1. Vectors and Matrices

1A. Vectors

Definition. *A* **direction** *is just a unit vector. The* **direction of A** *is defined by*

$$\text{dir } \mathbf{A} = \frac{\mathbf{A}}{|\mathbf{A}|}, \qquad (\mathbf{A} \neq \mathbf{0});$$

it is the unit vector lying along **A** *and pointed like* **A** *(not like* $-\mathbf{A}$*).*

1A-1 Find the magnitude and direction (see the definition above) of the vectors

a) $\mathbf{i} + \mathbf{j} + \mathbf{k}$ b) $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ c) $3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$

1A-2 For what value(s) of c will $\frac{1}{5}\mathbf{i} - \frac{1}{5}\mathbf{j} + c\,\mathbf{k}$ be a unit vector?

1A-3 a) If $P = (1, 3, -1)$ and $Q = (0, 1, 1)$, find $\mathbf{A} = PQ$, $|\mathbf{A}|$, and dir $\mathbf{A}$.

b) A vector $\mathbf{A}$ has magnitude 6 and direction $(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})/3$. If its tail is at $(-2, 0, 1)$, where is its head?

1A-4 a) Let P and Q be two points in space, and X the midpoint of the line segment PQ. Let O be an arbitrary fixed point; show that as vectors, $OX = \frac{1}{2}(OP + OQ)$.

b) With the notation of part (a), assume that X divides the line segment PQ in the ratio $r : s$, where $r + s = 1$. Derive an expression for OX in terms of OP and OQ.

1A-5 What are the $\mathbf{i}\,\mathbf{j}$-components of a plane vector $\mathbf{A}$ of length 3, if it makes an angle of 30^o with $\mathbf{i}$ and 60^o with $\mathbf{j}$. Is the second condition redundant?

1A-6 A small plane wishes to fly due north at 200 mph (as seen from the ground), in a wind blowing from the northeast at 50 mph. Tell with what vector velocity in the air it should travel (give the $\mathbf{i}\,\mathbf{j}$-components).

1A-7 Let $\mathbf{A} = a\,\mathbf{i} + b\,\mathbf{j}$ be a plane vector; find in terms of a and b the vectors $\mathbf{A}'$ and $\mathbf{A}''$ resulting from rotating $\mathbf{A}$ by 90^o a) clockwise b) counterclockwise.

(Hint: make $\mathbf{A}$ the diagonal of a rectangle with sides on the x and y-axes, and rotate the whole rectangle.)

c) Let $\mathbf{i}' = (3\,\mathbf{i} + 4\,\mathbf{j})/5$. Show that $\mathbf{i}'$ is a unit vector, and use the first part of the exercise to find a vector $\mathbf{j}'$ such that $\mathbf{i}', \mathbf{j}'$ forms a right-handed coordinate system.

1A-8 The direction (see definition above) of a space vector is in engineering practice often given by its **direction cosines**. To describe these, let $\mathbf{A} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ be a space vector, represented as an origin vector, and let α, β, and γ be the three angles ($\leq \pi$) that $\mathbf{A}$ makes respectively with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

a) Show that dir $\mathbf{A} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$. (The three coefficients are called the *direction cosines* of $\mathbf{A}$.)

b) Express the direction cosines of $\mathbf{A}$ in terms of a, b, c; find the direction cosines of the vector $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

c) Prove that three numbers t, u, v are the direction cosines of a vector in space if and only if they satisfy $t^2 + u^2 + v^2 = 1$.

1A-9 Prove using vector methods (without components) that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. (Call the two sides $\mathbf{A}$ and $\mathbf{B}$.)

1A-10 Prove using vector methods (without components) that the midpoints of the sides of a space quadrilateral form a parallelogram.

1A-11 Prove using vector methods (without components) that the diagonals of a parallelogram bisect each other. (One way: let X and Y be the midpoints of the two diagonals; show $X = Y$.)

1A-12* Label the four vertices of a parallelogram in counterclockwise order as OPQR. Prove that the line segment from O to the midpoint of PQ intersects the diagonal PR in a point X that is 1/3 of the way from P to R.

(Let $\mathbf{A} = $ OP, and $\mathbf{B} = $ OR; express everything in terms of $\mathbf{A}$ and $\mathbf{B}$.)

1A-13* a) Take a triangle PQR in the plane; prove that as vectors $PQ + QR + RP = \mathbf{0}$.

b) Continuing part a), let $\mathbf{A}$ be a vector the same length as PQ, but perpendicular to it, and pointing outside the triangle. Using similar vectors $\mathbf{B}$ and $\mathbf{C}$ for the other two sides, prove that $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$. (This only takes one sentence, and no computation.)

1A-14* Generalize parts a) and b) of the previous exercise to a closed polygon in the plane which doesn't cross itself (i.e., one whose interior is a single region); label its vertices $P_1, P_2, \ldots, P_n$ as you walk around it.

1A-15* Let $P_1, \ldots, P_n$ be the vertices of a regular n-gon in the plane, and O its center; show without computation or coordinates that $OP_1 + OP_2 + \ldots + OP_n = \mathbf{0}$,

a) if n is even; b) if n is odd.

1B. Dot Product

1B-1 Find the angle between the vectors

a) $\mathbf{i} - \mathbf{k}$ and $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ b) $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

1B-2 Tell for what values of c the vectors $c\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ will

a) be orthogonal b) form an acute angle

1B-3 Using vectors, find the angle between a longest diagonal PQ of a cube, and

a) a diagonal PR of one of its faces; b) an edge PS of the cube.

(Choose a size and position for the cube that makes calculation easiest.)

1B-4 Three points in space are $P : (a, 1, -1)$, $Q : (0, 1, 1)$, $R : (a, -1, 3)$. For what value(s) of a will PQR be

a) a right angle b) an acute angle ?

1B-5 Find the component of the force $\mathbf{F} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in

a) the direction $\dfrac{\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{3}}$ b) the direction of the vector $3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$.

1B-6 Let O be the origin, c a given number, and $\mathbf{u}$ a given direction (i.e., a unit vector). Describe geometrically the locus of all points P in space that satisfy the vector equation

$$OP \cdot \mathbf{u} \;=\; c|OP|\;.$$

In particular, tell for what value(s) of c the locus will be (Hint: divide through by $|OP|$):

a) a plane b) a ray (i.e., a half-line) c) empty

1B-7 a) Verify that $\mathbf{i}' = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$ and $\mathbf{j}' = \dfrac{-\mathbf{i}+\mathbf{j}}{\sqrt{2}}$ are perpendicular unit vectors that form a right-handed coordinate system

b) Express the vector $\mathbf{A} = 2\,\mathbf{i} - 3\,\mathbf{j}$ in the $\mathbf{i}'\mathbf{j}'$-system by using the dot product.

c) Do b) a different way, by solving for $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{i}'$ and $\mathbf{j}'$ and then substituting into the expression for $\mathbf{A}$.

1B-8 The vectors $\mathbf{i}' = \dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$, $\mathbf{j}' = \dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$, and $\mathbf{k}' = \dfrac{\mathbf{i}+\mathbf{j}-2\,\mathbf{k}}{\sqrt{6}}$ are three mutually perpendicular unit vectors that form a right-handed coordinate system.

a) Verify this. b) Express $\mathbf{A} = 2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$ in this system (cf. 1B-7b)

1B-9 Let $\mathbf{A}$ and $\mathbf{B}$ be two plane vectors, neither one of which is a multiple of the other. Express $\mathbf{B}$ as the sum of two vectors, one a multiple of $\mathbf{A}$, and the other perpendicular to $\mathbf{A}$; give the answer in terms of $\mathbf{A}$ and $\mathbf{B}$.

(Hint: let $\mathbf{u} = \text{dir } \mathbf{A}$; what's the $\mathbf{u}$-component of $\mathbf{B}$?)

1B-10 Prove using vector methods (without components) that the diagonals of a parallelogram have equal lengths if and only if it is a rectangle.

1B-11 Prove using vector methods (without components) that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides are equal.

1B-12 Prove using vector methods (without components) that an angle inscribed in a semicircle is a right angle.

1B-13 Prove the trigonometric formula: $\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$.

(Hint: consider two unit vectors making angles θ_1 and θ_2 with the positive x-axis.)

1B-14 Prove the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$ by using the algebraic laws for the dot product and its geometric interpretation.

1B-15* The **Cauchy-Schwarz inequality**

a) Prove from the geometric definition of the dot product the following inequality for vectors in the plane or space; under what circumstances does equality hold?

$$(*) \qquad |\mathbf{A} \cdot \mathbf{B}| \le |\mathbf{A}||\mathbf{B}|\;.$$

b) If the vectors are plane vectors, write out what this inequality says in terms of $\mathbf{i}\,\mathbf{j}$-components.

c) Give a different argument for the inequality (*) as follows (this argument generalizes to n-dimensional space):

i) for all values of t, we have $(\mathbf{A}+t\mathbf{B})\cdot(\mathbf{A}+t\mathbf{B}) \geq 0$;

ii) use the algebraic laws of the dot product to write the expression in (i) as a quadratic polynomial in t ;

iii) by (i) this polynomial has at most one zero; this implies by the quadratic formula that its coefficients must satisfy a certain inequality — what is it?

1C. Determinants

1C-1 Calculate the value of the determinants a) $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$ b) $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix}$

1C-2 Calculate $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix}$ using the Laplace expansion by the cofactors of:

a) the first row b) the first column

1C-3 Find the area of the plane triangle whose vertices lie at

a) $(0,0), (1,2), (1,-1)$; b) $(1,2), (1,-1), (2,3)$.

1C-4 Show that $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_1-x_2)(x_2-x_3)(x_3-x_1)$.

(This type of determinant is called a **Vandermonde** determinant.)

1C-5 a) Show that the value of a 2×2 determinant is unchanged if you add to the second row a scalar multiple of the first row.

b) Same question, with "row" replaced by "column".

1C-6 Use a Laplace expansion and Exercise 5a to show the value of a 3×3 determinant is unchanged if you add to the second row a scalar multiple of the third row.

1C-7 Let (x_1, y_1) and (x_2, y_2) both range over all unit vectors.
Find the maximum value of the function $f(x_1, x_2, y_1, y_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.

1C-8* The base of a parallelepiped is a parallelogram whose edges are the vectors **b** and **c**, while its third edge is the vector **a**. (All three vectors have their tail at the same vertex; one calls them "coterminal".)

a) Show that the volume of the parallelepiped **abc** is $\pm\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

b) Show that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) =$ the determinant whose rows are respectively the components of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(These two parts prove the volume interpretation of a 3×3 determinant.

1C-9 Use the formula in Exercise 1C-8 to calculate the volume of a tetrahedron having as vertices $(0,0,0),(0,-1,2),(0,1,-1),(1,2,1)$. (The volume of a tetrahedron is $\frac{1}{3}$(base)(height).)

1C-10 Show by using Exercise 8 that if three origin vectors lie in the same plane, the determinant having the three vectors as its three rows has the value zero.

1D. Cross Product

1D-1 Find $\mathbf{A} \times \mathbf{B}$ if

a) $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \mathbf{B} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ b) $\mathbf{A} = 2\mathbf{i} - 3\mathbf{k},\ \mathbf{B} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

1D-2 Find the area of the triangle in space having its vertices at the points

$$P:(2,0,1),\ Q:(3,1,0),\ R:(-1,1,-1).$$

1D-3 Two vectors $\mathbf{i}'$ and $\mathbf{j}'$ of a right-handed coordinate system are to have the directions respectively of the vectors $\mathbf{A} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find all three vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$.

1D-4 Verify that the cross product $\times$ does not in general satisfy the associative law, by showing that for the particular vectors $\mathbf{i}, \mathbf{i}, \mathbf{j}$, we have $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} \neq \mathbf{i} \times (\mathbf{i} \times \mathbf{j})$.

1D-5 What can you conclude about $\mathbf{A}$ and $\mathbf{B}$

a) if $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|$; b) if $|\mathbf{A} \times \mathbf{B}| = \mathbf{A} \cdot \mathbf{B}$.

1D-6 Take three faces of a unit cube having a common vertex P; each face has a diagonal ending at P; what is the volume of the parallelepiped having these three diagonals as coterminous edges?

1D-7 Find the volume of the tetrahedron having vertices at the four points

$$P:(1,0,1),\ Q:(-1,1,2),\ R:(0,0,2),\ S:(3,1,-1).$$

Hint: volume of tetrahedron $= \frac{1}{6}$(volume of parallelepiped with same 3 coterminous edges)

1D-8 Prove that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$, by using the determinantal formula for the scalar triple product, and the algebraic laws of determinants in Notes D.

1D-9 Show that the area of a triangle in the xy-plane having vertices at (x_i, y_i), for $i = 1,2,3$, is given by the determinant $\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. Do this two ways:

a) by relating the area of the triangle to the volume of a certain parallelepiped

b) by using the laws of determinants (p. L.1 of the notes) to relate this determinant to the 2×2 determinant that would normally be used to calculate the area.

1E. Equations of Lines and Planes

1E-1 Find the equations of the following planes:

a) through $(2, 0, -1)$ and perpendicular to $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
b) through the origin, $(1, 1, 0)$, and $(2, -1, 3)$
c) through $(1, 0, 1)$, $(2, -1, 2)$, $(-1, 3, 2)$
d) through the points on the x, y and z-axes where $x = a$, $y = b$, $z = c$ respectively (give the equation in the form $Ax + By + Cz = 1$ and remember it)
e) through $(1, 0, 1)$ and $(0, 1, 1)$ and parallel to $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

1E-2 Find the dihedral angle between the planes $2x - y + z = 3$ and $x + y + 2z = 1$.

1E-3 Find in parametric form the equations for

a) the line through $(1, 0, -1)$ and parallel to $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$
b) the line through $(2, -1, -1)$ and perpendicular to the plane $x - y + 2z = 3$
c) all lines passing through $(1, 1, 1)$ and lying in the plane $x + 2y - z = 2$

1E-4 Where does the line through $(0, 1, 2)$ and $(2, 0, 3)$ intersect the plane $x + 4y + z = 4$?

1E-5 The line passing through $(1, 1, -1)$ and perpendicular to the plane $x + 2y - z = 3$ intersects the plane $2x - y + z = 1$ at what point?

1E-6 Show that the distance D from the origin to the plane $ax + by + cz = d$ is given by the formula $D = \dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.

(Hint: Let $\mathbf{n}$ be the unit normal to the plane. and P be a point on the plane; consider the component of OP in the direction $\mathbf{n}$.)

1E-7* Formulate a general method for finding the distance between two skew (i.e., non-intersecting) lines in space, and carry it out for two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).

(Hint: the shortest line segment joining the two skew lines will be perpendicular to both of them (if it weren't, it could be shortened).)

1F. Matrix Algebra

1F-1* Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ -3 & 4 \\ 1 & 1 \end{pmatrix}$. Compute

a) $B + C$, $B - C$, $2B - 3C$.

b) AB, AC, BA, CA, BC^T, CB^T

c) $A(B + C)$, $AB + AC$; $(B + C)A$, $BA + CA$

1F-2* Let A be an arbitrary $m \times n$ matrix, and let I_k be the identity matrix of size k. Verify that $I_m A = A$ and $AI_n = A$.

1F-3 Find all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

1F-4* Show that matrix multiplication is not in general commutative by calculating for each pair below the matrix $AB - BA$:

a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{pmatrix}$

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1F-5 a) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Compute A^2, A^3. b) Find A^2, A^3, A^n if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

1F-6* Let A, A', B, B' be 2×2 matrices, and O the 2×2 zero matrix. Express in terms of these five matrices the product of the 4×4 matrices $\begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} A' & O \\ O & B' \end{pmatrix}$.

1F-7* Let $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Show there are no values of a and b such that $AB - BA = I_2$.

1F-8 a) If $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, what is the 3×3 matrix A?

b)* If $A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$, what is A?

1F-9 A square $n \times n$ matrix is called **orthogonal** if $A \cdot A^T = I_n$. Show that this condition is equivalent to saying that

a) each row of A is a row vector of length 1,

b) two different rows are orthogonal vectors.

1F-10* Suppose A is a 2×2 orthogonal matrix, whose first entry is $a_{11} = \cos\theta$. Fill in the rest of A. (There are four possibilities. Use Exercise 9.)

1F-11* Show that if $A + B$ and AB are defined, then
a) $(A+B)^T = A^T + B^T$, b) $(AB)^T = B^T A^T$.

1G. Solving Square Systems; Inverse Matrices

For each of the following, solve the equation $A\mathbf{x} = \mathbf{b}$ by finding A^{-1}.

1G-1* $A = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}$.

1G-2* a) $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$; b) $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

1G-3 $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$. Solve $A\mathbf{x} = \mathbf{b}$ by finding A^{-1}.

1G-4 Referring to Exercise 3 above, solve the system

$$x_1 - x_2 + x_3 = y_1, \quad x_2 + x_3 = y_2 \quad -x_1 - x_2 + 2x_3 = y_3$$

for the x_i as functions of the y_i.

1G-5 Show that $(AB)^{-1} = B^{-1}A^{-1}$, by using the definition of inverse matrix.

1G-6* **Another calculation of the inverse matrix.**

If we know A^{-1}, we can solve the system $A\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by writing $\mathbf{x} = A^{-1}\mathbf{y}$. But conversely, if we can solve by some other method (elimination, say) for $\mathbf{x}$ in terms of $\mathbf{y}$, getting $\mathbf{x} = B\mathbf{y}$, then the matrix $B = A^{-1}$, and we will have found A^{-1}.

This is a good method if A is an upper or lower triangular matrix — one with only zeros respectively below or above the main diagonal. To illustrate:

a) Let $A = \begin{pmatrix} -1 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$; find A^{-1} by solving $\begin{aligned} -x_1 + x_2 + 3x_3 &= y_1 \\ 2x_2 - x_3 &= y_2 \\ x_3 &= y_3 \end{aligned}$ for the x_i in terms of the y_i (start from the bottom and proceed upwards).

b) Calculate A^{-1} by the method given in the notes.

1G-7* Consider the rotation matrix $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ corresponding to rotation of the x and y axes through the angle θ. Calculate A_θ^{-1} by the adjoint matrix method, and explain why your answer looks the way it does.

1G-8* a) Show: A is an orthogonal matrix (cf. Exercise 1F-9) if and only if $A^{-1} = A^T$.

b) Illustrate with the matrix of exercise 7 above.

c) Use (a) to show that if A and B are $n \times n$ orthogonal matrices, so is AB.

1G-9* a) Let A be a 3×3 matrix such that $|A| \neq 0$. The notes construct a right-inverse A^{-1}, that is, a matrix such that $A \cdot A^{-1} = I$. Show that every such matrix A also has a left inverse B (i.e., a matrix such that $BA = I$.)

(Hint: Consider the equation $A^T(A^T)^{-1} = I$; cf. Exercise 1F-11.)

b) Deduce that $B = A^{-1}$ by a one-line argument.

(This shows that the right inverse A^{-1} is automatically the left inverse also. So if you want to check that two matrices are inverses, you only have to do the multiplication on one side — the product in the other order will automatically be I also.)

1G-10* Let A and B be two $n \times n$ matrices. Suppose that $B = P^{-1}AP$ for some invertible $n \times n$ matrix P. Show that $B^n = P^{-1}A^nP$. If $B = I_n$, what is A?

1G-11* Repeat Exercise 6a and 6b above, doing it this time for the general 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, assuming $|A| \neq 0$.

1H. Theorems about Square Systems

1H-1 Use Cramer's rule to solve for x in the following:

$$\text{(a)}\quad \begin{aligned} 3x - y + z &= 1 \\ -x + 2y + z &= 2 \\ x - y + z &= -3 \end{aligned}\ , \qquad \text{(b)}\quad \begin{aligned} x - y + z &= 0 \\ x - z &= 1 \\ -x + y + z &= 2 \end{aligned}.$$

(*We did not cover Cramer's rule in this course.*)

1H-2 Using Cramer's rule, give another proof that if A is an $n \times n$ matrix whose determinant is non-zero, then the equations $A\mathbf{x} = 0$ have only the trivial solution.
(*We did not cover Cramer's rule in this course.*)

1H-3 a) For what c-value(s) will $\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ -x_1 + cx_2 + 2x_3 &= 0 \end{aligned}$ have a non-trivial solution?

b) For what c-value(s) will $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = c\begin{pmatrix} x \\ y \end{pmatrix}$ have a non-trivial solution? (Write it as a system of homogeneous equations.)

c) For each value of c in part (a), find a non-trivial solution to the corresponding system. (Interpret the equations as asking for a vector orthogonal to three given vectors; find it by using the cross product.)

d)* For each value of c in part (b), find a non-trivial solution to the corresponding system.

1H-4* Find all solutions to the homogeneous system $\begin{aligned} x - 2y + z &= 0 \\ x + y - z &= 0 \\ 3x - 3x + z &= 0 \end{aligned}$; use the method suggested in Exercise 3c above.

1H-5 Suppose that for the system $\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$ we have $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. Assume that $a_1 \neq 0$. Show that the system is consistent (i.e., has solutions) if and only if $c_2 = \dfrac{a_2}{a_1}c_1$.

1H-6* Suppose $|A| = 0$, and that $\mathbf{x}_1$ is a particular solution of the system $A\mathbf{x} = B$. Show that any other solution $\mathbf{x}_2$ of this system can be written as $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{x}_0$, where $\mathbf{x}_0$ is a solution of the system $A\mathbf{x} = \mathbf{0}$.

1H-7 Suppose we want to find a pure oscillation (sine wave) of frequency 1 passing through two given points. In other words, we want to choose constants a and b so that the function

$$f(x) = a\cos x + b\sin x$$

has prescribed values at two given x-values: $f(x_1) = y_1,\ f(x_2) = y_2$.

a) Show this is possible in one and only one way, if we assume that $x_2 \neq x_1 + n\pi$, for every integer n.

b) If $x_2 = x_1 + n\pi$ for some integer n, when can a and b be found?

1H-8* The method of partial fractions, if you do it by undetermined coefficients, leads to a system of linear equations. Consider the simplest case:

$$\frac{ax+b}{(x-r_1)(x-r_2)} = \frac{c}{x-r_1} + \frac{d}{x-r_2}, \qquad (a, b, r_1, r_2 \text{ given; } c, d \text{ to be found});$$

what are the linear equations which determine the constants c and d? Under what circumstances do they have a unique solution?

(If you are ambitious, try doing this also for three roots r_i, $i = 1, 2, 3$. Evaluate the determinant by using column operations to get zeros in the top row.)

1I. Vector Functions and Parametric Equations

1I-1 The point P moves with constant speed v in the direction of the constant vector $a\,\mathbf{i} + b\,\mathbf{j}$. If at time $t = 0$ it is at (x_0, y_0), what is its position vector function $\mathbf{r}(t)$?

1I-2 A point moves *clockwise* with constant angular velocity ω on the circle of radius a centered at the origin. What is its position vector function $\mathbf{r}(t)$, if at time $t = 0$ it is at
(a) $(a, 0)$ (b) $(0, a)$

1I-3 Describe the motions given by each of the following position vector functions, as t goes from $-\infty$ to ∞. In each case, give the xy-equation of the curve along which P travels, and tell what part of the curve is actually traced out by P.

a) $\mathbf{r} = 2\cos^2 t\,\mathbf{i} + \sin^2 t\,\mathbf{j}$ b) $\mathbf{r} = \cos 2t\,\mathbf{i} + \cos t\,\mathbf{j}$ c) $\mathbf{r} = (t^2+1)\,\mathbf{i} + t^3\,\mathbf{j}$
d) $\mathbf{r} = \tan t\,\mathbf{i} + \sec t\,\mathbf{j}$

1I-4 A roll of plastic tape of outer radius a is held in a fixed position while the tape is being unwound counterclockwise. The end P of the unwound tape is always held so the unwound portion is perpendicular to the roll. Taking the center of the roll to be the origin O, and the end P to be initially at $(a, 0)$, write parametric equations for the motion of P.
(Use vectors; express the position vector OP as a vector function of one variable.)

1I-5 A string is wound clockwise around the circle of radius a centered at the origin O; the initial position of the end P of the string is $(a, 0)$. Unwind the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of P.
(Use vectors; express the position vector OP as a vector function of one variable.)

1I-6 A bow-and-arrow hunter walks toward the origin along the positive x-axis, with unit speed; at time 0 he is at $x = 10$. His arrow (of unit length) is aimed always toward a rabbit hopping with constant velocity $\sqrt{5}$ in the first quadrant along the line $y = 2x$; at time 0 it is at the origin.

a) Write down the vector function $\mathbf{A}(t)$ for the arrow at time t.

b) The hunter shoots (and misses) when closest to the rabbit; when is that?

1I-7 The cycloid is the curve traced out by a fixed point P on a circle of radius a which rolls along the x-axis in the positive direction, starting when P is at the origin O. Find the vector function OP; use as variable the angle θ through which the circle has rolled.
(Hint: begin by expressing OP as the sum of three simpler vector functions.)

1J. Differentiation of Vector Functions

1J-1 1. For each of the following vector functions of time, calculate the velocity, speed $|ds/dt|$, unit tangent vector (in the direction of velocity), and acceleration.

a) $e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$ b) $t^2\,\mathbf{i} + t^3\,\mathbf{j}$ c) $(1-2t^2)\,\mathbf{i} + t^2\,\mathbf{j} + (-2+2t^2)\,\mathbf{k}$

1J-2 Let $OP = \dfrac{1}{1+t^2}\,\mathbf{i} + \dfrac{t}{1+t^2}\,\mathbf{j}$ be the position vector for a motion.

a) Calculate $\mathbf{v}$, $|ds/dt|$, and $\mathbf{T}$.

b) At what point in the speed greatest? smallest?

c) Find the xy-equation of the curve along which the point P is moving, and describe it geometrically.

1J-3 Prove the rule for differentiating the scalar product of two plane vector functions:

$$\frac{d}{dt}\,\mathbf{r}\cdot\mathbf{s} = \frac{d\mathbf{r}}{dt}\cdot\mathbf{s} + \mathbf{r}\cdot\frac{d\mathbf{s}}{dt}\,,$$

by calculating with components, letting $\mathbf{r} = x_1\,\mathbf{i} + y_1\,\mathbf{j}$ and $\mathbf{s} = x_2\,\mathbf{i} + y_2\,\mathbf{j}$.

1J-4 Suppose a point P moves on the surface of a sphere with center at the origin; let

$$OP = \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}\,.$$

Show that the velocity vector $\mathbf{v}$ is always perpendicular to $\mathbf{r}$ two different ways:

a) using the x, y, z-coordinates

b) without coordinates (use the formula in **1J-3**, which is valid also in space).

c) Prove the converse: if $\mathbf{r}$ and $\mathbf{v}$ are perpendicular, then the motion of P is on the surface of a sphere centered at the origin.

1J-5 a) Suppose a point moves with constant speed. Show that its velocity vector and acceleration vector are perpendicular. (Use the formula in **1J-3**.)

b) Show the converse: if the velocity and acceleration vectors are perpendicular, the point P moves with constant speed.

1J-6 For the helical motion $r(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$,

a) calculate $\mathbf{v}$, $\mathbf{a}$, $\mathbf{T}$, $|ds/dt|$

b) show that $\mathbf{v}$ and $\mathbf{a}$ are perpendicular; explain using **1J-5**

1J-7 a) Suppose you have a differentiable vector function $\mathbf{r}(t)$. How can you tell if the parameter t is the arclength s (measured from some point in the direction of increasing t) without actually having to calculate s explicitly?

b) How should a be chosen so that t is the arclength if $\mathbf{r}(t) = (x_0+at)\,\mathbf{i} + (y_0+at)\,\mathbf{j}$?

c) How should a and b be chosen so that t is the arclength in the helical motion described in Exercise **1J-6**?

1J-8 a) Prove the formula $\displaystyle \frac{d}{dt}\, u(t)\mathbf{r}(t) \;=\; \frac{du}{dt}\,\mathbf{r}(t) + u(t)\frac{d\mathbf{r}}{dt}.$

(You may assume the vectors are in the plane; calculate with the components.)

b) Let $\mathbf{r}(t) = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j}$, the exponential spiral. Use part (a) to find the speed of this motion.

1J-9 A point P is moving in space, with position vector

$$\mathbf{r} = OP = 3\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} + 4\cos t\,\mathbf{k}$$

a) Show it moves on the surface of a sphere.
b) Show its speed is constant.
c) Show the acceleration is directed toward the origin.
d) Show it moves in a plane through the origin.
e) Describe the path of the point.

1J-10 The **positive curvature** κ of the vector function $\mathbf{r}(t)$ is defined by $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$.

a) Show that the helix of **1J-6** has constant curvature. (It is not necessary to calculate s explicitly; calculate $d\mathbf{T}/dt$ instead and relate it to κ by using the chain rule.)

b) What is this curvature if the helix is reduced to a circle in the xy-plane?

1K. Kepler's Second Law

1K-1 (Same as 1J-3). Prove the product rule for differentiating the dot product of two plane vectors: do the calculation using an $\mathbf{i}\,\mathbf{j}$-coordinate system.
(Let $\mathbf{r}(t) = x_1(t)\,\mathbf{i} + y_1(t)\,\mathbf{j}$ and $\mathbf{s}(t) = x_2(t)\,\mathbf{i} + y_2(t)\,\mathbf{j}$.)

1K-2 Let $\mathbf{s}(t)$ be a vector function. Prove by using components that

$$\frac{d\mathbf{s}}{dt} = \mathbf{0} \quad\Rightarrow\quad \mathbf{s}(t) = \mathbf{K}, \quad \text{where } \mathbf{K} \text{ is a constant vector.}$$

1K-3 In our proof that Kepler's second law is equivalent to the force being central, used the following steps to show the second law implies a central force. Kepler's second law says the motion is in a plane and

$$2\frac{dA}{dt} = |\mathbf{r}\times\mathbf{v}| \text{ is constant.} \tag{2}$$

This implies $\mathbf{r}\times\mathbf{v}$ is constant. So,

$$0 = \frac{d}{dt}\,(\mathbf{r}\times\mathbf{v}) \;=\; \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\mathbf{a} = \mathbf{r}\times\mathbf{a}.$$

This implies $\mathbf{a}$ and $\mathbf{r}$ are parallel, i.e. the force is central.

Reverse these steps to prove the converse: for motion under any type of central force, the path of motion will lie in a plane and area will be swept out by the radius vector at a constant rate. You will need the statement in exercise 1K-2.

2. Partial Differentiation

2A. Functions and Partial Derivatives

2A-1 Sketch five level curves for each of the following functions. Also, for a-d, sketch the portion of the graph of the function lying in the first octant; include in your sketch the traces of the graph in the three coordinate planes, if possible.

a) $1-x-y$ b) $\sqrt{x^2+y^2}$ c) x^2+y^2 d) $1-x^2-y^2$ e) x^2-y^2

2A-2 Calculate the first partial derivatives of each of the following functions:

a) $w = x^3y - 3xy^2 + 2y^2$ b) $z = \dfrac{x}{y}$ c) $\sin(3x+2y)$ d) e^{x^2y}
e) $z = x\ln(2x+y)$ f) $x^2z - 2yz^3$

2A-3 Verify that $f_{xy} = f_{yx}$ for each of the following:

a) x^my^n, $(m, n$ positive integers) b) $\dfrac{x}{x+y}$ c) $\cos(x^2+y)$
d) $f(x)g(y)$, for any differentiable f and g

2A-4 By using $f_{xy} = f_{yx}$, tell for what value of the constant a there exists a function $f(x,y)$ for which $f_x = axy + 3y^2$, $f_y = x^2 + 6xy$, and then using this value, find such a function by inspection.

2A-5 Show the following functions $w = f(x,y)$ satisfy the equation $w_{xx} + w_{yy} = 0$ (called the two-dimensional *Laplace equation*):

a) $w = e^{ax}\sin ay$ (a constant) b) $w = \ln(x^2+y^2)$

2B. Tangent Plane; Linear Approximation

2B-1 Give the equation of the tangent plane to each of these surfaces at the point indicated.

a) $z = xy^2$, $(1,1,1)$ b) $w = y^2/x$, $(1,2,4)$

2B-2 a) Find the equation of the tangent plane to the cone $z = \sqrt{x^2+y^2}$ at the point $P_0 : (x_0, y_0, z_0)$ on the cone.

b) Write parametric equations for the ray from the origin passing through P_0, and using them, show the ray lies on both the cone and the tangent plane at P_0.

2B-3 Using the approximation formula, find the approximate change in the hypotenuse of a right triangle, if the legs, initially of length 3 and 4, are each increased by .010 .

2B-4 The combined resistance R of two wires in parallel, having resistances R_1 and R_2 respectively, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}\ .$$

If the resistance in the wires are initially 1 and 2 ohms, with a possible error in each of $\pm.1$ ohm, what is the value of R, and by how much might this be in error? (Use the approximation formula.)

2B-5 Give the linearizations of each of the following functions at the indicated points:

a) $(x+y+2)^2$ at $(0,0)$; at $(1,2)$ b) $e^x\cos y$ at $(0,0)$; at $(0,\pi/2)$

2B-6 To determine the volume of a cylinder of radius around 2 and height around 3, about how accurately should the radius and height be measured for the error in the calculated volume not to exceed .1 ?

2B-7 a) If x and y are known to within .01, with what accuracy can the polar coordinates r and θ be calculated? Assume $x = 3,\ y = 4$.

b) At this point, are r and θ more sensitive to small changes in x or in y? Draw a picture showing x, y, r, θ and confirm your results by using geometric intuition.

2B-8* Two sides of a triangle are a and b, and θ is the included angle. The third side is c.

a) Give the approximation for Δc in terms of a, b, c, θ, and $\Delta a, \Delta b, \Delta\theta$.

b) If $a = 1,\ b = 2,\ \theta = \pi/3$, is c more sensitive to small changes in a or b?

2B-9 a) Around the point $(1, 0)$, is $w = x^2(y + 1)$ more sensitive to changes in x or in y?

b) What should the ratio of Δy to Δx be in order that small changes with this ratio produce *no* change in w, i.e., no first-order change — of course w will change a little, but like $(\Delta x)^2$, not like Δx.

2B-10* a) If $|a|$ is much larger than $|b|, |c|$, and $|d|$, to which entry is the value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ most sensitive?

b) Given a 3×3 determinant, how would you determine to which entry the value of the determinant is most sensitive? (Consider the various Laplace expansions by the cofactors of a given row or column.)

2C. Differentials; Approximations

2C-1 Find the differential (dw or dz). Make the answer look as neat as possible.

a) $w = \ln(xyz)$ b) $w = x^3y^2z$ c) $z = \dfrac{x-y}{x+y}$ d) $w = \sin^{-1}\dfrac{u}{t}$ (use $\sqrt{t^2 - u^2}$)

2C-2 The dimensions of a rectangular box are 5, 10, and 20 cm., with a possible measurement error in each side of $\pm.1$ cm. Use differentials to find what possible error should be attached to its volume.

2C-3 Two sides of a triangle have lengths respectively a and b, with θ the included angle. Let A be the area of the triangle.

a) Express dA in terms of the variables and their differentials.

b) If $a = 1,\ b = 2,\ \theta = \pi/6$, to which variable is A most sensisitve? least sensitive?

c) Using the values in (b), if the possible error in each value is .02, what is the possible error in A, to two decimal places?

2C-4 The pressure, volume, and temperature of an ideal gas confined to a container are related by the equation $PV = kT$, where k is a constant depending on the amount of gas and the units. Calculate dP two ways:

a) Express P in terms of V and T, and calculate dP as usual.

b) Calculate the differential of both sides of the equation, getting a "differential equation", and then solve it algebraically for dP.

c) Show the two answers agree.

2C-5 The following equations define w implicitly as a function of the other variables. Find dw in terms of all the variables by taking the differential of both sides and solving algebraically for dw.

a) $\dfrac{1}{w} = \dfrac{1}{t} + \dfrac{1}{u} + \dfrac{1}{v}$ b) $u^2 + 2v^2 + 3w^2 = 10$

2D. Gradient and Directional Derivative

2D-1 In each of the following, a function f, a point P, and a vector $\mathbf{A}$ are given. Calculate the gradient of f at the point, and the directional derivative $\left.\dfrac{df}{ds}\right|_{\mathbf{u}}$ at the point, in the direction $\mathbf{u}$ of the given vector $\mathbf{A}$.

a) $x^3 + 2y^3$; $(1,1)$, $\mathbf{i} - \mathbf{j}$ b) $w = \dfrac{xy}{z}$; $(2,-1,1)$, $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

c) $z = x\sin y + y\cos x$; $(0,\pi/2)$, $-3\mathbf{i} + 4\mathbf{j}$ d) $w = \ln(2t + 3u)$; $(-1,1)$, $4\mathbf{i} - 3\mathbf{j}$

e) $f(u,v,w) = (u + 2v + 3w)^2$; $(1,-1,1)$, $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

2D-2 For the following functions, each with a given point P,

(i) find the maximum and minimum values of $\left.\dfrac{df}{ds}\right|_{\mathbf{u}}$, as $\mathbf{u}$ varies;

(ii) tell for which directions the maximum and minimum occur;

(iii) find the direction(s) $\mathbf{u}$ for which $\left.\dfrac{df}{ds}\right|_{\mathbf{u}} = 0$.

a) $w = \ln(4x - 3y)$, $(1,1)$ b) $w = xy + yz + xz$, $(1,-1,2)$

c) $z = \sin^2(t - u)$, $(\pi/4, 0)$

2D-3 By viewing the following surfaces as a contour surface of a function $f(x,y,z)$, find its tangent plane at the given point.

a) $xy^2z^3 = 12$, $(3,2,1)$; b) the ellipsoid $x^2 + 4y^2 + 9z^2 = 14$, $(1,1,1)$

c) the cone $x^2 + y^2 - z^2 = 0$, (x_0, y_0, z_0) (simplify your answer)

2D-4 The function $T = \ln(x^2 + y^2)$ gives the temperature at each point in the plane (except $(0,0)$).

a) At the point $P : (1,2)$, in which direction should you go to get the most rapid increase in T?

b) At P, about how far should you go in the direction found in part (a) to get an increase of .20 in T?

c) At P, approximately how far should you go in the direction of $\mathbf{i} + \mathbf{j}$ to get an increase of about .12?

d) At P, in which direction(s) will the rate of change of temperature be 0?

2D-5 The function $T = x^2 + 2y^2 + 2z^2$ gives the temperature at each point in space.

a) What shape are the isotherms?.

b) At the point $P : (1,1,1)$, in which direction should you go to get the most rapid decrease in T?

c) At P, about how far should you go in the direction of part (b) to get a decrease of 1.2 in T?

d) At P, approximately how far should you go in the direction of $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ to get an increase of .10?

2D-6 Show that $\nabla(uv) = u\nabla v + v\nabla u$, and deduce that $\left.\dfrac{d(uv)}{ds}\right|_{\mathbf{u}} = u\left.\dfrac{dv}{ds}\right|_{\mathbf{u}} + v\left.\dfrac{du}{ds}\right|_{\mathbf{u}}$.
(Assume that u and v are functions of two variables.)

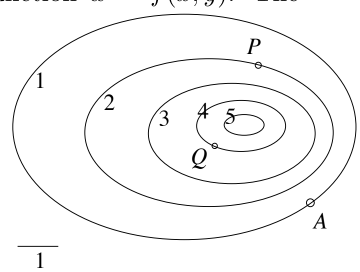
2D-7 Suppose $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 2$, $\left.\dfrac{dw}{ds}\right|_{\mathbf{v}} = 1$ at P, where $\mathbf{u} = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$, $\mathbf{v} = \dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$. Find $(\nabla w)_P$.

(This illustrates that the gradient can be calculated knowing the directional derivatives in any two non-parallel directions, not just the two standard directions $\mathbf{i}$ and $\mathbf{j}$.)

2D-8 The atmospheric pressure in a region of space near the origin is given by the formula $P = 30 + (x+1)(y+2)e^z$. Approximately where is the point closest to the origin at which the pressure is 31.1?

2D-9 The accompanying picture shows the level curves of a function $w = f(x,y)$. The value of w on each curve is marked. A unit distance is given.

a) Draw in the gradient vector at A.
b) Find a point B where $w = 3$ and $\partial w/\partial x = 0$.
c) Find a point C where $w = 3$ and $\partial w/\partial y = 0$.
d) At the point P estimate the value of $\partial w/\partial x$ and $\partial w/\partial y$.
e) At the point Q, estimate dw/ds in the direction of $\mathbf{i} + \mathbf{j}$
f) At the point Q, estimate dw/ds in the direction of $\mathbf{i} - \mathbf{j}$.
g) Approximately where is the gradient $\mathbf{0}$?



2E. Chain Rule

2E-1 In the following, find $\dfrac{df}{dt}$ for the composite function $f(x(t), y(t), z(t))$ in two ways:
(i) use the chain rule, then express your answer in terms of t by using $x = x(t)$, etc.;
(ii) express the composite function f in terms of t, and differentiate.

a) $w = xyz$, $x = t$, $y = t^2$, $z = t^3$ b) $w = x^2 - y^2$, $x = \cos t$, $y = \sin t$
c) $w = \ln(u^2 + v^2)$, $u = 2\cos t$, $v = 2\sin t$

2E-2 In each of these, information about the gradient of an unknown function $f(x,y)$ is given; x and y are in turn functions of t. Use the chain rule to find out additional information about the composite function $w = f\big(x(t), y(t)\big)$, without trying to determine f explicitly.

a) $\nabla w = 2\,\mathbf{i} + 3\,\mathbf{j}$ at $P:(1,0)$; $x = \cos t$, $y = \sin t$. Find the value of $\dfrac{dw}{dt}$ at $t = 0$.

b) $\nabla w = y\,\mathbf{i} + x\,\mathbf{j}$; $x = \cos t$, $y = \sin t$. Find $\dfrac{dw}{dt}$ and tell for what t-values it is zero.

c) $\nabla f = \langle 1, -1, 2\rangle$ at $(1,1,1)$. Let $x = t$, $y = t^2$, $z = t^3$; find $\dfrac{df}{dt}$ at $t = 1$.

d) $\nabla f = \langle 3x^2 y, x^3 + z, y\rangle$; $x = t$, $y = t^2$, $z = t^3$. Find $\dfrac{df}{dt}$.

2E-3 a) Use the chain rule for $f(u,v)$, where $u = u(t)$, $v = v(t)$, to prove the product rule

$$D(uv) = v\,Du + u\,Dv, \qquad \text{where } D = \frac{d}{dt}.$$

b) Using the chain rule for $f(u,v,w)$, derive a similar product rule for $\dfrac{d}{dt}(uvw)$, and use it to differentiate $te^{2t}\sin t$.

c)* Derive similarly a rule for the derivative $\dfrac{d}{dt}u^v$, and use it to differentiate $(\ln t)^t$.

2E-4 Let $w = f(x,y)$, and assume that $\nabla w = 2\,\mathbf{i} + 3\,\mathbf{j}$ at $(0,1)$. If $x = u^2 - v^2,\ y = uv$, find $\dfrac{\partial w}{\partial u},\ \dfrac{\partial w}{\partial v}$ at $u = 1,\ v = 1$.

2E-5 Let $w = f(x,y)$, and suppose we change from rectangular to polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$.

a) Show that $(w_x)^2 + (w_y)^2 = (w_r)^2 + \dfrac{1}{r^2}(w_\theta)^2$.

b) Suppose $\nabla w = 2\,\mathbf{i} - \mathbf{j}$ at the point $x = 1,\ y = 1$. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$ when $r = \sqrt{2},\ \theta = \pi/4$, and verify the relation in part (a), at the point.

2E-6 Let $w = f(x,y)$, and make the change of variables $x = u^2 - v^2,\ y = 2uv$. Show

$$(w_x)^2 + (w_y)^2 = \frac{(w_u)^2 + (w_v)^2}{4(u^2 + v^2)}$$

2E-7 The Jacobian matrix for the change of variables $x = x(u,v),\ y = y(u,v)$ is defined to be $J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$. Let $\nabla f(x,y)$ be represented as the row vector $\langle f_x\ f_y \rangle$. Show that

$$\nabla f\big(x(u,v), y(u,v)\big) \;=\; \nabla f(x,y)\,\cdot\,J \qquad \text{(matrix multiplication).}$$

2E-8 a) Let $w = f(y/x)$; i.e., w is the composite of the functions $w = f(u),\ u = y/x$. Show that w satisfies the PDE (partial differential equation) $\quad x\,\dfrac{\partial w}{\partial x} + y\,\dfrac{\partial w}{\partial y} = 0.$

b)* Let $w = f(x^2 - y^2)$; show that w satisfies the PDE $\quad y\,\dfrac{\partial w}{\partial x} + x\,\dfrac{\partial w}{\partial y} = 0.$

c)* Let $w = f(ax + by)$; show that w satisfies the PDE $\quad b\,\dfrac{\partial w}{\partial x} - a\,\dfrac{\partial w}{\partial y} = 0.$

2F. Maximum-minimum Problems

2F-1 Find the point(s) on each of the following surfaces which is closest to the origin. (Hint: it's easier to minimize the square of the distance, rather than the distance itself.)

a) $xyz^2 = 1$ b) $x^2 - yz = 1$

2F-2 A rectangular produce box is to be made of cardboard; the sides of single thickness, the front and back of double thickness, and the bottom of triple thickness, with the top left open. Its volume is to be 1 cubic foot; what proportions for the sides will use the least cardboard?

2F-3* Consider all planes passing through the point (2,1,1) and such that the intercepts on the three coordinate axes are all positive. For which of these planes is the product of the three intercepts smallest? (Hint: take the plane in the form $z = ax + by + c$, where a and b are the independent variables.)

2F-4* Find the extremal point of $x^2 + 2xy + 4y^2 + 6$ and show it is a minimum point by completing the square.

2F-5 A drawer in a chest has an open top; the bottom and back are made of cheap wood costing \$1/sq. ft; the sides have to be thicker, and cost \$2/sq.ft., while the front costs \$4/sq.ft. for the better quality wood and finishing. The volume is to be 2.5 cu. ft. What dimensions will produce the drawer costing the least to manufacture?

2G. Least-squares Interpolation

2G-1 Find by the method of least squares the line which best fits the three data points given. Do it from scratch, using the formula

$$D = \sum_{i=1}^{n} (y_i - (ax_i + b))^2,$$

which was in the reading on least squares, and differentiation (use the chain rule). Sketch the line and the three points as a check.

a)* $(0,0),(0,2),(1,3)$ b)* $(0,0),(1,2),(2,1)$ c) $(1,1),(2,3),(3,2)$

2G-2* Show that the equations (4) for the method of least squares have a unique solution, unless all the x_i are equal. Explain geometrically why this exception occurs.

> Hint: use the fact that for all values of u, we have $\sum_1^n (x_i - u)^2 \geq 0$, since squares are always non-negative. Write the left side as a quadratic polynomial in u. Usually it has no roots. What does this imply about the coefficients? When does it have a root? (Answer these two questions by using the quadratic formula.)

2G-3* Use least squares to fit a second degree polynomial exactly through the points $(-1,-1),(0,0),(1,3)$ (you might want to go back and read the last section in the note about least squares).

2G-4 What linear equations in a, b, c does the method of least squares lead to, when you use it to fit a linear function $z = a + bx + cy$ to a set of data points (x_i, y_i, z_i), $i = 1, \ldots, n$?

2G-5* What equations are you led to for determining a when you try to fit the exponential curve $y = e^{ax}$ to two data points $(1, y_1), (2, y_2)$ by the method of least squares?

> The moral is: don't do it this way. In general to fit an exponential $y = ce^{ax}$ to a set of data points (x_i, y_i), take the log of both sides:
>
> $$\ln y \;=\; a\,x + \ln c$$
>
> This gives a linear function in the variables x and $\ln y$, whose coefficients a and $\ln c$ can be determined by applying the method of least squares to fit the data points $(x_i, \ln y_i)$.

2H. Max-min: 2nd Derivative Criterion; Boundary Curves

2H-1 For each of the following functions, find the critical points, and classify them using the 2nd-derivative criterion.

a) $x^2 - xy - 2y^2 - 3x - 3y + 1$ b) $3x^2 + xy + y^2 - x - 2y + 4$ c) $2x^4 + y^2 - xy + 1$

d) $x^3 - 3xy + y^3$ e) $(x^3 + 1)(y^3 + 1)$

2H-2* Use the 2nd-derivative criterion to verify that the critical point (m_0, b_0) determining the regression (= least-squares) line $y = m_0 x + b_0$ really minimizes the function $D(m, b)$ giving the sum of the squares of the deviations. (You will need the inequality in problem 1B-15, for n-vectors $\mathbf{A} = \langle a_1, a_2, \ldots, a_n \rangle$, defining $|A| = \sqrt{\sum a_i^2}$ and $\mathbf{A} \cdot \mathbf{B} = \sum a_i b_i$.)

2H-3 Find the maximum and minimum of the function $f(x, y) = x^2 + y^2 + 2x + 4y - 1$ in the right half-plane R bounded by the diagonal line $y = -x$.

2H-4 Find the maximum and minimum points of the function $xy - x - y + 2$ on
a) the first quadrant b) the square $R: \ 0 \le x \le 2, \ 0 \le y \le 2$;
(study its values at the unique critical point and on the boundary lines) point.
c) use the data to guess the critical point type, and confirm it by the second derivative test.

2H-5 Find the maximum and minimum points of the function $f(x,y) = x + \sqrt{3}\,y + 2$ on the unit disc $R: x^2 + y^2 \le 1$.

2H-6 a) Two wires of length 4 are cut in the same way into three pieces, of length x, y and z; the four x, y pieces are used as the four sides of a rectangle; the two z pieces are bent at the middle and joined at the ends to make a square of side $z/2$. Find the rectangle and square made this way which together have the largest and the smallest total area.
Using the answer, tell what type the critical point is.
b) Confirm the critical point type by using the second derivative test.

2H-7 a) Find the maximum and minimum points of the function $2x^2 - 2xy + y^2 - 2x$ on the rectangle $R: \ 0 \le x \le 2; \ -1 \le y \le 2$; using this information, determine the type of the critical point.
b) Confirm the critical point type by using the second derivative test.

2I. Lagrange Multipliers

2I-1 A rectangular box is placed in the first octant so that one corner Q is at the origin and the three sides adjacent to Q lie in the coordinate planes. The corner P diagonally opposite Q lies on the surface $f(x,y,z) = c$. Using Lagrange multipliers, tell for which point P the box will have the largest volume, and tell how you know it gives a maximum point, if the surface is

a) the plane $x + 2y + 3z = 18$ b) the ellipsoid $x^2 + 2y^2 + 4z^2 = 12$.

2I-2 Using Lagrange multipliers, tell which point P in the first octant and on the surface $x^3y^2z = 6\sqrt{3}$ is closest to the origin. (As usual, it is easier algebraically to minimize $|OP|^2$ rather than $|OP|$.)

2I-3 (Repeat of 2F-2, but this time use Lagrange multipliers.) A rectangular produce box is to be made of cardboard; the sides of single thickness, the ends of double thickness, and the bottom of triple thickness, with the top left open. Its volume is to be 1 cubic foot; what should be its proportions in order to use the least cardboard?

2I-4 In an open-top wooden drawer, the two sides and back cost \$2/sq.ft., the bottom \$1/sq.ft. and the front \$4/sq.ft. Using Lagrange multipliers, show that the following problems lead to the same set of three equations in λ, plus a different fourth equation, and they have the same solution.

a) Find the dimensions of the drawer with largest capacity that can be made for a total wood cost of \$72.

b) Find the dimensions of the most economical drawer having volume 24 cu. ft.

2J. Non-independent Variables

All references are to the Examples and numbered equations in Notes N.

2J-1 In Example 1, calculate by direct substitution: a) $\left(\dfrac{\partial w}{\partial y}\right)_z$ b) $\left(\dfrac{\partial w}{\partial z}\right)_y$.

2J-2 Calculate the two derivatives in 2J-1 by using
(i) the chain rule (differentiate $z = x^2 + y^2$ implicitly) (ii) differentials

2J-3 In Example 2, using the chain rule calculate, in terms of x, y, z, t, the derivatives

a) $\left(\dfrac{\partial w}{\partial t}\right)_{x,z}$ b) $\left(\dfrac{\partial w}{\partial z}\right)_{x,y}$

2J-4 Repeat 2J-3, doing the calculation using differentials.

2J-5 Let $S = S(p, v, T)$ be the entropy of a gas, assumed to obey the ideal gas law (1). Give expressions in terms of the formal partial derivatives S_p, S_v, and S_T for

a) $\left(\dfrac{\partial S}{\partial p}\right)_v$ b) $\left(\dfrac{\partial S}{\partial T}\right)_v$

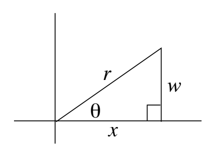
2J-6 If $w = u^3 - uv^2$, $u = xy$, $v = u + x$, find $\left(\dfrac{\partial w}{\partial u}\right)_x$ and $\left(\dfrac{\partial w}{\partial x}\right)_u$ using
a) the chain rule b) differentials .

2J-7 Let P be the point $(1, -1, 1)$, and assume $z = x^2 + y + 1$, and that $f(x, y, z)$ is a differentiable function for which $\nabla f(x, y, z) = 2\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$ at P.

Let $g(x, z) = f(x, y(x, z), z)$; find ∇g at the point $(1, 1)$, i.e., $x = 1$, $z = 1$.

2J-8 Interpreting r, θ as polar coordinates, let $w = \sqrt{r^2 - x^2}$.

a) Calculate $\left(\dfrac{\partial w}{\partial r}\right)_\theta$, by first writing w in terms of r and θ.



b)* Then calculate it by substituting into the final formula given in Example 6.

c)* Finally, obtain the answer by intuitive geometrical reasoning (see picture).

2K. Partial Differential Equations

2K-1. Show that $w = \ln r$, where $r = \sqrt{x^2 + y^2}$ is the usual polar coordinate, satisfies the two-dimensional Laplace equation (Notes P (1), without z), if $(x, y) \neq (0, 0)$. What's wrong with $(0, 0)$?

(The calculation will go faster if you remember that $\ln \sqrt{a} = \frac{1}{2} \ln a$.)

2K-2. For what value(s) of n will $w = (x^2 + y^2 + z^2)^n$ solve the 3-dimensional Laplace equation (Notes P, (1))? Where have you seen this function in physics?

2K-3. The solutions in exercises 2K-1 and 2K-2 have circular and spherical symmetry, respectively. But there are many other solutions. For example

a) find all solutions of the two-dimensional Laplace equation (see 2K-1) of the form

$$w = ax^2 + bxy + cy^2$$

and show they can all be written in the form $c_1 f_1(x,y) + c_2 f_2(x,y)$, where c_1, c_2 are arbitrary constants, and f_1, f_2 are two particular polynomials — that is, all such solutions are linear combinations of two particular polynomial solutions.

b)* Find and derive the analogue of part (a) for all of the cubic polynomial solutions $ax^3 + bx^2y + cxy^2 + dy^3$ to the two-dimensional Laplace equation.

2K-4. Show that the one-dimensional wave equation (Notes P, (4), first equation) is satisfied by any function of the form

$$w = f(x + ct) + g(x - ct) ,$$

where $f(u)$ and $g(u)$ are arbitrary twice-differentiable functions of one variable.

Take $g(u) = 0$, and interpret physically the solution $w = f(x + ct)$. What does $f(x)$ represent? What is the relation of $f(x + ct)$ to it?

> Note how this exercise shows that a solution to the wave equation can involve completely arbitrary functions; this is also clear from the remarks about the Laplace equation being solved by any gravitational or electrostatic potential function in a mass- or charge-free region of space.

2K-5. Find solutions to the one-dimensional heat equation (Notes P, (5), first equation) having the form

$$w = e^{rt} \sin kx \qquad k, r \text{ constants}$$

satisfying the additional conditions for all t:

$$w(0,t) = 0, \qquad w(1,t) = 0 .$$

Interpret your solutions physically. What happens to the temperature as $t \to \infty$?

3. Double Integrals

3A. Double Integrals in Rectangular Coordinates

3A-1 Evaluate each of the following iterated integrals:

a) $\displaystyle\int_0^2 \int_{-1}^1 (6x^2 + 2y)\, dy\, dx$ b) $\displaystyle\int_0^{\pi/2} \int_0^\pi (u \sin t + t \cos u)\, dt\, du$

c) $\displaystyle\int_0^1 \int_{\sqrt{x}}^{x^2} 2x^2 y\, dy\, dx$ d) $\displaystyle\int_0^1 \int_0^u \sqrt{u^2+4}\, dv\, du$

3A-2 Express each double integral over the given region R as an iterated integral, using the given order of integration. Use the method described in Notes I to supply the limits of integration. For some of them, it may be necessary to break the integral up into two parts. In each case, begin by sketching the region.

a) R is the triangle with vertices at the origin, $(0, 2)$, and $(-2, 2)$.
Express as an iterated integral: i) $\displaystyle\iint_R dy\, dx$ ii) $\displaystyle\iint_R dx\, dy$

b) R is the finite region between the parabola $y = 2x - x^2$ and the x-axis.
Express as an iterated integral: i) $\displaystyle\iint_R dy\, dx$ ii) $\displaystyle\iint_R dx\, dy$

c) R is the sector of the circle with center at the origin and radius 2 lying between the x-axis and the line $y = x$.
Express as an iterated integral: i) $\displaystyle\iint_R dy\, dx$ ii) $\displaystyle\iint_R dx\, dy$

d)* R is the finite region lying between the parabola $y^2 = x$ and the line through $(2, 0)$ having slope 1.
Express as an iterated integral: i) $\displaystyle\iint_R dy\, dx$ ii) $\displaystyle\iint_R dx\, dy$

3A-3 Evaluate each of the following double integrals over the indicated region R. Choose whichever order of integration seems easier — given the integrand, and the shape of R.

a) $\displaystyle\iint_R x\, dA$; R is the finite region bounded by the axes and $2y + x = 2$

b) $\displaystyle\iint_R (2x + y^2)\, dA$; R is the finite region in the first quadrant bounded by the axes and $y^2 = 1 - x$; ($dx\, dy$ is easier).

c) $\displaystyle\iint_R y\, dA$; R is the triangle with vertices at $(\pm 1, 0)$, $(0, 1)$.

3A-4 Find by double integration the volume of the following solids.

a) the solid lying under the graph of $z = \sin^2 x$ and over the region R bounded below by the x-axis and above by the central arch of the graph of $\cos x$

b) the solid lying over the finite region R in the first quadrant between the graphs of x and x^2, and underneath the graph of $z = xy$..

c) the finite solid lying underneath the graph of $x^2 - y^2$, above the xy-plane, and between the planes $x = 0$ and $x = 1$

3A-5 Evaluate each of the following iterated integrals, by changing the order of integration (begin by figuring out what the region R is, and sketching it).

a) $\displaystyle\int_0^2 \int_x^2 e^{-y^2}\, dy\, dx$ b) $\displaystyle\int_0^{1/4} \int_{\sqrt{t}}^{1/2} \frac{e^u}{u}\, du\, dt$ c) $\displaystyle\int_0^1 \int_{x^{1/3}}^1 \frac{1}{1+u^4}\, du\, dx$

3A-6 Each integral below is over the disc consisting of the interior R of the unit circle, centered at the origin. For each integral, use the symmetries of R and the integrand

i) to identify its value as zero; or if its value is not zero,

ii) to find a double integral which is equivalent (i.e., has the same value), but which has a simpler integrand and/or is taken over the first quadrant (if possible), or over a half-disc. (Do not evaluate the integral.)

$$\iint_R x\, dA; \quad \iint_R e^x\, dA; \quad \iint_R x^2\, dA; \quad \iint_R x^2 y\, dA; \quad \iint_R (x^2+y) dA; \quad \iint_R xy\, dA$$

3A-7 By using the inequality $f \le g$ on $R \Rightarrow \iint_R f\, dA \le \iint_R g\, dA$, show the following estimates are valid:

a) $\displaystyle\iint_R \frac{dA}{1+x^4+y^4} \le$ area of R

b) $\displaystyle\iint_R \frac{x\, dA}{1+x^2+y^2} < .35,$ R is the square $0 \le x, y \le 1$.

3B. Double Integrals in Polar Coordinates

In evaluating the integrals, the following definite integrals will be useful:

$$\int_0^{\pi/2} sin^n x\, dx = \int_0^{\pi/2} cos^n x\, dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(n-1)}{2\cdot 4\cdot\cdots\cdot n}\,\dfrac{\pi}{2}, & \text{if } n \text{ is an even integer} \ge 2 \\ \dfrac{2\cdot 4\cdot\cdots\cdot(n-1)}{1\cdot 3\cdot\cdots\cdots n}, & \text{if } n \text{ is an odd integer} \ge 3. \end{cases}$$

For example: $\displaystyle\int_0^{\pi/2} \sin^2 x\, dx = \frac{\pi}{4}$, $\displaystyle\int_0^{\pi/2} \sin^3 x\, dx = \frac{2}{3}$, $\displaystyle\int_0^{\pi/2} \sin^4 x\, dx = \frac{3\pi}{16}$,

and the same holds if $\cos x$ is substituted for $\sin x$.

3B-1 Express each double integral over the given region R as an iterated integral in polar coordinates. Use the method described in Notes I to supply the limits of integration. For some of them, it may be necessary to break the integral up into two parts. In each case, begin by sketching the region.

a) The region lying inside the circle with center at the origin and radius 2. and to the left of the vertical line through $(-1, 0)$.

b)* The circle of radius 1, and center at $(0, 1)$.

c) The region inside the cardioid $r = 1 - \cos\theta$ and outside the circle of radius $3/2$ and center at the origin.

d) The finite region bounded by the y-axis, the line $y = a$, and a quarter of the circle of radius a and center at $(a, 0)$.

3B-2 Evaluate by iteration the double integrals over the indicated regions. Use polar coordinates.

a) $\displaystyle\iint_R \frac{dA}{r}$; R is the region inside the first-quadrant loop of $r = \sin 2\theta$.

b) $\displaystyle\iint_R \frac{dx\,dy}{1+x^2+y^2}$; R is the first-quadrant portion of the interior of $x^2+y^2=a^2$

c) $\displaystyle\iint_R \tan^2\theta\, dA$; R is the triangle with vertices at $(0,0),\ (1,0),\ (1,1)$.

d) $\displaystyle\iint_R \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$; R is the right half-disk of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$.

3B-3 Find the volumes of the following domains by integrating in polar coordinates:

a) a solid hemisphere of radius a (place it so its base lies over the circle $x^2+y^2=a^2$)

b) the domain under the graph of xy and over the quarter-disc region R of 3B-2b

c) the domain lying under the cone $z=\sqrt{x^2+y^2}$ and over the circle of radius one and center at $(0,1)$

d) the domain lying under the paraboloid $z = x^2+y^2$ and over the interior of the right-hand loop of $r^2=\cos\theta$.

3B-4* Sometimes students wonder if you can do a double integral in polar coordinates iterating in the opposite order: $\iint_R d\theta\, dr$. Though this is uncommon, just to see if you can carry out in a new situation the basic procedure for putting in the limits, try supplying the limits for this integral over the region bounded above by the lines $x=1$ and $y=1$, and below by a quarter of the circle of radius 1 and center at the origin.

3C. Applications of Double Integration

If no coordinate system is specified for use, you can use either rectangular or polar coordinates, whichever is easier. In some of the problems, a good placement of the figure in the coordinate system simplifies the integration a lot.

3C-1 Let R be a right triangle, with legs both of length a, and density 1. Find the following ((b) and (c) can be deduced from (a) with no further calculation)
a) its moment of inertia about a leg;
b) its polar moment of inertia about the right-angle vertex;
c) its moment of inertia about the hypotenuse.

3C-2 Find the center of mass of the region inside one arch of $\sin x$, if: a) $\delta = 1$ b) $\delta = y$

3C-3 D is a diameter of a disc of radius a, and C is a chord parallel to D with distance c from it. C divides the disc into two segments; let R be the smaller one. Assuming $\delta = 1$, find the moment of R about D, giving the answer in simplest form, and using
(a) rectangular coordinates; (b) polar coordinates.

3C-4 Find the center of gravity of a sector of a circular disc of radius a, whose vertex angle is 2α. Take $\delta = 1$.

3C-5 Find the polar moment of inertia of one loop of the lemniscate $r^2 = a^2 \cos 2\theta$ about the origin. Take $\delta = 1$.

3D. Changing Variables in Multiple Integrals

3D-1 Evaluate $\displaystyle\iint_R \frac{x-3y}{2x+y}\,dx\,dy$, where R is the parallelogram bounded on the sides by $y = -2x+1$ and $y = -2x+4$, and above and below by $y = x/3$ and $y = (x-7)/3$. Use a change of variables $u = x-3y$, $v = 2x+y$.

3D-2 Evaluate $\displaystyle\iint_R \cos\left(\frac{x-y}{x+y}\right)\,dx\,dy$ by making the change of variables $u = x+y$, $v = x-y$; take as the region R the triangle with vertices at the origin, (2,0) and (1,1).

3D-3 Find the volume underneath the surface $z = 16 - x^2 - 4y^2$ and over the xy-plane; simplify the integral by making the change of variable $u = x,\ v = 2y$.

3D-4 Evaluate $\displaystyle\iint_R (2x-3y)^2(x+y)^2\ dx\,dy$, where R is the triangle bounded by the positive x-axis, negative y-axis, and line $2x-3y = 4$, by making a change of variable $u = x+y,\quad v = 2x-3y$.

3D-5 Set up an iterated integral for the polar moment of inertia of the finite "triangular" region R bounded by the lines $y = x$ and $y = 2x$, and a portion of the hyperbola $xy = 3$. Use a change of coordinates which makes the boundary curves grid curves in the new coordinate system.

3D-6* Verify that the Jacobian gives the right volume element in spherical coordinates. Recall spherical coordinates have

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

and the volume element is $\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

3D-7* Using the coordinate change $u = xy,\ v = y/x$, set up an iterated integral for the polar moment of inertia of the region bounded by the hyperbola $xy = 1$, the x-axis, and the two lines $x = 1$ and $x = 2$. Choose the order of integration which makes the limits simplest.

3D-8 For the change of coordinates in 3D-7, give the uv-equations of the following curves:
a) $y = x^2$ b) $x^2 + y^2 = 1$.

3D-9* Prove the relation between Jacobians: $\displaystyle\frac{\partial(x,y)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(x,y)} = 1$;
use the chain rule for partial differentiation, and the rule for multiplying determinants: $|AB| = |A||B|$, where A and B are square matrices of the same size.

3D-10* Let $u = x+y$ and $v = x-y$; change $\displaystyle\int_0^1\int_0^x dy\,dx$ to an iterated integral in the order $\displaystyle\iint_R dv\,du$, and check your work by evaluating it. (You will have to break the region up into two pieces, using different limits of integration for the pieces.)

4. Line Integrals in the Plane

4A. Plane Vector Fields

4A-1 Describe geometrically how the vector fields determined by each of the following vector functions looks. Tell for each what the largest region in which $\mathbf{F}$ is continuously differentiable is.

a) $a\,\mathbf{i} + b\,\mathbf{j}$, $\quad a, b$ constants $\qquad$ b) $-x\,\mathbf{i} - y\,\mathbf{j}$ $\qquad$ c) $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r}$ $\qquad$ d) $\dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{r}$

4A-2 Write down the gradient field ∇w for each of the following:

a) $w = ax + by$ $\qquad$ b) $w = \ln r$ $\qquad$ c) $w = f(r)$

4A-3 Write down an explicit expression for each of the following fields:

a) Each vector has the same direction and magnitude as $\mathbf{i} + 2\,\mathbf{j}$.

b) The vector at (x, y) is directed radially in towards the origin, with magnitude r^2.

c) The vector at (x, y) is tangent to the circle through (x, y) with center at the origin, clockwise direction, magnitude $1/r^2$.

d) Each vector is parallel to $\mathbf{i} + \mathbf{j}$, but the magnitude varies.

4A-4 The electromagnetic force field of a long straight wire along the z-axis, carrying a uniform current, is a two-dimensional field, tangent to horizontal circles centered along the z-axis, in the direction given by the right-hand rule (thumb pointed in positive z-direction), and with magnitude k/r. Write an expression for this field.

4B. Line Integrals in the Plane

4B-1 For each of the fields $\mathbf{F}$ and corresponding curve C or curves C_i, evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

Use any convenient parametrization of C, unless one is specified. . Begin by writing the integral in the differential form $\displaystyle\int_C M\,dx + N\,dy$.

a) $\mathbf{F} = (x^2 - y)\,\mathbf{i} + 2x\,\mathbf{j}$; $\quad C_1$ and C_2 both run from $(-1, 0)$ to $(1, 0)$:
C_1 : the x-axis $\qquad C_2$: the parabola $y = 1 - x^2$

b) $\mathbf{F} = xy\,\mathbf{i} - x^2\,\mathbf{j}$; $\qquad C$: the quarter of the unit circle running from $(0, 1)$ to $(1, 0)$.

c) $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$; $\quad C$: the triangle with vertices at $(0, 0)$, $(0, 1)$, $(1, 0)$, oriented clockwise.

d) $\mathbf{F} = y\,\mathbf{i}$; $\qquad C$ is the ellipse $x = 2\cos t$, $\;y = \sin t$, oriented counterclockwise.

e) $\mathbf{F} = 6y\,\mathbf{i} + x\,\mathbf{j}$; $\qquad C$ is the curve $x = t^2$, $\;y = t^3$, running from $(1, 1)$ to $(4, 8)$.

f) $\mathbf{F} = (x + y)\,\mathbf{i} + xy\,\mathbf{j}$; $\quad C$ is the broken line running from $(0, 0)$ to $(0, 2)$ to $(1, 2)$.

4B-2 For the following fields $\mathbf{F}$ and curves C, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ without any formal calculation, appealing instead to the geometry of $\mathbf{F}$ and C.

a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$; $\qquad C$ is the counterclockwise circle, center at $(0, 0)$, radius a.

b) $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$; $\qquad C$ is the counterclockwise circle, center at $(0, 0)$, radius a.

4B-3 Let $\mathbf{F} = \mathbf{i} + \mathbf{j}$. How would you place a directed line segment C of length one so that the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ would be

a) a maximum; b) a minimum; c) zero;

d) what would the maximum and minimum values of the integral be?

4C. Gradient Fields and Exact Differentials

4C-1 Let $f(x,y) = x^3y+y^3$, and C be $y^2 = x$, between $(1,-1)$ and $(1,1)$, directed upwards.

a) Calculate $F = \nabla f$.

b) Calculate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ three different ways:
(i) directly;
(ii) by using path-independence to replace C by a simpler path
(iii) by using the Fundamental Theorem for line integrals.

4C-2 Let $f(x,y) = x\,e^{xy}$, and C be the path $y = 1/x$ from $(1,1)$ to $(0,\infty)$.

a) Calculate $\mathbf{F} = \nabla f$.

b) Calculate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$
(i) directly;
(ii) by using the Fundamental Theorem for line integrals.

4C-3 Let $f(x,y) = \sin x \cos y$.

a) Calculate $\mathbf{F} = \nabla f$.

b) What is the maximum value $\int_C \mathbf{F} \cdot d\mathbf{r}$ can have over all possible paths C in the plane? Give a path C for which this maximum value is attained.

4C-4* The Fundamental Theorem for line integrals should really be called the First Fundamental Theorem. There is an analogue for line integrals of the Second Fundamental Theorem also, where you first integrate, then differentiate; it provides the justification for Method 1 in the reading. It runs:

If $\displaystyle\int_C M\,dx + N\,dy$ is path-independent, and $f(x,y) \;=\; \displaystyle\int_{(x_0,y_0)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$, then $\nabla f = M\,\mathbf{i} + N\,\mathbf{j}$.

The conclusion says $f_x = M, f_y = N$; prove the second of these. (Hint: use the Second Fundamental Theorem of Calculus.)

4C-5 For each of the following, tell for what value of the constants the field will be a gradient field, and for this value, find the corresponding (mathematical) potential function.

a) $\mathbf{F} = (y^2 + 2x)\,\mathbf{i} + axy\,\mathbf{j}$ b) $\mathbf{F} = e^{x+y}\big((x+a)\,\mathbf{i} + x\,\mathbf{j}\big)$

4C-6 Decide which of the following differentials is exact. For each one that is exact, express it in the form df.

a) $y\,dx - x\,dy$ b) $y(2x+y)\,dx + x(2y+x)\,dy$

c)* $x \sin y\,dx + y \sin x\,dy$ d)* $\dfrac{y\,dx - x\,dy}{(x+y)^2}$

4D. Green's Theorem

4D-1 For each of the following fields **F** and closed positively oriented curves C, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ both directly, as a line integral, and also by applying Green's theorem and calculating a double integral.

a) $\mathbf{F}:\ 2y\,\mathbf{i} + x\,\mathbf{j}\,, \quad C:\ x^2 + y^2 = 1$

b) $\mathbf{F}:\ x^2(\,\mathbf{i} + \mathbf{j}\,) \quad C$: rectangle joining (0,0), (2,0), (0,1), (2,1)

c) $\mathbf{F}:\ xy\,\mathbf{i} + y^2\,\mathbf{j}\,, \quad C: y = x^2$ and $y = x,\ 0 \le x \le 1$

4D-2 Show that $\oint_C 4x^3y dx + x^4 dy = 0$ for all closed curves C.

4D-3 Find the area inside the hypocycloid $x^{2/3} + y^{2/3} = 1$, by using Green's theorem. (This curve can be parametrized by $x = \cos^3\theta, y = \sin^3\theta$, between suitable limits on θ.)

4D-4 Show that the value of $\oint_C -y^3dx + x^3dy$ around any positively oriented simple closed curve C is always positive.

4D-5 Show that the value of $\oint_C xy^2dx + (x^2y + 2x)dy$ around any square C in the xy-plane depends only on the size of the square, and not upon its position.

4D-6* Show that $\oint_C -x^2y\,dx + xy^2\,dy\ >\ 0$ around any simple closed curve C.

4D-7* Show that the value of $\oint_C y(y+3)\,dx + 2xy\,dy$ around any equilateral triangle C depends only on the size of the triangle, and not upon its position in the xy-plane.

4E. Two-dimensional Flux

4E-1 Let $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$. Recalling the interpretation of this field as the velocity field of a rotating fluid or just by remembering how it looks geometrically, evaluate with little or no calculation the flux integral $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$, where
a) C is a circle of radius a centered at $(0,0)$, directed counterclockwise.
b) C is the line segment running from $(-1,0)$ to $(1,0)$
c) C is the line running from $(0,0)$ to $(1,0)$.

4E-2 Let **F** be the constant vector field $\mathbf{i} + \mathbf{j}$. Where would you place a directed line segment C of length one in the plane so that the flux across C would be

a) maximal b) minimal c) zero d) -1 e) what would the maximal and minimal values be?

4E-3 Let $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$. Evaluate $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$ if C is given by $r(t) = (t+1)\,\mathbf{i} + t^2\,\mathbf{j}$, where $0 \le t \le 1$; the positive direction on C is the direction of increasing t.

4E-4 Take C to be the square of side 1 with opposite vertices at $(0,0)$ and $(1,1)$, directed *clockwise.* Let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$; find the flux across C.

4E-5 Let $\mathbf{F}$ be defined everywhere except at the origin by the description:

dir $\mathbf{F}$ = radially outward, $|\mathbf{F}| = r^m$, m an integer.

a) Evaluate the flux of $\mathbf{F}$ across a circle of radius a and center at the origin, directed counterclockwise.

b) For which value(s) of m will the flux be independent of a?

4E-6* Let $\mathbf{F}$ be a constant vector field, and let C be a closed polygon, directed counterclockwise. Show that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$. (Hint evaluate the integral along one of the directed sides; then add up the integrals over the successive sides, using properties of vectors.)

4E-7* Let $\mathbf{F}$ be a constant vector field, and C a closed polygon, as in the preceding exercise. Show that $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$.

4F. Green's Theorem in Normal Form

4F-1 Calculate the functions div $\mathbf{F}$ and curl $\mathbf{F}$ for each of the following fields.

a) $a\,\mathbf{i} + b\,\mathbf{j}$ $(a, b$ constants$)$ b) $x^2\,\mathbf{i} + y^2\,\mathbf{j}$ c) $xy(\,\mathbf{i} + \mathbf{j}\,)$

4F-2 Let $\mathbf{F} = \omega(-y\,\mathbf{i} + x\,\mathbf{j}\,)$ be the vector field giving the velocity of a rotating fluid (reading V1, Example 4).

a) Calculate div $\mathbf{F}$ and curl $\mathbf{F}$.

b) Using the physical interpretation of this vector field, explain why it is reasonable that div $\mathbf{F} = 0$.

c) Using the physical interpretation of curl $\mathbf{F}$, explain why it is reasonable that curl $\mathbf{F} = 2\omega$ at the origin.

4F-3 Verify Green's theorem in the normal form by calculating both sides and showing they are equal if $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$, and C is formed by the upper half of the unit circle and the x-axis interval $[-1, 1]$.

4F-4 Verify Green's theorem in the normal form by calculating both sides and showing they are equal if $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$, and C is the square with opposite vertices at $(0, 0)$ and $(1, 1)$.

4F-5 Calculate div $\mathbf{F}$ and curl $\mathbf{F}$ for $\mathbf{F} = r^n(x\,\mathbf{i} + y\,\mathbf{j}\,)$. (Simplify the differentiation by using $r_x = x/r$, $r_y = y/r$.)

For which value(s) of n is div $\mathbf{F} = 0$? For which value(s) of n is curl $\mathbf{F} = 0$?

4F-6* a) Suppose that all the vectors of a field $\mathbf{F}$ point radially outward and their magnitude is a differentiable function $f(r)$ of r alone. Show that curl $\mathbf{F} = 0$.

b) Suppose all the vectors of a field $\mathbf{F}$ are parallel. Reasoning from the physical interpretation of curl $\mathbf{F}$, would you expect it to be zero everywhere? Illustrate your answer by an example.

4G. Simply-connected Regions.

4G-1 Using the criterion of this section, tell which of the following fields and differentials definitely are respectively conservative or exact, which of them are definitely not, and for which of them the criterion fails.

a) $(y^2+2)\,\mathbf{i} + 2xy\,\mathbf{j}$ b) $x(\cos y)\,dx + y(\cos x)\,dy$

c) $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{\sqrt{r^2-1}}$ d) $\dfrac{x\,dx + y\,dy}{\sqrt{1-r^2}}$ e) $\sqrt{x}\,\mathbf{i} + \sqrt{y}\,\mathbf{j}$

4G-2 For each of the following fields $\mathbf{F}$, find $f(x,y)$ such that $\mathbf{F} = \nabla f$.

a) the field of 1a b) the field of 1e

c) the field of 1d; use polar coordinates.

4G-3 Evaluate $\displaystyle\int_{(1,1)}^{(3,4)} \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r^3}$. Use the results of Example 3 in Notes V5.

4G-4 Even though the field of Example 1, $\mathbf{F} = xy\,\mathbf{i} + x^2\,\mathbf{j}$, is not a gradient field, show that $\oint \mathbf{F}\cdot d\mathbf{r} = 0$ around every simple closed curve which is symmetric about the y-axis.

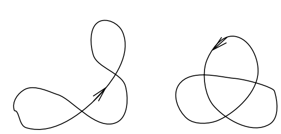
4G-5 Which of the following regions are simply-connected?
a) the half-plane lying above the x-axis
b) the plane minus the line segment joining (0,0) and (0,1)
c) the plane minus the positive x-axis
d) the plane minus the entire x-axis
e) in polar coordinates, the region where $r > 0,\ 0 < \theta < \theta_0$
f) the region between two concentric circles
g) the region in the plane between the two branches of the hyperbola $xy = 1$

4G-6 For which of the following vector fields is the domain where it is defined and continuously differentiable a simply-connected region?

a) $\sqrt{x}\,\mathbf{i} + \sqrt{y}\,\mathbf{j}$ b) $\dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{1-x^2-y^2}}$ c) $\dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{x^2+y^2-1}}$

d) $\dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{r}$ e) $(\mathbf{i}+\mathbf{j})\ln(x^2+y^2)$

4G-7* By following the method outlined in the proof of equation (3) in reading V5, show that if curl $\mathbf{F} = 0$ in the whole xy-plane, then $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ over each of the following closed paths (break them into as few pieces as possible):



5. Triple Integrals

5A. Triple integrals in rectangular and cylindrical coordinates

5A-1 Evaluate: a) $\displaystyle\int_0^2\int_{-1}^1\int_0^1 (x+y+z)dx\,dy\,dz$ b) $\displaystyle\int_0^2\int_0^{\sqrt{y}}\int_0^{xy} 2xy^2z\,dz\,dx\,dy$

5A-2. Follow the three steps in the notes to supply limits for the triple integrals over the following regions of 3-space.

a) The rectangular prism having as its two bases the triangle in the yz-plane cut out by the two axes and the line $y+z=1$, and the corresponding triangle in the plane $x=1$ obtained by adding 1 to the x-coordinate of each point in the first triangle. Supply limits for three different orders of integration:

(i) $\displaystyle\iiint dz\,dy\,dx$ (ii) $\displaystyle\iiint dx\,dz\,dy$ (iii) $\displaystyle\iiint dy\,dx\,dz$

b)* The tetrahedron having its four vertices at the origin, and the points on the three axes where respectively $x=1$, $y=2$, and $z=2$. Use the order $\displaystyle\iiint dz\,dy\,dx$.

c) The quarter of a solid circular cylinder of radius 1 and height 2 lying in the first octant, with its central axis the interval $0 \le y \le 2$ on the y-axis, and base the quarter circle in the xz-plane with center at the origin, radius 1, and lying in the first quadrant. Integrate with respect to y first; use suitable cylindrical coordinates.

d) The region bounded below by the cone $z^2 = x^2+y^2$, and above by the sphere of radius $\sqrt{2}$ and center at the origin. Use cylindrical coordinates.

5A-3 Find the center of mass of the tetrahedron D in the first octant formed by the coordinate planes and the plane $x+y+z=1$. Assume $\delta=1$.

5A-4 A solid right circular cone of height h with 90^0 vertex angle has density at point P numerically equal to the distance from P to the central axis. Choosing the placement of the cone which will give the easiest integral, find

a) its mass b) its center of mass

5A-5 An engine part is a solid S in the shape of an Egyptian-type pyramid having height 2 and a square base with diagonal D of length 2. Inside the engine it rotates about D. Set up (but do not evaluate) an iterated integral giving its moment of inertia about D. Assume $\delta=1$. (Place S so the positive z axis is its central axis.)

5A-6 Using cylindrical coordinates, find the moment of inertia of a solid hemisphere D of radius a about the central axis perpendicular to the base of D. Assume $\delta = 1$..

5A-7 The paraboloid $z = x^2+y^2$ is shaped like a wine-glass, and the plane $z=2x$ slices off a finite piece D of the region above the paraboloid (i.e., inside the wine-glass). Find the moment of inertia of D about the z-axis, assuming $\delta=1$.

5B. Triple Integrals in Spherical Coordinates

5B-1 Supply limits for iterated integrals in spherical coordinates $\iiint d\rho\, d\phi\, d\theta$ for each of the following regions. (No integrand is specified; $d\rho\, d\phi\, d\theta$ is given so as to determine the order of integration.)

a) The region of 5A-2d: bounded below by the cone $z^2 = x^2 + y^2$, and above by the sphere of radius $\sqrt{2}$ and center at the origin.

b) The first octant.

c) That part of the sphere of radius 1 and center at $z = 1$ on the z-axis which lies above the plane $z = 1$.

5B-2 Find the center of mass of a hemisphere of radius a, using spherical coordinates. Assume the density $\delta = 1$.

5B-3 A solid D is bounded below by a right circular cone whose generators have length a and make an angle $\pi/6$ with the central axis. It is bounded above by a portion of the sphere of radius a centered at the vertex of the cone. Find its moment of inertia about its central axis, assuming the density δ at a point is numerically equal to the distance of the point from a plane through the vertex perpendicular to the central axis.

5B-4 Find the average distance of a point in a solid sphere of radius a from
a) the center b) a fixed diameter c) a fixed plane through the center

5C. Gravitational Attraction

5C-1.* Find the gravitational attraction of the solid V bounded by a right circular cone of vertex angle 60^o and slant height a, surmounted by the cap of a sphere of radius a centered at the vertex of the cone; take the density to be

(a) 1 (b) the distance from the vertex. Ans.: a) $\pi Ga/4$ b) $\pi Ga^2/8$

5C-2. Find the gravitational attraction of the region bounded above by the plane $z = 2$ and below by the cone $z^2 = 4(x^2 + y^2)$, on a unit mass at the origin; take $\delta = 1$.

5C-3. Find the gravitational attraction of a solid sphere of radius 1 on a unit point mass Q on its surface, if the density of the sphere at $P(x, y, z)$ is $|PQ|^{-1/2}$.

5C-4. Find the gravitational attraction of the region which is bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the sphere $x^2 + y^2 + z^2 = 2z$, on a unit mass at the origin. (Take $\delta = 1$.)

5C-5.* Find the gravitational attraction of a solid hemisphere of radius a and density 1 on a unit point mass placed at its pole. Ans: $2\pi Ga(1 - \sqrt{2}/3)$

5C-6.* Let V be a uniform solid sphere of mass M and radius a. Place a unit point mass a distance b from the center of V. Show that the gravitational attraction of V on the point mass is

a) GM/b^2, if $b \geq a$; b) GM'/b^2, if $b \leq a$, where $M' = \dfrac{b^3}{a^3} M$.

Part (a) is Newton's theorem, described in the Remark. Part (b) says that the outer portion of the sphere—the spherical shell of inner radius b and outer radius a —exerts no force on the test mass: all of it comes from the inner sphere of radius b, which has total mass $\dfrac{b^3}{a^3} M$.

5C-7.* Use Problem 6b to show that if we dig a straight hole through the earth, it takes a point mass m a total of $\pi\sqrt{R/g} \approx 42$ minutes to fall from one end to the other, no matter what the length of the hole is.

(Write $\mathbf{F} = m\mathbf{a}$, letting x be the distance from the middle of the hole, and obtain an equation of simple harmonic motion for $x(t)$. Here

$R =$ earth's radius, $M =$ earth's mass, $g = GM/R^2$.)

6. Vector Integral Calculus in Space

6A. Vector Fields in Space

6A-1 Describe geometrically the following vector fields: a) $\dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho}$ b) $-x\,\mathbf{i} - z\,\mathbf{k}$

6A-2 Write down the vector field where each vector runs from (x, y, z) to a point half-way towards the origin.

6A-3 Write down the velocity field $\mathbf{F}$ representing a rotation about the x-axis in the direction given by the right-hand rule (thumb pointing in positive x-direction), and having constant angular velocity ω.

6A-4 Write down the most general vector field all of whose vectors are parallel to the plane $3x - 4y + z = 2$.

6B. Surface Integrals and Flux

6B-1 Without calculating, find the flux of $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through the sphere of radius a and center at the origin. Take $\mathbf{n}$ pointing outward.

6B-2 Without calculation, find the flux of $\mathbf{k}$ through the infinite cylinder $x^2 + y^2 = 1$. (Take $\mathbf{n}$ pointing outward.)

6B-3 Without calculation, find the flux of $\mathbf{i}$ through that portion of the plane $x+y+z = 1$ lying in the first octant (take $\mathbf{n}$ pointed away from the origin).

6B-4 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\,\mathbf{j}$, and S = the half of the sphere $x^2 + y^2 + z^2 = a^2$ for which $y \geq 0$, oriented so that $\mathbf{n}$ points away from the origin.

6B-5 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where where $\mathbf{F} = z\,\mathbf{k}$, and S is the surface of Exercise 6B-3 above.

6B-6 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and S is the part of the paraboloid $z = x^2 + y^2$ lying underneath the plane $z = 1$, with $\mathbf{n}$ pointing generally upwards. Explain geometrically why your answer is negative.

6B-7* Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{x^2 + y^2 + z^2}$, and S is the surface of Exercise 6B-2.

6B-8 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\,\mathbf{j}$ and S is that portion of the cylinder $x^2 + y^2 = a^2$ between the planes $z = 0$ and $z = h$, and to the right of the xz-plane; $\mathbf{n}$ points outwards.

6B-9* Find the center of gravity of a hemispherical shell of radius a. (Assume the density is 1, and place it so its base is on the xy-plane.

6B-10* Let S be that portion of the plane $-12x + 4y + 3z = 12$ projecting vertically onto the plane region $(x - 1)^2 + y^2 \leq 4$. Evaluate

a) the area of S b) $\displaystyle\iint_S z\,dS$ c) $\displaystyle\iint_S (x^2 + y^2 + 3z)\,dS$

6B-11* Let S be that portion of the cylinder $x^2 + y^2 = a^2$ bounded below by the xy-plane and above by the cone $z = \sqrt{(x-a)^2 + y^2}$.

a) Find the area of S. Recall that $\sqrt{1-\cos\theta} = \sqrt{2}\sin(\theta/2)$. (Hint: remember that the upper limit of integration for the z-integral will be a function of θ determined by the intersection of the two surfaces.)

b) Find the moment of inertia of S about the z-axis. There should be nothing to calculate once you've done part (a).

c) Evaluate $\iint_S z^2\, dS$.

6B-12 Find the average height above the xy-plane of a point chosen at random on the surface of the hemisphere $x^2 + y^2 + z^2 = a^2,\ z \geq 0$.

6C. Divergence Theorem

6C-1 Calculate div **F** for each of the following fields

a) $x^2y\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$ b)* $3x^2yz\,\mathbf{i} + x^3z\,\mathbf{j} + x^3y\,\mathbf{k}$ c)* $\sin^3 x\,\mathbf{i} + 3y\cos^3 x\,\mathbf{j} + 2x\,\mathbf{k}$

6C-2 Calculate div **F** if $\mathbf{F} = \rho^n(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$, and tell for what value(s) of n we have div $\mathbf{F} = 0$. (Use $\rho_x = x/\rho$, etc.)

6C-3 Verify the divergence theorem when $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the surface composed of the upper half of the sphere of radius a and center at the origin, together with the circular disc in the xy-plane centered at the origin and of radius a.

6C-4* Verify the divergence theorem if **F** is as in Exercise 3 and S is the surface of the unit cube having diagonally opposite vertices at (0,0,0) and (1,1,1), with three sides in the coordinate planes. (All the surface integrals are easy and do not require any formulas.)

6C-5 By using the divergence theorem, evaluate the surface integral giving the flux of $\mathbf{F} = x\,\mathbf{i} + z^2\,\mathbf{j} + y^2\,\mathbf{k}$ over the tetrahedron with vertices at the origin and the three points on the positive coordinate axes at distance 1 from the origin.

6C-6 Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ over the closed surface S formed below by a piece of the cone $z^2 = x^2 + y^2$ and above by a circular disc in the plane $z = 1$; take **F** to be the field of Exercise 6B-5; use the divergence theorem.

6C-7 Verify the divergence theorem when S is the closed surface having for its sides a portion of the cylinder $x^2 + y^2 = 1$ and for its top and bottom circular portions of the planes $z = 1$ and $z = 0$; take **F** to be

a) $x^2\,\mathbf{i} + xy\,\mathbf{j}$ b)* $zy\,\mathbf{k}$ c)* $x^2\,\mathbf{i} + xy\,\mathbf{j} + zy\,\mathbf{k}$ (use (a) and (b))

6C-8 Suppose div $\mathbf{F} = 0$ and S_1 and S_2 are the upper and lower hemispheres of the unit sphere centered at the origin. Direct both hemispheres so that the unit normal is "up", i.e., has positive **k**-component.

a) Show that $\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$, and interpret this physically in terms of flux.

b) State a generalization to an arbitrary closed surface S and a field **F** such that div $\mathbf{F} = 0$.

6C-9* Let **F** be the vector field for which all vectors are aimed radially away from the origin, with magnitude $1/\rho^2$.

a) What is the domain of **F**?

b) Show that div $\mathbf{F} = 0$.

c) Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is a sphere of radius a centered at the origin. Does the fact that the answer is not zero contradict the divergence theorem? Explain.

d) Prove using the divergence theorem that $\iint_S \mathbf{F} \cdot d\mathbf{S}$ over a positively oriented closed surface S has the value zero if the surface does not contain the origin, and the value 4π if it does.

(**F** is the vector field for the flow arising from a *source of strength* 4π at the origin.)

6C-10 A flow field **F** is said to be *incompressible* if $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for all closed surfaces S. Assume that **F** is continuously differentiable. Show that

$$\mathbf{F} \textit{ is the field of an incompressible flow} \quad \Longleftrightarrow \quad \operatorname{div} \mathbf{F} = 0 \ .$$

6C-11 Show that the flux of the position vector $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ outward through a closed surface S is three times the volume contained in that surface.

6D. Line Integrals in Space

6D-1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following fields **F** and curves C:

a) $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} - x\,\mathbf{k}$; C is the twisted cubic curve $x = t,\ y = t^2,\ z = t^3$ running from $(0,0,0)$ to $(1,1,1)$.

b) **F** is the field of (a); C is the line running from $(0,0,0)$ to $(1,1,1)$

c) **F** is the field of (a); C is the path made up of the succession of line segments running from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$.

d) $\mathbf{F} = zx\,\mathbf{i} + zy\,\mathbf{j} + x\,\mathbf{k}$; C is the helix $x = \cos t,\ y = \sin t,\ z = t,$ running from $(1,0,0)$ to $(1,0,2\pi)$.

6D-2 Let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any curve C lying on a sphere of radius a centered at the origin.

6D-3* a) Let C be the directed line segment running from P to Q, and let **F** be a constant vector field. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot PQ$.

b) Let C be a closed space polygon $P_1P_2 \ldots P_nP_1$, and let **F** be a constant vector field. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. (Use part (a).)

c) Let C be a closed space curve, **F** a constant vector field. Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. (Use part (b).)

6D-4 a) Let $f(x,y,z) = x^2 + y^2 + z^2$; calculate $\mathbf{F} = \nabla f$.

b) Let C be the helix of 6D-1d above, but running from $t = 0$ to $t = 2n\pi$. Calculate the work done by $\mathbf{F}$ moving a unit point mass along C; use three methods:

(i) directly

(ii) by using the path-independence of the integral to replace C by a simpler path

(iii) by using the first fundamental theorem for line integrals.

6D-5 Let $\mathbf{F} = \nabla f$, where $f(x,y,z) = \sin(xyz)$. What is the maximum value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ over all possible paths C? Give a path C for which this maximum value is attained.

6D-6* Let $\mathbf{F} = \nabla f$, where $f(x,y,z) = \dfrac{1}{x+y+z+1}$. Find the work done by $\mathbf{F}$ carrying a unit point mass from the origin out to ∞ along a ray.
(Take the ray to be $x = at, y = bt, z = ct$.)

6E. Gradient Fields in Space

6E-1 Which of the following differentials are exact? For each one which is, express it in the form *df* for a suitable function $f(x,y,z)$, using one of the systematic methods.

a) $x^2\,dx + y^2\,dy + z^2\,dz$ b) $y^2z\,dx + 2xyz\,dy + xy^2\,dz$

c) $y(6x^2 + z)\,dx + x(2x^2 + z)\,dy + xy\,dz$

6E-2 Find curl $\mathbf{F}$, if $\mathbf{F} = x^2y\,\mathbf{i} + yz\,\mathbf{j} + xyz^2\,\mathbf{k}$.

6E-3 The fields $\mathbf{F}$ below are defined for all x, y, z. For each,

a) show that curl $\mathbf{F} = \mathbf{0}$;

b) find a potential function $f(x,y,z)$, using either method, or inspection.

(i) $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ (ii) $(2xy + z)\,\mathbf{i} + x^2\,\mathbf{j} + x\,\mathbf{k}$ (iii) $yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$

6E-4 Show that if $f(x,y,z)$ and $g(x,y,z)$ are two functions having the same gradient, then $f = g + c$ for some constant c. (Use the Fundamental Theorem for Line Integrals.)

6E-5 For what values of a and b will $\mathbf{F} = yz^2\,\mathbf{i} + (xz^2 + ayz)\,\mathbf{j} + (bxyz + y^2)\,\mathbf{k}$ be a conservative field? Using these values, find the corresponding potential function $f(x,y,z)$ by one of the systematic methods.

6E-6 a) Define what it means for $Mdx + Ndy + Pdz$ to be an exact differential.

b) Find all values of a, b, c for which

$$(axyz + y^3z^2)dx + (a/2)x^2z + 3xy^2z^2 + byz^3)dy + (3x^2y + cxy^3z + 6y^2z^2)dz$$

will be exact.

c) For those values of a, b, c, express the differential as *df* for a suitable $f(x,y,z)$.

6F. Stokes' Theorem

6F-1 Verify Stokes' theorem when S is the upper hemisphere of the sphere of radius one centered at the origin and C is its boundary; i.e., calculate both integrals in the theorem and show they are equal. Do this for the vector fields

a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; b) $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$.

6F-2 Verify Stokes' theorem if $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ and S is the portion of the plane $x + y + z = 0$ cut out by the cylinder $x^2 + y^2 = 1$, and C is its boundary (an ellipse).

6F-3 Verify Stokes' theorem when S is the rectangle with vertices at (0,0,0), (1,1,0), (0,0,1), and (1,1,1), and $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.

6F-4* Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, where M, N, P have continuous second partial derivatives.

a) Show by direct calculation that div(curl F) = 0.

b) Using (a), show that $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = 0$ for any closed surface S.

6F-5 Let S be the surface formed by the cylinder $x^2 + y^2 = a^2$, $0 \le z \le h$, together with the circular disc forming its top, oriented so the normal vector points up or out. Let $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + x^2\,\mathbf{k}$. Find the flux of $\nabla \times \mathbf{F}$ through S

(a) directly, by calculating two surface integrals;

(b) by using Stokes' theorem.

6G. Topological Questions

6G-1 Which regions are simply-connected?

a) first octant b) exterior of a torus c) region between two concentric spheres

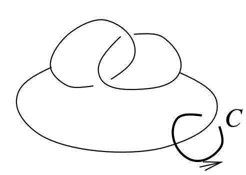
d) three-space with one of the following removed:

i) a line ii) a point iii) a circle iv) the letter H v) the letter R vi) a ray

6G-2 Show that the fields $\mathbf{F} = \rho^n(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$, where $\rho = \sqrt{x^2 + y^2 + z^2}$, are gradient fields for any value of the integer n. (Use $\rho_x = x/\rho$, etc.)

Then, find the potential function $f(x, y, z)$. (It is easiest to phrase the question in terms of differentials: one wants $df = \rho^n(x\,dx + y\,dy + z\,dz)$; for $n = 0$, you can find f by inspection; from this you can guess the answer for $n \neq 0$ as well. The case $n = -2$ is an exception, and must be handled separately. The printed solutions use this method, somewhat more formally phrased using the fundamental theorem of line integrals.)

6G-3* If D is taken to be the exterior of the wire link shown, then the little closed curve C cannot be shrunk to a point without leaving D, i.e., without crossing the link. Nonetheless, show that C is the boundary of a two-sided surface lying entirely inside D. (So if $\mathbf{F}$ is a field in D such that curl $\mathbf{F} = \mathbf{0}$, the above considerations show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.)



6G-4* In cylindrical coordinates r, θ, z, let $\mathbf{F} = \nabla\varphi$, where $\varphi = \tan^{-1}\dfrac{z}{r-1}$.

a) Interpret φ geometrically. What is the domain of $\mathbf{F}$?

b) From the geometric interpretation what will be the value of $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around a closed path C that links with the unit circle in the xy-plane (for example, take C to be the circle in the yz-plane with radius 1 and center at $(0,1,0)$?

6H. Applications to Physics

6H-1 Prove that $\nabla\cdot\nabla\times\mathbf{F} = 0$. What are the appropriate hypotheses about the field $\mathbf{F}$?

6H-2 Show that for any closed surface S, and continuously differentiable vector field $\mathbf{F}$,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} \;=\; 0\ .$$

Do it two ways: a) using the divergence theorem; b) using Stokes' theorem.

6H-3* Prove each of the following (ϕ is a (scalar) function):

a) $\nabla\cdot(\phi\,\mathbf{F}) \;=\; \phi\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla\phi$

b) $\nabla\times(\phi\mathbf{F}) \;=\; \phi\nabla\times\mathbf{F} + (\nabla\phi)\times\mathbf{F}$

c) $\nabla\cdot(\mathbf{F}\times\mathbf{G}) \;= \mathbf{G}\cdot\nabla\times\mathbf{F} - \mathbf{F}\cdot\nabla\times\mathbf{G}$

6H-4* The *normal derivative.* If S is an oriented surface with unit normal vector $\mathbf{n}$, and ϕ is a function defined and differentiable on some domain containing S, then the **normal derivative** of ϕ on S is defined to be the directional derivative of ϕ in the direction $\mathbf{n}$. In symbols (on the left is the notation for the normal derivative):

$$\frac{\partial\phi}{\partial n} \;=\; \nabla\phi\cdot\mathbf{n}.$$

Prove that if S is closed and D its interior, and if ϕ has continuous second derivatives inside D, then

$$\iint_S \frac{\partial\phi}{\partial n}\,dS \;=\; \iiint_D \nabla^2\phi\,dV\ .$$

(This shows for example that if you are trying to find a *harmonic* function ϕ defined in D and having a prescribed normal derivative on S, you must be sure that $\dfrac{\partial\phi}{\partial n}$ has been prescribed so that $\displaystyle\iint_S \frac{\partial\phi}{\partial n}\,dS \;=\; 0.$

6H-5* Formulate and prove the analogue of the preceding exercise for the plane.

6H-6* Prove that, if S is a closed surface with interior D, and ϕ has continuous second derivatives in D, then

$$\iint_S \phi\frac{\partial\phi}{\partial n}\,dS \;=\; \iiint_D \phi(\nabla^2\phi) + (\nabla\phi)^2\,dV\ .$$

6H-7* Formulate and prove the analogue of the preceding exercise for a plane.

6H-8 A boundary value problem.* Suppose you want to find a function ϕ defined in a domain containing a closed surface S and its interior D, such that (i) ϕ is harmonic in D and (ii) $\phi = 0$ on S.

a) Show that the two conditions imply that $\phi = 0$ on all of D. (Use Exercise 6.)

b) Instead of assuming (ii), assume instead that the values of ϕ on S are prescribed as some continuous function on S. Prove that if a function ϕ exists which is harmonic in D and has these prescribed boundary values, then it is unique — there is only one such function. (In other words, the values of a harmonic function function on the boundary surface S determine its values everywhere inside S.) (Hint: Assume there are two such functions and consider their difference.)

6H-9 Vector potential* In the same way that $\mathbf{F} = \nabla\phi \;\Rightarrow\; \nabla \times \mathbf{F} = \mathbf{0}$ has the partial converse

$$\nabla \times \mathbf{F} = 0 \quad \text{in a simply-connected region} \quad \Rightarrow \quad \mathbf{F} = \nabla f,$$

so the theorem $\quad \mathbf{F} = \nabla \times \mathbf{G} \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = 0 \quad$ has the partial converse

$$(*) \qquad \nabla \cdot \mathbf{F} = 0 \quad \text{in a suitable region} \quad \Rightarrow \quad \mathbf{F} = \nabla \times \mathbf{G}, \quad \text{for some } \mathbf{G}.$$

$\mathbf{G}$ is called a **vector potential** for $\mathbf{F}$. A suitable region is one with this property: whenever P lies in the region, the whole line segment joining P to the origin lies in the region. (Instead of the origin, one could use some other fixed point.) For instance, a sphere, a cube, or all of 3-space would be suitable regions.

Suppose for instance that $\nabla \cdot \mathbf{F} = 0$ in all of 3-space. Then $\mathbf{G}$ exists in all of 3-space, and is given by the formula

$$(**) \qquad \mathbf{G} \;=\; \int_0^1 t\,\mathbf{F}(tx, ty, tz) \times \mathbf{R}\,dt, \qquad \mathbf{R} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

The integral means: integrate separately each component of the vector function occurring in the integrand, and you'll get the corresponding component of $\mathbf{G}$.

We shall not prove this formula here; the proof depends on Leibniz' rule for differentiating under an integral sign. We can however try out the formula.

a) Let $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$. Check that div $\mathbf{F}$=0, find $\mathbf{G}$ from the formula (**), and check your answer by verifying that $\mathbf{F} = \text{curl}\ \mathbf{G}$.

b) Show that $\mathbf{G}$ is unique up to the addition of an arbitrary gradient field; i.e., if $\mathbf{G}$ is one such field, then all others are of the form

$$(***) \qquad \mathbf{G}' = \mathbf{G} + \nabla f,$$

for an arbitrary function $f(x, y, z)$. (Show that if $\mathbf{G}'$ has the form (***), then $\mathbf{F} = \text{curl}\ \mathbf{G}'$; then show conversely that if $\mathbf{G}'$ is a field such that curl $\mathbf{G}' = \mathbf{F}$, then $\mathbf{G}'$ has the form (***).)

6H-10 Let **B** be a magnetic field produced by a moving electric field **E**. Assume there are no charges in the region. Then one of Maxwell's equations in differential form reads

$$\nabla \times \mathbf{B} \;=\; \frac{1}{c}\,\frac{\partial \mathbf{E}}{\partial t}\;.$$

What is the integrated form of this law? Prove your answer, as in the notes; you can assume that the partial differentiation can be moved outside of the integral sign.

6H-11* In the preceding problem if we also allow for a field **j** which gives the current density at each point of space, we get Ampere's law in differential form (as modified by Maxwell):

$$\nabla \times \mathbf{B} \;=\; \frac{1}{c}\left(4\pi\,\mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}\right).$$

Give the integrated form of this law, and deduce it from the differential form, as done in the notes.