S. 18.02 Solutions to Exercises

1. Vectors and Matrices

1A. Vectors

1A-1 a) $|\mathbf{A}| = \sqrt{3}$, $\quad$ dir $\mathbf{A} = \mathbf{A}/\sqrt{3}$ $\qquad$ b) $|\mathbf{A}| = 3$, $\quad$ dir $\mathbf{A} = \mathbf{A}/3$
c) $|\mathbf{A}| = 7$, $\quad$ dir $\mathbf{A} = \mathbf{A}/7$

1A-2 $1/25 + 1/25 + c^2 = 1 \;\Rightarrow\; c = \pm\sqrt{23}/5$

1A-3 a) $\mathbf{A} = -\,\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$, $\quad |\mathbf{A}| = 3$, $\quad$ dir $\mathbf{A} = \mathbf{A}/3$.

b) $\mathbf{A} = |\mathbf{A}|$ dir $\mathbf{A} = 2\,\mathbf{i} + 4\,\mathbf{j} - 4\,\mathbf{k}$. Let P be its tail and Q its head. Then $OQ = OP + \mathbf{A} = 4\,\mathbf{j} - 3\,\mathbf{k}$; therefore $Q = (0, 4, -3)$.

1A-4 a) $OX = OP + PX = OP + \frac{1}{2}(PQ) = OP + \frac{1}{2}(OQ - OP) = \frac{1}{2}(OP + OQ)$
b) $OX = s\,OP + r\,OQ$; replace $\frac{1}{2}$ by r in above; use $1 - r = s$.

1A-5 $\mathbf{A} = \frac{3}{2}\sqrt{3}\,\mathbf{i} + \frac{3}{2}\,\mathbf{j}$. The condition is not redundant since there are two vectors of length 3 making an angle of 30^o with $\mathbf{i}$.

1A-6 wind $\mathbf{w} = 50(-\,\mathbf{i} - \mathbf{j})/\sqrt{2})$, $\quad \mathbf{v} + \mathbf{w} = 200\,\mathbf{j} \;\Rightarrow\; \mathbf{v} = 50/\sqrt{2}\,\mathbf{i} + (200 + 50/\sqrt{2})\,\mathbf{j}$.

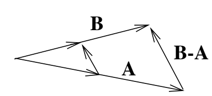
1A-7 a) $b\,\mathbf{i} - a\,\mathbf{j}$ $\quad$ b) $-b\,\mathbf{i} + a\,\mathbf{j}$ $\quad$ c) $(3/5)^2 + (4/5)^2 = 1$; $\quad \mathbf{j}' = -(4/5)\,\mathbf{i} + (3/5)\,\mathbf{j}$

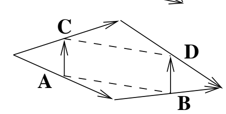
1A-8 a) is elementary trigonometry;

b) $\cos\alpha = a/\sqrt{a^2 + b^2 + c^2}$, etc.; dir $\mathbf{A} = (-1/3, 2/3, 2/3)$

c) if t, u, v are direction cosines of some $\mathbf{A}$, then $t\,\mathbf{i} + u\,\mathbf{j} + v\,\mathbf{k} =$ dir $\mathbf{A}$, a unit vector, so $t^2 + u^2 + v^2 = 1$; conversely, if this relation holds, then $t\,\mathbf{i} + u\,\mathbf{j} + v\,\mathbf{k} = \mathbf{u}$ is a unit vector, so dir $\mathbf{u} = \mathbf{u}$ and t, u, v are the direction cosines of $\mathbf{u}$.

1A-9 Letting $\mathbf{A}$ and $\mathbf{B}$ be the two sides, the third side is $\mathbf{B} - \mathbf{A}$; the line joining the two midpoints is $\frac{1}{2}\mathbf{B} - \frac{1}{2}\mathbf{A}$, which $= \frac{1}{2}(\mathbf{B} - \mathbf{A})$, a vector parallel to the third side and half its length.





1A-10 Letting $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be the four sides; then if the vectors are suitably oriented, we have $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$.

The vector from the midpoint of $\mathbf{A}$ to the midpoint of $\mathbf{C}$ is $\frac{1}{2}\mathbf{C} - \frac{1}{2}\mathbf{A}$; similarly, the vector joining the midpoints of the other two sides is $\frac{1}{2}\mathbf{B} - \frac{1}{2}\mathbf{D}$, and

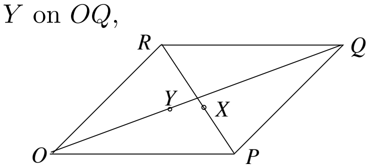
$$\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D} \quad\Rightarrow\quad \mathbf{C} - \mathbf{A} = \mathbf{B} - \mathbf{D} \quad\Rightarrow\quad \tfrac{1}{2}(\mathbf{C} - \mathbf{A}) = \tfrac{1}{2}(\mathbf{B} - \mathbf{D})\ ;$$

thus two opposite sides are equal and parallel, which shows the figure is a parallelogram.

1A-11 Letting the four vertices be O, P, Q, R, with X on PR and Y on OQ,

$$\begin{aligned} OX &= OP + PX \;=\; OP + \tfrac{1}{2}PR \\ &= OP + \tfrac{1}{2}(OR - OP) \\ &= \tfrac{1}{2}(OR + OP) \;=\; \tfrac{1}{2}OQ \;=\; OY; \end{aligned}$$

therefore $X = Y$.



1B. Dot Product

1B-1 a) $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \dfrac{4+2}{\sqrt{2}\cdot 6} = \dfrac{1}{\sqrt{2}}, \quad \theta = \dfrac{\pi}{4}$ b) $\cos\theta = \dfrac{3}{\sqrt{6}\cdot\sqrt{6}} = \dfrac{1}{2}, \quad \theta = \dfrac{\pi}{3}$.

1B-2 $\mathbf{A}\cdot\mathbf{B} = c-4$; therefore (a) orthogonal if $c=4$,

b) $\cos\theta = \dfrac{c-4}{\sqrt{c^2+5}\sqrt{6}}$; the angle θ is acute if $\cos\theta > 0$, i.e., if $c > 4$.

1B-3 Place the cube in the first octant so the origin is at one corner P, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are three edges. The longest diagonal $PQ = \mathbf{i}+\mathbf{j}+\mathbf{k}$; a face diagonal $PR = \mathbf{i}+\mathbf{j}$.

a) $\cos\theta = \dfrac{PQ\cdot PR}{|PQ|\cdot|PR|} = \dfrac{2}{\sqrt{3}\sqrt{2}}; \quad \theta = \cos^{-1}\sqrt{2/3}$

b) $\cos\theta = \dfrac{PQ\cdot\mathbf{i}}{|PQ||\mathbf{i}|} = \dfrac{1}{\sqrt{3}}, \quad \theta = \cos^{-1} 1/\sqrt{3}$.

1B-4 $QP = (a,0,-2), \quad QR = (a,-2,2)$, therefore

a) $QP\cdot QR = a^2-4$; therefore PQR is a right angle if $a^2-4=0$, i.e., if $a=\pm 2$.

b) $\cos\theta = \dfrac{a^2-4}{\sqrt{a^2+4}\sqrt{a^2+8}}$; the angle is acute if $\cos\theta > 0$, i.e., if $a^2-4>0$, or $|a|>2$, i.e., $a>2$ or $a<-2$.

1B-5 a) $\mathbf{F}\cdot\mathbf{u} = -1/\sqrt{3}$ b) $\mathbf{u} = \text{dir } \mathbf{A} = \mathbf{A}/7$, so $\mathbf{F}\cdot\mathbf{u} = -4/7$

1B-6 After dividing by $|OP|$, the equation says $\cos\theta = c$, where θ is the angle between OP and $\mathbf{u}$; call its solution $\theta_0 = \cos^{-1} c$. Then the locus is the nappe of a right circular cone with axis in the direction $\mathbf{u}$ and vertex angle $2\theta_0$. In particular this cone is

a) a plane if $\theta_0 = \pi/2$, i.e., if $c=0$ b) a ray if $\theta_0 = 0,\ \pi$, i.e., if $c=\pm 1$.

c) Locus is the origin, if $c>1$ or $c<-1$ (division by $|OP|$ is illegal, notice).

1B-7 a) $|\mathbf{i}'| = |\mathbf{j}'| = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$; a picture shows the system is right-handed.

b) $\mathbf{A}\cdot\mathbf{i}' = -1/\sqrt{2}; \qquad \mathbf{A}\cdot\mathbf{j}' = -5/\sqrt{2}$;

since they are perpendicular unit vectors, $\mathbf{A} = \dfrac{-\mathbf{i}'-5\mathbf{j}'}{\sqrt{2}}$.

c) Solving, $\mathbf{i} = \dfrac{\mathbf{i}'-\mathbf{j}'}{\sqrt{2}}, \quad \mathbf{j} = \dfrac{\mathbf{i}'+\mathbf{j}'}{\sqrt{2}}$;

thus $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} = \dfrac{2(\mathbf{i}'-\mathbf{j}')}{\sqrt{2}} - \dfrac{3(\mathbf{i}'+\mathbf{j}')}{\sqrt{2}} = \dfrac{-\mathbf{i}'-5\mathbf{j}'}{\sqrt{2}}$, as before.

1B-8 a) Check that each has length 1, and the three dot products $\mathbf{i}'\cdot\mathbf{j}'$, $\mathbf{i}'\cdot\mathbf{k}'$, $\mathbf{j}'\cdot\mathbf{k}'$ are 0; make a sketch to check right-handedness.

b) $\mathbf{A}\cdot\mathbf{i}' = \sqrt{3}, \quad \mathbf{A}\cdot\mathbf{j}' = 0, \quad \mathbf{A}\cdot\mathbf{k}' = \sqrt{6}$, therefore, $\mathbf{A} = \sqrt{3}\,\mathbf{i}' + \sqrt{6}\,\mathbf{k}'$.

1B-9 Let $\mathbf{u} = \text{dir } \mathbf{A}$, then the vector $\mathbf{u}$-component of $\mathbf{B}$ is $(\mathbf{B}\cdot\mathbf{u})\mathbf{u}$. Subtracting it off gives a vector perpendicular to $\mathbf{u}$ (and therefore also to $\mathbf{A}$); thus

$$\mathbf{B} = (\mathbf{B}\cdot\mathbf{u})\mathbf{u} + \big(\mathbf{B} - (\mathbf{B}\cdot\mathbf{u})\mathbf{u}\big)$$

or in terms of A, remembering that $|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}$,

$$\mathbf{B} \;=\; \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\,\mathbf{A} + \left(\mathbf{B} \;-\; \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\,\mathbf{A}\right)$$

.

1B-10 Let two adjacent edges of the parallelogram be the vectors $\mathbf{A}$ and $\mathbf{B}$; then the two diagonals are $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$. Remembering that for any vector $\mathbf{C}$ we have $\mathbf{C}\cdot\mathbf{C} = |\mathbf{C}|^2$, the two diagonals have equal lengths

$$\begin{aligned}
\Leftrightarrow \quad (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) &= (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) \\
\Leftrightarrow \quad (\mathbf{A}\cdot\mathbf{A}) + 2(\mathbf{A}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\mathbf{A}) - 2(\mathbf{A}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{B}) \\
\Leftrightarrow \quad \mathbf{A}\cdot\mathbf{B} &= 0,
\end{aligned}$$

which says the two sides are perpendicular, i.e., the parallelogram is a rectangle.

1B-11 Using the notation of the previous exercise, we have successively,

$$\begin{aligned}
(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) &= \mathbf{A}\cdot\mathbf{A} - \mathbf{B}\cdot\mathbf{B}; \quad \text{therefore,} \\
(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = 0 \quad \Leftrightarrow \quad \mathbf{A}\cdot\mathbf{A} &= \mathbf{B}\cdot\mathbf{B},
\end{aligned}$$

i.e., the diagonals are perpendicular if and only if two adjacent edges have equal length, in other words, if the parallelogram is a rhombus.

1B-12 Let O be the center of the semicircle, Q and R the two ends of the diameter, and P the vertex of the inscribed angle; set $\mathbf{A} = QO = OR$ and $\mathbf{B} = OP$; then $|\mathbf{A}| = |\mathbf{B}|$.

The angle sides are $QP = \mathbf{A}+\mathbf{B}$ and $PR = \mathbf{A}-\mathbf{B}$; they are perpendicular since

$$\begin{aligned}
(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) &= \mathbf{A}\cdot\mathbf{A} - \mathbf{B}\cdot\mathbf{B} \\
&= 0, \qquad \text{since} \;\; |\mathbf{A}| = |\mathbf{B}|.
\end{aligned}$$

1B-13 The unit vectors are $\mathbf{u}_i = \cos\theta_i\,\mathbf{i} + \sin\theta_i\,\mathbf{j}$, for $i = 1, 2$; the angle between them is $\theta_2 - \theta_1$. We then have by the geometric definition of the dot product

$$\begin{aligned}
\cos(\theta_2 - \theta_1) &= \frac{\mathbf{u}_1\cdot\mathbf{u}_2}{|\mathbf{u}_1||\mathbf{u}_2|}, \\
&= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2,
\end{aligned}$$

according to the formula for evaluating the dot product in terms of components.

1B-14 Let the coterminal vectors $\mathbf{A}$ and $\mathbf{B}$ represent two sides of the triangle, and let θ be the included angle. Suitably directed, the third side is then $\mathbf{C} = \mathbf{A} - \mathbf{B}$, and

$$\begin{aligned}
|\mathbf{C}|^2 \;=\; (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) &= \mathbf{A}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B} - 2\mathbf{A}\cdot\mathbf{B} \\
&= |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos\theta,
\end{aligned}$$

by the geometric interpretation of the dot product.

1C. Determinants

1C-1 a) $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} = -1-8=-9$ b) $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix} = -10.$

1C-2 $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix} = 2+0-8-(24+4+0) = -34.$

a) By the cofactors of row one: $= -1\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} + 4\cdot\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -34$

b) By the cofactors of column one: $= -1\cdot\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 1\cdot\begin{vmatrix} 0 & 4 \\ -2 & -1 \end{vmatrix} + 3\cdot\begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix} = -34.$

1C-3 a) $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -3;$ so area of the parallelogram is 3, area of the triangle is 3/2

b) sides are $PQ = (0,-3)$, $PR = (1,1)$, $\begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix} = 3$, so area of the parallelogram is 3, area of the triangle is 3/2

1C-4 $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = x_2x_3^2 + x_1x_2^2 + x_1^2x_3 - x_1^2x_2 - x_2^2x_3 - x_1x_3^2$

$(x_1-x_2)(x_1-x_3)(x_2-x_3) = x_1^2x_2 - x_1^2x_3 - x_1x_3x_2 + x_1x_3^2 - x_2^2x_1 + x_2x_1x_3 + x_2^2x_3 - x_2x_3^2.$

Two terms cancel, and the other six are the same as those above, except they have the opposite sign.

1C-5 a) $\begin{vmatrix} x_1 & y_1 \\ x_2+ax_1 & y_2+ay_1 \end{vmatrix} = x_1y_2 + ax_1y_1 - x_2y_1 - ay_1x_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.

b) is similar.

1C-6 Use the Laplace expansion by the cofactors of the first row.

1C-7 The heads of two vectors are on the unit circle. The area of the parallelogram they span is biggest when the vectors are perpendicular, since area $= ab\sin\theta = 1\cdot 1\cdot\sin\theta$, and $\sin\theta$ has its maximum when $\theta = \pi/2$.

Therefore the maximum value of $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} =$ area of unit square $= 1$.

1C-9 $PQ = (0,-1,2)$, $PR = (0,1,-1)$, $PS = (1,2,1)$;

$$\text{volume parallelepiped} = \pm\begin{vmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \pm(-1) = 1\ .$$

vol. tetrahedron $= \frac{1}{3}$(base)(ht.) $= \frac{1}{3}\cdot\frac{1}{2}$ (p'piped base)(ht.) $= \frac{1}{6}$(vol. p'piped) $= 1/6$.

1C-10 Thinking of them all as origin vectors, **A** lies in the plane of **B** and **C**, therefore the volume of the parallelepiped spanned by the three vectors is zero.

1D. Cross Product

1D-1 a) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{vmatrix} = 3\,\mathbf{i} - (-3)\,\mathbf{j} + 3\,\mathbf{k}$ b) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 1 & 1 & -1 \end{vmatrix} = 3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$

1D-2 $PQ = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $PR = -3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, so $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -3 & 1 & -2 \end{vmatrix} = -1\,\mathbf{i} + 5\,\mathbf{j} + 4\,\mathbf{k}$;

area of the triangle $= \frac{1}{2}|PQ \times PR| = \frac{1}{2}\sqrt{42}$.

1D-3 We get a third vector (properly oriented) perpendicular to $\mathbf{A}$ and $\mathbf{B}$ by using $\mathbf{A} \times \mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = -\,\mathbf{i} - 2\mathbf{j} + 5\,\mathbf{k}$$

Make these unit vectors: $\mathbf{i}' = \mathbf{A}/\sqrt{5}$, $\mathbf{j}' = \mathbf{B}/\sqrt{6}$, $\mathbf{k}' = (-\,\mathbf{i} - 2\,\mathbf{j} + 5\,\mathbf{k})/\sqrt{30}$.

1D-4 $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$; $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

1D-5 For both, use $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin\theta$, where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

a) $\sin\theta = 1 \Rightarrow \theta = \pi/2$; the two vectors are orthogonal.

b) $|\mathbf{A}||\mathbf{B}| \sin\theta = |\mathbf{A}||\mathbf{B}| \cos\theta$, therefore $\tan\theta = 1$, so $\theta = \pi/4$

1D-6 Taking the cube so P is at the origin and three coterminous edges are $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the three diagonals of the faces are $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$, $\mathbf{i} + \mathbf{k}$, so

$$\text{volume of parallelepiped spanned by diagonals} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2\,.$$

1D-7 We have $PQ = (-2, 1, 1)$, $PR = (-1, 0, 1)$, $PS = (2, 1, -2)$;

$$\text{volume parallelepiped} = \pm \begin{vmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \pm 1 = 1; \quad \text{volume tetrahedron} = \frac{1}{6}.$$

1D-8 One determinant has rows in the order $\mathbf{A}, \mathbf{B}, \mathbf{C}$, the other represents $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$, and therefore has its rows in the order $\mathbf{C}, \mathbf{A}, \mathbf{B}$.

To change the first determinant into the second, interchange the second and third rows, then the first and second row; each interchange multiplies the determinant by -1, (See rule **D-1** in reading D1), therefore the net effect of two successive interchanges is to leave its value unchanged; thus $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.

1D-9 a) Lift the triangle up into the plane $z = 1$, so its vertices are at the three points $P_i = (x_i, y_i, 1)$, $i = 1, 2, 3$.

volume tetrahedron $OP_1P_2P_3 = \frac{1}{3}$(height)(base) $= \frac{1}{3}1\cdot$(area of triangle);

volume tetrahedron $OP_1P_2P_3 = \frac{1}{6}$ (volume parallelepiped spanned by the OP_i)

$= \frac{1}{6}$ (determinant);

Therefore: area of triangle $= \frac{1}{2}$ (determinant)

b) Subtracting the first row from the second, and the first row from the third does not alter the value of the determinant, (see rule **D-4** in reading D1), and gives

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix},$$

using the Laplace expansion by the cofactors of the last column; but this 2×2 determinant gives the area of the parallelogram spanned by the vectors representing two sides of the plane triangle, and the triangle has half this area.

1E. Lines and Planes

1E-1 a) $(x-2) + 2y - 2(z+1) = 0 \quad \Rightarrow \quad x + 2y - 2z = 4.$

b) $\mathbf{N} = (1,1,0) \times (2,-1,3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & -1 & 3 \end{vmatrix} = 3\,\mathbf{i} - 3\,\mathbf{j} - 3\,\mathbf{k}$

Removing the common factor 3, the equation is $x - y - z = 0$.

c) Calling the points respectrively P, Q, R, we have $PQ = (1,-1,1)$, $PR = (-2,3,1)$;

$\mathbf{N} = PQ \times PR = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ -2 & 3 & 1 \end{vmatrix} = -4\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}$

Equation (through $P : (1,0,1)$): $-4(x-1) - 3y + (z-1) = 0$, or $-4x - 3y + z = -3$.

d) $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1.$

e) $\mathbf{N}$ must be perpendicular to both $\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$ and $PQ = (-1,1,0)$. Therefore

$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = -2\,\mathbf{i} - 2\,\mathbf{j}$; a plane through $(1,0,1)$ with this $\mathbf{N}$ is then $x + y = 1$.

1E-2 The dihedral angle between two planes is the same as the angle θ between their normal vectors. The normal vectors to the planes are respectively $(2,-1,1)$ and $(1,1,2)$; therefore $\cos\theta = \dfrac{3}{\sqrt{6}\sqrt{6}} = \dfrac{1}{2}$, so that $\theta = 60^o$ or $\pi/3$.

1E-3 a) $x = 1 + 2t,\ y = -t,\ z = -1 + 3t$.

b) $x = 2 + t,\ y = -1 - t,\ z = -1 + 2t$, since the line has the direction of the normal to the plane.

c) The direction vector of the line should be parallel to the plane, i.e., perpendicular to its normal vector $\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$; so the answer is

$x = 1 + at,\ y = 1 + bt,\ z = 1 + ct$, where $a + 2b - c = 0$, a, b, c not all 0, or better,

$x = 1 + at,\ y = 1 + bt,\ z = 1 + (a + 2b)t$ for any constants a, b.

1E-4 The line has direction vector $PQ = (2, -1, 1)$, so its parametric equations are:

$$x = 2t,\ y = 1 - t,\ z = 2 + t.$$

Substitute these into the equation of the plane to find a point that lies in both the line and the plane:

$$2t + 4(1 - t) + (2 + t) = 4, \quad \text{or} \quad -t + 6 = 4;$$

therefore $t = 2$, and the point is (substituting into the parametric equations): $(4, -1, 4)$.

1E-5 The line has the direction of the normal to the plane, so its parametric equations are

$$x = 1 + t,\ y = 1 + 2t,\ z = -1 - t;$$

substituting, it intersects the plane when

$$2(1 + t) - (1 + 2t) + (-1 - t) = 1, \quad \text{or} \quad -t = 1;$$

therefore, at $(0, 1, 0)$.

1E-6 Let $P_0 : (x_0, y_0, z_0)$ be a point on the plane, and $\mathbf{N} = (a, b, c)$ be a normal vector to the plane. The distance we want is the length of that origin vector which is perpendicular to the plane; but this is exactly the component of OP_0 in the direction of $\mathbf{N}$. So we get (choose the sign which makes it positive):

$$\text{distance from point to plane} = \pm\, OP_0 \cdot \frac{\mathbf{N}}{|\mathbf{N}|} = \pm\, (x_0, y_0, z_0) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_0 + by_0 + cz_0|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}};$$

the last equality holds since the point satisfies the equation of the plane.

1F. Matrix Algebra

1F-3 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{pmatrix}$

We want all four entries of the product to be zero; this gives the equations:

$$a^2 = -bc, \qquad b(a + d) = 0, \qquad c(a + d) = 0, \qquad d^2 = -bc.$$

case 1: $a + d \neq 0$; then $b = 0$ and $c = 0$; thus $a = 0$ and $d = 0$.

case 2: $a + d = 0$; then $d = -a,\ bc = -a^2$

Answer: $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $bc = -a^2$, i.e., $\begin{vmatrix} a & b \\ c & -a \end{vmatrix} = 0.$

1F-5 a) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}; \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

Guess: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$; Proof by induction:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}.$$

1F-8 a) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ d \\ g \end{pmatrix}$; $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ e \\ h \end{pmatrix}$; etc.

Answer: $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 0 & 1 \\ 1 & 4 & -1 \end{pmatrix}$ (See p. 13 for a more .

1F-9 For the entries of the product matrix $A \cdot A^T = C$, we have

$$c_{ij} = \begin{cases} 0 & \text{if } i \neq j; \\ 1 & \text{if } i = j, \end{cases} \qquad \text{since } A \cdot A^T = I.$$

On the other hand, by the definition of matrix multiplication,

$$c_{ij} = (i\text{-th row of } A)\cdot(j\text{-th column of } A^T) = (i\text{-th row of } A)\cdot(j\text{-th row of } A).$$

Since the right-hand sides of the two expressions for c_{ij} must be equal, when $j = i$ it shows that the i-th row has length 1; while for $j \neq i$, it shows that different rows are orthogonal to each other.

1G. Solving Square Systems; Inverse Matrices

1G-3 $M = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 3 & 2 \\ -2 & -1 & 1 \end{pmatrix}$; $M^T = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$; $A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$,

where M is the matrix of cofactors (watch the signs), M^T is its transpose (the adjoint matrix), and we calculated that $\det A = 5$, to get A^{-1} . Thus

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{5}\begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 0 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

1G-4 The system is $A\mathbf{x} = \mathbf{y}$; the solution is $\mathbf{x} = A^{-1}\mathbf{y}$, or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$;

written out, this is the system of equations

$$x_1 = \tfrac{3}{5}y_1 + \tfrac{1}{5}y_2 - \tfrac{2}{5}y_3, \qquad x_2 = -\tfrac{1}{5}y_1 + \tfrac{3}{5}y_2 - \tfrac{1}{5}y_3, \qquad x_3 = \tfrac{1}{5}y_1 + \tfrac{2}{5}y_2 + \tfrac{1}{5}y_3 .$$

1G-5 Using in turn the associative law, definition of the inverse, and identity law,

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Similarly, $(AB)(B^{-1}A^{-1}) = I$. Therefore, $B^{-1}A^{-1}$ is the inverse to AB.

1H. Theorems about Square Systems

1H-1 b) $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 2;$ $\qquad x = \dfrac{1}{|A|}\begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{vmatrix} = \dfrac{1}{2}\cdot 4 = 2.$

1H-2 Using Cramer's rule, the determinants in the numerators for x, y, and z all have a column of zeros, therefore have the value zero, by the determinant law **D-2**.

1H-3 a) The condition for it to have a non-zero solution is $\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & c & 2 \end{vmatrix} = 0$; expanding,

$2 + 2c + 1 - (-1 + c - 4) = 0$, or $c = -8$.

b) $\begin{cases} (2-c)x + y = 0 \\ (-1-c)y = 0 \end{cases}$ has a nontrivial solution if $\begin{vmatrix} 2-c & 1 \\ 0 & -1-c \end{vmatrix} = 0$, i.e.,

if $(2-c)(-1-c) = 0$, or $c = 2$, $c = -1$.

c) Take $c = -8$. The equations say we want a vector (x_1, x_2, x_3) which is orthogonal to the three vectors

$$(1, -1, 2), \quad (2, 1, 1), \quad (-1, -8, 2)\ .$$

A vector orthogonal to the first two is $(1, -1, 1) \times (2, 1, 1) = (-2, 1, 3)$ (by calculation). And this is orthogonal to $(-1, -8, 2)$ also: $(-2, 1, 3) \cdot (-1, -8, 2) = 0$.

1H-5 If (x_0, y_0) is a solution, then $\begin{cases} a_1 x_0 + b_1 y_0 = c_1 \\ a_2 x_0 + b_2 y_0 = c_2 \end{cases}$.

Eliminating x_0 gives $(a_2 b_1 - a_1 b_2) y_0 = a_2 c_1 - a_1 c_2$.

The left side is zero by hypothesis, so the right side is also zero: $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = 0$.

Conversely, if this holds, then a solution is $x_0 = \dfrac{c_1}{a_1}$, $y_0 = 0$ (or $x_0 = \dfrac{c_2}{a_2}$, if $a_1 = 0$).

1H-7 a) $\begin{cases} a\cos x_1 + b\sin x_1 &= y_1 \\ a\cos x_2 + b\sin x_2 &= y_2; \end{cases}$ has a unique solution if $\begin{vmatrix} \cos x_1 & \sin x_1 \\ \cos x_2 & \sin x_2 \end{vmatrix} \neq 0$, i.e.,

if $\cos x_1 \sin x_2 - \cos x_2 \sin x_1 \neq 0$, or equivalently, $\sin(x_2 - x_1) \neq 0$, and this last holds if and only if $x_2 - x_1 \neq n\pi$, for any integer n.

b) Since $\cos(x + n\pi) = (-1)^n \cos x$ and $\sin(x + n\pi) = (-1)^n \sin x$, the equations are solvable if and only if $y_2 = (-1)^n y_1$.

1I. Vector Functions and Parametric Equations

1I-1 Let $\mathbf{u} = \text{dir}\,(a\,\mathbf{i} + b\,\mathbf{j}) = \dfrac{a\,\mathbf{i} + b\,\mathbf{j}}{\sqrt{a^2 + b^2}}$, and $\mathbf{x}_0 = x_0\,\mathbf{i} + y_0\,\mathbf{j}$. Then

$$\mathbf{r}(t) = \mathbf{x}_0 + vt\,\mathbf{u}$$

1I-2 a) Since the motion is the reflection in the x-axis of the usual counterclockwise motion, $\mathbf{r} = a\,\cos(\omega t)\,\mathbf{i} - a\,\sin(\omega t)\,\mathbf{j}$. (This is a little special; part (b) illustrates an approach more generally applicable.)

b) The position vector is $\mathbf{r} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}$. At time $t = 0$, the angle $\theta = \pi/2$; then it decreases linearly at the rate ω. Therefore $\theta = \pi/2 - \omega t$; substituting and then using the trigonometric identities for $\cos(A + B)$ and $\sin(A + B)$, we get

$$\mathbf{r} = a\cos(\pi/2 - \omega t)\,\mathbf{i} + a\sin(\pi/2 - \omega t)\,\mathbf{j} = a\sin\omega t\,\mathbf{i} + a\cos\omega t\,\mathbf{j}$$

(In retrospect, we could have given another "special" derivation by observing that this motion is the reflection in the diagonal line $y = x$ of the usual counterclockwise motion starting at $(a, 0)$, so we get its position vector $\mathbf{r}(t)$ by interchanging the x and y in the usual position vector function $\mathbf{r} = \cos(\omega t)\,\mathbf{i} + \sin(\omega t)\,\mathbf{j}$.)

1I-3 a) $x = 2\cos^2 t,\ y = \sin^2 t,$ so $x + 2y = 2$; only the part of this line between $(0, 1)$ and $(2, 0)$ is traced out, back and forth.

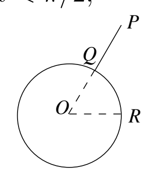
b) $x = \cos 2t,\ y = \cos t$; eliminating t to get the xy-equation, we have

$$\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 \quad \Rightarrow \quad x = 2y^2 - 1\ ;$$

only the part of this parabola between $(1, 1)$ and $(1, -1)$ is traced out, back and forth.

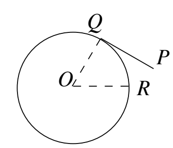
c) $x = t^2 + 1,\ y = t^3$; eliminating t, we get $y^2 = (x - 1)^3$; the entire curve is traced out as t increases, with y going from $-\infty$ to ∞.

d) $x = \tan t,\ y = \sec t$; eliminate t via the trigonometric identity $\tan^2 t + 1 = \sec^2 t$, getting $y^2 - x^2 = 1$. This is a hyperbola; the upper branch is traced out for $-\pi/2 < t < \pi/2$, the lower branch for $\pi/2 < t < 3\pi/2$. Then it repeats.



1I-4 $OP = |OP| \cdot \text{dir}\, OP$; $\text{dir}\, OP = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$;
$|OP| = |OQ| + |QP| = a + a\theta$, since $|QP| = \text{arc}\ QR = a\theta$.

So $OP = a(1 + \theta)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$ or $x = a(1 + \theta)\cos\theta,\ y = a(1 + \theta)\sin\theta$.



1I-5 $OP = OQ + QP$; $OQ = a(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$
$QP = |QP|\ \text{dir}\ QP = a\theta(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})$, since $|QP| = \text{arc}\ QR = a\theta$
(cf. Exer. 1A-7a for dir QP)

Therefore, $OP = \mathbf{r} = a(\cos\theta + \theta\sin\theta)\,\mathbf{i} + a(\sin\theta - \theta\cos\theta)\,\mathbf{j}$

1I-6 a) $\mathbf{r}_1(t) = (10 - t)\,\mathbf{i}$ (hunter); $\mathbf{r}_2(t) = t\,\mathbf{i} + 2t\,\mathbf{j}$ (rabbit; note that $\mathbf{v} = \mathbf{i} + 2\,\mathbf{j}$, so indeed $|\mathbf{v}| = \sqrt{5}$)

$$\begin{aligned} \text{Arrow} = HA &= \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}, \text{ since the arrow has unit length} \\ &= \frac{(t-5)\,\mathbf{i} + t\,\mathbf{j}}{\sqrt{2t^2 - 10t + 25}}, \text{ after some algebra.} \end{aligned}$$

b) It is easier mathematically to minimize the square of the distance between hunter and rabbit, rather than the distance itself; you get the same t-value in either case.

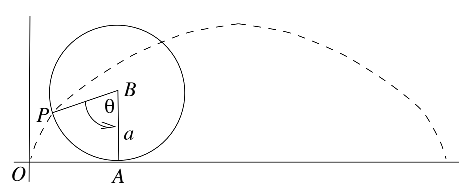
Let $f(t) = |\mathbf{r}_2 - \mathbf{r}_1|^2 = 2t^2 - 10t + 25$; then $f'(t) = 4t - 10 = 0$ when $t = 2.5$.

1I-7 $OP = OA + AB + BP$;

$OA = \text{arc}AP = a\theta$; $AB = a\,\mathbf{j}$;

$BP = a(-\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})$

Therefore, $OP = a(\theta - \sin\theta)\,\mathbf{i} + a(1 - \cos\theta)\,\mathbf{j}$.



1J. Vector Derivatives

1J-1 a) $\mathbf{r} = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$; $\quad \mathbf{v} = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}$, $\quad |\mathbf{v}| = \sqrt{e^{2t} + e^{-2t}}$, $\quad \mathbf{T} = \dfrac{e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$,

$\mathbf{a} = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$

b) $\mathbf{r} = t^2\,\mathbf{i} + t^3\,\mathbf{j}$; $\quad \mathbf{v} = 2t\,\mathbf{i} + 3t^2\,\mathbf{j}$; $\quad |\mathbf{v}| = t\sqrt{4 + 9t^2}$; $\quad \mathbf{T} = \dfrac{2\,\mathbf{i} + 3t\,\mathbf{j}}{\sqrt{4 + 9t^2}}$;

$\mathbf{a} = 2\,\mathbf{i} + 6t\,\mathbf{j}$

c) $\mathbf{r} = (1 - 2t^2)\,\mathbf{i} + t^2\,\mathbf{j} + (-2 + 2t^2)\,\mathbf{k}$; $\quad \mathbf{v} = 2t(-2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k})$; $\quad |\mathbf{v}| = 6t$;

$\mathbf{T} = \frac{1}{3}(-2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k})$; $\quad \mathbf{a} = 2(-2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k})$

1J-2 a) $\mathbf{r} = \dfrac{1}{1+t^2}\,\mathbf{i} + \dfrac{t}{1+t^2}\,\mathbf{j}$; $\quad \mathbf{v} = \dfrac{-2t\,\mathbf{i} + (1-t^2)\,\mathbf{j}}{(1+t^2)^2}$; $\quad |\mathbf{v}| = \dfrac{1}{1+t^2}$; $\quad \mathbf{T} = \dfrac{-2t\,\mathbf{i} + (1-t^2)\,\mathbf{j}}{1+t^2}$

b) $|\mathbf{v}|$ is largest when $t = 0$, therefore at the point $(1, 0)$. There is no point at which $|\mathbf{v}|$ is smallest; as $t \to \infty$ or $t \to -\infty$, the point $P \to (0, 0)$, and $|\mathbf{v}| \to 0$.

c) The position vector shows $y = tx$, so $t = y/x$; substituting into $x = 1/(1 + t^2)$ yields after some algebra the equation $x^2 + y^2 - x = 0$; completing the square gives the equation $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$, which is a circle with center at $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$.

1J-3 $\dfrac{d}{dt}(x_1y_1 + x_2y_2) = \begin{cases} x_1'y_1 + x_1y_1' + \\ x_2'y_2 + x_2y_2' \end{cases}$

Adding the columns, we get: $(x_1, x_2)' \cdot (y_1, y_2) + (x_1, x_2) \cdot (y_1, y_2)' = \dfrac{d\mathbf{r}}{dt} \cdot \mathbf{s} + \mathbf{r} \cdot \dfrac{d\mathbf{s}}{dt}$.

1J-4 a) Since P moves on a sphere, say of radius a,

$$x(t)^2 + y(t)^2 + z(t)^2 = a^2;$$

Differentiating,

$$2xx' + 2yy' + 2zz' = 0,$$

which says that $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \cdot x'\,\mathbf{i} + y'\,\mathbf{j} + z'\,\mathbf{k} = 0$ for all t, i.e., $\mathbf{r} \cdot \mathbf{r}' = 0$.

b) Since by hypothesis, $\mathbf{r}(t)$ has length a, for all t, we get the chain of implications

$$|\mathbf{r}| = a \quad \Rightarrow \quad \mathbf{r} \cdot \mathbf{r} = a^2 \quad \Rightarrow \quad 2r \cdot \frac{d\mathbf{r}}{dt} = 0 \quad \Rightarrow \quad \mathbf{r} \cdot \mathbf{v} = 0.$$

c) Using first the result in Exercise **1J-3**, then $\mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2$, we have

$$\mathbf{r} \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{d}{dt}\mathbf{r} \cdot \mathbf{r} = 0 \quad \Rightarrow \quad \mathbf{r} \cdot \mathbf{r} = c, \text{ a constant}, \quad \Rightarrow \quad |\mathbf{r}| = \sqrt{c},$$

which shows that the head of $\mathbf{r}$ moves on a sphere of radius $\sqrt{c}$.

1J-5 a) $|\mathbf{v}| = c \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{v} = c^2 \quad \Rightarrow \quad \dfrac{d}{dt}\mathbf{v} \cdot \mathbf{v} = 2\mathbf{v} \cdot \mathbf{a} = 0$, by **1J-3**. Therefore the velocity and acceleration vectors are perpendicular.

b) $\mathbf{v} \cdot \mathbf{a} = 0 \quad \Rightarrow \quad \frac{d}{dt}\mathbf{v} \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{v} = a \quad \Rightarrow \quad |\mathbf{v}| = \sqrt{a}$, which shows the speed is constant.

1J-6 a) $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$; $\quad \mathbf{v} = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k}$; $\quad |\mathbf{v}| = \sqrt{a^2 + b^2}$;

$\mathbf{T} = \dfrac{\mathbf{v}}{\sqrt{a^2 + b^2}}$; $\quad \mathbf{a} = -a(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$

b) By direct calculation using the components, we see that $\mathbf{v}\cdot\mathbf{a} = 0$; this also follows theoretically from Exercise **1J-5b**, since the speed is constant.

1J-7 a) The criterion is $|\mathbf{v}| = 1$; namely, if we measure arclength s so $s = 0$ when $t = 0$, then since s increases with t,

$$|\mathbf{v}| = 1 \quad\Rightarrow\quad ds/dt = 1 \quad\Rightarrow\quad s = t + c \quad\Rightarrow\quad s = t\ .$$

b) $\mathbf{r} = (x_0 + at)\,\mathbf{i} + (y_0 + at)\,\mathbf{j}$; $\mathbf{v} = a(\,\mathbf{i} + \mathbf{j}\,)$; $|\mathbf{v}| = a\sqrt{2}$; therefore choose $a = 1/\sqrt{2}$.

c) Choose a and b to be non-negative numbers such that $a^2 + b^2 = 1$; then $\mathbf{v}| = 1$.

1J-8 a) Let $\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$; then $u(t)\mathbf{r}(t) = ux\,\mathbf{i} + uy\,\mathbf{j}$, and differentiation gives

$(u\mathbf{r})' = (ux)'\,\mathbf{i} + (uy)'\,\mathbf{j} = (u'x+ux')\,\mathbf{i} + (u'y+uy')\,\mathbf{j} = u'(x\,\mathbf{i} + y\,\mathbf{j}) + u(x'\,\mathbf{i} + y'j) = u'\mathbf{r} + u\mathbf{r}'$.

b)
$$\begin{aligned}\frac{d}{dt}\,e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j}) &= e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j}) + e^t(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\\ &= e^t\big((\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j}\big).\end{aligned}$$

Therefore $|\mathbf{v}| \;=\; e^t\big|(\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j}\big| = 2e^t$, after calculation.

1J-9 a) $\mathbf{r} = 3\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} + 4\cos t\,\mathbf{k} \quad\Rightarrow\quad |\mathbf{r}| = \sqrt{25\cos^2 t + 25\sin^2 t} = 5.$

b) $\mathbf{v} = -3\sin t\,\mathbf{i} + 5\cos t\,\mathbf{j} - 4\sin t\,\mathbf{k}$; therefore $|\mathbf{v}| = \sqrt{25\cos^2 t + 25\sin^2 t} = 5$.

c) $\mathbf{a} = d\mathbf{v}/dt = -3\cos t\,\mathbf{i} - 5\sin t\,\mathbf{j} - 4\cos t\,\mathbf{k} = -\mathbf{r}$

d) By inspection, the x, y, z coordinates of p satisfy $4x - 3z = 0$, which is a plane through the origin.

e) Since by part (a) the point P moves on the surface of a sphere of radius 5 centered at the origin, and by part (d) also in a plane through the origin, its path must be the intersection of these two surfaces, which is a great circle of radius 5 on the sphere.

1J-10 a) Use the results of Exercise **1J-6**:

$$\mathbf{T} \;=\; \frac{-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k}}{\sqrt{a^2+b^2}}; \qquad |\mathbf{v}| \;=\; \sqrt{a^2+b^2}\ .$$

By the chain rule,

$$\left|\frac{d\mathbf{T}}{dt}\right| \;=\; \left|\frac{d\mathbf{T}}{ds}\right|\left|\frac{ds}{dt}\right|;$$

therefore

$$\frac{|-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}\,|}{\sqrt{a^2+b^2}} \;=\; \kappa\sqrt{a^2+b^2}\ ;$$

since the numerator on the left has the value $|a|$, we get

$$\kappa \;=\; \frac{|a|}{a^2+b^2}\ .$$

b) If $b = 0$, the helix is a circle of radius $|a|$ in the xy-plane, and $\kappa = \dfrac{1}{|a|}$.

1K. Kepler's Second Law

1K-1 $\dfrac{d}{dt}(x_1y_1 + x_2y_2) \;=\; \begin{cases} x_1'y_1 + x_1y_1' + \\ x_2'y_2 + x_2y_2' \end{cases}$

Adding the columns, we get: $\langle x_1, x_2\rangle' \cdot \langle y_1, y_2\rangle + \langle x_1, x_2\rangle \cdot \langle y_1, y_2\rangle' \;=\; \dfrac{d\mathbf{r}}{dt} \cdot \mathbf{s} + \mathbf{r} \cdot \dfrac{d\mathbf{s}}{dt}.$

1K-2 In two dimensions, $\mathbf{s}(t) = \langle x(t), y(t)\rangle$, $\mathbf{s}'(t) = \langle x'(t), y'(t)\rangle$.

Therefore $\mathbf{s}'(t) = \mathbf{0} \;\Rightarrow\; x'(t) = 0,\; y'(t) = 0 \;\Rightarrow\; x(t) = k_1,\; y(t) = k_2 \;\Rightarrow\; \mathbf{s}(t) = (k_1, k_2)$ where k_1, k_2 are constants.

1K-3 Since $\mathbf{F}$ is central, we have $\mathbf{F} = c\mathbf{r}$; using Newton's law, $\mathbf{a} = \mathbf{F}/m \;=\; (c/m)\mathbf{r}$; so

$$\begin{aligned} \mathbf{F} = c\mathbf{r} \quad &\Rightarrow \quad \mathbf{r} \times \mathbf{a} \;=\; \mathbf{r} \times \frac{c}{m}\mathbf{r} \;=\; \mathbf{0}, \\ &\Rightarrow \quad \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) \;=\; \mathbf{r} \times \mathbf{a} = \; \mathbf{0} \\ (*) \qquad &\Rightarrow \quad \mathbf{r} \times \mathbf{v} \;=\; \mathbf{K}, \quad \text{a constant vector, by Exercise K-2.} \end{aligned}$$

This last line (*) shows that $\mathbf{r}$ is perpendicular to $\mathbf{K}$, and therefore its head (the point P) lies in the plane through the origin which has $\mathbf{K}$ as normal vector. Also, since

$$\begin{aligned} |\mathbf{r} \times \mathbf{v}| \;&=\; 2\,\frac{dA}{dt}, \quad \text{by (2) in the problem statement,} \\ |\mathbf{r} \times \mathbf{v}| \;&=\; |K|, \quad \text{by (*)}, \end{aligned}$$

we conclude that

$$\frac{dA}{dt} \;=\; \frac{1}{2}\,|K|\;,$$

which shows the area is swept out at a constant rate.

2. Partial Differentiation

2A. Functions and Partial Derivatives

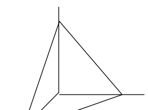
2A-1 In the pictures below, not all of the level curves are labeled. In (c) and (d), the picture is the same, but the labelings are different. In more detail:

b) the origin is the level curve 0; the other two unlabeled level curves are .5 and 1.5;

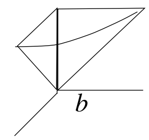
c) on the left, two level curves are labeled; the unlabeled ones are 2 and 3; the origin is the level curve 0;

d) on the right, two level curves are labeled; the unlabeled ones are -1 and -2; the origin is the level curve 1;

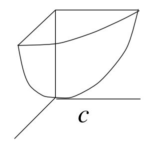
The crude sketches of the graph in the first octant are at the right.



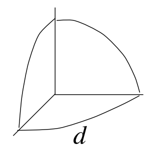
a



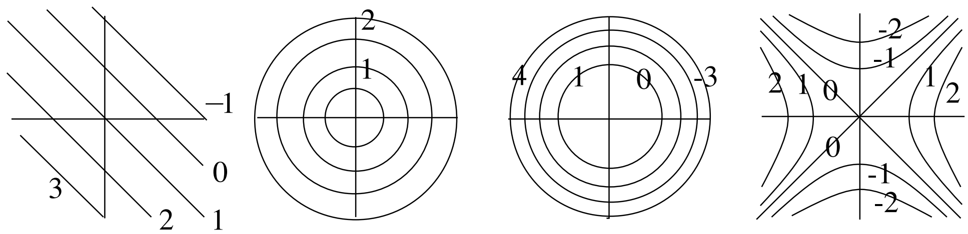
b



c



d



a *b* *c, d* *e*

2A-2 a) $f_x = 3x^2y - 3y^2, \quad f_y = x^3 - 6xy + 4y$ b) $z_x = \dfrac{1}{y}, \quad z_y = -\dfrac{x}{y^2}$

c) $f_x = 3\cos(3x + 2y), \quad f_y = 2\cos(3x + 2y)$

d) $f_x = 2xye^{x^2y}, \quad f_y = x^2e^{x^2y}$ e) $z_x = \ln(2x + y) + \dfrac{2x}{2x + y}, \quad z_y = \dfrac{x}{2x + y}$

f) $f_x = 2xz, \quad f_y = -2z^3, \quad f_z = x^2 - 6yz^2$

2A-3 a) both sides are $mnx^{m-1}y^{n-1}$

b) $f_x = \dfrac{y}{(x + y)^2}, \quad f_{xy} = (f_x)_y = \dfrac{x - y}{(x + y)^3}; \quad f_y = \dfrac{-x}{(x + y)^2}, \quad f_{yx} = \dfrac{-(y - x)}{(x + y)^3}.$

c) $f_x = -2x\sin(x^2 + y), \quad f_{xy} = (f_x)_y = -2x\cos(x^2 + y);$
$f_y = -\sin(x^2 + y), \quad f_{yx} = -\cos(x^2 + y) \cdot 2x.$

d) both sides are $f'(x)g'(y)$.

2A-4 $(f_x)_y = ax + 6y, \quad (f_y)_x = 2x + 6y$; therefore $f_{xy} = f_{yx} \Leftrightarrow a = 2$. By inspection, one sees that if $a = 2, \quad f(x, y) = x^2y + 3xy^2$ is a function with the given f_x and f_y.

2A-5

a) $w_x = ae^{ax}\sin ay, \quad w_{xx} = a^2e^{ax}\sin ay;$
$w_y = e^{ax}a\cos ay, \quad w_{yy} = e^{ax}a^2(-\sin ay);$ therefore $w_{yy} = -w_{xx}$.

b) We have $w_x = \dfrac{2x}{x^2 + y^2}, \quad w_{xx} = \dfrac{2(y^2 - x^2)}{(x^2 + y^2)^2}$. If we interchange x and y, the function $w = \ln(x^2 + y^2)$ remains the same, while w_{xx} gets turned into w_{yy}; since the interchange just changes the sign of the right hand side, it follows that $w_{yy} = -w_{xx}$.

2B. Tangent Plane; Linear Approximation

2B-1 a) $z_x = y^2, \quad z_y = 2xy$; therefore at (1,1,1), we get $z_x = 1, \quad z_y = 2$, so that the tangent plane is $z = 1 + (x - 1) + 2(y - 1)$, or $z = x + 2y - 2$.

b) $w_x = -y^2/x^2$, $w_y = 2y/x$; therefore at (1,2,4), we get $w_x = -4$, $w_y = 4$, so that the tangent plane is $w = 4 - 4(x-1) + 4(y-2)$, or $w = -4x + 4y$.

2B-2 a) $z_x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{z}$; by symmetry (interchanging x and y), $z_y = \dfrac{y}{z}$; then the tangent plane is $z = z_0 + \dfrac{x_0}{z_0}(x-x_0) + \dfrac{y_0}{z_0}(y-y_0)$, or $z = \dfrac{x_0}{z_0}x + \dfrac{y_0}{z_0}y$, since $x_0^2+y_0^2 = z_0^2$.

b) The line is $x = x_0 t$, $y = y_0 t$, $z = z_0 t$; substituting into the equations of the cone and the tangent plane, both are satisfied for all values of t; this shows the line lies on both the cone and tangent plane (this can also be seen geometrically).

2B-3 Letting x, y, z be respectively the lengths of the two legs and the hypotenuse, we have $z = \sqrt{x^2+y^2}$; thus the calculation of partial derivatives is the same as in **2B-2**, and we get $\Delta z \approx \dfrac{3}{5}\Delta x + \dfrac{4}{5}\Delta y$. Taking $\Delta x = \Delta y = .01$, we get $\Delta z \approx \dfrac{7}{5}(.01) = .014$.

2B-4 From the formula, we get $R = \dfrac{R_1R_2}{R_1+R_2}$. From this we calculate

$$\frac{\partial R}{\partial R_1} = \left(\frac{R_2}{R_1+R_2}\right)^2, \text{ and by symmetry, } \frac{\partial R}{\partial R_2} = \left(\frac{R_1}{R_1+R_2}\right)^2.$$

Substituting $R_1 = 1$, $R_2 = 2$ the approximation formula then gives $\Delta R = \dfrac{4}{9}\Delta R_1 + \dfrac{1}{9}\Delta R_2$. By hypothesis, $|\Delta R_i| \le .1$, for $i = 1, 2$, so that $|\Delta R| \le \dfrac{4}{9}(.1) + \dfrac{1}{9}(.1) = \dfrac{5}{9}(.1) \approx .06$; thus

$$R = \frac{2}{3} = .67 \ \pm .06.$$

2B-5 a) We have $f(x,y) = (x+y+2)^2$, $f_x = 2(x+y+2)$, $f_y = 2(x+y+2)$. Therefore

at $(0,0)$, $f_x(0,0) = f_y(0,0) = 4$, $f(0,0) = 4$; linearization is $4 + 4x + 4y$;

at $(1,2)$, $f_x(1,2) = f_y(1,2) = 10$, $f(1,2) = 25$;
linearization is $10(x-1) + 10(y-2) + 25$, or $10x + 10y - 5$.

b) $f = e^x \cos y$; $f_x = e^x \cos y$; $f_y = -e^x \sin y$.

linearization at $(0,0)$: $1 + x$; linearization at $(0, \pi/2)$: $-(y - \pi/2)$

2B-6 We have $V = \pi r^2 h$, $\dfrac{\partial V}{\partial r} = 2\pi r h$, $\dfrac{\partial V}{\partial h} = \pi r^2$; $\Delta V \approx \left(\dfrac{\partial V}{\partial r}\right)_0 \Delta r + \left(\dfrac{\partial V}{\partial h}\right)_0 \Delta h$. Evaluating the partials at $r = 2$, $h = 3$, we get

$$\Delta V \approx 12\pi \Delta r + 4\pi \Delta h.$$

Assuming the same accuracy $|\Delta r| \le \epsilon$, $|\Delta h| \le \epsilon$ for both measurements, we get

$$|\Delta V| \le 12\pi\,\epsilon + 4\pi\,\epsilon = 16\pi\,\epsilon, \quad \text{which is } < .1 \text{ if } \epsilon < \frac{1}{160\pi} < .002 \ .$$

2B-7 We have $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$; $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

Therefore at $(3,4)$, $r = 5$, and $\Delta r \approx \frac{3}{5}\Delta x + \frac{4}{5}\Delta y$. If $|\Delta x|$ and $|\Delta y|$ are both $\le .01$, then

$$|\Delta r| \le \tfrac{3}{5}|\Delta x| + \tfrac{4}{5}|\Delta y| = \tfrac{7}{5}(.01) = .014 \text{ (or .02)}.$$

Similarly, $\dfrac{\partial \theta}{\partial x} = \dfrac{-y}{x^2+y^2}$; $\dfrac{\partial \theta}{\partial y} = \dfrac{x}{x^2+y^2}$, so at the point $(3,4)$,

$$|\Delta\theta| \;\le\; |\tfrac{-4}{25}\Delta x| + |\tfrac{3}{25}\Delta y| \;\le\; \tfrac{7}{25}(.01) \;=\; .0028 \text{ (or } .003).$$

Since at $(3,4)$ we have $|r_y| > |r_x|$, r is more sensitive there to changes in y; by analogous reasoning, θ is more sensitive there to x.

2B-9 a) $w = x^2(y+1)$; $w_x = 2x(y+1) = 2$ at $(1,0)$, and $w_y = x^2 = 1$ at $(1,0)$; therefore w is more sensitive to changes in x around this point.

b) To first order approximation, $\Delta w \approx 2\Delta x + \Delta y$, using the above values of the partial derivatives.

If we want $\Delta w = 0$, then by the above, $2\Delta x + \Delta y = 0$, or $\Delta y/\Delta x = -2$.

2C. Differentials; Approximations

2C-1 a) $dw = \dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z}$ b) $dw = 3x^2y^2z\,dx + 2x^3yz\,dy + x^3y^2dz$

c) $dz = \dfrac{2y\,dx - 2x\,dy}{(x+y)^2}$ d) $dw = \dfrac{t\,du - u\,dt}{t\sqrt{t^2-u^2}}$

2C-2 The volume is $V = xyz$; so $dV = yz\,dx + xz\,dy + xy\,dz$. For $x = 5$, $y = 10$, $z = 20$,

$$\Delta V \approx dV = 200\,dx + 100\,dy + 50\,dz,$$

from which we see that $|\Delta V| \le 350(.1)$; therefore $V = 1000 \pm 35$.

2C-3 a) $A = \frac{1}{2}ab\sin\theta$. Therefore, $dA = \frac{1}{2}(b\sin\theta\,da + a\sin\theta\,db + ab\cos\theta\,d\theta)$.

b) $dA = \frac{1}{2}(2\cdot\frac{1}{2}\,da + 1\cdot\frac{1}{2}\,db + 1\cdot 2\cdot\frac{1}{2}\sqrt{3}\,d\theta) = \frac{1}{2}(da + \frac{1}{2}\,db + \sqrt{3}\,d\theta)$;
therefore most sensitive to θ, least senstitive to b, since $d\theta$ and db have respectively the largest and smallest coefficients.

c) $dA = \frac{1}{2}(.02 + .01 + 1.73(.02) \approx \frac{1}{2}(.065) \approx .03$

2C-4 a) $P = \dfrac{kT}{V}$; therefore $dP = \dfrac{k}{V}\,dT - \dfrac{kT}{V^2}\,dV$

b) $V\,dP + P\,dV = k\,dT$; therefore $dP = \dfrac{k\,dT - P\,dV}{V}$.

c) Substituting $P = kT/V$ into (b) turns it into (a).

2C-5 a) $-\dfrac{dw}{w^2} = -\dfrac{dt}{t^2} - \dfrac{du}{u^2} - \dfrac{dv}{v^2}$; therefore $dw = w^2\left(\dfrac{dt}{t^2} + \dfrac{du}{u^2} + \dfrac{dv}{v^2}\right)$.

b) $2u\,du + 4v\,dv + 6w\,dw = 0$; therefore $dw = -\dfrac{u\,du + 2v\,dv}{3w}$.

2D. Gradient; Directional Derivative

2D-1 a) $\nabla f = 3x^2\,\mathbf{i} + 6y^2\,\mathbf{j}$; $(\nabla f)_P = 3\,\mathbf{i} + 6\,\mathbf{j}$; $\left.\dfrac{df}{ds}\right|_{\mathbf{u}} = (3\,\mathbf{i} + 6\,\mathbf{j})\cdot\dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{2}} = -\dfrac{3\sqrt{2}}{2}$

b) $\nabla w = \dfrac{y}{z}\,\mathbf{i} + \dfrac{x}{z}\,\mathbf{j} - \dfrac{xy}{z^2}\,\mathbf{k}$; $(\nabla w)_P = -\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$; $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (\nabla w)_P\cdot\dfrac{\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}}{3} = -\dfrac{1}{3}$

c) $\nabla z = (\sin y - y\sin x)\,\mathbf{i} + (x\cos y + \cos x)\,\mathbf{j}$; $(\nabla z)_P = \mathbf{i} + \mathbf{j}$;

$$\left.\frac{dz}{ds}\right|_{\mathbf{u}} = (\,\mathbf{i} + \mathbf{j}\,)\cdot\frac{-3\,\mathbf{i} + 4\,\mathbf{j}}{5} = \frac{1}{5}$$

d) $\nabla w = \dfrac{2\,\mathbf{i} + 3\,\mathbf{j}}{2t+3u}$; $\qquad (\nabla w)_P = 2\,\mathbf{i} + 3\,\mathbf{j}$; $\qquad \left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (2\,\mathbf{i} + 3\,\mathbf{j}) \cdot \dfrac{4i - 3\,\mathbf{j}}{5} = -\dfrac{1}{5}$

e) $\nabla f = 2(u+2v+3w)(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,)$; $\qquad (\nabla f)_P = 4(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,)$

$$\left.\frac{df}{ds}\right|_{\mathbf{u}} = 4(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,) \cdot \frac{-2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}}{3} = -\frac{4}{3}$$

2D-2 a) $\nabla w = \dfrac{4\,\mathbf{i} - 3\,\mathbf{j}}{4x-3y}$; $\qquad (\nabla w)_P = 4\,\mathbf{i} - 3\,\mathbf{j}$

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (4\,\mathbf{i} - 3\,\mathbf{j}\,) \cdot \mathbf{u}$ has maximum 5, in the direction $\mathbf{u} = \dfrac{4\,\mathbf{i} - 3\,\mathbf{j}}{5}$,
and minimum -5 in the opposite direction.

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\pm\dfrac{3\,\mathbf{i} + 4\,\mathbf{j}}{5}$.

b) $\nabla w = \langle y+z, x+z, x+y\rangle$; $\qquad (\nabla w)_P = \langle 1, 3, 0\rangle$;

max $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = \sqrt{10}$, direction $\dfrac{\mathbf{i} + 3\,\mathbf{j}}{\sqrt{10}}$; $\qquad$ min $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = -\sqrt{10}$, direction $-\dfrac{\mathbf{i} + 3\,\mathbf{j}}{\sqrt{10}}$;

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\mathbf{u} = \pm\dfrac{-3\,\mathbf{i} + \mathbf{j} + c\,\mathbf{k}}{\sqrt{10+c^2}}$ (for all c)

c) $\nabla z = 2\sin(t-u)\cos(t-u)(\,\mathbf{i} - \mathbf{j}\,) = \sin 2(t-u)(\,\mathbf{i} - \mathbf{j}\,)$; $\qquad (\nabla z)_P = \mathbf{i} - \mathbf{j}$;

max $\left.\dfrac{dz}{ds}\right|_{\mathbf{u}} = \sqrt{2}$, direction $\dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$; $\qquad$ min $\left.\dfrac{dz}{ds}\right|_{\mathbf{u}} = -\sqrt{2}$, direction $-\dfrac{-\,\mathbf{i} + \mathbf{j}}{\sqrt{2}}$;

$\left.\dfrac{dz}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\pm\dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$

2D-3 a) $\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$; $\qquad (\nabla f)_P = \langle 4, 12, 36\rangle$; $\quad$ normal at P: $\langle 1, 3, 9\rangle$;
tangent plane at P: $x + 3y + 9z = 18$

b) $\nabla f = \langle 2x, 8y, 18z\rangle$; $\quad$ normal at P: $\langle 1, 4, 9\rangle$, $\quad$ tangent plane: $x + 4y + 9z = 14$.

c) $(\nabla w)_P = \langle 2x_0, 2y_0, -2z_0\rangle$; $\quad$ tangent plane: $x_0(x-x_0) + y_0(y-y_0) - z_0(z-z_0) = 0$,
or $x_0x + y_0y - z_0z = 0$, since $x_0^2 + y_0^2 - z_0^2 = 0$.

2D-4 a) $\nabla T = \dfrac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{x^2+y^2}$; $\qquad (\nabla T)_P = \dfrac{2\,\mathbf{i} + 4\,\mathbf{j}}{5}$;

T is increasing at P most rapidly in the direction of $(\nabla T)_P$, which is $\dfrac{\mathbf{i} + 2\,\mathbf{j}}{\sqrt{5}}$.

b) $|\nabla T| = \dfrac{2}{\sqrt{5}}$ = rate of increase in direction $\dfrac{\mathbf{i} + 2\,\mathbf{j}}{\sqrt{5}}$. Call the distance to go Δs, then

$$\frac{2}{\sqrt{5}}\,\Delta s = .20 \quad\Rightarrow\quad \Delta s = \frac{.2\sqrt{5}}{2} = \frac{\sqrt{5}}{10} \approx .22.$$

c) $\left.\dfrac{dT}{ds}\right|_{\mathbf{u}} = (\nabla T)_P \cdot \mathbf{u} = \dfrac{2\,\mathbf{i} + 4\,\mathbf{j}}{5} \cdot \dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \dfrac{6}{5\sqrt{2}}$;

$$\frac{6}{5\sqrt{2}}\,\Delta s = .12 \quad\Rightarrow\quad \Delta s = \frac{5\sqrt{2}}{6}\,(.12) \;\approx\; (.10)(\sqrt{2}) \;\approx\; .14$$

d) In the directions orthogonal to the gradient: $\pm\dfrac{2\,\mathbf{i} - \mathbf{j}}{\sqrt{5}}$

2D-5 a) isotherms = the level surfaces $x^2 + 2y^2 + 2z^2 = c$, which are ellipsoids.

b) $\nabla T = \langle 2x, 4y, 4z\rangle$; $(\nabla T)_P = \langle 2, 4, 4\rangle$; $|(\nabla T)_P| = 6$;
for most rapid decrease, use direction of $-(\nabla T)_P$: $-\frac{1}{3}\langle 1, 2, 2\rangle$

c) let Δs be distance to go; then $-6(\Delta s) = -1.2$; $\Delta s \approx .2$

d) $\left.\dfrac{dT}{ds}\right|_{\mathbf{u}} = (\nabla T)_P \cdot \mathbf{u} = \langle 2, 4, 4\rangle \cdot \dfrac{\langle 1, -2, 2\rangle}{3} = \dfrac{2}{3}$; $\dfrac{2}{3}\Delta s \approx .10 \Rightarrow \Delta s \approx .15.$

2D-6 $\nabla uv = \langle (uv)_x, (uv)_y\rangle = \langle uv_x + vu_x, uv_y + vu_y\rangle = \langle uv_x, uv_y\rangle + \langle vu_x + vu_y\rangle = u\nabla v + v\nabla u$

$\nabla(uv) = u\nabla v + v\nabla u \quad\Rightarrow\quad \nabla(uv)\cdot\mathbf{u} = u\nabla v\cdot\mathbf{u} + v\nabla u\cdot\mathbf{u} \quad\Rightarrow\quad \left.\dfrac{d(uv)}{ds}\right|_{\mathbf{u}} = u\left.\dfrac{dv}{ds}\right|_{\mathbf{u}} + v\left.\dfrac{du}{ds}\right|_{\mathbf{u}}.$

2D-7 At P, let $\nabla w = a\,\mathbf{i} + b\,\mathbf{j}$. Then

$$a\,\mathbf{i} + b\,\mathbf{j}\cdot\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}} = 2 \quad\Rightarrow\quad a + b = 2\sqrt{2}$$
$$a\,\mathbf{i} + b\,\mathbf{j}\cdot\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}} = 1 \quad\Rightarrow\quad a - b = \sqrt{2}$$

Adding and subtracting the equations on the right, we get $a = \frac{3}{2}\sqrt{2}$, $b = \frac{1}{2}\sqrt{2}$.

2D-8 We have $P(0,0,0) = 32$; we wish to decrease it to 31.1 by traveling the shortest distance from the origin $\mathbf{0}$; for this we should travel in the direction of $-(\nabla P)_{\mathbf{0}}$.

$\nabla P = \langle (y+2)e^z, (x+1)e^z, (x+1)(y+2)e^z\rangle$; $(\nabla P)_{\mathbf{0}} = \langle 2, 1, 2\rangle$. $|(\nabla P)_{\mathbf{0}}| = 3$.

Since $(-3)\cdot(\Delta s) = -.9 \Rightarrow \Delta s = .3$, we should travel a distance .3 in the direction of $-(\nabla P)_{\mathbf{0}}$. Since $|-\langle 2,1,2\rangle| = 3$, the distance .3 will be $\frac{1}{10}$ of the distance from $(0,0,0)$ to $(-2,-1,-2)$, which will bring us to $(-.2, -.1, -.2)$.

2D-9 In these, we use $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} \approx \dfrac{\Delta w}{\Delta s}$: we travel in the direction $\mathbf{u}$ from a given point P to the nearest level curve C; then Δs is the distance traveled (estimate it by using the unit distance), and Δw is the corresponding change in w (estimate it by using the labels on the level curves).

a) The *direction* of ∇f is perpendicular to the level curve at A, in the increasing sense (the "uphill" direction). The *magnitude* of ∇f is the directional derivative in that direction: from the picture, $\dfrac{\Delta w}{\Delta s} \approx \dfrac{1}{.5} = 2.$

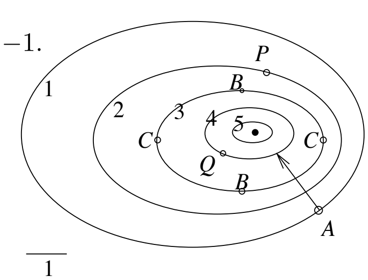
b), c) $\dfrac{\partial w}{\partial x} = \left.\dfrac{dw}{ds}\right|_{\mathbf{i}}$, $\dfrac{\partial w}{\partial y} = \left.\dfrac{dw}{ds}\right|_{\mathbf{j}}$, so B will be where $\mathbf{i}$ is tangent to the level curve and C where $\mathbf{j}$ is tangent to the level curve.

d) At P, $\dfrac{\partial w}{\partial x} = \left.\dfrac{dw}{ds}\right|_{\mathbf{i}} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{5/3} = -.6$; $\dfrac{\partial w}{\partial y} = \left.\dfrac{dw}{ds}\right|_{\mathbf{j}} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{1} = -1.$

e) If $\mathbf{u}$ is the direction of $\mathbf{i} + \mathbf{j}$, we have $\left.\dfrac{dw}{ds}\right|_{u} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{1}{.5} = 2$

f) If $\mathbf{u}$ is the direction of $\mathbf{i} - \mathbf{j}$, we have $\left.\dfrac{dw}{ds}\right|_{u} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{5/4} = -.8$

g) The gradient is 0 at a local extremum point: here at the point marked giving the location of the hilltop.



2E. Chain Rule

2E-1

a) (i) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = yz \cdot 1 + xz \cdot 2t + xy \cdot 3t^2 = t^5 + 2t^5 + 3t^5 = 6t^5$

(ii) $w = xyz = t^6$; $\quad \dfrac{dw}{dt} = 6t^5$

b) (i) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = 2x(-\sin t) - 2y(\cos t) = -4\sin t\cos t$

(ii) $w = x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t$; $\quad \dfrac{dw}{dt} = -2\sin 2t$

c) (i) $\dfrac{dw}{dt} = \dfrac{2u}{u^2+v^2}(-2\sin t) + \dfrac{2v}{u^2+v^2}(2\cos t) = -\cos t\sin t + \sin t\cos t = 0$

(ii) $w = \ln(u^2+v^2) = \ln(4\cos^2 t + 4\sin^2 t) = \ln 4$; $\quad \dfrac{dw}{dt} = 0.$

2E-2 a) The value $t = 0$ corresponds to the point $(x(0), y(0)) = (1, 0) = P$.

$$\left.\frac{dw}{dt}\right|_0 = \left.\frac{\partial w}{\partial x}\right|_P \left.\frac{dx}{dt}\right|_0 + \left.\frac{\partial w}{\partial y}\right|_P \left.\frac{dy}{dt}\right|_0 = -2\,\sin t + 3\,\cos t\Big]_0 \;=\; 3.$$

b) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = y(-\sin t) + x(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t.$

$\dfrac{dw}{dt} = 0$ when $2t = \dfrac{\pi}{2} + n\pi$, therefore when $t = \dfrac{\pi}{4} + \dfrac{n\pi}{2}$.

c) $t = 1$ corresponds to the point $(x(1), y(1), z(1)) = (1, 1, 1)$.

$$\left.\frac{df}{dt}\right|_1 = 1 \cdot \left.\frac{dx}{dt}\right|_1 - 1 \cdot \left.\frac{dy}{dt}\right|_1 + 2 \cdot \left.\frac{dz}{dt}\right|_1 = 1 \cdot 1 - 1 \cdot 2 + 2 \cdot 3 = 5.$$

d) $\dfrac{df}{dt} = 3x^2y\dfrac{dx}{dt} + (x^3+z)\dfrac{dy}{dt} + y\dfrac{dz}{dt} = 3t^4 \cdot 1 + 2x^3 \cdot 2t + t^2 \cdot 3t^2 = 10t^4.$

2E-3 a) Let $w = uv$, where $u = u(t)$, $v = v(t)$; $\quad \dfrac{dw}{dt} = \dfrac{\partial w}{\partial u}\dfrac{du}{dt} + \dfrac{\partial w}{\partial v}\dfrac{dv}{dt} = v\,\dfrac{du}{dt} + u\,\dfrac{dv}{dt}.$

b) $\dfrac{d(uvw)}{dt} = vw\dfrac{du}{dt} + uw\dfrac{dv}{dt} + uv\dfrac{dw}{dt}$; $\quad e^{2t}\sin t + 2te^{2t}\sin t + te^{2t}\cos t$

2E-4 The values $u = 1$, $v = 1$ correspond to the point $x = 0$, $y = 1$. At this point,

$$\frac{\partial w}{\partial u} \;=\; \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \;=\; 2 \cdot 2u + 3 \cdot v \;=\; 2 \cdot 2 + 3 \;=\; 7.$$

$$\frac{\partial w}{\partial v} \;=\; \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} \;=\; 2 \cdot (-2v) + 3 \cdot u \;=\; 2 \cdot (-2) + 3 \cdot 1 \;=\; -1.$$

2E-5 a) $w_r = w_x x_r + w_y y_r = w_x \cos\theta + w_y \sin\theta$

$w_\theta = w_x x_\theta + w_y y_\theta = w_x(-r\sin\theta) + w_y(r\cos\theta)$

Therefore,

$$\begin{aligned}
&(w_r)^2 + (w_\theta/r)^2 \\
&\quad = (w_x)^2(\cos^2\theta + \sin^2\theta) + (w_y)^2(\sin^2\theta + \cos^2\theta) + 2w_xw_y\cos\theta\sin\theta - 2w_xw_y\sin\theta\cos\theta \\
&\quad = (w_x)^2 + (w_y)^2.
\end{aligned}$$

b) The point $r=\sqrt{2}$, $\theta=\pi/4$ in polar coordinates corresponds in rectangular coordinates to the point $x=1,\ y=1$. Using the chain rule equations in part (a),

$$w_r = w_x\cos\theta + w_y\sin\theta; \qquad w_\theta = w_x(-r\sin\theta)+w_y(r\cos\theta)$$

but evaluating all the partial derivatives at the point, we get

$$w_r = 2\cdot\tfrac{1}{2}\sqrt{2} - 1\cdot\tfrac{1}{2}\sqrt{2} = \tfrac{1}{2}\sqrt{2}; \qquad \frac{w_\theta}{r} = 2(-\tfrac{1}{2})\sqrt{2} - \tfrac{1}{2}\sqrt{2} = -\tfrac{3}{2}\sqrt{2};$$

$$(w_r)^2 + \frac{1}{r}(w_\theta)^2 = \tfrac{1}{2}+\tfrac{9}{2} = 5; \qquad (w_x)^2+(w_y)^2 = 2^2+(-1)^2 = 5.$$

2E-6 $w_u = w_x\cdot 2u + w_y\cdot 2v$; $w_v = w_x\cdot(-2v)+w_y\cdot 2u$, by the chain rule.
Therefore

$$\begin{aligned}(w_u)^2+(w_v)^2 &= [4u^2(w_x)+4v^2(w_y)^2+4uvw_xw_y]+[4v^2(w_x)+4u^2(w_y)^2-4uvw_xw_y]\\ &= 4(u^2+v^2)[(w_x)^2+(w_y)^2].\end{aligned}$$

2E-7 By the chain rule, $f_u = f_xx_u+f_yy_u$, $f_v = f_xx_v+f_yy_v$; therefore

$$\langle f_u\ f_v\rangle = \langle f_x\ f_y\rangle\begin{pmatrix} x_u & x_v\\ y_u & y_v\end{pmatrix}$$

2E-8 a) By the chain rule for functions of one variable,

$$\frac{\partial w}{\partial x} = f'(u)\cdot\frac{\partial u}{\partial x} = f'(u)\cdot -\frac{y}{x^2}; \qquad \frac{\partial w}{\partial y} = f'(u)\cdot\frac{\partial u}{\partial y} = f'(u)\cdot\frac{1}{x};$$

Therefore,

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = f'(u)\cdot -\frac{y}{x} + f'(u)\cdot\frac{y}{x} = 0.$$

2F. Maximum-minimum Problems

2F-1 In these, denote by $D = x^2+y^2+z^2$ the square of the distance from the point (x,y,z) to the origin; then the point which minimizes D will also minimize the actual distance.

a) Since $z^2 = \dfrac{1}{xy}$, we get on substituting, $D = x^2+y^2+\dfrac{1}{xy}$. with x and y independent; setting the partial derivatives equal to zero, we get

$$D_x = 2x - \frac{1}{x^2y} = 0; \quad D_y = 2y-\frac{1}{y^2x} = 0; \quad \text{or} \quad 2x^2 = \frac{1}{xy}, \quad 2y^2 = \frac{1}{xy}.$$

Solving, we see first that $x^2 = \dfrac{1}{2xy} = y^2$, from which $y=\pm x$.

If $y=x$, then $x^4 = \frac{1}{2}$ and $x=y=2^{-1/4}$, and so $z = 2^{1/4}$; if $y=-x$, then $x^4 = -\frac{1}{2}$ and there are no solutions. Thus the unique point is $(1/2^{1/4}, 1/2^{1/4}, 2^{1/4})$.

b) Using the relation $x^2 = 1+yz$ to eliminate x, we have $D = 1+yz+y^2+z^2$, with y and z independent; setting the partial derivatives equal to zero, we get

$$D_y = 2y+z=0, \quad D_z = 2z+y=0;$$

solving, these equations only have the solution $y=z=0$; therefore $x=\pm1$, and there are two points: $(\pm1,0,0)$, both at distance 1 from the origin.

2F-2 Letting x be the length of the ends, y the length of the sides, and z the height, we have

$$\text{total area of cardboard} \quad A = 3xy+4xz+2yz, \qquad \text{volume} \quad V = xyz = 1.$$

Eliminating z to make the remaining variables independent, and equating the partials to zero, we get

$$A = 3xy + \frac{4}{y} + \frac{2}{x}; \qquad A_x = 3y - \frac{2}{x^2} = 0, \quad A_y = 3x - \frac{4}{y^2} = 0.$$

From these last two equations, we get

$$3xy = \frac{2}{x}, \quad 3xy = \frac{4}{y} \quad \Rightarrow \quad \frac{2}{x} = \frac{4}{y} \quad \Rightarrow \quad y = 2x$$

$$\Rightarrow \quad 3x^3 = 1 \quad \Rightarrow \quad x = \frac{1}{3^{1/3}}, \ \ y = \frac{2}{3^{1/3}}, \ \ z = \frac{1}{xy} = \frac{3^{2/3}}{2} = \frac{3}{2 \cdot 3^{1/3}};$$

therefore the proportions of the most economical box are $x : y : z = 1 : 2 : \frac{3}{2}$.

2F-5 The cost is $C = xy + xz + 4yz + 4xz$, where the successive terms represent in turn the bottom, back, two sides, and front; i.e., the problem is:

minimize: $C = xy + 5xz + 4yz$, with the constraint: $xyz = V = 2.5$

Substituting $z = V/xy$ into C, we get

$$C = xy + \frac{5V}{y} + \frac{4V}{x}; \qquad \frac{\partial C}{\partial x} = y - \frac{4V}{x^2}, \quad \frac{\partial C}{\partial y} = x - \frac{5V}{y^2}.$$

We set the two partial derivatives equal to zero and solving the resulting equations simultaneously, by eliminating y; we get $x^3 = \dfrac{16V}{5} = 8$, (using $V = 5/2$), so $x = 2$, $y = \frac{5}{2}$, $z = \frac{1}{2}$.

2G. Least-squares Interpolation

2G-1 Find $y = mx + b$ that best fits $(1,1)$, $(2,3)$, $(3,2)$.

$$\begin{aligned}
D &= (m+b-1)^2 + (2m+b-3)^2 + (3m+b-2)^2 \\
\frac{\partial D}{\partial m} &= 2(m+b-1) + 4(2m+b-3) + 6(3m+b-2) \ = \ 2(14m+6b-13) \\
\frac{\partial D}{\partial b} &= 2(m+b-1) + 2(2m+b-3) + 2(3m+b-2) \ = \ 2(6m+3b-6).
\end{aligned}$$

Thus the equations $\dfrac{\partial D}{\partial m} = 0$ and $\dfrac{\partial D}{\partial b} = 0$ are $\begin{cases} 14m + 6b = 13 \\ 6m + 3b = 6 \end{cases}$, whose solution is $m = \frac{1}{2}$, $b = 1$, and the line is $y = \frac{1}{2}x + 1$.

2G-4 $D = \sum_i (a + bx_i + cy_i - z_i)^2$. The equations are

$$\begin{aligned}
\partial D/\partial a &= \sum 2(a + bx_i + cy_i - z_i) \ = \ 0 \\
\partial D/\partial b &= \sum 2x_i(a + bx_i + cy_i - z_i) \ = \ 0 \\
\partial D/\partial c &= \sum 2y_i(a + bx_i + cy_i - z_i) \ = \ 0
\end{aligned}$$

Cancel the 2's; the equations become (on the right, $\mathbf{x} = [x_1, \dots, x_n]$, $\mathbf{1} = [1, \dots, 1]$, etc.)

$$\begin{aligned}
na + (\textstyle\sum x_i)b + (\sum y_i)c &= \textstyle\sum z_i & & & n\,a + (\mathbf{x}\cdot\mathbf{1})\,b + (\mathbf{y}\cdot\mathbf{1})\,c &= \mathbf{z}\cdot\mathbf{1} \\
(\textstyle\sum x_i)a + (\sum x_i^2)b + (\sum x_i y_i)c &= \textstyle\sum x_i z_i & \quad \text{or} \quad & & (\mathbf{x}\cdot\mathbf{1})\,a + (\mathbf{x}\cdot\mathbf{x})\,b + (\mathbf{x}\cdot\mathbf{y})\,c &= \mathbf{x}\cdot\mathbf{z} \\
(\textstyle\sum y_i)a + (\sum x_i y_i)b + (\sum y_i^2)c &= \textstyle\sum y_i z_i & & & (\mathbf{y}\cdot\mathbf{1})\,a + (\mathbf{x}\cdot\mathbf{y})\,b + (\mathbf{y}\cdot\mathbf{y})\,c &= \mathbf{y}\cdot\mathbf{z}
\end{aligned}$$

2H. Max-min: 2nd Derivative Criterion; Boundary Curves

2H-1

a) $f_x = 0:\ 2x - y = 3;\quad f_y = 0:\ -x - 4y = 3\qquad$ critical point: $(1, -1)$
$A = f_{xx} = 2;\ B = f_{xy} = -1;\ C = f_{yy} = -4;\qquad AC - B^2 = -9 < 0$; saddle point

b) $f_x = 0:\ 6x + y = 1;\quad f_y = 0:\ x + 2y = 2\qquad$ critical point: $(0, 1)$
$A = f_{xx} = 6;\ B = f_{xy} = 1;\ C = f_{yy} = 2;\qquad AC - B^2 = 11 > 0$; local minimum

c) $f_x = 0:\ 8x^3 - y = 0;\quad f_y = 0:\ 2y - x = 0;\qquad$ eliminating y, we get

$16x^3 - x = 0$, or $x(16x^2 - 1) = 0\ \Rightarrow\ x = 0,\ x = \frac{1}{4},\ x = -\frac{1}{4}$, giving the critical points

$$(0,0),\quad (\tfrac{1}{4}, \tfrac{1}{8}),\quad (-\tfrac{1}{4}, -\tfrac{1}{8}).$$

Since $f_{xx} = 24x^2,\quad f_{xy} = -1,\quad f_{yy} = 2$, we get for the three points respectively:

$(0,0):\ \Delta = -1$ (saddle); $\quad (\frac{1}{4}, \frac{1}{8}):\ \Delta = 2$ (minimum); $\qquad (-\frac{1}{4}, -\frac{1}{8}):\ \Delta = 2$ (minimum)

d) $f_x = 0:\quad 3x^2 - 3y = 0;\qquad f_y = 0:\quad -3x + 3y^2 = 0$. Eliminating y gives

$-x + x^4 = 0$, or $x(x^3 - 1) = 0\quad\Rightarrow\quad x = 0,\ y = 0$ or $x = 1,\ y = 1$.

Since $f_{xx} = 6x,\quad f_{xy} = -3,\quad f_{yy} = 6y$, we get for the two critical points respectively:

$(0,0):\quad AC - B^2 = -9$ (saddle); $\qquad (1,1):\quad AC - B^2 = 27$ (minimum)

e) $f_x = 0:\quad 3x^2(y^3 + 1) = 0;\qquad f_y = 0:\quad 3y^2(x^3 + 1) = 0$; solving simultaneously, we get from the first equation that either $x = 0$ or $y = -1$; finding in each case the other coordinate then leads to the two critical points $(0,0)$ and $(-1,-1)$.

Since $f_{xx} = 6x(y^3 + 1),\quad f_{xy} = 3x^2 \cdot 3y^2,\quad f_{yy} = 6y(x^3 + 1)$, we have

$(-1,-1):\quad AC - B^2 = -9$ (saddle); $\qquad (0,0):\quad AC - B^2 = 0$, test fails.

(By studying the behavior of $f(x,y)$ on the lines $y = mx$, for different values of m, it is possible to see that also $(0,0)$ is a saddle point.)

2H-3 The region R has no critical points; namely, the equations $f_x = 0$ and $f_y = 0$ are

$$2x + 2 = 0,\quad 2y + 4 = 0\quad\Rightarrow\quad x = -1,\ y = -2,$$

but this point is not in R. We therefore investigate the diagonal boundary of R, using the parametrization $x = t,\ y = -t$. Restricted to this line, $f(x,y)$ becomes a function of t alone, which we denote by $g(t)$, and we look for its maxima and minima.

$$g(t) = f(t, -t) = 2t^2 - 4t - 1;\qquad g'(t) = 4t - 2, \text{ which is 0 at } t = 1/2.$$

This point is evidently a minimum for $g(t)$; there is no maximum: $g(t)$ tends to ∞. Therefore for $f(x,y)$ on R, the minimum occurs at the point $(1/2, -1/2)$, and there is no maximum; $f(x,y)$ tends to infinity in different directions in R.

2H-4 We have $f_x = y - 1,\quad f_y = x - 1,$ so the only critical point is at $(1, 1)$.

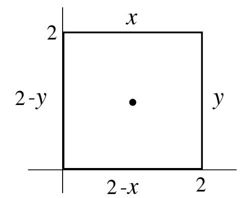
a) On the two sides of the boundary, the function $f(x, y)$ becomes respectively

$$y = 0:\quad f(x, y) = -x + 2; \qquad x = 0:\quad f(x, y) = -y + 2.$$

Since the function is linear and decreasing on both sides, it has no minimum points (informally, the minimum is $-\infty$). Since $f(1, 1) = 1$ and $f(x, x) = x^2 - 2x + 2 \to \infty$ as $x \to \infty$, the maximum of f on the first quadrant is ∞.

b) Continuing the reasoning of (a) to find the maximum and minimum points of $f(x, y)$ on the boundary, on the other two sides of the boundary square, the function $f(x, y)$ becomes

$$y = 2:\quad f(x, y) = x \qquad x = 2:\quad f(x, y) = y$$



Since $f(x, y)$ is thus increasing or decreasing on each of the four sides, the maximum and minimum points on the boundary square R can only occur at the four corner points; evaluating $f(x, y)$ at these four points, we find

$$f(0, 0) = 2; \qquad f(2, 2) = 2; \qquad f(2, 0) = 0; \qquad f(0, 2) = 0.$$

As in (a), since $f(1, 1) = 1$,

maximum points of f on R: $(0, 0)$ and $(2, 2)$; minimum points: $(2, 0)$ and $(0, 2)$.

c) The data indicates that $(1, 1)$ is probably a saddle point. Confirming this, we have $f_{xx} = 0,\ f_{xy} = 1,\ f_{yy} = 0$ for all x and y; therefore $AC - B^2 = -1 < 0$, so $(1, 1)$ is a saddle point, by the 2nd-derivative criterion.

2H-5 Since $f(x, y)$ is linear, it will not have critical points: namely, for all x and y we have $f_x = 1,\ f_y = \sqrt{3}$. So any maxima or minima must occur on the boundary circle.

We parametrize the circle by $x = \cos\theta,\ y = \sin\theta$; restricted to this boundary circle, $f(x, y)$ becomes a function of θ alone which we call $g(\theta)$:

$$g(\theta) = f(\cos\theta, \sin\theta) = \cos\theta + \sqrt{3}\sin\theta + 2.$$

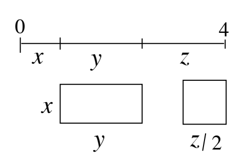
Proceeding in the usual way to find the maxima and minima of $g(\theta)$, we get

$$g'(\theta) = -\sin\theta + \sqrt{3}\cos\theta = 0, \qquad \text{or} \qquad \tan\theta = \sqrt{3}.$$

It follows that the two critical points of $g(\theta)$ are $\theta = \dfrac{\pi}{3}$ and $\dfrac{4\pi}{3}$; evaluating g at these two points, we get $g(\pi/3) = 4$ (the maximum), and $g(4\pi/3) = 0$ (the minimum).

Thus the maximum of $f(x, y)$ in the circular disc R is at $(1/2, \sqrt{3}/2)$, while the minimum is at $(-1/2, -\sqrt{3}/2)$.

2H-6 a) Since $z = 4 - x - y$, the problem is to find on R the maximum and minimum of the total area



$$f(x, y) = xy + \tfrac{1}{4}(4 - x - y)^2$$

where R is the triangle given by$R:\ 0 \le x, \quad 0 \le y, \quad x + y \le 4.$

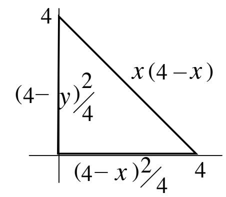
To find the critical points of $f(x, y)$, the equations $f_x = 0$ and $f_y = 0$ are respectively

$$y - \tfrac{1}{2}(4 - x - y) = 0; \quad x - \tfrac{1}{2}(4 - x - y) = 0,$$

which imply first that $x = y$, and from this, $x - \frac{1}{2}(4 - 2x)$; the unique solution is $x = 1,\ y = 1$.

The region R is a triangle, on whose sides $f(x,y)$ takes respectively the values

$$\text{bottom: } y=0;\ f=\tfrac{1}{4}(4-x)^2; \qquad \text{left side: } x=0;\ f=\tfrac{1}{4}(4-y)^2;$$
$$\text{diagonal } y=4-x;\ f=x(4-x).$$

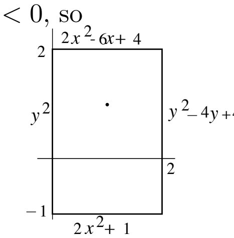


On the bottom and side, f is decreasing; on the diagonal, f has a maximum at $x=2,\ y=2$. Therefore we need to examine the three corner points and $(2,2)$ as candidates for maximum and minimum points, as well as the critical point $(1,1)$. We find

$$f(0,0)=4; \quad f(4,0)=0; \quad f(0,4)=0; \quad f(2,2)=4 \quad f(1,1)=2.$$

It follows that the critical point is just a saddle point; to get the maximum total area 4, make $x=y=0,\ z=4$, or $x=y=2,\ z=0$, either of which gives a point "rectangle" and a square of side 2; for the minimum total area 0, take for example $x=0, y=4, z=0$, which gives a "rectangle" of length 4 with zero area, and a point square.

b) We have $f_{xx}=\frac{1}{2},\ f_{xy}=\frac{3}{2},\ f_{yy}=\frac{1}{2}$ for all x and y; therefore $AC-B^2=-2<0$, so $(1,1)$ is a saddle point, by the 2nd-derivative criterion.



2H-7 a) $f_x=4x-2y-2, \quad f_y=-2x+2y$; setting these $=0$ and solving simultaneously, we get $x=1,\ y=1$, which is therefore the only critical point.

On the four sides of the boundary rectangle R, the function $f(x,y)$ becomes:

on $y=-1$: $f(x,y) = 2x^2+1$; on $y=2$: $f(x,y)=2x^2-6x+4$

on $x=0$: $f(x,y) = y^2$; on $x=2$: $f(x,y)=y^2-4y+4$

By one-variable calculus, $f(x,y)$ is increasing on the bottom and decresing on the right side; on the left side it has a minimum at $(0,0)$, and on the top a minimum at $(\frac{3}{2},2)$. Thus the maximum and minimum points on the boundary rectangle R can only occur at the four corner points, or at $(0,0)$ or $(\frac{3}{2},2)$. At these we find:

$$f(0,-1)=1; \quad f(0,2)=4; \quad f(2,-1)=9; \quad f(2,2)=0; \quad f(\tfrac{3}{2},2)=-\tfrac{1}{2}, \quad f(0,0)=0.$$

At the critical point $f(1,1)=-1$; comparing with the above, it is a minimum; therefore,

maximum point of $f(x,y)$ on R: $(2,-1)$ minimum point of $f(x,y)$ on R: $(1,1)$

b) We have $f_{xx}=4,\ f_{xy}=-2,\ f_{yy}=2$ for all x and y; therefore $AC-B^2=4>0$ and $A=4>0$, so $(1,1)$ is a minimum point, by the 2nd-derivative criterion.

2I. Lagrange Multipliers

2I-1 Letting $P:(x,y,z)$ be the point, in both problems we want to maximize $V=xyz$, subject to a constraint $f(x,y,z)=c$. The Lagrange equations for this, in vector form, are

$$\nabla(xyz)=\lambda\cdot\nabla f(x,y,z), \qquad f(x,y,z)=c.$$

a) Here $f=c$ is $x+2y+3z=18$; equating components, the Lagrange equations become

$$yz=\lambda, \quad xz=2\lambda, \quad xy=3\lambda; \qquad x+2y+3z=18.$$

To solve these symmetrically, multiply the left sides respectively by x, y, and z to make them equal; this gives

$$\lambda x=2\lambda y=3\lambda z, \quad \text{or} \quad x=2y=3z=6, \text{ since the sum is 18.}$$

We get therefore as the answer $x = 6, \quad y = 3, \quad z = 2$. This is a maximum point, since if P lies on the triangular boundary of the region in the first octant over which it varies, the volume of the box is zero.

b) Here $f = c$ is $x^2 + 2y^2 + 4z^2 = 12$; equating components, the Lagrange equations become

$$yz = \lambda \cdot 2x, \quad xz = \lambda \cdot 4y, \quad xy = \lambda \cdot 8z; \qquad x^2 + 2y^2 + 4z^2 = 12.$$

To solve these symmetrically, multiply the left sides respectively by x, y, and z to make them equal; this gives

$$\lambda \cdot 2x^2 = \lambda \cdot 4y^2 = \lambda \cdot 8z^2, \quad \text{or} \quad x^2 = 2y^2 = 4z^2 = 4, \quad \text{since the sum is 12.}$$

We get therefore as the answer $x = 2, \quad y = \sqrt{2}, \quad z = 1$. This is a maximum point, since if P lies on the boundary of the region in the first octant over which it varies (1/8 of the ellipsoid), the volume of the box is zero.

2I-2 Since we want to minimize $x^2 + y^2 + z^2$, subject to the constraint $x^3y^2z = 6\sqrt{3}$, the Lagrange multiplier equations are

$$2x = \lambda \cdot 3x^2y^2z, \quad 2y = \lambda \cdot 2x^3yz, \quad 2z = \lambda \cdot x^3y^2; \quad x^3y^2z = 6\sqrt{3}.$$

To solve them symmetrically, multiply the first three equations respectively by x, y, and z, then divide them through respectively by 3, 2, and 1; this makes the right sides equal, so that, after canceling 2 from every numerator, we get

$$\frac{x^2}{3} = \frac{y^2}{2} = z^2; \quad \text{therefore } x = z\sqrt{3},\ y = z\sqrt{2}.$$

Substituting into $x^3y^2z = 6\sqrt{3}$, we get $3\sqrt{3}z^3 \cdot 2z^2 \cdot z = 6\sqrt{3}$, which gives as the answer, $x = \sqrt{3},\ y = \sqrt{2},\ z = 1$.

This is clearly a minimum, since if P is near one of the coordinate planes, one of the variables is close to zero and therefore one of the others must be large, since $x^3y^2z = 6\sqrt{3}$; thus P will be far from the origin.

2I-3 Referring to the solution of 2F-2, we let x be the length of the ends, y the length of the sides, and z the height, and get

$$\text{total area of cardboard } A = 3xy + 4xz + 2yz, \qquad \text{volume } V = xyz = 1.$$

The Lagrange multiplier equations $\nabla A = \lambda \cdot \nabla(xyz)$; $xyz = 1$, then become

$$3y + 4z = \lambda yz, \quad 3x + 2z = \lambda xz, \quad 4x + 2y = \lambda xy, \quad xyz = 1.$$

To solve these equations for x, y, z, λ, treat them symmetrically. Divide the first equation through by yz, and treat the next two equations analogously, to get

$$3/z + 4/y = \lambda, \quad 3/z + 2/x = \lambda, \quad 4/y + 2/x = \lambda,$$

which by subtracting the equations in pairs leads to $3/z = 4/y = 2/x$; setting these all equal to k, we get $x = 2/k, y = 4/k, z = 3/k$, which shows the proportions using least cardboard are $x : y : z = 2 : 4 : 3$.

To find the actual values of x, y, and z, we set $1/k = m$; then substituting into $xyz = 1$ gives $(2m)(4m)(3m) = 1$, from which $m^3 = 1/24$, $m = 1/2 \cdot 3^{1/3}$, giving finally

$$x = \frac{1}{3^{1/3}}, \quad y = \frac{2}{3^{1/3}}, \quad z = \frac{3}{2 \cdot 3^{1/3}}.$$

2I-4 The equations for the cost C and the volume V are $xy+4yz+6xz = C$ and $xyz = V$. The Lagrange multiplier equations for the two problems are

a) $yz = \lambda(y+6z), \quad xz = \lambda(x+4z), \quad xy = \lambda(4y+6x); \qquad xy+4yz+6xz = 72$

b) $y+6z = \mu \cdot yz, \quad x+4z = \mu \cdot xz, \quad 4y+6x = \mu \cdot xy; \qquad xyz = 24$

The first three equations are the same in both cases, since we can set $\mu = 1/\lambda$. Solving the first three equations in (a) symmetrically, we multiply the equations through by x, y, and z respectively, which makes the left sides equal; since the right sides are therefore equal, we get after canceling the λ,

$$xy+6xz = xy+4yz = 4yz+6xz, \quad \text{which implies} \quad xy = 4yz = 6xz.$$

a) Since the sum of the three equal products is 72, by hypothesis, we get

$$xy = 24, \quad yz = 6, \quad xz = 4;$$

from the first two we get $x = 4z$, and from the first and third we get $y = 6z$, which lead to the solution $x = 4$, $y = 6$, $z = 1$.

b) Dividing $xy = 4yz = 6xz$ by xyz leads after cross-multiplication to $x = 4z$, $y = 6z$; since by hypothesis, $xyz = 24$, again this leads to the solution $x = 4$, $y = 6$, $z = 1$.

2J. Non-independent Variables

2J-1 a) $\left(\dfrac{\partial w}{\partial y}\right)_z$ means that x is the dependent variable; get rid of it by writing $w = (z-y)^2 + y^2 + z^2 = z + z^2$. This shows that $\left(\dfrac{\partial w}{\partial y}\right)_z = 0$.

b) To calculate $\left(\dfrac{\partial w}{\partial z}\right)_y$, once again x is the dependent variable; as in part (a), we have $w = z + z^2$ and so $\left(\dfrac{\partial w}{\partial z}\right)_y = 1 + 2z$.

2J-2 a) Differentiating $z = x^2 + y^2$ w.r.t. y: $\quad 0 = 2x\left(\dfrac{\partial x}{\partial y}\right)_z + 2y$; so $\left(\dfrac{\partial x}{\partial y}\right)_z = -\dfrac{y}{x}$;

By the chain rule, $\left(\dfrac{\partial w}{\partial y}\right)_z = 2x\left(\dfrac{\partial x}{\partial y}\right)_z + 2y = 2x\left(\dfrac{-y}{x}\right) + 2y = 0.$

Differentiating $z = x^2 + y^2$ with respect to z: $\quad 1 = 2x\left(\dfrac{\partial x}{\partial z}\right)_y$; so $\left(\dfrac{\partial x}{\partial z}\right)_y = \dfrac{1}{2x}$;

By the chain rule, $\left(\dfrac{\partial w}{\partial z}\right)_y = 2x\left(\dfrac{\partial x}{\partial z}\right)_y + 2z = 1 + 2z.$

b) Using differentials, $\quad dw = 2x\,dx + 2y\,dy + 2z\,dz, \qquad dz = 2x\,dx + 2y\,dy;$
since the independent variables are y and z, we eliminate dx by substracting the second equation from the first, which gives $\quad dw = 0\,dy + (1+2z)\,dz;$
therefore by **D2**, we get $\quad \left(\dfrac{\partial w}{\partial y}\right)_z = 0, \quad \left(\dfrac{\partial w}{\partial z}\right)_y = 1 + 2z.$

2J-3 a) To calculate $\left(\frac{\partial w}{\partial t}\right)_{x,z}$, we see that y is the dependent variable; solving for it, we get $y = \frac{zt}{x}$; using the chain rule, $\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^3\left(\frac{\partial y}{\partial t}\right)_{x,z} - z^2 = x^3\frac{z}{x} - z^2 = x^2z - z^2$.

b) Similarly, $\left(\frac{\partial w}{\partial z}\right)_{x,y}$ means that t is the dependent variable; since $t = \frac{xy}{z}$, we have by the chain rule, $\left(\frac{\partial w}{\partial z}\right)_{x,y} = -2zt - z^2\left(\frac{\partial t}{\partial z}\right)_{x,y} = -2zt - z^2\cdot\frac{-xy}{z^2} = -zt$.

2J-4 The differentials are calculated in equation (4).

a) Since x, z, t are independent, we eliminate dy by solving the second equation for $x\,dy$, substituting this into the first equation, and grouping terms:
$dw = 2x^2y\,dx + (x^2z - z^2)dt + (x^2t - 2zt)dz$, which shows by **D2** that $\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^2z - z^2$.

b) Since x, y, z are independent, we eliminate dt by solving the second equation for $z\,dt$, substituting this into the first equation, and grouping terms:
$dw = (3x^2y - zy)dx + (x^3 - zx)dy - zt\,dz$, which shows by **D2** that $\left(\frac{\partial w}{\partial z}\right)_{x,y} = -zt$.

2J-5 a) If $pv = nRT$, then $\left(\frac{\partial S}{\partial p}\right)_v = S_p + S_T\cdot\left(\frac{\partial T}{\partial p}\right)_v = S_p + S_T\cdot\frac{v}{nR}$.

b) Similarly, we have $\left(\frac{\partial S}{\partial T}\right)_v = S_T + S_p\cdot\left(\frac{\partial p}{\partial T}\right)_v = S_T + S_p\cdot\frac{nR}{v}$.

2J-6 a) $\left(\frac{\partial w}{\partial u}\right)_x = 3u^2 - v^2 - u\cdot 2v\left(\frac{\partial v}{\partial u}\right)_x = 3u^2 - v^2 - 2uv$.

$\left(\frac{\partial w}{\partial x}\right)_u = -u\cdot 2v\left(\frac{\partial v}{\partial x}\right)_u = -2uv$.

b) $dw = (3u^2 - v^2)du - 2uvdv$; $du = x\,dy + y\,dx$; $dv = du + dx$;
for both derivatives, u and x are the independent variables, so we eliminate dv, getting

$$dw = (3u^2 - v^2)du - 2uv(du + dx) = (3u^2 - v^2 - 2uv)du - 2uv\,dx,$$

whose coefficients by **D2** are $\left(\frac{\partial w}{\partial u}\right)_x$ and $\left(\frac{\partial w}{\partial x}\right)_u$.

2J-7 Since we need both derivatives for the gradient, we use differentials.

$df = 2dx + dy - 3dz$ at P; $dz = 2x\,dx + dy = 2\,dx + dy$ at P;

the independent variables are to be x and z, so we eliminate dy, getting

$df = 0\,dx - 2\,dz$ at the point $(x, z) = (1, 1)$. So $\nabla g = \langle 0, -2\rangle$ at $(1, 1)$.

2J-8 To calculate $\left(\frac{\partial w}{\partial r}\right)_\theta$, note that r and θ are independent. Therefore,
$\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{\partial w}{\partial r} + \frac{\partial w}{\partial x}\cdot\left(\frac{\partial x}{\partial r}\right)_\theta$. Now, $x = r\cos\theta$, so $\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$. Therefore
$\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{r}{\sqrt{r^2 - x^2}} + \frac{-x}{\sqrt{r^2 - x^2}}\cdot\cos\theta = \frac{r - x\cos\theta}{\sqrt{r^2 - x^2}}$

$$= \frac{r - r\cos^2\theta}{r|\sin\theta|} = \frac{r\sin^2\theta}{r|\sin\theta|} = |\sin\theta|.$$

2K. Partial Differential Equations

2K-1 $w = \frac{1}{2}\ln(x^2+y^2)$. If $(x,y) \neq (0,0)$, then

$$w_{xx} = \frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2},$$

$$w_{yy} = \frac{\partial}{\partial y}(w_y) = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2},$$

Therefore w satisfies the two-dimensional Laplace equation, $w_{xx} + w_{yy} = 0$; we exclude the point $(0,0)$ since $\ln 0$ is not defined.

2K-2 If $w = (x^2+y^2+z^2)^n$, then $\frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x}\big(2x \cdot n(x^2+y^2+z^2)^{n-1}\big)$

$$= 2n(x^2+y^2+z^2)^{n-1} + 4x^2n(n-1)(x^2+y^2+z^2)^{n-2}$$

We get w_{yy} and w_{zz} by symmetry; adding and combining, we get

$$w_{xx}+w_{yy}+w_{zz} = 6n(x^2+y^2+z^2)^{n-1} + 4(x^2+y^2+z^2)n(n-1)(x^2+y^2+z^2)^{n-2}$$

$= 2n(2n+1)(x^2+y^2+z^2)^{n-1}$, which is identically zero if $n=0$, or if $n=-1/2$.

2K-3 a) $w = ax^2+bxy+cy^2$; $\quad w_{xx} = 2a, \quad w_{yy} = 2c.$

$$w_{xx}+w_{yy} = 0 \quad\Rightarrow\quad 2a+2c = 0, \text{ or } c = -a.$$

Therefore all quadratic polynomials satisfying the Laplace equation are of the form

$$ax^2+bxy-ay^2 = a(x^2-y^2)+bxy;$$

i.e., linear combinations of the two polynomials $f(x,y) = x^2-y^2$ and $g(x,y) = xy$.

2K-4 The one-dimensional wave equation is $w_{xx} = \frac{1}{c^2}w_{tt}$. So

$$\begin{aligned} w = f(x+ct)+g(x-ct) \quad &\Rightarrow \quad w_{xx} = f''(x+ct)+g''(x-ct) \\ &\Rightarrow \quad w_t = cf'(x+ct)+-cg'(x-ct). \\ &\Rightarrow \quad w_{tt} = c^2f''(x+ct)+c^2g''(x-ct) = c^2w_{xx}, \end{aligned}$$

which shows w satisfies the wave equation.

2K-5 The one-dimensional heat equation is $w_{xx} = \frac{1}{\alpha^2}w_t$. So if $w(x,t) = \sin kxe^rt$, then

$$w_{xx} = e^{rt}\cdot k^2(-\sin kx) = -k^2\,w.$$
$$w_t = re^{rt}\sin kx = r\,w.$$

Therefore, we must have $-k^2w = \frac{1}{\alpha^2}r\,w$, or $r = -\alpha^2k^2$.

However, from the additional condition that $w=0$ at $x=1$, we must have

$$\sin k\; e^{rt} = 0\;;$$

Therefore $\sin k = 0$, and so $k = n\pi$, where n is an integer.

To see what happens to w as $t\to\infty$, we note that since $|\sin kx| \le 1$,

$$|w| = e^{rt}|\sin kx| \le e^{rt}.$$

Now, if $k \neq 0$, then $r = -\alpha^2k^2$ is negative and $e^{rt}\to 0$ as $t\to\infty$; therefore $|w|\to 0$.

Thus w will be a solution satisfying the given side conditions if $k = n\pi$, where n is a non-zero integer, and $r = -\alpha^2 k^2$.

3. Double Integrals

3A. Double integrals in rectangular coordinates

3A-1

a) Inner: $6x^2y + y^2\Big]_{y=-1}^{1} = 12x^2$; Outer: $4x^3\Big]_0^2 = 32$.

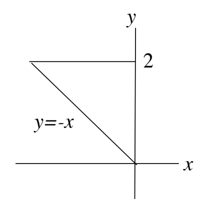
b) Inner: $-u\cos t + \frac{1}{2}t^2\cos u\Big]_{t=0}^{\pi} = 2u + \frac{1}{2}\pi^2\cos u$

Outer: $u^2 + \frac{1}{2}\pi^2\sin u\Big]_0^{\pi/2} = (\frac{1}{2}\pi)^2 + \frac{1}{2}\pi^2 = \frac{3}{4}\pi^2$.

c) Inner: $x^2y^2\Big]_{\sqrt{x}}^{x^2} = x^6 - x^3$; Outer: $\frac{1}{7}x^7 - \frac{1}{4}x^4\Big]_0^1 = \frac{1}{7} - \frac{1}{4} = -\frac{3}{28}$

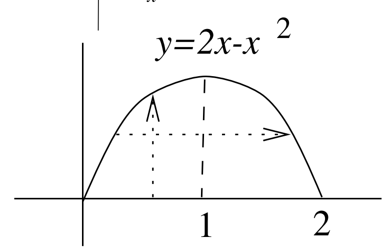
d) Inner: $v\sqrt{u^2+4}\Big]_0^u = u\sqrt{u^2+4}$; Outer: $\frac{1}{3}(u^2+4)^{3/2}\Big]_0^1 = \frac{1}{3}(5\sqrt{5} - 8)$

3A-2



a) (i) $\displaystyle\iint_R dy\,dx = \int_{-2}^{0}\int_{-x}^{2} dy\,dx$ (ii) $\displaystyle\iint_R dx\,dy = \int_0^2\int_{-y}^{0} dx\,dy$

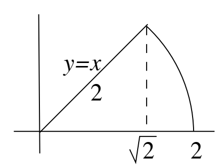
b) i) The ends of R are at 0 and 2, since $2x - x^2 = 0$ has 0 and 2 as roots.



$$\iint_R dydx = \int_0^2\int_0^{2x-x^2} dydx$$

ii) We solve $y = 2x - x^2$ for x in terms of y: write the equation as $x^2 - 2x + y = 0$ and solve for x by the quadratic formula, getting $x = 1 \pm \sqrt{1-y}$. Note also that the maximum point of the graph is $(1,1)$ (it lies midway between the two roots 0 and 2). We get

$$\iint_R dxdy = \int_0^1\int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dxdy,$$



c) (i) $\displaystyle\iint_R dy\,dx = \int_0^{\sqrt{2}}\int_0^x dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} dy\,dx$

(ii) $\displaystyle\iint_R dx\,dy = \int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} dx\,dy$

d) Hint: First you have to find the points where the two curves intersect, by solving simultaneously $y^2 = x$ and $y = x - 2$ (eliminate x).

The integral $\displaystyle\iint_R dy\,dx$ requires two pieces; $\displaystyle\iint_R dx\,dy$ only one.

3A-3 a) $\displaystyle\iint_R x\,dA = \int_0^2\int_0^{1-x/2} x\,dy\,dx$;

Inner: $x(1 - \frac{1}{2}x)$ Outer: $\frac{1}{2}x^2 - \frac{1}{6}x^3\Big]_0^2 = \frac{4}{2} - \frac{8}{6} = \frac{2}{3}$.

b) $\displaystyle\iint_R (2x+y^2)\,dA = \int_0^1 \int_0^{1-y^2} (2x+y^2)\,dx\,dy$

Inner: $x^2 + y^2x\Big]_0^{1-y^2} = 1-y^2;$ Outer: $y - \frac{1}{3}y^3\Big]_0^1 = \frac{2}{3}.$

c) $\displaystyle\iint_R y\,dA = \int_0^1 \int_{y-1}^{1-y} y\,dx\,dy$

Inner: $xy\Big]_{y-1}^{1-y} = y[(1-y)-(y-1)] = 2y-2y^2$ Outer: $y^2 - \frac{2}{3}y^3\Big]_0^1 = \frac{1}{3}.$

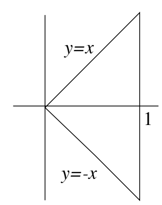
3A-4 a) $\displaystyle\iint_R \sin^2 x\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} \sin^2 x\,dy\,dx$

Inner: $y\,\sin^2 x\Big]_0^{\cos x} = \cos x \sin^2 x$ Outer: $\frac{1}{3}\sin^3 x\Big]-\pi/2^{\pi/2} = \frac{1}{3}(1-(-1)) = \frac{2}{3}.$

b) $\displaystyle\iint_R xy\,dA = \int_0^1 \int_{x^2}^{x} (xy)\,dy\,dx.$

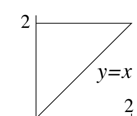
Inner: $\frac{1}{2}xy^2\Big]_{x^2}^{x} = \frac{1}{2}(x^3-x^5)$ Outer: $\displaystyle\frac{1}{2}\left(\frac{x^4}{4}-\frac{x^6}{6}\right)_0^1 = \frac{1}{2}\cdot\frac{1}{12} = \frac{1}{24}.$

c) The function x^2-y^2 is zero on the lines $y=x$ and $y=-x$, and positive on the region R shown, lying between $x=0$ and $x=1$. Therefore

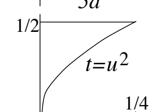


Volume $= \displaystyle\iint_R (x^2-y^2)\,dA = \int_0^1 \int_{-x}^{x} (x^2-y^2)\,dy\,dx.$

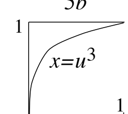
Inner: $x^2y - \frac{1}{3}y^3\Big]_{-x}^{x} = \frac{4}{3}x^3;$ Outer: $\frac{1}{3}x^4\Big]_0^1 = \frac{1}{3}.$



5a



5b



5c

3A-5 a) $\displaystyle\int_0^2\int_x^2 e^{-y^2}\,dy\,dx = \int_0^2\int_0^y e^{-y^2}\,dx\,dy = \int_0^2 e^{-y^2}\,y\,dy = -\frac{1}{2}e^{-y^2}\Big]_0^2 = \frac{1}{2}(1-e^{-4})$

b) $\displaystyle\int_0^{\frac{1}{4}}\int_{\sqrt{t}}^{\frac{1}{2}} \frac{e^u}{u}\,du\,dt = \int_0^{\frac{1}{2}}\int_0^{u^2} \frac{e^u}{u}\,dt\,du = \int_0^{\frac{1}{2}} u\,e^u\,du = (u-1)e^u\Big]_0^{\frac{1}{2}} = 1-\frac{1}{2}\sqrt{e}$

c) $\displaystyle\int_0^1\int_{x^{1/3}}^1 \frac{1}{1+u^4}\,du\,dx = \int_0^1\int_0^{u^3} \frac{1}{1+u^4}\,dx\,du = \int_0^1 \frac{u^3}{1+u^4}\,du = \frac{1}{4}\ln(1+u^4)\Big]_0^1 = \frac{\ln 2}{4}.$

3A-6 0; $\displaystyle 2\iint_S e^x\,dA$, $S =$ upper half of R; $\displaystyle 4\iint_Q x^2\,dA$, $Q =$ first quadrant

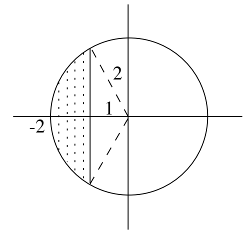
0; $\displaystyle 4\iint_Q x^2\,dA$; 0

3A-7 a) $x^4+y^4 \ge 0 \;\Rightarrow\; \dfrac{1}{1+x^4+y^4} \le 1$

b) $\displaystyle\iint_R \frac{x\,dA}{1+x^2+y^2} \le \int_0^1\int_0^1 \frac{x}{1+x^2}\,dx\,dy = \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{\ln 2}{2} < \frac{.7}{2}.$

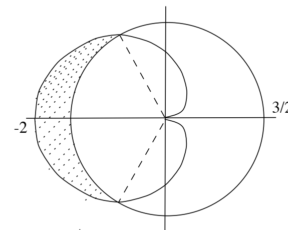
3B. Double Integrals in polar coordinates

3B-1



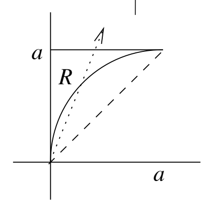
a) In polar coordinates, the line $x = -1$ becomes $r\cos\theta = -1$, or $r = -\sec\theta$. We also need the polar angle of the intersection points; since the right triangle is a 30-60-90 triangle (it has one leg 1 and hypotenuse 2), the limits are (no integrand is given):

$$\iint_R dr\, d\theta = \int_{2\pi/3}^{4\pi/3}\int_{-\sec\theta}^{2} dr\, d\theta.$$



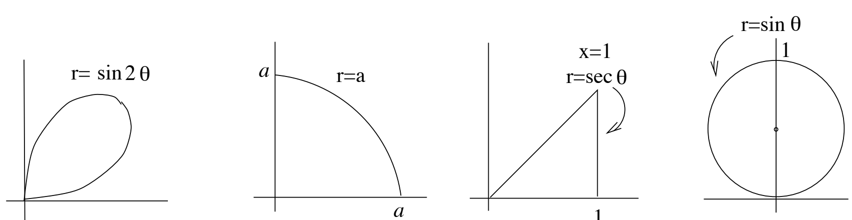
c) We need the polar angle of the intersection points. To find it, we solve the two equations $r = \frac{3}{2}$ and $r = 1 - \cos\theta$ simultanously. Eliminating r, we get $\frac{3}{2} = 1 - \cos\theta$, from which $\theta = 2\pi/3$ and $4\pi/3$. Thus the limits are (no integrand is given):

$$\iint_R dr\, d\theta = \int_{2\pi/3}^{4\pi/3}\int_{3/2}^{1-\cos\theta} dr\, d\theta.$$



d) The circle has polar equation $r = 2a\cos\theta$. The line $y = a$ has polar equation $r\sin\theta = a$, or $r = a\csc\theta$. Thus the limits are (no integrand):

$$\iint_R dr\, d\theta = \int_{\pi/4}^{\pi/2}\int_{2a\cos\theta}^{a\csc\theta} dr\, d\theta.$$



2a 2b 2c 2d

3B-2 a) $\displaystyle\int_0^{\pi/2}\int_0^{\sin 2\theta}\frac{r\, dr\, d\theta}{r} = \int_0^{\pi/2}\sin 2\theta\, d\theta = -\frac{1}{2}\cos 2\theta\Big]_0^{\pi/2} = -\frac{1}{2}(-1-1) = 1.$

b) $\displaystyle\int_0^{\pi/2}\int_0^{a}\frac{r}{1+r^2}\, dr\, d\theta = \frac{\pi}{2}\cdot\frac{1}{2}\ln(1+r^2)\Big]_0^{a} = \frac{\pi}{4}\ln(1+a^2).$

c) $\displaystyle\int_0^{\pi/4}\int_0^{\sec\theta}\tan^2\theta\cdot r\, dr\, d\theta = \frac{1}{2}\int_0^{\pi/4}\tan^2\theta\,\sec^2\theta\, d\theta = \frac{1}{6}\tan^3\theta\Big]_0^{\pi/4} = \frac{1}{6}.$

d) $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta}\frac{r}{\sqrt{1-r^2}}\, dr\, d\theta$

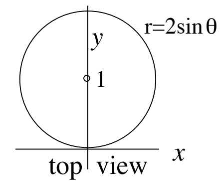
Inner: $\displaystyle -\sqrt{1-r^2}\Big]_0^{\sin\theta} = 1 - \cos\theta$ Outer: $\displaystyle \theta - \sin\theta\Big]_0^{\pi/2} = \pi/2 - 1.$

3B-3 a) the hemisphere is the graph of $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$, so we get

$$\iint_R \sqrt{a^2-r^2}\,dA = \int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\pi\cdot -\frac{1}{3}(a^2-r^2)^{3/2}\Big]_0^a = 2\pi\cdot\frac{1}{3}a^3 = \frac{2}{3}\pi a^3.$$

b) $\displaystyle\int_0^{\pi/2}\int_0^a (r\cos\theta)(r\sin\theta)r\,dr\,d\theta = \int_0^a r^3\,dr\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{a^4}{4}\cdot\frac{1}{2} = \frac{a^4}{8}.$

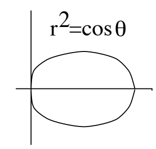
c) In order to be able to use the integral formulas at the beginning of 3B, we use symmetry about the y-axis to compute the volume of just the right side, and double the answer.



$$\iint_R \sqrt{x^2+y^2}\,dA = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r\,r\,dr\,d\theta = 2\int_0^{\pi/2}\frac{1}{3}(2\sin\theta)^3\,d\theta$$

$= 2\cdot\dfrac{8}{3}\cdot\dfrac{2}{3} = \dfrac{32}{9}$, by the integral formula at the beginning of **3B**.

d) $\displaystyle 2\int_0^{\pi/2}\int_0^{\sqrt{\cos\theta}} r^2\,r\,dr\,d\theta = 2\int_0^{\pi/2}\frac{1}{4}\cos^2\theta\,d\theta = 2\cdot\frac{1}{4}\cdot\frac{\pi}{4} = \frac{\pi}{8}.$

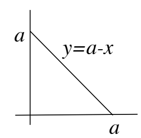


3C. Applications of Double Integration

3C-1 Placing the figure so its legs are on the positive x- and y-axes,

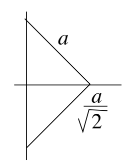
a) M.I. $= \displaystyle\int_0^a\int_0^{a-x} x^2\,dy\,dx$ Inner: $x^2y\Big]_0^{a-x} = x^2(a-x)$; Outer: $\displaystyle\frac{1}{3}x^3a - \frac{1}{4}x^4\Big]_0^a = \frac{1}{12}a^4.$

b) $\displaystyle\iint_R (x^2+y^2)\,dA = \iint_R x^2\,dA + \iint_R y^2\,dA = \frac{1}{12}a^4 + \frac{1}{12}a^4 = \frac{1}{6}a^4.$



c) Divide the triangle symmetrically into two smaller triangles, their legs are $\dfrac{a}{\sqrt{2}}$;

Using the result of part (a), M.I. of R about hypotenuse $= 2\cdot\dfrac{1}{12}\left(\dfrac{a}{\sqrt{2}}\right)^4 = \dfrac{a^4}{24}$



3C-2 In both cases, $\bar{x}$ is clear by symmetry; we only need $\bar{y}$.

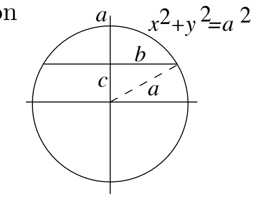
a) Mass is $\displaystyle\iint_R dA = \int_0^\pi \sin x\,dx = 2$

y-moment is $\displaystyle\iint_R y\,dA = \int_0^\pi\int_0^{\sin x} y\,dy\,dx = \frac{1}{2}\int_0^\pi \sin^2 x\,dx = \frac{\pi}{4}$; therefore $\bar{y} = \dfrac{\pi}{8}$.

b) Mass is $\displaystyle\iint_R y\,dA = \frac{\pi}{4}$, by part (a). Using the formulas at the beginning of **3B**,

y-moment is $\displaystyle\iint_R y^2\,dA = \int_0^\pi\int_0^{\sin x} y^2\,dy\,dx = 2\int_0^{\pi/2}\frac{\sin^3 x}{3}\,dx = 2\cdot\frac{1}{3}\cdot\frac{2}{3} = \frac{4}{9},$

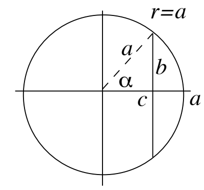
Therefore $\bar{y} = \dfrac{4}{9}\cdot\dfrac{4}{\pi} = \dfrac{16}{9\pi}$.



3C-3 Place the segment either horizontally or vertically, so the diameter is respectively on the x or y axis. Find the moment of half the segment and double the answer.

(a) (Horizontally, using rectangular coordinates) Note that $a^2 - c^2 = b^2$.

$$\int_0^b \int_c^{\sqrt{a^2-x^2}} y\,dy\,dx = \int_0^b \frac{1}{2}(a^2 - x^2 - c^2)\,dx = \frac{1}{2}\Big[b^2 x - \frac{x^3}{3}\Big]_0^b = \frac{1}{3}b^3; \quad \text{ans: } \frac{2}{3}b^3.$$



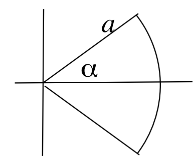
(b) (Vertically, using polar coordinates). Note that $x = c$ becomes $r = c\sec\theta$.

$$\text{Moment } = \int_0^\alpha \int_{c\sec\theta}^a (r\cos\theta)\,r\,dr\,d\theta \qquad \text{Inner: } \tfrac{1}{3}r^3\cos\theta\Big]_{c\sec\theta}^a = \tfrac{1}{3}(a^3\cos\theta - c^3\sec^2\theta)$$

$$\text{Outer: } \tfrac{1}{3}\Big[a^3\sin\theta - c^3\tan\theta\Big]_0^\alpha = \tfrac{1}{3}(a^2 b - c^2 b) = \tfrac{1}{3}b^3; \quad \text{ans: } \frac{2}{3}b^3.$$

3C-4 Place the sector so its vertex is at the origin and its axis of symmetry lies along the positive x-axis. By symmetry, the center of mass lies on the x-axis, so we only need find $\bar{x}$.

Since $\delta = 1$, the area and mass of the disc are the same: $\pi a^2 \cdot \dfrac{2\alpha}{2\pi} = a^2\alpha$.

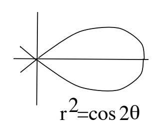


$$x\text{-moment: } 2\int_0^\alpha \int_0^a r\cos\theta \cdot r\,dr\,d\theta \qquad \text{Inner: } \tfrac{2}{3}r^3\cos\theta\Big]_0^a;$$

$$\text{Outer: } \tfrac{2}{3}a^3\sin\theta\Big]_0^\alpha = \tfrac{2}{3}a^3\sin\alpha \qquad \bar{x} = \frac{\tfrac{2}{3}a^3\sin\alpha}{a^2\alpha} = \frac{2}{3}\cdot a\cdot\frac{\sin\alpha}{\alpha}.$$

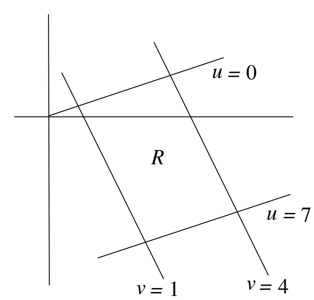
3C-5 By symmetry, we use just the upper half of the loop and double the answer. The upper half lies between $\theta = 0$ and $\theta = \pi/4$.

$$2\int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r^2\,r\,dr\,d\theta = 2\int_0^{\pi/4} \frac{1}{4}a^4\cos^2 2\theta\,d\theta$$



$$\text{Putting } u = 2\theta, \text{ the above } = \frac{a^4}{2\cdot 2}\int_0^{\pi/2}\cos^2 u\,du = \frac{a^4}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^4}{16}.$$

3D. Changing Variables



3D-1 Let $u = x - 3y$, $v = 2x + y$; $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 7$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{7}$.

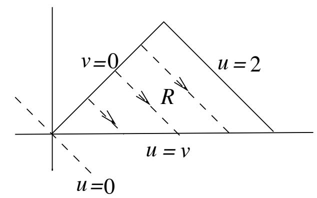
$$\iint_R \frac{x-3y}{2x+y}\,dx\,dy \quad = \quad \frac{1}{7}\int_0^7 \int_1^4 \frac{u}{v}\,dv\,du$$

$$\text{Inner: } u\ln v\Big]_1^4 = u\ln 4; \quad \text{Outer: } \tfrac{1}{2}u^2\ln 4\Big]_0^7 = \frac{49\ln 4}{2}; \quad \text{Ans: } \frac{1}{7}\frac{49\ln 4}{2} \;=\; 7\ln 2$$

3D-2 Let $u = x + y$, $v = x - y$. Then $\dfrac{\partial(u,v)}{\partial(x,y)} = 2$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2}$.

To get the uv-equation of the bottom of the triangular region:

$$y = 0 \quad \Rightarrow \quad u = x,\ v = x \quad \Rightarrow \quad u = v.$$



$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dx\,dy \;=\; \frac{1}{2}\int_0^2 \int_0^u \cos\frac{v}{u}\,dv\,du$$

Inner: $u \sin \dfrac{v}{u}\Big]_0^u = u \sin 1$ Outer: $\frac{1}{2}u^2 \sin 1\Big]_0^2 = 2\sin 1$ Ans: $\sin 1$

3D-3 Let $u = x$, $v = 2y$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \dfrac{1}{2}$

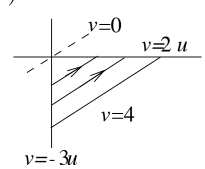
Letting R be the elliptical region whose boundary is $x^2 + 4y^2 = 16$ in xy-coordinates, and $u^2 + v^2 = 16$ in uv-coordinates (a circular disc), we have

$$\iint_R (16 - x^2 - 4y^2)\,dy\,dx \;=\; \frac{1}{2}\iint_R (16 - u^2 - v^2)\,dv\,du$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^4 (16 - r^2)\,r\,dr\,d\theta \;=\; \pi\left(16\frac{r^2}{2} - \frac{r^4}{4}\right)_0^4 \;=\; 64\pi.$$

3D-4 Let $u = x + y$, $v = 2x - 3y$; then $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{5}$.

We next express the boundary of the region R in uv-coordinates.
For the x-axis, we have $y = 0$, so $u = x$, $v = 2x$, giving $v = 2u$.
For the y-axis, we have $x = 0$, so $u = y$, $v = -3y$, giving $v = -3u$.



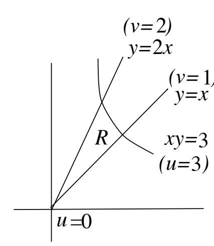
It is best to integrate first over the lines shown, $v = c$; this means v is held constant, i.e., we are integrating first with respect to u. This gives

$$\iint_R (2x - 3y)^2(x + y)^2 dx\,dy \;=\; \int_0^4 \int_{-v/3}^{v/2} v^2u^2\frac{du\,dv}{5}.$$

Inner: $\dfrac{v^2}{15}u^3\Big]_{-v/3}^{v/2} = \dfrac{v^2}{15}v^3\left(\dfrac{1}{8} - \dfrac{-1}{27}\right)$ Outer: $\dfrac{v^6}{6\cdot 15}\left(\dfrac{1}{8} + \dfrac{1}{27}\right)_0^4 = \dfrac{4^6}{6\cdot 15}\left(\dfrac{1}{8} + \dfrac{1}{27}\right)$.

3D-5 Let $u = xy$, $v = y/x$; in the other direction this gives $y^2 = uv$, $x^2 = u/v$.

We have $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \dfrac{2y}{x} = 2v$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2v}$; this gives



$$\iint_R (x^2 + y^2)\,dx\,dy \;=\; \int_0^3 \int_1^2 \left(\frac{u}{v} + uv\right)\frac{1}{2v}\,dv\,du.$$

Inner: $\dfrac{-u}{2v} + \dfrac{u}{2}v\Big]_1^2 = u\left(-\dfrac{1}{4} + 1 + \dfrac{1}{2} - \dfrac{1}{2}\right) = \dfrac{3u}{4}$; Outer: $\dfrac{3}{8}u^2\Big]_0^3 = \dfrac{27}{8}$.

3D-8 a) $y = x^2$; therefore $u = x^3$, $v = x$, which gives $u = v^3$.

b) We get $\dfrac{u}{v} + uv = 1$, or $u = \dfrac{v}{v^2+1}$; (cf. 3D-5)

4. Line Integrals in the Plane

4A. Plane Vector Fields

4A-1

a) All vectors in the field are identical; continuously differentiable everywhere.
b) The vector at P has its tail at P and head at the origin; field is cont. diff. everywhere.
c) All vectors have unit length and point radially outwards; cont. diff. except at $(0,0)$.
d) Vector at P has unit length, and the clockwise direction perpendicular to OP.

4A-2 a) $a\,\mathbf{i}+b\,\mathbf{j}$ b) $\dfrac{x\,\mathbf{i}+y\,\mathbf{j}}{r^2}$ c) $f'(r)\dfrac{x\,\mathbf{i}+y\,\mathbf{j}}{r}$

4A-3 a) $\mathbf{i}+2\mathbf{j}$ b) $-r(x\,\mathbf{i}+y\,\mathbf{j})$ c) $\dfrac{y\,\mathbf{i}-x\,\mathbf{j}}{r^3}$ d) $f(x,y)(\,\mathbf{i}+\mathbf{j}\,)$

4A-4 $k\cdot\dfrac{-y\,\mathbf{i}+x\,\mathbf{j}}{r^2}$

4B. Line Integrals in the Plane

4B-1

a) On C_1: $y=0,\ dy=0$; therefore $\displaystyle\int_{C_1}(x^2-y)\,dx+2x\,dy=\int_{-1}^{1}x^2\,dx=\frac{x^3}{3}\Big]_{-1}^{1}=\frac{2}{3}.$

On C_2: $y=1-x^2,\ dy=-2x\,dx$; $\displaystyle\int_{C_2}(x^2-y)\,dx+2x\,dy=\int_{-1}^{1}(2x^2-1)\,dx-4x^2\,dx$

$$=\int_{-1}^{1}(-2x^2-1)\,dx=-\left[\frac{2}{3}x^3+x\right]_{-1}^{1}=-\frac{4}{3}-2=-\frac{10}{3}.$$

b) C: use the parametrization $x=\cos t,\ y=\sin t$; then $dx=-\sin t\,dt,\ dy=\cos t\,dt$

$$\int_C xy\,dx-x^2\,dy=\int_{\pi/2}^{0}-\sin^2 t\cos t\,dt-\cos^2 t\cos t\,dt=-\int_{\pi/2}^{0}\cos t\,dt=-\sin t\Big]_{\pi/2}^{0}=1.$$

c) $C=C_1+C_2+C_3$; $C_1: x=dx=0$; $C_2:\ y=1-x$; $C_3: y=dy=0$

$$\int_C y\,dx-x\,dy\ =\ \int_{C_1}0+\int_0^1(1-x)dx-x(-dx)+\int_{C_3}0\ =\ \int_0^1 dx\ =\ 1.$$

d) $C: x=2\cos t,\ y=\sin t$; $dx=-2\sin t\,dt$ $\displaystyle\int_C y\,dx=\int_0^{2\pi}-2\sin^2 t\,dt=-2\pi.$

e) $C:\ x=t^2,\ y=t^3$; $dx=2t\,dt,\ dy=3t^2\,dt$

$$\int_C 6y\,dx+x\,dy=\int_1^2 6t^3(2t\,dt)+t^2(3t^2\,dt)=\int_1^2(15t^4)\,dt=3t^5\Big]_1^2=3\cdot 31.$$

f) $\displaystyle\int_C(x+y)dx+xy\,dy=\int_{C_1}0+\int_0^1(x+2)dx=\frac{x^2}{2}+2x\Big]_0^1=\frac{5}{2}.$

4B-2 a) The field $\mathbf{F}$ points radially outward, the unit tangent $\mathbf{t}$ to the circle is always perpendicular to the radius; therefore $\mathbf{F}\cdot\mathbf{t}=0$ and $\int_C\mathbf{F}\cdot d\mathbf{r}=\int_C\mathbf{F}\cdot\mathbf{t}\,ds=0$

b) The field $\mathbf{F}$ is always tangent to the circle of radius a, in the clockwise direction, and of magnitude a. Therefore $\mathbf{F}=-a\mathbf{t}$, so that $\int_C\mathbf{F}\cdot d\mathbf{r}=\int_C\mathbf{F}\cdot\mathbf{t}\,ds=-\int_C a\,ds=-2\pi a^2$.

4B-3 a) maximum if C is in the direction of the field: $C = \dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$

b) minimum if C is in the opposite direction to the field: $C = -\dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$

c) zero if C is perpendicular to the field: $C = \pm\dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$

d) max $= \sqrt{2}$, min $= -\sqrt{2}$: by (a) and (b), for the max or min $\mathbf{F}$ and C have respectively the same or opposite constant direction, so $\int_C \mathbf{F} \cdot d\mathbf{r} = \pm|\mathbf{F}| \cdot |C| = \pm\sqrt{2}$.

4C. Gradient Fields and Exact Differentials

4C-1 a) $\mathbf{F} = \nabla f = 3x^2y\,\mathbf{i} + (x^3 + 3y^2)\,\mathbf{j}$

b) (i) Using y as parameter, C_1 is: $x = y^2$, $y = y$; thus $dx = 2y\,dy$, and

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} 3(y^2)^2 y \cdot 2y\,dy + [(y^2)^3 + 3y^2]\,dy = \int_{-1}^{1} (7y^6 + 3y^2)\,dy = (y^7 + y^3)\Big]_{-1}^{1} = 4.$$

b) (ii) Using y as parameter, C_2 is: $x = 1$, $y = y$; thus $dx = 0$, and

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} (1 + 3y^2)\,dy = (y + y^3)\Big]_{-1}^{1} = 4.$$

b) (iii) By the Fundamental Theorem of Calculus for line integrals,

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

Here $A = (1, -1)$ and $B = (1, 1)$, so that $\displaystyle\int_C \nabla f \cdot d\mathbf{r} = (1 + 1) - (-1 - 1) = 4$.

4C-2 a) $\mathbf{F} = \nabla f = (xye^{xy} + e^{xy})\,\mathbf{i} + (x^2e^{xy})\,\mathbf{j}$.

b) (i) Using x as parameter, C is: $x = x$, $y = 1/x$, so $dy = -dx/x^2$, and so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^0 (e + e)\,dx + (x^2e)(-dx/x^2) = (2ex - ex)\Big]_1^0 = -e.$$

b) (ii) Using the F.T.C. for line integrals, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, \infty) = 0 - e = -e$.

4C-3 a) $\mathbf{F} = \nabla f = (\cos x \cos y)\,\mathbf{i} - (\sin x \sin y)\,\mathbf{j}$.

b) Since $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is path-independent, for any C connecting $A : (x_0, y_0)$ to $B : (x_1, y_1)$, we have by the F.T.C. for line integrals,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \sin x_1 \cos y_1 - \sin x_0 \cos y_0$$

This difference on the right-hand side is maximized if $\sin x_1 \cos y_1$ is maximized, and $\sin x_0 \cos y_0$ is minimized. Since $|\sin x \cos y| = |\sin x||\cos y| \le 1$, the difference on the right hand side has a maximum of 2, attained when $\sin x_1 \cos y_1 = 1$ and $\sin x_0 \cos y_0 = -1$.

(For example, a C running from $(-\pi/2, 0)$ to $(\pi/2, 0)$ gives this maximum value.)

4C-5 a) $\mathbf{F}$ is a gradient field only if $M_y = N_x$, that is, if $2y = ay$, so $a = 2$.

By inspection, the potential function is $f(x, y) = xy^2 + x^2 + c$; you can check that $\mathbf{F} = \nabla f$.

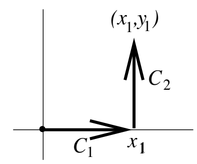
b) The equation $M_y = N_x$ becomes $e^{x+y}(x+a) = xe^{x+y} + e^{x+y}$, which $= e^{x+y}(x+1)$. Therefore $a = 1$.

To find the potential function $f(x, y)$, using Method 2 we have

$f_x = e^y e^x(x + 1) \;\Rightarrow\; f(x, y) = e^y xe^x + g(y)$.

Differentiating, and comparing the result with N, we find

$f_y = e^y xe^x + g'(y) = xe^{x+y}$; therefore $g'(y) = 0$, so $g(y) = c$ and $f(x, y) = x\, e^{x+y} + c$.



4C-6 a) $ydx - xdy$ is not exact, since $M_y = 1$ but $N_x = -1$.

b) $y(2x + y)\, dx + x(2y + x)\, dy$ is exact, since $M_y = 2x + 2y = N_x$.

Using Method 1 to find the potential function $f(x, y)$, we calculate the line integral over the standard broken line path shown, $C = C_1 + C_2$.

$$f(x_1, y_1) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0)}^{(x_1,y_1)} y(2x + y)\, dx + x(2y + x)\, dy.$$

On C_1 we have $y = 0$ and $dy = 0$, so $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$.

On C_2, we have $x = x_1$ and $dx = 0$, so $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{y_1} x_1(2y + x_1)\, dx = x_1 y_1^2 + x_1^2 y_1$.

Therefore, $f(x, y) = x^2 y + xy^2$; to get all possible functions, add $+c$.

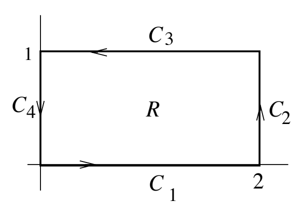
4D. Green's Theorem

4D-1 a) Evaluating the line integral first, we have $C:\; x = \cos t,\; y = \sin t,$ so

$$\oint_C 2y\, dx + x\, dy = \int_0^{2\pi} (-2\sin^2 t + \cos^2 t)\, dt = \int_0^{2\pi} (1 - 3\sin^2 t)\, dt = t - 3\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Bigg]_0^{2\pi} = -\pi.$$

For the double integral over the circular region R inside the C, we have

$$\iint_R (N_x - M_y)\, dA = \iint_R (1 - 2)\, dA = -\text{ area of } R = -\pi.$$



b) Evaluating the line integral, over the indicated path $C = C_1 + C_2 + C_3 + C_4$,

$$\oint_C x^2 dx + x^2 dy = \int_0^2 x^2 dx + \int_0^1 4\, dy + \int_2^0 x^2 dx + \int_1^0 0\, dy \;=\; 4,$$

since the first and third integrals cancel, and the fourth is 0.

For the double integral over the rectangle R,

$$\iint_R 2x\, dA = \int_0^2 \int_0^1 2x\, dy dx = x^2\Big]_0^2 = 4.$$

c) Evaluating the line integral over $C = C_1 + C_2$, we have

$$C_1: \; x = x,\; y = x^2; \quad \int_{C_1} xy\,dx + y^2 dy = \int_0^1 x \cdot x^2 dx + x^4 \cdot 2x\,dx = \frac{x^4}{4} + \frac{x^6}{3}\Big]_0^1 = \frac{7}{12}$$

$$C_2: \; x = x,\; y = x; \quad \int_{C_2} xy\,dx + y^2 dy = \int_1^0 (x^2 dx + x^2 dx) = \frac{2}{3}x^3\Big]_1^0 = -\frac{2}{3}.$$

Therefore, $\displaystyle\oint_C xy\,dx + y^2 dy = \frac{7}{12} - \frac{2}{3} = -\frac{1}{12}.$

Evaluating the double integral over the interior R of C, we have

$$\iint_R -x\,dA = \int_0^1 \int_{x^2}^{x} -x\,dy dx;$$

evaluating: Inner: $\displaystyle -xy\Big]_{y=x^2}^{y=x} = -x^2 + x^3;$ Outer: $\displaystyle -\frac{x^3}{3} + \frac{x^4}{4}\Big]_0^1 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}.$

4D-2 By Green's theorem, $\displaystyle\oint_C 4x^3 y\,dx + x^4\,dy = \iint (4x^3 - 4x^3)\,dA = 0.$

This is true for every closed curve C in the plane, since M and N have continuous derivatives for all x, y.

4D-3 We use the symmetric form for the integrand since the parametrization of the curve does not favor x or y; this leads to the easiest calculation.

$$\text{Area} = \frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} 3\sin^4 t\cos^2 t\,dt + 3\sin^2 t\cos^4 t\,dt = \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\,dt$$

Using $\sin^2 t\cos^2 t = \frac{1}{4}(\sin 2t)^2 = \frac{1}{4}\cdot\frac{1}{2}(1 - \cos 4t)$, the above $\displaystyle = \frac{3}{8}\left(\frac{t}{2} - \frac{\sin 4t}{8}\right)_0^{2\pi} = \frac{3\pi}{8}.$

4D-4 By Green's theorem, $\displaystyle\oint_C -y^3 dx + x^3 dy = \iint_R (3x^2 + 3y^2)\,dA > 0$, since the integrand is always positive outside the origin.

4D-5 Let C be a square, and R its interior. Using Green's theorem,

$$\oint_C xy^2 dx + (x^2 y + 2x)\,dy = \iint_R (2xy + 2 - 2xy)\,dA = \iint_R 2\,dA = 2(\text{area of } R).$$

4E. Two-dimensional Flux

4E-1 The vector $\mathbf{F}$ is the velocity vector for a rotating disc; it is at each point tangent to the circle centered at the origin and passing through that point.

a) Since $\mathbf{F}$ is tangent to the circle, $\mathbf{F}\cdot\mathbf{n} = 0$ at every point on the circle, so the flux is 0.

b) $\mathbf{F} = x\,\mathbf{j}$ at the point $(x, 0)$ on the line. So if $x_0 > 0$, the flux at x_0 has the same magnitude as the flux at $-x_0$ but the opposite sign, so the net flux over the line is 0.

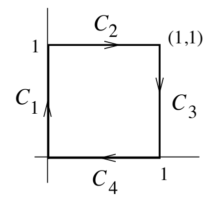
c) $\mathbf{n} = -\mathbf{j}$, so $\mathbf{F}\cdot\mathbf{n} = x\,\mathbf{j}\cdot -\mathbf{j} = -x$. Thus $\displaystyle\int \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 -x\,dx = -\frac{1}{2}.$

4E-2 All the vectors of $\mathbf{F}$ have length $\sqrt{2}$ and point northeast. So the flux across a line segment C of length 1 will be

a) maximal, if C points northwest;
b) minimal, if C point southeast;
c) zero, if C points northeast or southwest;
d) -1, if C has the direction and magnitude of $\mathbf{i}$ or $-\mathbf{j}$; the corresponding normal vectors are then respectively $-\mathbf{j}$ and $-\mathbf{i}$, by convention, so that $\mathbf{F}\cdot\mathbf{n} = (\mathbf{i}+\mathbf{j})\cdot -\mathbf{j} = -1$. or $(\mathbf{i}+\mathbf{j})\cdot -\mathbf{i} = -1$.
e) respectively $\sqrt{2}$ and $-\sqrt{2}$, since the angle θ between $\mathbf{F}$ and n is respectively 0 and π, so that respectively $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}|\cos\theta = \pm\sqrt{2}$.

4E-3 $\displaystyle\int_C M\,dy - N\,dx = \int_C x^2 dy - xy\,dx = \int_0^1 (t+1)^2 2t\,dt - (t+1)t^2\,dt$

$$= \int_0^1 (t^3 + 3t^2 + 2t)\,dt = \frac{t^4}{4} + t^3 + t^2\Big]_0^1 = \frac{9}{4}.$$



4E-4 Taking the curve $C = C_1 + C_2 + C_3 + C_4$ as shown,

$$\int_C x\,dy - y\,dx = \int_{C_1} 0 + \int_0^1 -dx + \int_1^0 dy + \int_{C_4} 0 = -2.$$

4E-5 Since $\mathbf{F}$ and $\mathbf{n}$ both point radially outwards, $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}| = a^m$, at every point of the circle C of radius a centered at the origin.

a) The flux across C is $a^m \cdot 2\pi a = 2\pi a^{m+1}$.
b) The flux will be independent of a if $m = -1$.

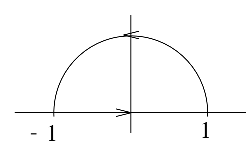
4F. Green's Theorem in Normal Form

4F-1 a) both are 0 b) div $\mathbf{F} = 2x + 2y$; curl $\mathbf{F} = 0$ c) div $\mathbf{F} = x + y$; curl $\mathbf{F} = y - x$

4F-2 a) div $\mathbf{F} = (-\omega y)_x + (\omega x)_y = 0$; curl $\mathbf{F} = (\omega x)_x - (-\omega y)_y = 2\omega$.

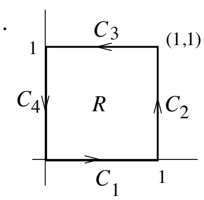
b) Since $\mathbf{F}$ is the velocity field of a fluid rotating with constant angular velocity (like a rigid disc), there are no sources or sinks: fluid is not being added to or subtracted from the flow at any point.

c) A paddlewheel placed at the origin will clearly spin with the same angular velocity ω as the rotating fluid, so by Notes V4,(11), the curl should be 2ω at the origin. (It is much less clear that the curl is 2ω at all other points as well.)



4F-3 The line integral for flux is $\displaystyle\int_C x\,dy - y\,dx$; its value is 0 on any segment of the x-axis since $y = dy = 0$; on the upper half of the unit semicircle (oriented counterclockwise), $\mathbf{F}\cdot\mathbf{n} = 1$, so the flux is the length of the semicircle: π.

Letting R be the region inside C, $\displaystyle\iint_R \text{div}\,\mathbf{F}\,dA = \iint_R 2\,dA = 2(\pi/2) = \pi$.



4F-4 For the flux integral $\displaystyle\oint_C x^2 dy - xy\,dx$ over $C = C_1 + C_2 + C_3 + C_4$,

we get for the four sides respectively $\displaystyle\int_{C_1} 0 + \int_0^1 dy + \int_1^0 -x\,dx + \int_{C_4} 0 = \frac{3}{2}$.

For the double integral, $\displaystyle\iint_R \text{div}\,\mathbf{F}\,dA = \iint_R 3x\,dA = \int_0^1\int_0^1 3x\,dydx = \frac{3}{2}x^2\Big]_0^1 = \frac{3}{2}.$

4F-5 $r = (x^2+y^2)^{1/2} \Rightarrow r_x = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \dfrac{x}{r}$; by symmetry, $r_y = \dfrac{y}{r}$.

To calculate div $\mathbf{F}$, we have $M = r^n x$ and $N = r^n y$; therefore by the chain rule, and the above values for r_x and r_y, we have

$M_x = r^n + nr^{n-1}x\cdot\dfrac{x}{r} = r^n + n\,r^{n-2}x^2$; similarly (or by symmetry),

$N_y = r^n + nr^{n-1}y\cdot\dfrac{y}{r} = r^n + n\,r^{n-2}y^2$, so that

div $\mathbf{F} = M_x + N_y = 2r^n + nr^{n-2}(x^2+y^2) = r^n(2+n)$, which $=0$ if $n=-2$.

To calculate curl $\mathbf{F}$, we have by the chain rule

$N_x = nr^{n-1}\cdot\dfrac{x}{r}\cdot y;\quad M_y = nr^{n-1}\cdot\dfrac{y}{r}\cdot x,$ so that curl $\mathbf{F} = N_x - M_y = 0$, for all n.

4G. Simply-connected Regions

4G-1 Hypotheses: the region R is simply connected, $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ has continuous derivatives in R, and curl $\mathbf{F} = 0$ in R.

Conclusion: $\mathbf{F}$ is a gradient field in R (or, $M\,dx + N\,dy$ is an exact differential).

a) curl $\mathbf{F} = 2y - 2y = 0$, and R is the whole xy-plane. Therefore $\mathbf{F} = \nabla f$ in the plane.

b) curl $\mathbf{F} = -y\sin x - x\sin y \neq 0$, so the differential is not exact.

c) curl $\mathbf{F} = 0$, but R is the exterior of the unit circle, which is not simply-connected; criterion fails.

d) curl $\mathbf{F} = 0$, and R is the interior of the unit circle, which is simply-connected, so the differential is exact.

e) curl $\mathbf{F} = 0$ and R is the first quadrant, which is simply-connected, so $\mathbf{F}$ is a gradient field.

4G-2 a) $f(x,y) = xy^2 + 2x$ b) $f(x,y) = \frac{2}{3}x^{3/2} + \frac{2}{3}y^{3/2}$

c) Using Method 1, we take the origin as the starting point and use the straight line to (x_1,y_1) as the path C. In polar coordinates, $x_1 = r_1\cos\theta_1,\ y_1 = r_1\sin\theta_1$; we use r as the parameter, so the path is $C: x = r\cos\theta_1,\ y = r\sin\theta_1,\ 0\le r\le r_1$. Then

$$f(x_1,y_1) = \int_C \frac{x\,dx + y\,dy}{\sqrt{1-r^2}} = \int_0^{r_1}\frac{r\cos^2\theta_1 + r\sin^2\theta_1}{\sqrt{1-r^2}}\,dr$$

$$= \int_0^{r_1}\frac{r}{\sqrt{1-r^2}}dr = -\sqrt{1-r^2}\Big]_0^{r_1} = -\sqrt{1-{r_1}^2} + 1.$$

Therefore, $\dfrac{x\,dx + y\,dy}{\sqrt{1-r^2}} = d(-\sqrt{1-r^2}).$

Another approach: $x\,dx + y\,dy = \frac{1}{2}d(r^2)$; therefore $\dfrac{x\,dx+y\,dy}{\sqrt{1-r^2}} = \dfrac{1}{2}\dfrac{d(r^2)}{\sqrt{1-r^2}} = d(-\sqrt{1-r^2}).$ (Think of r^2 as a new variable u, and integrate.)

4G-3 By Example 3 in Notes V5, we know that $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r^3} = \nabla\left(-\dfrac{1}{r}\right)$.

Therefore, $\displaystyle\int_{(1,1)}^{(3,4)} = -\frac{1}{r}\Big]_{\sqrt{2}}^{5} = \frac{1}{\sqrt{2}} - \frac{1}{5}$.

4G-4 By Green's theorem $\displaystyle\oint_C xy\,dx + x^2\,dy = \iint_R x\,dA$.

For any plane region of density 1, we have $\iint_R x\,dA = \bar{x}\cdot(\text{area of } R)$, where $\bar{x}$ is the x-component of its center of mass. Since our region is symmetric with respect to the y-axis, its center of mass is on the y-axis, hence $\bar{x} = 0$ and so $\iint_R x\,dA = 0$.

4G-5

a) yes
b) no (a circle surrounding the line segment lies in R, but its interior does not)
c) yes (no finite curve could surround the entire positive x-axis)
d) no (the region does not consist of one connected piece)
e) yes if $\theta_0 < 2\pi$; no if $\theta_0 \geq 2\pi$, since then R is the plane with $(0,0)$ removed
f) no (a circle between the two boundary circles lies in R, but its interior does not)
g) yes

4G-6

a) continuously differentiable for $x, y > 0$; thus R is the first quadrant without the two axes, which is simply-connected.

b) continuous differentiable if $r < 1$; thus R is the interior of the unit circle, and is simply-connected.

c) continuously differentiable if $r > 1$; thus R is the exterior of the unit circle, and is not simply-connected.

d) continuously differentiable if $r \neq 0$; thus R is the plane with the origin removed, and is not simply-connected.

e) continuously differentiable if $r \neq 0$; same as (d).

4H. Multiply-connected Regions

4H-1 a) 0; 0 b) 2; 4π c) -1; -2π d) -2; -4π

4H-2 In each case, the winding number about each of the points is given, then the value of the line integral of $\mathbf{F}$ around the curve.

a) $(1, -1, 1)$; $2 - \sqrt{2} + \sqrt{3}$
b) $(-1, 0, 1)$; $-2 + \sqrt{3}$
c) $(-1, 0, 0)$; -2
d) $(-1, -2, 1)$; $-2 - 2\sqrt{2} + \sqrt{3}$

5. Triple Integrals

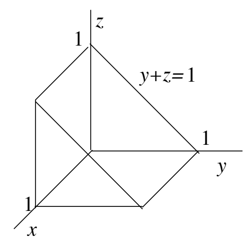
5A. Triple integrals in rectangular and cylindrical coordinates

5A-1 a) $\displaystyle\int_0^2\int_{-1}^1\int_0^1 (x+y+z)dx\,dy\,dz$ Inner: $\frac{1}{2}x^2 + x(y+z)\Big]_{x=0}^1 = \frac{1}{2}+y+z$

Middle: $\frac{1}{2}y+\frac{1}{2}y^2+yz\Big]_{y=-1}^1 = 1+z-(-z) = 1+2z$ Outer: $z+z^2\Big]_0^2 = 6$

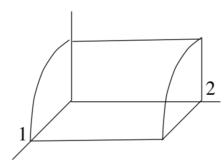
b) $\displaystyle\int_0^2\int_0^{\sqrt{y}}\int_0^{xy} 2xy^2z\,dz\,dx\,dy$ Inner: $xy^2z^2\Big]_0^{xy} = x^3y^4$

Middle: $\frac{1}{4}x^4y^4\Big]_0^{\sqrt{y}} = \frac{1}{4}y^6$ Outer: $\frac{1}{28}y^7\Big]_0^2 = \frac{32}{7}$.



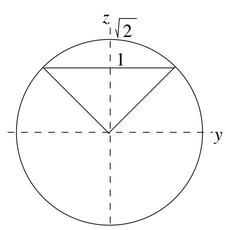
5A-2

a) (i) $\displaystyle\int_0^1\int_0^1\int_0^{1-y} dz\,dy\,dx$ (ii) $\displaystyle\int_0^1\int_0^{1-y}\int_0^1 dx\,dz\,dy$ (iii) $\displaystyle\int_0^1\int_0^1\int_0^{1-z} dy\,dx\,dz$



c) In cylindrical coordinates, with the polar coordinates r and θ in xz-plane, we get

$$\iiint_R dy\,dr\,d\theta = \int_0^{\pi/2}\int_0^1\int_0^2 dy\,dr\,d\theta$$



cross-section view

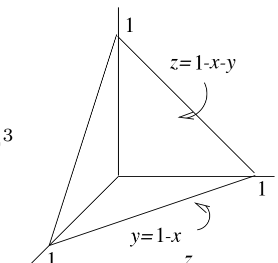
d) The sphere has equation $x^2+y^2+z^2 = 2$, or $r^2+z^2 = 2$ in cylindrical coordinates.

The cone has equation $z^2 = r^2$, or $z = r$. The circle in which they intersect has a radius r found by solving the two equations $z = r$ and $z^2+r^2 = 2$ simultaneously; eliminating z we get $r^2 = 1$, so $r = 1$. Putting it all together, we get

$$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta.$$

5A-3 By symmetry, $\bar{x} = \bar{y} = \bar{z}$, so it suffices to calculate just one of these, say $\bar{z}$. We have

$$z\text{-moment} = \iiint_D z\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx$$

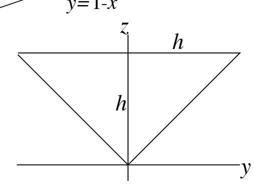


Inner: $\frac{1}{2}z^2\Big]_0^{1-x-y} = \frac{1}{2}(1-x-y)^2$ Middle: $-\frac{1}{6}(1-x-y)^3\Big]_0^{1-x} = \frac{1}{6}(1-x)^3$

Outer: $-\frac{1}{24}(1-x)^4\Big]_0^1 = \frac{1}{24} = \bar{z}$ moment.

mass of D = volume of D = $\frac{1}{3}$(base)(height) $= \frac{1}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{6}$.

Therefore $\bar{z} = \frac{1}{24}/\frac{1}{6} = \frac{1}{4}$; this is also $\bar{x}$ and $\bar{y}$, by symmetry.



vertical cross-section

5A-4 Placing the cone as shown, its equation in cylindrical coordinates is $z = r$ and the density is given by $\delta = r$. By the geometry, its projection onto the xy-plane is the interior R of the origin-centered circle of radius h.

a) Mass of solid $D = \iiint_D \delta\, dV = \int_0^{2\pi}\int_0^h\int_r^h r\cdot r\, dz\, dr\, d\theta$

Inner: $(h-r)r^2$; Middle: $\displaystyle \frac{hr^3}{3} - \frac{r^4}{4}\Big]_0^h = \frac{h^4}{12}$; Outer: $\displaystyle\frac{2\pi h^4}{12}$

b) By symmetry, the center of mass is on the z-axis, so we only have to compute its z-coordinate, $\bar{z}$.

z-moment of $D = \iiint_D z\,\delta\, dV = \int_0^{2\pi}\int_0^h\int_r^h zr\cdot r\, dz\, dr\, d\theta$

Inner: $\frac{1}{2}z^2r^2\Big]_r^h = \frac{1}{2}(h^2r^2 - r^4)$ Middle: $\displaystyle\frac{1}{2}\left(h^2\frac{r^2}{3} - \frac{r^5}{5}\right)_0^h = \frac{1}{2}h^5\cdot\frac{2}{15}$

Outer: $\displaystyle\frac{2\pi h^5}{15}$. Therefore, $\displaystyle\bar{z} = \frac{\frac{2}{15}\pi h^5}{\frac{2}{12}\pi h^4} = \frac{4}{5}h.$

5A-5 Position S so that its base is in the xy-plane and its diagonal D lies along the x-axis (the y-axis would do equally well). The octants divide S into four tetrahedra, which by symmetry have the same moment of inertia about the x-axis; we calculate the one in the first octant. (The picture looks like that for 5A-3, except the height is 2.)

The top of the tetrahedron is a plane intersecting the x- and y-axes at 1, and the z-axis at 2. Its equation is therefore $x + y + \frac{1}{2}z = 1$.

The square of the distance of a point (x, y, z) to the axis of rotation (i.e., the x-axis) is given by $y^2 + z^2$. We therefore get:

$$\text{moment of inertia} = 4\int_0^1\int_0^{1-x}\int_0^{2(1-x-y)}(y^2+z^2)\, dz\, dy\, dx.$$

5A-6 Placing D so its axis lies along the positive z-axis and its base is the origin-centered disc of radius a in the xy-plane, the equation of the hemisphere is $z = \sqrt{a^2 - x^2 - y^2}$, or $z = \sqrt{a^2 - r^2}$ in cylindrical coordinates. Doing the inner and outer integrals mentally:

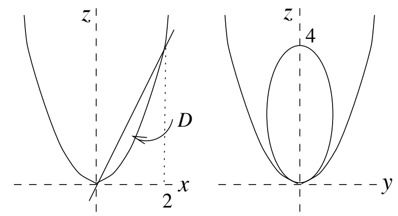
$$z\text{-moment of inertia of } D = \iiint_D r^2\, dV = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} r^2\, dz\, r\, dr\, d\theta = 2\pi\int_0^a r^3\sqrt{a^2-r^2}dr.$$

The integral can be done using integration by parts (write the integrand $r^2\cdot r\sqrt{a^2-r^2}$), or by substitution; following the latter course, we substitute $r = a\sin u$, $dr = a\cos u\, du$, and get (using the formulas at the beginning of exercises 3B)

$$\int_0^a r^3\sqrt{a^2-r^2}dr = \int_0^{\pi/2} a^3\sin^3 u\cdot a^2\cos^2 u\, du$$

$$= a^5\int_0^{\pi/2}\left(\sin^3 u - \sin^5 u\right)du = a^5\left(\frac{2}{3} - \frac{2\cdot 4}{1\cdot 3\cdot 5}\right) = \frac{2}{15}a^5. \qquad \text{Ans: } \frac{4\pi}{15}a^5.$$

5A-7 The solid D is bounded below by $z = x^2 + y^2$ and above by $z = 2x$. The main problem is determining the projection R of D to the xy-plane, since we need to know this before we can put in the limits on the iterated integral.



The outline of R is the projection (i.e., vertical shadow) of the curve in which the paraboloid and plane intersect. This curve is made up of the points in which the graphs of $z = 2x$ and $z = x^2 + y^2$ intersect, i.e., the simultaneous solutions of the two equations. To project the curve, we omit the z-coordinates of the points on it. Algebraically, this amounts to solving the equations simultaneously by eliminating z from the two equations; doing this, we get as the outline of R the curve

$$x^2 + y^2 = 2x \quad \text{or, completing the square,} \quad (x-1)^2 + y^2 = 1.$$

This is a circle of radius 1 and center at $(1, 0)$, whose polar equation is therefore $r = 2\cos\theta$.

We use symmetry to calculate just the right half of D and double the answer:

$$z\text{-moment of inertia of } D = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_{x^2+y^2}^{2x} r^2\,dz\,r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_{r^2}^{2r\cos\theta} r^3\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^3(2r\cos\theta - r^2)\,dr\,d\theta$$

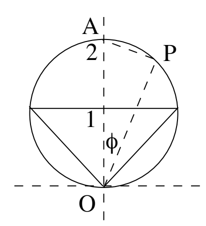
Inner: $\frac{2}{5}r^5\cos\theta - \frac{1}{6}r^6\Big]_0^{2\cos\theta} = \frac{2}{5}\cdot 32\cos^6\theta - \frac{1}{3}\cdot 32\cos^6\theta$

Outer: $\cdot\frac{32}{15}\int_0^{\pi/2}\cos^6\theta\,d\theta = \cdot\frac{32}{15}\cdot\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{\pi}{2} = \frac{\pi}{3}$. Ans: $\frac{2\pi}{3}$

5B. Triple Integrals in spherical coordinates

5B-1 a) The angle between the central axis of the cone and any of the lines on the cone is $\pi/4$; the sphere is $\rho = \sqrt{2}$; so the limits are (no integrand given):: $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} d\rho\,d\phi\,d\theta$.

b) The limits are (no integrand is given): $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\infty} d\rho\,d\phi\,d\theta$



cross-section

c) To get the equation of the sphere in spherical coordinates, we note that AOP is always a right triangle, for any position of P on the sphere. Since $AO = 2$ and $OP = \rho$, we get according to the definition of the cosine, $\cos\phi = \rho/2$, or $\rho = 2\cos\phi$. (The picture shows the cross-section of the sphere by the plane containing P and the central axis AO.)

The plane $z = 1$ has in spherical coordinates the equation $\rho\cos\phi = 1$, or $\rho = \sec\phi$. It intersects the sphere in a circle of radius 1; this shows that $\pi/4$ is the maximum value of ϕ for which the ray having angle ϕ intersects the region.. Therefore the limits are (no integrand is given):

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi}^{2\cos\phi} d\rho\,d\phi\,d\theta.$$

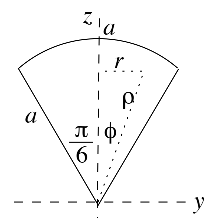
5B-2 Place the solid hemisphere D so that its central axis lies along the positive z-axis and its base is in the xy-plane. By symmetry, $\bar{x} = 0$ and $\bar{y} = 0$, so we only need $\bar{z}$. The integral for it is the product of three separate one-variable integrals, since the integrand is the product of three one-variable functions and the limits of integration are all constants.

$$\bar{z}\text{-moment} = \iiint_D z\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\cdot\left(\frac{1}{4}\rho^4\right)_0^a\cdot\left(\frac{1}{2}\sin^2\phi\right)_0^{\pi/2} = 2\pi\cdot\frac{1}{4}a^4\cdot\frac{1}{2} = \frac{\pi a^4}{4}.$$

Since the mass is $\frac{2}{3}\pi a^3$, we have finally $\bar{z} = \dfrac{\pi a^4/4}{2\pi a^3/3} = \dfrac{3}{8}a.$

5B-3 Place the solid so the vertex is at the origin, and the central axis lies along the positive z-axis. In spherical coordinates, the density is given by $\delta = z = \rho\cos\phi$, and referring to the picture, we have



$$\text{M. of I.} = \iiint_D r^2\cdot z\,dV = \iiint_D (\rho\sin\phi)^2(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/6}\int_0^a \rho^5\sin^3\phi\cos\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\cdot\frac{a^6}{6}\cdot\frac{1}{4}\sin^4\phi\Big]_0^{\pi/6} = 2\pi\cdot\frac{a^6}{6}\cdot\frac{1}{4}\left(\frac{1}{2}\right)^4 = \frac{\pi a^6}{2^6\cdot 3}.$$

5B-4 Place the sphere so its center is at the origin. In each case the iterated integral can be expressed as the product of three one-variable integrals (which are easily calculated), since the integrand is the product of one-variable functions and the limits are constants.

a) $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\cdot 2\cdot\frac{1}{4}a^4 = \pi a^4;$ average $= \dfrac{\pi a^4}{4\pi a^3/3} = \dfrac{3a}{4}.$

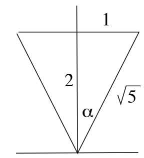
b) Use the z-axis as diameter. The distance of a point from the z-axis is $r = \rho\sin\phi$.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho\sin\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\cdot\frac{\pi}{2}\cdot\frac{1}{4}a^4 = \frac{\pi^2a^4}{4};\qquad \text{average} = \frac{\pi^2a^4/4}{4\pi a^3/3} = \frac{3\pi a}{16}.$$

c) Use the xy-plane and the upper solid hemisphere. The distance is $z = \rho\cos\phi$.

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho\cos\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\cdot\frac{1}{2}\cdot\frac{1}{4}a^4 = \frac{\pi a^4}{4};\qquad \text{average} = \frac{\pi a^4/4}{2\pi a^3/3} = \frac{3a}{8}.$$

5C. Gravitational Attraction



5C-2 The top of the cone is given by $z = 2$; in spherical coordinates: $\rho\cos\phi = 2$. Let α be the angle between the axis of the cone and any of its generators. The density $\delta = 1$. Since the cone is symmetric about its axis, the gravitational attraction has only a k-component, and is

$$G\int_0^{2\pi}\int_0^{\alpha}\int_0^{2/\cos\phi}\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta.$$

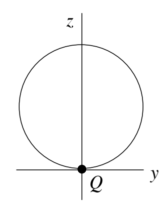
Inner: $\dfrac{2}{\cos\phi}\sin\phi\cos\phi$ Middle: $-2\cos\phi\Big]_0^{\alpha} = -2\cos\alpha + 2$ Outer: $2\pi\cdot 2(1-\cos\alpha)$

Ans: $4\pi G\left(1-\dfrac{2}{\sqrt{5}}\right)$.

5C-3 Place the sphere as shown so that Q is at the origin. Since it is rotationally symmetric about the z-axis, the force will be in the $\mathbf{k}$-direction.

Equation of sphere: $\rho = 2\cos\phi$ Density: $\delta = \rho^{-1/2}$

$$F_z = G\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\phi}\rho^{-1/2}\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta$$



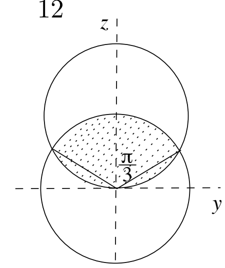
Inner: $\cos\phi\sin\phi\;2\rho^{1/2}\Big]_0^{2\cos\phi} = 2\sqrt{2}\;\cos^{3/2}\phi\;\sin\phi$

Middle: $2\sqrt{2}\left[-\dfrac{2}{5}\cos^{5/2}\phi\right]_0^{\pi/2} = \dfrac{4\sqrt{2}}{5}$ Outer: $2\pi G\dfrac{4\sqrt{2}}{5} = \dfrac{8\sqrt{2}}{5}\pi G$.

5C-4 Referring to the figure, the total gravitational attraction (which is in the $\mathbf{k}$ direction, by rotational symmetry) is the sum of the two integrals

$$G\int_0^{2\pi}\int_0^{\pi/3}\int_0^{1}\cos\phi\,\sin\phi\,d\rho\,d\phi\,d\theta \quad + \quad G\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2\cos\phi}\cos\phi\,\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \;2\pi G\cdot\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 \;+\; 2\pi G\cdot\frac{2}{3}\left(\frac{1}{2}\right)^3 \;=\; \frac{3}{4}\pi G + \frac{1}{6}\pi G = \frac{11}{12}\pi G.$$



The two spheres are shown in cross-section. The spheres intersect at the points where $\phi = \pi/3$.

The first integral respresents the gravitational attraction of the top part of the solid, bounded below by the cone $\phi = \pi/3$ and above by the sphere $\rho = 1$.

The second integral represents the bottom part of the solid, bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone.

6. Vector Integral Calculus in Space

6A. Vector Fields in Space

6A-1 a) the vectors are all unit vectors, pointing radially outward.
b) the vector at P has its head on the y-axis, and is perpendicular to it

6A-2 $\frac{1}{2}(-x\,\mathbf{i} - y\,\mathbf{j} - z\,\mathbf{k})$

6A-3 $\omega(-z\,\mathbf{j} + y\,\mathbf{k})$

6A-4 A vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is parallel to the plane $3x - 4y + z = 2$ if it is perpendicular to the normal vector to the plane, $3\,\mathbf{i} - 4\,\mathbf{j} + \mathbf{k}$: the condition on M, N, P therefore is $3M - 4N + P = 0$, or $P = 4N - 3M$.

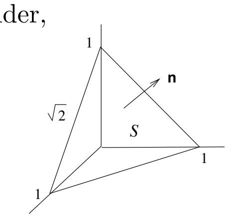
The most general such field is therefore $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + (4N - 3M)\,\mathbf{k}$, where M and N are functions of x, y, z.

6B. Surface Integrals and Flux

6B-1 We have $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$; therefore $\mathbf{F} \cdot \mathbf{n} = a$.

Flux through $S = \displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = a(\text{area of } S) = 4\pi\, a^3$.

6B-2 Since $\mathbf{k}$ is parallel to the surface, the field is everywhere tangent to the cylinder, hence the flux is 0.



6B-3 $\dfrac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$ is a normal vector to the plane, so $\mathbf{F} \cdot \mathbf{n} = \dfrac{1}{\sqrt{3}}$.

Therefore, flux $= \dfrac{\text{area of region}}{\sqrt{3}} = \dfrac{\frac{1}{2}(\text{base})(\text{height})}{\sqrt{3}} = \dfrac{\frac{1}{2}(\sqrt{2})(\frac{\sqrt{3}}{2}\sqrt{2})}{\sqrt{3}} = \dfrac{1}{2}$.

6B-4 $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$; $\mathbf{F} \cdot \mathbf{n} = \dfrac{y^2}{a}$. Calculating in spherical coordinates,

flux $= \displaystyle\iint_S \frac{y^2}{a}\, dS = \frac{1}{a}\int_0^\pi \int_0^\pi a^4 \sin^3\phi \sin^2\theta\, d\phi\, d\theta = a^3 \int_0^\pi \int_0^\pi \sin^3\phi \sin^2\theta\, d\phi d\theta.$

Inner integral: $\sin^2\theta(-\cos\phi + \frac{1}{3}\cos^3\phi)\Big]_0^\pi = \frac{4}{3}\sin^2\theta;$

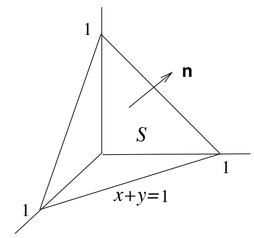
Outer integral: $\frac{4}{3}a^3(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta)\Big]_0^\pi = \frac{2}{3}\pi a^3.$

6B-5 $\mathbf{n} = \dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$; $\mathbf{F}\cdot\mathbf{n} = \dfrac{z}{\sqrt{3}}$.

flux $= \displaystyle\iint_S \frac{z}{\sqrt{3}}\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \frac{1}{\sqrt{3}}\iint_S (1-x-y)\frac{dx\,dy}{1/\sqrt{3}} = \int_0^1\int_0^{1-y}(1-x-y)\,dx\,dy.$

Inner integral: $= x - \frac{1}{2}x^2 - xy\Big]_0^{1-y} = \frac{1}{2}(1-y)^2.$

Outer integral: $= \displaystyle\int_0^1 \frac{1}{2}(1-y)^2dy = \frac{1}{2}\cdot -\frac{1}{3}\cdot(1-y)^3\Big]_0^1 = \frac{1}{6}.$



6B-6 $z = f(x,y) = x^2+y^2$ (a paraboloid). By (13) in Notes V9,

$$d\mathbf{S} = (-2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k})\,dx\,dy.$$

(This points generally "up", since the $\mathbf{k}$ component is positive.) Since $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_R (-2x^2 - 2y^2 + z)\,dxdy \qquad ,$$

where R is the interior of the unit circle in the xy-plane, i.e., the projection of S onto the xy-plane). Since $z = x^2+y^2$, the above integral

$$= -\iint_R (x^2+y^2)\,dxdy = -\int_0^{2\pi}\int_0^1 r^2\cdot r\,dr\,d\theta = -2\pi\cdot\frac{1}{4} = -\frac{\pi}{2}\ .$$

The answer is negative since the positive direction for flux is that of $\mathbf{n}$, which here points into the inside of the paraboloidal cup, whereas the flow $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ is generally from the inside toward the outside of the cup, i.e., in the opposite direction.

6B-8 On the cylindrical surface, $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{a}$, $\mathbf{F}\cdot\mathbf{n} = \dfrac{y^2}{a}$.
In cylindrical coordinates, since $y = a\sin\theta$, this gives us $\mathbf{F}\cdot d\mathbf{S} = \mathbf{F}\cdot\mathbf{n}\,dS = a^2\sin^2\theta\,dz\,d\theta$.

Flux $= \displaystyle\int_{-\pi/2}^{\pi/2}\int_0^k a^2\sin^2\theta\,dz\,d\theta = a^2h\int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta = a^2h\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_{-\pi/2}^{\pi/2} = \frac{\pi}{2}a^2h\ .$

6B-12 Since the distance from a point $(x,y,0)$ up to the hemispherical surface is z,

$$\text{average distance} = \frac{\iint_S z\,dS}{\iint_S dS}.$$

In spherical coordinates, $\displaystyle\iint_S z\,dS = \int_0^{2\pi}\int_0^{\pi/2} a\cos\phi\cdot a^2\sin\phi\,d\phi\,d\theta.$

Inner: $= a^3\displaystyle\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi = a^3(\frac{\sin^2\phi}{2}\Big]_0^{\pi/2} = \frac{a^3}{2}.$ Outer: $= \displaystyle\frac{a^3}{2}\int_0^{2\pi}d\theta = \pi a^3.$

Finally, $\displaystyle\iint_S dS =$ area of hemisphere $= 2\pi a^2$, so average distance $= \dfrac{\pi a^3}{2\pi a^2} = \dfrac{a}{2}.$

6C. Divergence Theorem

6C-1a div $\mathbf{F} = M_x + N_y + P_z = 2xy + x + x = 2x(y+1)$.

6C-2 Using the product and chain rules for the first, symmetry for the others,

$$(\rho^n x)_x = n\rho^{n-1}\,\frac{x}{\rho}\,x + \rho^n, \quad (\rho^n y)_y = n\rho^{n-1}\,\frac{y}{\rho}\,y + \rho^n, \quad (\rho^n z)_z = n\rho^{n-1}\,\frac{z}{\rho}\,z + \rho^n;$$

adding these three, we get $\quad \text{div}\,\mathbf{F} = n\rho^{n-1}\dfrac{x^2+y^2+z^2}{\rho} \;+\; 3\rho^n \;=\; \rho^n(n+3).$

Therefore, div $\mathbf{F} = 0 \;\Leftrightarrow\; n = -3$.

6C-3 Evaluating the triple integral first, we have div $\mathbf{F} = 3$, therefore

$$\iiint_D \text{div}\,\mathbf{F}\,dV = 3(\text{vol.of } D) = 3\,\frac{2}{3}\pi a^3 = 2\pi a^3.$$

To evaluate the double integral over the closed surface $S_1 + S_2$, the normal vectors are:

$$\mathbf{n}_1 = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a} \quad (\text{hemisphere } S_1), \qquad \mathbf{n}_2 = -\,\mathbf{k} \quad (\text{disc } S_2);$$

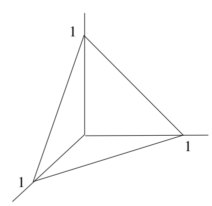
using these, the surface integral for the flux through S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \frac{x^2+y^2+z^2}{a}\,dS + \iint_{S_2} -z\,dS = \iint_{S_1} a\,dS,$$

since $x^2+y^2+z^2 = \rho^2 = a^2$ on S_1, and $z = 0$ on S_2. So the value of the surface integral is

$$a(\text{area of } S_1) = a(2\pi a^2) = 2\pi a^3,$$

which agrees with the triple integral above.



6C-5 The divergence theorem says $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \text{div}\,\mathbf{F}\,dV.$

Here div $\mathbf{F} = 1$, so that the right-hand integral is just the volume of the tetrahedron, which is $\frac{1}{3}$(base)(height)$= \frac{1}{3}(\frac{1}{2})(1) = \frac{1}{6}$.

6C-6 The divergence theorem says $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \text{div}\,\mathbf{F}\,dV.$

Here div $\mathbf{F} = 1$, so the right-hand integral is the volume of the solid cone, which has height 1 and base radius 1; its volume is $\frac{1}{3}$(base)(height)$= \pi/3$.

6C-7a Evaluating the triple integral first, over the cylindrical solid D, we have

$$\text{div}\,\mathbf{F} = 2x + x = 3x; \qquad \iiint_D 3x\,dV = 0,$$

since the solid is symmetric with respect to the yz-plane. (Physically, assuming the density is 1, the integral has the value $\bar{x}$(mass of D), where $\bar{x}$ is the x-coordinate of the center of mass; this must be in the yz plane since the solid is symmetric with respect to this plane.)

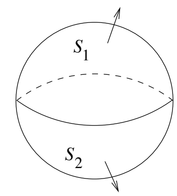
To evaluate the double integral, note that $\mathbf{F}$ has no $\mathbf{k}$-component, so there is no flux across the two disc-like ends of the solid. To find the flux across the cylindrical side,

$$\mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j}, \qquad \mathbf{F}\cdot\mathbf{n} = x^3 + xy^2 = x^3 + x(1-x^2) = x,$$

since the cylinder has radius 1 and equation $x^2+y^2=1$. Thus

$$\iint_S x\,dS = \int_0^{2\pi}\int_0^1 \cos\theta\,dz\,d\theta = \int_0^{2\pi}\cos\theta\,d\theta = 0.$$

6C-8 a) Reorient the lower hemisphere S_2 by reversing its normal vector; call the reoriented surface S_2'. Then $S = S_1 + S_2'$ is a closed surface, with the normal vector pointing outward everywhere, so by the divergence theorem,



$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2'}\mathbf{F}\cdot d\mathbf{S} = \iiint_D \operatorname{div}\mathbf{F}\,dV = 0,$$

since by hypothesis $\operatorname{div}\mathbf{F}=0$. The above shows

$$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = -\iint_{S_2'}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S},$$

since reversing the orientation of a surface changes the sign of the flux through it.

b) The same statement holds if S_1 and S_2 are two oriented surfaces having the same boundary curve, but not intersecting anywhere else, and oriented so that S_1 and S_2' (i.e., S_2 with its orientation reversed) together make up a closed surface S with outward-pointing normal.

6C-10 If $\operatorname{div}\mathbf{F}=0$, then for any closed surface S, we have by the divergence theorem

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \operatorname{div}\mathbf{F}\,dV = 0.$$

Conversely: $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$ for every closed surface $S \quad\Rightarrow\quad \operatorname{div}\mathbf{F}=0.$

For suppose there were a point P_0 at which $(\operatorname{div}\mathbf{F})_0 \neq 0$ — say $(\operatorname{div}\mathbf{F})_0 > 0$. Then by continuity, $\operatorname{div}\mathbf{F} > 0$ in a very small spherical ball D surrounding P_0, so that by the divergence theorem (S is the surface of the ball D),

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \operatorname{div}\mathbf{F}\,dV \quad > \quad 0.$$

But this contradicts our hypothesis that $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$ for every closed surface S.

6C-11 flux of $\mathbf{F} = \displaystyle\iint_S \mathbf{F}\cdot d\mathbf{n} = \iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D 3\,dV = 3(\text{vol. of } D).$

6D. Line Integrals in Space

6D-1 a) $C: \quad x=t,\ dx=dt;\quad y=t^2,\ dy=2t\,dt;\quad z=t^3,\ dz=3t^2\,dt;$

$$\begin{aligned}\int_C y\,dx + z\,dy - x\,dz \;&=\; \int_0^1 (t^2)dt + t^3(2t\,dt) - t(3t^2\,dt)\\ &= \int_0^1 (t^2+2t^4-3t^3)dt \;=\; \frac{t^3}{3}+\frac{2t^5}{5}-\frac{3t^4}{4}\Big]_0^1 \;=\; \frac13+\frac25-\frac34 \;=\; -\frac{1}{60}.\end{aligned}$$

b) $C: \quad x=t,\ y=t,\ z=t;\qquad \displaystyle\int_C y\,dx + z\,dy - x\,dz \;=\; \int_0^1 t\,dt \;=\; \frac12.$

c) $C = C_1 + C_2 + C_3$; $\quad C_1 : y = z = 0$; $\quad C_2 : x = 1,\ z = 0$; $\quad C_3 : x = 1,\ y = 1$

$$\int_C y\,dx + z\,dy - x\,dz \;=\; \int_{C_1} 0 + \int_{C_2} 0 + \int_0^1 -dz \;=\; -1.$$

d) $C : x = \cos t,\ y = \sin t,\ z = t$; $\quad \displaystyle\int_C zx\,dx + zy\,dy + x\,dz$

$$= \int_0^{2\pi} t\cos t(-\sin t\,dt) + t\sin t(\cos t\,dt) + \cos t\,dt \;=\; \int_0^{2\pi} \cos t\,dt \;=\; 0.$$

6D-2 The field $\mathbf{F}$ is always pointed radially outward; if C lies on a sphere centered at the origin, its unit tangent $\mathbf{t}$ is always tangent to the sphere, therefore perpendicular to the radius; this means $\mathbf{F}\cdot\mathbf{t} = 0$ at every point of C. Thus $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{t}\,ds = 0$.

6D-4 a) $\mathbf{F} = \nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$.

b) (i) Directly, letting C be the helix: $x = \cos t,\ y = \sin t,\ z = t$, from $t = 0$ to $t = 2n\pi$,

$$\int_C M dx + N dy + P dz = \int_0^{2n\pi} 2\cos t(-\sin t)dt + 2\sin t(\cos t)dt + 2t\,dt = \int_0^{2n\pi} 2t\,dt = (2n\pi)^2.$$

b) (ii) Choose the vertical path $x = 1,\ y = 0,\ z = t$; then

$$\int_C M dx + N dy + P dz = \int_0^{2n\pi} 2t\,dt = (2n\pi)^2.$$

b) (iii) By the First Fundamental Theorem for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,0,2n\pi) - f(1,0,0) = 91^2 + (2n\pi)^2) - 1^2 = (2n\pi)^2$$

6D-5 By the First Fundamental Theorem for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \sin(xyz)\Big|_Q - \sin(xyz)\Big|_P,$$

where C is any path joining P to Q. The maximum value of this difference is $1 - (-1) = 2$, since $\sin(xyz)$ ranges between -1 and 1.

For example, any path C connecting $P : (1, 1, -\pi/2)$ to $Q : (1, 1, \pi/2)$ will give this maximum value of 2 for $\int_C \mathbf{F}\cdot d\mathbf{r}$.

6E. Gradient Fields in Space

6E-1 a) Since $M = x^2,\ N = y^2,\ P = z^2$ are continuously differentiable, the differential is exact because $N_z = P_y = 0,\quad M_z = P_x = 0,\quad M_y = N_x = 0;\ f(x,y,z) = (x^3 + y^3 + z^3)/3$.

b) Exact: M, N, P are continuously differentiable for all x, y, z, and

$$N_z = P_y = 2xy, \quad M_z = P_x = y^2, \quad M_y = N_x = 2yz; \quad f(x,y,z) = xy^2\ .$$

c) Exact: M, N, P are continuously differentiable for all x, y, z, and

$$N_z = P_y = x, \quad M_z = P_x = y, \quad M_y = N_x = 6x^2 + z;\ f(x,y,z) = 2x^3y + xyz.$$

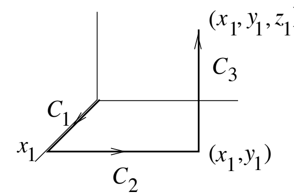
6E-2 curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2y & yz & xyz^2 \end{vmatrix} = (xz^2 - y)\,\mathbf{i} - yz^2\,\mathbf{j} - x^2\,\mathbf{k}$.

6E-3 a) It is easily checked that curl $\mathbf{F} = 0$.

b) (i) using method I:

$$f(x_1, y_1, z_1) = \int_{(0,0,0)}^{(x_1,y_1,z_1)} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^{x_1} x\,dx + \int_0^{y_1} y\,dy + \int_0^{z_1} z\,dz = \frac{1}{2}x_1^2 + \frac{1}{2}y_1^2 + \frac{1}{2}z_2^2.$$



Therefore $f(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2) + c$.

(ii) Using method II: We seek $f(x, y, z)$ such that $f_x = 2xy + z$, $f_y = x^2$, $f_z = x$.

$f_x = 2xy + z \quad \Rightarrow \quad f = x^2y + xz + g(y, z).$
$f_y = x^2 + g_y = x^2 \quad \Rightarrow \quad g_y = 0 \quad \Rightarrow \quad g = h(z)$
$f_z = x + h'(z) = x \quad \Rightarrow \quad h' = 0 \quad \Rightarrow \quad h = c$

Therefore $f(x, y, z) = x^2y + xz + c$.

(iii) If $f_x = yz$, $f_y = xz$, $f_z = xy$, then by inspection, $f(x, y, z) = xyz + c$.

6E-4 Let $F = f - g$. Since ∇ is a linear operator, $\nabla F = \nabla f - \nabla g = \mathbf{0}$

We now show: $\nabla F = \mathbf{0} \Rightarrow F = c$.

Fix a point $P_0 : (x_0, y_0, z_0)$. Then by the Fundamental Theorem for line integrals,

$$F(P) - F(P_0) = \int_{P_0}^{P} \nabla F \cdot d\mathbf{r} = 0.$$

Therefore $F(P) = F(P_0)$ for all P, i.e., $F(x, y, z) = F(x_0, y_0, z_0) = c$.

6E-5 $\mathbf{F}$ is a gradient field only if these equations are satisfied:

$$N_z = P_y : \ 2xz + ay = bxz + 2y \qquad M_z = P_x : \ 2yz = byz \qquad M_y = N_x : \ z^2 = z^2.$$

Thus the conditions are: $a = 2$, $b = 2$.

Using these values of a and b we employ Method 2 to find the potential function f:

$f_x = yz^2 \quad \Rightarrow \quad f = xyz^2 + g(y, z);$
$f_y = xz^2 + g_y = xz^2 + 2yz \quad \Rightarrow \quad g_y = 2yz \quad \Rightarrow \quad g = y^2z + h(z)$
$f_z = 2xyz + y^2 + h'(z) = 2xyz + y^2 \quad \Rightarrow \quad h = c;$

therefore, $f(x, y, z) = xyz^2 + y^2z + c$.

6E-6 a) $Mdx + Ndy + Pdz$ is an exact differential if there exists some function $f(x, y, z)$ for which $df = Mdx + Ndy + Pdz$; that, is, for which $f_x = M$, $f_y = N$, $f_z = P$.

b) The given differential is exact if the following equations are satisfied:

$N_z = P_y : \quad (a/2)x^2 + 6xy^2z + 3byz^2 = 3x^2 + 3cxy^2z + 12yz^2;$
$M_z = P_x : \quad axy + 2y^3z = 6xy + cy^3z$
$M_y = N_x : \quad axz + 3y^2z^2 = axz + 3y^2z^2.$

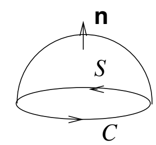
Solving these, we find that the differential is exact if $a = 6$, $b = 4$, $c = 2$.

c) We find $f(x, y, z)$ using method 2:

$$f_x = 6xyz + y^3z^2 \quad \Rightarrow \quad f = 3x^2yz + xy^3z^2 + g(y,z);$$
$$f_y = 3x^2z + 3xy^2z^2 + g_y = 3x^2z + 3xy^2z^2 + 4yz^3 \quad \Rightarrow \quad g_y = 4yz^3 \quad \Rightarrow \quad g = 2y^2z^3 + h(z)$$
$$f_z = 3x^2y + 2xy^3z + 6y^2z^2 + h'(z) = 3x^2y + 2xy^3z + 6y^2z^2 \quad \Rightarrow \quad h'(z) = 0 \quad \Rightarrow \quad h = c.$$

Therefore, $\quad f(x, y, z) = 3x^2yz + xy^3z^2 + 2y^2z^3 + c.$

6F. Stokes' Theorem



6F-1 a) For the line integral, $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C x\,dx + y\,dy + z\,dz = 0$, since the differential is exact.

For the surface integral, $\quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = \mathbf{0}$, and therefore $\displaystyle\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = 0.$

b) Line integral: $\displaystyle\oint_C y\,dx + z\,dy + x\,dz = \oint_C y\,dx$, since $z = 0$ and $dz = 0$ on C.

Using $x = \cos t, \quad y = \sin t, \quad \displaystyle\int_0^{2\pi} -\sin^2 t\,dt = -\int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt = -\pi.$

Surface integral: curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = -\,\mathbf{i} - \mathbf{j} - \mathbf{k}\,; \qquad \mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$$\iint_S \nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = -\iint_S (x+y+z)\,dS.$$

To evaluate, we use $x = r\cos\theta, \quad y = r\sin\theta, \quad z = \rho\cos\phi. \quad r = \rho\sin\phi, \quad dS = \rho^2\sin\phi\,d\phi d\theta$; note that $\rho = 1$ on S. The integral then becomes

$$-\int_0^{2\pi}\int_0^{\pi/2} [\sin\phi(\cos\theta + \sin\theta) + \cos\phi]\sin\phi\,d\phi\,d\theta$$

Inner: $\displaystyle -\left[(\cos\theta + \sin\theta)(\frac{\phi}{2} - \frac{\sin 2\phi}{4}) + \frac{1}{2}\sin^2\phi\right]_0^{\pi/2} \quad = \quad -\left[(\cos\theta+\sin\theta)\frac{\pi}{4} + \frac{1}{2}\right]\,;$

Outer: $\displaystyle\int_0^{2\pi}\left(-\frac{1}{2} - (\cos\theta+\sin\theta)\frac{\pi}{4}\right)d\theta = -\pi.$

6F-2 The surface S is: $\; z = -x - y$, so that $f(x, y) = -x - y$.

$$\mathbf{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy \;=\; \langle 1,1,1\rangle\,dx\,dy.$$

(Note the signs: $\mathbf{n}$ points upwards, and therefore should have a positive k-component.)

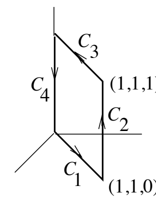
$$\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = -\,\mathbf{i} - \mathbf{j} - \mathbf{k}$$

Therefore $\displaystyle\iint_S \text{curl }\mathbf{F}\cdot\mathbf{n}\,dS = -\iint_{S'} 3\,dA = -3\pi$, where S' is the projection of S, i.e., the interior of the unit circle in the xy-plane.

As for the line integral, we have $C: \quad x = \cos t, \quad y = \sin t \quad z = -\cos t - \sin t$, so that

$$\oint_C ydx + zdy + xdz = \int_0^{2\pi}\left[-\sin^2 t - (\cos^2 t + \sin t\cos t) + \cos t(\sin t - \cos t)\right] dt$$
$$= \int_0^{2\pi}(-\sin^2 t - \cos^2 t - \cos^2 t)\, dt = \int_0^{2\pi}\left[-1 - \frac{1}{2}(1+\cos 2t)\right] dt = -\frac{3}{2}\cdot 2\pi = -3\pi.$$

6F-3 Line integral: $\oint_C yz\,dx + xz\,dy + xy\,dz$ over the path $C = C_1 + \ldots + C_4$:



$\int_{C_1} = 0,\quad$ since $z = dz = 0$ on C_1;

$\int_{C_2} = \int_0^1 1\cdot 1\,dz = 1,\quad$ since $x=1,\ y=1,\ dx=0,\ dy=0$ on C_2;

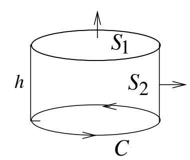
$\int_{C_3} ydx + xdy = \int_1^0 xdx + xdx = -1,\quad$ since $y=x,\ z=1,\ dz=0$ on C_3;

$\int_{C_4} = 0,\quad$ since $x=0,\ y=0$ on C_4.

Adding up, we get $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0.$ For the surface integral,

$$\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{vmatrix} = \mathbf{i}\,(x-x) - \mathbf{j}\,(y-y) + \mathbf{k}\,(z-z) = \mathbf{0};\ \text{ thus } \iint \text{curl }\mathbf{F}\cdot d\mathbf{S} = 0.$$

6F-5 Let S_1 be the top of the cylinder (oriented so $\mathbf{n} = \mathbf{k}$), and S_2 the side.



a) We have $\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & x^2 \end{vmatrix} = -2x\,\mathbf{j} + 2\,\mathbf{k}$.

For the top: $\iint_{S_1} \text{curl }\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} 2\,dS = 2(\text{area of } S_1) = 2\pi a^2.$

For the side: we have $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{a}$, and $dS = dz\cdot a\,d\theta$, so that

$$\iint_{S_2} \text{curl }\mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^h \frac{-2xy}{a}a\,dz\,d\theta = \int_0^{2\pi} -2h(a\cos\theta)(a\sin\theta)\,d\theta = -ha^2\sin^2\theta\Big]_0^{2\pi} = 0.$$

Adding, $\iint_S \text{curl }\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} + \iint_{S_2} = 2\pi a^2.$

b) Let C be the circular boundary of S, parameterized by $x = a\cos\theta,\ y = a\sin\theta,\ z = 0.$ Then using Stokes' theorem,

$$\iint_S \text{curl }\mathbf{F}\cdot d\mathbf{S} = \oint_C -y\,dx + x\,dy + x^2\,dz = \int_0^{2\pi}(a^2\sin^2\theta + a^2\cos^2\theta)\,d\theta = 2\pi a^2.$$

6G. Topological Questions

6G-1 a) yes b) no c) yes d) no; yes; no; yes; no; yes

6G-2 Recall that $\rho_x = x/\rho$, etc. Then, using the chain rule,

$$\text{curl }\mathbf{F} = (n\rho^{n-1}z\,\frac{y}{\rho} - n\rho^{n-1}y\,\frac{z}{\rho})\,\mathbf{i} + (n\rho^{n-1}z\,\frac{x}{\rho} - n\rho^{n-1}x\,\frac{z}{\rho})\,\mathbf{j} + (n\rho^{n-1}y\,\frac{x}{\rho} - n\rho^{n-1}x\,\frac{y}{\rho})\,\mathbf{k}\,.$$

Therefore curl $\mathbf{F} = \mathbf{0}$. To find the potential function, we let P_0 be any convenient starting point, and integrate along some path to $P_1 : (x_1, y_1, z_1)$. Then, if $n \neq -2$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{P_0}^{P_1} \rho^n (x\,dx + y\,dy + z\,dz) = \int_{P_0}^{P_1} \rho^n \frac{1}{2} d(\rho^2)$$
$$= \int_{P_0}^{P_1} \rho^{n+1} d\rho = \frac{\rho^{n+2}}{n+2}\bigg]_{P_0}^{P_1} = \frac{\rho_1^{n+2}}{n+2} - \frac{\rho_0^{n+2}}{n+2} = \frac{\rho_1^{n+2}}{n+2} + c, \text{ since } P_0 \text{ is fixed.}$$

Therefore, we get $\mathbf{F} = \nabla \dfrac{\rho^{n+2}}{n+2}$, if $n \neq -2$.

If $n = -2$, the line integral becomes $\displaystyle\int_{P_0}^{P_1} \frac{d\rho}{\rho} = \ln \rho_1 + c$, so that $\mathbf{F} = \nabla(\ln \rho)$.

6H. Applications and Further Exercises

6H-1 Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$. By the definition of curl $\mathbf{F}$, we have

$$\nabla \times \mathbf{F} = (P_y - N_z)\,\mathbf{i} + (M_z - P_x)\,\mathbf{j} + (N_x - M_y)\,\mathbf{k}\,,$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = (P_{yx} - N_{zx}) + (M_{zy} - P_{xy}) + (N_{xz} - M_{yz})$$

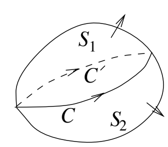
If all the mixed partials exist and are continuous, then $P_{xy} = P_{yx}$, etc. and the right-hand side of the above equation is zero: div (curl $\mathbf{F}$) $= 0$.

6H-2 a) Using the divergence theorem, and the previous problem, (D is the interior of S),

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iiint_D \text{div curl } \mathbf{F}\, dV = \iiint_D 0\, dV = 0.$$

b) Draw a closed curve C on S that divides it into two pieces S_1 and S_2 both having C as boundary. Orient C compatibly with S_1, then the curve C' obtained by reversing the orientation of C will be oriented compatibly with S_2. Using Stokes' theorem,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} + \oint_{C'} \mathbf{F} \cdot d\mathbf{r} = 0,$$



since the integral on C' is the negative of the integral on C.

Or more simply, consider the limiting case where C has been shrunk to a point; even as a point, it can still be considered to be the boundary of S. Since it has zero length, the line integral around it is zero, and therefore Stokes' theorem gives

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \;=\; \oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; 0.$$

6H-10 Let C be an oriented closed curve, and S a compatibly-oriented surface having C as its boundary. Using Stokes' theorem and the Maxwell equation, we get respectively

$$\iint_S \nabla\times\mathbf{B}\cdot d\mathbf{S} = \oint_C \mathbf{B}\cdot d\mathbf{r} \qquad \text{and} \qquad \iint_S \nabla\times\mathbf{B}\cdot d\mathbf{S} = \iint_S \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{S} = \frac{1}{c}\frac{d}{dt}\iint_S E\cdot d\mathbf{S}.$$

Since the two left sides are the same, we get $\displaystyle\oint_C \mathbf{B} \cdot d\mathbf{r} = \frac{1}{c}\frac{d}{dt}\iint_S \mathbf{E} \cdot d\mathbf{S}.$

In words: for the magnetic field $\mathbf{B}$ produced by a moving electric field $\mathbf{E}(t)$, the magnetomotive force around a closed loop C is, up to a constant factor depending on the units, the time-rate at which the electric flux through C is changing.

18.02 Exam 1 Solutions

Problem 1.

a) $P = (1,0,0)$, $Q = (0,2,0)$ and $R = (0,0,3)$. Therefore $\overrightarrow{QP} = \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$ and $\overrightarrow{QR} = -2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}$.

b) $\cos\theta = \dfrac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QP}\right|} = \dfrac{\langle 1,-2,0\rangle\cdot\langle 0,-2,3\rangle}{\sqrt{1^2+2^2}\sqrt{2^2+3^2}} = \dfrac{4}{\sqrt{65}}$

Problem 2.

a) $\overrightarrow{PQ} = \langle -1,2,0\rangle$, $\overrightarrow{PR} = \langle -1,0,3\rangle$.

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}.$$

Then $\quad area(\Delta) = \dfrac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \dfrac{1}{2}\sqrt{6^2+3^2+2^2} = \dfrac{1}{2}\sqrt{49} = \dfrac{7}{2}.$

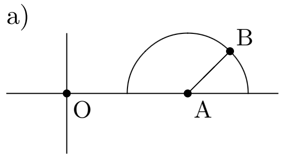
b) A normal to the plane is given by $\overrightarrow{N} = \overrightarrow{PQ}\times\overrightarrow{PR} = \langle 6,3,2\rangle$. Hence the equation has the form $6x+3y+2z = d$. Since P is on the plane $d = 6\cdot 1 + 3\cdot 1 + 2\cdot 1 = 11$. In conclusion the equation of the plane is

$$6x+3y+2z = 11.$$

c) The line is parallel to $\langle 2-1, 2-2, 0-3\rangle = \langle 1,0,-3\rangle$. Since $\overrightarrow{N}\cdot\langle 1,0,-3\rangle = 6-6 = 0$, the line is parallel to the plane.

Problem 3.

a)



$\overrightarrow{OA} = \langle 10t, 0\rangle$ and $\overrightarrow{AB} = \langle \cos t, \sin t\rangle$, hence

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \langle 10t + \cos t, \sin t\rangle.$$

The rear bumper is reached at time $t = \pi$ and the position of B is $(10\pi - 1, 0)$.

b) $\overrightarrow{V} = \langle 10 - \sin t, \cos t\rangle$, thus

$$|\overrightarrow{V}|^2 = (10-\sin t)^2 + \cos^2 t = 100 - 20\sin t + \sin^2 t + \cos^2 t = 101 - 20\sin t.$$

The speed is then given by $\sqrt{101 - 20\sin t}$. The speed is smallest when $\sin t$ is largest i.e. $\sin t = 1$. It occurs when $t = \pi/2$. At this time, the position of the bug is $(5\pi, 1)$. The speed is largest when $\sin t$ is smallest; that happens at the times $t = 0$ or π for which the position is then $(0,0)$ and $(10\pi - 1, 0)$.

Problem 4.

a) $|M| = -12$.

b) $a = -5$, $b = 7$.

c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{12}\begin{bmatrix} 1 & 1 & 4 \\ -5 & 7 & -8 \\ 7 & -5 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} t/12 + 1 \\ 7t/12 - 2 \\ -5t/12 + 1 \end{bmatrix}$

d) $\dfrac{d\vec{\boldsymbol{r}}}{dt} = \left\langle \dfrac{1}{12}, \dfrac{7}{12}, -\dfrac{5}{12}\right\rangle.$

Problem 5.

a) $\overrightarrow{N}\cdot\vec{r}(t) = 6$, where $\overrightarrow{N} = \langle 4,-3,-2\rangle$.

b) We differentiate $\overrightarrow{N}\cdot\vec{r}(t) = 6$:

$0 = \dfrac{d}{dt}\left(\overrightarrow{N}\cdot\vec{r}(t)\right) = \dfrac{d}{dt}\overrightarrow{N}\cdot\vec{r}(t) + \overrightarrow{N}\cdot\dfrac{d}{dt}\vec{r}(t) = \vec{0}\cdot\vec{r}(t) + \overrightarrow{N}\cdot\dfrac{d}{dt}\vec{r}(t)$ $\quad$ and hence $\overrightarrow{N}\perp\frac{d}{dt}\vec{r}(t)$.

18.02SC Multivariable Calculus
Fall 2010

18.02 Exam 2 – Solutions

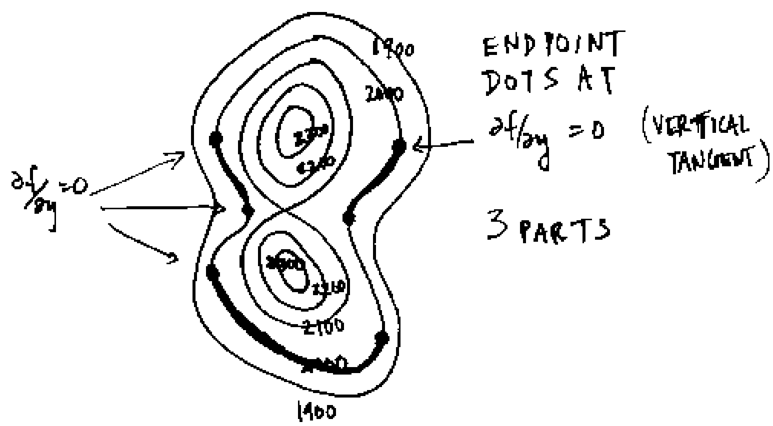
Problem 1. a) $\nabla f = \langle 2xy^2 - 1, 2x^2y\rangle = \langle 3, 8\rangle = 3\hat{\imath} + 8\hat{\jmath}$.

b) $z - 2 = 3(x-2) + 8(y-1) \quad$ or $\quad z = 3x + 8y - 12$.

c) $\Delta x = 1.9 - 2 = -1/10$ and $\Delta y = 1.1 - 1 = 1/10$. So $z \simeq 2 + 3\Delta x + 8\Delta y = 2 - 3/10 + 8/10 = 2.5$

d) $\left.\dfrac{df}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u} = \langle 3, 8\rangle \cdot \dfrac{\langle -1, 1\rangle}{\sqrt{2}} = \dfrac{-3+8}{\sqrt{2}} = \dfrac{5}{\sqrt{2}}$

Problem 2.



Problem 3. a) $\left.\begin{array}{lcl} w_x = -6x - 4y + 16 = 0 & \Rightarrow & -3x - 2y + 8 = 0 \\ w_y = -4x - 2y - 12 = 0 & \Rightarrow & 4x + 2y + 12 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = -20 \\ y = 34 \end{array}\right.$

Therefore there is just one critical point at $(-20, 34)$. Since

$$w_{xx}w_{yy} - w_{xy}^2 = (-6)\,(-2) - (-4)^2 = 12 - 16 = -4 < 0,$$

the critical point is a saddle point.

b) There is no critical point in the first quadrant, hence the maximum must be at infinity or on the boundary of the first quadrant.

The boundary is composed of two half-lines:

- $x = 0$ and $y \geq 0$ on which $w = -y^2 - 12y$. It has a maximum ($w = 0$) at $y = 0$.
- $y = 0$ and $x \geq 0$, where $w = -3x^2 + 16x$. (The graph is a parabola pointing downwards). Maximum: $w_x = -6x + 16 = 0 \quad \Rightarrow x = 8/3$. Hence w has a maximum at $(8/3, 0)$ and $w = -3(8/3)^2 + 16 \cdot 8/3 = 64/3 > 0$.

We now check that the maximum of w is not at infinity:

- If $y \geq 0$ and $x \to +\infty$ then $w \leq -3x^2 + 16x$, which tends to $-\infty$ as $x \to +\infty$.
- If $0 \leq x \leq C$ and $y \to +\infty$, then $w \leq -y^2 + 16C$, which tends to $-\infty$ as $y \to +\infty$.

We conclude that the maximum of w in the first quadrant is at $(8/3, 0)$.

Problem 4. a) $\left\{\begin{array}{lcl} w_x & = u_x w_u + v_x w_v & = & -\frac{y}{x^2} w_u + 2x w_v \\ w_y & = u_y w_u + v_y w_v & = & \frac{1}{x} w_u + 2y w_v \end{array}\right.$

b)$x w_x + y w_y = x(-\dfrac{y}{x^2} w_u + 2x w_v) + y(\dfrac{1}{x} w_u + 2y w_v) = (-\dfrac{y}{x} + \dfrac{y}{x}) w_u + (2x^2 + 2y^2) w_v = 2v w_v$.

c)$x w_x + y w_y = 2v w_v = 2v \cdot 5v^4 = 10v^5$.

Problem 5. a) $f(x,y,z) = x$; the constraint is $g(x,y,z) = x^4 + y^4 + z^4 + xy + yz + zx = 6$. The Lagrange multiplier equation is:

$$\nabla f = \lambda \nabla g \quad \Leftrightarrow \quad \begin{cases} 1 & = & \lambda(4x^3 + y + z) \\ 0 & = & \lambda(4y^3 + x + z) \\ 0 & = & \lambda(4z^3 + x + y) \end{cases}$$

b) The level surfaces of f and g are tangent at (x_0, y_0, z_0), so they have the same tangent plane. The level surface of f is the plane $x = x_0$; hence this is also the tangent plane to the surface $g = 6$ at (x_0, y_0, z_0).

Second method: at (x_0, y_0, z_0), we have

$$\left.\begin{array}{rl} 1 = & \lambda g_x \\ 0 = & \lambda g_y \\ 0 = & \lambda g_z \end{array}\right\} \quad \Rightarrow \quad \lambda \neq 0 \text{ and } \langle g_x, g_y, g_z \rangle = \langle \frac{1}{\lambda}, 0, 0 \rangle.$$

So $\langle \frac{1}{\lambda}, 0, 0 \rangle$ is perpendicular to the tangent plane at (x_0, y_0, z_0); the equation of the tangent plane is then $\frac{1}{\lambda}(x - x_0) = 0$, or equivalently $x = x_0$.

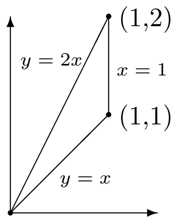
Problem 6.

a) Taking the total differential of $x^2 + y^3 - z^4 = 1$, we get: $2x\,dx + 3y^2\,dy - 4z^3\,dz = 0$. Similarly, from $z^3 + zx + xy = 3$, we get: $(y + z)\,dx + x\,dy + (3z^2 + x)\,dz = 0$.

b) At $(1,1,1)$ we have: $2\,dx + 3\,dy - 4\,dz = 0$ and $2\,dx + dy + 4\,dz = 0$. We eliminate dz (by adding these two equations): $4\,dx + 4\,dy = 0$, so $dy = -dx$, and hence $dy/dx = -1$.

18.02 Exam 3 – Solutions

1. a)



b) $\displaystyle\int_0^1 \int_{y/2}^{y} dxdy + \int_1^2 \int_{y/2}^{1} dxdy.$

(the first integral corresponds to the bottom half $0 \le y \le 1$, the second integral to the top half $1 \le y \le 2$.)

2. a) $\delta dA = \dfrac{r\sin\theta}{r^2} r dr d\theta = \sin\theta dr d\theta.$

$$M = \iint_R \delta dA = \int_0^\pi \int_1^3 \sin\theta \; dr d\theta = \int_0^\pi 2\sin\theta d\theta = \big[-2\cos\theta\big]_0^\pi = 4.$$

b) $\displaystyle \bar{x} = \frac{1}{M}\iint_R x\delta dA = \frac{1}{4}\int_0^\pi \int_1^3 r\cos\theta\sin\theta dr d\theta$

The reason why one knows that $\bar{x} = 0$ without computation is that the region **and the density** are symmetric with respect to the y-axis ($\delta(x, y) = \delta(-x, y)$).

3. a) $N_x = -12y = M_y$, hence $\mathbf{F}$ is conservative.

b) $f_x = 3x^2 - 6y^2 \Rightarrow f = x^3 - 6y^2x + c(y) \Rightarrow f_y = -12xy + c'(y) = -12xy + 4y$. So $c'(y) = 4y$, thus $c(y) = 2y^2 \quad$ (+ constant). In conclusion

$$f = x^3 - 6xy^2 + 2y^2 \quad (+\text{ constant}).$$

c) The curve C starts at $(1, 0)$ and ends at $(1, 1)$, therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(1,0) = (1 - 6 + 2) - 1 = -4.$$

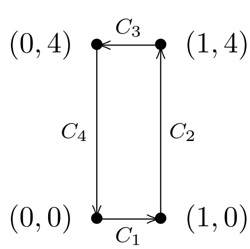
4. a) The parametrization of the circle C is $x = \cos t$, $y = \sin t$, for $0 \le t < 2\pi$; then $dx = -\sin t dt$, $dy = \cos t dt$ and

$$W = \int_0^{2\pi} (5\cos t + 3\sin t)(-\sin t)dt + (1 + \cos(\sin t))\cos t dt.$$

b) Let R be the unit disc inside C;

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (N_x - M_y)dA = \iint_R (0-3)dA = -3 \text{ area}(R) = -3\pi.$$

5. a)



$$\begin{aligned}
\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\, ds &= \iint_R \operatorname{div}\mathbf{F}\, dxdy \\
&= \iint_R (y + \cos x\cos y - \cos x\cos y)dxdy = \iint_R y dxdy \\
&= \int_0^4\int_0^1 y dxdy = \int_0^4 y dy = [y^2/2]_0^4 = 8.
\end{aligned}$$

b) On C_4, $x = 0$, so $\mathbf{F} = -\sin y\,\hat{\mathbf{j}}$, whereas $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$. Hence $\mathbf{F} \cdot \hat{\mathbf{n}} = 0$. Therefore the flux of $\mathbf{F}$ through C_4 equals 0. Thus

$$\int_{C_1+C_2+C_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds - \int_{C_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds\ ;$$

and the total flux through $C_1 + C_2 + C_3$ is equal to the flux through C.

6. Let $u = 2x - y$ and $v = x + y - 1$. The Jacobian $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3.$

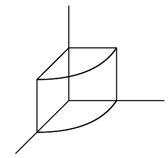
Hence $dudv = 3dxdy$ and $dxdy = \dfrac{1}{3}dudv$, so that

$$\begin{aligned}
V &= \iint\limits_{(2x-y)^2+(x+y-1)^2<4} (4 - (2x-y)^2 - (x+y-1)^2)\, dxdy \\
&= \iint\limits_{u^2+v^2<4} (4 - u^2 - v^2)\frac{1}{3}dudv \\
&= \int_0^{2\pi}\int_0^2 (4 - r^2)\frac{1}{3}rdrd\theta = \int_0^{2\pi} \left[\frac{2}{3}r^2 - \frac{1}{12}r^4\right]_0^2 d\theta \\
&= \int_0^{2\pi} \frac{4}{3}d\theta = \frac{8\pi}{3}.
\end{aligned}$$

18.02SC Multivariable Calculus
Fall 2010

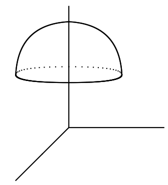
18.02 Exam 4 – Solutions

Problem 1.



$$\int_0^{\pi/2}\int_0^1\int_0^1 r^2\, r\, dz\, dr\, d\theta.$$

Problem 2.

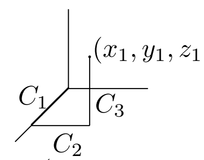


a) sphere: $\rho = 2a\cos\phi$. b) plane: $\rho = a\sec\phi$.

c) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_{a\sec\phi}^{2a\cos\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$

Problem 3.

a) $\frac{\partial}{\partial y}(2xy+z^3) = 2x = \frac{\partial}{\partial x}(x^2+2yz)$; $\frac{\partial}{\partial z}(2xy+z^3) = 3z^2 = \frac{\partial}{\partial x}(y^2+3xz^2-1)$;
$\frac{\partial}{\partial z}(x^2+2yz) = 2y = \frac{\partial}{\partial y}(y^2+3xz^2-1)$; so $\vec F$ is conservative.



b) <u>Method 1:</u> $f(x,y,z) = \int_{C_1+C_2+C_3}\vec F\cdot d\vec r$;
$\int_{C_1}\vec F\cdot d\vec r = \int_0^{x_1}(2xy+z^3)\,dx = \int_0^{x_1} 0\,dx = 0 \quad (y=0,\ z=0)$
$\int_{C_2}\vec F\cdot d\vec r = \int_0^{y_1}(x^2+2yz)\,dy = \int_0^{y_1} x_1^2\,dy = x_1^2 y_1 \quad (x = x_1,\ z=0)$
$\int_{C_3}\vec F\cdot d\vec r = \int_0^{z_1}(y^2+3xz^2-1)\,dz = \int_0^{z_1}(y_1^2+3x_1z^2-1)\,dz = y_1^2z_1 + x_1z_1^3 - z_1 \quad (x=x_1,\ y=y_1)$
So $f(x,y,z) = x^2y + y^2z + xz^3 - z + c$.

<u>Method 2:</u> $\frac{\partial f}{\partial x} = 2xy + z^3$, so $f(x,y,z) = x^2y + xz^3 + g(y,z)$.
$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 + 2yz$, so $\frac{\partial g}{\partial y} = 2yz$.
Therefore $g(y,z) = y^2z + h(z)$, and $f(x,y,z) = x^2y + xz^3 + y^2z + h(z)$.
$\frac{\partial f}{\partial z} = 3xz^2 + y^2 + h'(z) = y^2 + 3xz^2 - 1$, so $h'(z) = -1$.
Therefore $h(z) = -z + c$, and $f(x,y,z) = x^2y + xz^3 + y^2z - z + c$.

Problem 4.

a) S is the graph of $z = f(x,y) = 1 - x^2 - y^2$, so $\hat{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dA = \langle 2x, 2y, 1\rangle\,dA$.
Therefore $\iint_S \vec F\cdot\hat n\,dS = \iint_S\langle x, y, 2(1-z)\rangle\cdot\langle 2x,2y,1\rangle\,dA = \iint_S 2x^2+2y^2+2(1-z)\,dA = \iint_S 4x^2 + 4y^2\,dA$ (since $z = 1-x^2-y^2$).
Shadow = unit disc $x^2+y^2\le 1$; switching to polar coordinates, we have
$\iint_S \vec F\cdot\hat n\,dS = \int_0^{2\pi}\int_0^1 4r^2\,r\,dr\,d\theta = \int_0^{2\pi}\left[r^4\right]_0^1 d\theta = 2\pi.$
b) Let $T=$ unit disc in the xy-plane, with normal vector pointing down ($\hat n = -\hat k$). Then
$\iint_T \vec F\cdot\hat n\,dS = \iint_T\langle x,y,2\rangle\cdot(-\hat k)\,dS = \iint_T -2\,dS = -2\,\text{Area} = -2\pi$. By divergence theorem,
$\iint_{S+T}\vec F\cdot\hat n\,dS = \iiint_D \operatorname{div}\vec F\,dV = 0$, since $\operatorname{div}\vec F = 1+1-2 = 0$. Therefore $\iint_S = -\iint_T = +2\pi$.

Problem 5.

a) $\operatorname{curl}\vec F = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial_x & \partial_y & \partial_z\\ -2xz & 0 & y^2\end{vmatrix} = 2y\hat\imath - 2x\hat\jmath.$

b) On the unit sphere, $\hat n = x\hat\imath + y\hat\jmath + z\hat k$, so $\operatorname{curl}\vec F\cdot\hat n = \langle 2y, -2x, 0\rangle\cdot\langle x,y,z\rangle = 2xy - 2xy = 0$; therefore $\iint_R \operatorname{curl}\vec F\cdot\hat n\,dS = 0$.

c) By Stokes, $\oint_C \vec F\cdot d\vec r = \iint_R\operatorname{curl}\vec F\cdot\hat n\,dS$, where R is the region delimited by C on the unit sphere. Using the result of b), we get $\oint_C\vec F\cdot d\vec r = \iint_R\operatorname{curl}\vec F\cdot\hat n\,dS = 0$.

18.02 Final Exam Solutions

Problem 1.

a) Line L has direction vector $\mathbf{v} = \langle -1, 2, -3 \rangle$ which lies in $\mathcal{P}$.

To get a point P_0 on L take $t = 0 \Rightarrow P_0 = (1, 1, 2)$.

$\Rightarrow \overrightarrow{\mathbf{P_0Q}} = \langle -1, 1, 2 \rangle - \langle 1, 1, 2 \rangle = \langle -2, 0, 0 \rangle$ also lies in $\mathcal{P}$.

$\Rightarrow$ A normal to $\mathcal{P}$ is

$$\mathbf{n} = \mathbf{v} \times \overrightarrow{\mathbf{P_0Q}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ -2 & 0 & 0 \end{vmatrix} = \mathbf{i}(0) - \mathbf{j}(-6) + \mathbf{k}(4) = \langle 0, 6, 4 \rangle.$$

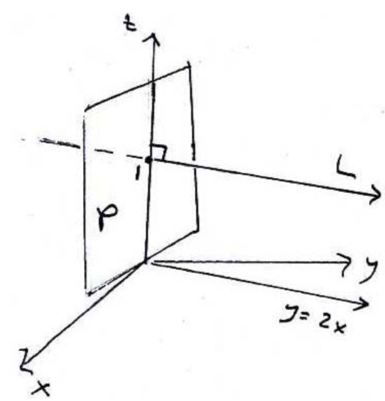
So, the equation of $\mathcal{P}$ is

$$0(x-1) + 6(y-1) + 4(z-2) = 0 \quad \text{or} \quad 6y = 4z = 14 \quad \text{or} \quad 3y + 2z = 7.$$

b) $\mathbf{n}_Q = \langle 2, 1, 1 \rangle \Rightarrow \widehat{\mathbf{n}} = \frac{1}{\sqrt{6}}\langle 2, 1, 1 \rangle, \quad \mathbf{v} = \langle -1, 2, -3 \rangle \Rightarrow \widehat{\mathbf{v}} = \frac{1}{\sqrt{14}}\langle -1, 2, -3 \rangle$

Component of $\widehat{\mathbf{n}}$ on $\widehat{\mathbf{v}}$ is

$$\widehat{\mathbf{n}} \cdot \widehat{\mathbf{v}} = \frac{1}{\sqrt{6 \cdot 14}}(2 + 2 - 3) = -\frac{3}{\sqrt{84}}$$



Problem 2.

a) Direction vector for L: $\mathbf{v} = \langle 1, 2, 0 \rangle$.

$P_0 = (0, 0, 1) \Rightarrow$ equation for L:

$$\mathbf{r} = \langle x, y, z \rangle = \langle 0, 0, 1 \rangle + t\langle 1, 2, 0 \rangle$$

or

$$x = t, \quad y = 2t, \quad z = 1.$$

b) $\mathbf{n}$ = normal vector for $\mathcal{P} = \langle 1, 2, 0 \rangle$ since $L \perp \mathcal{P}$.

$P_0 = (0, 0, 1) \Rightarrow 1(x - 0) + 2(y - 0) + 0(z - 1)$ or $x + 2y = 0$.

c) P on $L \Rightarrow P = (t, 2t, 1)$ for some $t \neq 0$ (part (a))

$P^* = (-t, -2t, 1)$ since then $\text{dist}(P_0, P) = \text{dist}(P_0, P^*) = |t|\sqrt{5}$.

Problem 3.

a) $$\begin{vmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} = 3 - 3 = 0.$$

b) To get a non-zero solution take the cross-product of any two rows of A_2; for example

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ -2 & 1 & -1 \end{vmatrix} = \langle -3, -5, 1 \rangle$$

This implies all solutions to $A\mathbf{x} = \mathbf{0}$ are $\mathbf{x} = t \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix}$.

c)

$$A_1^{-1} A_1 = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} * & * & * \\ -3 & p & 5 \\ * & * & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ -3-2p-5 & * & * \\ * & * & * \end{pmatrix} \Rightarrow -8-2p = 0 \Rightarrow p = -4.$$

Problem 4.

a) $\mathbf{r}'(t) = \langle -\sin(\mathrm{e}^t)\mathrm{e}^t, \cos(\mathrm{e}^t)\mathrm{e}^t, \mathrm{e}^t \rangle \Rightarrow |\mathbf{r}'(t)| = \mathrm{e}^t\sqrt{1+1} = \mathrm{e}^t\sqrt{2}$

$\Rightarrow \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{2}}\langle -\sin(\mathrm{e}^t), \cos(\mathrm{e}^t), 1 \rangle.$

b)

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos(\mathrm{e}^t), -\sin(\mathrm{e}^t), 0 \rangle = -\frac{\mathrm{e}^t}{\sqrt{2}}\langle \cos(\mathrm{e}^t), \sin(\mathrm{e}^t), 0 \rangle.$$

Problem 5.

a) $F_x = \dfrac{\partial F}{\partial x} = \dfrac{xz}{(x^2+y)^{1/2}} \Rightarrow F_x(1,3,23) = 2/2 = 1.$

$F_y = \dfrac{\partial F}{\partial y} = \dfrac{z}{2(x^2+y)^{1/2}} + \dfrac{2}{z} \Rightarrow F_y(1,3,2) = 3/2.$

$F_z = \dfrac{\partial F}{\partial z} = (x^2+y)^{1/2} - \dfrac{2y}{z^2} \Rightarrow F_z(1,3,2) = \dfrac{1}{2}.$

$\mathbf{n} = \boldsymbol{\nabla} F(1,3,2) = \left\langle 1, \dfrac{3}{2}, \dfrac{1}{2} \right\rangle$, $P_0 = (1,3,2)$

$\Rightarrow$ tangent plane equation

$$1(x-1) + \frac{3}{2}(y-3) + \frac{1}{2}(z-2) \quad \text{or } 2x + 3y + z = 13.$$

b) At $P_0 = (1,3,2)$ we have $|F_y| = 3/2 > |F_x|, |F_z|$. So, a change in y produces the largest change in F.

$\Delta F = F_y \Delta y = \dfrac{3}{2}(0.1) = 0.15.$

c) $\left.\dfrac{df}{ds}\right|_{P_0,\mathbf{u}} = \widehat{\mathbf{u}} \cdot \boldsymbol{\nabla} F(P_0) = \pm\dfrac{1}{3}\langle -2, 2, 1 \rangle \cdot \langle 1, 3/2, 1/2 \rangle = \pm\dfrac{1}{3}(-2+3-1/2) = \pm\dfrac{1}{6}.$

$\Delta F \approx \left.\dfrac{dF}{ds}\right|_{P_0,\widehat{\mathbf{u}}} \Delta s \Rightarrow 0.1 = \dfrac{1}{6}\Delta s \Rightarrow \boxed{\Delta s = 0.6}$

Problem 6. a)

$$\left.\begin{array}{rcl} f_x &=& 1-2/(x^2y)=0 \\ f_y &=& 4-2/(xy^2)=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x^2y=2 \\ xy^2=1/2 \end{array}\right\} \Rightarrow x=4y$$

$$\Rightarrow 4y^3=\frac{1}{2} \Rightarrow y^3=\frac{1}{8} \Rightarrow y=\frac{1}{2} \Rightarrow x=2.$$

There is one critical point at $(x,y)=(2,1/2)$.

b) $f_{xx}=4/(x^2y)$, $f_{yy}=4/(xy^3)$, $f_{xy}=f_{yx}=2/(x^2y^2)$
$A=f_{xx}(2,1/2)=1$, $C=f_{yy}(2,1/2)=16$, $B=f_{xy}(2,1/2)=2$
$\Rightarrow AC-B^2=12>0$, $A>0 \Rightarrow f$ has a relative minimum at $(2,1/2)$.

Problem 7.
$f(x,y,z)=\text{dist}^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$
subject to $g(x,y,z)=Ax+By+Cz=D$.
$\boldsymbol{\nabla} f=2\langle x-x_0,y-y_0,z-z_0\rangle$, $\quad \boldsymbol{\nabla} g=\langle A,B,C\rangle$.

$$\boldsymbol{\nabla} f=\lambda\boldsymbol{\nabla} g, \text{ and } g=D \Rightarrow \begin{array}{l} 2(x-x_0)=\lambda A \\ 2(y-y_0)=\lambda B \\ 2(z-z_0)=\lambda C \\ Ax+By+Cz=D. \end{array}$$

Problem 8. a)

$$\frac{\partial F}{\partial \phi}=\frac{\partial F}{\partial x}\frac{\partial x}{\partial \phi}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial \phi}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial \phi}$$

$$\frac{\partial x}{\partial \phi}=\rho\cos\phi\cos\theta \Rightarrow x_\phi(2,\pi/4,-\pi/4)=2\cos(\pi/4)\cos(-\pi/4)=1.$$

$$\frac{\partial y}{\partial \phi}=\rho\cos\phi\sin\theta \Rightarrow y_\phi(2,\pi/4,-\pi/4)=2\cos(\pi/4)\sin(-\pi/4)=-1.$$

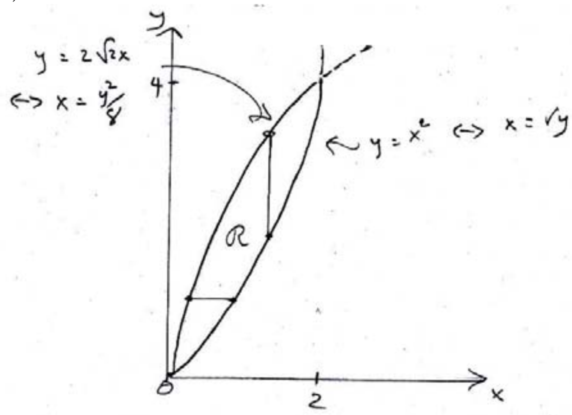
$$\frac{\partial z}{\partial \phi}=-\rho\sin\phi \Rightarrow z_\phi(2,\pi/4,-\pi/4)=-2\sin(\pi/4)=-\sqrt{2}.$$

b) **NOT Possible**
$\langle -y,x\rangle$ is *not* a gradient field. (Test: $\langle -y,x\rangle=\langle P,Q\rangle$: $\quad P_y=-1\neq Q_x=1$.)

Problem 9.

a)

$$R = \begin{cases} x^2 \le y \le 2\sqrt{x} \\ 0 \le x \le 2 \end{cases}$$



b)

$$R = \begin{cases} y^2/8 \le x \le \sqrt{y} \\ 0 \le y \le 4 \end{cases} \quad \Rightarrow \quad \int\int_R f\,dA = \int_0^4 \int_{y^2/8}^{\sqrt{y}} f(x,y)\,dx\,dy.$$

Problem 10.

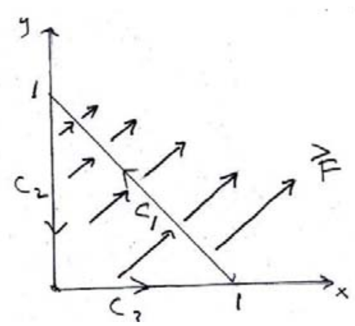
$$R_{u,v} : \begin{cases} 4 \le u \le 9 \\ 1 \le v \le 2 \end{cases}$$

Jacobian $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} u^{-2/3}v^{-1/3}/3 & -u^{1/3}v^{-4/3}/3 \\ u^{-2/3}v^{2/3}/3 & 2u^{1/3}v^{-1/3}/3 \end{vmatrix} = \left(\frac{2}{9} + \frac{1}{9}\right) u^{-1/3}v^{-2/3} = \frac{1}{3}u^{-1/3}v^{-2/3}.$

$$\int_R f(x,y)\,dA = int_1^2 \int_4^9 f(u^{1/3}v^{-1/3}, u^{1/3}v^{2/3})(\frac{1}{3}u^{-1/3}v^{-2/3})\,du\,dv.$$

Problem 11.

a) Net flux out of R will be positive (more flow out than into R)



b)

$$\int_C \mathbf{F} \cdot \widehat{\mathbf{n}}\,ds = \int_C -N\,dx + M\,dy = \int_C -x\,dx + x\,dy$$

$C_1 : x = 1 - t,\ y = t \ \Rightarrow\ dx = -dt,\ dy = dt$

$$\Rightarrow \int_{C_1} \mathbf{F} \cdot \widehat{\mathbf{n}}\,ds = \int_0^1 -(1-t)(-1) + (1-t)(1)\,dt = -(1-t)^2\Big|_0^1 = 1.$$

$C_2 : x = 0 \;\Rightarrow\; \displaystyle\int_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds = 0.$

$C_3 : y = 0, dy = 0 \;\Rightarrow\; \displaystyle\int_{C_3} \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 -x\,dx = -\frac{x^2}{2} = -\frac{1}{2}.$

Thus,

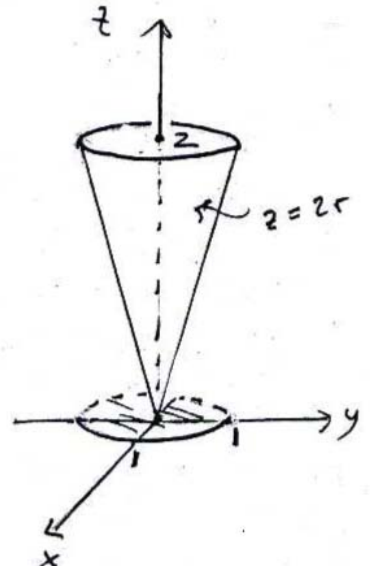
$$\oint_C \mathbf{F}\cdot\widehat{\mathbf{n}}\,ds = \int_{C_1+C_2+C_3} = 1 + 0 + (-1/2) = \frac{1}{2}.$$

c)

$$\operatorname{div}(\mathbf{F}) = M_x + N_y = 1 \;\Rightarrow\; \int\int_R \operatorname{div}(\mathbf{F})\,dA = \int\int_R dA = area(R) = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$$

Problem 12.

a) Limits on G are $2r \le z \le 2;\quad 0 \le r \le 1;\quad 0 \le \theta \le 2\pi.$



In cylindrical coordinates $dV = dz\,r\,dr\,d\theta$.

Thus

$$M = \int\int\int_G z\,dV = \int_0^{2\pi}\int_0^1\int_{2r}^2 z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2(1-r^2)\,r\,dr\,d\theta = 4\pi\cdot\frac{1}{4} = \pi.$$

b)

$$\bar{z} = \frac{1}{M}\int\int\int_G z\cdot\delta\,dV = \frac{1}{\pi}\int_0^{2\pi}\int_0^1\int_{2r}^2 z^2\,dz\,r\,dr\,d\theta.$$

c) In spherical coordinates: $z = 2 \;\Rightarrow\; \rho\cos\phi = 2 \;\Rightarrow\; \rho = 2\sec\phi$.

Limits on G: $0 \le \rho \le 2\sec\phi;\quad 0 \le \phi \le \tan^{-1}(1/2);\quad 0 \le \theta \le 2\pi.$

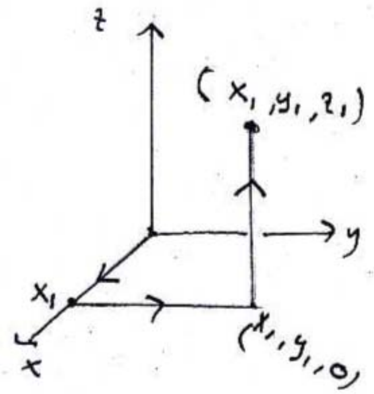
In spherical coordinates: $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ and $z = \rho\cos\phi$, so

$$\bar{z} = \frac{1}{M}\int\int\int_G z\cdot\delta\,dV = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\tan^{-1}(1/2)}\int_0^{2\sec\phi}(\rho\cos\phi)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Problem 13.
a) We have $\mathbf{F} = \langle P, Q, R \rangle$, where $P = y + y^2 z, \quad Q = x - z + 2xyz, \quad R = -y + xy^2$.

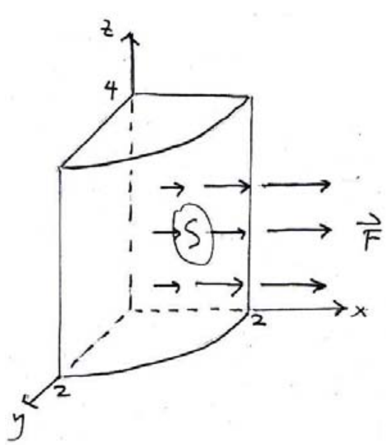
$$\frac{\partial P}{\partial z} = y^2 = \frac{\partial R}{\partial x}; \quad \frac{\partial Q}{\partial z} = -1 + 2xy = \frac{\partial R}{\partial y}; \quad \frac{\partial P}{\partial y} = 1 + 2yz = \frac{\partial Q}{\partial x}.$$

b)



$f(x_1, y_1, z_1) = \displaystyle\int_0^{x_1} P(x,0,0)\,dx + \int_0^{y_1} Q(x_1, y, 0)\,dy + \int_0^{z_1} Q(x_1, y_1, z)\,dz.$
$P(x,0,0) = 0; \quad Q(x_1, y, 0) = x_1; \quad R(x_1, y_1, z) = -y_1 + x_1 y_1^2.$
$f(x_1, y_1, z_1) = 0 + \displaystyle\int_0^{y_1} x_1\,dy + \int_0^{z_1} (-y_1 + x_1 y_1^2)\,dz$
$\Rightarrow \ f(x_1, y_1, z_1) = x_1 y_1 - y_1 z_1 + x_1 y_1^2 z_1 \ \Rightarrow \ f(x,y,z) = xy - yz + xy^2 z + C.$
c) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,-1,2) - f(2,2,1) = -10 + 3 = -7.$

Problem 14. a)



b) $\widehat{\mathbf{n}} = \frac{1}{2}\langle x, y, 0 \rangle, \quad \mathbf{F} = \langle x, 0, 0 \rangle.$
Thus, $\mathbf{F} \cdot \widehat{\mathbf{n}} = \dfrac{1}{2} x^2$ and in cylindrical coordinates $dS = 2dz\,d\theta$.

On the surface $x = 2\cos\theta$ and the limits of integration are $0 \le z \le 4$, and $0 \le \theta \le \pi/2$

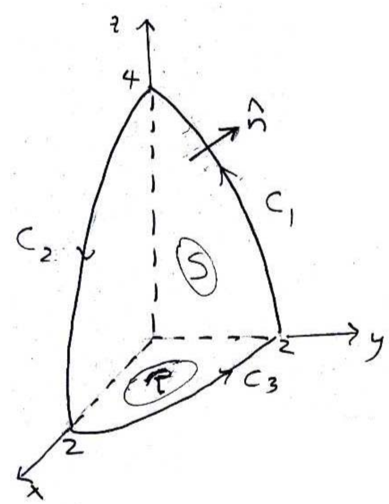
$$\int\int_S \mathbf{F}\cdot\widehat{\mathbf{n}}\, dS = \frac{1}{2}\int\int_S x^2\, dS = \frac{1}{2}\int_0^{\pi/2}\int_0^4 (2\cos\theta)^2\, dz\, 2d\theta = 4\int_0^4 dz \int_0^{\pi/2} \cos^2(\theta)\, d\theta = 16\cdot\frac{\pi}{4}.$$

(We used the half angle formula $\cos^2\theta = \dfrac{1}{2}(1+\cos\theta)$.

c) $\operatorname{div}(\mathbf{F}) = \boldsymbol{\nabla}\cdot\mathbf{F} = 1 \;\Rightarrow\; \displaystyle\int\int\int_G \boldsymbol{\nabla}\cdot\mathbf{F}\, dV = \int\int\int_G 1\, dV = \mathrm{Vol}(G) = \frac{1}{4}\pi 2^2\cdot 4 = 4\pi.$

d) Flux of $\mathbf{F}$ across all four flat faces of G is zero.
Check:
face on xz-plane: $\widehat{\mathbf{n}} = -\mathbf{j} \;\Rightarrow\; \mathbf{F}\cdot\widehat{\mathbf{n}} = 0.$
face on yz-plane: $\widehat{\mathbf{n}} = -\mathbf{i} \;\Rightarrow\; \mathbf{F}\cdot\widehat{\mathbf{n}} = -x = 0$ on $yz-$ plane.
faces on xy-plane and plane $z=4$: $\widehat{\mathbf{n}} = -\mathbf{k}$ and $\mathbf{k}$ respectively, in either case $\mathbf{F}\cdot\widehat{\mathbf{n}}$.

Problem 15. a)



$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xy & 1 \end{vmatrix} = \mathbf{i}(x) - j(-y) + \mathbf{k}(-2z) = \langle x, y, -2z\rangle.$$

$\widehat{\mathbf{n}}\, dS = \langle -z_x, -z_y, 1\rangle\, dA = \langle 2x, 2y, 1\rangle\, dA.$

$(\boldsymbol{\nabla}\times\mathbf{F})\cdot\widehat{\mathbf{n}}\, dS = (2x^2+2y^2-2z)\, dA =$ (subst. for z) $= (2x^2+2y^2-2(4-x^2-y^2))\, dA = 4(x^2+y^2-2)\, dA.$
Limits of integration on R are $0 \le r \le 2$; $\quad 0 \le \theta \le \pi/2$.

$$\int\int_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\widehat{\mathbf{n}}\, dS = 4\int\int_R (x^2+y^2-2)\, dA = 4\int_0^{\pi/2}\int_0^2 (r^2-2)r\, dr\, d\theta = 4\cdot\frac{\pi}{2}\Big(\frac{r^4}{4} - r^2\Big|_0^2 = 2\pi(4-4) = 0.$$

b) $\mathbf{F} = \langle yz, -xz, 1\rangle \;\Rightarrow\; \displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (yz)\, dx - (xz)\, dy + 1\, dz.$

C_1 is in the yz-plane: $x = 0$, $dx = 0$, $y = t$, $z = 4 - t^2$, $dz = (-2t)\,dt$ t goes from 2 to 0.

$$\Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} 1\,dz = \int_2^0 (-2t)\,dt = 4.$$

C_2 is in the xz-plane: $y = 0$, $dy = 0$, $x = t$, $z = 4 - t^2$, $dz = (-2t)\,dt$ t goes from 0 to 2.

$$\Rightarrow \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} 1\,dz = \int_0^2 (-2t)\,dt = -4.$$

C_3 is in the xy-plane: $z = 0$, $dz = 0$

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0.$$

Thus,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1+C_2+C_2} \mathbf{F} \cdot d\mathbf{r} = 4 + (-4) = 0.$$