Basics of Dynamical and Control System (CS—)

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The Concept of Stability: Physical Significance

- The notion of stability is very old and has a clear intuitive meaning.
- Let us take an ordinary pendulum and put it in the lowest position, in which it is stable. Now, put it in the utmost upper position where it is unstable.
- Stable and unstable situations can be seen everywhere - in mechanical motion, in technical devices, in medical treatment (stable or unstable state of the patient) and so on.

The Concept of Stability: Physical Significance

The concept of stability can be illustrated by a cone placed on a plane horizontal surface:

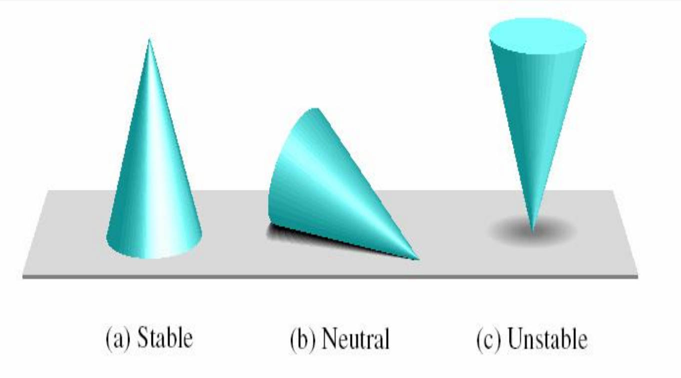


Figure: Conceptual description of linear system stability.

Ref: nptel.ac.in

The Concept of Stability: Why Important?

(a) Opening day of the Tacoma Narrows Bridge, Tacoma, Washington, July 1, 1940. The bridge was found to oscillate whenever the wind used to blow.

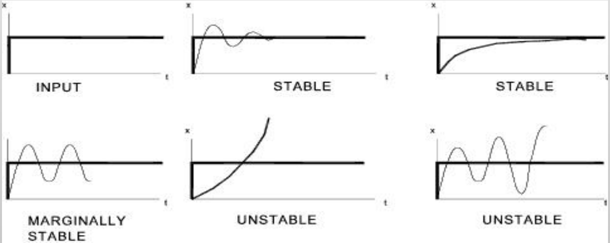
(b) The 1940 Tacoma Narrows Bridge collapsing in a 42 miles per hour (68 km/h) gust on November 7, 1940

Figure: An example of unstable system in real life.

Ref: https://en.wikipedia.org/wiki/

Definitions of Stability

- A system is Stable when its natural response goes to zero as time approaches infinity.
- A system is Unstable when its natural response goes to infinity as time approaches infinity.
- A system is Marginally Stable when its natural response remains constant or oscillates within a bound.



Ref: nptel.ac.in

Definitions of Stability

- BIBO stability: A system is said to be BIBO stable if for any bounded input, its output is also bounded.
- Thus for any bounded input the output either remain constant or decrease with time.

- Absolute stability: Stable /Unstable
- Relative stability: Degree of stability (i.e. how far from instability). Relative system stability can be measured by observing the relative real part of each root. In the following diagram $r2$ is relatively more stable than the pair of roots labeled as $r1$.

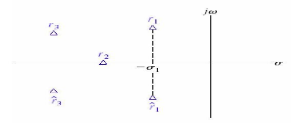


Figure: Example of relative system stability.

Ref: www.calvin.edu

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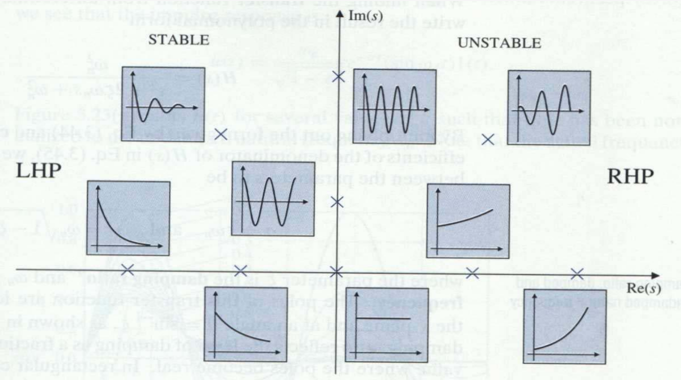
Stability: S-Plane and Transient Response



A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

Ref: nptel.ac.in

Stability Analysis from Closed Loop Transfer function

- Stable systems have closed-loop transfer functions whose poles reside only in the left half-plane.
- Unstable systems have closed-loop transfer functions with at least one pole in the right half plane and/or poles of multiplicity greater than one on the imaginary axis.
- Marginally Stable systems have closed-loop transfer functions with only imaginary axis poles of multiplicity 1 and poles in the left half-plane.

Stability Analysis

Let us consider the transfer function of a closed-loop system:

$$G(S) = \frac{C(S)}{R(S)} = \frac{\sum_{i=0}^{m} c_i s^{m-i}}{\sum_{i=0}^{n} r_i s^{n-i}}$$

Conditions for Stability

- Necessary condition for stability:
  – All coefficients of $R(s)$ have the same sign.
- Necessary and sufficient condition for stability:
  – All poles of G(s) reside in the left-half-plane (LHP)

$$i.e. \quad R(s) \neq 0 \;\; for \;\; Re[s] \geq 0$$

Stability Analysis

Necessary condition for stability

$$
\begin{aligned}
R(s) &= r_0 s^n + r_1 s^{n-1} + \ldots + r_{n-1} s + r_n \\
&= r_0(s+p_1)(s+p_2)\ldots(s+p_n) \\
&= r_0 s^n + r_0(p_1+p_2+\ldots+p_n)s^{n-1} \\
&\quad + r_0(p_1p_2+\ldots+p_{n-1}p_n)s^{n-2} \\
&\quad \ldots + r_0(p_1p_2\ldots p_n)
\end{aligned}
$$

$-p_1$ to $-p_n$ are the poles of the system.

Therefore, given a system to be stable:

- All poles of the system must have negative real parts.
- The coefficients of the polynomial should have the same sign.

Examples

$$R(s) = s^3 + s^2 + s + 1 \qquad \text{can be stable or unstable}$$
$$R(s) = s^3 - s^2 + s + 1 \qquad \text{is unstable}$$

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Stability of Linear Time Invariant (LTI) Systems

Let us consider a general linear (time-invariant) system given by:

$$\dot{x} = Ax \quad \text{and} \quad x(0) = X_0, \qquad x \in \Re^m$$

It may represent the closed or open loop system where $A \in \Re^{m \times m}$.

- Eigenvalues of A are the "Poles" of the given system.
- We can define the nature of the solution, without solving the system model.
- The nature of the solution is governed only by the locations of its poles.

Stability Properties of a Linear System

Considering the linear system defined earlier, and for each eigenvalue λ of A, suppose that m_λ denotes the algebraic multiplicity of λ and d_λ the geometric multiplicity of λ.

We can conclude the following:

- The system is asymptotically stable if and only if A is a **stability matrix**; i.e., every eigenvalue of A has a negative real part.
- The system is neutrally stable if and only if
  - Every eigenvalue of A has a nonpositive real part, *and*
  - At least one eigenvalue has a zero real part, and $d_\lambda = m_\lambda$ for every eigenvalue λ with a zero real part.
- The system is unstable if and only if
  - Some eigenvalue of A has a positive real part, *or*
  - There is an eigenvalue λ with a zero real part and $d_\lambda < m_\lambda$.

Ref: https://www.ru.ac.za @Claudiu C. Remsing, 2006.

Motivation for Stability Analysis of Non-Linear Systems

- Eigenvalue analysis concept is not suitable for nonlinear systems.
- Non-linear systems can have multiple equilibrium points and limit cycles.
- Stability behaviour of nonlinear systems need not be always global (unlike linear systems).

Stability of Non-Linear Systems

Basic Concepts

Let us consider a nonlinear dynamical system $\sum$ defined by

$$\dot{x} = F(t, x), \quad x \in \Re^m$$

where $x(\cdot)$ is a curve in the state space $\Re^m$ and F is a vector-valued mapping having components F_i, $i = 1, 2, ..., m$.

- Here, we will assume that the components F_i are continuous and satisfy standard conditions.
- From a geometric point of view, the right-hand side (RHS) F can be interpreted as a **time-dependent** vector field on $\Re^m$.
- If the functions F_i do not depend explicitly on t, then system $\sum$ is called **autonomous** (or *time-independent*); otherwise, **nonautonomous** (or *time-dependent*).

Stability of Non-Linear Systems

Equilibrium states

If $F(t, c) = 0$ for all t, then $c \in \Re^m$ is said to be an **equilibrium (or critical) state**.

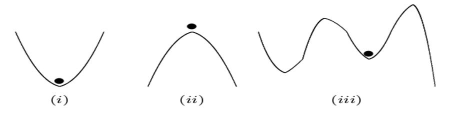
It follows that (for an equilibrium state c) if $x(t_0) = c$, then $x(t) = c$ for all $t \geq t_0$. Thus solution curves starting at c remain there.

The intuitive idea of stability in a dynamical setting is that for *"small" perturbations from the equilibrium state at some time t_0, subsequent motions $t \to x(t)$, $t \geq t_0$ should not be too "large"*.

Stability of Non-Linear Systems

Equilibrium states

Consider a ball resting in equilibrium on a sheet of metal bent into various shapes with cross-sections as shown below:



If frictional forces can be neglected, then small perturbations lead to :

- oscillatory motion about equilibrium (case (i)) ;
- the ball moving away without returning to equilibrium (case (ii));
- oscillatory motion about equilibrium, unless the initial perturbation is so large that the ball is forced to oscillate about a new equilibrium position (case (iii)).

Ref: https://www.ru.ac.za @Claudiu C. Remsing, 2006.

Stability in the Sense of Lyapunov

Let us Consider the general nonautonomous system:

$$\dot{x} = f(x,t), \quad x(t_0) = x_0 \in \Re^n$$

where the control input $u(t) = h(x(t),t)$, has been combined into the system function f. Without loss of generality, let us assume that the origin $x = 0$ is the system equilibrium of interest.

This system is said to be stable **in the sense of Lyapunov** with respect to the equilibrium $x^* = 0$, if for any $\epsilon > 0$ and any initial time $t_0 \geq 0$, there exists a constant, $\delta = \delta(\epsilon, t_0) > 0$, such that

$$||x(t_0)|| < \delta \Rightarrow ||x(t)|| < \epsilon \quad \text{for all } t \geq t_0$$

This stability is illustrated by the following figure:

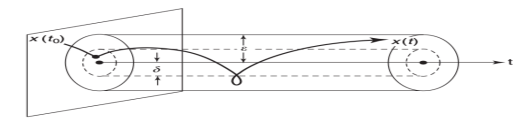


Figure: Geometric meaning of stability in the sense of Lyapunov.

Ref: www.ee.cityu.edu.hk/ gchen/pdf/C-Encyclopedia04

Asymptotic Stability

Consider the same general nonautonomous system:

$$\dot{x} = f(x,t), \qquad x(t_0) = x_0 \in \Re^n$$

This system is said to be **asymptotically stable** about its equilibrium $x^* = 0$, if it is stable in the sense of Lyapunov and, furthermore, there exists a constant, $\delta = \delta(t_0) > 0$, such that

$$||x(t_0)|| < \delta \Rightarrow ||x(t)|| \to 0 \quad \text{as } t \to \infty$$

This stability can be visualized by the following figure:

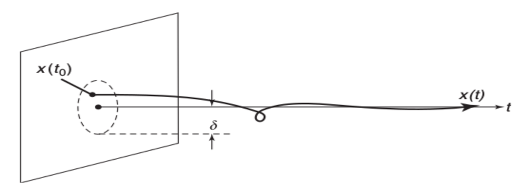


Figure: Geometric meaning of the asymptotic stability.

Ref: www.ee.cityu.edu.hk/ gchen/pdf/C-Encyclopedia04

Asymptotic Stability: Classification

Uniform Asymptotic Stability

The asymptotic stability is said to be uniform if the existing constant δ is independent of t_0 over $[0, \infty)$.

Global Asymptotic Stability

The asymptotic stability is said to be global if the convergence, $||x|| \rightarrow 0$, is independent of the initial state $x(t_0)$ over the entire spatial domain on which the system is defined (e.g., when $\delta = \infty$).

Ref: www.ee.cityu.edu.hk/ gchen/pdf/C-Encyclopedia04

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What is Stability ??

Stability Analysis from Closed Loop Transfer function in S-Plane

Stability Analysis of Linear and Nonlinear Systems

Exponential Stability

Considering the same general nonautonomous system defined earlier, the equilibrium state is said to be **exponentially stable** if

- it is stable in the sense of Lyapunov and,
- there exists two positive constants c and σ, such that

$$||x(t_0)|| < \delta \Rightarrow ||x(t)|| \leq ce^{-\sigma t}$$

This stability is visualized by the following figure:

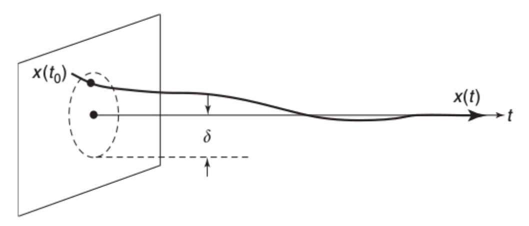

Figure: Geometric meaning of the exponential stability.

Ref: www.ee.cityu.edu.hk/ gchen/pdf/C-Encyclopedia04

Asymptotic and Exponential Stabilities

We can conclude that exponential stability implies asymptotic stability, and asymptotic stability implies the stability in the sense of Lyapunov, but the reverse need not be true.

Examples

1. Let us take a system which has output trajectory $x_1(t) = x_0 sin(t)$; it is stable in the sense of Lyapunov about 0, but is not asymptotically stable.

2. A system with output trajectory $x_2(t) = x_0(1 + t - t_0)^{-1}$ is asymptotically stable (so also is stable in the sense of Lyapunov) if $t_0 < 1$ but is not exponentially stable about 0.

3. A system $x_3(t) = x_0 e^{-t}$ is exponentially stable (hence, is both asymptotically stable and stable in the sense of Lyapunov).

Ref: www.ee.cityu.edu.hk/ gchen/pdf/C-Encyclopedia04

Lyapunov Theory

The so-called **"direct" method of Lyapunov** in relation to the nonlinear autonomous dynamical system $\sum$ given by

$$\dot{x} = F(x), \quad x(0) = x_0 \in \Re^m; \quad F(0) = 0.$$

To deal with the (nonautonomous) case some changes are required:

$$\dot{x} = F(t, x), \quad x(t_0) = x_0$$

- Lyapunov theory is used to *determine the stability nature of the equilibrium state (at the origin) of system* $\sum$ *without obtaining the solution* $x(\cdot)$.
- The main idea is to generalize the concept of energy V for a conservative system in mechanics, where a well-known result states that **an equilibrium point is stable if the energy is minimum**.
- Hence, V is a positive function which has $\dot{V}$ negative in the neighbourhood of a stable equilibrium point.

Ref: https://www.ru.ac.za @Claudiu C. Remsing, 2006.

Lyapunov function

Definition

We define a **Lyapunov function** $V : \Re^m \rightarrow \Re$ as follows:

- V and all its partial derivatives $\frac{\partial V}{\partial x_i}$ are continuous;
- V is *positive definite* (**PD**); i.e.,
  i. $V(x) \geq 0$ for all x
  ii. $V(x) = 0$ if and only if $x = 0$
  iii. all sublevel sets of V are bounded, which is equivalent to $V(x) \rightarrow \infty$ as $x \rightarrow \infty$

Lyapunov function

The (directional) derivative of V with respect to the vector field, F can be written as

$$\dot{V} = \frac{\partial V}{\partial x} F = \begin{bmatrix} \frac{\partial V}{\partial x_1} \cdots \frac{\partial V}{\partial x_m} \end{bmatrix} \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}$$

$$= \frac{\partial V}{\partial x_1} F_1 + \frac{\partial V}{\partial x_2} F_2 + \frac{\partial V}{\partial x_m} F_m.$$

A Lyapunov function V for the defined system $\sum$ is termed as

- **strong** if the derivative $\dot{V}$ is **negative definite**; i.e., $\dot{V}(0) = 0$ and $\dot{V}(x) < 0$ for $x \neq 0$ such that $||x|| \leq k$.
- **weak** if the derivative $\dot{V}$ is **negative semi-definite**; i.e., $\dot{V}(0) = 0$ and $\dot{V}(x) \leq 0$ for all x such that $||x|| \leq k$.

Ref: https://www.ru.ac.za @Claudiu C. Remsing, 2006.

The Lyapunov stability theorems

The two basic theorems of Lyapunov are:

Lyapunovs First Theorem

Suppose that there is a **strong Lyapunov function** V for system $\sum$. Then system $\sum$ is **asymptotically stable**.

Lyapunovs Second Theorem

Suppose that there is a **weak Lyapunov function** V for system $\sum$. Then system $\sum$ is **stable**.

Ref: https://www.ru.ac.za @Claudiu C. Remsing, 2006.

Instability theorem

Let us consider an autonomous dynamical system and assume an equilibrium point to be at $X = 0$. Now if the Lyapunov function($V : D \rightarrow \Re$) for the system has the following properties:

i. $V(0) = 0$

ii. $\exists X_0 \in \Re^n$ (arbitrarily close to X $= 0$), such that $V(X_0) > 0$

iii. $\dot{V} > 0 \;\; \forall X \in U$, where the set U is defined by

$$U = \{X \in D : ||X|| \leq \epsilon \;\; and \;\; V(X) > 0\}$$

Under all these conditions, the equilibrium state X $= 0$ is said to be unstable.

Example

Let us Consider a unit mass suspended from a fixed support by a spring, z being the displacement from the equilibrium. If first the spring is assumed to obey Hookes law, then the equation of motion is

$$\ddot{z} + kz = 0 \quad \text{where} \quad k \quad \text{is the spring constant.}$$

Now if we assume $x_1 \rightarrow z$ and $x_2 \rightarrow \dot{z}$, the equation of motion can be defined as

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -kx_1 \end{cases}$$

Since the system is conservative, the total energy

$$E = \frac{1}{2}kx_1^2 + \frac{1}{2}x_2^2$$

is a Lyapunov function(V) and it is easy to observe that

$$\dot{E} = kx_1x_2 - kx_2x_1 = 0$$

Hence, by **Lyapunov's Second Theorem** the origin of the system is **stable**.

Ref: https://www.ru.ac.za @Claudiu C. Remsing, 2006.