e_2 = \lambda e_1 \;\text{ and }\; -C^T Ce_1 - A^T e_2 = \lambda e_2. \quad (\star)$$

By left-multiplying e_2^* and e_1^* we infer

$$e_2^* Ae_1 - \|e_2^* BU^{-1}\|^2 = \lambda e_2^* e_1 \;\text{ and }\; -\|Ce_1\|^2 - e_1^* A^T e_2 = \lambda e_1^* e_2.$$

The conjugate of the latter is $-\|Ce_1\|^2 - e_2^* Ae_1 = \bar{\lambda} e_2^* e_1$. Adding to the first and exploiting $\bar{\lambda} + \lambda = 0$ (since $\lambda \in \mathbb{C}^0$) implies $\|e_2^* BU^{-1}\|^2 + \|Ce_1\|^2 = 0$ and thus $e_2^* B = 0$ and $Ce_1 = 0$; therefore $Qe_1 = 0$. By (3) we hence have $(A - \lambda I)e_1 = 0$ and $(A^T - \bar{\lambda} I)e_2 = 0$. Since either $e_1 \neq 0$ or $e_2 \neq 0$, λ is either an unobservable mode of (A, Q) or an uncontrollable mode of (A, B). The **converse** is shown by reversing the arguments.

Riccati Theory: Main Result II

Theorem 12 *If $Q \succcurlyeq 0$, the ARE $A^T P + PA - PBR^{-1}B^T P + Q = 0$ has a stabilizing solution if and only if (A, B) is stabilizable and (A, Q) has no unobservable modes on the imaginary axis.*

Proof. If the ARE has a stabilizing solution, Theorem 7 implies that (A, B) is stabilizable and H has no eigenvalues in $\mathbb{C}^0$; by Theorem 11 we infer that (A, Q) cannot have unobservable modes in $\mathbb{C}^0$.

If (A, B) is stabilizable and (A, C) has no unobservable modes in $\mathbb{C}^0$, then H has no eigenvalues in $\mathbb{C}^0$ by Theorem 11; hence Theorem 7 shows the existence of the stabilizing solution of the ARE. ∎

Remark. The unobservable modes of (A, Q) coincide with those of (A, C) in case that $Q = C^T C$; hence the existence of the stabilizing ARE solution is guaranteed if (A, B) is stabilizable and (A, C) is detectable.

Riccati Theory: Additional Fact II

Theorem 13 *If* $Q \succcurlyeq 0$*, the stabilizing solution* P *of (ARE) (if existing) is positive semi-definite. If, in addition,* (A, Q) *is observable, then* $P \succ 0$.

Proof. If P is any symmetric solution of the ARE we infer

$$(A - BR^{-1}B^TP)^TP + P(A - BR^{-1}B^TP) = -PBR^{-1}B^TP - Q.$$

With $\hat{A} = A - BR^{-1}B^TP$ and $\hat{Q} = -PBR^{-1}B^TP - Q$ we infer

$$\hat{A}^TP + P\hat{A} = \hat{Q}.$$

If P is the stabilizing solution then $\hat{A}$ is Hurwitz. Since $Q \succcurlyeq 0$, we have $\hat{Q} \preccurlyeq 0$ and thus $P \succcurlyeq 0$ by Theorem 3.

If (A, Q) is observable then $(\hat{A}, \hat{Q})$ is: $\hat{A}x = \lambda x$, $\hat{Q}x = 0$ implies $x^*\hat{Q}x = 0$ and thus $B^TPx =$ and $Qx = 0$; the former implies $Ax = \lambda x$; since (A, Q) is observable we get together with the latter $x = 0$.

Then Theorem 3 even allows to conclude $P \succ 0$.

Riccati Theory: Additional Fact III

Theorem 14 *Suppose that $P \succcurlyeq 0$ satisfies (ARE). If (A, Q) is detectable then P is the stabilizing solution.*

Proof. As above we re-arrange the ARE to

$$(A - BR^{-1}B^T P)^T P + P(A - BR^{-1}B^T P) = -PBR^{-1}B^T P - Q.$$

Now suppose that $(A - BR^{-1}B^T P)x = \lambda x$ with $x \neq 0$. Left-multiplication with x^* and right-multiplication with x leads to

$$\text{Re}(\lambda) x^* P x = -x^* PBR^{-1}B^T P x - x^* Q x.$$

If $x^* P x = 0$ we infer $Px = 0$ and thus $x^* Q x = 0$ and thus $Qx = 0$. Due to $Ax = \lambda x$ we infer with the Hautus-test that $\text{Re}(\lambda) < 0$.

If $x^* P x > 0$ we directly get $\text{Re}(\lambda) \leq 0$; the latter inequality is strict since $\text{Re}(\lambda) = 0$ leads to a contradiction if following the arguments above.

Riccati Theory: Summary

Let us summarize all individual statements for the ARE

$$A^T P + PA - PBR^{-1}B^T P + C^T C = 0$$

under the hypothesis that (A, B) is stabilizable and (A, C) is detectable.

- The ARE has a unique stabilizing solution.
- The stabilizing solution is largest among all other solutions.
- The stabilizing solution is positive semi-definite.
- If P is positive semi-definite, it is the stabilizing solution.

These are the standard hypothesis as they are often formulated in the literature and used in applications.

Solution of the LQ-Problem: Summary

Suppose that (A, B) is stabilizable and that (A, Q) with $Q \succcurlyeq 0$ has no unobservable modes on the imaginary axis.

- Then one can compute the unique solution $P \succcurlyeq 0$ of the ARE

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

  for which $A - BR^{-1}B^T P$ is Hurwitz.

- The LQ-optimal control problem has a unique solution.
- The optimal value is $\xi^T P \xi$ and the optimal control strategy can be implemented as a static state-feedback controller:

$$u = -R^{-1}B^T P x.$$

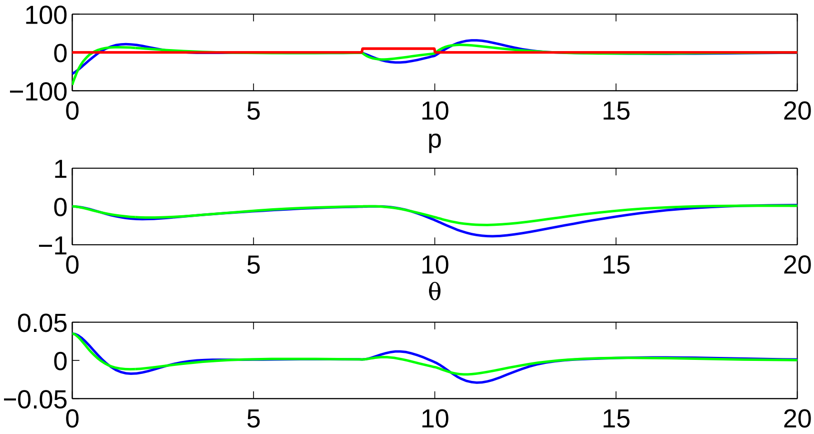
  The closed-loop eigenvalues are equal to those eigenvalues of the Hamiltonian that are contained in the open left half-plane.

This fundamental result follows directly from page 17 and Theorem 12.

In Matlab the solution is made available with the command `lqr`.

Example: Segway

With the data of [AM] p. 189 and the linearization in the upright position (zero input), we designed a static state-feedback controller by pole-placement in Lecture 3 (blue). With $R = 0.1$, $Q = \mathrm{diag}(100, 1, 1, 1)$ the LQ-responses (green) are improved:



Recap: Schur-Complement

For a block-matrix with invertible D we have

$$\begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & 0 \\ C & D \end{pmatrix}.$$

Schur-determinant-formula: If D is invertible then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\det(D).$$

If $A = A^T$, $C = B^T$ and $D = D^T$ is invertible then

$$\begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}\begin{pmatrix} I & 0 \\ -D^{-1}B^T & I \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}.$$

Schur-complement-lemma:

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \succ 0 \quad\Longleftrightarrow\quad D \succ 0 \text{ and } A - BD^{-1}C \succ 0.$$

Different variants are easily proved similarly.

Varying Input Weight

The closed-loop eigenvalues for the LQ-optimal gain are equal to the eigenvalues of the Hamiltonian in the open left half-plane.

With some fixed positive definite matrix R_0 suppose that we choose $R = \rho R_0$ for some scalar $\rho \in (0, \infty)$ to get

$$H = \begin{pmatrix} A & -\frac{1}{\rho} B R_0^{-1} B^T \\ -Q & -A^T \end{pmatrix}.$$

For large ρ we try to keep the control effort small. Since $-\frac{1}{\rho} B R_0^{-1} B^T$ approaches 0 for $\rho \to \infty$, the limiting closed-loop eigenvalues are equal to the stable eigenvalues of

$$H = \begin{pmatrix} A & 0 \\ -Q & -A^T \end{pmatrix}.$$

Hence they equal the stable eigenvalues of A (open-loop eigenvalues) and of $-A^T$ (open-loop eigenvalues **mirrored on imaginary axis**).

Cheap Control

For small ρ we allow for a large control effort (i.e. control is "cheap"). Let us use

$Q = C^T C, \quad R_0^{-1} = U_0 U_0^T$ (U_0 invertible), $\quad G(s) = C(sI - A)^{-1} B U_0.$

With the Schur-determinant formula we get

$$\det(sI - H) = \det(sI - A)\det(sI + A^T - Q(sI - A)^{-1} B R_0^{-1} B^T/\rho)$$

$$= \det(sI - A)\det(sI + A^T)\det(I - (sI + A^T)^{-1} C^T G(s) U_0^T B^T/\rho) =$$

$$= \det(sI - A)\det(sI + A^T)\det(I - U_0^T B^T (sI + A^T)^{-1} C^T G(s)/\rho) =$$

$$= \det(sI - A)\det(sI + A^T)\det(I + \frac{1}{\rho} G(-s)^T G(s)).$$

In general the zeros of this polynomial are not easy to analyze for $\rho \to 0$. One can show that some zeros move off to ∞, and others move to the zeros of $\det(G(-s)^T G(s))$ if this polynomial does not vanish identically.

Cheap Control - Butterworth Pattern

If $G(s)$ is SISO define $d(s) = \det(sI - A)$ with zeros $p_1, ..., p_n$ and $n(s) = d(s)G(s)$ with zeros $z_1, ..., z_m$. We need to analyze the zeros of

$$d(-s)d(s) + \frac{1}{\rho}n(-s)n(s) = 0. \qquad (\star)$$

For $\rho \to 0$ the following holds (Kwakernaak, Sivan, 1972):

- $2m$ zeros of $(\star)$ approach $\pm z_1, \ldots, \pm z_m$.
- $2(n-m)$ move to ∞ asymptotically along straight lines through the origin with the following angles to the positive real axis:

$$\frac{k\pi}{n-m}, \quad k = 0, 1, \ldots, 2n-2m-1, \quad n-m \text{ odd}$$

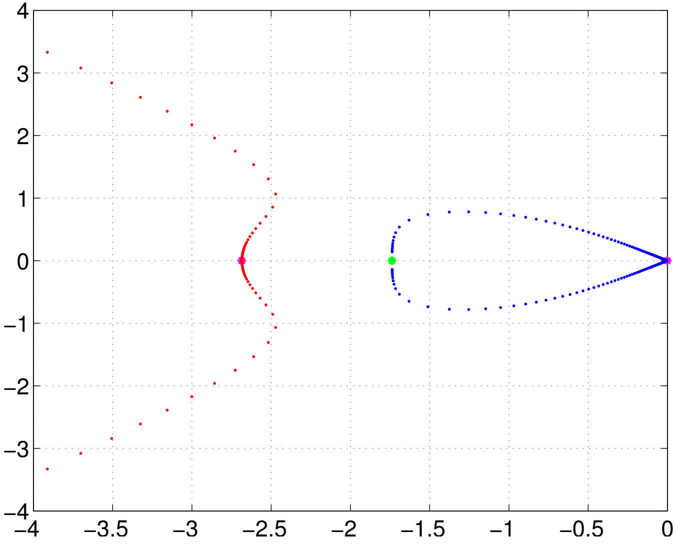
$$\frac{(k+\frac{1}{2})\pi}{n-m}, \quad k = 0, 1, \ldots, 2n-2m-1, \quad n-m \text{ even.}$$

Those in the open left half-plane are the closed-loop eigenvalues.

Example

Segway with $Q = C^T C$ and $C = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}$ as well as $R_0 = 1$.

Magenta: Zeros $d(s)$. Green: Zeros $n(s)$. Eigenvalues for $\rho \in (10^{-6}, 100)$:



A Version of Barbalat's Lemma

It is convenient to use $L_2^n := L_2([0,\infty),\mathbb{R}^n)$. Moreover let $L_{2,\mathrm{loc}}^n$ be all measurable $x : [0,\infty) \to \mathbb{R}^n$ with $x(.) \in L_2([0,T],\mathbb{R}^n)$ for every $T > 0$.

Lemma 15 *Suppose that $x \in L_2^n$ is locally absolutely continuous and that $\dot{x} \in L_2^n$. Then $x(t) \to 0$ for $t \to \infty$.*

Proof. By partial integration we have

$$2\int_0^t x(\tau)^T \dot{x}(\tau)\, d\tau = \|x(t)\|^2 - \|x(0)\|^2.$$

Since $x(.)^T\dot{x}(.) \in L_1[0,\infty)$, the left-hand side has a limit for $t \to \infty$. This implies that there exists $\alpha \geq 0$ with $\|x(t)\| \to \alpha$ for $t \to \infty$.

If $\alpha > 0$, there exists $T > 0$ such that

$$\|x(t)\|^2 \geq \alpha^2/2 \;\text{ for }\; t \in [T,\infty)$$

which clearly contradicts the fact that $x(.)$ has a finite L_2-norm. Hence $\alpha = 0$ and thus $x(t) \to 0$ for $t \to \infty$.

Young's Inequality for Convolutions

Lemma 16 *For* $1 \le p \le q \le \infty$ *choose the unique* $a \in [1, \infty]$ *with*

$$\frac{1}{p} + \frac{1}{a} = 1 + \frac{1}{q}.$$

If $M \in L_a([0,\infty), \mathbb{R}^{k\times m})$ *and* $u \in L_p([0,\infty), \mathbb{R}^m)$ *define*

$$y(\bullet) = \int_0^\bullet M(\bullet - \tau) u(\tau)\, d\tau \quad \text{on} \quad [0,\infty).$$

Then $y \in L_q([0,\infty), \mathbb{R}^k)$ *and*

$$\|y\|_q \le \|M\|_a \|u\|_p.$$

With the spectral norm $\|.\|$ for matrices and if $a < \infty$ we clearly use

$$\|M\|_a := \begin{cases} \sqrt[a]{\int_0^\infty \|M(t)\|^a\, dt} & \text{for } a < \infty \\ \text{ess sup}_{t\in[0,\infty)} \|M(t)\| & \text{for } a = \infty, \end{cases}$$

with the corresponding specializations to vector-valued functions.

A Useful Auxiliary Result

Lemma 17 *Suppose that* $\dot{x} = Ax + Bu$, $y = Cx$ *is detectable. If* $(x(.), u(.), y(.))$ *is any trajectory such that* $u \in L_2^m$ *and* $y \in L_2^k$ *then* $x(t) \to 0$ *for* $t \to \infty$.

Proof. Choose L such that $A - LC$ is Hurwitz. Then the trajectories also satisfy the relations

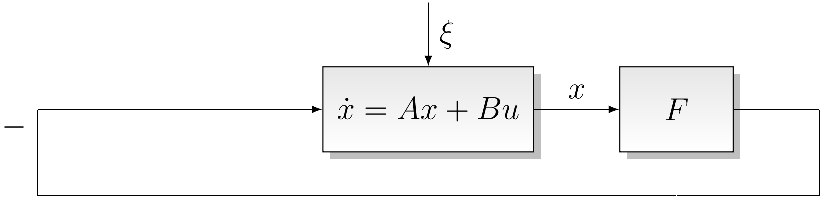
$$\dot{x}(t) = (A - LC)x(t) + v(t) \quad \text{for} \quad v(.) := Ly(.) + Bu(.).$$

Note that $v \in L_2^n$. Since $A - LC$ is Hurwitz, we certainly have $e^{(A-LC)\bullet} \in L_1([0,\infty), \mathbb{R}^{n\times n}) \cap L_2([0,\infty), \mathbb{R}^{n\times n})$.

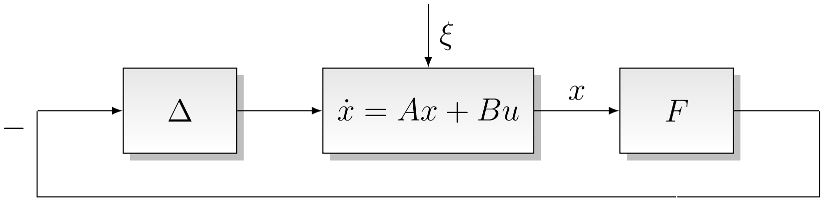
The Variation-of-Constants formula and Young's inequality imply that $x(.) \in L_2^n$ and the differential equation then immediately reveals that $\dot{x}(.) \in L_2^n$ as well. By applying our version of Barbalat's lemma, the claim is proved.

Robustness Properties

A perfect implementation of a state-feedback controller leads to



In a non-ideal situation, the signal sent to the system might get distorted. This is modeled by a filter Δ which is just another dynamical system:



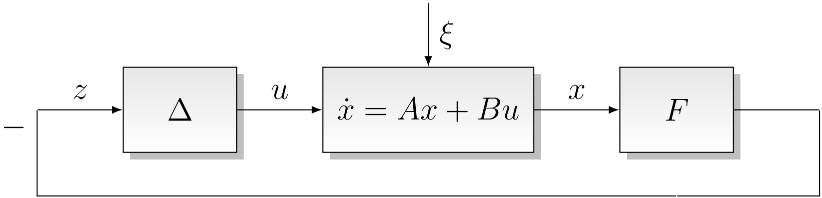
Typically examples: Actuator or transmission channel dynamics, delays

Robustness Properties

If Δ equals the static gain matrix $I \in \mathbb{R}^{m\times m}$, we know that the system is stable in the sense that $\lim_{t\to\infty} x(t) = 0$ for all $\xi \in \mathbb{R}^n$.

Question: How much can Δ deviate from I without loosing stability?

For example, one could consider static gain perturbations $\Delta \in \mathbb{R}^{m\times m}$.



The interconnection is then compactly described as

$$\begin{pmatrix} \dot{x} \\ z \end{pmatrix} = \begin{pmatrix} A & B \\ -F & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \quad x(0) = \xi, \quad u = \Delta z.$$

Robustness Properties

Let P satisfy (ARE) and set $F = R^{-1}B^TP$. We infer

$$0 = \begin{pmatrix} A^TP + PA - PBR^{-1}B^TP + Q & 0 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} A^TP + PA + Q - F^TRF & PB - F^TR \\ B^TP - RF & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} A^TP + PA & PB \\ B^TP & 0 \end{pmatrix} + \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} F^TRF & F^TR \\ RF & 0 \end{pmatrix} \succcurlyeq$$

$$\succcurlyeq \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^T \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} -F & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} -R & R \\ R & 0 \end{pmatrix} \begin{pmatrix} -F & 0 \\ 0 & I \end{pmatrix}$$

Why did we derive this inequality?

It leads to a crucial inequality that allows us to specify a very large class of Δ's which do not destroy stability of the loop.

Robustness Properties

For any systems trajectory and for all $t \geq 0$ we easily conclude:

$$0 \geq \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^T \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} +$$

$$+ \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \begin{pmatrix} -F & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} -R & R \\ R & 0 \end{pmatrix} \begin{pmatrix} -F & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} =$$

$$= \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}^T \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} + \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} -R & R \\ R & 0 \end{pmatrix} \begin{pmatrix} z(t) \\ u(t) \end{pmatrix} =$$

$$= \frac{d}{dt} x(t)^T P x(t) + \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} -R & R \\ R & 0 \end{pmatrix} \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}.$$

For any system trajectory and $T > 0$ the following inequality holds

$$x(T)^T P x(T) + \int_0^T \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} -R & R \\ R & 0 \end{pmatrix} \begin{pmatrix} z(t) \\ u(t) \end{pmatrix} dt \leq \xi^T P \xi. \quad (1)$$

Robustness Properties: Main Result III

Let $\Delta : L_{2,\text{loc}}^m \to L_{2,\text{loc}}^m$ have the following properties: There exist real constants γ and $\epsilon > 0$ such that for all $z \in L_{2,\text{loc}}^m$ and for all $T > 0$:

$$\int_0^T \|\Delta(z)(t)\|^2 \, dt \leq \gamma^2 \int_0^T \|z(t)\|^2 \, dt \tag{2}$$

and

$$\int_0^T \begin{pmatrix} z(t) \\ \Delta(z)(t) \end{pmatrix}^T \begin{pmatrix} -R & R \\ R & 0 \end{pmatrix} \begin{pmatrix} z(t) \\ \Delta(z)(t) \end{pmatrix} dt \geq \epsilon \int_0^T \|z(t)\|^2 \, dt. \tag{3}$$

Theorem 18 *Let $P \succcurlyeq 0$ be the stabilizing solution of (ARE) for $Q \succcurlyeq 0$. With any Δ as above and any $\xi \in \mathbb{R}^n$, all responses $x \in L_{2,\text{loc}}^m$ of the interconnection*

$$\begin{pmatrix} \dot{x} \\ z \end{pmatrix} = \begin{pmatrix} A & B \\ -F & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \quad x(0) = \xi, \quad u = \Delta(z) \tag{4}$$

satisfy $\lim_{t\to\infty} x(t) = 0$.

Proof

Consider any response $x \in L_{2,\mathrm{loc}}^n$ of the interconnection. We infer that $z(.) = -Fx(.) \in L_{2,\mathrm{loc}}^m$ and thus $u(.) = \Delta(z(.)) \in L_{2,\mathrm{loc}}^m$. We can hence merge (1) and (3) to infer

$$\epsilon \int_0^T \|z(t)\|^2 \, dt \leq \xi^T P \xi$$

for all $T > 0$. This implies $z(.) \in L_2^m$!

Then (2) leads to

$$\int_0^T \|u(t)\|^2 \, dt \leq \gamma^2 \int_0^T \|z(t)\|^2 \, dt \leq \gamma^2 \int_0^\infty \|z(t)\|^2 \, dt < \infty$$

for all $T > 0$ and hence $u(.) \in L_2^m$.

Since $A - BF$ is Hurwitz, $(A, -F)$ is detectable. Then the statement follows from Lemma 17.

Example: Static Gains

Let $D \in \mathbb{R}^{m\times m}$ be a static gain-matrix such that

$$\begin{pmatrix} I \\ D \end{pmatrix}^T \begin{pmatrix} -R & R \\ R & 0 \end{pmatrix} \begin{pmatrix} I \\ D \end{pmatrix} \succ 0.$$

Then Δ defined through $\Delta(z)(t) := Dz(t)$ for $t \geq 0$ satisfies all required hypotheses. Hence (4) described in this case through

$$\dot{x} = (A - BDF)x, \quad x(0) = \xi$$

satisfies, for any initial condition, $x(t) \to 0$ for $t \to \infty$.

The inequality characterizing allowed D's can be interpreted in various fashions. Here is a simple one: We can allow for $D = dI$ with $d \in (\frac{1}{2}, \infty)$; classically this means that the system has an impressive **gain-margin**.

If F is the optimal LQ-gain, it can be changed to dF for $d \in (\frac{1}{2}, \infty)$ without endangering stability of the closed-loop system.

Example: Static Nonlinearities

Let $N : \mathbb{R}^m \to \mathbb{R}^m$ be Lipschitz-continuous and suppose there exist $\gamma \in \mathbb{R}$ and $\epsilon > 0$ such that for all $z \in \mathbb{R}^m$:

$$\|N(z)\| \leq \gamma \|z\| \quad \text{and} \quad \begin{pmatrix} z \\ N(z) \end{pmatrix}^T \begin{pmatrix} -R & R \\ R & 0 \end{pmatrix} \begin{pmatrix} z \\ N(z) \end{pmatrix} \succ \epsilon \|z\|^2$$

Then Δ defined through $\Delta(z)(t) := N(z(t))$ for $t \geq 0$ satisfies all required hypotheses. Hence (4) described in this case through

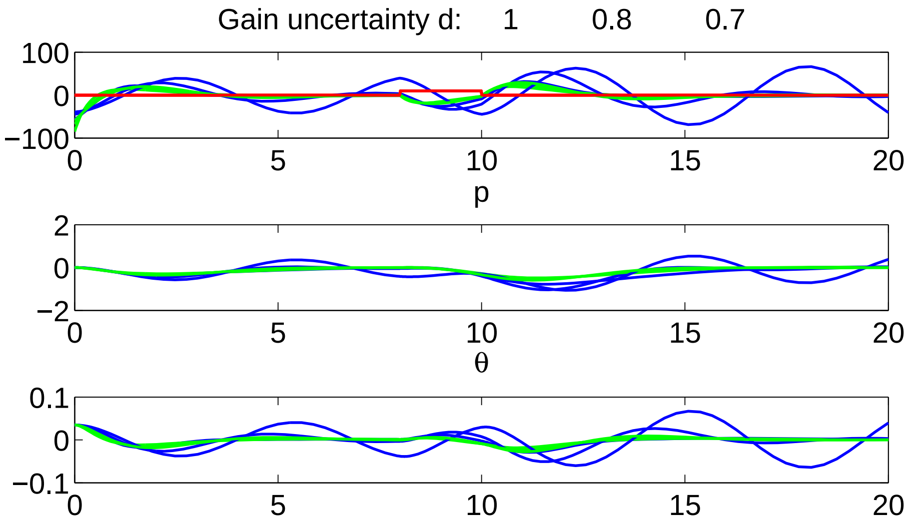
$$\dot{x} = Ax - BN(Fx), \quad x(0) = \xi$$

satisfies, for any initial condition, $x(t) \to 0$ for $t \to \infty$.

- Lipschitz-continuity of $N(.)$ implies that the ivp has a unique solution. The conditions then also guarantee that there is no finite escape time.
- Our result covers a much larger class of Δ's, that can be generated by finite- or infinite-dimensional dynamical systems. We just scratched the surface of an area which is called robust control.

Example: Segway

If compared to pole-placement controller (blue), the LQ-controller (green) leads to better robustness properties.



Example: Segway

The margin $d = 0.5$ is tight, as seen from the next simulations for the LQ-controller (green) only.