Contents

Section outline

Sets

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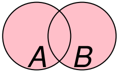

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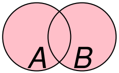

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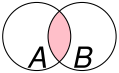

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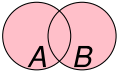

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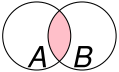

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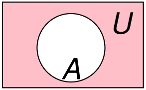

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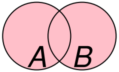

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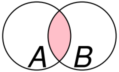

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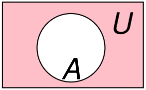
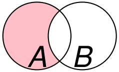


- Set difference: $A - B = A \cap \overline{B}$



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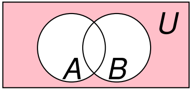

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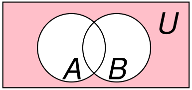

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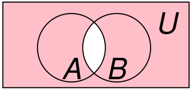

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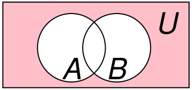

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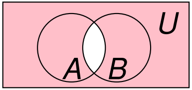

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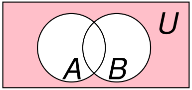

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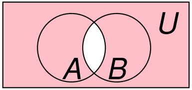

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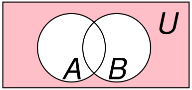

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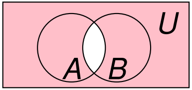

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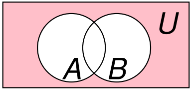

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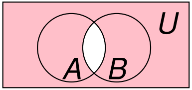

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  $A \cap \overline{B}$, $B \cap \overline{A}$, $A \cap B$ and $\overline{A \cup B}$ constitute a partition of U

Set algebra

Idempotence	$A \cup A = A$	$A \cap A = A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cup C = A \cap (B \cup C)$
Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	$A \cup \{\} = A, A \cup U = U$	$A \cap \{\} = \{\}, A \cup U = A$
Involution	$\bar{\bar{A}} = A$	
Complements	$\bar{U} = \{\}, A \cup \bar{A} = U$	$\overline{\{\}} = U, A \cap \bar{A} = \{\}$
DeMorgan	$\overline{A \cup B} = \bar{A} \cap \bar{B}$	$\overline{A \cap B} = \bar{A} \cup \bar{B}$

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- An equivalence relation induces a partition and vice versa

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- $\mathcal{R} \subseteq A \times A$ is reflexive if $\forall x \in A.\ x\mathcal{R}x$
- $\mathcal{R} \subseteq A \times A$ is symmetric if $\forall x, y \in A.\ x\mathcal{R}y \Rightarrow y\mathcal{R}x$
- $\mathcal{R} \subseteq A \times A$ is transitive if $\forall x, y, z \in A.\ x\mathcal{R}y \wedge y\mathcal{R}z \Rightarrow x\mathcal{R}z$
- $\mathcal{R} \subseteq A \times A$ is antisymmetric if $\forall x, y \in A.\ x\mathcal{R}y \wedge y\mathcal{R}x \Rightarrow x = y$
- Equivalence relation: $\mathcal{R}$ is reflexive, symmetric and transitive
- An equivalence relation induces a partition and vice versa
- Partial order: $\mathcal{R}$ is reflexive, antisymmetric and transitive

Relations (contd.)

- Connected relation: $\forall x, y \in A$, either $x\mathcal{R}y$ or $y\mathcal{R}x$

Relations (contd.)

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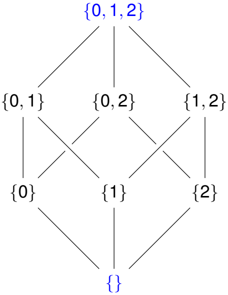

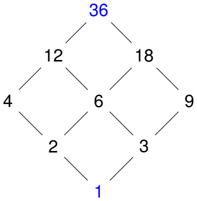

Divisors of 36 (60?)

Suppose $\langle A, \leq\rangle$ is a poset, $M, \in A$ $(m, \in A)$, $S \subseteq A$

M (m) is a maximal (minimal) element of S iff $M \in S$ $(m \in S)$ and $\nexists x \in S$ st $M < x$ $(x < m)$

M (m) is a maximum (minimum) of S iff $M \in S$ $(m \in S)$ and $\forall x \in S,\ x \leq M$ $(m \leq x)$

A PO can be embedded in a total order

Relations (contd.)

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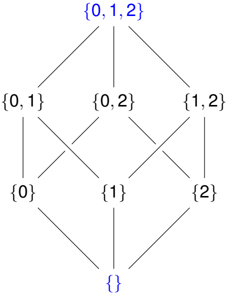

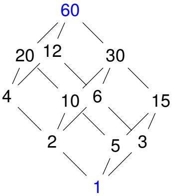

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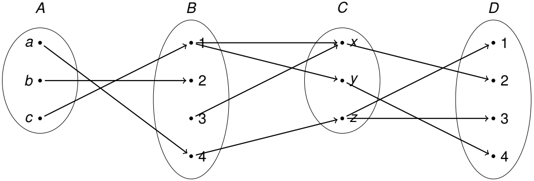
Composition of relations

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{S}$$



- $R \circ S = \{(a, c) | \; \exists b \in B \text{ st } \langle a, b\rangle \in R \wedge \langle b, c\rangle \in S\}$

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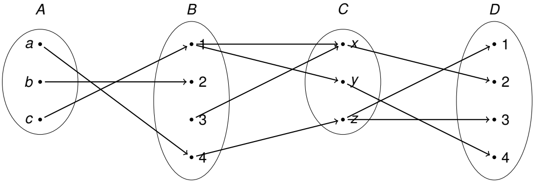

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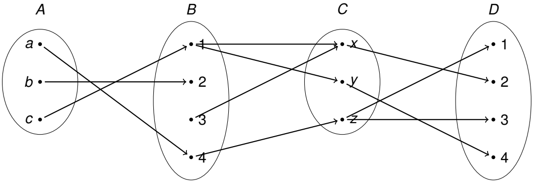

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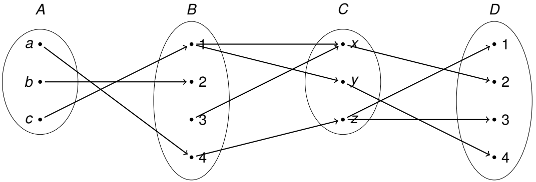

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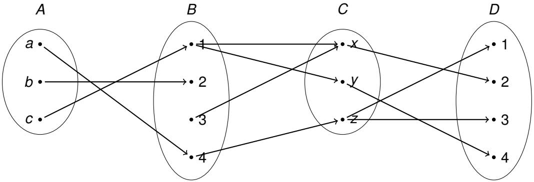

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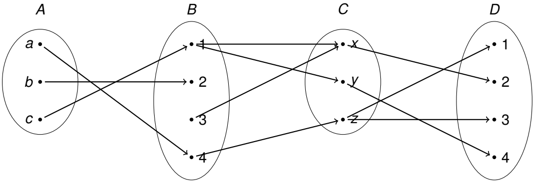

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Lattices

Let $\langle A, \leq \rangle$ be a poset, let $x, y \in A$

- The *meet* of x and y ($x \wedge y$), is the maximum of all lower bounds for x and y: $x \wedge y = \max \{w \in A : w \leq x, w \leq y\}$, *glb* for x and y
- The *join* of x and y ($x \vee y$), is the minimum of all upper bounds for x and y; $x \vee y = \min \{z \in A : x \leq z, y \leq z\}$, *lub* for x and y

A poset $\langle A, \leq \rangle$ is a lattice iff every pair of elements in A have both a meet and a join

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A poset $\langle A, \leq \rangle$ is a lattice iff every pair of elements in A have both a meet and a join

---

Basic order properties of meet and join

- $x \wedge y \leq \{x, y\} \leq x \vee y$
- $x \leq y$ iff $x \wedge y = x$
- $x \leq y$ iff $x \vee y = y$
- If $x \leq y$, then $x \wedge z \leq y \wedge z$ and $x \vee z \leq y \vee z$
- If $x \leq y$ and $z \leq w$, then $x \wedge z \leq y \wedge w$ and $x \vee z \leq y \vee w$

Lattices (contd.)

Commutativity $x \wedge y = y \wedge x, \quad x \vee y = y \vee x$

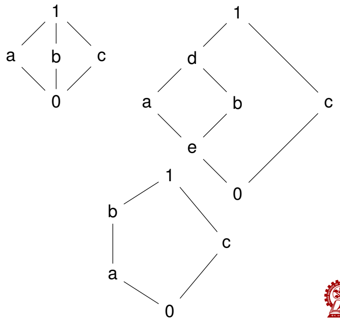
Associativity $(x \wedge y) \wedge z = x \wedge (y \wedge z), \quad (x \vee y) \vee z = x \vee (y \vee z)$

Idempotence $x \wedge x = x, \quad x \vee x = x$

Absorption $x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x$

Distributive lattice:
$\forall x, y, z \in A,\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Are these lattices distributive?



Bounded lattice: It has a maximum element (1) and a minimum element (0), in which case

- $0 \vee x = x = x \vee 0, \quad 1 \wedge x = x = x \wedge 1$
- $0 \wedge x = 0 = x \wedge 0, \quad 1 \vee x = 1 = x \vee 1$

Complemented lattice

- Complement in a bounded lattice: Complement of a x is z st $x \wedge z = 0$ and $x \vee z = 1$
- Bounded complemented lattice: every element has a complement
- In a bounded distributive lattice with minimum 0 and maximum 1, the complements of elements are unique, provided they exist – let $\bar{x}$ and z be complements of x ...

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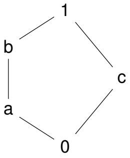
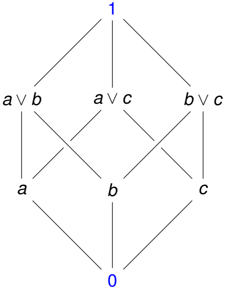
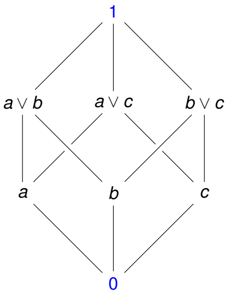
- In a bounded distributive lattice with minimum 0 and maximum 1, the complements of elements are unique, provided they exist – let $\bar{x}$ and z be complements of x ...
  - $\bar{x} = \bar{x} \wedge 1 =$
  - $\bar{x} \wedge (x \vee z) =$
  - $(\bar{x} \wedge x) \vee (\bar{x} \wedge z) =$
  - $0 \vee (\bar{x} \wedge z) =$
  - $(x \wedge z) \vee (\bar{x} \wedge z) =$
  - $(x \vee \bar{x}) \wedge z =$
  - $1 \wedge z = z$





- Complemented distributive lattice are called *Boolean* lattices – elements have unique complements and $\langle \wedge, \vee \rangle$ satisfy De Morgan's laws (to be proven)

Boolean lattices

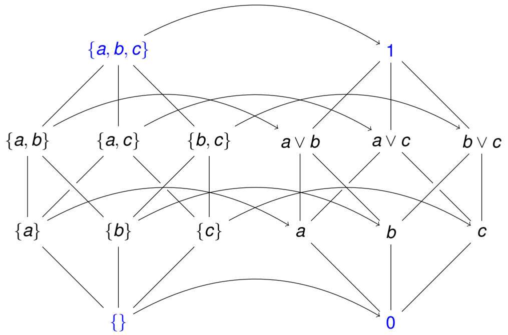


Atom of a Boolean lattice: Non-trivial minimal element of $A \setminus \{0\}$

$|A| = 2^n$ for some n for a Boolean lattice, its structure is that of the power set of the atomic elements

- Non-trivial atomic elements are present for $|A| > 1$ directly above level 0, let those be $S = \{a_1, \ldots, a_n\}$, akin to $\{a_1\}, \{a_2\}, \ldots \{a_n\}$
- Join of pairs of elements Y_1, Y_2 at level i $(n > i > 1)$ st $|Y_1 - Y_2| = |Y_2 - Y_1| = 1$ at level $i + 1$ is $Y = Y_1 \cup Y_2$
- Meet of pairs of elements X_1, X_2 at level i $(n > i > 1)$ st $|X_1 - X_2| = |X_2 - X_1| = 1$ at level $i - 1$ is $Y = Y_1 \cap Y_2$
- There will be $\binom{n}{i}$ such sets in level i, totaling to $\sum_0^n \binom{n}{i} = 2^n$

Homomorphism from PS to BL



- $f : P \rightarrow A$
- $f : \{a_1, a_2, \ldots, a_m\} \mapsto a_1 \vee a_2 \ldots a_m$
- $f(S_1 \cup S_2) = f(S_1) \vee f(S_2)$
- $f(S_1 \cap S_2) = f(S_1) \wedge f(S_2)$
- f is a homomorphism from the power set lattice to the Boolean lattice with atoms a, b and c

Boolean algebra axioms

Commutative Laws

1. $x + y = y + x$
2. $x \cdot y = y \cdot x$

Associative Laws

1. $(x + y) + z = x + (y + z)$
2. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Distributive Laws

1. $x \cdot (y + z) = x \cdot y + x \cdot z$
2. $x + (y \cdot z) = (x + y) \cdot (x + z)$

Identity Laws

1. $x + 0 = x = 0 + x$
2. $x \cdot 1 = x = 1 \cdot x$

Complementation Laws

1. $x + \bar{x} = 1 = \bar{x} + x$
2. $x \cdot \bar{x} = 0 = \bar{x} \cdot x$

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- Is the powerset lattice a BA?
- Is $\langle \{0, 1\}, \max, \min, \bar{x} \equiv 1 - x \rangle$ a BA?

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- Is the powerset lattice a BA?
- Is $\langle \{0, 1\}, \max, \min, \bar{x} \equiv 1 - x \rangle$ a BA?

Is a Boolean algebra a Boolean lattice? – to prove:

- Idempotence: $x + x = x$, $x \cdot x = x$
- Absorption: $x + xy = x$, $x \cdot (x + y) = x$, $x + \bar{x}y = x + y$, $x \cdot (\bar{x} + y) = xy$
- Boundedness: $x + 1 = 1$, $x \cdot 0 = 0$

Boolean algebra (contd.)

Idempotence

- $x + x = (x + x) \cdot 1$
- $= (x + x) \cdot (x + \bar{x})$
- $= x + (x \cdot \bar{x})$
- $= x + 0 = x$

Absorption

- $x + xy = (x \cdot 1) + xy$
- $= x(1 + y) = x(y + 1)$
- $= x \cdot 1 = x$

Boundedness

- $x + 1 = 1 \cdot (x + 1)$
- $= (x + \bar{x}) \cdot (x + 1)$
- $= x + (\bar{x} \cdot 1)$
- $= x + \bar{x} = 1$

$\bar{0} = 0 + \bar{0} = 1$

Functions

- $f : X \rightarrow Y$ maps each element from a domain X to a range in Y

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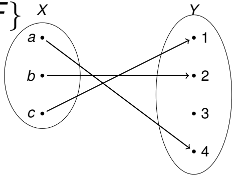

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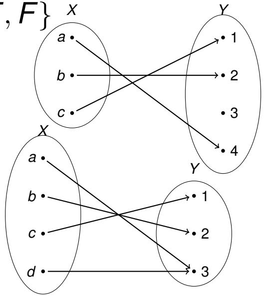
- Injective (one-to-one, no two domain elements mapped to same element in range, $\forall x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$)
- Surjective (onto, a domain element is mapped to each element in the range)



Functions

- $f : X \rightarrow Y$ maps each element from a domain X to a range in Y
- k-ary if $f : X_1 \times X_2 \times \ldots \times X_k \rightarrow Y$
- k-ary predicate if $f : X_1 \times X_2 \times \ldots \times X_k \rightarrow \{T, F\}$
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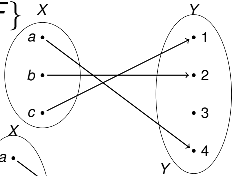

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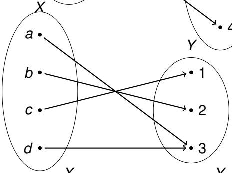
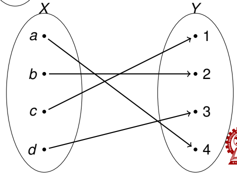


- Bijective (both injective and surjective, so f^{-1} exists)



Peano axioms

- Peano axioms to generate $\mathbb{N}$
  1. For every $x \in \mathbb{N},\ x = x$ [Reflexivity of $=$]
  2. For every $x, y \in \mathbb{N},\ x = y \Rightarrow y = x$ [Symmetry of $=$]
  3. For every $x, y, z \in \mathbb{N},\ x = y \wedge y = z \Rightarrow x = z$ [Transitivity of $=$]
  4. For every x, y, if $x \in \mathbb{N} \wedge x = y \Rightarrow y \in \mathbb{N}$ [Closure of $=$]
  5. $0 \in \mathbb{N}$ [Seeding $\mathbb{N}$]
  6. if $x \in \mathbb{N}$ then $S(x) \in \mathbb{N}$ [Need a successor function $S : \mathbb{N} \to \mathbb{N}$]
  7. For every $x \in \mathbb{N},\ S(x) \neq 0$ [Prevent $S(0) = 0$,]
  8. For every $x, y \in \mathbb{N},\ S(x) = S(y) \Rightarrow x = y$ [Prevent loop back of $S(x)$, now $\{0, S(0) = 1, S^2(0) = 2, S^3(0) = 3, \ldots\} \subset \mathbb{N}$]
     A set V is inductive if: $0 \in V$ and if $x \in V \Rightarrow S(x) \in V$
  9. If V is an inductive set, then $\mathbb{N} \subset V$ [Avoid extra elements in $\mathbb{N}$, now $\mathbb{N} \subset \{0, 1, 2, 3, \ldots\}$, together $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$]
- Addition: $a + 0 = 0,\ a + S(b) = S(a + b)$
- Multiplication: $a \cdot 0 = 0,\ a \cdot S(b) = a + a \times b$

Cardinality and Countability

- Which set is "bigger," the set of all integers or the set of non-negative integers?

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- What about: $\{2x|x \in Z^{\geq}\}$, $Z^{\geq} - \{3x|x \in Z^{\geq}\}$, the set of positive fractions ($\mathcal{Q}^{+,\leq 1}$), $\mathbb{Q}^{+}$ which is the set of positive rational numbers?

Cardinality and Countability (contd.)

- As such cardinality indicates the number of elements in a set
- Problem with a definition of this kind is that the set in question might have more elements than the numbers available for counting
- So, an indirect method is employed through the notion of mappings

$|A| = |B|$ if a bijection exists from $AtoB$, denoted as $A \approx B$, A is equinumerous to B

$|A| \leq |B|$ if there exists an injective function from A into B, cardinality of A is less than or equal to cardinality of B

$|A| < |B|$ if there exists an injective function from A into B but no bijective function from A to B, cardinality of A is strictly less than the cardinality of B

Cardinality and Countability (contd.)

$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\cdots$
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$\frac{1}{4}$	$\frac{2}{5}$	$\frac{3}{6}$	$\frac{4}{7}$	$\frac{5}{8}$	$\frac{6}{9}$	$\cdots$
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$\frac{1}{7}$	$\frac{2}{8}$	$\frac{3}{9}$	$\frac{4}{10}$	$\frac{5}{11}$	$\frac{6}{12}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

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$$f(\langle i,j\rangle) = \begin{cases} \frac{j(j-1)}{2} + i & \gcd(i,j) = 1 \\ f(\frac{i}{n}, \frac{j}{n}) & n = \gcd(i,j), n \neq 1 \end{cases}$$

This is an injection from $\mathcal{Q}^{+,\leq 1}$ to Z^+, so $\mathcal{Q}^{+,\leq 1} \leq Z^+$

Now, $g(i) = \langle i, i+1\rangle$ is an injection from Z^+ to $\mathcal{Q}^{+,\leq 1}$, so $Z^+ \leq \mathcal{Q}^{+,\leq 1}$

Together, $\mathcal{Q}^{+,\leq 1} \approx Z^+$ (by the Schröder-Bernstein Theorem)

Cantor's theorem

Theorem

If A is any set, cardinality of A is strictly less than the cardinality of $\mathcal{P}(A)$

Proof.

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The key step was to prove absence of a bijection

Uncountability of the Reals ($\mathbb{N} \not\approx \mathbb{R}$)

0	d_1	d_2	d_2	d_2	d_2	d_2	...
r_1	*2*	3	4	5	6	7	...
r_2	3	*9*	4	5	6	3	...
r_3	1	4	*5*	3	4	2	...
r_4	9	8	3	*1*	3	1	...
r_5	4	5	2	1	*1*	4	...
r_6	2	2	3	3	7	*7*	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

- Assume $\mathbb{N} \approx \mathbb{R}$
- Should be possible to tabulate all the real numbers (as shown)
- Construct a real number r as follows: for r_i, if $d_i \neq 1$, d_i for r is 1 and 5 otherwise
- For the tabulated numbers, $r = 111551\ldots$ and r is not in the table, as it differs in the i^{th} position of each r_i
- Hence, $\mathbb{N} \not\approx \mathbb{R}$ (by the diagonalisation argument)

Inclusion-exclusion principle

- If A and B are disjoint and finite: $|A \cup B| = |A| + |B|$
- In general (for finite sets): $|A \cup B| = |A| + |B| - |A \cap B|$
- For finite A, B, C:
  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- For n sets: $\left|\left(\bigcup_{i=1}^{n} A_i\right)\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} \left|A_i \cap A_j\right| - \ldots +$
  $\sum_{i<j<k} \left|A_i \cap A_j \cap A_k\right| - \cdots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|$
  - Let an element occur in precisely r ($r > 0$) of the sets – intersections of $r+1$ or more sets will be empty
  - By this formula its count for that element is $\binom{r}{1} - \binom{r}{2} + \cdots + (-1)^{r-1}\binom{r}{r}$ and should be $1 = \binom{r}{0}$
  - But, $(1-1)^r = \binom{r}{0} - \binom{r}{1} + \binom{r}{2} + \cdots + (-1)^r\binom{r}{r} = 0$ [binomial thm]
- $\left|\bigcap_{i=1}^{n} \bar{A}_i\right| = \left|U - \bigcup_{i=1}^{n} A_i\right| = |U| - \sum_{i=1}^{n} |A_i| + \sum_{i<j} \left|A_i \cap A_j\right| + \ldots -$
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  $1 - \left(\frac{364}{365}\right)^{\frac{n(n-1)}{2}}$

Examples problems for inclusion-exclusion

- How many solutions does $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2 and x_3 are nonnegative integers with $x_1 \leq 3, x_2 \leq 4$ and $x_3 \leq 6$?
- In a month of 30 days, a football team plays at least one game daily and at most 45 games in the whole month. Show that cumulative games played on various days must differ by 14 within the month.
- Among any six people, there are three who are mutually aquaintainted, or there are three who are mutually not aquaintainted.

Propositional logic

- Boolean operations:

and/conjunction: $\wedge$	or/disjunction: $\vee$
not/negation: $\neg$	xor/exclusive-or: $\oplus$
implies: $\Rightarrow$	equality/two-way-implication: $\Leftrightarrow$

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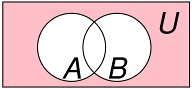

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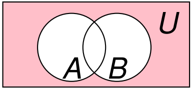

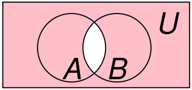

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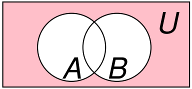

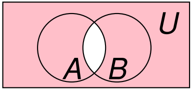

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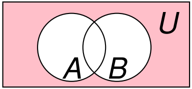

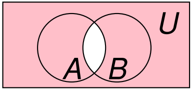

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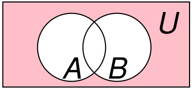

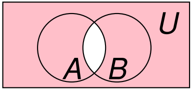

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Axiomatisation of PL (Jan Lukasiewicz)

- L has connectives $\neg, \Rightarrow, (,)$ and statements A_i $(A_1, A_2, A_3, \ldots)$
  1. All statement letters are wfs.
  2. If B and C are wfs, then so are $\neg B$ and $(B \Rightarrow C)$
- If B, C and D are wfs of L, then the following are axioms of L
  - **A** $(B \Rightarrow (C \Rightarrow B))$
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L is sound and complete, a wff of L is a theorem iff it's a tautology

Deduction theorem

If Γ is a set of wfs and B and C are wfs and $\Gamma, B \models C$, then $\Gamma, \models B \Rightarrow C$

- Let $C_1, C_2, \ldots, C_n = C$ be a proof from $\Gamma \cup B$
- C_1 must be an axiom or $C \in \Gamma$ or $C = B$
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- Inductive hypothesis: $\Gamma \models B \Rightarrow C_k$ for $k < j$
- For C_n, additionally, C_j follows by MP from C_l and $C_m = C_l \Rightarrow C_j$
- By inductive hypothesis, $\Gamma \models B \Rightarrow C_l$ and $\Gamma \models B \Rightarrow (C_l \Rightarrow C_j)$
- By A2, $B \Rightarrow (C_l \Rightarrow C_j) \Rightarrow ((B \Rightarrow C_l) \Rightarrow (B \Rightarrow C_j))$
- Thus, we have $\Gamma \models B \Rightarrow C_j$

Practice examples for propositional logic

Prove or disprove the following:

1. $((p \Rightarrow q) \Rightarrow r)$ and $p \Rightarrow (q \Rightarrow r)$ are logically equivalent
2. $\neg(p \iff q) \iff (p \iff \neg q)$
3. $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

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- $\exists y.\exists x.(P(f(x)) \vee P(g(f(y))))$
- From $\forall x.(\text{IsMan}(x) \Rightarrow \text{IsMortal}(x))$ and IsMan(Socrates) deduce IsMortal(Socrates)

Practice examples for FOL

Translate to first order logic formulae

1. Everybody likes somebody
2. There is somebody whom everybody likes
3. There is exactly one person whom everbody likes

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Proof by contradiction Claim: P is true
proved as follows: Assume P is false
proceeding with this assumption, conclude that some absurdity must be true, through some logical reasoning
thereby conclude that P cannot be assumed false, so must be true

Theorems and proofs (contd.)

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Establish the base case, show $P(0)$ is true

Do the inductive step, assuming $P(i)$ is true, show through logical reasoning that $P(i+1)$ is true

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Proof by structural induction P is true for all structures ordered by some structural parameter x

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Do the inductive step, assuming $P(i)$ is true for $x = i$, show through logical reasoning that $P(i+1)$ is true for $x = i + 1$

thereby conclude that P is true for all non-negative values of x

Practice examples for MI

Prove the following using mathematical induction:

1. $1 + 2 + 3 + ... + n = \frac{n \cdot (n+1)}{2}$
2. $n^3 - n$ is divisible by 3, $n > 0$
3. $2^n > n^2$ if n is an integer greater than 4

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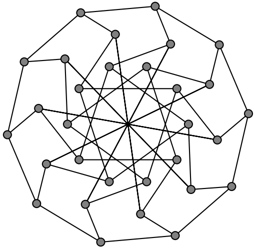

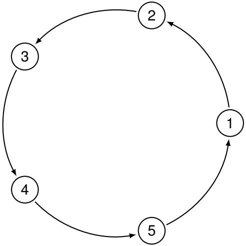

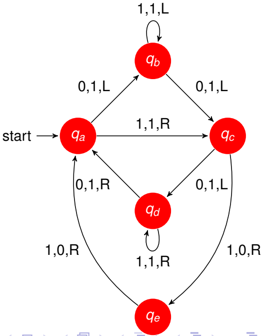

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- A *connected component* (or just component) of an undirected graph is a subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the supergraph

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- In a graph $G(V, E)$ a walk is an alternating sequence of vertices and edges $v_0, e_1, \ldots, e_n, v_k$, such that e_i is an edge from v_{i-1} to v_i; length of the walk is k

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- A *circuit* is a closed trail. That is, a circuit has no repeated edges but may have repeated vertices

Graphs (contd.)

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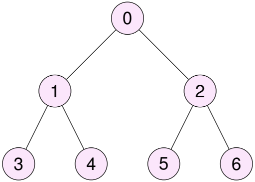

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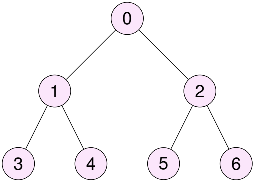

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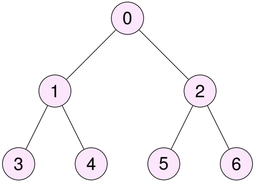

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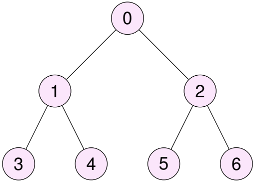
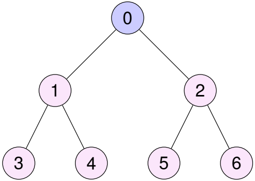
- A *rooted tree* is a tree where one of the vertices is designated as the root





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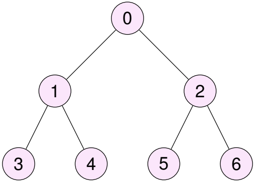
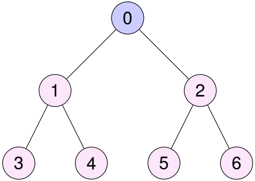
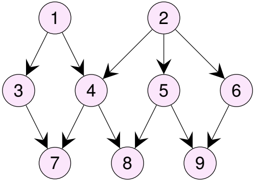


- A *rooted tree* is a tree where one of the vertices is designated as the root



- A *directed acyclic graph* (DAG) is a cycle free directed graph



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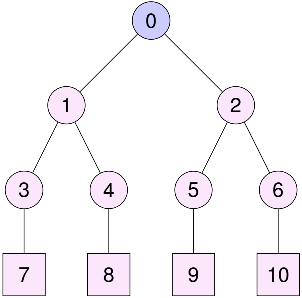
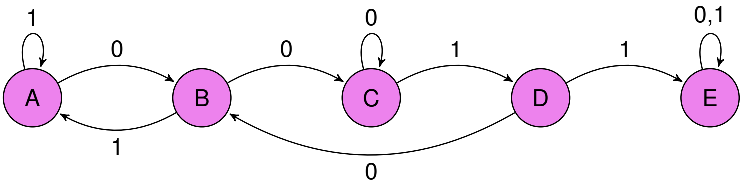


- What is the maximum number of nodes in such a tree of height h?
- For such a tree of n leaf nodes, what is the minimum height?

Confluence of concepts



- For this graph, identify walks of lengths 4

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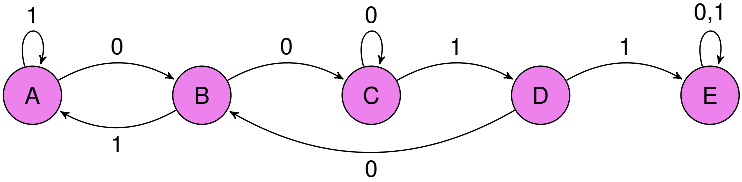

- For this graph, identify walks of lengths 4, 5

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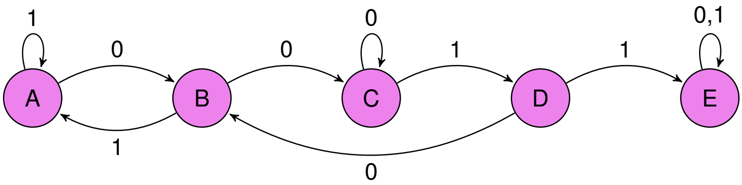

- For this graph, identify walks of lengths 4, 5, 6

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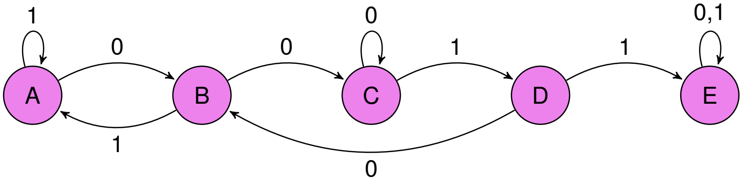

- For this graph, identify walks of lengths 4, 5, 6, 7

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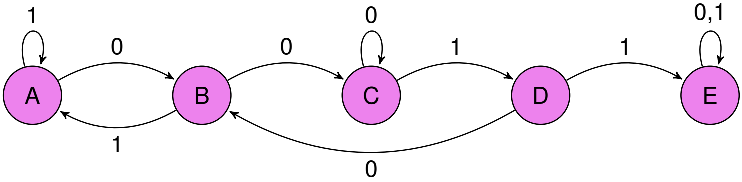

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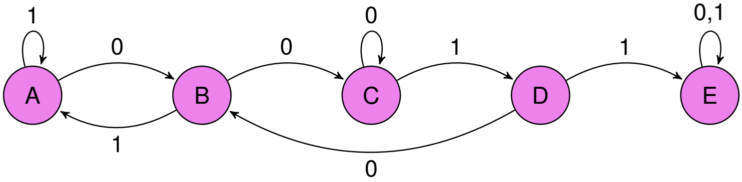

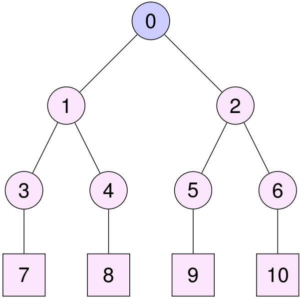

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- For a tree of *n* leaf nodes, as shown, at least how many node labels are needed to ensure that each path has a distinct label?

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