18.02 Exam 1

Problem 1.

Let P, Q and R be the points at 1 on the x-axis, 2 on the y-axis and 3 on the z-axis, respectively.

a) (6) Express $\overrightarrow{QP}$ and $\overrightarrow{QR}$ in terms of $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$.

b) (9) Find the cosine of the angle PQR.

Problem 2. Let $P = (1, 1, 1)$, $Q = (0, 3, 1)$ and $R = (0, 1, 4)$.

a) (10) Find the area of the triangle PQR.

b) (5) Find the plane through P, Q and R, expressed in the form $ax + by + cz = d$.

c) (5) Is the line through $(1, 2, 3)$ and $(2, 2, 0)$ parallel to the plane in part (b)? Explain why or why not.

Problem 3. A ladybug is climbing on a Volkswagen Bug (= VW). In its starting position, the the surface of the VW is represented by the unit semicircle $x^2 + y^2 = 1$, $y \geq 0$ in the xy-plane. The road is represented as the x-axis. At time $t = 0$ the ladybug starts at the front bumper, $(1, 0)$, and walks counterclockwise around the VW at unit speed relative to the VW. At the same time the VW moves to the right at speed 10.

a) (15) Find the parametric formula for the trajectory of the ladybug, and find its position when it reaches the rear bumper. (At $t = 0$, the rear bumper is at $(-1, 0)$.)

b) (10) Compute the speed of the bug, and find where it is largest and smallest. Hint: It is easier to work with the square of the speed.

Problem 4.

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & -1 & -1 \end{pmatrix} \qquad\qquad M^{-1} = \frac{1}{12}\begin{pmatrix} 1 & 1 & 4 \\ a & 7 & -8 \\ b & -5 & 4 \end{pmatrix}$$

(a) (5) Compute the determinant of M.

b) (10) Find the numbers a and b in the formula for the matrix M^{-1}.

c) (10) Find the solution $\vec{\boldsymbol{r}} = \langle x, y, z\rangle$ to $\begin{aligned} x + 2y + 3z &= 0 \\ 3x + 2y + z &= t \\ 2x - y - z &= 3 \end{aligned}$ as a function of t.

d) (5) Compute $\dfrac{d\vec{\boldsymbol{r}}}{dt}$.

Problem 5.

(a) (5) Let $P(t)$ be a point with position vector $\vec{\boldsymbol{r}}(t)$. Express the property that $P(t)$ lies on the plane $4x - 3y - 2z = 6$ in vector notation as an equation involving $\vec{\boldsymbol{r}}$ and the normal vector to the plane.

(b) (5) By differentiating your answer to (a), show that $\dfrac{d\vec{\boldsymbol{r}}}{dt}$ is perpendicular to the normal vector to the plane.

18.02 Exam 2

Problem 1. Let $f(x,y) = x^2y^2 - x$.

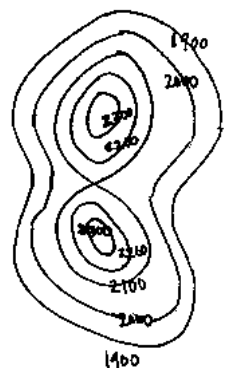
a) (5) Find ∇f at $(2,1)$

b) (5) Write the equation for the tangent plane to the graph of f at $(2,1,2)$.

c) (5) Use a linear approximation to find the approximate value of $f(1.9, 1.1)$.

d) (5) Find the directional derivative of f at $(2,1)$ in the direction of $-\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

Problem 2. (10) On the contour plot below, mark the portion of the level curve $f = 2000$ on which $\dfrac{\partial f}{\partial y} \geq 0$.



Problem 3. a) (10) Find the critical points of

$$w = -3x^2 - 4xy - y^2 - 12y + 16x$$

and say what type each critical point is.

b) (10) Find the point of the first quadrant $x \geq 0$, $y \geq 0$ at which w is largest. Justify your answer.

Problem 4. Let $u = y/x$, $v = x^2 + y^2$, $w = w(u,v)$.

a) (10) Express the partial derivatives w_x and w_y in terms of w_u and w_v (and x and y).

b) (7) Express $xw_x + yw_y$ in terms of w_u and w_v. Write the coefficients as functions of u and v.

c) (3) Find $xw_x + yw_y$ in case $w = v^5$.

Problem 5. a) (10) Find the Lagrange multiplier equations for the point of the surface

$$x^4 + y^4 + z^4 + xy + yz + zx = 6$$

at which x is largest. (Do not solve.)

b) (5) Given that x is largest at the point (x_0, y_0, z_0), find the equation for the tangent plane to the surface at that point.

Problem 6. Suppose that $x^2 + y^3 - z^4 = 1$ and $z^3 + zx + xy = 3$.

a) (8) Take the total differential of each of these equations.

b) (7) The two surfaces in part (a) intersect in a curve along which y is a function of x. Find dy/dx at $(x,y,z) = (1,1,1)$.

18.02 Exam 3

Problem 1. a) Draw a picture of the region of integration of $\displaystyle\int_0^1 \int_x^{2x} dydx$.

b) Exchange the order of integration to express the integral in part (a) in terms of integration in the order $dxdy$. Warning: your answer will have two pieces.

Problem 2. a) Find the mass M of the upper half of the annulus $1 < x^2 + y^2 < 9$ $(y \geq 0)$ with density $\delta = \dfrac{y}{x^2 + y^2}$.

b) Express the x-coordinate of the center of mass, $\bar{x}$, as an iterated integral. (Write explicitly the integrand and limits of integration.) Without evaluating the integral, explain why $\bar{x} = 0$.

Problem 3. a) Show that $\mathbf{F} = (3x^2 - 6y^2)\hat{\mathbf{i}} + (-12xy + 4y)\hat{\mathbf{j}}$ is conservative.

b) Find a potential function for $\mathbf{F}$.

c) Let C be the curve $x = 1 + y^3(1 - y)^3$, $0 \leq y \leq 1$. Calculate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

Problem 4. a) Express the work done by the force field $\mathbf{F} = (5x+3y)\hat{\mathbf{i}}+(1+\cos y)\hat{\mathbf{j}}$ on a particle moving counterclockwise once around the unit circle centered at the origin in the form $\displaystyle\int_a^b f(t)dt$. (Do not evaluate the integral; don't even simplify $f(t)$.)

b) Evaluate the line integral using Green's theorem.

Problem 5. Consider the rectangle R with vertices $(0, 0)$, $(1, 0)$, $(1, 4)$ and $(0, 4)$. The boundary of R is the curve C, consisting of C_1, the segment from $(0, 0)$ to $(1, 0)$, C_2, the segment from $(1, 0)$ to $(1, 4)$, C_3 the segment from $(1, 4)$ to $(0, 4)$ and C_4 the segment from $(0, 4)$ to $(0, 0)$. Consider the vector field

$$\mathbf{F} = (xy + \sin x \cos y)\hat{\mathbf{i}} - (\cos x \sin y)\hat{\mathbf{j}}$$

a) Find the flux of $\mathbf{F}$ out of R through C. Show your reasoning.

b) Is the total flux out of R through C_1, C_2 and C_3, more than, less than or equal to the flux out of R through C? Show your reasoning.

Problem 6. Find the volume of the region enclosed by the plane $z = 4$ and the surface

$$z = (2x - y)^2 + (x + y - 1)^2.$$

(Suggestion: change of variables.)

18.02 Exam 4

Problem 1. (10 points)

Let C be the portion of the cylinder $x^2+y^2 \leq 1$ lying in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) and below the plane $z = 1$. Set up a triple integral in *cylindrical coordinates* which gives the moment of inertial of C about the z-axis; assue the density to be $\delta = 1$.
(Give integrand and limits of integration, but *do not evaluate.*)

Problem 2. (20 points: 5, 15)

a) A solid sphere S of radius a is placed above the xy-plane so it is tangent at the origin and its diameter lies along the z-axis. Give its equation in *spherical coordinates.*

b) Give the equation of the horizontal plane $z = a$ in spherical coordinates.

c) Set up a triple integral in spherical coordinates which gives the volume of the portion of the sphere S lying *above* the plane $z = a$. (Give integrand and limits of integration, but *do not evaluate.*)

Problem 3. (20 points: 5, 15)

Let $\vec{F} = (2xy + z^3)\,\hat{\mathbf{i}} + (x^2 + 2yz)\,\hat{\mathbf{j}} + (y^2 + 3xz^2 - 1)\,\hat{\mathbf{k}}$.

a) Show $\vec{F}$ is conservative.

b) Using a systematic method, find a potential function $f(x, y, z)$ such that $\vec{F} = \vec{\nabla} f$. Show your work even if you can do it mentally.

Problem 4.(25 points: 15, 10)

Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, and let $\vec{F} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + 2(1 - z)\,\hat{\mathbf{k}}$.

Calculate the flux of $\vec{F}$ across S, taking the upward direction as the one for which the flux is positive. Do this in two ways:

a) by direct calculation of $\int\int_S \vec{F} \cdot \hat{\mathbf{n}}\, dS$;

b) by computing the flux across a simpler surface and using the divergence theorem.

Problem 5. (25 points: 10, 8, 7)

Let $\vec{F} = -2xz\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{k}}$.

a) Calculate $\operatorname{curl} \vec{F}$.

b) Show that $\int\int_R \operatorname{curl} \vec{F} \cdot \hat{\mathbf{n}}\, dS = 0$ for any finite portion R of the unit sphere $x^2 + y^2 + z^2 = 1$ (take the normal vector $\hat{\mathbf{n}}$ pointing outward).

c) Show that $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any simple closed curve C on the unit sphere $x^2 + y^2 + z^2 = 1$.

18.02 Final Exam

No books, notes or calculators.
15 problems, 250 points.

Useful formula: $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$

Problem 1. (20 points)
a) (15 pts.) Find the equation in the form $Ax + By + Cz = D$ of the plane $\mathcal{P}$ which contains the line L given by $x = 1 - t,\ \ y = 1 + 2t,\ \ z = 2 - 3t$ and the point (- 1, 1, 2).

b) (5 pts.) Let $\mathcal{Q}$ be the plane $2x + y + z = 4$. Find the component of a unit normal vector for $\mathcal{Q}$ projected on a unit direction vector for the line L of part(a).

Problem 2. (15 points)
Let L denote the line which passes through (0,0,1) and is parallel to the line in the xy-plane given by $y = 2x$.

a) (5 pts.) Sketch L and give its equation in vector-parametric form.

b) (5 pts.) Let $\mathcal{P}$ be the plane which passes through (0,0,1) and is *perpendicular* to the line L of part(a). Sketch in $\mathcal{P}$ (above) and give its equation in point-normal form.

c) (5 pts.) Suppose that the point $P \neq (0, 0, 1)$ lies on L. Write down the method or formula you would use to find the point P^* which is: (i) on L; (ii) the same distance away from the point (0,0,1) as P; and is (iii) on the *other* side of $\mathcal{P}$ from P.

Problem 3. (20 points)

Given the 3×3 matrix: $A_a = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & a \end{bmatrix}$:

a) (5 pts.) Let $a = 2$: show that $|\ A_2\ | = 0$

b) (7 pts.) Find the line of solutions to $A_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

c) (8 pts.) Suppose now that $a = 1$, and that $\mathrm{A_1}^{-1} = \begin{bmatrix} * & * & * \\ -3 & p & 5 \\ * & * & * \end{bmatrix}$. Find p.

Problem 4. (10 points)
Let $\mathbf{r}(t) = \langle \cos(e^t),\, \sin(e^t),\, e^t \rangle$.

a) (5 pts.) Compute and simplify the unit tangent vector $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\ \mathbf{r}'(t)\ |}$.

b) (5 pts.) Compute $\mathbf{T}'(t)$

Problem 5. (20 points)
Consider the function $F(x,y,z) = z\sqrt{x^2+y} + 2\,\dfrac{y}{z}$:

a) (10 pts.) The point $P_0 : (1,3,2)$ lies on the surface $F(x,y,z) = 7$. Find the equation of the tangent plane to the surface $F = 7$ at P_0.

b) (5 pts.) If starting at P_0 a small change were to be made in only *one* of the variables, which one would produce the largest change (in absolute value) in F ? If the change is this variable was of size 0.1, approximately how large would the change in F be ?

c) (5 pts.) What distance from P_0 in the direction $\pm\langle -2, 2, -1\rangle$ will produce an approximate change in F of size 0.1 units, according to the (already computed) linearization of F?

Problem 6. (15 points)
Let $f(x,y) = x + 4y + \dfrac{2}{xy}$.

a) (10 pts.) Find the critical point(s) of $f(x,y)$

b) (5 pts.) Use the second-derivative test to test the critical point(s) found in part(a).

Problem 7. (10 points)
Let $\mathcal{P}$ be the plane with equation $Ax+By+Cz = D$ and $P_0 = (x_0, y_0, z_0)$ be a point which is *not* on $\mathcal{P}$.
Use the Lagrange multiplier method to *set up* the equations satisfied by the point (x,y,z) on $\mathcal{P}$ which is *closest* to P_0. (Do not solve.)

Problem 8. (15 points)
a) (10 pts.) Let $F(x,y,z)$ be a smooth function of three variables for which $\nabla F(1,-1,\sqrt{2}) = \langle 1, 2, -2\rangle$.

Use the Chain Rule to evaluate $\dfrac{\partial F}{\partial \phi}$ at $(\rho,\phi,\theta) = (2, \frac{\pi}{4}, -\frac{\pi}{4})$.

(Use $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.)

b) (5 pts.) Suppose $f(x,y)$ is a smooth, non-constant function. Is it possible that, for *all* points (x,y), the gradient of f at the point (x,y) is equal to the vector $\langle -y, x\rangle$? Justify (briefly).

Problem 9. (10 points)

$$\iint_R f\,dA = \int_0^2 \int_{x^2}^{2\sqrt{2x}} f(x,y)\,dy\,dx \ .$$

a) (5 pts.) Sketch the region R.

b) (5 pts.) Rewrite the double integral as an iterated integral with the order interchanged.

Problem 10. (15 points)
Set up the integral $\displaystyle\iint_R f(x,y)\,dA$ where R is the region bounded by the four curves $x^2y = 4$, $x^2y = 9$, $\frac{y}{x} = 1$, and $\frac{y}{x} = 2$ as a double integral in the variables $u = x^2y$ and $v = \frac{y}{x}$. (Your answer should be completely ready to integrate, once the function f is given.)
Note: the inverse transformation is given by $x = u^{\frac{1}{3}}\,v^{-\frac{1}{3}}$, $y = u^{\frac{1}{3}}\,v^{\frac{2}{3}}$.

Problem 11. (15 points)
$\mathbf{F}(x,y) = x\,(\mathbf{i} + \mathbf{j})$, and let C be the closed curve in the xy-plane formed by the triangle with vertices at the origin and the points (1,0) and (0,1).

a) (5 pts.) Give a rough sketch of the field $\mathbf{F}$ in the first quadrant, and use it to predict whether the net flux out of the region $R =$ the interior of C will be positive or negative.

b) (5 pts.) Compute the flux integral $\displaystyle\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds$ directly.
(Specify which orientation you are using for C.)

c) (5 pts.) Compute the flux integral $\displaystyle\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds$ using the appropriate *double* integral. (Set up, then using short-cut is ok.)

Problem 12. (20 points)
Let G be the solid 3-D cone bounded by the lateral surface given by $z = 2\sqrt{x^2+y^2}$ and by the plane $z = 2$. The problem is to compute
$\bar{z} =$ the z-coordinate of the center of mass of G, in the case where the density is equal to the height above the xy-plane.

a) (5 pts.) Find the mass of G using cylindrical coordinates

(b) (5 pts.) Set up the calculation for $\bar{z}$ using cylindrical coordinates
(Answers should be ready to integrate out – but *do not evaluate.*)

(c) (10 pts.) Set up the calculation for $\bar{z}$ using spherical coordinates.
(Answers should be ready to integrate out – but *do not evaluate.*)

Problem 13. (15 points)
$\mathbf{F}(x,y,z) = \left(y + y^2z\right)\,\mathbf{i} + (x - z + 2xyz)\,\mathbf{j} + \left(-y + xy^2\right)\,\mathbf{k}$

a) (3 pts.) Show that $\mathbf{F}(x,y,z)$ is a gradient field using the derivative conditions.

b) (10 pts.) Find a potential function $f(x,y,z)$ for $\mathbf{F}(x,y,z)$, using any *systematic* method. Show the method used and all work clearly.

c) (2 pts.) Find $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the straight line joining the points (2, 2, 1) and (1, -1, 2) (in that order), using as little computation as possible.

Problem 14. (25 points)
In this problem S is the surface given by the quarter of the right-circular cylinder centered on the z-axis, of radius 2 and height 4, which lies in the first octant. The field $\mathbf{F}(x, y, z) = x\,\mathbf{i}$.

a) (5 pts.) Sketch the surface S and the field $\mathbf{F}$.
(Suggestion: use a coordinate system with y pointing out of the paper.)

b) (10 pts.) Compute the flux integral $\displaystyle\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$.
(Use the normal which points 'outward' from S, i.e. on the side away from the z-axis.)

c) (5 pts.) G be the 3D solid in the first octant given by the interior of the quarter cylinder defined above. Use the divergence theorem to compute the flux of the field $\mathbf{F} = x\,\mathbf{i}$ out of the region G.

d) (5 pts.) The boundary surface of G is comprised of S together with four other faces. What is the flux outward through these four faces, and why ? Use the answers to parts(b) and (c), and also verify using the sketch of part(a).

Problem 15. (25 points)
$\mathbf{F}(x, y, z) = (yz)\,\mathbf{i} + (-xz)\,\mathbf{j} + \mathbf{k}$. Let S be the portion of surface of the paraboloid $z = 4 - x^2 - y^2$ which lies above the first octant; and let C be the closed curve $C = C_1 + C_2 + C_3$, where the curves C_1, C_2 and C_3 are the three curves formed by intersecting S with the xy, yz and xz planes respectively (so that C is the boundary of S). Orient C so that it is traversed CCW when seen from above in the first octant.

a) (15 pts.) Use Stokes' Theorem to compute the loop integral $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$ by using the *surface integral* over the capping surface S.

b) (10 pts.) Set up and evaluate the loop integral $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly by parametrizing each piece of the curve C and then adding up the three line integrals.