18.03 EXERCISES

1. First-order ODE's

1A. Introduction; Separation of Variables

1A-1. Verify that each of the following ODE's has the indicated solutions (c_i, a are constants):

a) $y'' - 2y' + y = 0, \qquad y = c_1 e^x + c_2 x e^x$

b) $xy' + y = x \sin x, \qquad y = \dfrac{\sin x + a}{x} - \cos x$

1A-2. On how many arbitrary constants (also called *parameters*) does each of the following families of functions depend? (There can be less than meets the eye...; a, b, c, d, k are constants.)

a) $c_1 e^{kx}$ b) $c_1 e^{x+a}$ c) $c_1 + c_2 \cos 2x + c_3 \cos^2 x$ d) $\ln(ax + b) + \ln(cx + d)$

1A-3. Write down an explicit solution (involving a definite integral) to the following initial-value problems (IVP's):

a) $y' = \dfrac{1}{y^2 \ln x}, \quad y(2) = 0$ b) $y' = \dfrac{ye^x}{x}, \quad y(1) = 1$

1A-4. Solve the IVP's (initial-value problems):

a) $y' = \dfrac{xy + x}{y}, \quad y(2) = 0$ b) $\dfrac{du}{dt} = \sin t \cos^2 u, \quad u(0) = 0$

1A-5. Find the general solution by separation of variables:

a) $(y^2 - 2y)\, dx + x^2 dy = 0$ b) $x \dfrac{dv}{dx} = \sqrt{1 - v^2}$

c) $y' = \left(\dfrac{y-1}{x+1}\right)^2$ d) $\dfrac{dx}{dt} = \dfrac{\sqrt{1+x}}{t^2 + 4}$

1B. Standard First-order Methods

1B-1. Test the following ODE's for exactness, and find the general solution for those which are exact.

a) $3x^2 y\, dx + (x^3 + y^3)\, dy = 0$ b) $(x^2 - y^2)\, dx + (y^2 - x^2)\, dy = 0$

c) $ve^{uv} du + ye^{uv} dv = 0$ d) $2xy\, dx - x^2 dy = 0$

1B-2. Find an integrating factor and solve:

a) $2x\, dx + \dfrac{x^2}{y}\, dy = 0$ b) $y\, dx - (x + y)\, dy = 0, \quad y(1) = 1$

c) $(t^2 + 4)\, dt + t\, dx = x\, dt$ d) $u(du - dv) + v(du + dv) = 0. \quad v(0) = 1$

1B-3. Solve the homogeneous equations

a) $y' = \dfrac{2y-x}{y+4x}$ b) $\dfrac{dw}{du} = \dfrac{2uw}{u^2-w^2}$

c) $xy\,dy - y^2dx = x\sqrt{x^2-y^2}\,dx$

1B-4. Show that a change of variable of the form $u = \dfrac{y}{x^n}$ turns $y' = \dfrac{4+xy^2}{x^2y}$ into an equation whose variables are separable, and solve it.

(Hint: as for homogeneous equations, since you want to get rid of y and y', begin by expressing them in terms of u and x.)

1B-5. Solve each of the following, finding the general solution, or the solution satisfying the given initial condition.

a) $xy' + 2y = x$ b) $\dfrac{dx}{dt} - x\tan t = \dfrac{t}{\cos t}$, $x(0) = 0$

c) $(x^2-1)y' = 1 - 2xy$ d) $3v\,dt = t(dt - dv)$, $v(1) = \frac{1}{4}$

1B-6. Consider the ODE $\dfrac{dx}{dt} + ax = r(t)$, where a is a positive constant, and $\lim_{t\to\infty} r(t) = 0$. Show that if $x(t)$ is any solution, then $\lim_{t\to\infty} x(t) = 0$. (Hint: use L'Hospital's rule.)

1B-7. Solve $y' = \dfrac{y}{y^3+x}$. Hint: consider $\dfrac{dx}{dy}$.

1B-8. The **Bernouilli** equation. This is an ODE of the form $y' + p(x)y = q(x)y^n$, $n \neq 1$. Show it becomes linear if one makes the change of dependent variable $u = y^{1-n}$.

(Hint: begin by dividing both sides of the ODE by y^n.)

1B-9. Solve these Bernouilli equations using the method decribed in 1B-8:

a) $y' + y = 2xy^2$ b) $x^2y' - y^3 = xy$

1B-10. The **Riccati** equation. After the linear equation $y' = A(x) + B(x)\,y$, in a sense the next simplest equation is the Riccati equation

$$y' = A(x) + B(x)y + C(x)y^2,$$

where the right-hand side is now a quadratic function of y instead of a linear function. In general the Riccati equation is not solvable by elementary means. However,

a) show that if $y_1(x)$ is a solution, then the general solution is

$$y = y_1 + u,$$

where u is the general solution of a certain Bernouilli equation (cf. 1B-8).

b) Solve the Riccati equation $y' = 1 - x^2 + y^2$ by the above method.

1B-11. Solve the following second-order autonomous equations ("autonomous" is an important word; it means that the independent variable does not appear explicitly in the equation — it does lurk in the derivatives, of course.)

a) $y'' = a^2 y$ b) $yy'' = y'^2$ c) $y'' = y'(1+3y^2)$, $y(0)=1,\ y'(0)=2$

1B-12. For each of the following, tell what type of ODE it is — i.e., what method you would use to solve it. (Don't actually carry out the solution.) For some, there are several methods which could be used.

1. $(x^3+y)\,dx + x\,dy = 0$
2. $\dfrac{dy}{dt} + 2ty - e^{-t} = 0$
3. $y' = \dfrac{x^2-y^2}{5xy}$
4. $(1+2p)\,dq + (2-q)\,dp = 0$
5. $\cos x\,dy = (y\sin x + e^x)\,dx$
6. $x(\tan y)y' = -1$
7. $y' = \dfrac{y}{x} + \dfrac{1}{y}$
8. $\dfrac{dv}{du} = e^{2u+3v}$
9. $xy' = y + xe^{y/x}$
10. $xy' - y = x^2 \sin x$
11. $y' = (x+e^y)^{-1}$
12. $y' + \dfrac{2y}{x} - \dfrac{y^2}{x} = 0$
13. $\dfrac{dx}{dy} = -x\left(\dfrac{2x^2y+\cos y}{3x^2y^2+\sin y}\right)$
14. $y' + 3y = e^{-3t}$
15. $x\dfrac{dy}{dx} - y = \sqrt{x^2+y^2}$
16. $\dfrac{y'-1}{x^2} = 1$
17. $xy' - 2y + y^2 = x^4$
18. $y'' = \dfrac{y(y+1)}{y'}$
19. $t\dfrac{ds}{dt} = s(1 - \ln t + \ln s)$
20. $\dfrac{dy}{dx} = \dfrac{3-2y}{2x+y+1}$
21. $x^2y' + xy + y^2 = 0$
22. $y'\tan(x+y) = 1 - \tan(x+y)$
23. $y\,ds - 3s\,dy = y^4\,dy$
24. $du = -\dfrac{1+u\cos^2 t}{t\cos^2 t}\,dt$
25. $y' + y^2 + (2x+1)y + 1 + x + x^2 = 0$
26. $y'' + x^2y' + 3x^3 = \sin x$

1C. Graphical and Numerical Methods

1C-1. For each of the following ODE's, draw a direction field by using about five isoclines; the picture should be square, using the intervals between -2 and 2 on both axes. Then sketch in some integral curves, using the information provided by the direction field. Finally, do whatever else is asked.

a) $y' = -\dfrac{y}{x}$; solve the equation exactly and compare your integral curves with the correct ones.

b) $y' = 2x + y$; find a solution whose graph is also an isocline, and verify this fact analytically (i.e., by calculation, not from a picture).

c) $y' = x - y$; same as in (b).

d) $y' = x^2 + y^2 - 1$

e) $y' = \dfrac{1}{x+y}$; use the interval -3 to 3 on both axes; draw in the integral curves that pass respectively through $(0,0)$, $(-1,1)$, $(0,-2)$. Will these curves cross the line

$y = -x - 1$? Explain by using the Intersection Principle (Notes G, (3)).

1C-2. Sketch a direction field, concentrating on the first quadrant, for the ODE

$$y' = \frac{-y}{x^2 + y^2} \ .$$

Explain, using it and the ODE itself how one can tell that the solution $y(x)$ satisfying the initial condition $y(0) = 1$

a) is a decreasing function for $x > 0$;
b) is always positive for $x > 0$.

1C-3. Let $y(x)$ be the solution to the IVP $y' = x - y, \quad y(0) = 1.$

a) Use the Euler method and the step size $h = .1$ to to find an approximate value of $y(x)$ for $x = .1, .2, .3$. (Make a table as in notes G).
Is your answer for $y(.3)$ too high or too low, and why?

b) Use the Modified Euler method (also called Improved Euler, or Heun's method) and the step size $h = .1$ to determine the approximate value of $y(.1)$. Is the value for $y(.1)$ you found in part (a) corrected in the right direction — e.g., if the previous value was too high, is the new one lower?

1C-4. Use the Euler method and the step size .1 on the IVP $y' = x + y^2, \quad y(0) = 1,$ to calculate an approximate value for the solution $y(x)$ when $x = .1, .2, .3$. (Make a table as in Notes G.) Is your answer for $y(.3)$ too high or too low?

1C-5. Prove that the Euler method converges to the exact value for $y(1)$ as the progressively smaller step sizes $h = 1/n, \quad n = 1, 2, 3, \ldots$ are used, for the IVP

$$y' = x - y, \qquad y(0) = 1 \ .$$

(First show by mathematical induction that the approximation to y(1) gotten by using the step size $1/n$ is

$$y_n = 2(1 - h)^n - 1 + nh \ .$$

The exact solution is easily found to be $y = 2e^{-x} + x - 1$.)

1C-6. Consider the IVP $y' = f(x), \quad y(0) = y_0.$

We want to calculate $y(2nh)$, where h is the step size, using n steps of the Runge-Kutta method.

The exact value, by Chapter D of the notes, is $y(2nh) = y_0 + \displaystyle\int_0^{2nh} f(x)\, dx$.

Show that the value for $y(2nh)$ produced by Runge-Kutta is the same as the value for $y(2nh)$ obtained by using Simpson's rule to evaluate the definite integral.

1C-7. According to the existence and uniqueness theorem, under what conditions on $a(x), b(x)$, and (x) will the IVP

$$a(x)\, y' + b(x)\, y = c(x), \quad y(x_0) = y_0$$

have a unique solution in some interval $[x_0 - h, x_0 + h]$ centered around x_0?

1D. Geometric and Physical Applications

1D-1. Find all curves $y = y(x)$ whose graphs have the indicated geometric property. (Use the geometric property to find an ODE satisfied by $y(x)$, and then solve it.)

a) For each tangent line to the curve, the segment of the tangent line lying in the first quadrant is bisected by the point of tangency.

b) For each normal to the curve, the segment lying between the curve and the x-axis las constant length 1.

c) For each normal to the curve, the segment lying between the curve and the x-axis is bisected by the y-axis.

d) For a fixed a, the area under the curve between a and x is proportional to $y(x) - y(a)$.

1D-2. For each of the following families of curves,

(i) find the ODE satisfied by the family (i.e., having these curves as its integral curves);
(ii) find the orthogonal trajectories to the given family;
(iii) sketch both the original family and the orthogonal trajectories.

a) all lines whose y-intercept is twice the slope
b) the exponential curves $y = ce^x$
c) the hyperbolas $x^2 - y^2 = c$
d) the family of circles centered on the y-axis and tangent to the x-axis.

1D-3. Mixing A container holds V liters of salt solution. At time $t = 0$, the salt concentration is c_0 g/liter. Salt solution having concentration c_1 is added at the rate of k liters/min, with instantaneous mixing, and the resulting mixture flows out of the container at the same rate. How does the salt concentration in the tank vary with time?

Let $x(t)$ be the *amount* of salt in the tank at time t. Then $c(t) = \dfrac{x(t)}{V}$ is the concentration of salt at time t.

a) Write an ODE satisfied by $x(t)$, and give the initial condition.

b) Solve it, assuming that it is pure water that is being added. (Lump the constants by setting $a = k/V$.)

c) Solve it, assuming that c_1 is constant; determine $c(t)$ and find $\lim_{t\to\infty} c(t)$. Give an intuitive explanation for the value of this limit.

d) Suppose now that c_1 is not constant, but is decreasing exponentially with time:

$$c_1 = c_0 e^{-\alpha t}, \qquad \alpha > 0.$$

Assume that $a \neq \alpha$ (cf. part (b)), and determine $c(t)$, by solving the IVP. Check your answer by putting $\alpha = 0$ and comparing with your answer to (c).

1D-4. Radioactive decay A radioactive substance **A** decays into **B**, which then further decays to **C**.

a) If the decay constants of **A** and **B** are respectively λ_1 and λ_2 (the decay constant is by definition $(\ln 2/\text{half-life})$), and the initial amounts are respectively A_0 and B_0, set up an ODE for determining $B(t)$, the amount of **B** present at time t, and solve it. (Assume $\lambda_1 \neq \lambda_2$.)

b) Assume $\lambda_1 = 1$ and $\lambda_2 = 2$. Tell when $B(t)$ reaches a maximum.

1D-5. Heat transfer According to Newton's Law of Cooling, the rate at which the temperature T of a body changes is proportional to the difference between T and the external temperature.

At time $t = 0$, a pot of boiling water is removed from the stove. After five minutes, the water temperature is 80^oC. If the room temperature is 20^oC, when will the water have cooled to 60^oC? (Set up and solve an ODE for $T(t)$.)

1D-6. Motion A mass m falls through air under gravity. Find its velocity $v(t)$ and its terminal velocity (that is, $\lim_{t\to\infty} v(t)$) assuming that

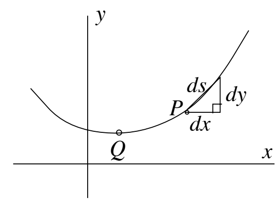
a) air resistance is kv (k constant; this is valid for small v);

b) air resistance is kv^2 (k constant; this is valid for high v).

Call the gravitational constant g. In part (b), lump the constants by introducing a parameter $a = \sqrt{gm/k}$.

1D-7. A loaded cable is hanging from two points of support, with Q the lowest point on the cable. The portion QP is acted on by the total load W on it, the constant tension T_Q at Q, and the variable tension T at P. Both W and T vary with the point P.

Let s denote the length of arc QP.



a) Show that $\dfrac{dx}{T_Q} = \dfrac{dy}{W} = \dfrac{ds}{T}$.

b) Deduce that if the cable hangs under its own weight, and $y(x)$ is the function whose graph is the curve in which the cable hangs, then

(i) $y'' = k\sqrt{1+y'^2}$, $\quad$ k constant

(ii) $y = \sqrt{s^2+c^2} + c_1$, $\quad$ c, c_1 constants

c) Solve the suspension bridge problem: the cable is of negligible weight, and the loading is of constant horizontal density. ("Solve" means: find $y(x)$.)

d) Consider the "Marseilles curtain" problem: the cable is of negligible weight, and loaded with equally and closely spaced vertical rods whose bottoms lie on a horizontal line. (Take the x-axis as the line, and show $y(x)$ satisfies the ODE $y'' = k^2 y$.)

1E. First-order autonomous ODE's

1E-1. For each of the following autonomous equations $dx/dt = f(x)$, obtain a qualitative picture of the solutions as follows:

(i) draw horizontally the axis of the dependent variable x, indicating the critical points of the equation; put arrows on the axis indicating the direction of motion between the critical points; label each critical point as stable, unstable, or semi-stable. Indicate where this information comes from by including in the same picture the graph of $f(x)$, drawn in dashed lines;

(ii) use the information in the first picture to make a second picture showing the tx-plane, with a set of typical solutions to the ODE: the sketch should show the main qualitative features (e.g., the constant solutions, asymptotic behavior of the non-constant solutions).

a) $x' = x^2 + 2x$
b) $x' = -(x-1)^2$
c) $x' = 2x - x^2$
d) $x' = (2-x)^3$

2. Higher-order Linear ODE's

2A. Second-order Linear ODE's: General Properties

2A-1. On the right below is an abbreviated form of the ODE on the left:

$$(*) \qquad y'' + p(x)y' + q(x)y = r(x) \qquad\qquad Ly = r(x) \ ;$$

where L is the *differential operator*:

$$L = D^2 + p(x)D + q(x) \ .$$

a) If u_1 and u_2 are any two twice-differentiable functions, and c is a constant, then

$$L(u_1 + u_2) = L(u_1) + L(u_2) \qquad \text{and} \qquad L(c\,u) = c\,L(u).$$

Operators which have these two properties are called **linear** . Verify that L is linear, i.e., that the two equations are satisfied.

b) Show that if y_p is a solution to (*), then all other solutions to (*) can be written in the form

$$y = y_c + y_p \ ,$$

where y_c is a solution to the *associated homogeneous equation* $Ly = 0$.

2A-2.

a) By eliminating the constants, find a second-order linear homogeneous ODE whose general solution is $y = c_1e^x + c_2e^{2x}$.

b) Verify for this ODE that the IVP consisting of the ODE together with the initial conditions

$$y(x_0) = y_0, \qquad y'(x_0) = y_0' \qquad y_0,\ y_0' \ \text{ constants}$$

is always solvable.

2A-3.

a) By eliminating the constants, find a second-order linear homogeneous ODE whose general solution is $y = c_1x + c_2x^2$.

b) Show that there is no solution to the ODE you found in part (a) which satisfies the initial conditions $y(0) = 1, \quad y'(0) = 1$.

c) Why doesn't part (b) contradict the existence theorem for solutions to second-order linear homogeneous ODE's? (Book: Theorem 2, p. 110.)

2A-4. Consider the ODE $y'' + p(x)y' + q(x)y = 0$.

a) Show that if p and q are continuous for all x, a solution whose graph is tangent to the x-axis at some point must be identically zero, i.e., zero for all x.

b) Find an equation of the above form having x^2 as a solution, by calculating its derivatives and finding a linear equation connecting them. Why isn't part (a) contradicted, since the function x^2 has a graph tangent to the x axis at 0?

2A-5. Show that the following pairs of functions are linearly independent, by calculating their Wronskian.

a) $e^{m_1 x}$, $e^{m_2 x}$, $m_1 \neq m_2$ b) e^{mx}, xe^{mx} (can $m = 0$?)

2A-6. Consider $y_1 = x^2$ and $y_2 = x|x|$. (Sketch the graph of y_2.)

a) Show that $W(y_1, y_2) \equiv 0$ (i.e., is identically zero).

b) Show that y_1 and y_2 are not linearly dependent on any interval (a, b) containing 0. Why doesn't this contradict theorem 3b, p. 116 in your book?

2A-7. Let y_1 and y_2 be two solutions of $y'' + p(x)y' + q(x)y = 0$.

a) Prove that $\dfrac{dW}{dx} = -p(x)W$, where $W = W(y_1, y_2)$, the Wronskian.

b) Prove that if $p(x) = 0$, then $W(y_1, y_2)$ is always a constant.

c) Verify (b) by direct calculation for $y'' + k^2 y = 0$, $k \neq 0$, whose general solution is $y_1 = c_1 \sin kx + c_2 \cos kx$.

2B. Reduction of Order

2B-1. Find a second solution y_2 to $y'' - 2y' + y = 0$, given that one solution is $y_1 = e^x$, by three methods:

a) putting $y_2 = ue^x$ and determining $u(x)$ by substituting into the ODE;

b) determining $W(y_1, y_2)$ using Exercise 2A-7a, and from this getting y_2;

c) by using the general formula $y_2 = y_1 \displaystyle\int \frac{1}{y_1^2} e^{-\int p\,dx} dx$.

d) If you don't get the same answer in each case, account for the differences. (What is the most general form for y_2?)

2B-2. In Exercise 2B-1, prove that the general formula in part (c) for a second solution gives a function y_2 such that y_1 and y_2 are linearly independent. (Calculate their Wronskian.)

2B-3. Use the method of reduction of order (as in 2B-1a) to find a second solution to

$$x^2 y'' + 2xy' - 2y = 0 \ ,$$

given that one solution is $y_1 = x$.

2B-4. Find the general solution on the interval $(-1, 1)$ to the ODE

$$(1 - x^2)y'' - 2xy' + 2y = 0,$$

given that $y_1 = x$ is a solution.

2C. Second-order Linear ODE's with Constant Coefficients

2C-1. Find the general solution, or the solution satisfying the given initial conditions, to each of the following:

a) $y'' - 3y' + 2y = 0$

b) $y'' + 2y' - 3y = 0; \quad y(0) = 1,\ y'(0) = -1$

c) $y'' + 2y' + 2y = 0$

d) $y'' - 2y' + 5y = 0; \quad y(0) = 1,\ y'(0) = -1$

e) $y'' - 4y' + 4y = 0; \quad y(0) = 1,\ y'(0) = 1$

2C-2. Show by using the Wronskian criterion that $e^{ax}\cos bx$ and $e^{ax}\sin bx$ are linearly independent. Are there any restrictions on the constants a and b ?

2C-3. Consider $y'' + cy' + 4y = 0$, c constant. For each statement below, tell for what value(s) of c it holds (indicate reasoning):

a) the equation has oscillatory solutions

b) all solutions are damped oscillations

2C-4. **Euler's equidimensional equation** is the ODE

$$x^2y'' + pxy' + qy = 0; \qquad p,\ q \text{ constants.}$$

a) Show that setting $x = e^t$ changes it into an equation with constant coefficients.

b) Use this to find the general solution to $x^2y'' + xy' + y = 0$.

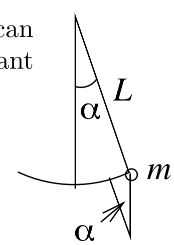
2C-5. The equation $mx'' + cx' + kx = 0$ represents the motion of a damped spring-mass system. (The independent variable is the time t.)

How are the constants m, c, k related if the system is critically damped (i.e., just on the edge of being oscillatory)?

2C-6. Show that the angle α of the pendulum swinging with small amplitude (so you can use the approximation $\sin\alpha \approx \alpha$) approximately obeys a second-order ODE with constant coefficients. Use

$$L = \text{length}, \qquad m = \text{ mass}, \qquad \text{damping } = mc\frac{d\alpha}{dt}, \quad \text{for some constant } c$$

If the motion is undamped, i.e., $c = 0$, express the period in terms of L, m, and the gravitational constant g.



2C-7. For each of the following, tell what you would use as the trial solution in determining a particular solution by the method of undetermined coefficients

a) $y'' + 2y' + 2y = x + e^x$

b) $y'' - 4y' = \cos 2x$

c) $y'' + 4y = 3\cos 2x$

d) $y'' - 2y' + y = 3e^x$

e) $y'' - 3y' + 2y = e^{-x} + 3e^{2x}$

f) $y'' - 6y' + 9y = 2xe^{3x}$

2C-8. Find the general solution, or the solution satisfying the given initial conditions:

a) $y'' - 6y' + 5y = e^x$

b) $y'' + 4y = 2\cos x, \quad y(0) = 0,\ y'(0) = 1$

c) $y'' + y' + y = 2x\,e^x$

d) $y'' - y = x^2, \quad y(0) = 0,\ y'(0) = -1$

2C-9. Consider the ODE $y'' + p(x)y' + q(x)y = r(x)$.

a) Prove that if y_i is a particular solution when $r = r_i(x)$, $(i = 1, 2)$, then $y_1 + y_2$ is a particular solution when $r = r_1 + r_2$. (Use the ideas of Exercise 2A-1.)

b) Use part (a) to find a particular solution to $y'' + 2y' + 2y = 2x + \cos x$.

2C-10. A series RLC-circuit is modeled by either of the ODE's (the second equation is just the derivative of the first)

$$Lq'' + Rq' + \frac{q}{C} = \mathcal{E},$$

$$Li'' + Ri' + \frac{i}{C} = \mathcal{E}',$$

where $q(t)$ is the charge on the capacitor, and $i(t)$ is the current in the circuit; $\mathcal{E}(t)$ is the applied electromotive force (from a battery or generator), and the constants L, R, C are respectively the inductance of the coil, the resistance, and the capacitance, measured in some compatible system of units.

a) Show that if $R = 0$ and $\mathcal{E} = 0$, then $q(t)$ varies periodically, and find the period. (Assume $L \neq 0$.)

b) Assume $\mathcal{E} = 0$; how must R, L, C be related if the current oscillates?

c) If $R = 0$ and $\mathcal{E} = E_0 \sin \omega t$, then for a certain ω_0, the current will have large amplitude whenever $\omega \approx \omega_0$. What is the value of ω_0. (Indicate reason.)

2D. Variation of Parameters

2D-1. Find a particular solution by variation of parameters:

a) $y'' + y = \tan x$

b) $y'' + 2y' - 3y = e^{-x}$

c) $y'' + 4y = \sec^2 2x$

2D-2. Bessel's equation of order p is $x^2y'' + xy' + (x^2 - p^2)y = 0$.

For $p = \frac{1}{2}$, two independent solutions for $x > 0$ are

$$y_1 = \frac{\sin x}{\sqrt{x}} \quad \text{and} \quad y_2 = \frac{\cos x}{\sqrt{x}}, \quad x > 0.$$

Find the general solution to

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})\, y = x^{3/2} \cos x \ .$$

2D-3. Consider the ODE $y'' + p(x)y' + q(x)y = r(x)$.

a) Show that the particular solution obtained by variation of parameters can be written as the definite integral

$$y \;=\; \int_a^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{W(y_1(t), y_2(t))}\, r(t)\, dt \ .$$

(Write the functions v_1 and v_2 (in the Variation of Parameters formula) as definite integrals.)

b) If instead the particular solution is written as an indefinite integral, there are arbitrary constants of integration, so the particular solution is not precisely defined. Explain why this doesn't matter.

2D-4. When *must* you use variation of parameters to find a particular solution, rather than the method of undetermined coefficients?

2E. Complex Numbers

All references are to Notes C: Complex Numbers

2E-1. Change to polar form: a) $-1+i$ b) $\sqrt{3}-i$.

2E-2. Express $\dfrac{1-i}{1+i}$ in the form $a+bi$ by two methods: one using the Cartesian form throughout, and one changing numerator and denominator to polar form. Show the two answers agree.

2E-3.* Show the distance between any two complex points z_1 and z_2 is given by $|z_2-z_1|$.

2E-4. Prove two laws of complex conjugation:

for any complex numbers z and w, a) $\overline{z+w}=\overline{z}+\overline{w}$ b) $\overline{zw}=\overline{z}\overline{w}$.

2E-5.* Suppose $f(x)$ is a polynomial with *real* coefficients. Using the results of 2E-4, show that if $a+ib$ is a zero, then the complex conjugate $a-ib$ is also a zero. (Thus, complex roots of a real polynomial occur in conjugate pairs.)

2E-6.* Prove the formula $e^{i\theta}e^{i\theta'}=e^{i(\theta+\theta')}$ by using the definition (Euler's formula (9)), and the trigonometric addition formulas.

2E-7. Calculate each of the following two ways: by changing to polar form and using DeMoivre's formula (13), and also by using the binomial theorem.

a) $(1-i)^4$ b) $(1+i\sqrt{3})^3$

2E-8.* By using DeMoivre's formula (13) and the binomial theorem, express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

2E-9. Express in the form $a+bi$ the six sixth roots of 1.

2E-10. Solve the equation $x^4+16=0$.

2E-11.* Solve the equation $x^4+2x^2+4=0$, expressing the four roots in both the polar form and the Cartesian form $a+bi$.

2E-12.* Calculate A and B explicitly in the form $a+bi$ for the cubic equation on the first page of Notes C, and then show that $A+B$ is indeed real, and a root of the equation.

2E-13.* Prove the law of exponentials (16), as suggested there.

2E-14. Express $\sin^4 x$ in terms of $\cos 4x$ and $\cos 2x$, using (18) and the binomial theorem. Why would you not expect $\sin 4x$ or $\sin 2x$ in the answer?

2E-15. Find $\displaystyle\int e^{2x}\sin x\,dx$ by using complex exponentials.

2E-16. Prove (18): a) $\cos x = \frac{1}{2}(e^{ix}+e^{-ix})$, b) $\sin x = \frac{1}{2i}(e^{ix}-e^{-ix})$.

2E-17.* Derive formula (20): $D(e^{(a+ib)x})=(a+ib)e^{(a+ib)x}$ from the definition of complex exponential and the derivative formula (19): $D(u+iv)=Du+iDv$.

2E-18.* Find the three cube roots of unity in the $a+bi$ form by locating them on the unit circle and using elementary geometry.

2F. Linear Operators and Higher-order ODE's

2F-1. Find the general solution to each of the following ODE's:

a) $(D-2)^3(D^2+2D+2)\,y=0$

b) $(D^8+2D^4+1)\,y=0$

c) $y^{(4)}+y=0$

d) $y^{(4)}-8y''+16y=0$

e) $y^{(6)}-y=0$ (use 2E-9)

f) $y^{(4)}+16y=0$ (use 2E-10)

2F-2. Find the solution to $y^{(4)}-16y=0$, which in addition satisfies the four side conditions $y(0)=0$, $y'(0)=0$, $y(\pi)=1$, and $|y(x)|<K$ for some constant K and all $x>0$.

2F-3. Find the general solution to

a) $(D^3-D^2+2D-2)\,y=0$

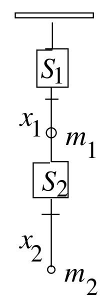
b) $(D^3+D^2-2)\,y=0$

c) $y^{(3)}-2y'-4=0$

d) $y^{(4)}+2y''+4y=0$

(By high-school algebra, if m is a zero of a polynomial $p(D)$, then $(D-m)$ is a factor of $p(D)$. If the polynomial has integer coefficients and leading coefficient 1, then any integer zeros of $p(D)$ must divide the constant term.)

2F-4. A system consisting of two coupled springs is modeled by the pair of ODE's (we take the masses and spring constants to be 1; in the picture the S_i are springs, the m_i are the masses, and x_i represents the distance of mass m_i from its equilibrium position (represented here by a short horisontal line)):



$$x_1''+2x_1-x_2=0, \qquad x_2''+x_2-x_1=0\ .$$

a) Eliminate x_1 to get a 4th order ODE for x_2.

b) Solve it to find the general solution.

2F-5. Let $y=e^{2x}\cos x$. Find y'' by using operator formulas.

2F-6. Find a particular solution to

a) $(D^2+1)y=4e^x$

b) $y^{(3)}+y''-y'+2y=2\cos x$

c) $y''-2y'+4y=e^x\cos x$

d) $y''-6y'+9y=e^{3x}$

(Use the methods in Notes O; use complex exponentials where possible.)

2F-7. Find a particular solution to the general first-order linear equation with constant coefficients, $y'+ay=f(x)$, by assuming it is of the form $y_p=e^{-ax}u$, and applying the exponential-shift formula.

2G. Stability of Linear ODE's with Constant Coefficients

2G-1. For the equation $y'' + 2y' + cy = 0$, c constant,

(i) tell which values of c correspond to each of the three cases in Notes S, p.2;

(ii) for the case of two real roots, tell for which values of c both roots are negative, both roots are positive, or the roots have different signs.

(iii) Summarize the above information by drawing a c-axis, and marking the intervals on it corresponding to the different possibilities for the roots of the characteristic equation.

(iv) Finally, use this information to mark the interval on the c-axis for which the corresponding ODE is stable. (The stability criterion using roots is what you will need.)

2G-2. Prove the stability criterion (coefficient form) (Notes S,(8)), in the direction $\Longrightarrow$.

(You can assume that $a_0 > 0$, after multiplying the characteristic equation through by -1 if necessary. Use the high-school algebra relations which express the coefficients in terms of the roots.)

2G-3. Prove the stability criterion in the coefficient form (Notes S,(8)) in the direction $\Longleftarrow$. Use the quadratic formula, paying particular attention to the case of two real roots.

2G-4.* *Note: in what follows, formula references (11), (12), etc. are to Notes S.*

(a) Prove the higher-order stability criterion in the coefficient form (12).

(You can use the fact that a real polynomial factors into linear and quadratic factors, corresponding respectively to its real roots and its pairs of complex conjugate roots. You will need (11) and the stability criterion in the coefficient form for second-order equations.)

(b) Prove that the converse to (12) is true for those equations all of whose characteristic roots are real.

(Use an indirect proof — assume it is false and derive a contradiction.)

(c) To illustrate that the converse to (12) is in general false, show by using the criterion (11) that the equation $y''' + y'' + y' + 6y = 0$ is not stable. (Find a root of the characteristic equation by inspection, then use this to factor the characteristic polynomial.)

2G-5.* (a) Show when n = 2, the Routh-Hurwitz conditions (Notes S, (13)) are the same as the conditions given for second-order ODE's in (8).

(b) For the ODE $y''' + y'' + y' + cy = 0$, use the Routh-Hurwitz conditions to find all values of c for which the ODE is stable.

2G-6.* Take as the input $r(t) = At$, where A is a constant, in the ODE

$$ay'' + by' + cy = r(t), \qquad a,\ b,\ c \text{ constants}, \quad t = \text{time}. \tag{1}$$

a) Assume $a, b, c > 0$ and find by undetermined coefficients the steady-state solution. Express it in the form $K(t-d)$, where K and d are constants depending on the parameter A and on the coefficients of the equation.

b) We may think of d as the "time-delay". Going back to the two physical interpretations of (1) (i.e., springs and circuits), for each interpretation, express d in terms of the usual constants of the system (m-b-k, or R-L-C, depending on the interpretation).

2H. Impulse Response and Convolution

2H-1. Find the unit impulse response $w(t)$ to $y'' - k^2 y = f(t)$.

2H-2.* a) Find the unit impulse response $w(t)$ to $y'' - (a+b)y' + aby = f(t)$.

b) As $b \to a$, the associated homogeneous system turns into one having the repeated characteristic root a, and te^{at} as its weight function, according to Example 2 in the Notes. So the weight function $w(t)$ you found in part (a) should turn into $t\,e^{at}$, even though the two functions look rather different.

Show that indeed, $\lim\limits_{b\to a} w(t) \;=\; t\,e^{at}$. (Hint: write $b = a+h$ and find $\lim_{h\to 0}$.)

2H-3. a) Use (10) in Notes I to solve $y'' + 4y' + 4y = f(x)$, $y(0) = y'(0) = 0$, $x \ge 0$, where $f(x) = e^{-2x}$.
Check your answer by using the method of undetermined coefficients.

b)* Build on part (a) by using (10) to solve the IVP if $f(x) = \begin{cases} e^{-2x}, & 0 \le x \le 1; \\ 0, & x > 1\ . \end{cases}$

2H-4. Let $\phi(x) = \displaystyle\int_0^x (2x+3t)^2 dt$. Calculate $\phi'(x)$ two ways:

a) by using Leibniz' formula
b) directly, by calculating $\phi(x)$ explicitly, and differentiating it.

2H-5.* Using Leibniz' formula, verify directly that these IVP's have the solution given:

a) $y'' + y = f(x)$, $y(0) = y'(0) = 0$; $\qquad y_p = \dfrac{1}{k}\displaystyle\int_0^x \sin k(x-t)\, f(t)\, dt.$

b) $y'' - 2ky' + k^2 y = f(x)$, $y(0) = y'(0) = 0$; $\qquad y_p = \displaystyle\int_0^x (x-t)e^{k(x-t)}\, f(t)\, dt.$

2H-6.* Find the following convolutions, as explicit functions $f(x)$:

a) $e^{ax} * e^{ax} = xe^{ax}$ (cf. (15)) b) $1 * x$ c) $x * x^2$

2H-7.* Give, with reasoning, the solution to Example 7.

2H-8.* Show $y' + ay = r(x)$, $y(0) = 0$ has the solution $y_p = e^{-ax} * r(x)$ by
a) Leibniz' formula
b) solving the IVP by the first-order method, using a definite integral (cf. Notes D).

2H-9.* There is an analogue of (10) for the IVP with non-constant coefficients:

$$(*) \qquad y'' + p(x)y' + q(x)y = f(x), \quad y(0) = y'(0) = 0.$$

It assumes you know the complementary function: $y_c = c_1 u(x) + c_2 v(x)$. It says

$$y(x) \;=\; \int_0^x g(x,t)\, f(t)\, dt, \qquad \text{where } g(x,t) = \frac{\begin{vmatrix} u(t) & v(t) \\ u(x) & v(x) \end{vmatrix}}{\begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix}}.$$

By using Leibniz' formula, prove this solves the IVP (*).

18.03 Differential Equations
Spring 2010

3. Laplace Transform

3A. Elementary Properties and Formulas

3A-1. Show from the definition of Laplace transform that $\mathcal{L}(t) = \dfrac{1}{s^2}$, $s > 0$.

3A-2. Derive the formulas for $\mathcal{L}(e^{at}\cos bt)$ and $\mathcal{L}(e^{at}\sin bt)$ by assuming the formula

$$\mathcal{L}(e^{\alpha t}) = \frac{1}{s-\alpha}$$

is also valid when α is a complex number; you will also need

$$\mathcal{L}(u+iv) = \mathcal{L}(u) + i\mathcal{L}(v),$$

for a complex-valued function $u(t) + iv(t)$.

3A-3. Find $\mathcal{L}^{-1}\big(F(s)\big)$ for each of the following, by using the Laplace transform formulas. (For (c) and (e) use a partial fractions decomposition.)

a) $\dfrac{1}{\frac{1}{2}s+3}$ b) $\dfrac{3}{s^2+4}$ c) $\dfrac{1}{s^2-4}$ d) $\dfrac{1+2s}{s^3}$ e) $\dfrac{1}{s^4-9s^2}$

3A-4. Deduce the formula for $\mathcal{L}(\sin at)$ from the definition of Laplace transform and the formula for $\mathcal{L}(\cos at)$, by using integration by parts.

3A-5. a) Find $\mathcal{L}(\cos^2 at)$ and $\mathcal{L}(\sin^2 at)$ by using a trigonometric identity to change the form of each of these functions.

b) Check your answers to part (a) by calculating $\mathcal{L}(\cos^2 at) + \mathcal{L}(\sin^2 at)$. By inspection, what should the answer be?

3A-6. a) Show that $\mathcal{L}\left(\dfrac{1}{\sqrt{t}}\right) = \sqrt{\dfrac{\pi}{s}}$, $s > 0$, by using the well-known integral

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\,.$$

(Hint: Write down the definition of the Laplace transform, and make a change of variable in the integral to make it look like the one just given. Throughout this change of variable, s behaves like a constant.)

b) Deduce from the above formula that $\mathcal{L}(\sqrt{t}) = \dfrac{\sqrt{\pi}}{2s^{3/2}}$, $s > 0$.

3A-7. Prove that $\mathcal{L}(e^{t^2})$ does not exist for any interval of the form $s > a$.
(Show the definite integral does not converge for any value of s.)

3A-8. For what values of k will $\mathcal{L}(1/t^k)$ exist? (Write down the definition of this Laplace transform, and determine for what k it converges.)

3A-9. By using the table of formulas, find: a) $\mathcal{L}(e^{-t}\sin 3t)$ b) $\mathcal{L}(e^{2t}(t^2-3t+2))$

3A-10. Find $\mathcal{L}^{-1}\big(F(s)\big)$, if $F(s) =$

a) $\dfrac{3}{(s-2)^4}$ b) $\dfrac{1}{s(s-2)}$ c) $\dfrac{s+1}{s^2-4s+5}$

3B. Derivative Formulas; Solving ODE's

3B-1. Solve the following IVP's by using the Laplace transform:

a) $y' - y = e^{3t}, \quad y(0) = 1$ b) $y'' - 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 1$

c) $y'' + 4y = \sin t, \quad y(0) = 1,\ y'(0) = 0$ d) $y'' - 2y' + 2y = 2e^t, \quad y(0) = 0,\ y'(0) = 1$

e) $y'' - 2y' + y = e^t, \quad y(0) = 1,\ y'(0) = 0$.

3B-2. Without referring to your book or to notes, derive the formula for $\mathcal{L}(f'(t))$ in terms of $\mathcal{L}(f(t))$. What are the assumptions on $f(t)$ and $f'(t)$?

3B-3. Find the Laplace transforms of the following, using formulas and tables:

a) $t\cos bt$ b) $t^n e^{kt}$ (two ways) c) $e^{at} t \sin t$

3B-4. Find $\mathcal{L}^{-1}(F(s))$ if $F(s) =$ a) $\dfrac{s}{(s^2+1)^2}$ b) $\dfrac{1}{(s^2+1)^2}$

3B-5. Without consulting your book or notes, derive the formulas

a) $\mathcal{L}\big(e^{at} f(t)\big) = F(s-a)$ b) $\mathcal{L}\big(t\, f(t)\big) = -F'(s)$

3B-6. If $y(t)$ is a solution to the IVP $y'' + ty = 0, \quad y(0) = 1,\ y'(0) = 0,$ what ODE is satisfied by the function $Y(s) = \mathcal{L}(y(t))$?

(The solution $y(t)$ is called an *Airy function*; the ODE it satisfies is the *Airy equation*.)

3C. Discontinuous Functions

3C-1. Find the Laplace transforms of each of the following functions; do it as far as possible by expressing the functions in terms of known functions and using the tables, rather than by calculating from scratch. In each case, sketch the graph of $f(t)$. (Use the unit step function $u(t)$ wherever possible.)

a) $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -1, & 1 < t \le 2 \\ 0, & \text{otherwise} \end{cases}$ b) $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$

c) $f(t) = |\sin t|, \quad t \ge 0.$

3C-2. Find $\mathcal{L}^{-1}$ for the following: a) $\dfrac{e^{-s}}{s^2+3s+2}$ b) $\dfrac{e^{-s} - e^{-3s}}{s}$ (sketch answer)

3C-3. Find $\mathcal{L}(f(t))$ for the square wave $f(t) = \begin{cases} 1, & 2n \le t \le 2n+1,\ n = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$

a) directly from the definition of Laplace transform;

b) by expressing $f(t)$ as the sum of an infinite series of functions, taking the Laplace transform of the series term-by-term, and then adding up the infinite series of Laplace transforms.

3C-4. Solve by the Laplace transform the following IVP, where $h(t) = \begin{cases} 1, & \pi \le t \le 2\pi, \\ 0, & \text{otherwise} \end{cases}$

$$y'' + 2y' + 2y = h(t), \quad y(0) = 0,\ y'(0) = 1;$$

write the solution in the format used for $h(t)$.

3C-5. Solve the IVP: $y'' - 3y' + 2y = r(t)$, $y(0) = 1,\ y'(0) = 0$, where $r(t) = u(t)\,t$, the ramp function.

3D. Convolution and Delta Function

3D-1. Solve the IVP: $y'' + 2y' + y = \delta(t) + u(t-1)$, $y(0) = 0,\ y'(0^-) = 1$.

Write the answer in the "cases" format $y(t) = \begin{cases} \cdots, & 0 \le t \le 1 \\ \cdots, & t > 1 \end{cases}$

3D-2. Solve the IVP: $y''+y = r(t)$, $y(0) = 0,\ y'(0) = 1$, where $r(t) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

Write the answer in the "cases" format (see 3D-1 above).

3D-3. If $f(t+c) = f(t)$ for all t, where c is a fixed positive constant, the function $f(t)$ is said to be *periodic*, with period c. (For example, $\sin x$ is periodic, with period 2π.)

a) Show that if $f(t)$ is periodic with period c, then its Laplace transform is

$$F(s) = \frac{1}{1-e^{-cs}} \int_0^c e^{-st} f(t)\, dt\ .$$

b) Do Exercise 3C-3, using the above formula.

3D-4. Find $\mathcal{L}^{-1}$ by using the convolution: a) $\dfrac{s}{(s+1)(s^2+4)}$ b) $\dfrac{1}{(s^2+1)^2}$

Your answer should not contain the convolution $*$.

3D-5. Assume $f(t) = 0$, for $t \le 0$. Show informally that $\delta(t) * f(t) = f(t)$, by using the definition of convolution; then do it by using the definition of $\delta(t)$.

(See (5), section 4.6 of your book; $\delta(t)$ is written $\delta_0(t)$ there.)

3D-6. Prove that $f(t) * g(t) = g(t) * f(t)$ directly from the definition of convolution, by making a change of variable in the convolution integral.

3D-7. Show that the IVP: $y'' + k^2 y = r(t)$, $y(0) = 0,\ y'(0) = 0$ has the solution

$$y(t) = \frac{1}{k} \int_0^t r(u) \sin k(t-u)\, du\ ,$$

by using the Laplace transform and the convolution.

3D-8. By using the Laplace transform and the convolution, show that in general the IVP (here a and b are constants):

$$y'' + ay' + by = r(t), \qquad y(0) = 0,\ y'(0) = 0,$$

has the solution

$$y(t) = \int_0^t w(t-u) r(u)\, du\ ,$$

where $w(t)$ is the solution to the IVP: $y'' + ay' + by = 0$, $y(0) = 0,\ y'(0) = 1$.

(The function $w(t-u)$ is called the **Green's function** for the linear operator $D^2 + aD + b$.)

4. Linear Systems

4A. Review of Matrices

4A-1. Verify that $\begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -2 & -6 \end{pmatrix}$.

4A-2. If $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$, show that $AB \neq BA$.

4A-3. Calculate A^{-1} if $A = \begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix}$, and check your answer by showing that $AA^{-1} = I$ and $A^{-1}A = I$.

4A-4. Verify the formula given in Notes LS.1 for the inverse of a 2×2 matrix.

4A-5. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Find A^3 $(= A \cdot A \cdot A)$.

4A-6. For what value of c will the vectors $\mathbf{x}_1 = (1, 2, c)$, $\mathbf{x}_2 = (-1, 0, 1)$, and $\mathbf{x}_3 = (2, 3, 0)$ be linearly dependent? For this value, find by trial and error (or otherwise) a linear relation connecting them, i.e., one of the form $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$

4B. General Systems; Elimination; Using Matrices

4B-1. Write the following equations as equivalent first-order systems:

a) $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + tx^2 = 0$ b) $y'' - x^2y' + (1 - x^2)y = \sin x$

4B-2. Write the IVP

$$y^{(3)} + p(t)y'' + q(t)y' + r(t)y = 0, \qquad y(0) = y_0,\ y'(0) = y_0',\ y''(0) = y_0''$$

as an equivalent IVP for a system of three first-order linear ODE's. Write this system both as three separate equations, and in matrix form.

4B-3. Write out $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ as a system of two first-order equations.

a) Eliminate y so as to obtain a single second-order equation for x.

b) Take the second-order equation and write it as an equivalent first-order system. This isn't the system you started with, but show a change of variables converts one system into the other.

4B-4. For the system $x' = 4x - y$, $y' = 2x + y$,

a) using matrix notation, verify that $x = e^{3t}$, $y = e^{3t}$ and $x = e^{2t}$, $y = 2e^{2t}$ are solutions;

b) verify that they form a fundamental set of solutions — i.e., that they are linearly independent;

c) write the general solution to the system in terms of two arbitrary constants c_1 and c_2; write it both in vector form, and in the form $x = \dots$, $y = \dots$.

4B-5. For the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$,

a) show that $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$ form a fundamental set of solutions (i.e., they are linearly independent and solutions);

b) solve the IVP: $\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

4B-6. Solve the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}$ in two ways:

a) Solve the second equation, substitute for y into the first equation, and solve it.

b) Eliminate y by solving the first equation for y, then substitute into the second equation, getting a second order equation for x. Solve it, and then find y from the first equation. Do your two methods give the same answer?

4B-7. Suppose a radioactive substance R decays into a second one S which then also decays. Let x and y represent the amounts of R and S present at time t, respectively.

a) Show that the physical system is modeled by a system of equations

$$\mathbf{x}' = A\mathbf{x}, \qquad \text{where } A = \begin{pmatrix} -a & 0 \\ a & -b \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad a,\ b \text{ constants.}$$

b) Solve this sytem by either method of elimination described in 4B-6.

c) Find the amounts present at time t if initially only R is present, in the amount x_0.

Remark. The method of elimination which was suggested in some of the preceding problems (4B-3,6,7; book section 5.2) is always available. Other than in these exercises, we will not discuss it much, as it does not give insights into systems the way the methods will decribe later do.

Warning. Elimination sometimes produces extraneous solutions — extra "solutions" that do not actually solve the original system. Expect this to happen when you have to differentiate both equations to do the elimination. (Note that you also get extraneous solutions when doing elimination in ordinary algebra, too.) If you get more independent solutions than the order of the system, they must be tested to see if they actually solve the original system. (The order of a system of ODE's is the sum of the orders of each of the ODE's in it.)

4C. Eigenvalues and Eigenvectors

4C-1. Solve $\mathbf{x}' = A\mathbf{x}$, if A is: a) $\begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$ c) $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$.

(First find the eigenvalues and associated eigenvectors, and from these construct the normal modes and then the general solution.)

4C-2. Prove that $m = 0$ is an eigenvalue of the $n \times n$ constant matrix A if and only if A is a singular matrix ($\det A = 0$). (You can use the characteristic equation, or you can use the definition of eigenvalue.)

4C-3. Suppose a 3×3 matrix is upper triangular. (This means it has the form below, where $*$ indicates an arbitrary numerical entry.)

$$A = \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix}$$

Find its eigenvalues. What would be the generalization to an $n \times n$ matrix?

4C-4. Show that if $\vec{\alpha}$ is an eigenvector of the matrix A, associated with the eigenvalue m, then $\vec{\alpha}$ is also an eigenvector of the matrix A^2, associated this time with the eigenvalue m^2. (Hint: use the eigenvector equation in 4F-3.)

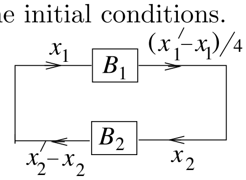
4C-5. Solve the radioactive decay problem (4B-7) using eigenvectors and eigenvalues.

4C-6. Farmer Smith has a rabbit colony in his pasture, and so does Farmer Jones. Each year a certain fraction a of Smith's rabbits move to Jones' pasture because the grass is greener there, and a fraction b of Jones' rabbits move to Smith's pasture (for the same reason). Assume (foolishly, but conveniently) that the growth rate of rabbits is 1 rabbit (per rabbit/per year).

a) Write a system of ODE's for determining how S and J, the respective rabbit populations, vary with time t (years).

b) Assume $a = b = \frac{1}{2}$. If initially Smith has 20 rabbits and Jones 10 rabbits, how do the two populations subsequently vary with time?

c) Show that S and J never oscillate, regardless of a, b and the initial conditions.



4C-7. The figure shows a simple feedback loop.

Black box B_1 inputs $x_1(t)$ and outsputs $\frac{1}{4}(x_1' - x_1)$.

Black box B_2 inputs $x_2(t)$ and outputs $x_2' - x_2$.

If they are hooked up in a loop as shown, and initially $x_1 = 1, x_2 = 0$, how do x_1 and x_2 subsequently vary with time t? (If it helps, you can think of x_1 and x_2 as currents, for instance, or as the monetary values of trading between two countries, or as the number of times/minute Punch hits Judy and vice-versa.)

4D. Complex and Repeated Eigenvalues

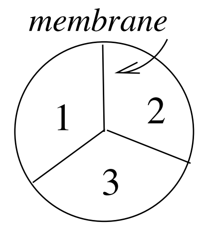
4D-1. Solve the system $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} \mathbf{x}$.

4D-2. Solve the system $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \mathbf{x}$.

4D-3. Solve the system $\mathbf{x}' = \begin{pmatrix} 2 & 3 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$.

4D-4. Three identical cells are pictured, each containing salt solution, and separated by identical semi-permeable membranes. Let A_i represent the amount of salt in cell i ($i = 1, 2, 3$), and let

$$x_i = A_i - A_0$$



be the difference bwetween this amount and some standard reference amount A_0. Assume the rate at which salt diffuses across the membranes is proportional to the difference in concentrations, i.e. to the difference in the two values of A_i on either side, since we are supposing the cells identical. Take the constant of proportionality to be 1.

a) Derive the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$.

b) Find three normal modes, and interpret each of them physically. (To what initial conditions does each correspond — is it reasonable as a solution, in view of the physical set-up?)

4E. Decoupling

4E-1. A system is given by $x' = 4x + 2y$, $y' = 3x - y$. Give a new set of variables, u and v, linearly related to x and y, which decouples the system. Then verify by direct substitution that the system becomes decoupled when written in terms of u and v.

4E-2. Answer the same questions as in the previous problem for the system in 4D-4. (Use the solution given in the Notes to get the normal modes. In the last part of the problem, do the substitution by using matrices.)

4F. Theory of Linear Systems

4F-1. Take the second-order equation $x'' + p(t)x' + q(t)x = 0$.

a) Change it to a first-order system $\mathbf{x}' = A\mathbf{x}$ in the usual way.

b) Show that the Wronskian of two solutions x_1 and x_2 of the original equation is the same as the Wronskian of the two corresponding solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the system.

4F-2. Let $\mathbf{x}_1 = \begin{pmatrix} t \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ be two vector functions.

a) Prove by using the definition that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

b) Calculate the Wronskian $W(\mathbf{x}_1, \mathbf{x}_2)$.

c) How do you reconcile (a) and (b) with Theorem 5C in Notes LS.5?

d) Find a linear system $\mathbf{x}' = A\mathbf{x}$ having $\mathbf{x}_1$ and $\mathbf{x}_2$ as solutions, and confirm your answer to (c). (To do this, treat the entries of A as unknowns, and find a system of equations whose solutions will give you the entries. A will be a matrix function of t, i.e., its entries will be functions of t.)

4F-3. Suppose the 2×2 constant matrix A has two distinct eigenvectors m_1 and m_2, with associated eigenvectors respectively $\vec{\alpha}_1$ and $\vec{\alpha}_2$. Prove that the corresponding vector functions

$$\mathbf{x}_1 = \vec{\alpha}_1 e^{m_1 t}, \qquad \mathbf{x}_2 = \vec{\alpha}_2 e^{m_2 t}$$

are linearly independent, as follows:

a) using the determinantal criterion, show they are linearly independent if and only if $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are linearly independent;

b) then show that $c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 = 0 \quad \Rightarrow \quad c_1 = 0,\ c_2 = 0$. (Use the eigenvector equation $(A - m_i I)\vec{\alpha}_i = 0$ in the form: $A\vec{\alpha}_i = m_i\vec{\alpha}_i$.)

4F-4. Suppose $\mathbf{x}' = A\mathbf{x}$, where A is a nonsingular constant matrix. Show that if $\mathbf{x}(t)$ is a solution whose velocity vector $\mathbf{x}'(t)$ is $\mathbf{0}$ at time t_0, then $\mathbf{x}(t)$ is identically zero for all t. What is the minimum hypothesis on A that is needed for this result to be true? Can A be a function of t, for example?

4G. Fundamental Matrices

4G-1. Two independent solutions to $\mathbf{x}' = A\mathbf{x}$ are $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$.

a) Find the solutions satisfying $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

b) Using part (a), find in a simple way the solution satisfying $x(0) = \begin{pmatrix} a \\ b \end{pmatrix}$.

4G-2. For the system $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$,

a) find a fundamental matrix, using the normal modes, and use it to find the solution satisfying $x(0) = 2,\ y(0) = -1$;

b) find the fundamental matrix normalized at $t = 0$, and solve the same IVP as in part (a) using it.

4G-3.* Same as 4G-2, using the matrix $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ instead.

4H. Exponential Matrices

4H-1. Calculate e^{At} if $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Verify directly that $\mathbf{x} = e^{At}\mathbf{x}_0$ is the solution to $\mathbf{x}' = A\mathbf{x},\ \ \mathbf{x}(0) = \mathbf{x}_0$.

4H-2. Calculate e^{At} if $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; then answer same question as in 4H-1.

4H-3. Calculate e^{At} directly from the infinite series, if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; then answer same question as in 4H-1.

4H-4. What goes wrong with the argument justifying the e^{At} solution of $\mathbf{x}' = A\mathbf{x}$ if A is not a constant matrix, but has entries which depend on t?

4H-5. Prove that a) $e^{kIt} = Ie^{kt}$. b) $Ae^{At} = e^{At}A$.

(Here k is a constant, I is the identity matrix, A any square constant matrix.)

4H-6. Calculate the exponential matrix in 4H-3, this time using the third method in the Notes (writing $A = B + C$).

4H-7. Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Calculate e^{At} three ways:

a) directly, from its definition as an infinite series;

b) by expressing A as a sum of simpler matrices, as in Notes LS.6, Example 6.3C;

c) by solving the system by elimination so as to obtain a fundamental matrix, then normalizing it.

4I. Inhomogeneous Systems

4I-1. Solve $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -8 \end{pmatrix} - \begin{pmatrix} 5 \\ 8 \end{pmatrix}t$, by variation of parameters.

4I-2. a) Solve $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$ by variation of parameters.

b) Also do it by undetermined coefficients, by writing the forcing term and trial solution respectively in the form:

$$\begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-2t} + \begin{pmatrix} 0 \\ -2 \end{pmatrix}e^{t}; \qquad \mathbf{x}_p = \vec{c}e^{-2t} + \vec{d}e^{t}\ .$$

4I-3.* Solve $\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ -5 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix}$ by undetermined coefficients.

4I-4. Solve $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t}$ by undetermined coefficients.

4I-5. Suppose $\mathbf{x}' = A\mathbf{x} + \mathbf{x}_0$ is a first-order order system, where A is a nonsingular $n \times n$ constant matrix, and $\mathbf{x}_0$ is a constant n-vector. Find a particular solution $\mathbf{x}_p$.

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Section 5. Graphing Systems

5A. The Phase Plane

5A-1. Find the critical points of each of the following non-linear autonomous systems.

a) $$\begin{aligned} x' &= x^2 - y^2 \\ y' &= x - xy \end{aligned}$$ b) $$\begin{aligned} x' &= 1 - x + y \\ y' &= y + 2x^2 \end{aligned}$$

5A-2. Write each of the following equations as an equivalent first-order system, and find the critical points.

a) $x'' + a(x^2 - 1)x' + x = 0$ b) $x'' - x' + 1 - x^2 = 0$

5A-3. In general, what can you say about the relation between the trajectories and the critical points of the system on the left below, and those of the two systems on the right?

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$ a) $$\begin{aligned} x' &= -f(x,y) \\ y' &= -g(x,y) \end{aligned}$$ b) $$\begin{aligned} x' &= g(x,y) \\ y' &= -f(x,y) \end{aligned}$$

5A-4. Consider the autonomous system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} ; \qquad \text{solution : } \quad \mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

a) Show that if $\mathbf{x}_1(t)$ is a solution, then $\mathbf{x}_2(t) = \mathbf{x}_1(t - t_0)$ is also a solution. What is the geometric relation between the two solutions?

b) The existence and uniqueness theorem for the system says that if f and g are continuously differentiable everywhere, there is one and only one solution $\mathbf{x}(t)$ satisfying a given initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

Using this and part (a), show that two trajectories cannot intersect anywhere.

(Note that if two trajectories intersect at a point (a, b), the corresponding solutions $\mathbf{x}(t)$ which trace them out may be at (a, b) at different times. Part (a) shows how to get around this difficulty.)

5B. Sketching Linear Systems

5B-1. Follow the Notes (GS.2) for graphing the trajectories of the system $\begin{cases} x' = -x \\ y' = -2y \end{cases}$.

a) Eliminate t to get one ODE $\dfrac{dy}{dx} = F(x, y)$. Solve it and sketch the solution curves.

b) Solve the original system (by inspection, or using eigenvalues and eigenvectors), obtaining the equations of the trajectories in parametric form: $x = x(t), y = y(t)$. Using these, put the direction of motion on your solution curves for part (a). What new trajectories are there which were not included in the curves found in part (a)?

c) How many trajectories are needed to cover a typical solution curve found in part (a)? Indicate them on your sketch.

d) If the system were $x' = x, \quad y' = 2y$ instead, how would your picture be modified? (Consider both parts (a) and (b).)

5B-2. Answer the same questions as in 5B-1 for the system $x' = y, \quad y' = x$. (For part (d), use $-y$ and $-x$ as the two functions on the right.)

5B-3. Answer the same question as in 5B-1a,b for the system $x' = y, \quad y' = -2x$.

For part (b), put in the direction of motion on the curves by making use of the vector field corresponding to the system.

5B-4. For each of the following linear systems, carry out the graphing program in Notes GS.4; that is,

(i) find the eigenvalues of the associated matrix and from this determine the geometric type of the critical point at the origin, and its stability;

(ii) if the eigenvalues are real, find the associated eigenvectors and sketch the corresponding trajectories, showing the direction of motion for increasing t; then draw in some nearby trajectories;

(iii) if the eigenvalues are complex, obtain the direction of motion and the approximate shape of the spiral by sketching in a few vectors from the vector field defined by the system.

a) $\begin{aligned} x' &= 2x - 3y \\ y' &= x - 2y \end{aligned}$ b) $\begin{aligned} x' &= 2x \\ y' &= 3x + y \end{aligned}$ c) $\begin{aligned} x' &= -2x - 2y \\ y' &= -x - 3y \end{aligned}$

d) $\begin{aligned} x' &= x - 2y \\ y' &= x + y \end{aligned}$ e) $\begin{aligned} x' &= x + y \\ y' &= -2x - y \end{aligned}$

5B-5. For the damped spring-mass system modeled by the ODE

$$mx'' + cx' + kx = 0, \qquad m, c, k > 0 \ ,$$

a) write it as an equivalent first-order linear system;

b) tell what the geometric type of the critical point at $(0, 0)$ is, and determine its stability, in each of the following cases. Do it by the methods of Sections GS.3 and GS.4, and check the result by physical intuition.

(i) $c = 0$ (ii) $c \approx 0;\ m, k \gg 1.$ (iii) Can the geometric type be a saddle? Explain.

5C. Sketching Non-linear Systems

5C-1. For the following system, the origin is clearly a critical point. Give its geometric type and stability, and sketch some nearby trajectories of the system.

$$\begin{aligned} x' &= x - y + xy \\ y' &= 3x - 2y - xy \end{aligned}$$

5C-2. Repeat 5C-1 for the system $\begin{cases} x' = x + 2x^2 - y^2 \\ y' = x - 2y + x^3 \end{cases}$

5C-3. Repeat 5C-1 for the system $\begin{cases} x' = 2x + y + xy^3 \\ y' = x - 2y - xy \end{cases}$

5C-4. For the following system, carry out the program outlined in Notes GS.6 for sketching trajectories — find the critical points, analyse each, draw in nearby trajectories, then add some other trajectories compatible with the ones you have drawn; when necessary, put in a vector from the vector field to help.

$$\begin{aligned} x' &= 1 - y \\ y' &= x^2 - y^2 \end{aligned}$$

5C-5. Repeat 5C-4 for the system $\begin{cases} x' = x - x^2 - xy \\ y' = 3y - xy - 2y^2 \end{cases}$

5D. Limit Cycles

5D-1. In Notes LC, Example 1,

a) Show that $(0,0)$ is the only critical point (hint: show that if (x,y) is a non-zero critical point, then $y/x = -x/y$; derive a contradiction).

b) Show that $(\cos t, \sin t)$ is a solution; it is periodic: what is its trajectory?

c) Show that all other non-zero solutions to the system get steadily closer to the solution in part (b). (This shows the solution is an asymptotically stable limit cycle, and the only periodic solution to the system.)

5D-2. Show that each of these systems has no closed trajectories in the region R (this is the whole xy-plane, except in part (c)).

a) $\begin{aligned} x' &= x + x^3 + y^3 \\ y' &= y + x^3 + y^3 \end{aligned}$ b) $\begin{aligned} x' &= x^2 + y^2 \\ y' &= 1 + x - y \end{aligned}$ c) $\begin{aligned} x' &= 2x + x^2 + y^2 \\ y' &= x^2 - y^2 \end{aligned}$ $R = $ half-plane $x < -1$

d) $\begin{aligned} x' &= ax + bx^2 - 2cxy + dy^2 \\ y' &= ex + fx^2 - 2bxy + cy^2 \end{aligned}$: find the condition(s) on the six constants that guarantees no closed trajectories in the xy-plane

5D-3. Show that Lienard's equation (Notes LC, (6)) has no periodic solution if either

a) $u(x) > 0$ for all x b) $v(x) > 0$ for all x .

(Hint: consider the corresponding system, in each case.)

5D-4.* a) Show van der Pol's equation (Notes LC.4) satisfies the hypotheses of the Levinson-Smith theorem (this shows it has a unique limit cycle).

b) The corresponding system has a unique critical point at the origin; show this and determine its geometric type and stability. (Its type depends on the value of the parameter).

5D-5.* Consider the following system (where $r = \sqrt{x^2 + y^2}$): $\begin{aligned} x' &= -y + xf(r) \\ y' &= x + yf(r) \end{aligned}$

a) Show that if $f(r)$ has a positive zero a, then the system has a circular periodic solution.

b) Show that if $f(r)$ is a decreasing function for $r \approx a$, then this periodic solution is actually a stable limit cycle. (Hint: how does the direction field then look?)

5E. Structural stability; Volterra's Principle

5E-1. Each of the following systems has a critical point at the origin. For this critical point, find the geometric type and stability of the corresponding linearized system, and then tell what the possibilities would be for the corresponding critical point of the given non-linear system.

a) $x' = x - 4y - xy^2, \quad y' = 2x - y + x^2y$

b) $x' = 3x - y + x^2 + y^2, \quad y' = -6x + 2y + 3xy$

5E-2. Each of the following systems has one critical point whose linearization is not structurally stable. In each case, sketch several pictures showing the different ways the trajectories of the non-linear system might look.

Begin by finding the critical points and determining the type of the corresponding linearized system at each of the critical points.

a) $x' = y, \quad y' = x(1 - x)$

b) $x' = x^2 - x + y, \quad y' = -yx^2 - y$

5E-3. The main tourist attraction at Monet Gardens is Pristine Acres, an expanse covered with artfully arranged wildflowers. Unfortunately, the flower stems are the favorite food of the Kandinsky borer; the flower and borer populations fluctuate cyclically in accordance with Volterra's predator-prey equations. To boost the wildflower level for the tourists, the director wants to fertilize the Acres, so that the wildflower growth will outrun that of the borers.

Assume that fertilizing would boost the wildflower growth rate (in the absence of borers) by 25 percent. What do you think of this proposal?

Using suitable units, let x be the wildflower population and y be the borer population.

Take the equations to be $x' = ax - pxy, \qquad y' = -by + qxy,$ where a, b, p, q are constants.

6. Power Series

6A. Power Series Operations

6A-1. Find the radius of convergence for each of the following:

a) $\displaystyle\sum_0^\infty n\,x^n$ b) $\displaystyle\sum_0^\infty \frac{x^{2n}}{n2^n}$ c) $\displaystyle\sum_1^\infty n!\,x^n$ d) $\displaystyle\sum_0^\infty \frac{(2n)!}{(n!)^2}\,x^n$

6A-2. Starting from the series $\displaystyle\sum_0^\infty x^n = \frac{1}{1-x}$ and $\displaystyle\sum_0^\infty \frac{x^n}{n!} = e^x$,
by using operations on series (substitution, addition and multiplication, term-by-term differentiation and integration), find series for each of the following

a) $\displaystyle\frac{1}{(1-x)^2}$ b) xe^{-x^2} c) $\tan^{-1} x$ d) $\ln(1+x)$

6A-3. Let $\displaystyle y = \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$. Show that

a) y is a solution to the ODE $y'' - y = 0$ b) $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$.

6A-4. Find the sum of the following power series (using the operations in 6A-2 as a help):

a) $\displaystyle\sum_0^\infty x^{3n+2}$ b) $\displaystyle\sum_0^\infty \frac{x^n}{n+1}$ c) $\displaystyle\sum_0^\infty nx^n$

6B. First-order ODE's

6B-1. For the nonlinear IVP $y' = x + y^2$, $y(0) = 1$, find the first four nonzero terms of a series solution $y(x)$ two ways:

a) by setting $y = \sum_0^\infty a_n x^n$ and finding in order $a_0, a_1, a_2, \ldots$, using the initial condition and substituting the series into the ODE;

b) by differentiating the ODE repeatedly to obtain $y(0), y'(0), y''(0), \ldots$, and then using Taylor's formula.

6B-2. Solve the following linear IVP by assuming a series solution

$$y = \sum_0^\infty a_n x^n ,$$

substituting it into the ODE and determining the a_n recursively by the method of undetermined coefficients. Then sum the series to obtain an answer in closed form, if possible (the techniques of 6A-2,4 will help):

a) $y' = x + y$, $y(0) = 0$ b) $y' = -xy$, $y(0) = 1$ c) $(1-x)y' - y = 0$, $y(0) = 1$

6C. Solving Second-order ODE's

6C-1. Express each of the following as a power series of the form $\sum_N^\infty b_n x^n$. Indicate the value of N, and express b_n in terms of a_n.

a) $\sum_1^\infty a_n x^{n+3}$ b) $\sum_0^\infty n(n-1)a_n x^{n-2}$ c) $\sum_1^\infty (n+1)a_n x^{n-1}$

6C-2. Find two independent power series solutions $\sum a_n x^n$ to $y'' - 4y = 0$, by obtaining a recursion formula for the a_n.

6C-3. For the ODE $y'' + 2xy' + 2y = 0$,

a) find two independent series solutions y_1 and y_2;

b) determine their radius of convergence;

c) express the solution satisfying $y(0) = 1$, $y'(0) = -1$ in terms of y_1 and y_2;

d) express the series in terms of elementary functions (i.e., sum the series to an elementary function).

(One of the two series is easily recognizable; the other can be gotten using the operations on series, or by using the known solution and the method of reduction of order—see Exercises 2B.)

6C-4. Hermite's equation is $y'' - 2xy' + ky = 0$. Show that if k is a positive even integer $2m$, then one of the power series solutions is a polynomial of degree m.

6C-5. Find two independent series solutions in powers of x to the Airy equation: $y'' = xy$.

Determine their radius of convergence. For each solution, give the first three non-zero terms and the general term.

6C-6. Find two independent power series solutions $\sum a_n x^n$ to

$$(1 - x^2)y'' - 2xy' + 6y = 0 \ .$$

Determine their radius of convergence R. To what extent is R predictable from the original ODE?

6C-7. If the recurrence relation for the a_n has three terms instead of just two, it is more difficult to find a formula for the general term of the corresponding series. Give the recurrence relation and the first three nonzero terms of two independent power series solutions to

$$y'' + 2y' + (x - 1)y = 0 \ .$$

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7. Fourier Series

Based on exercises in Chap. 8, Edwards and Penney, Elementary Differential Equations

7A. Fourier Series

7A-1. Find the smallest period for each of the following periodic functions:

a) $\sin \pi t/3$ b) $|\sin t|$ c) $\cos^2 3t$

7A-2. Find the Fourier series of the function $f(t)$ of period 2π which is given over the interval $-\pi < t \le \pi$ by

a) $f(t) = \begin{cases} 0, & -\pi < t \le 0; \\ 1, & 0 < t \le \pi \end{cases}$ b) $f(t) = \begin{cases} -t, & -\pi < t < 0; \\ t, & 0 \le t \le \pi \end{cases}$

7A-3. Give another proof of the orthogonality relations $\displaystyle\int_{-\pi}^{\pi} \cos mt \, \cos nt \, dt = \begin{cases} 0, & m \ne n; \\ \pi, & m = n\ . \end{cases}$

by using the trigonometric identity: $\cos A \cos B = \frac{1}{2}\big(\cos(A+B) + \cos(A-B)\big)$.

7A-4. Suppose that $f(t)$ has period P. Show that $\int_I f(t)\,dt$ has the same value over any interval I of length P, as follows:

a) Show that for any a, we have $\displaystyle\int_P^{a+P} f(t)\,dt = \int_0^a f(t)\,dt$. (Make a change of variable.)

b) From part (a), deduce that $\displaystyle\int_a^{a+P} f(t)\,dt = \int_0^P f(t)\,dt$.

7B. Even and Odd Series; Boundary-value Problems

7B-1. a) Find the Fourier cosine series of the function $1 - t$ over the interval $0 < t < 1$, and then draw over the interval $[-2, 2]$ the graph of the function $f(t)$ which is the sum of this Fourier cosine series.

b) Answer the same question for the Fourier sine series of $1 - t$ over the interval $(0, 1)$.

7B-2. Find a formal solution as a Fourier series, for these boundary-value problems (you can use any Fourier series derived in the book's Examples):

a) $x'' + 2x = 1, \quad x(0) = x(\pi) = 0;$
b) $x'' + 2x = t, \quad x'(0) = x'(\pi) = 0$ (use a cosine series)

7B-3. Assume $a > 0$; show that $\displaystyle\int_{-a}^{0} f(t)\,dt = \pm \int_0^a f(t)\,dt$, according to whether $f(t)$ is respectively an even function or an odd function.

7B-4. The Fourier series of the function $f(t)$ having period 2, and for which $f(t) = t^2$ for $0 < t < 2$, is

$$f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_1^\infty \frac{\cos n\pi t}{n^2} - \frac{4}{\pi}\sum_1^\infty \frac{\sin n\pi t}{n}\ .$$

Differentiate this series term-by-term, and show that the resulting series does not converge to $f'(t)$.

7C. Applications to resonant frequencies

7C-1. For each spring-mass system, find whether pure resonance occurs, without actually calculating the solution.

a) $2x'' + 10x = F(t)$; $F(t) = 1$ on $(0,1)$, $F(t)$ is odd, and of period 2;
b) $x'' + 4\pi^2 x = F(t)$; $F(t) = 2t$ on $(0,1)$, $F(t)$ is odd, and of period 2;
c) $x'' + 9x = F(t)$; $F(t) = 1$ on $(0,\pi)$, $F(t)$ is odd, and of period 2π.

7C-2. Find a periodic solution as a Fourier series to $x'' + 3x = F(t)$, where $F(t) = 2t$ on $(0,\pi)$, $F(t)$ is odd, and has period 2π.

7C-3. For the following two lightly damped spring-mass systems, by considering the form of the Fourier series solution and the frequency of the corresponding undamped system, determine what term of the Fourier series solution should dominate — i.e., have the biggest amplitude.

a) $2x'' + .1x' + 18x = F(t)$; $F(t)$ is as in 7C-2.
b) $3x'' + x' + 30x = F(t)$; $F(t) = t - t^2$ on $(0,1)$, $F(t)$ is odd, with period 2.