Section 1 SOLUTIONS

SOLUTIONS - SECTION 1

$\boxed{1A-1}$ a) $y = c_1e^x + c_2xe^x$

$\times -2$ $\quad y' = (c_1+c_2)e^x + c_2xe^x$

$y'' = (c_1+2c_2)e^x + c_2xe^x$

Add $\quad y'' - 2y' + y = 0$ ✓ (easily checked)

b) $y' = -\dfrac{(\sin x + a)}{x^2} + \dfrac{\cos x}{x} + \sin x$

$\dfrac{y}{x} = \dfrac{\sin x + a}{x^2} - \dfrac{\cos x}{x}$

$\therefore \quad y' + \dfrac{y}{x} = \sin x$

$\boxed{1A-2}$ a) c_1e^{kx} and $c_1'e^{k'x}$ are the same only if $c_1 = c_1'$, $k = k'$

b) let $k = c_1e^a$

then $y = ke^x$

c) $\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$

$\therefore \ y = c_1 + c_2(2\cos^2 x - 1) + c_3\cos^2 x$
$= (c_1 - c_2) + (2c_2 + c_3)\cos^2 x$
$= k_1 + k_2\cos^2 x$

d) $y = \ln(ax+b)(cx+d)$
$= \ln(acx^2 + (ad+bc)x + bd)$

$\therefore \ y = \ln(k_1x^2 + k_2x + k_3)$

$\boxed{1A-3}$ a) Separating variables gives

$y^2\,dy = \dfrac{dx}{\ln x}$ Integrate both sides from 2 to x:

$\left.\dfrac{y^3}{3}\right]_2^x = \displaystyle\int_2^x \frac{dt}{\ln t}$. Now use $y(2)=0$:

$\dfrac{y(x)^3}{3} - \dfrac{0^3}{3} = \displaystyle\int_2^x \frac{dt}{\ln t}$

$\therefore \ y = \left[3\displaystyle\int_2^x \frac{dt}{\ln t}\right]^{1/3}$.

b) Separate variables: $\dfrac{dy}{y} = \dfrac{e^x}{x}\,dx$

Can either use same method as in (a), or else: integrate both sides, using a definite integral as the antiderivative on the right:

$\ln y + c = \displaystyle\int_1^x \frac{e^t}{t}\,dt$ ⊛

Evaluate c by using $y(1) = 1$. This gives

$\underbrace{\ln y(1)}_{0} + c = \displaystyle\int_1^1 \frac{e^t}{t}\,dt = 0$

$\therefore \ c = 0$

So from ⊛ $\quad y = e^{\int_1^x e^t/t\,dt}$

$\boxed{1A-4}$ a) $\dfrac{y\,dy}{y+1} = x\,dx$ Integrate, noting that $\dfrac{y}{y+1} = 1 - \dfrac{1}{y+1}$

$\therefore \ dy - \dfrac{dy}{y+1} = x\,dx$

$y - \ln(y+1) = c + \frac{1}{2}x^2$ Put $x = 2$ to evaluate c: $[y(2)=0]$

$0 - \ln(1) = c + \frac{1}{2}\cdot 2^2$

$\therefore \ c = -2$

soln: $\boxed{y - \ln(y+1) = \frac{1}{2}x^2 - 2}$

b) $\sec^2 u\,du = \sin t\,dt$

$\therefore \ \tan u = -\cos t + c$ Put $t=0$: $u(0)=0$

$\therefore \ \tan 0 = -1 + c$

so $c = 1$

soln: $\boxed{u = \tan^{-1}(1 - \cos t)}$

1A-5 a) $\dfrac{dy}{y^2-2y} = -\dfrac{dx}{x^2}$ Integrate left side by partial fractions

$$\frac{1}{2}\frac{dy}{y-2} - \frac{1}{2}\frac{dy}{y} = \frac{-dx}{x^2}$$

$$\frac{1}{2}\ln\left(\frac{y-2}{y}\right) = c_1 + \frac{1}{x}$$ Multiply by 2, exponentiate

$$= 1-\frac{2}{y} \rightarrow \frac{y-2}{y} = c_2 e^{2/x}$$

$$\therefore\ y = \frac{2}{1-c_2e^{2/x}}$$ algebra; (replace left side by $1-\frac{2}{y}$)

b) $\dfrac{dv}{\sqrt{1-v^2}} = \dfrac{dx}{x}$

$$\sin^{-1}v = \ln x + c$$

$$v = \sin(\ln x + c)$$

c) $\dfrac{dy}{(y-1)^2} = \dfrac{dx}{(x+1)^2}$

$$-\frac{1}{y-1} = \frac{-1}{x+1} + c$$

Solve for y by ordinary algebra.

$$y = 1 + \frac{x+1}{1-c(x+1)}$$

d) $\dfrac{dx}{\sqrt{1+x}} = \dfrac{dt}{t^2+4}$

$$2\sqrt{1+x} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c$$

$$\therefore\ x = \frac{1}{4}\left(\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+c\right)^2 - 1$$

These problems all take for granted that you know the standard integration formulas and methods from 18.01. Review them if you are having trouble.

You need also the laws of exponentials and logarithms.

1B-1

a) $\dfrac{\partial M}{\partial y} = 3x^2 = \dfrac{\partial N}{\partial x}$ $\therefore$ exact.   what's $f(x,y)$?

$$\frac{\partial f}{\partial x} = 3x^2y \quad \therefore f = x^3y + g(y)$$

$$\frac{\partial f}{\partial y} = x^3 + g'(y) = x^3+y^3. \quad \therefore g = \tfrac{1}{4}y^4 + c$$

so that $f = x^3y + \frac{1}{4}y^4 + c$.

Solution: $\boxed{x^3y + y^4/4 = c_1}$

b) $\dfrac{\partial M}{\partial y} = -2y,\quad \dfrac{\partial N}{\partial x} = -2x$ not exact.

c) $\dfrac{\partial M}{\partial v} = e^{uv} + ve^{uv} = \dfrac{\partial N}{\partial u}$ $\therefore$ exact

$$\frac{\partial f}{\partial u} = ve^{uv},\ \therefore\ f = e^{uv} + g(v)$$

$$\frac{\partial f}{\partial v} = ue^{uv} + g'(v) = ue^{uv}. \quad \therefore g = c$$

so $f = e^{uv} + c$.   Soln: $e^{uv} = c_1$

or taking ln of both sides:

$$\boxed{uv = c_2}$$

d) $\dfrac{\partial M}{\partial y} = 2x,\quad \dfrac{\partial N}{\partial x} = -2x$ not exact.

1B-2

a) Multiply by y — this gives

$$2xy\,dx + x^2dy = 0$$

or $d(x^2y) = 0$ $\therefore\ x^2y = c$

so $\boxed{y = c/x^2}$

b) Integrating factor is $\dfrac{1}{y^2}$:

$$\frac{y\,dx - x\,dy}{y^2} - \frac{dy}{y} = 0$$

$$d\left(\frac{x}{y}\right) - d(\ln y) = 0$$

(*) $\dfrac{x}{y} - \ln y = c.$

Evaluate c by setting $x=1$ (so $y(1)=1$)

$$\therefore\ \frac{1}{1} - \underbrace{\ln 1}_{0} = c,\quad \text{so } c = 1$$

$$\therefore\ x - y\ln y = y$$

or $\boxed{x = y(\ln y + 1)}$

1B-2

c) Divide by t^2 (so integrating factor is $1/t^2$)

$$\left(1+\frac{4}{t^2}\right)dt = \frac{x\,dt - t\,dx}{t^2}$$

$$\therefore\ d\left(t-\frac{4}{t}\right) = d\left(\frac{-x}{t}\right)$$

$$t - 4/t = -\frac{x}{t} + c$$

$$\therefore\ \boxed{x = 4 - t^2 + ct}$$

d) $\frac{1}{u^2+v^2}$ is an integrating factor:

$$\frac{u\,du + v\,dv}{u^2+v^2} + \frac{v\,du - u\,dv}{u^2+v^2} = 0$$

$$\tfrac{1}{2}\ln(u^2+v^2) + \tan^{-1}\left(\frac{u}{v}\right) = c$$

when $u=0$, $v=1$; $\tfrac{1}{2}\ln 1 + \tan^{-1}(0) = c$
$\therefore\ c = 0$

$$\boxed{\tfrac{1}{2}\ln(u^2+v^2) + \tan^{-1}\left(\frac{u}{v}\right) = 0}$$

(substitute $r = \sqrt{u^2+v^2}$, $\theta = \tan^{-1}\frac{u}{v}$ to get polar coords)

equation becomes $\ln r + \theta = 0$
or $\boxed{r = e^{-\theta}}$

1B-3

a) $z = y/x$ $\therefore y = zx$, $y' = z'x + z$

Substituting:

$$z'x + z = \frac{2z-1}{z+4},\ \therefore z'x = -\frac{(z+1)^2}{z+4}$$

Sep. variables:

$$\frac{z+4}{(z+1)^2}\,dz = \frac{-dx}{x}$$

For ease, write $z+1 = u$

$$\left(\frac{u+3}{u^2}\right)du = \frac{-dx}{x}$$

integrate:

$$\ln u - \frac{3}{u} = -\ln x + c$$

To improve this:

$$\ln u + \ln x = \frac{3}{u} + c$$

Combine & exponentiate: $ux = ke^{3/u}$

Finally: $u = z+1 = \frac{y}{x} + 1 = \frac{y+x}{x}$

$$\therefore\ \boxed{y + x = ke^{3x/(y+x)}}$$

b) let $z = \frac{w}{u}$, so $w = zu$, $w' = z'u + z$

Substituting:

$$z'u + z = \frac{2z}{1-z^2}$$

$$\therefore\ z'u = \frac{z(1+z^2)}{1-z^2},$$ after a little algebra

Separate variables:

$$(*)\quad \frac{1-z^2}{z(1+z^2)}\,dz = \frac{du}{u}.$$

Use partial fractions on the left; result

$$\frac{1-z^2}{z(1+z^2)} = \frac{1}{z} + \frac{-2z}{z^2+1}$$

Integrating $(*)$:

$$\ln z - \ln(z^2+1) = \ln u + c$$

Combine and exponentiate both sides:

$$\frac{z}{z^2+1} = ku$$

Finally, put $z = w/u$; result is

$$\boxed{\frac{w}{w^2+u^2} = k}$$

as the solution (you could also solve for u in terms of w)

c) Put $z = y/x$; so $y = zx$, $y' = z'x + z$

Here $\frac{dy}{dx} = \frac{y^2 + x\sqrt{x^2-y^2}}{xy}$ — Substitute $y = zx$

$$z'x + z = \frac{z^2 + \sqrt{1-z^2}}{z}.$$

$$\therefore\ z'x = \frac{\sqrt{1-z^2}}{z}$$

Separate variables

$$\frac{z\,dz}{\sqrt{1-z^2}} = \frac{dx}{x}$$

$$-\sqrt{1-z^2} = \ln x + c$$

$$\boxed{\sqrt{1 - y^2/x^2} = c_1 - \ln x}$$

This can be solved explicitly for y: square both sides, etc...

$$\boxed{y = x\sqrt{1 - (c_1 - \ln x)^2}}$$

1B-4

$y = ux^n$

$\therefore\ y' = x^n u' + nx^{n-1}u$

$$x^n u' + nx^{n-1}u = \frac{4 + x^{2n+1}u^2}{x^{n+2}u}$$

$$\therefore\ u' = \frac{4 + (1-n)x^{2n+1}u^2}{x^{2n+2}u}$$

If $n=1$, we can separate vars:

$$u\,du = \frac{4\,dx}{x^4}$$

$$\therefore\ \frac{u^2}{2} = \frac{-4}{3}\cdot\frac{1}{x^3} + c$$

Since $n=1$, $u = y/x$

$$\therefore\ \boxed{y^2 = -\frac{8}{3x} + 2cx^2}$$

1B-5

a) $y' + \frac{2}{x}y = 1$ when written in normal form for linear eq'n.

Integ. factor: $e^{\int 2/x\,dx} = e^{2\ln x} = x^2$

$\therefore\ x^2y' + 2xy = x^2$

or $(x^2y)' = x^2$

$x^2 y = \frac{1}{3}x^3 + c$

$$\boxed{y = \frac{x}{3} + \frac{c}{x^2}}.$$

b) In standard form;

integ. factor is $e^{\int -\tan t\,dt} = e^{\ln(\cos t)} = \cos t$

$\therefore\ \cos t\,\frac{dx}{dt} - x\sin t = t$

or $(x\cos t)' = t$

$x\cos t = \frac{t^2}{2} + c$

Since $x(0)=0$, putting $t=0$ shows $c=0$.

$$\therefore\ \boxed{x = \frac{t^2}{2}\sec t}$$

1B-5

c) $(x^2-1)y' + 2xy = 1$ LHS is already exact!

$[(x^2-1)y]' = 1$

$(x^2-1)y = x + c$

$$\therefore\ y = \frac{x+c}{x^2-1}$$

d) Writing it in standard linear form

$$\frac{dv}{dt} + \frac{3v}{t} = 1$$

Integrating factor: $e^{\int 3/t\,dt} = e^{3\ln t} = t^3$

$\therefore\ t^3v' + 3t^2v = t^3$

$(t^3v)' = t^3$

$t^3 v = \frac{1}{4}t^4 + c$

$v(1) = \frac{1}{4} \Rightarrow c = 0$ (put $t=1$)

$$\therefore\ \boxed{v = \frac{1}{4}t}$$

1B-6

The integrating factor for this linear equation is $e^{\int a\,dt} = e^{at}$

$(xe^{at})' = e^{at}r(t)$

$$x = e^{-at}\left[\int_0^t e^{as}r(s)\,ds\right] + c$$

$$x = \frac{\int_0^t e^{as}r(s)\,ds}{e^{at}} + \frac{c}{e^{at}}$$

To find $\lim_{t\to\infty} x(t)$, use L'Hospital's rule, (∞/∞) [note that $c/e^{at} \to 0$] differentiating top and bottom

$$\therefore\ \lim_{t\to\infty} x(t) = \lim_{t\to\infty}\frac{e^{at}r(t)}{ae^{at}} = \lim_{t\to\infty}\frac{r(t)}{a}$$

$= 0$ by hypothesis

[Where did we need the hypothesis $a>0$?]

[We used, in connection with L'H rule, the result $\frac{d}{dt}\int_0^t e^{as}r(s)\,ds = e^{at}r(t)$.

This follows from the 2^{nd} Fundamental Theorem of Calculus].

1B-7

$$\frac{dy}{dx} = \frac{y}{y^3+x} \Rightarrow \frac{dx}{dy} = \frac{y^3+x}{y}$$

$$\therefore \frac{dx}{dy} - \frac{1}{y}x = y^2$$

This is now a linear equation in x.

Integ. factor: $e^{-\int \frac{dy}{y}} = e^{-\ln y} = y^{-1}$

$\therefore$ multiply by $\frac{1}{y}$:

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = y$$

or

$$\frac{d}{dy}\left(\frac{x}{y}\right) = y$$

$$\frac{x}{y} = \frac{y^2}{2} + c$$

$$\boxed{x = y^3/2 + cy}$$

1B-8

The systematic procedure — it always works, though it's a bit longer in this case —: since we want to substitute for y, y', begin by expressing them in terms of u.

(Don't just differentiate $u = y^{1-n}$ as is).

$$y = u^{1/1-n}$$

$$y' = \frac{1}{1-n}u^{\left(\frac{1}{1-n}-1\right)} \cdot u' = \frac{1}{1-n}u^{\frac{n}{1-n}}u'$$

Substitute into the ODE:

$$\frac{1}{1-n}u^{\frac{n}{1-n}}u' + pu^{\frac{1}{1-n}} = qu^{\frac{n}{1-n}}$$

Divide through by $u^{\frac{n}{1-n}}$:

$$\boxed{\frac{1}{1-n}u' + pu = q}$$

[Note: in this particular case, it's actually easier just to bumble around, but in general, this only leads to a mess.

However: $y' + py = qy^n$

Divide: $\frac{y'}{y^n} + \frac{p}{y^{n-1}} = q$ (*)

Put $u = y^{1-n} = \frac{1}{y^{n-1}}$

$$u' = (1-n)\cdot\frac{1}{y^n}\cdot y'$$

$\therefore$ (*) becomes

$$\frac{u'}{1-n} + pu = q, \text{ as before.}]$$

1B-9

$n=2$, so $u = y^{1-2} = y^{-1}$ (by problem 13)

Since we want to substitute for y, y', express them in terms of u and u':

$$y = \frac{1}{u}, \quad y' = -\frac{1}{u^2}\cdot u'$$

$\therefore$ the ODE becomes

$$-\frac{u'}{u^2} + \frac{1}{u} = 2x\cdot\frac{1}{u^2}$$

or $\boxed{u' - u = -2x}$ in standard linear eqn form.

Integ. factor: $e^{\int -dx} = e^{-x}$

Eq'n becomes

$$(e^{-x}u)' = -2xe^{-x} \leftarrow \text{integrate by parts}$$

$$\therefore e^{-x}u = 2xe^{-x} + 2e^{-x} + c$$

$$u = 2x + 2 + Ce^x$$

$$\therefore \boxed{y = \frac{1}{2x+2+Ce^x}}$$

1B-9

$y' - y$ Here $n=3$, so by prob. 13,

$$u = y^{1-3} = y^{-2}$$

As above, calculate y, y' in terms of u and u' (not other way around)

$$y = \frac{1}{\sqrt{u}}, \quad y' = -\frac{1}{2}u^{-3/2}\cdot u'$$

Substitute into the ODE:

$$-x^2\cdot\frac{u'}{2u^{3/2}} - \frac{1}{u^{3/2}} = \frac{x}{u^{1/2}}$$

$$\therefore \boxed{u' + \frac{2u}{x} = -\frac{2}{x^2}}$$

This is linear ODE; integ. factor is

$$e^{\int\frac{2dx}{x}} = e^{2\ln x} = x^2$$

ODE becomes

$$x^2u' + 2xu = -2$$

$$(x^2u)' = -2$$

$$x^2u = -2x + c$$

$$u = \frac{c-2x}{x^2}$$

$$\boxed{y = \frac{\pm x}{\sqrt{c-2x}}}$$

1B-10

a) $y = y_1 + u$

$y' = y_1' + u' = A + By_1 + Cy_1^2 + u'$

Substituting into the ODE:

$$A + By_1 + Cy_1^2 + u' = A + B(y_1+u) + C(y_1+u)^2$$

After some algebra,

$$u' = Bu + 2Cy_1u + Cu^2$$

$$\therefore\ u' - (B + 2Cy_1)u = Cu^2$$

This is a Bernouilli eq'n (problem 13) with $n = 2$.

b) By inspection, $y_1 = x$ is a sol'n to the ODE. $\therefore$ put $y = x + u$

$y' = 1 + u'$;

Substitution into the ODE gives

$$1 + u' = 1 - x^2 + (x+u)^2$$

$$\therefore\ \boxed{u' - 2xu = u^2},$$

a Bernouilli equation with $n = 2$.

Put $w = u^{1-2} = u^{-1}$

$\therefore\ u = \frac{1}{w},\quad u' = \frac{-w'}{w^2}$

Substituting,

$$\frac{-w'}{w^2} - \frac{2x}{w} = \frac{1}{w^2}$$

or $\boxed{w' + 2xw = -1}$

Linear ODE with integrating factor

$$e^{\int 2x\,dx} = e^{x^2}$$

$$\therefore\ (e^{x^2}w)' = -e^{x^2}$$

$$e^{x^2}w = -\int e^{x^2}dx + C$$

$$\boxed{w = -e^{-x^2}\int e^{x^2}dx + Ce^{-x^2}}$$

Finally:

$$y = x + u = x + \frac{1}{w}$$

$$\therefore\ y = x + \frac{e^{x^2}}{C - \int e^{x^2}dx}$$

(Actually, no value for C gives the original sol'n $y = x$; we have to take "$C = \infty$", or simply add $y = x$ to the above family.

1B-11

a) $y' = z$

$$y'' = \frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} = \frac{dz}{dy}\cdot z$$

Substitute into the ODE:

$$\frac{dz}{dy}\cdot z = a^2y\ ; \quad \text{Sep. vars:}$$

$$z\,dz = a^2y\,dy$$

$$z^2 = a^2y^2 + K$$

$$z = \sqrt{a^2y^2 + K}$$

$$\therefore\ y' = \sqrt{a^2y^2+K}$$

Separate variables again:

$$\frac{dy}{\sqrt{y^2 + K/a^2}} = a\,dx$$

Look this integral up!

$$\cosh^{-1}\left(\frac{ay}{\sqrt{K}}\right) = ax + C$$

$$y = \frac{\sqrt{K}}{a}\cosh(ax + C)$$

$$\therefore\ \boxed{y = C_1\cosh(ax + C)}$$

$\boxed{1B-11}$
16 b)

Let $y' = z$

$$y'' = \frac{dz}{dx} = \frac{dz}{dy}\cdot z$$

Substituting, $y\cdot\frac{dz}{dy}\cdot z = z^2$

$\therefore \frac{dz}{z} = \frac{dy}{y}$ $\quad\therefore \ln z = \ln y + \text{const.}$

$$\therefore \quad z = y' = Ky$$

Then $\frac{dy}{y} = K\,dx$

$$\therefore \quad \ln y = Kx + c$$

or $y = e^{Kx+c}$ is the solution

$\boxed{1B-11}$
(c)

let $y' = z$

$$y'' = \frac{dz}{dy}\cdot z$$

Substituting, $\frac{dz}{dy}\cdot z = z(1+3y^2)$

$$\therefore \quad dz = (1+3y^2)\,dy$$

$\therefore \quad z = y + y^3 + c$ Using the initial conditions, $c = 0$

$\therefore \frac{dy}{y+y^3} = dx$ (remember: $z = \frac{dy}{dx}$)

Integrating by partial fractions:

$$\frac{1}{y+y^3} = \frac{1}{y(y^2+1)} = \frac{1}{y} - \frac{y}{y^2+1}$$

$$\therefore \frac{dy}{y} - \frac{y\,dy}{y^2+1} = dx$$

$$\ln y - \frac{1}{2}\ln(y^2+1) = x + c$$

Exponentiating both sides,

$$\frac{y}{\sqrt{y^2+1}} = Ke^x$$

Using the initial conditions,

$$\frac{1}{\sqrt{2}} = K$$

$\therefore$ soln: $\to \boxed{\frac{y}{\sqrt{y^2+1}} = \frac{e^x}{\sqrt{2}}}$ (can solve for y in terms of x, if desired)

$\searrow \quad y = \frac{e^x}{\sqrt{2-e^{2x}}}$ $\quad\swarrow$ (by squaring both sides)

$\boxed{1B-12}$

1. Exact; also linear (divide by dx)

2. Linear; (integ. factor is e^{t^2})

3. Homogeneous: put $z = y/x$, get an ODE for z where you separate variables.

4. Separate variables; also linear in q and linear in p.

5. Exact; also linear.

6. Separate variables.

7. Bernoulli equation: $n = -1$
put $u = y^{1-(-1)} = y^2 \ldots$

8. Separate variables: $\dfrac{dv}{e^{3v}} = e^{2u}du$

9. Divide by x – this make it homogeneous, so put $z = y/x \ldots$

10. Linear equation (integ. factor is $\frac{1}{x^2}$)

11. Think of y as indep't variable, x as dep't variable; then equation is $\dfrac{dx}{dy} = x + e^y$, which is linear in x.

12. Separate variables; also a Bernoulli equation (exercise 13)

13. When written in the form $P(x,y)\,dx + Q(x,y)\,dy = 0$, it becomes exact.

14. Linear, with int. factor e^{3x}

15. Divide by x — it becomes homogeneous, so put $z = y/x$, etc.

16. Separate variables

17. Riccati equation (exercise 15a)
A particular sol'n is $y_1 = x^2$;
make the substitution $u = y - y_1$,
get Bernoulli eq'n in u $(n=2)$, etc.

18. Autonomous – x missing.
Put $y' = v$, $y'' = v\dfrac{dv}{dy}$; separate variables

19. homogeneous – put $z = s/t$
($\ln s - \ln t = \ln s/t$, notice)

20. Exact when written as $P\,dy + Q\,dx = 0$

21. Bernoulli eqn with $n=2$. (ex. 13)

22. Make change of variable
$u = x + y$
(so $u' = 1 + y'$)
Then you can separate variables

23. Becomes linear if you think of y as indep't variable, s as dependent variable.

24. Linear (u dep't variable t indep't variable)

25. $y_1 = -x$ is a particular sol'n.
Riccati equation (ex. 15a) –
put $u = y - y_1$, ...

OR BETTER:
write as $y' + (x+y)^2 + (x+y) + 1 = 0$.
and put $u = x + y$, $u' = 1 + y'$
leads to separation of variables.

26. Put $y' = v$ (so $y'' = v'$)
Get a first order linear eq'n in v.

1C-1

(a)

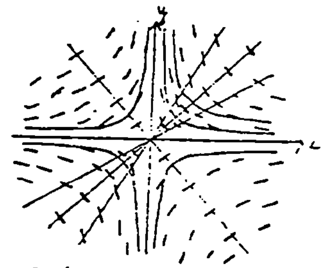
Isoclines: $\frac{-y}{x} = c$

Exact solution:

$$\frac{dy}{y} = \frac{-dx}{x}$$

$$\therefore \ln y = -\ln x + K'$$

$$\therefore y = \frac{K'}{x}$$



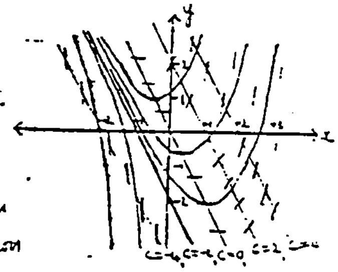
(Sol'n curves are hyperbolas)

(b) Isoclines:

$2x + y = c$.

This is a solution if $y' = -2 = c$;

i.e. $y + 2x + 2 = 0$ is an isocline which is a solution

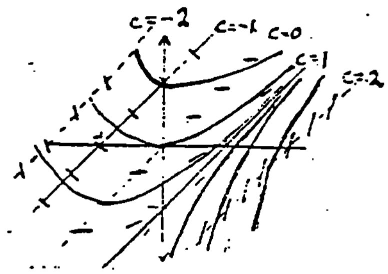


(c) Isoclines:

$x - y = c$

This is a solution if $y' = 1 = c$;

i.e., $x - y = 1$ is an isocline which is a solution.

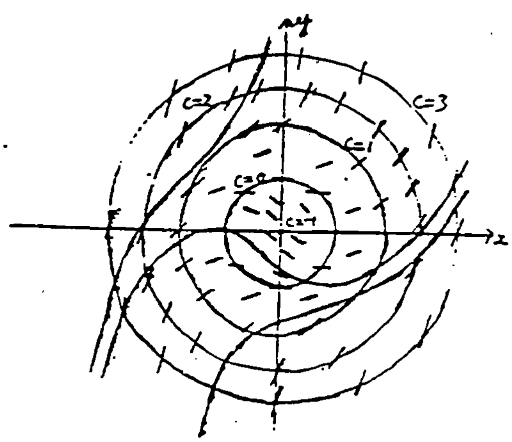


1C-1

d)

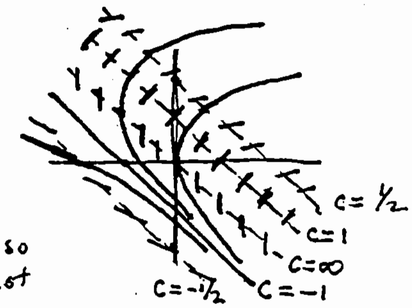
Isoclines: $x^2 + y^2 - 1 = c$

i.e. circles centre $(0,0)$, radius $\sqrt{1+c}$



1C-1

e) Isoclines $x+y=1/c$ or $y=-x+\frac{1}{c}$



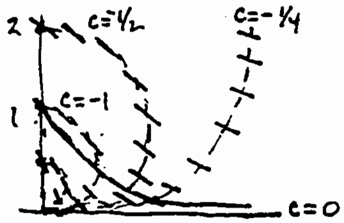
$y=-x-1$ is an integral curve, so other sol'ns cannot cross it.

1C-2

Isoclines: $x^2+y^2+\frac{y}{c}=0$, or completing the square:

$$x^2+\left(y+\frac{1}{2c}\right)^2=\left(\frac{1}{2c}\right)^2$$

(Circles, center at $(0,-1/2c)$.)



a) decreasing, since $y'=\dfrac{-y}{x^2+y^2}<0$ when $y>0$

b) soln must have $y>0$ for $x>0$ since it cannot cross the integral curve $y=0$.

1C-3

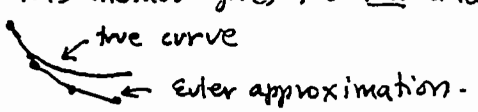
a) Using $\Delta y_n = hf(x_n,y_n) = h(x_n-y_n)$, get $y_{n+1} = y_n + h(x_n-y_n)$.

Table entries:

x	0	.1	.2	.3
y	1	.9	.82	.758

For example,
$$y_1 = y_0 + h(x_0-y_0) = 1 + .1(-1) = .9$$
$$y_2 = y_1 + h(x_1-y_1) = .9+.1(.1-.9) = .82$$
$$y_3 = .82+.1(.2-.82) = .758$$

some isoclines $x-y=c$ are drawn.
sol'n curve through $(0,1)$ is convex (= "concave up"); thus Euler's method gives too <u>low</u> a result:



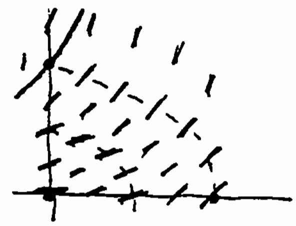
1C-4

Euler method formula:
$$y_{n+1} = y_n + hf(x_n,y_n)$$

x_n	y_n	$f(x_n,y_n)$	$hf(x_n,y_n)$
0	1	1	.1
.1	1.1	1.31	.131
.2	1.231	1.72	.172
.3	1.403		

$h=.1$

$f(x,y) = x+y^2$



isoclines $x+y^2=c$ (parabolas)))

Solution curve through $(0,1)$ is convex ("concave up"), $\therefore$ Euler method gives too <u>low</u> a result (same reasoning as in 19a)

1C-3

b) $\Delta y_n = \frac{h}{2}\left[f(x_n,y_n)+f(x_{n+1},\bar{y}_{n+1})\right]$

$y_{n+1}-y_n$

For this ODE, $f(x,y)=x-y$

($\bar{y}_{n+1}$ is the value given by the next step of Euler's method).

Thus.

So, $y_0=1$, $\bar{y}_1 = .9$ (from part a)

$$\therefore\ y_1-y_0 = \frac{.1}{2}\left[f(0,1)+f(.1,.9)\right] = \frac{.1}{2}\left[-1-.8\right] = -.09$$

$$\therefore\ y_1 = y_0 - .09$$
$$y_1 = 1-.09 = \boxed{.91}$$

This does correct the Euler value ($\bar{y}_1=.9$) in the right direction, since we predicted it would be too low. (.910 is actually the correct value of the sol'n to 3 places.)

1C-5

By the formula in 19a,

$$y_n = y_{n-1} + h(x_{n-1} - y_{n-1})$$
$$= (1-h)y_{n-1} + h\,x_{n-1}.$$

But for $x_0 = 0$, we get $x_1 = h$, $x_2 = 2h$, and in general $x_{n-1} = (n-1)h$.

$$\therefore \quad \boxed{y_n = (1-h)\,y_{n-1} + h^2(n-1)} \quad (**)$$

We prove by induction that the explicit formula for y_n is:

$$(*) \quad \boxed{y_n = 2(1-h)^n - 1 + nh}$$

a) it's true if $n=0$, since
$y_0 = 2(1-h)^0 - 1 + 0 = 1$ ✓

b) if true for y_n, it's true for y_{n+1}: since, using (*),

$$y_{n+1} = (1-h)y_n + h^2 n$$
$$= 2(1-h)^{n+1} + (1-h)(-1+nh) + h^2 n$$
$$\therefore \quad y_{n+1} = (1-h)^{n+1} - 1 + (n+1)h \quad ✓$$

[Note: (**) is called a "difference equation" — there are standard ways to solve such things; here (*) is the solution].

Continuing, in our case $h = 1/n$

$$\therefore \quad y_n = 2\left(1-\tfrac{1}{n}\right)^n - 1 + 1$$
$$= 2\left(1-\tfrac{1}{n}\right)^n.$$

$$\lim_{n\to\infty} y_n = 2e^{-1} \quad \left(\text{since } \lim_{k\to\infty}\left(1+\tfrac{1}{k}\right)^k = e; \text{ put } k = -n\right)$$

The exact sol'n to the equation is

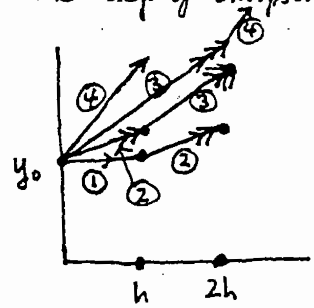
$$y = 2e^{-x} - 1 + x.$$

so $y(1) = 2e^{-1} - 1 + 1 = 2e^{-1}$,

which checks.

1C-6

It suffices to prove this is true for one step of the Runge-Kutta method and one step of Simpson's rule.



We calculate, in R-K method, the 4 slopes marked ①→④

Then we use a weighted average of them to find $y(2h)$:

$$y_{2h} = y_0 + 2h\cdot\left(\frac{① + 2\cdot② + 2\cdot③ + ④}{6}\right)$$

Since the ODE is simply:

$$y' = f(x),$$

from the picture

slope ① $= f(0)$
slope ② $= f(h)$
slope ③ $= f(h)$
slope ④ $= f(2h)$

$$\therefore \quad y_{2h} = y_0 + \frac{2h}{6}\left(f(0) + 4f(h) + f(2h)\right)$$

Contrast this with the exact formula: $y_{2h} = y_0 + \int_0^{2h} f(x)\,dx$

Evaluating the integral approximately by one step of Simpson's rule:

$$y_{2h} = y_0 + \frac{2h}{6}\left(f(0) + 4f(h) + f(2h)\right),$$

same as what Runge-Kutta gives.

1C-7

The existence and uniqueness theorem requires the equation to be written in the form

$$y' = f(x,y).$$

Doing this, we get

$$y' = -\frac{b(x)}{a(x)}\,y + \frac{c(x)}{a(x)}.$$

The conditions then are:

"$f(x,y)$ continuous", which will be so if

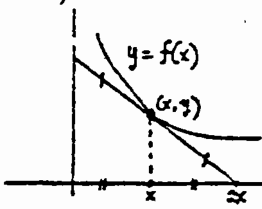
$$\boxed{\begin{array}{l} a(x),\ b(x),\ c(x) \text{ continuous (on an interval } [x_0-h, x_0+h]) \\ \text{and } a(x) \neq 0 \text{ in this interval.} \end{array}}$$

"$f_y(x,y)$ continuous", which will be so if $\frac{b(x)}{a(x)}$ is continuous, — ~~any~~ and this is already implied by the above condition.

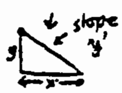
[Note that we must have $a(x) \neq 0$, a condition which is often missed.]

1D-1

a)



If (x,y) is a point on the curve, the geometric condition translates to:

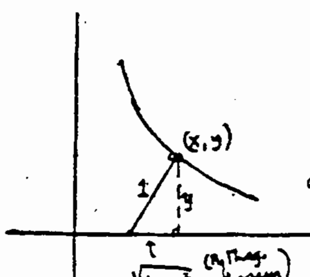


$$\text{slope of tan. line} = \frac{y}{-x}$$

$$\therefore\ y' = -\frac{y}{x}$$

The solution (sep. of vars.) is $\boxed{y = \frac{c}{x}}$ [hyperboles]

b)



Since the normal is $\perp$ to the tangent, its slope is the negative reciprocal.

$$\therefore\ \frac{y}{\sqrt{1-y^2}} = -\frac{1}{y'}$$

Solve by sep. of variables: $-\frac{y\,dy}{\sqrt{1-y^2}} = dx$

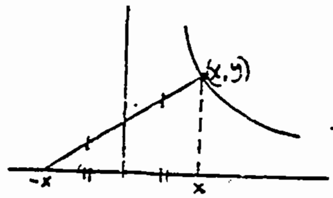
$\therefore\ \sqrt{1-y^2} = x + c$ or

$(x+c)^2 + y^2 = 1$ (Circles, radius 1, centre on x-axis — obvious, what?)

$y = \pm 1$ are also solutions to the problem (above assumed implicitly that $y \neq \pm 1$)

1D-1

(c)



Equating slopes of normal:

$$\frac{y}{2x} = -\frac{1}{y'} \quad \text{(neg. recip. of slope of tangent)}$$

Solve by sep. vars,

get $\boxed{\frac{1}{2}y^2 + x^2 = C}$ (ellipses)

(d)

The required property translates mathematically into:

$$\int_a^x y(t)\,dt = k\big(y(x) - y(a)\big)$$

k = constant of proportionality

Differentiate this to get an ODE for $y(x)$:

$$y(x) = k\,y'(x)$$

(by 2nd Fund. Thm of Calculus)

solution: $\boxed{y = ce^{x/k}}$ This is the general exponential curve.

1D-2

(a)

(i) The y-intercept of line $y = mx + c$ is $(0, c)$ $\therefore\ c = 2m$

i.e. $y = mx + 2m = m(x+2)$

Differentiating $\Rightarrow y' = m$

Eliminate m: $\therefore\ \boxed{y' = \frac{y}{x+2}}$ ODE of family

(ii) Orthogonal trajectories satisfy: $-\frac{1}{y'} = \frac{y}{x+2}$

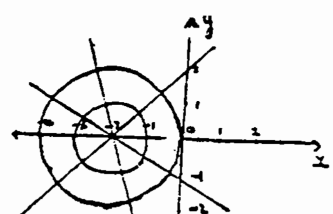
$\Rightarrow \frac{-dx}{dy} = y/x+2 \quad \Rightarrow y\,dy = -x\,dx + 2\,dx$

$\therefore\ \frac{y^2}{2} + \frac{x^2}{2} + 2x = \text{constant}$

$\therefore\ (x+2)^2 + y^2 = k$

i.e. Circles centre $(-2, 0)$, variable radii

(iii)



Original family: Lines thro' $(-2, 0)$

Orthogonal trajectories: Circles centre $(-2, 0)$

1D-2

(b)

$$y = ce^x$$
$$y' = ce^x = y$$

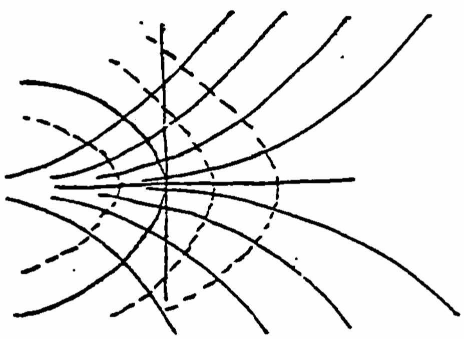
Equation of the orthogonal family:

$$y' = -\frac{1}{y}$$

To find the curves, solve by separation of variables:

$$y\,dy = -dx$$
$$\tfrac{1}{2}y^2 = -x + c \qquad \underline{\text{parabolas}}$$

(all translations of one fixed parabola $\tfrac{1}{2}y^2 = -x$ along the x-axis)



1D-2

(c)

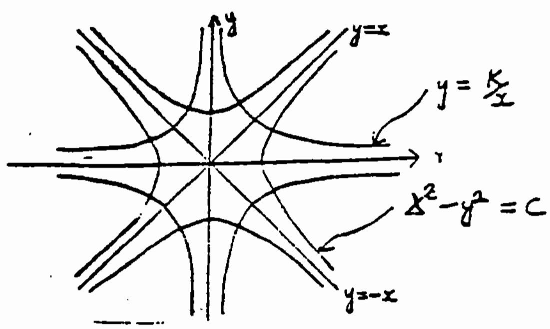
(i) Differentiating gives

$$2x - 2yy' = 0$$
$$\therefore \quad y' = \frac{x}{y} \quad \text{is required ODE}$$

(ii) Orthogonal trajectories satisfy

$$-\frac{1}{y'} = \frac{x}{y}$$
$$\therefore \quad \frac{-dy}{y} = \frac{dx}{x}$$
$$\therefore \quad -\ln y = \ln x + C_1$$
$$\therefore \quad y = K/x$$

(iii)



1D-2

(d) Circles with centre on y-axis have equation $x^2 + (y-k)^2 = r^2$

Circle tangent to x-axis

$$\Rightarrow \quad r = \pm k \quad \therefore \quad r^2 = k^2$$
$$\therefore \quad x^2 + y^2 - 2yk = 0$$
$$\therefore \quad \frac{x^2+y^2}{2y} = k.$$

Differentiate w.r.t. x:

$$\therefore \quad \frac{2x + 2yy'}{2y} - \frac{(x^2+y^2)y'}{2y^2} = 0$$
$$\therefore \quad 2xy + 2y^2y' - x^2y' - y^2y' = 0$$
$$\text{i.e.} \quad y' = \frac{2xy}{x^2-y^2}$$

(ii) Orthogonal trajectories satisfy

$$-\frac{1}{y'} = \frac{2xy}{x^2-y^2}$$
$$\text{i.e.} \quad y' = \frac{y^2-x^2}{2xy} \quad \leftarrow \text{a \underline{homogeneous} equation}$$

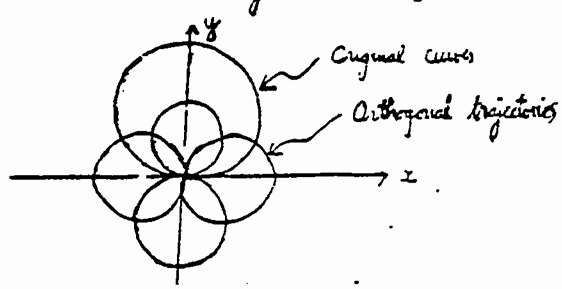
Let $y = zx \quad \therefore \quad z = y/x$

Then $y' = xz' + z$

$$\therefore \quad xz' + z = \frac{z^2x^2 - x^2}{2zx^2} = \frac{z^2-1}{2z}$$
$$\therefore \quad xz' = \frac{-(z^2+1)}{2z} \quad \text{i.e.} \quad \frac{2z\,dz}{z^2+1} = \frac{-dx}{x}$$
$$\therefore \quad \ln(z^2+1) = -\ln x + C$$
$$\therefore \quad z^2+1 = \frac{2K}{x} \qquad (2K = e^C)$$
$$\therefore \quad y^2 + x^2 = 2Kx$$

These are circles with centre on the x-axis and tangent to y-axis

(iii)



1D-3

a) $\frac{dx(t)}{dt}$ = rate at which salt flows in − rate of salt outflow

$= \begin{pmatrix}\text{flow}\\ \text{rate}\\ \text{in}\end{pmatrix}\cdot\begin{pmatrix}\text{conc.}\\ \text{salt}\\ \text{in added}\\ \text{sol'n}\end{pmatrix} - \begin{pmatrix}\text{flow}\\ \text{rate}\\ \text{out}\end{pmatrix}\cdot\begin{pmatrix}\text{conc.}\\ \text{of salt}\\ \text{in tank}\end{pmatrix}$

$$\boxed{x' = kc_1 - k\cdot\frac{x}{V}}$$

b) $x' + ax = 0$ (since $c_1 = 0$)
$x(0) = Vc_0$ $(a = k/V)$

Solution is, by sep. of variables
$x = Vc_0e^{-at}$ $(a = k/V)$

c) The general case is $\begin{cases} x' + ax = kc_1, \\ x(0) = Vc_0\end{cases}$ which can be solved by separating variables, or as a linear equation.

Separating variables:

$$\frac{dx}{dt} = kc_1 - ax$$

$$\frac{dx}{kc_1 - ax} = dt$$

$$-\frac{1}{a}\ln(kc_1 - ax) = t + A \quad \text{const. of integration}$$

or $kc_1 - ax = A_1e^{-at}$ A_1 = arbitrary constant

Using the initial condition to find A_1:

$$kc_1 - aVc_0 = A_1 \quad (\text{note that } aV = k)$$

$$\therefore\ k(c_1 - c_0) = A_1$$

so soln is (note that $k/a = V$)

$$x = Vc_1 - V(c_1 - c_0)e^{-at}$$

or in terms of the concentration $c(t)$:

$$\boxed{c = c_1 - (c_1 - c_0)e^{-at}}$$

As $t \to \infty$, $e^{-at} \to 0$, so $\boxed{c \to c_1}$

d) If $c_1 = c_0e^{-\alpha t}$, then the ODE (IVP) becomes (as in (c))

$$\begin{cases} x' + ax = kc_0e^{-\alpha t} \\ x(0) = Vc_0\end{cases}$$

This must be solved as a linear equation. The integrating factor is e^{at}:

$$x'e^{at} + axe^{at} = kc_0e^{(a-\alpha)t}$$

or $(xe^{at})' = kc_0e^{(a-\alpha)t}$ ⊛

Integrating,

$$xe^{at} = \frac{kc_0}{a-\alpha}e^{(a-\alpha)t} + A \quad \text{const. of integ.}$$

Using the initial condition to find A:

$$Vc_0 = A + \frac{kc_0}{a-\alpha}$$

$$\therefore\ x = \frac{kc_0}{a-\alpha}e^{-\alpha t} + \left(Vc_0 - \frac{kc_0}{a-\alpha}\right)e^{-at}$$

Dividing by V to get concentration:

$$c = \frac{ac_0}{a-\alpha}e^{-\alpha t} + \left(c_0 - \frac{ac_0}{a-\alpha}\right)e^{-at}$$

[If $\alpha = 0$, then $c_1 = c_0$, and this agrees with part (c)]

1D-4

$$\frac{dA}{dt} = -\lambda_1 A, \quad \lambda_1 = \frac{\ln 2}{\text{half-life}}$$

$\frac{dB}{dt}$ = rate at which B produced by decay of A − rate at which B is lost by decay of B

$$\therefore\ \frac{dB}{dt} = \lambda_1 A - \lambda_2 B$$

$\therefore$ From the first equation, $A = A_0e^{-\lambda_1 t}$

$$\therefore\ \boxed{\frac{dB}{dt} + \lambda_2 B = \lambda_1 A_0 e^{-\lambda_1 t}} \quad \text{ODE for } B(t)$$

Solve it as a linear equation, using $e^{+\lambda_2 t}$ as integrating factor, and $B(0) = B_0$ as initial condition. Solution is

$$B(t) = \frac{\lambda_1 A_0}{\lambda_2 - \lambda_1}e^{-\lambda_1 t} + \left(B_0 - \frac{\lambda_1 A_0}{\lambda_2 - \lambda_1}\right)e^{-\lambda_2 t}$$

Taking $\lambda_1 = 1$, $\lambda_2 = 2$,

$$B(t) = A_0e^{-t} + (B_0 - A_0)e^{-2t}$$

Differentiating to see when $B(t)$ is maximum:

$$0 = B'(t) = -A_0e^{-t} - 2(B_0 - A_0)e^{-2t}$$

Solving for t: $\frac{A_0}{2(A_0 - B_0)} = e^{-t}$

If $A_0 > 2B_0$, then $t = -\ln\left(\frac{A_0}{2(A_0 - B_0)}\right) > 0$

If $A_0 \le 2B_0$, no solution (the maximum is at $t = 0$).

1D-5

By Newtons cooling law

$$\frac{dT}{dt} = K(T-20)$$

(K a constant of proportionality)

Solving this (by sep. of variables) gives

$$T = \alpha e^{Kt} + 20 \quad (\alpha \text{ another constant})$$

$$T(0) = 100$$

$$\therefore \quad \alpha + 20 = 100$$

$$\therefore \quad \alpha = 80$$

$$T(5) = \alpha e^{5K} + 20 = 80$$

$$\therefore \quad \alpha e^{5K} = 60$$

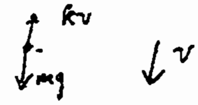
$$\therefore \quad K = \tfrac{1}{5}\ln\left(\tfrac{60}{\alpha}\right) = \tfrac{1}{5}\ln\left(\tfrac{3}{4}\right) < 0.$$

$$\therefore \quad T = 80\, e^{-\frac{1}{5}\ln(\frac{4}{3})t} + 20$$

When $T = 60$ we then find

$$t = \frac{5\ln 2}{\ln(\frac{4}{3})} \simeq 12 \text{ mins.}$$

1D-6

a) 

Downwards force $= m\frac{dv}{dt} = mg - kv$

$$\therefore \quad \frac{dv}{dt} + \frac{k}{m}v = g$$

Solving this by separation of variables (or as a linear equation), we get

$$v = \alpha e^{-k/m\, t} + \frac{mg}{k} \quad (\alpha \text{ constant})$$

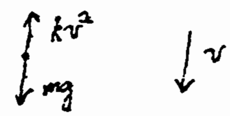
Using the initial condition

$$v(0) = 0. \quad \therefore \quad \frac{mg}{k} + \alpha = 0.$$

$$\therefore \quad v = \frac{mg}{k}\left(1 - e^{-kt/m}\right) \qquad \text{SOLN.}$$

terminal velocity:

$$\lim_{t\to\infty} v(t) = \frac{mg}{k} \qquad \text{(constant)}$$

b) 

Downwards force $= m\frac{dv}{dt} = mg - kv^2$

$$\therefore \quad \frac{dv}{v^2 - \frac{mg}{k}} = -\frac{k}{m}\,dt$$

But

$$\frac{1}{v^2 - \frac{mg}{k}} = \frac{1}{v^2 - a^2} = \frac{1}{2a}\left[\frac{1}{v-a} - \frac{1}{v+a}\right]$$

where $a \equiv \sqrt{\frac{mg}{k}}$

$$\therefore \quad \frac{dv}{v-a} - \frac{dv}{v+a} = -\frac{2ak}{m}\,dt$$

$$\therefore \quad \ln\left|\frac{v-a}{v+a}\right| = c - \frac{2ak}{m}t$$

But $v(0) = 0 \quad \therefore \ln 1 = c$ i.e., $c = 0$

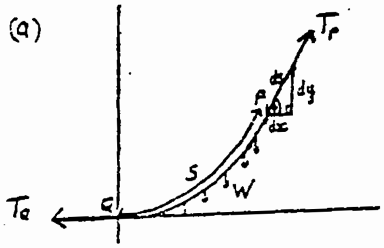
$$\therefore \quad \frac{a-v}{a+v} = e^{-\frac{2akt}{m}} \quad \text{(since LHS} > 0 \text{ (at least near } t=0\text{)}$$

$$\therefore \quad v = a\left(\frac{1 - e^{-\frac{2akt}{m}}}{1 + e^{-\frac{2akt}{m}}}\right)$$

$$\therefore \lim_{t\to\infty} v(t) = a = \sqrt{\frac{mg}{k}}$$

1D-7

(a)



$\tan\phi = \frac{dy}{dx}$

Balancing forces horizontally

$$T_a = T_p \cos\phi = T_p \frac{dx}{ds}$$

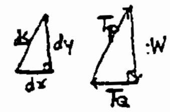
$$\therefore \quad \frac{ds}{T_p} = \frac{dx}{T_a} \qquad (i)$$

Balancing forces vertically

$$W = T_p \sin\phi = T_p \frac{dy}{ds}$$

$$\therefore \quad \frac{ds}{T_p} = \frac{dy}{W} \quad (ii) \quad \text{as required.}$$

OR: the Δs are similar:



(Δ of forces is closed since cable is in equilibrium)

$\therefore$

$$\frac{dx}{T_a} = \frac{dy}{W} = \frac{ds}{T_p}$$

(corresponding sides)

(b) Suppose the cable hangs under its own weight and has constant density ρ per unit length.

Then $W = \rho s$

Now $\frac{dx}{T_a} = \frac{dy}{W} = \frac{dy}{\rho s}$

$\therefore \quad \frac{dy}{dx} = ks$ (where $k = \frac{\rho}{T_a}$ is a constant)

Then $\frac{d^2y}{dx^2} = k\frac{ds}{dx} = k\frac{\sqrt{(dx)^2+(dy)^2}}{dx}$

$$= k\sqrt{1+(y')^2} \quad \text{which proves (i)}$$

Also, $\frac{dy}{W} = \frac{ds}{T_p}$; but $T_p = \sqrt{W^2+T_a^2}$ (from the force triangle)

$\therefore \quad \frac{dy}{\rho s} = \frac{ds}{\sqrt{\rho^2 s^2 + T_a^2}}$

$\therefore \quad \frac{dy}{ds} = \frac{s}{\sqrt{s^2+c^2}}$ where $c = T_a/\rho$

$\therefore \quad y = \sqrt{s^2+c^2} + c_1$,

which proves (ii)

(c) Let λ be the constant weight per unit horizontal length

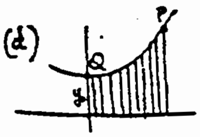
$$\therefore \quad W = \lambda x$$

Then $\frac{dy}{dx} = \frac{W}{T_a} = \frac{\lambda x}{T_a}$

$$\therefore \quad y = \frac{\lambda}{T_a}\frac{x^2}{2} + y_0$$

Thus the cable takes the form of a parabola.

(d)



Here $W = k_1 \cdot (\text{area under } \widehat{QP})$ since rods are equally and closely spaced.

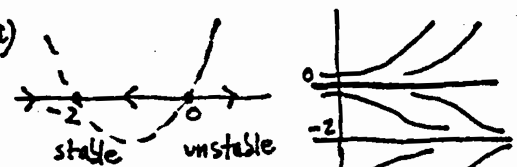
So $\frac{dy}{dx} = \frac{W}{T_a} = \frac{k_1}{T_a}\int_0^x y(t)\,dt$

$\therefore \quad \frac{d^2y}{dx^2} = k^2 y$, by the 2nd Fund. Thm. of Calculus. $(k^2 = k_1/T_a > 0)$

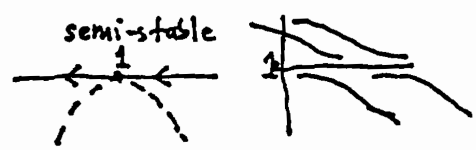
[The curve is once again of the form $y = \cosh(kx) + c_1$]

1E-1

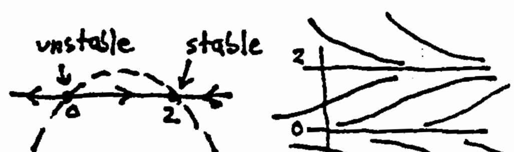
a)



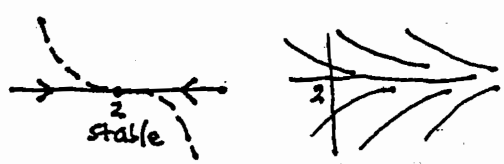
b)



c)



d)



(write: $(2-x)^3 = -(x-2)^3$)

Section II Solutions

$\boxed{2A-1}$ a) This is true because D^2, pD, and multiplich by q are all linear operators:

$$q(y_1+y_2) = qy_1 + qy_2 \quad (1)$$

$$pD(y_1+y_2) = p(Dy_1+Dy_2)$$

$$\therefore \quad = pDy_1 + pDy_2 \quad (2)$$

$$D^2(y_1+y_2) = D^2y_1 + D^2y_1 \quad (3)$$

Adding ①, ② ③ gives

$$L(y_1+y_2) = Ly_1 + Ly_2$$

The proof for $L(cy_1) = cLy_1$ is similar.

b) (i) $Ly_h = 0$ since y_h solves the eqn $Ly=0$

$Ly_p = r$ since y_p solves the original eqn.

Adding, using linearity of L: $L(y_h+y_p) = r$ $\quad \therefore y_h+y_p$ is a soln.

(ii) if y_1 is any soln, then

$$L(y_1-y_p) = Ly_1 - Ly_p = r-r=0$$

$$\therefore \quad y_1 - y_p = y_h \quad (\text{a sol'n of } Ly=0)$$

$$\therefore \quad y_1 = y_h + y_p.$$

Parts (i) + (ii) together show all solns are of the form $y_h + y_p$.

$\boxed{2A-2}$ a)

$$\left.\begin{array}{l} y = c_1e^x + c_2e^{2x} \\ y' = c_1e^x + 2c_2e^{2x} \end{array}\right\} \quad y'-y = c_2e^{2x}$$

$$\left.\begin{array}{l} y' = c_1e^x + 2c_2e^{2x} \\ y'' = c_1e^x + 4c_2e^{2x} \end{array}\right\} \quad y''-y' = 2c_2e^{2x}$$

$$\therefore \quad y''-y' = 2(y'-y)$$

or: $\boxed{y''-3y'+2y=0}$.

b) The question is whether we can find values for c_1, c_2 such that

$$c_1e^{x_0} + c_2e^{2x_0} = y_0$$

$$c_1e^{x_0} + 2c_2e^{2x_0} = y_0'$$

These equations can be solved (by Cramer's rule) for c_1, c_2 provided that $\begin{vmatrix} e^{x_0} & e^{2x_0} \\ e^{x_0} & 2e^{2x_0} \end{vmatrix} \neq 0$. (coefficient determinant)

But this det $= e^{3x_0} \neq 0$ for any x_0. ✓

$\boxed{2A-3}$ a)

$$y = c_1x + c_2x^2$$

$$y' = c_1 + 2c_2x$$

$$y'' = 2c_2$$

You want to eliminate c_1, c_2. One way —:

$$\begin{cases} c_2 = y''/2 & \text{from last eqn} \\ c_1 = y' - y''x & \text{from } 2^{nd} + 3^{rd} \text{ eqn.} \end{cases}$$

Substitute into 1st eqn, get

$$y = (y'-y''x)x + \frac{y''}{2}x^2,$$

which by algebra becomes

$$\boxed{x^2y'' - 2xy' + 2y = 0}$$

b) all solns $y = c_1x + c_2x^2$ satisfy $y(0)=0$

c) This theorem requires that when eqn is written $y'' + p(x)y' + q(x)y = 0$, that p, q be continuous functions. But here, the ODE in standard form is

$$y'' - \frac{2}{x}y' + \frac{2}{x^2} = 0;$$

coefficients are discontinuous at $x=0$.

$\boxed{2A-4}$ a) Suppose y_1 is a solution to

$$y'' + p(x)y' + q(x)y = 0 \quad ⊛$$

tangent to x-axis at the pt. x_0. Then

$$y_1(x_0) = 0$$

$$y_1'(x_0) = 0.$$

But $y_2(x) \equiv 0$ is another soln to ⊛ with this same property:

$$y_2(x_0) = 0$$

$$y_2'(x_0) = 0$$

$\therefore$ by the uniqueness theorem,

$$y_1 \equiv y_2 \quad \text{for all } x,$$

i.e., $y_1 \equiv 0$.

b)

$$y = x^2$$

$$y' = 2x$$

$$y'' = 2$$

$\therefore \boxed{xy'' - y' = 0}$ is such an equation or: $\boxed{y'' - \frac{1}{x}y' = 0}$

Part (a) is not contradicted, since the coefficient $\frac{1}{x}$ is discontinuous at $x=0$.

2A-5 a) $W(e^{m_1x}, e^{m_2x}) = \begin{vmatrix} e^{m_1x} & e^{m_2x} \\ m_1e^{m_1x} & m_2e^{m_2x} \end{vmatrix}$

$$= (m_2 - m_1)e^{(m_1+m_2)x};$$

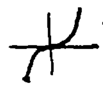
since $e^x \neq 0$ for all x, this is never 0 if $m_1 \neq m_2$. $\therefore$ functions are lin. ind.

b) $W(e^{mx}, xe^{mx}) = \begin{vmatrix} e^{mx} & xe^{mx} \\ me^{mx} & mxe^{mx} + e^{mx} \end{vmatrix}$

$$= e^{2mx} \neq 0 \text{ for any } x.$$

(This holds true even if $m=0$).

$\therefore$ the functions are lin. indep't.

2A-6 (The graph of $x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$: )

a) If $x \geq 0$, $W = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} \equiv 0$

if $x \leq 0$, $W = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} \equiv 0$

b) Suppose they were linearly dependent on an interval (a, b) containing 0, that is, suppose there are c_1, c_2 such that $\boxed{c_1y_1 + c_2y_2 = 0 \text{ for all } x \in (a,b)}$.

Then if $x \geq 0$, $y_1 = y_2$, $\therefore c_1 = -c_2$

if $x < 0$, $y_1 = -y_2$, $\therefore c_1 = c_2$

Thus $c_1 = 0$ and $c_2 = 0$, so that y_1 and y_2 are not lin. dep't on (a, b).

Since $y_2' = 2x$ for $x > 0$,

$y_2' = -2x$ for $x < 0$

graph of y_2' is 

Thus y_2'' does not exist at $x = 0$, so it cannot be the solution to a 2^{nd} order equation $y'' + p(x)y' + q(x)y = 0$ on the interval (a,b) containing 0.

Thus thm in the book ($W \equiv 0$ for 2 solns to ODE $\Rightarrow$ solns are lin dept) is not contradicted.

2A-7 a) This can be done directly, by differentiating $y_1y_2' - y_1'y_2$. (*see below)

An elegant way to do it is to use the formula for differentiating a determinant: diff. one row at a time, then add:

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}' = \begin{vmatrix} u_1' & u_2' \\ v_1 & v_2 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 \\ v_1' & v_2' \end{vmatrix}$$

(this works for dets. of any size).

Applying this to the Wronskian:

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}' = \underbrace{\begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix}}_{=0} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix};$$

Since y_1 and y_2 solve $y'' = -py' - qy$, we get the above right-hand det.

$$= \begin{vmatrix} y_1 & y_2 \\ -py_1' - qy_1 & -py_2' - qy_2 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ -py_1' & -py_2' \end{vmatrix}$$

(adding $q \cdot (1^{st}$ row) to 2^{nd} doesn't change value of the determinant)

$$= -p\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -pW.$$

(* you also have to use that y_1, y_2 are solns, i.e., that $y_1'' = -py_1' - qy_1$, $y_2'' = -py_2' - qy_2$).

b) From part (a), if $p(x) = 0$, then $\frac{dW}{dx} = 0$, so $W(y_1, y_2) = c$.

c) $y'' + k^2y = 0$ Here $p = 0$

$W(\cos kx, \sin kx)$

$$= \begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{vmatrix}$$

$$= k(\cos^2 kx + \sin^2 kx)$$

$$= k, \text{ a constant.}$$

$\boxed{2B-1}$

a) $y_2 = ue^x$

$\underline{x\ -2]}$ $\quad y_2' = u'e^x + ue^x$

$y_2'' = u''e^x + 2u'e^x + ue^x$

Multiply second row by -2 and add:

$y_2'' - 2y_2' + y_2 = u''e^x$ (all other terms cancel out)

If y_2 is a soln to the ODE, the left-hand side must be 0. Therefore we must have

$$u''e^x = 0$$

so $\quad u'' = 0,$

$\therefore\ u = ax + b$

and $\quad \therefore\ y_2 = (ax+b)e^x$

Any of these for which $a \neq 0$ gives a second solution — for ex., $\boxed{y_2 = xe^x}$.

b) From II/7a, $\frac{dW}{dx} = -pW = 2W$

$\therefore\ W(y_1, y_2) = ce^{2x},\ c \neq 0$

But $\quad W(y_1, y_2) = \begin{vmatrix} e^x & y_2 \\ e^x & y_2' \end{vmatrix}$

Equating these two expressions for W,

$$e^x(y_2' - y_2) = ce^{2x}$$

$$\therefore\ y_2' - y_2 = ce^x$$

(c can have any $\neq 0$ value)

Solving this ODE gives (it's a linear equation)

$y_2 = e^x(x + c_1)$ as a family of second solutions.

c) $y_2 = e^x \int \frac{1}{e^{2x}} e^{-\int -2dx} dx$

$= e^x \int 1 \cdot dx = e^x(x+c)$

[more generally: $e^{\int 2dx} = e^{2x+c_1}$

$\therefore\ y_2 = e^x \int (e^{c_1}) dx$ $\quad$ put $c_2 = e^{c_1}$

$= e^x(c_2 x + c)$]

d) All the solutions are the same — the most general form is

$y_2 = e^x(c_1 x + c_2)$, with $c_1 \neq 0$

(if $c_1 = 0$, we just get y_1 back)

$\boxed{2B-2}$

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^x(ax+b) \\ e^x & e^x(ax+b) + ae^x \end{vmatrix}$$

$= ae^{2x},\ \neq 0$ if $a \neq 0$.

[This shows it for the special equation only].

In general:

$W[y_1, y_2] = y_1 y_2' - y_2 y_1'$

$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p\,dx} dx$

$\therefore\ y_2' = y_1' \int \frac{1}{y_1^2} e^{-\int p\,dx} + y_1 \cdot \frac{1}{y_1^2} e^{-\int p dx}$

$= y_1' y_2 / y_1 + \frac{1}{y_1} e^{-\int p dx}$

$\therefore\ W(y_1, y_2) = y_1' y_2 + e^{-\int p dx} - y_1' y_2$

$= e^{-\int p dx} \neq 0$

[Note that this same formula for the Wronskian follows from II/7a].

$\boxed{2B-3}$

let $y_2 = x \cdot u$, so that

$y_2' = u + xu',\quad y_2'' = 2u' + xu''$.

Substituting into $x^2y'' + 2xy' - 2y = 0$ gives after cancellation and dividing by x^2:

$xu'' + 4u' = 0$ $\quad$ Put $v = u'$.

$x\frac{dv}{dx} + 4v = 0$ or $\boxed{\frac{dv}{v} = \frac{-4dx}{x}}$

Solving, $\quad v = \frac{c}{x^4}$, or $u' = \frac{c}{x^4}$

$\therefore\ u = \frac{c}{-3x^3} + c_0 = \frac{c_1}{x^3} + c_0$

$\therefore\ \boxed{y_2 = \frac{c_1}{x^2} + c_0 x}$, a second sol'n (if $c_1 \neq 0$)

[can also use the general formula given in II/8c]

$\boxed{2B-4}$

Using the general formula [II/8c]:

Find: $e^{-\int p dx}$ $\quad \int p\,dx = \int \frac{-2x}{1-x^2} dx = \ln(1-x^2)$

$= \frac{1}{1-x^2}$

$\therefore\ \int \frac{1}{x^2} e^{-\int p dx} = \int \frac{dx}{x^2(1-x^2)}$

we do this by partial fractions →

(cont'd)

2B-4

(cont'd)

$$\frac{1}{x^2(1-x^2)} = \frac{1}{x^2(1-x)(1+x)} = \frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{1+x}$$

$$\therefore \int \frac{dx}{x^2(1-x^2)} = -\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) = -\frac{1}{x} + \frac{1}{2}\ln\frac{(1+x)}{(1-x)}$$

$$\therefore y_2 = \underset{=x}{y_1} \int \frac{1}{y_1^2} e^{-\int p\,dx} = \boxed{-1 + \frac{x}{2}\ln\frac{1+x}{1-x}}$$

The general solution is now

$$c_1 y_1 + c_2 y_2$$

or $\boxed{c_1 x + c_2\left(-1 + \frac{x}{2}\ln\frac{1+x}{1-x}\right)}$

2C-1

a) Char eq'n: $\lambda^2 - 3\lambda + 2 = 0$

or $(\lambda - 1)(\lambda - 2) = 0$

roots: $\lambda = 1, 2$

$$\therefore \boxed{y = c_1 e^x + c_2 e^{2x}}$$

b) Char eq'n: $r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0$

$\therefore y = c_1 e^x + c_2 e^{-3x}$. Put in initial conditions:

$y(0)=1 \Rightarrow c_1 + c_2 = 1$

$y'(0)=1 \Rightarrow c_1 - 3c_2 = -1$

} solve for c_1, c_2

$c_1 = 1/2$, $c_2 = 1/2$

$$\therefore \boxed{y = \frac{1}{2}e^x + \frac{1}{2}e^{-3x}}$$

c) Char. eqn $r^2 + 2r + 2 = 0$

By quad. formula: $r = -1 \pm i$

$$\therefore y = e^{-x}(c_1\cos x + c_2 \sin x)$$

[using as y_1, y_2 the real + imaginary parts of the cx. soln $y = e^{(-1+i)x}$
$= e^{-x}(\cos x + i\sin x)$]

2C-1

d) Char. eqn: $r^2 - 2r + 5 = 0$

By quad. fmla: $r = 1 \pm 2i$

Gen'l soln: $y = e^x(c_1\cos 2x + c_2 \sin 2x)$

Putting in initial cond'ns (you'll have to find y' first!)

$y(0) = 1 \Rightarrow c_1 = 1$

$y'(0) = 1 \Rightarrow c_1 + 2c_2 = -1, \quad \therefore c_2 = -1$

so $y = e^x(\cos 2x - \sin 2x)$

e) Char. eqn: $r^2 - 4r + 4 = 0$

or $(r-2)^2 = 0$; $r = 2$ double root

$\therefore y = e^{2x}(c_1 x + c_2)$

is the general solution. Put in initial conditions:

$y(0) = 1 \Rightarrow c_2 = 1$

$y'(0) = 1 \Rightarrow 2c_2 + c_1 = 1, \quad \therefore c_1 = -1$

so sol'n is: $y = (1-x)e^{2x}$

2C-2

$$W = \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ e^{ax}(a\cos bx - b\sin bx) & e^{ax}(a\sin bx + b\cos bx) \end{vmatrix}$$

$$= \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ -e^{ax} b\sin bx & e^{ax}(b\cos bx) \end{vmatrix},$$

(by subtracting $a\cdot$(1st row) from 2nd row);

$= e^{2ax}(b\cos^2 bx + b\sin^2 bx) = e^{2ax}\cdot b$

$\neq 0$ if $\boxed{b \neq 0}$ (no restriction on a)

2C-3

Char. eqn: $r^2 + cr + 4 = 0$

roots: $r = \dfrac{-c \pm \sqrt{c^2 - 16}}{2}$

a) has oscillatory solns $\Leftrightarrow$ r is complex (so soln has sin + cos terms);

$\Leftrightarrow c^2 - 16 < 0$, or $\boxed{-4 < c < 4}$

b) if the solutions oscillate, above shows that $r = -\frac{c}{2} \pm i\beta$ $(\beta \neq 0)$

and solns are $y = e^{-\frac{cx}{2}}(c_1\cos\beta x + c_2\sin\beta x)$.

<u>Damped</u> oscillations $\Leftrightarrow$ $c > 0$ (so $y \to 0$ as $t \to \infty$)

$\therefore \boxed{0 < c < 4}$ is condition.

2C-4

a) [Use y' for $\frac{dy}{dx}$, $\dot{y}$ for $\frac{dy}{dt}$.]

We have $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$ $\quad x = e^t$, $\frac{dx}{dt} = e^t$, $\frac{dt}{dx} = e^{-t}$

$$\therefore\ y' = \dot{y}\,e^{-t}$$

$$y'' = \frac{d}{dt}(\dot{y}e^{-t})\cdot\frac{dt}{dx}$$

$$= (\ddot{y}e^{-t} - \dot{y}e^{-t})e^{-t}$$

$$= (\ddot{y} - \dot{y})e^{-2t}$$

Substituting into the ODE:

$x^2y'' + pxy' + qy = 0$ becomes

$$(\ddot{y} - \dot{y}) + p\dot{y} + qy = 0$$

b) $p = q = 1$, so we get $\ddot{y} + y = 0$, whose solution are $y = c_1\cos t + c_2\sin t$

$\left.\begin{matrix} x = e^t \\ \therefore t = \ln x \end{matrix}\right\}$ gives $y = c_1\cos(\ln x) + c_2\sin(\ln x)$

2C-5

Char. eqn is $\boxed{Mr^2 + cr + k = 0}$

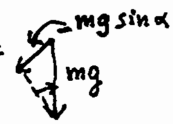
For critical damping, it should have two equal roots; by quadratic formula

$r = \frac{-c \pm \sqrt{c^2 - 4Mk}}{2M}$, $\quad\therefore \boxed{c^2 - 4mk = 0}$ is condition

(when $c^2 - 4mk < 0$, get oscillations).

2G-6

Force triangle



$\vec{F} = ma$ becomes:

$$\underset{\text{(grav.)}}{-mg\sin\alpha} \underset{\text{(air res.)}}{- mc\frac{d\alpha}{dt}} = m\ell\frac{d^2\alpha}{dt^2}$$

$\therefore \boxed{\ddot{\alpha} + \frac{c}{\ell}\dot{\alpha} + \frac{g}{\ell}\sin\alpha = 0}$ If α small, $\sin\alpha \approx \alpha$

If undamped, $c = 0$, get approx.

$\boxed{\ddot{\alpha} + \frac{g}{\ell}\alpha = 0}$ $\quad$ [char eqn is $r^2 + g/\ell = 0$]

$\therefore$ Solns are $y = c_1\cos\sqrt{\frac{g}{\ell}}\,t + c_2\sin(\sqrt{\frac{g}{\ell}}\,t)$

The period $= \frac{2\pi}{\sqrt{g/\ell}} = 2\pi\sqrt{\frac{\ell}{g}}$

(so as length increases, so does the period; on the moon (less g), it swings slower (bigger period))

2C-7

a) $a + bx + ce^x$ $\quad$ b) $a\cos 2x + b\sin 2x$

c) $ax\cos 2x + bx\sin 2x$

d) ax^2e^x (1 is a double root of the char. eqn)

e) $ae^{-x} + bxe^{2x}$ (2 is a root of char. eqn)

f) $(ax^3 + bx^2)e^{3x}$ (3 is double root of char. eqn)

2C-8

b) $y_h = a_1\cos 2x + a_2\sin 2x$

To find y_p, use undet. coefficients:

$y_p = c_1\cos x + c_2\sin x$ $\quad$ $\lfloor\times 4$ (mult. factor)

$\therefore\ y_p'' = -c_1\cos x - c_2\sin x$ $\quad$ and add: LHS is by hypothesis: $y_p'' + 4y_p = 2\cos x$

$2\cos x = 3c_1\cos x + 3c_2\sin x$

$\therefore c_1 = 2/3 \quad c_2 = 0$

So $\boxed{y = a_1\cos 2x + a_2\sin 2x + \frac{2}{3}\cos x}$

$y(0) = 0 \Rightarrow a_1 + 2/3 = 0$ $\quad\therefore \boxed{\begin{matrix} a_1 = -2/3 \\ a_2 = 1/2 \end{matrix}}$

$y'(0) = 1 \Rightarrow 2a_2 = 1$

2C-8

a) $y_h = a_1e^x + a_2e^{5x}$, as usual.

Try $y_p = cxe^x$ $\quad$ $\lfloor\times 5$ $\}$ multiplic'n factors

$\therefore\ y_p' = ce^x(x+1)$ $\quad$ $\lfloor\times -6$

$y_p'' = ce^x(x+2)$ $\quad$ then add:

$$e^x = e^x(-6c + 2c) + xe^x(\underbrace{5c - 6c + c}_{0})$$

$\therefore -4c = 1$

$c = -1/4$

$$\boxed{y = a_1e^x + a_2e^{5x} - \frac{1}{4}xe^x}$$

c) Char eqn: $r^2 + r + 1 = 0$, $\quad r = \frac{-1 \pm \sqrt{-3}}{2}$

$\therefore y_h = e^{-x/2}(a_1\cos\frac{\sqrt{3}}{2}x + a_2\sin\frac{\sqrt{3}}{2}x)$

Try $y_p = c_1xe^x + c_2e^x$

$y_p' = c_1e^x(x+1) + c_2e^x$ $\quad$ Add the eqns:

$y_p'' = c_1e^x(x+2) + c_2e^x$

$2xe^x = 3c_1xe^x + (3c_1 + 3c_2)e^x$

$\therefore c_1 = 2/3, \quad c_2 = -2/3$

$$y = e^{-x/2}(a_1\cos\tfrac{\sqrt{3}}{2}x + a_2\sin\tfrac{\sqrt{3}}{2}x) + \tfrac{2}{3}e^x(x-1)$$

2C-8

d) $y_h = a_1e^x + a_2e^{-x}$

Try: $y_p = c_1x^2 + c_2x + c_3$ $\underline{L\cdot -1}$

$y_p'' = 2c_1$ Add:

$$x^2 = -c_1x^2 + c_2x + 2c_1 - c_3$$

$\therefore c_1 = -1,\ c_2 = 0,\ 2c_1 - c_3 = 0$
$c_3 = -2$

$$\boxed{y = a_1e^x + a_2e^{-x} - x^2 - 2}$$

$y(0) = 0 \Rightarrow a_1 + a_2 - 2 = 0$

$y'(0) = -1 \Rightarrow a_1 - a_2 = -1$

solving, $\boxed{a_1 = 1/2,\ a_2 = 3/2}$

$\therefore\ y = \frac{1}{2}e^x + \frac{3}{2}e^{-x} - x^2 - 2$

2C-9

a) Write the ODE as $\boxed{Ly = r}$,
where L is the linear operator
$$L = D^2 + pD + q$$

• By hypothesis,

$Ly_1 = r_1$ ← (i.e., y_1 is a solution to $Ly = r_1$)

$Ly_2 = r_2$ ← (similarly)

Adding, $L(y_1 + y_2) = r_1 + r_2$

(using the linearity of L: $L(y_1+y_2) = Ly_1 + Ly_2$)

$\therefore\ y_1 + y_2$ solves $Ly = r_1 + r_2$.

b) First consider $y'' + 2y' + 2y = 2x$

Try $y_1 = c_1x + c_2$ $\underline{L\cdot 2}$

$y_1' = c_1$ $\underline{L\cdot 2}$

$y_1'' = 0$ Add

$$2x = 2c_1x + (2c_2 + 2c_1)$$

$\therefore c_1 = 1,\ c_2 = -1$ $\boxed{y_1 = x - 1}$

Then: $y'' + 2y' + 2y = \cos x$

Try $y_2 = a_1\cos x + a_2\sin x$ $\underline{L\cdot 2}$

$y_2' = -a_1\sin x + a_2\cos x$ $\underline{L\cdot 2}$

$y_2'' = -a_1\cos x - a_2\sin x$ Add

$$\cos x = \cos x\,(2a_1 + 2a_2 - a_1) + \sin x\,(2a_2 - 2a_1 - a_2)$$

$\therefore \left.\begin{array}{l} a_1 + 2a_2 = 1 \\ -2a_1 + a_2 = 0 \end{array}\right\}\ \therefore\ \begin{array}{l} a_2 = 2/5 \\ a_1 = 1/5 \end{array}$ $y_2 = \frac{1}{5}\cos x + \frac{2}{5}\sin x$

2C-10

a) $R = 0,\ E = 0$

Eqn is $Lq'' + \frac{q}{C} = 0$ or $q'' + \frac{q}{LC} = 0$

Solving as usual,

$$\boxed{q = c_1\cos\frac{1}{\sqrt{LC}}t + c_2\sin\frac{1}{\sqrt{LC}}t}$$

Period is $2\pi\sqrt{LC}$ ($= 2\pi/\text{frequency}$)
frequency $= 1/\sqrt{LC}$

b) Char. eqn is $Lr^2 + Rr + \frac{1}{C} = 0$

roots: $r = \dfrac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$

oscillates if $\boxed{R^2 - \frac{4L}{C} < 0}$

c) $Li'' + \frac{i}{C} = \omega E_0\cos\omega t$

Solns of homog. eqn are
$$i = a_1\cos\frac{1}{\sqrt{LC}}t + a_2\sin\frac{1}{\sqrt{LC}}t$$

The particular soln i_p will have form $c_1\cos\omega t + c_2\sin\omega t$ <u>unless</u> $\omega = \frac{1}{\sqrt{LC}}$, in which case it will be $c_1t\cos\omega t + c_2t\sin\omega t$, which ~~gets~~ large as $t \to \infty$.

Thus if $\boxed{\omega \approx \frac{1}{\sqrt{LC}}}$, solns will be large in amplitude

$\therefore$ this is ω_0

---o---

> The advantage of this method (divide and conquer!) is that we don't have to assume
> $$y_p = d_1x + d_2 + d_3\cos x + d_4\sin x,$$
> which would give 4 equations in 4 unknowns to solve...

Using part (a), the particular solution to
$$y'' + 2y' + 2y = 2x + \cos x$$
is
$$y = y_1 + y_2 = x - 1 + \frac{1}{5}\cos x + \frac{2}{5}\cos x$$

2D-1

a) $y_h = c_1 \underset{y_1}{\cos x} + c_2 \underset{y_2}{\sin x}$, as usual.

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Let $y_p = u_1 y_1 + u_2 y_2$

The equations for variation of pars. are:

$$u_1' \cos x + u_2' \sin x = 0$$
$$u_1'(-\sin x) + u_2' \cos x = \tan x$$

Either by elimination, or by Cramer's rule, we get as sol'n: (the denom. is $W(y_1, y_2)$)

$$u_1' = \frac{-y_2 f(x)}{W(y_1, y_2)} = -\sin x \tan x = \cos x - \sec x$$

(so it can be integrated)

$$u_2' = \frac{y_1 f(x)}{W(y_1, y_2)} = \cos x \tan x = \sin x$$

(from tables)

$$\therefore \quad u_1 = \sin x - \ln|\sec x + \tan x|$$
$$u_2 = -\cos x$$

$$\therefore \quad y_p = \left(\sin x - \ln|\sec x + \tan x|\right)\cos x - \cos x \sin x$$

i.e., $\boxed{y_p = -\cos x\,(\ln|\sec x + \tan x|)}$

b) Two indept sol'ns of the assoc. homog. eqn are: $y_1 = e^x$, $y_2 = e^{-3x}$ (method as usual)

$$W(y_1, y_2) = -4e^{-2x} \quad \left(= \begin{vmatrix} e^x & e^{-3x} \\ e^x & -3e^{-3x} \end{vmatrix}\right)$$

$$y_p = u_1 y_1 + u_2 y_2 \qquad (*)$$

The eqn's for variation of parameters are:

$$u_1' e^x + u_2' e^{-3x} = 0$$
$$u_1' e^x - u_2'(3e^{-3x}) = e^{-x}$$

$f(x)$, (from orig. eqn)

Solve them by elimination, or by Cramer's rule; following the latter, we get as sol'n

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{1}{4}e^{-2x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{e^x \cdot e^{-x}}{-4e^{-2x}} = -\frac{1}{4}e^{2x}$$

$$\therefore \quad u_1 = -\frac{1}{8}e^{-2x}, \qquad u_2 = -\frac{1}{8}e^{2x}$$

and so $y_p = -\frac{1}{8}e^{-2x} \cdot e^x - \frac{1}{8}e^{2x} \cdot e^{-3x}$, by $(*)$;

or: $\boxed{y_p = -\frac{1}{4}e^{-x}}$

c) Two indept sol'ns of the assoc. homog. eq'n are: $y_1 = \cos 2x$, $y_2 = \sin 2x$ (by the usual method)

$$W(y_1, y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Let $y_p = u_1 y_1 + u_2 y_2$

Then
$$\begin{cases} u_1' \cos 2x + u_2' \sin 2x = 0 \\ u_1'(-2\sin 2x) + u_2'(2\cos 2x) = \sec^2 2x \end{cases}$$

are the eqn's for the method of var. of pars.

Solving them by elimination, or by Cramer's rule:

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{-\sin 2x}{2\cos^2 2x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{\cos 2x}{2\cos^2 2x} = \frac{\sec 2x}{2}$$

Integrating,

$$u_1 = -\frac{1}{4} \cdot \frac{1}{\cos 2x}$$

$$u_2 = \frac{1}{4}\ln|\sec 2x + \tan 2x|$$

$$\therefore \quad \boxed{y_p = -\frac{1}{4} + \frac{1}{4}\ln|\sec x + \tan x| \cdot \sin 2x}$$

2D-2

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{1}{x}$, after some calculation.

$$y_p = u_1 y_1 + u_2 y_2$$

Equations for method of var. of pars. are:

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = \frac{\cos x}{\sqrt{x}}$$

(note: the ODE must be written $y'' + \frac{1}{x}y' + (\ldots)y = \frac{\cos x}{\sqrt{x}}$) $f(x)$

Solving these by Cramer's rule:

$$u_1' = \frac{-y_2 f(x)}{W} = \cos^2 x$$

$$u_2' = \frac{y_1 f(x)}{W} = -\sin x \cos x$$

$$\therefore \quad u_1 = \frac{x}{2} + \frac{\sin 2x}{4}, \qquad u_2 = \frac{\cos 2x}{4}$$

and so (using identities):

$$y_p = \frac{\sin x}{\sqrt{x}}\left(\frac{x}{2} + \frac{2\sin x \cos x}{4}\right) + \frac{\cos x}{\sqrt{x}}\left(\frac{\cos^2 x - \sin^2 x}{4}\right)$$

so $y_p = \frac{x \sin x}{2\sqrt{x}} + \frac{1}{4}\frac{\cos x}{\sqrt{x}}$

(The term $\frac{1}{4}\frac{\cos x}{\sqrt{x}}$ is part of the general sol'n $y = y_p + c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\sin x}{\sqrt{x}}$; so it can be omitted:

$\boxed{y_p = \frac{\sqrt{x}\sin x}{2}}$ is the best answer)

2D-3

a) Let y_1, y_2 be indep't solutions of the associated homogeneous equation.

$$y_p = u_1 y_1 + u_2 y_2, \qquad W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x), & y_2'(x) \end{vmatrix}$$

and the eqn's for the method of var. of pars. are

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f(x)$$

Solving by Cramer's rule gives

$$u_1' = \frac{-y_2(x) f(x)}{W[y_1(x), y_2(x)]}, \qquad u_2' = \frac{y_1(x) f(x)}{W[y_1(x), y_2(x)]}$$

so that (use definite integrals so as to get a definite function)

$$u_1(x) = \int_a^x \frac{-y_2(t) f(t)\, dt}{W[y_1(t), y_2(t)]}, \qquad u_2(x) = \int_a^x \frac{y_1(t) f(t)\, dt}{W[y_1(t), y_2(t)]}$$

Thus: $y_p(x) = u_1(x) \cdot y_1(x) + u_2(x) y_2(x)$ — we can put $y_1(x)$ and $y_2(x)$ inside the integral sign because they are "constants" — the integration is with respect to t, not x; then we can add the integrands. The result is:

$$y_p = \int_a^x \frac{-y_1(x)\, y_2(t) + y_2(x)\, y_1(t)}{W[y_1(t), y_2(t)]} \cdot f(t)\, dt$$

or

$$y_p = \int_a^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix} f(t)\, dt}{W(y_1(t), y_2(t))}$$

b) The arbitrary constants of integration — call them a_1 and a_2, — will change u_1 and u_2 by an additive constant:

$$u_1 + a_1, \qquad u_2 + a_2$$

leading to the particular soln:

$$y_p = (u_1 + a_1) y_1 + (u_2 + a_2) y_2$$

$$\circledast \quad y_p = \boxed{u_1 y_1 + u_2 y_2} + a_1 y_1 + a_2 y_2$$

The boxed part is the particular solution of part (a); the part added on is in the general soln y_h to the associated homog. eqn, hence the particular soln $\circledast$ is just as good a particular soln as the previous one.

2D-4

It depends on the ODE form — (it must be linear!)

Undetermined coefficients requires

① The ODE is linear, with constant coefficients

② The inhomogeneous term $f(x)$ has a special form: a sum of terms of the form

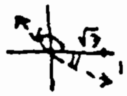
$$(\text{polynomial}) \cdot e^{ax} \cdot \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}.$$

↑ can be 1 — ↑ a can be 0 — ↑ b can be 0

If the coeffs. are not constant, or $f(x)$ is not of the above form, you must use variation of parameters to find y_p.

Drawback: you must be able to find y_1, y_2 first — i.e., solve the assoc. homog. eq'n.

(Note that finding y_p by undet. coeffs. does not require you to solve for y_1, y_2 first (unless you are unlucky and $f(x)$ is a soln of the assoc. homog. eqn --- but you can always test this without solving the eqn.)

2E-1 

$-1+i = \sqrt{2}\, e^{i3\pi/4}$

$\sqrt{3}-i = 2e^{-i\pi/6}$

2E-2 $\dfrac{1-i}{1+i} = \dfrac{(1-i)^2}{(1+i)(1-i)} = \dfrac{-2i}{2} = -i$

Other way:

$1-i = \sqrt{2}\, e^{-i\pi/4}$

$1+i = \sqrt{2}\, e^{i\pi/4}$

$\therefore \dfrac{1-i}{1+i} = \dfrac{\sqrt{2}}{\sqrt{2}} \cdot e^{i(-\pi/4-\pi/4)}$

$= e^{-i\pi/2} = -i$

2E-4 $z = a+bi, \quad w = c+di$

$zw = (ac-bd) + i(ad+bc)$

$\therefore \overline{zw} = (ac-bd) - i(ad+bc)$

$\bar{z}\,\bar{w} = (a-bi)(c-di),$

$= (ac-bd) - i(ad+bc)$

2E-7 a) $(1-i)^4 = 1 + 4(-i) + 6(-i)^2 + 4(-i)^3 + (-i)^4$

$= 1-6+1 + i(-4+4) = \boxed{-4}$

By DeMoivre:

$1-i = \sqrt{2}\, e^{-i\pi/4}$

$(1-i)^4 = (\sqrt{2})^4 e^{-i\pi} = 4\cdot(-1)$

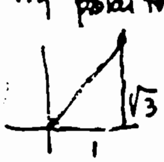
$= -4.$

b) $(1+i\sqrt{3})^3 = 1 + 3(i\sqrt{3}) + 3(i\sqrt{3})^2 + (i\sqrt{3})^3$

$= 1 + 3i\sqrt{3} + 3\cdot -3 + i^3 3\sqrt{3}$

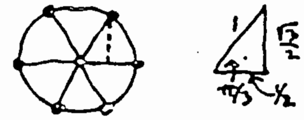
$= -8 + i(3\sqrt{3} - 3\sqrt{3}) = -8$

By polar form:



$1+i\sqrt{3} = 2e^{i\pi/3}$

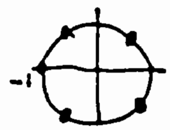
$(1+i\sqrt{3})^3 = 8e^{i\pi} = -8$



2E-9 The sixth roots of 1 are $e^{\frac{i\pi}{3}k}$ where $k = 0, 1, 2, \dots, 5$

get $\therefore\ 1, -1, \dfrac{\pm 1 \pm i\sqrt{3}}{2}$.

2E-10 $\sqrt[4]{-16} = 2\cdot\sqrt[4]{-1}$.



The 4th roots of -1 are on the picture: $\dfrac{\pm 1 \pm i}{\sqrt{2}}$

$\therefore \sqrt{2}\cdot(\pm 1 \pm i)$ are the roots of $x^4+16=0$.

2E-14 $\sin^4 x = \left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)^4$; by bin. thm, this

$= \frac{1}{16}\left(e^{4ix} - 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} - 4e^{ix}e^{-3ix} + e^{-4ix}\right)$

$= \frac{1}{16}\left(e^{4ix}+e^{-4ix}\right) - \frac{4}{16}\left(e^{2ix}+e^{-2ix}\right) + \frac{6}{16}\cdot 1$

$= \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}$.

Since $\sin^4 x$ is an even function, the answer should not contain the odd functions $\sin 4x$, $\sin 2x$.

2E-15 $e^{(2+i)x} = e^{2x}(\cos x + i\sin x)$

So $e^{2x}\sin x = \operatorname{Im} e^{(2+i)x}$.

$\int e^{(2+i)x}\,dx = \frac{1}{2+i}e^{(2+i)x}$; $\quad \frac{1}{2+i}\cdot\frac{2-i}{2-i} = \frac{2-i}{5}$;

$= \frac{2-i}{5}\left(e^{2x}\cos x + ie^{2x}\sin x\right)$

We want just the imaginary part:

$\therefore \int e^{2x}\sin x\,dx = e^{2x}\left(\frac{2}{5}\sin x - \frac{1}{5}\cos x\right)$

2E-16 $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$ $\quad$ since $\cos(-x)=\cos x$ + $\sin(-x) = -\sin x$,

Adding: $\dfrac{e^{ix}+e^{-ix}}{2} = \cos x$

Subtract: $\dfrac{e^{ix}-e^{-ix}}{2i} = \sin x$

2F-1

a) $D^2+2D+2=0$ has roots $-1\pm i$

$\therefore\ y = e^{2x}(c_1+c_2x+c_3x^2) + e^{-x}(c_4\cos x + c_5 \sin x)$

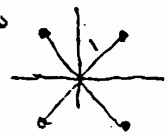
b) $D^8-2D^4+1 = (D^4-1)^2 = [(D^2-1)(D^2+1)]^2$

$= (D-1)^2(D+1)^2(D^2+1)^2$

$\therefore\ y = e^x(c_1+c_2x) + e^{-x}(c_3+c_4x) + \cos x(c_5+c_6x) + \sin x(c_7+c_8x)$

c) Characteristic eq'n is $\boxed{z^4+1=0}$

Roots are $\sqrt[4]{-1}$



$\frac{1\pm i}{\sqrt2}$ and $\frac{-1\pm i}{\sqrt2}$

letting $a = 1/\sqrt2$, get $\therefore$

$y = e^{ax}(c_1\cos ax + c_2 \sin ax) + e^{-ax}(c_3\cos ax + c_4 \sin ax)$

d) Char. eq'n is $\boxed{z^4-8z^2+16=0}$

which factors as

$(z^2-4)^2$ or $(z+2)^2(z-2)^2$

$\therefore$ has double roots at $2, -2$

so

$y = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + c_4xe^{-2x}$

e) $y = c_1e^x + c_2e^{-x} + e^{x/2}(c_3\cos\frac{\sqrt3}{2}x + c_4\sin\frac{\sqrt3}{2}x) + e^{-x/2}(c_5\cos\frac{\sqrt3}{2}x + c_6\sin\frac{\sqrt3}{2}x)$

[using roots as given in soln to 2E-9]

f) $y = e^{\sqrt2 x}(c_1\cos\sqrt2 x + c_2\sin\sqrt2 x) + e^{-\sqrt2 x}(c_3\cos\sqrt2 x + c_4\sin\sqrt2 x)$

2F-2

$y'''' - 16y = 0$

characteristic equation $\boxed{z^4-16=0}$

roots: $2,\ 2i,\ -2,\ -2i$

(one real root is 2, so the others are all of the form $2\cdot\sqrt[4]{1}$, where $\sqrt[4]{1} = 1, i, -1, -i$)

From roots, general sol'n is

$y = c_1e^{2x} + c_2e^{-2x} + c_3\sin 2x + c_4\cos 2x$

Putting in conditions:

$\boxed{c_1=0}$ since $|y(x)| < K$ for all $x>0$

($\because |c_1e^{2x}| \to \infty$ unless $c_1=0$)

$y(0)=0 \Rightarrow c_2+c_4=0 \quad \therefore c_4=-c_2$

$y'(0)=0 \Rightarrow -2c_2+2c_3=0 \quad \therefore c_3=c_2$

$\therefore$ sol'n is – so far –

$\boxed{y = c_2(e^{-2x} + \sin 2x - \cos 2x)}$

Finally

$y(\pi)=1 \Rightarrow c_2(e^{-2\pi}-1) = 1$,

$\therefore\ \boxed{c_2 = \frac{1}{e^{-2\pi}-1}}$.

2F-3

a) $\boxed{z^3-z^2+2z-2=0}$ is char. eq'n.

1 is a root, $\therefore z-1$ is factor

get $(z-1)(z^2+2)$ roots: $1, i\sqrt2, -i\sqrt2$

$y = c_1e^x + c_2\cos\sqrt2 x + c_3\sin\sqrt2 x$

b) $\boxed{z^3+z^2-2=0} = (z-1)(z^2+2z+2)$

roots $1,\ -1\pm i$

$\therefore\ y = c_1e^x + c_2e^{-x}\cos x + c_3e^{-x}\sin x$

c) $(D^3-2D-4) = (D-2)(D^2+2D+2)$

$\therefore\ y = c_1e^{2x} + e^{-x}(c_2\cos x + c_3\sin x)$

roots are $-1\pm i$

d)

d): $x^4+2x^2+4=0$; $\therefore\ x^2 = \frac{-2\pm\sqrt{2^2-4\cdot4}}{2} = -1\pm\sqrt{-3} = -1\pm\sqrt3\, i$;

2F-8

changing to polar representation: $= 2e^{2\pi/3\, i},\ 2e^{4\pi/3\, i}$

$\therefore\ x = \sqrt2\, e^{\pi/3\, i},\ \sqrt2\, e^{4\pi/3\, i}$ (square roots of the first ↗)

$= \sqrt2\, e^{2\pi/3\, i},\ \sqrt2\, e^{5\pi/3\, i}$ (" " " " others →)

↖ and ↙ are conjugates

Using therefore just $\sqrt2\, e^{\pi/3\, i}$ and $\sqrt2\, e^{\frac{2\pi}{3}i}$, :

$\sqrt2\, e^{\pi/3\, i} = \sqrt2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = \sqrt2(\frac12 + i\frac{\sqrt3}{2})$; similarly, get $\sqrt2(-\frac12 + i\frac{\sqrt3}{2})$

leading to: $y = e^{\sqrt2/2\, x}(c_1\cos\frac{\sqrt6}{2}x + c_2\sin\frac{\sqrt6}{2}x) + e^{-\sqrt2/2\, x}(c_3\cos\frac{\sqrt6}{2}x + c_4\sin\frac{\sqrt6}{2}x)$

2F-4

$$x_1'' + 2x_1 - x_2 = 0$$
$$x_2'' + x_2 - x_1 = 0$$

Eliminate x_1 by solving for x_1:

$$x_1 = x_2'' + x_2$$

substitute into first equation:

$$(x_2'' + x_2)'' + 2(x_2'' + x_2) - x_2 = 0$$

or $\quad x_2'''' + 3x_2'' + x_2 = 0$

char. eqn: $\boxed{z^4 + 3z^2 + 1 = 0}$

as quadratic eqn in z^2: solve, get

$z^2 = \dfrac{-3 \pm \sqrt{5}}{2}$: both nos. are real, + negative; call them $-a^2, -b^2$

$\therefore z^2 = -a^2, \quad z^2 = -b^2$
$z = \pm ia, \quad z = \pm ib$

so $X_2 = c_1 \cos at + c_2 \sin at + c_3 \cos bt + c_4 \sin bt$

2F-5

$$D^2 e^{2x} \cos x = e^{2x}(D+2)^2 \cos x$$
$$= e^{2x}(D^2 + 4D + 4)\cos x$$
$$= e^{2x}(3\cos x - 4\sin x)$$

2F-6

a) By (12) in notes, (see Example 2)

$$y_p = \frac{4}{1^2+1} e^x = 2e^x$$

b) $(D^3 + D^2 - D + 2)y = 2e^{ix}$

$$\therefore y_p = \frac{2e^{ix}}{i^3 + i^2 - i + 2} = \frac{2(1+2i)}{(1-2i)(1+2i)} e^{ix}$$

$\therefore y_p = \dfrac{2+4i}{5}(\cos x + i \sin x) \quad \therefore Re(y_p) = \boxed{\frac{2}{5}\cos x - \frac{4}{5}\sin x}$

c) $(D^2 - 2D + 4)y = e^{(1+i)x}$

$(1+i)^2 - 2(1+i) + 4 = 2 \qquad \therefore y_p = \dfrac{e^{(1+i)x}}{2}$

$$Re(y_p) = \boxed{\frac{1}{2} e^x \cos x}$$

d) $D^2 - 6D + 9 = (D-3)^2 \qquad \therefore y_p = cx^2 e^{3x}$

$(D-3)^2 y_p = ce^{3x} D^2 x^2 \quad$ (by exp-shift rule)

$= 2ce^{3x} = e^{3x}$ (from the ODE)

$\therefore c = 1/2, \quad y_p = \frac{1}{2} x^2 e^{3x}$

2F-7

$(D+a)e^{-ax}u = e^{-ax}Du = f(x)$

$\therefore Du = e^{ax}f(x), \quad u = \int e^{ax} f(x)\,dx$

$y_p = e^{-ax} \int e^{ax} f(x)\,dx$

2G-1

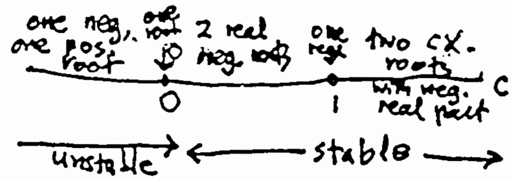
$$y'' + 2y' + cy = 0$$

Char. eqn is:

$$r^2 + 2r + c = 0$$

By quadratic formula:

$$\text{roots} = \frac{-2 \pm \sqrt{4-4c}}{2}$$
$$= -1 \pm \sqrt{1-c}\,.$$



2G-2

$$r^2 + \frac{b}{a}r + \frac{c}{a} = (r - r_1)(r - r_2)$$

$$\therefore \frac{b}{a} = -(r_1 + r_2) \qquad (*)$$
$$\frac{c}{a} = r_1 r_2$$

Real case: $r_1, r_2 < 0 \Rightarrow b/a > 0$, $c/a > 0$

$\therefore a, b, c$ have same sign.

Complex case:

$r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\alpha < 0 \Rightarrow \dfrac{b}{a} = -2\alpha > 0$

by $(*)$ $\dfrac{c}{a} = \alpha^2 + \beta^2 > 0.$

2G-3

Assume $a, b, c > 0$ (if not, multiply ODE through by -1).

$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are the roots.

If roots are real, $\dfrac{-b - \sqrt{\ }}{2a} < 0$

and $-b + \sqrt{b^2 - 4ac} < 0$, therefore (since $b^2 - 4ac < b^2$).

If roots are complex, $\dfrac{-b}{2a} < 0$

$\therefore$ in both cases, the char. roots have negative real part.

$\boxed{2H-1}$ $y'' - k^2 y = 0$, $y(0) = 0$, $y'(0) = 1$

$y_c = c_1 e^{kx} + c_2 e^{-kx}$

Soln to IVP is

$$w(t) = \frac{e^{kx} - e^{-kx}}{2k} = \frac{\sinh kx}{k}$$

$\boxed{2H-3a}$ By Example 2 (p. 2),

$$w(x) = xe^{-2x}$$

Therefore

$$y(x) = \int_0^x \underbrace{(x-t)e^{-2(x-t)}}_{w(x-t)} \cdot \underbrace{e^{-2t}}_{f(t)} dt$$

$$= e^{-2x} \int_0^x (x-t)\, dt$$

$$= e^{-2x}\left(xt - \frac{t^2}{2}\right)_0^x = \boxed{\frac{x^2}{2} e^{-2x}}$$

By undetermined coeffs, since $y_c = e^{-2x}(c_1 + c_2 x)$, try $c x^2 e^{-2x}$

$$(D+2)^2 c e^{-2x} x^2 = c e^{-2x} D^2 x^2$$

$$= c e^{-2x} \cdot 2$$

From the ODE,

$$c e^{-2x} \cdot 2 = e^{-2x}, \quad \boxed{c = \frac{1}{2}} \checkmark$$

$\boxed{2H-4}$ a) By Leibniz:

$$\phi'(x) = \frac{d}{dx} \int_0^x (2x+3t)^2 dt =$$

$$= (5x)^2 + \int_0^x 2 \cdot (2x+3t) \cdot 2\, dt$$

$$= (5x)^2 + 4\left(2xt + \frac{3t^2}{2}\right)\Big]_0^x = (5x)^2 + 14x^2$$

$$= \boxed{39x^2}$$

b) Directly:

$$\phi(x) = \frac{1}{9}(2x+3t)^3 \Big]_0^x = \frac{1}{9}\left((5x)^3 - (2x)^3\right)$$

$$= 13x^3$$

So. $\phi'(x) = 39x^2$. $\checkmark$

18.03 Differential Equations
Spring 2010

Section 3 Solutions

$\boxed{3A-1}$ $\mathcal{L}\{t\} = \int_0^\infty te^{-st}\,dt$. Integrate by parts:

$$= t\frac{e^{-st}}{-s}\Big]_{t=0}^{\infty} - \int_0^\infty \frac{e^{-st}}{-s}\,dt$$

Since $\lim_{t\to\infty} te^{-st} = 0$ $\boxed{\text{if } s>0,}$ the left-hand term is 0 at both endpoints. Integrating the right-hand term:

$$= -\frac{e^{-st}}{(-s)(-s)}\Big]_0^\infty = 0 - \left(\frac{-1}{s^2}\right) = \frac{1}{s^2}, \quad s>0.$$

$\boxed{3A-2}$ $\mathcal{L}\{e^{(a+ib)t}\} = \mathcal{L}\{e^{at}\cos bt\} + i\mathcal{L}\{e^{at}\sin bt\}$ ⊛

On the other hand,

$$\mathcal{L}\{e^{(a+ib)t}\} = \frac{1}{s-(a+ib)};$$ multiplying top + bottom by $(s-a)+ib$:

$$= \frac{(s-a)+ib}{(s-a)^2+b^2} = \frac{s-a}{(\cdots)} + \frac{ib}{(\cdots)} \quad ⊛⊛$$

$$\therefore \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}, \quad \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}.$$

[by equating real + imag. parts of ⊛ and ⊛⊛.]

$\boxed{3A-3}$ a) $\mathcal{L}^{-1}\left(\frac{1}{\frac{s}{2}+3}\right) = \mathcal{L}^{-1}\left(\frac{2}{s+6}\right) = 2e^{-6t}$

b) $\mathcal{L}^{-1}\left(\frac{3}{s^2+4}\right) = \frac{3}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \frac{3}{2}\sin 2t$

c) $\mathcal{L}^{-1}$: $\frac{1}{s^2-4} = \frac{1/4}{s-2} - \frac{1/4}{s+2}$ (partial fractions)

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2-4}\right) = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$$

d) $\frac{1+2s}{s^3} = \frac{1}{s^3} + \frac{2}{s^2}$

$$\therefore \mathcal{L}^{-1}\left(\frac{1+2s}{s^3}\right) = \frac{t^2}{2} + 2t$$

e) $\frac{1}{s^4-9s^2} = \frac{-1/9}{s^2} + \frac{0}{s} + \frac{1/54}{s-3} + \frac{-1/54}{s+3}$

$= \frac{1}{s^2(s-3)(s+3)}$ (by cover-up method. Find the 0 by putting $s=1$)

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^4-9s^2}\right) = -\frac{t}{9} + \frac{1}{54}\left(e^{3t} - e^{-3t}\right)$$

$\boxed{3A-4}$ $\mathcal{L}\{\sin at\} = \int_0^\infty \sin at\cdot e^{-st}\,dt$; Integrate by parts:

$$= \sin at\cdot\frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty a\cos at\cdot\frac{e^{-st}}{-s}\,dt$$

$$= 0 + \frac{a}{s}\mathcal{L}\{\cos at\}$$

$$= \frac{a}{s}\cdot\frac{s}{s^2+a^2}, \quad s>0$$

$$= \frac{a}{s^2+a^2}, \quad s>0.$$

$\boxed{3A-5}$ $\mathcal{L}\{\cos^2 at\} = \mathcal{L}\{\tfrac{1}{2} + \tfrac{1}{2}\cos 2at\}$

$$= \mathcal{L}\{\tfrac{1}{2}\} + \tfrac{1}{2}\mathcal{L}\{\cos 2at\}$$

$$= \frac{1}{2s} + \frac{1}{2}\left(\frac{s}{s^2+4a^2}\right).$$

$$\mathcal{L}\{\sin^2 at\} = \mathcal{L}\{\tfrac{1}{2} - \tfrac{1}{2}\cos 2at\}$$

$$= \frac{1}{2s} - \frac{1}{2}\left(\frac{s}{s^2+4a^2}\right)$$

$\mathcal{L}\{\cos^2 at + \sin^2 at\} = \frac{1}{s}$, from the above;

$$\overset{\shortparallel}{\mathcal{L}\{1\}} = \frac{1}{s} \;\checkmark.$$

$\boxed{3A-6}$ a) $\mathcal{L}\{\frac{1}{\sqrt{t}}\} = \int_0^\infty e^{-st}\frac{1}{\sqrt{t}}\,dt$, $(s>0)$

Put $x^2 = st$, so $t = \frac{x^2}{s}$

Then the integral becomes (in terms of s, x):

$$= \int_0^\infty e^{-x^2}\frac{\sqrt{s}}{x}\cdot\frac{2x\,dx}{s}$$

$$= \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\,dx = \frac{2}{\sqrt{s}}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi/s}.$$

b) $\mathcal{L}\{\sqrt{t}\} = \int_0^\infty e^{-st}\sqrt{t}\,dt$; integrate by parts:

$$= \sqrt{t}\,\frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\cdot\frac{1}{2\sqrt{t}}\,dt$$

$$= 0 + \frac{1}{2s}\int_0^\infty e^{-st}\cdot\frac{1}{\sqrt{t}}\,dt$$

$$= \frac{1}{2s}\mathcal{L}\{\frac{1}{\sqrt{t}}\} = \frac{1}{2s}\cdot\sqrt{\frac{\pi}{s}} = \frac{\sqrt{\pi}}{2s^{3/2}}.$$

$\boxed{3A-7}$ $\mathcal{L}\{e^{t^2}\} = \int_0^\infty e^{-st}\cdot e^{t^2}\,dt$
$= \int_0^\infty e^{t^2-st}\,dt$

This integral is infinite for every real value of s, no matter how large, since if $t>s$, $t^2-st>0$, and therefore
$$\int_0^\infty e^{t^2-st}dt > \int_s^\infty e^{t^2-st}dt > \int_s^\infty \underbrace{e^0 dt}_{=\infty},$$

$\boxed{3A-8}$ $\mathcal{L}\{\frac{1}{t^k}\} = \int_0^\infty e^{-st}\frac{1}{t^k}\,dt, \ (s>0)$

The trouble here is when $t=0$.
Near $t=0$, $e^{-st}\approx e^0 = 1$.
$\therefore$ the integral is like:
$$\int_0^a e^{-st}\frac{1}{t^k}\,dt \gtrsim \int_0^a \frac{dt}{t^k}$$
and this last integral converges only if $k<1$ $\left[\text{since it} \begin{cases} = \frac{t^{1-k}}{1-k}\Big|_0^a & \text{for } k\neq 1 \\ = \ln x\Big|_0^a & \text{for } k=1 \end{cases}\right]$

[At the upper limit ∞ the original integral always converges, if $s>0$].
$\therefore$ $\mathcal{L}\{\frac{1}{t^k}\}$ exists for $k<1$.

$\boxed{3A-9}$a) $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9} = F(s)$
By the exponential-shift formula,
$$\mathcal{L}\{e^{-t}\sin 3t\} = F(s+1) = \frac{3}{(s+1)^2+9}$$

b) $\mathcal{L}\{t^2-3t+2\} = \frac{2}{s^3}-\frac{3}{s^2}+\frac{2}{s} = F(s)$
By exponential-shift rule,
$$\mathcal{L}\{e^{2t}(t^2-3t+2)\} = F(s-2)$$
$$= \frac{2}{(s-2)^3} - \frac{3}{(s-2)^2} + \frac{2}{s-2}$$

$\boxed{3A-10}$ $\mathcal{L}^{-1}\{\frac{3}{(s-2)^4}\} = e^{2t}\mathcal{L}^{-1}\{\frac{3}{s^4}\} =$
$= e^{2t}\frac{t^3}{2}$

$\mathcal{L}^{-1}\{\frac{1}{s(s-2)}\} = \mathcal{L}^{-1}\{\frac{1/2}{s-2} - \frac{1/2}{s}\}$,
(by partial fractions)
$= \frac{1}{2}e^{2t} - \frac{1}{2}$.

$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s+5}\right\}$:
Complete the square in the denominator:
$\frac{s+1}{s^2-4s+5} = \frac{s+1}{(s-2)^2+1}$; express top in terms of $s-2$:
$$= \frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$$
$\therefore \mathcal{L}^{-1}(\cdots) = e^{2t}\cos t + 3e^{2t}\sin t$,
(by the exponential-shift rule).

3B-1 We use throughout the two formulas:

$$\mathcal{L}(y') = -y(0+) + sY \leftarrow (\mathcal{L}(y))$$

and

$$\mathcal{L}(y'') = -y'(0+) - sy(0+) + s^2 Y$$

[The $0+$ indicates that if $y(t)$ is discontinuous at 0, we $\lim_{t\to 0+} y(t)$, the righthand limit].

a) $y' - y = e^{3t}, \quad y(0) = 1$

$$(sY - 1) - Y = \frac{1}{s-3}$$

$$(s-1)Y = 1 + \frac{1}{s-3}$$

$$Y = \frac{1}{s-1} + \frac{1}{(s-3)(s-1)}$$

make partial fractions decomp; $= \dfrac{1/2}{s-1} + \dfrac{1/2}{s-3}$

$$\therefore \boxed{y = \tfrac{1}{2}e^t + \tfrac{1}{2}e^{3t}}$$

b) $y'' - 3y' + 2y = 0, \quad y(0) = 1, y'(0) = 1$

$$(s^2Y - s - 1) - 3(sY - 1) + 2Y = 0$$

$$\therefore (s^2 - 3s + 2)Y = s - 2$$

$$Y = \frac{1}{s-1}$$

$$\therefore \boxed{y = e^t}$$

c) $y'' + 4y = \sin t, \quad y(0) = 1, y'(0) = 0$

$$(s^2 Y - s) + 4Y = \frac{1}{s^2+1}$$

$$\therefore Y = \frac{1}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4} \quad (*)$$

Apply partial fractions ↑; treat s^2 as a single variable: i.e.,

$$\frac{1}{(u+1)(u+4)} = \frac{1/3}{u+1} - \frac{1/3}{u+4}; \quad \text{now put } u = s^2.$$

$$Y = \frac{1/3}{s^2+1} - \frac{1/3}{s^2+4} + \frac{s}{s^2+4}$$

$$\therefore \boxed{y = \tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t + \cos 2t}$$

(*) Note that it's easier not to combine terms at this point

d) $y'' - 2y' + 2y = 2e^t, \quad y(0) = 0, \ y'(0) = 1$

$$(s^2 Y - 1) - 2sY + 2Y = \frac{2}{s-1}$$

$$\therefore (s^2 - 2s + 2)Y = \frac{2}{s-1} + 1 = \frac{s+1}{s-1}$$

$$Y = \frac{s+1}{(s^2-2s+2)(s-1)}$$

By partial fractions:

$$Y = \frac{2}{s-1} + \frac{3-2s}{s^2-2s+2}; \quad \text{complete the square:}$$

$$= \frac{2}{s-1} - \frac{2(s-1)}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

(note how we write the 2nd term as an expression in $s-1$; the last term is what's left over.)

$$\therefore y = 2e^t - 2e^t\cos t + e^t \sin t$$

e) $y'' - 2y' + y = e^t, \quad y(0) = 1, \ y'(0) = 0.$

$$s^2 Y - s - 2(sY - 1) + Y = \frac{1}{s-1}$$

$$\underbrace{(s^2 - 2s + 1)}_{(s-1)^2} Y = \frac{1}{s-1} + s - 2$$

$$= \frac{1}{s-1} + (s-1) - 1$$

$$\therefore Y = \frac{1}{(s-1)^3} + \frac{1}{(s-1)} - \frac{1}{(s-1)^2}$$

$$\therefore \boxed{y = \tfrac{t^2}{2}e^t + e^t - te^t}$$

3B-2

12. $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$

Integ. by parts: $= e^{-st}f(t)\Big]_0^\infty - \int_0^\infty -se^{-st}f(t)\,dt$

see below ↓ (since $f(t)$ is of exp. order) $\Rightarrow 0 - f(0) + s\int_0^\infty e^{-st}f(t)\,dt$

$$\therefore \mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}.$$

Assumes:

$f'(t)$ piecewise continuous and exponential order (so $\int_0^\infty e^{-st}f'(t)\,dt$ exists) (i.e., $|f'(t)| < Ke^{at}$ if t is large).

$f(t)$ of exponential order, so $\mathcal{L}\{f\}$ exists. (It's continuous, since $f'(t)$ exists.)

3B-3

These use the formula:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \underset{=\mathcal{L}\{f(t)\}}{F(s)}$$

a)

$$\mathcal{L}\{t\cos bt\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)$$
$$= \frac{b^2-s^2}{(b^2+s^2)^2}$$

b) $\mathcal{L}\{t^n e^{kt}\}$: by the exp-shift rule,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$
$$\therefore \mathcal{L}\{t^n e^{kt}\} = \frac{n!}{(s-k)^{n+1}}.$$

By the above formula,

$$\mathcal{L}\{t^n e^{kt}\} = (-1)^n \frac{d^n}{ds^n}(s-k)^{-1}$$
$$= (-1)^n \cdot (-1)(-2)\cdots(-n)(s-k)^{-(n+1)}$$
$$= \frac{n!}{(s-k)^{n+1}}, \text{ as before.}$$

c) $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$$\mathcal{L}\{t\sin t\} = \frac{2s}{(s^2+1)^2} \quad \text{by the above fmla.}$$

$$\therefore \mathcal{L}\{te^{at}\sin t\} = \frac{2(s-a)}{((s-a)^2+1)^2}.$$

3B-4

a) $\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{t\sin t}{2}$, as in 13 (c) above

b) $\frac{1}{(s^2+1)^2}$ suggests some combination of $\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$ and $\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$

$$\mathcal{L}\{t\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{1}{s^2+1} - \underset{\text{what we want}}{\frac{2}{(s^2+1)^2}}$$
$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \longrightarrow$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}[\sin t - t\cos t]$$

3B-5

a) $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt$$
$$= F(s-a),$$

since $F(s) = \int_0^\infty e^{-st}f(t)\,dt$.

b) $F(s) = \int_0^\infty e^{-st}f(t)\,dt$

Differentiating under the integral sign, with respect to s: ⊛

$$F'(s) = \int_0^\infty -te^{-st}f(t)\,dt,$$

since t is a constant with respect to the differentiation;

$$= \mathcal{L}\{-tf(t)\}$$
$$= -\mathcal{L}\{tf(t)\}.$$

[⊛ this is legal if $f(t)$ is continuous and of exponential order].

3B-6

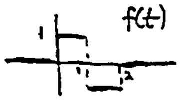
$$y'' + ty = 0, \quad y(0)=1, \; y'(0)=0$$

Take the Laplace transform:

$$(s^2 Y - s) - \frac{d}{ds}Y = 0$$
$$\frac{dY}{ds} - s^2 Y = -s,$$

(which is first order, linear).

3C-1

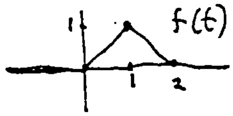
a) 

Using $u(t)$: $f(t) = u(t) - 2u(t-1) + u(t-2)$

$\therefore F(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1}{s}(1-e^{-s})^2$

Directly:

$F(s) = \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt = \frac{1}{s}(1-e^{-s})^2$ (by straight calc.)

b) 

Using $u(t)$: $f(t) = t\,u(t) - u(t-1)\cdot 2(t-1) + u(t-2)(t-2)$

$\therefore F(s) = \frac{1}{s^2}(1-2e^{-s}+e^{-2s})$ (*)

Directly:

$F(s) = \int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt$ [integrate each $\int$ by parts]

$= \frac{te^{-st}}{-s}\Big]_0^1 - \left[\frac{e^{-st}}{(-s)^2}\right]_0^1 + (2-t)\frac{e^{-st}}{-s}\Big]_1^2 - \frac{-e^{-st}}{(-s)^2}\Big]_1^2$ (which agrees with * after canceling terms)

c) 

$|\sin t| = (-1)^n \sin t,\quad n\pi \le t \le (n+1)\pi.$

This can be done directly,
(adding up the integrals over even + odd intervals):

$F(s) = \int_0^\infty |\sin t| e^{-st}dt = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} (-1)^n \sin t \cdot e^{-st}dt$

Change variable: $u = t - n\pi$

$= \sum_{n=0}^\infty \int_0^\pi (-1)^n \sin(u+n\pi)e^{-s(u+n\pi)}du$

$\sin(u+n\pi) = (-1)^n \sin u$; $e^{-sn\pi}$ is a "constant"

$= \sum_{n=0}^\infty e^{-sn\pi} \underbrace{\int_0^\pi \sin u \cdot e^{-su}du}_{\text{call it "K".}}$ Then $K = \frac{1+e^{-s\pi}}{1+s^2}$ (from tables)

$= K \cdot \sum_0^\infty e^{-sn\pi}$; adding up this geometric series gives

$= K \cdot \frac{1}{1-e^{-s\pi}}$ Ans: $\frac{1+e^{-s\pi}}{(1+s^2)(1-e^{-s\pi})}$

3C-2

a) $\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$ (Partial fractions)

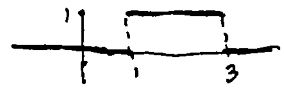
$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = e^{-t} - e^{-2t} = f(t)$

$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+3s+2}\right\} = u(t-1)f(t-1)$

$= u(t-1)\left(e^{1-t} - e^{2(1-t)}\right)$

b) $\mathcal{L}^{-1}\left(\frac{e^{-s}-e^{-3s}}{s}\right) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s}\right)$

$= u(t-1) - u(t-3)$



3C-3

a) $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$

$= \int_0^1 e^{-st}dt + \int_2^3 e^{-st}dt + \int_4^5 e^{-st}dt + \dots$

$= \frac{e^0 - e^{-s}}{s} + \frac{e^{-2s}-e^{-3s}}{s} + \frac{e^{-4s}-e^{-5s}}{s} + \dots$

$= \frac{1}{s}\cdot(e^0 - e^{-s} + e^{-2s} - e^{-3s} + \dots)$ geometric series, whose sum is

$= \frac{1}{s}\cdot\left(\frac{1}{1+e^{-s}}\right)$

b) $f(t) = u(t) - u(t-1) + u(t-2) - \dots$

$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \dots$

$= \frac{1}{s}(e^0 - e^{-s} + e^{-2s} - e^{-3s} \dots)$

$= \frac{1}{s}\cdot\frac{1}{1+e^{-s}}$, as before.

3C-4

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{u(t-\pi) - u(t-2\pi)\} = \frac{e^{-s\pi}-e^{-2s\pi}}{s}$$

The ODE is: $y''+2y'+2y = h(t)$, $\;y(0)=0$, $y'(0)=1$;

Laplace Transform is:

$$(s^2Y-1)+2(sY)+2Y = \frac{e^{-s\pi}-e^{-2s\pi}}{s}$$

$$(s^2+2s+2)Y = 1+\frac{e^{-s\pi}-e^{-2s\pi}}{s}$$

$$Y = \frac{1}{(s+1)^2+1}\left[1+\frac{e^{-s\pi}-e^{-2s\pi}}{s}\right]$$

By partial fractions

$$\frac{1}{(s^2+2s+2)s} = \frac{-s/2-1}{s^2+2s+2}+\frac{1/2}{s} = \frac{-\frac12(s+1)-\frac12}{(s+1)^2+1}+\frac{1/2}{s}$$

$$\therefore\; y = e^{-t}\sin t + \frac12\Big[1-e^{-(t-\pi)}\big(\underbrace{\sin(t-\pi)}_{=-\sin t}+\underbrace{\cos(t-\pi)}_{=-\cos t}\big)\Big]u(t-\pi) - \frac12\Big[1-e^{-(t-2\pi)}\big(\underbrace{\sin(t-2\pi)}_{=\sin t}+\underbrace{\cos(t-2\pi)}_{=\cos t}\big)\Big]u(t-2\pi)$$

$$\therefore\; y = \begin{cases} e^{-t}\sin t, & (0\le t\le \pi) \\ \frac12 + \left(1+\frac{e^{\pi}}{2}\right)e^{-t}\sin t + \frac{e^{\pi}}{2}e^{-t}\cos t, & (\pi\le t\le 2\pi) \\ \left(1+\frac{e^{\pi}}{2}+\frac{e^{2\pi}}{2}\right)e^{-t}\sin t + \left(\frac{e^{\pi}}{2}+\frac{e^{2\pi}}{2}\right)e^{-t}\cos t, & (2\pi\le t)\end{cases}$$

3C-5

$$\mathcal{L}\{\overset{=r(t)}{u(t)\cdot t}\} = \mathcal{L}\{t\} = \frac{1}{s^2}$$

$y''-3y'+2y = r(t)$, $\;y(0)=1$, $y'(0)=0$ gives:

$$(s^2Y-s)-3(sY-1)+2Y = \frac{1}{s^2}$$

$$(s^2-3s+2)Y = s-3+\frac{1}{s^2}$$

$$Y = \frac{s-3}{(s-2)(s-1)} + \frac{1}{s^2(s-2)(s-1)} = \frac{s^3-3s^2+1}{s^2(s-2)(s-1)}$$

cont'd above

3C-5 (cont'd) By partial fractions

$$Y = \frac{1}{s-1} - \frac{3/4}{s-2} + \frac{3/4}{s} + \frac{1/2}{s^2}$$

$$\therefore\; y = e^{t} - \frac34 e^{2t} + 3/4 + t/2$$

3D-1

$y''+2y'+y = \delta(t)+u(t-1)$, $\;y(0)=0$, $y'(0)=1$

$$(s^2Y-1)+2sY+Y = 1+\frac{e^{-s}}{s}$$

$$(s^2+2s+1)Y = 2+\frac{e^{-s}}{s};$$ Divide, use part. fractions:

$$Y = \frac{2}{(s+1)^2} + e^{-s}\left[\frac1s - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right]$$

$$y = 2te^{-t} + u(t-1)\left[1-e^{-(t-1)}-(t-1)e^{-(t-1)}\right]$$

$$= 2te^{-t} + u(t-1)\left[1-te^{1-t}\right]$$

$$\therefore\; y(t) = \begin{cases} 2te^{-t}, & 0\le t\le 1 \\ 1+(2-e)te^{-t}, & t\ge 1\end{cases}$$

3D-2

$y''+y = r(t)$, $\;y(0)=0$, $y'(0)=1$

$$r(t) = \begin{cases}1, & 0\le x\le\pi \\ 0 & \text{otherwise}\end{cases} = 1 - u(t-\pi)$$

$$\therefore\; \mathcal{L}\{r(t)\} = \frac1s - \frac{e^{-\pi s}}{s}$$

So

$$(s^2Y-1)+Y = \frac{1-e^{-\pi s}}{s}$$

$$Y = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-\pi s}}{s(s^2+1)}$$

$$\boxed{\frac{1}{s(s^2+1)} = \frac1s - \frac{s}{s^2+1}}$$

$$y = \sin t + 1 - \cos t - \big(1-\underbrace{\cos(t-\pi)}_{[=-\cos t]}\big)u(t-\pi)$$

$$\therefore\; y = \begin{cases} 1+\sin t-\cos t, & 0\le t\le\pi \\ \sin t - 2\cos t, & t\ge\pi\end{cases}$$

$\boxed{3D-3}$

ALSO: $\boxed{\text{SEE BELOW}}$

a) $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$= \sum_{n=0}^{\infty} \int_{nc}^{(n+1)c} e^{-st} f(t)\,dt$$

[breaking $[0,\infty)$ up into the intervals $[nc,(n+1)c]$.

Change variable: $u = t - nc$

$$\int_{nc}^{(n+1)c} e^{-st} f(t)\,dt = \int_0^c e^{-s(u+nc)} f(u)\,du,$$

since $f(u+nc) = f(u)$.

Therefore our sum becomes:

$$F(s) = \sum_{n=0}^{\infty} e^{-snc} \underbrace{\int_0^c e^{-su} f(u)\,du}_{\text{call this } K}$$

$$= K \sum_{n=0}^{\infty} (e^{-sc})^n \quad \leftarrow \text{a geometric series, whose sum is}$$

$$= K \cdot \frac{1}{1-e^{-sc}}$$

$$\therefore\ F(s) = \frac{1}{1-e^{-sc}} \cdot \int_0^c e^{-su} f(u)\,du$$

(FOR A BETTER WAY, SEE NEXT PAGE)

b) For problem 19, $c = 2$

$$\int_0^2 e^{-su} f(u)\,du = \int_0^1 e^{-su}\,du$$

$$= \frac{1-e^{-s}}{s}$$

$$\therefore\ F(s) = \frac{1}{1-e^{-2s}} \cdot \frac{1-e^{-s}}{s}$$

$$= \frac{1}{s\cdot(1+e^{-s})}, \text{ as before.}$$

$\boxed{3D-4}$

a) $$\frac{s}{(s+1)(s^2+4)} = \frac{1}{s+1} \cdot \frac{s}{s^2+4}$$

$$\therefore\ \mathcal{L}^{-1}\left(\frac{\cdots}{\cdots}\right) = e^{-t} * \cos 2t$$

$$= \int_0^t e^{-(t-u)} \cos 2u\,du$$

$$= e^{-t} \int_0^t e^{u} \cos 2u\,du$$

$$= e^{-t}\left[\frac{e^t}{5}(\cos 2t + 2\sin 2t) - \frac{1}{5}\right]$$

$$= \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t - \frac{1}{5}e^{-t}$$

b) $$\frac{1}{(s^2+1)^2} = \frac{1}{s^2+1} \cdot \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \sin t * \sin t$$

$$= \int_0^t \sin(t-u)\cdot \sin u\,du$$

Easiest is to use a trig identity:

$$= \int_0^t \tfrac{1}{2}\left[\cos(t-2u) - \cos t\right] du$$

$$= \frac{\sin t}{2} - \frac{t}{2}\cos t.$$

$\boxed{3D-5}$

a) $f(t) \underset{\mathcal{L}}{\rightsquigarrow} F(s)$, $\delta(t) \underset{\mathcal{L}}{\rightsquigarrow} 1$

$$\mathcal{L}\{\delta * f\} = 1\cdot F(s) = F(s)$$

$$\therefore\ \delta * f(t) = f(t)u(t) = f(t),$$

[THIS IS JUST FORMAL] since $f(t)=0$, $t\le 0$.

b) Using the definition of $*$:

$$\delta * f = \int_0^t \delta(t-u) f(u)\,du$$

$$= \int_{-\infty}^{\infty} \delta(t-u) f(u)\,du$$

(SHADY) $$= f(t)\int_{-\infty}^{\infty} \delta(t-u)\,du \quad \Big\}\ \text{since } \delta(t-u)=0 \text{ except if } u=t$$

$$= f(t)\cdot 1$$

✓

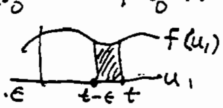
---

c) using the "definition" of $\delta(t)$

$$\delta * f(t) = \int_0^t \delta(t-u) f(u)\,du = \int_0^t \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[u(t-u_1) - u(t-u_1-\epsilon)\right] f(t)\,dt$$

$$= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^t \left[u(t-u_1) - u(t-u_1-\epsilon)\right] f(t)\,dt = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_0^t f(u_1)\,du_1 - \int_0^{t-\epsilon} f(u_1)\,du_1\right]$$

(SHADY!) $$= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_{t-\epsilon}^{t} f(u_1)\,du = f(t),$$ since



area $\approx f(t)\cdot\epsilon$

3D-6

$$(f * g)(t) = \int_0^t f(t-u)\,g(u)\,du$$

let $x = t-u$ (change variable u to the var. x in the integral)
$dx = -du$

limits:
when $u=0$, $x=t$ $\therefore$ integral becomes:
when $u=t$, $x=0$

$$= -\int_t^0 f(x)\,g(t-x)\,dx = \int_0^t g(t-x)\,f(x)\,dx$$

$$= (g*f)(t).$$

3D-7

Taking the Laplace Transform:

$$s^2Y + k^2Y = R(s),$$

where $R(s) = \mathcal{L}\{r(t)\}$.

$$\therefore\ Y = \frac{R(s)}{s^2+k^2} = \frac{1}{s^2+k^2}\cdot R(s)$$

$$\therefore\ y = \frac{1}{k}\sin kt * r(t)$$

$$= \frac{1}{k}\int_0^t \sin k(t-u)\cdot r(u)\,du.$$

3D-8

$$y'' + ay' + by = r(t), \qquad y(0)=0,\ y'(0)=0$$

$$s^2Y + asY + bY = R(s)$$

$$\therefore\ Y = \frac{1}{s^2+as+b}\cdot R(s)$$

$$\therefore\ y = g(t) * r(t), \quad \text{where } g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+as+b}\right\}$$

$$y = \int_0^t g(t-u)\,r(u)\,du.$$

To interpret $g(t)$, consider the ODE-IVP

$$y'' + ay' + by = 0, \qquad y(0)=0,\ y'(0)=1.$$

then $s^2Y - 1 + asY + bY = 0$

so $Y = \dfrac{1}{s^2+as+b}$

and $y = g(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+as+b}\right\}$. Thus $g(t)$ may be interpreted as the soln to this IVP.

(see above for another interp. of g)

3D-8

(continued)

$g(t)$ may also be interpreted as the solution to

$$y'' + ay' + by = \delta(t), \qquad y(0)=0,\ y'(0)=0$$

since this leads to

$$s^2Y + asY + bY = 1$$

or $Y = \dfrac{1}{s^2+as+b}$,

so that $y = g(t)$.

3D-3

a.

we have:

$$u(t-c)f(t-c) + f_0(t) = u(t)f(t),$$

where $f_0(t) = \begin{cases} f(t), & 0 \le t < c \\ 0 & \text{elsewhere} \end{cases}$.

$\therefore$ taking LT's:

$$e^{-cs}F(s) + \int_0^c e^{-st}f(t)\,dt = F(s).$$

Solve for $F(s)$:

$$F(s) = \frac{1}{1-e^{-cs}}\int_0^c e^{-st}f(t)\,dt.$$

$\boxed{4A-1}$ Product is $\begin{bmatrix}1&0&0\\3&-2&-6\end{bmatrix}$

$\boxed{4A-2}$ $AB=\begin{bmatrix}4&1\\-2&-4\end{bmatrix}$ $BA=\begin{bmatrix}-3&1\\5&3\end{bmatrix}$

$\boxed{4A-3}$ $A^{-1}=\frac{1}{-2}\cdot\begin{bmatrix}2&-2\\-3&2\end{bmatrix}$ by the formula

(since $|A|=-2$) $=\begin{bmatrix}-1&1\\3/2&-1\end{bmatrix}$

check: $\begin{bmatrix}2&2\\3&2\end{bmatrix}\begin{bmatrix}-1&1\\3/2&-1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\boxed{4A-4}$ $\frac{1}{|A|}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}\cdot\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$=\frac{1}{|A|}\cdot\begin{bmatrix}ad-bc&0\\0&ad-bc\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

(similarly, $\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}=\begin{bmatrix}|A|&0\\0&|A|\end{bmatrix}$)

$\boxed{4A-5}$ $\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}0&1\\1&1\end{bmatrix}=\begin{bmatrix}1&2\\1&2\end{bmatrix}=A^2$

$\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}1&2\\1&2\end{bmatrix}=\begin{bmatrix}1&2\\2&4\end{bmatrix}$

$\boxed{4A-6}$ Using determinantal criterion for lin. dependence,

we want $0=\begin{vmatrix}1&2&c\\-1&0&1\\2&3&0\end{vmatrix}=4-3c-3$

$\therefore -3c+1=0$

$c=1/3$

Adding: $(1\ \ 2\ \ \overset{1/3}{\overset{\shortparallel}{c}})\times 3$
$-(-1\ \ 0\ \ 1)$
$-(2\ \ 3\ \ 0)\times 2$
$(0\ \ 0\ \ 0)$

$\boxed{4B-1}$ a) $x''+5x'+tx^2=0 \rightarrow$ $x'=y$, $y'=-tx^2-5y$

b) $y''-x^2y'+(1-x^2)y=\sin x$

$\rightarrow$ $y'=z$
$z'=(x^2-1)y+x^2z+\sin x$

$\boxed{4B-2}$

$y'''+py''+qy'+ry=0$
let $y=y_1$

$y_1'=y_2$
$y_2'=y_3$
$y_3'=-py_3-qy_2-ry_1$

matrix form: $\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}'=\begin{bmatrix}0&1&0\\0&0&1\\-r&-q&-p\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$

$\boxed{4B-3}$

$\begin{cases}x'=x+y\\y'=4x+y\end{cases}$ To eliminate y: $y=x'-x$ from 1st eqn.

$\therefore (x'-x)'=4x+(x'-x)$ 2nd eq.

or $x''-x'=4x+x'-x$

or $\boxed{x''-2x'-3x=0}$

converting to system:
let $x_1=x$

system is $\begin{cases}x_1'=x_2\\x_2'=2x_2+3x_1\end{cases}$ $\rightarrow$ This system is not same as first, but is equivalent to it — just using different dep't variable.

The rel'n between the variables is:

$x_1=x$
$x_2=x+y$

or the other way: $\begin{bmatrix}x=x_1\\y=x_2-x_1\end{bmatrix}$

If you make this change of vars. the 1st system turns into the second.

$\boxed{4B-4}$

$\begin{bmatrix}1\\1\end{bmatrix}e^{3t}$ and $\begin{bmatrix}1\\2\end{bmatrix}e^{2t}$ solve $\vec{x}'=\begin{bmatrix}4&-1\\2&1\end{bmatrix}\vec{x}$:

a) vectorially: $\frac{d}{dt}\begin{bmatrix}1\\1\end{bmatrix}e^{3t}=3\begin{bmatrix}1\\1\end{bmatrix}e^{3t}$

$\begin{bmatrix}4&-1\\2&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}e^{3t}=\begin{bmatrix}3\\3\end{bmatrix}e^{3t}$

these are equal. Other goes same way.

components: $x=e^{3t}$, $y=e^{3t}$ solves $\begin{cases}x'=4x-y\\y'=2x+y\end{cases}$: just plug in + check it.

b) linearly indep't: $\begin{vmatrix}e^{3t}&e^{2t}\\e^{3t}&2e^{2t}\end{vmatrix}=e^{5t}\neq 0$.

c) gen soln: $c_1\begin{bmatrix}1\\1\end{bmatrix}e^{3t}+c_2\begin{bmatrix}1\\2\end{bmatrix}e^{2t}$ or $\begin{bmatrix}c_1e^{3t}+c_2e^{2t}\\c_1e^{3t}+2c_2e^{2t}\end{bmatrix}$

which is same as: $x=c_1e^{3t}+c_2e^{2t}$
$y=c_1e^{3t}+2c_2e^{2t}$.

$\boxed{4B\text{-}5}$

$\begin{bmatrix}1\\1\end{bmatrix}e^{4t}$ and $\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}$ solve $\begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}1&3\\3&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$.

(do it same as #3 above). Linear indep: $\begin{vmatrix} e^{4t} & e^{-2t}\\ e^{4t} & -e^{-2t}\end{vmatrix} = -2e^{2t}$

IVP: $\vec{x}(0) = \begin{bmatrix}5\\1\end{bmatrix}$, gives: (since $e^{4t}, e^{-2t} = 1$ when $t=0$)

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix} \quad \therefore \quad \begin{matrix}c_1+c_2=5\\ c_1-c_2=1\end{matrix} \quad \therefore \quad \begin{matrix}c_1=3\\ c_2=2\end{matrix}$$

soln: $3\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + 2\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t} = \vec{x}$.

$\boxed{4B\text{-}6}$

$\begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$. or $\begin{matrix}x' = x+y\\ y' = y\end{matrix}$

a) From second eqn, $y = c_1e^t$

$\therefore \; x' - x = c_1e^t$ soln: $x = c_2e^t + c_1te^t$, $y = c_1e^t$

b) Here we eliminate y instead:

$y = x' - x$ (1st eqn.) $\therefore \; (x'-x)' = x'-x$

$x'' - 2x' + x = 0 \quad \therefore \; x = c_1e^t + c_2te^t$

$(m-1)^2 = 0 \quad \therefore \; y = c_2e^t$ (since $y = x'-x$)

Same as before (just switch c_1 & c_2).

$\boxed{4B\text{-}7}$

$x' = -ax$ (straight decay)

$y' = \underbrace{-by}_{\text{decay rate}} + \underbrace{ax}_{\text{rate at which decay of } x \text{ produces } y}$

matrix: $\begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}-a&0\\a&-b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Soln by elimination: eliminate x: $x = \frac{1}{a}y' + \frac{b}{a}y$

subst. into 1st eqn, get

$$\frac{1}{a}y'' + \frac{b}{a}y' = -y' - by$$

$$y'' + (b+a)y' + bay = 0 \qquad m^2 + (a+b)m + ab = 0$$

$$m = -a,\; m = -b$$

$\therefore \begin{cases} y = c_1e^{-at} + c_2e^{-bt} \\ x = c_1\left(-1+\frac{b}{a}\right)e^{-at} \end{cases} \leftarrow \left[\; x = \frac{1}{a}(y'+by) = \frac{1}{a}\left(-ac_1e^{-at} - bc_2e^{-bt} + bc_1e^{-at} + bc_2e^{-bt}\right) \right.$

[NOTE: having found y, you can't just say $x' = -ax$, $\therefore \; x = c_3e^{-at}$, since c_3 is not arbitrary — x must also satisfy the 2nd eqn !!]

4C-1 a)

a) $\vec{x}' = \begin{bmatrix}-3 & 4\\-2 & 3\end{bmatrix}\vec{x}$ eigenvalues: $\begin{vmatrix}-3-m & 4\\-2 & 3-m\end{vmatrix} = 0$

$\therefore -(3+m)(3-m) + 8 = 0$

$m^2 - 1 = 0 \qquad m = \pm 1$

Eigenvectors:

if $m = 1$,

$\begin{cases}-4\alpha_1 + 4\alpha_2 = 0\\ -2\alpha_1 + 2\alpha_2 = 0\end{cases}$

$\therefore \begin{bmatrix}1\\1\end{bmatrix}$ and its mult. are soln: eigenvector.

if $m = -1$:

$-2\alpha_1 + 4\alpha_2 = 0$
$-2\alpha_1 + 4\alpha_2 = 0$

soln: $\begin{bmatrix}2\\1\end{bmatrix}$ eigenvector.

$\therefore \vec{x} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{t} + c_2\begin{bmatrix}2\\1\end{bmatrix}e^{-t}$

[NOTE: can also write down char. polyn. using: $m^2 - (a_1+b_2)m + \det A = 0$.

b) $\vec{x}' = \begin{bmatrix}4 & -3\\8 & -6\end{bmatrix}\vec{x}$ $\begin{vmatrix}4-m & -3\\8 & -6-m\end{vmatrix} = 0$ gives

$m^2 + 2m = 0$ $m = -2$, $m = 0$.

$m = 0$:

$4\alpha_1 - 3\alpha_2 = 0$
$8\alpha_1 - 6\alpha_2 = 0$

$\begin{bmatrix}3\\4\end{bmatrix}$ eigenvector.

$m = -2$

$6\alpha_1 - 3\alpha_2 = 0$
$8\alpha_1 - 4\alpha_2 = 0$

$\begin{bmatrix}1\\2\end{bmatrix}$ eigenv.

$\therefore \vec{x} = c_1\begin{bmatrix}3\\4\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix}e^{-2t}$

4C-1

c) eigenvalues: $\begin{vmatrix}1-m & -1 & 0\\1 & 2-m & 1\\-2 & 1 & -1-m\end{vmatrix} = -(1-m)(2-m)(1+m) + 2 + (m-1) - 1 - m = 0$

$\boxed{(1-m)(2-m)(1+m) = 0}$

eigenvalues $\therefore$ are $m = 1,\ m = 2,\ m = -1$

m = 1

$0\alpha_1 - \alpha_2 = 0$
$\alpha_1 + \alpha_2 + \alpha_3 = 0$
$-2\alpha_1 + \alpha_2 - 2\alpha_3 = 0$

$\therefore \alpha_2 = 0 \quad \alpha_3 = -\alpha_1$

soln: $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ eigenvector

m = 2

$-\alpha_1 - \alpha_2 = 0$
$\alpha_1 + \alpha_3 = 0$
$-2\alpha_1 + \alpha_2 - 3\alpha_3 = 0$

$\therefore \alpha_2 = -\alpha_1$
$\alpha_3 = -\alpha_1$

$\begin{bmatrix}1\\-1\\-1\end{bmatrix}$ eigenvector

m = -1

$+2\alpha_1 - \alpha_2 = 0$
$\alpha_1 + 3\alpha_2 + \alpha_3 = 0$
$-2\alpha_1 + \alpha_2 = 0$

$\alpha_2 = +2\alpha_1$
$\alpha_3 = -7\alpha_1$

$\begin{bmatrix}1\\+2\\-7\end{bmatrix}$ eigenvector.

soln:

$\vec{x} = c_1\begin{bmatrix}1\\0\\-1\end{bmatrix}e^{t} + c_2\begin{bmatrix}1\\-1\\-1\end{bmatrix}e^{2t} + c_3\begin{bmatrix}1\\2\\-7\end{bmatrix}e^{-t}$

4C-2

Proof #1: $\therefore$ 0 is an eigenvalue if and only if

$A\vec{\alpha} = 0\vec{\alpha}$ has a nontriv. soln for $\vec{\alpha}$

$\Leftrightarrow A\vec{\alpha} = \vec{0}$ " " " " "

$\Leftrightarrow \det A = 0$ (see notes p.2 (5)).

Proof #2: The characteristic equation is

$\det(A - mI) = 0.$

If $m = 0$ is a root, this says (substituting $m=0$)

$\det(A) = 0$

4C-3

$\begin{vmatrix}a-m & * & *\\0 & b-m & *\\0 & 0 & c-m\end{vmatrix} = (a-m)(b-m)(c-m) = 0$

$\therefore m = a, b, c$ are eigenvalues

This always holds: using a Laplace expansion by the minors of first column:

$\begin{vmatrix}a_1-m & * & \cdots & *\\0 & a_2-m & & \vdots\\ \vdots & & \ddots & \\0 & \cdots & & a_k-m\end{vmatrix} = (a_1-m)\begin{vmatrix}a_2-m & * & \cdots & *\\0 & \ddots & & \\0 & \cdots & 0 & a_k-m\end{vmatrix}$

$= (a_1-m)(a_2-m)\cdots(a_k-m)$

by mathematical induction on the size of matrix (i.e., k)

$\therefore$ eigenvalues are the roots:

$m = a_1, a_2, \ldots, a_k$ = diagonal elements.

4C-4

By hypothesis, $A\vec{\alpha} = m\vec{\alpha}$.

Multiply both sides by A:

$A A\vec{\alpha} = m A\vec{\alpha} = m(m\vec{\alpha})$

$\therefore A^2\vec{\alpha} = m^2\vec{\alpha}$ so $\vec{\alpha}$ is eigenv. of A^2, assoc. to eigenvalue m^2.

[Continuing, one sees that $A^k\vec{\alpha} = m^k\vec{\alpha}$ — the eigenvalues of A^k are the k^{th} powers of the eigenvalues of A].

4C-5

$\vec{x}' = \begin{bmatrix}-a & 0\\a & -b\end{bmatrix}\vec{x}$ Eigenvalues: $-a, -b$ (by previous problem, or directly)

$m = -a$

$a\alpha_1 + (b+a)\alpha_2 = 0$

soln: $\begin{bmatrix}\frac{a-b}{a}\\1\end{bmatrix}$ eigenvector

$m = -b$

$(-a-b)\alpha_1 = 0$
$a\alpha_1 + 0\alpha_2 = 0$

$\therefore \begin{bmatrix}0\\1\end{bmatrix}$ soln. eigenvector.

$\therefore \vec{x} = c_1\begin{bmatrix}\frac{a-b}{a}\\1\end{bmatrix}e^{-at} + c_2\begin{bmatrix}0\\1\end{bmatrix}e^{-bt}$

when written out with components, this is identical to our earlier solution...

4C-6

$S' = S - aS + bJ$ ← rabbits coming in from Jones

(natural growth rate; fraction that leave for Jones)

$J' = J - bJ + aS$

$\therefore S' = (1-a)S + bJ$
$J' = aS + (1-b)J.$

if $a = b = \frac{1}{2}$,

$$\begin{bmatrix} S \\ J \end{bmatrix}' = \begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}\begin{bmatrix} S \\ J \end{bmatrix}$$

Eigenvalues: $\begin{vmatrix} \frac12 - m & \frac12 \\ \frac12 & \frac12 - m \end{vmatrix} = m^2 - m = 0$

$m = 0,\ m = 1$ eigenvalues:

$\underline{m=0}$: $\frac12\alpha_1 + \frac12\alpha_2 = 0$: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underline{m=1}$: $-\frac12\alpha_1 + \frac12\alpha_2 = 0$, $\frac12\alpha_1 - \frac12\alpha_2 = 0$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \vec{x} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^t$

IVP: $c_1 + c_2 = 20$, $-c_1 + c_2 = 10$ (since $t=0$) $c_2 = 15$, $c_1 = 5$

Soln: $\therefore \begin{bmatrix} S \\ J \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 15\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^t$

Nonoscillation: Show this char. polynomial never has complex roots:
(Sorry this part should come later!!) $m^2 + (a+b-2)m + (1-a-b) = 0$

4C-7

From the "picture":

$\frac14(x_1' - x_1) = x_2$
$x_2' - x_2 = x_1$

$\therefore \begin{cases} x_1' = x_1 + 4x_2 \\ x_2' = x_1 + x_2 \end{cases}$

solving:

eigenvalues: $\begin{vmatrix} 1-m & 4 \\ 1 & 1-m \end{vmatrix} = (1-m)^2 - 4 = 0$ $\therefore 1 - m = \pm 2$ $\therefore m = 3, -1$

$\underline{m=3}$: $-2\alpha_1 + 4\alpha_2 = 0$ soln: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\underline{m=-1}$: $2\alpha_1 + 4\alpha_2 = 0$ soln: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore \vec{x} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-t}$

Initial condition: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\quad 2c_1 - 2c_2 = 1$, $c_1 + c_2 = 0$

Soln: $\vec{x} = \frac14\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3t} - \frac14\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-t}$ $\quad \therefore c_1 = \frac14 \quad c_2 = -\frac14$

4D-1

26. Characteristic equation: $m^2 + 4$; $m = 2i$ eigenvalue

Corresponding eigenvector:

$\begin{cases} (1-2i)\alpha_1 - 5\alpha_2 = 0 \\ \alpha_1 + (-1-2i)\alpha_2 = 0 \end{cases}$ these are multiples of each other

Possible choices for eigenvector: $\begin{bmatrix} 5 \\ 1-2i \end{bmatrix}$ or $\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$

The second choice gives as the soln $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}i\right)(\cos 2t + i\sin 2t)$

with real part $\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t$, imag. part: $\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = c_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t\right) + c_2\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right)$

$\therefore x = (c_1 + 2c_2)\cos 2t + (c_2 - 2c_1)\sin 2t$
$y = c_1\cos 2t + c_2\sin 2t$

The other choice of eigenvector leads to $x = 5a_1\cos 2t + 5a_2\sin 2t$
$y = (a_1 - 2a_2)\cos 2t + (2a_1 + a_2)\sin 2t$

(an equivalent solution).

4D-2

Characteristic equation is $m^2 - 6m + 25 = 0$

$\therefore m = 3 \pm 4i$, by quadratic formula

Using $3+4i$ as complex eigenvalue, corresponding eigenvector comes from equation $(3-m)\alpha_1 + 4\alpha_2 = 0$ $\therefore \vec{\alpha} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Corresponding solution is formed from real + imag. parts of $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}i\right)e^{3t}(\cos 4t + i\sin 4t)$, giving

$x = e^{3t}(c_1\cos 4t + c_2\sin 4t)$
$y = e^{3t}(c_1\sin 4t - c_2\cos 4t)$

4D-3

Char. equation is $(m-2)^2(m+1) = 0$

eigenvalue -1 gives eq'ns $3\alpha_1 + 3\alpha_2 + 3\alpha_3 = 0$, $-3\alpha_3 = 0$, $3\alpha_3 = 0$ $\quad \vec{\alpha} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

repeated eigenvalue 2 gives eq'ns $\begin{cases} 3\alpha_2 + 3\alpha_3 = 0 \\ -3\alpha_2 - 3\alpha_3 = 0 \\ 0 = 0 \end{cases}$ which have 2 lin. indep't sol'ns, namely $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ — thus 2 is a complete eigenvalue

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}e^{2t} + c_3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{2t}$

or $x = c_1e^{-t} + c_2e^{+2t}$
$y = -c_1e^{-t} + c_3e^{+2t}$
$z = -c_3e^{+2t}$

4D-4

a) $A_1' = (A_2 - A_1) + (A_3 - A_1)$ $\quad A_2 - A_1 = x_2 - x_1$, $A_1' = x_1'$

(rate of change of salt in cell 1; rate of diffusion from $2 \to 1$; rate of diffusion from $3 \to 1$)

$\therefore x_1' = x_2 - x_1 + x_3 - x_1 = -2x_1 + x_2 + x_3$

Similarly, $x_2' = x_1 - 2x_2 + x_3$
$x_3' = x_1 + x_2 - 2x_3$

b) Characteristic eq'n is $m^3 + 6m^2 + 9m = 0$
$= m(m+3)^2$

Eigenvalue 0 gives eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, normal mode is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
($e^{0t} = 1$, notice)

Eigenvalue -3 gives for eigenvector equations just
$\alpha_1 + \alpha_2 + \alpha_3 = 0$ (all 3 eq'ns are same)

This is a complete eigenvalue: it has multiplicity 2 and 2 lin indep sol'ns: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

normal modes: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-3t}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-3t}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: all 3 cells have same am't of salt — stays that way

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-3t}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-3t}$ — one cell is at standard concentration A_0 + stays that way; other two cells are equally above + below A_0 at start; salt flows from one to other until "at ∞" they all have A_0 salt in them.

$\boxed{4E-1}$ $\vec{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ Solving to get eigenvectors:

$\lambda^2 - 3\lambda - 10 = 0$ $\lambda = 5$ gives $\binom{2}{1}$

$(\lambda-5)(\lambda+2) = 0$ $\lambda = -2$ gives $\binom{1}{-3}$

[eq'ns are: $-a_1 + 2a_2 = 0$ and $6a_1 + 2a_2 = 0$, respectively]

$\therefore$ coörd. change is:

$$\binom{x}{y} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\binom{u}{v}$$

[can multiply each column by a constant and it's still OK]

Check it decouples: $x = 2u + v$
$y = u - 3v$

$\therefore$ substituting into system:

$$2u' + v' = 4(2u+v) + 2(u-3v) = 10 - 2v$$

$$u' - 3v' = 5u + 6v, \text{ similarly}$$

Multiply {top eqn by 3 and add; bot. eqn by 2 and subtract}

and you get $u' = 5u$ decoupled!
$v' = -2v$

$\boxed{4E-2}$ $\vec{x}' = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ use the eigenvectors given in 4D-4:

variable change matrix is:

$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$; $\vec{x} = E\vec{u}$ is the change of vars.

(cols are eigenvectors)

To check, use matrices: $\vec{u} = E^{-1}\vec{x}$ + subst. into system

$$\boxed{\vec{u}' = E^{-1}AE\,\vec{u}}$$

is the new system. Calculating:

$$\vec{u}' = \underbrace{\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix}}_{E^{-1}}\underbrace{\begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & -3 \\ 0 & 3 & 3 \end{bmatrix}}_{A\cdot E}\vec{u}$$

$$\vec{u}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}\vec{u}$$

So system is decoupled: $u_1' = 0$
$u_2' = -3u_2$
$u_3' = -3u_3$

$\boxed{4F-1}$ $x'' + px' + qx = 0$

a) $x' = y$ $\qquad \vec{x} = \binom{x}{y} = \binom{x}{x'}$

b) $y' = -qx - py$

$\therefore$ Wronskian of two solutions $\vec{x}_1$ and $\vec{x}_2$ is $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$, or $\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$, since $y_i = x_i'$,

which is the usual Wronskian of x_1 and x_2.

$\boxed{4F-2}$

a) Neither is a constant multiple of the other.

b) $W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2$

c) Since $W = 0$ when $t = 0$, $\vec{x}_1$ and $\vec{x}_2$ cannot be solutions of $\vec{x}' = A(t)\vec{x}$, where the entries of $A(t)$ are continuous.

d) To find $A(t)$ explicitly, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\boxed{\vec{x}' = A\vec{x}}$

Then since $\begin{bmatrix} t \\ 1 \end{bmatrix}$ is soln, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} t \\ 1 \end{bmatrix}$ or: $1 = at + b$, $0 = ct + d$

Since $\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ is soln, $\begin{bmatrix} 2t \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ or: $2t = at^2 + b2t$, $2 = ct^2 + d2t$

These are 4 equations for a, b, c, d. Solving:
$a = 0$, $b = 1$, $c = -2/t^2$ $d = 2/t$ so not contin. at $t=0$

$\boxed{4F-3}$

a) $\begin{vmatrix} \alpha_1 e^{m_1 t} & \alpha_2 e^{m_2 t} \\ \beta_1 e^{m_1 t} & \beta_2 e^{m_2 t} \end{vmatrix} = e^{(m_1+m_2)t}\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$

$\vec{\alpha}_1 = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$ $\qquad = 0 \Leftrightarrow \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0$

$\vec{\alpha}_2 = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$ $\qquad \Leftrightarrow \vec{\alpha}_1, \vec{\alpha}_2$ are lin. dep't

b) Suppose $c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 = \vec{0}$ Multiply by A:

$$c_1 A\vec{\alpha}_1 + c_2 A\vec{\alpha}_2 = A\vec{0}$$

$$\therefore c_1 m_1\vec{\alpha}_1 + c_2 m_2\vec{\alpha}_2 = \vec{0}$$

Multiply top eq'n by m_1, subtract from 3rd eq'n, get

$$c_2(m_2 - m_1)\vec{\alpha}_2 = \vec{0}$$

But $m_1 \neq m_2$, $\vec{\alpha}_2 \neq \vec{0}$ (since it's an eigenvector)

$\therefore$ $c_2 = 0$

$\therefore$ also $c_1 = 0$ (since $c_1\vec{\alpha}_1 = \vec{0}$ & $\vec{\alpha}_1 \neq \vec{0}$)

4F-4

If $\vec{x}'(0) = \vec{0}$, then since $\vec{x}' = A\vec{x}$, it follows that $A\vec{x}(0) = \vec{0}$, also.

Since A is nonsingular, we can multiply by A^{-1}, + get $\vec{x}(0) = \vec{0}$.

$\therefore$ by the uniqueness theorem, $\vec{x}(t) = \vec{0}$ for all t.

Hypotheses needed: A can be a function of t (with continuous entries); require only that at time $t=0$, $A(0)$ is nonsingular — then above reasoning still applies.

4G-1

a) Gen soln is: $\vec{x} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{3t} + c_2\begin{bmatrix}1\\2\end{bmatrix}e^{2t}$

$\vec{x}_2(0) = \begin{bmatrix}0\\1\end{bmatrix}$: $\begin{bmatrix}0\\1\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix}$

or: $c_1 + c_2 = 0$, $c_1 + 2c_2 = 1$ $\quad\therefore c_2 = 1,\ c_1 = -1$

$\therefore \vec{x}_2 = -\begin{bmatrix}1\\1\end{bmatrix}e^{3t} + \begin{bmatrix}1\\2\end{bmatrix}e^{2t}$: solves $\vec{x}_2(0) = \begin{bmatrix}0\\1\end{bmatrix}$

b) $\vec{x}_1 = 2\begin{bmatrix}1\\1\end{bmatrix}e^{3t} - \begin{bmatrix}1\\2\end{bmatrix}e^{2t}$ solves $\vec{x}_1(0) = \begin{bmatrix}1\\0\end{bmatrix}$

$\therefore$ soln to $\vec{x}(0) = \begin{bmatrix}a\\b\end{bmatrix}$ is: $\boxed{a\vec{x}_1 + b\vec{x}_2}$

(since $\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$)

$$a\vec{x}_1 + b\vec{x}_2 = (2a-b)\begin{bmatrix}1\\1\end{bmatrix}e^{3t} + (b-a)\begin{bmatrix}1\\2\end{bmatrix}e^{2t}$$

4G-2

a) $\vec{x}' = \begin{bmatrix}5 & -1\\3 & 1\end{bmatrix}\vec{x}$ $\quad$ Eigenvalues: $\begin{vmatrix}5-m & -1\\3 & 1-m\end{vmatrix} = m^2 - 6m + 8 = 0$, $m = 4, 2$

$m=4$: $\alpha_1 - \alpha_2 = 0$; $\begin{bmatrix}1\\1\end{bmatrix}$ eigenvector, $\begin{bmatrix}1\\1\end{bmatrix}e^{4t}$ sol'n

$m=2$: $3\alpha_1 - \alpha_2 = 0$; $\begin{bmatrix}1\\3\end{bmatrix}$ eigvector, $\begin{bmatrix}1\\3\end{bmatrix}e^{2t}$ solution.

Fund. matrix: $\begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix} = F(t)$ $\quad F(0) = \begin{bmatrix}1 & 1\\3 & 1\end{bmatrix}$ $\quad F(0)^{-1} = -\frac{1}{2}\begin{bmatrix}1 & -1\\-3 & 1\end{bmatrix}$

Soln to IVP:

$$= F(t)F(0)^{-1}\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\\3/2 & -1/2\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}$$

[see p.12, (34)]

$$= \begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix}\begin{bmatrix}-3/2\\7/2\end{bmatrix} = -\frac{3}{2}\begin{bmatrix}e^{2t}\\3e^{2t}\end{bmatrix} + \frac{7}{2}\begin{bmatrix}e^{4t}\\e^{4t}\end{bmatrix}$$

b) Normalized fund. mx: $\begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\\3/2 & -1/2\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}e^{2t}+\frac{3}{2}e^{4t} & \frac{1}{2}e^{2t}-\frac{1}{2}e^{4t}\\-\frac{3}{2}e^{2t}+\frac{3}{2}e^{4t} & \frac{3}{2}e^{2t}-\frac{1}{2}e^{4t}\end{bmatrix}$

Multiply this on right by $\begin{bmatrix}2\\-1\end{bmatrix}$ to get same answer.

4H-1

$A = \begin{bmatrix}a & 0\\0 & b\end{bmatrix}$, $A^2 = \begin{bmatrix}a^2 & 0\\0 & b^2\end{bmatrix}$, ... $A^n = \begin{bmatrix}a^n & 0\\0 & b^n\end{bmatrix}$ by rules for mx. mult.

$$\therefore e^{At} = I + At + A^2\frac{t^2}{2!} + \dots$$
$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}at & 0\\0 & bt\end{bmatrix} + \begin{bmatrix}\frac{a^2t^2}{2!} & 0\\0 & \frac{b^2t^2}{2!}\end{bmatrix} + \dots$$
$$= \begin{bmatrix}1 + at + \frac{a^2t^2}{2!} + \dots & 0\\0 & 1 + bt + \frac{b^2t^2}{2!} + \dots\end{bmatrix} = \begin{bmatrix}e^{at} & 0\\0 & e^{bt}\end{bmatrix}$$

$$\vec{x} = e^{At}\vec{x}_0 = \begin{bmatrix}e^{at} & 0\\0 & e^{bt}\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}k_1e^{at}\\k_2e^{bt}\end{bmatrix}$$

verify: $x = k_1e^{at}$, $y = k_2e^{bt}$ is sol'n of: $\begin{cases}x' = ax\\y' = by\end{cases}$ with $x(0) = k_1$, $y(0) = k_2$ $\quad$ obvious!

4H-2

$A = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ $\quad A^2 = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$, $A^3 = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$, $A^4 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

after this it repeats (since $A^4 = I$)

ie, $A^5 = A$, $A^6 = A^2$, etc.

$$e^{At} = \begin{bmatrix}1 - t^2/2! + t^4/4! \cdots & t - t^3/3! + t^5/5! - \cdots\\-t + t^3/3! \cdots & 1 - t^2/2! + t^4/4! \cdots\end{bmatrix}$$
$$= \begin{bmatrix}\cos t & \sin t\\-\sin t & \cos t\end{bmatrix}$$

$$\vec{x} = e^{At}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}\cos t & \sin t\\-\sin t & \cos t\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}k_1\cos t + k_2\sin t\\-k_1\sin t + k_2\cos t\end{bmatrix}$$

This obviously satisfies the system (I.V.P.): $x' = y$, $y' = -x$, $x(0) = k_1$, $y(0) = k_2$.

4H-4

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \dots \quad (*)$$

In general, for matrices B, C, (square),
$$\frac{d}{dt}B(t)C(t) = \frac{dB}{dt}C + B\frac{dC}{dt}$$
$$\therefore \frac{d}{dt}A(t)A(t) = \frac{dA}{dt}\cdot A + A\cdot\frac{dA}{dt}$$
$$\neq 2A\frac{dA}{dt}$$
since above two matrices are not $=$!!

$\therefore$ in general,
$$\frac{d}{dt}A^n(t) \neq nA^{n-1}\frac{dA}{dt},$$
and so you can't differentiate $(*)$ term-by-term to get Ae^{At}.

4H-7

a) $A = \begin{bmatrix}1&0\\2&1\end{bmatrix}$

$$A^2 = \begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}1&0\\2&1\end{bmatrix} = \begin{bmatrix}1&0\\4&1\end{bmatrix}$$

$$A^3 = \begin{bmatrix}1&0\\4&1\end{bmatrix}\begin{bmatrix}1&0\\2&1\end{bmatrix} = \begin{bmatrix}1&0\\6&1\end{bmatrix} \cdots$$

similarly, $A^n = \begin{bmatrix}1&0\\2n&1\end{bmatrix}$

$$\therefore e^{At} = \begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}t&0\\2t&t\end{bmatrix} + \begin{bmatrix}t^2/2!&0\\4t^2/2!&t^2/2!\end{bmatrix} + \cdots$$

$$= \begin{bmatrix}1+t+t^2/2!+\cdots & 0\\ 2t+4t^2/2!+6t^3/3!+\cdots & 1+t+t^2/2!+\cdots\end{bmatrix}$$

But lower-left corner

$$= 2t\left(1+\frac{2t}{2!}+\frac{3t^2}{3!}+\frac{4t^3}{4!}+\cdots\right) = 2te^t$$

$$\therefore e^{At} = \begin{bmatrix}e^t&0\\2te^t&e^t\end{bmatrix} \quad \circledast$$

b) $$e^{\begin{bmatrix}1&0\\2&1\end{bmatrix}t} = e^{\begin{bmatrix}1&0\\0&1\end{bmatrix}t} \cdot e^{\begin{bmatrix}0&0\\2&0\end{bmatrix}t}$$

$$= \begin{bmatrix}e^t&0\\0&e^t\end{bmatrix} \quad \left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&0\\2t&0\end{bmatrix}\right)$$

(see book ex. 1 p. 451) (higher power of mx. are 0)

$$= \begin{bmatrix}e^t&0\\0&e^t\end{bmatrix}\cdot\begin{bmatrix}1&0\\2t&1\end{bmatrix} = \circledast$$

c) Find F by solving the system:

$$\begin{aligned} x' &= x \\ y' &= 2x+y \end{aligned} \Rightarrow \begin{aligned} x &= c_1e^t \\ y'-y &= 2c_1e^t \end{aligned}$$

solving 2nd equation as a linear eqn:

$$(ye^{-t})' = 2c_1$$

$$ye^{-t} = 2c_1t + c_2$$

$$y = c_1\cdot 2te^t + c_2e^t$$

$$\therefore F = \begin{bmatrix}e^t&0\\2te^t&e^t\end{bmatrix} \qquad F(0)^{-1} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

so $$e^{At} = F\cdot F(0)^{-1} = \begin{bmatrix}e^t&0\\2te^t&e^t\end{bmatrix}$$

4I-1

$$\vec{x}' = \begin{bmatrix}1&1\\4&-2\end{bmatrix}\vec{x} + \begin{bmatrix}-5\\-8\end{bmatrix}t + \begin{bmatrix}2\\-8\end{bmatrix}$$

① Solve the reduced equation $\vec{x}' = A\vec{x}$

· char. eqn is $m^2+m-6=0$ roots: $m=-3$
$(m+3)(m-2)=0$ $m=2$

$\underline{m=-3}$: $4a_1+a_2=0$ soln: $\begin{bmatrix}1\\-4\end{bmatrix}e^{-3t}$ $\quad$ $\underline{m=2}$: $-a_1+a_2=0$ soln: $\begin{bmatrix}1\\1\end{bmatrix}e^{2t}$

Fund mx: $\begin{bmatrix}e^{-3t}&e^{2t}\\-4e^{-3t}&e^{2t}\end{bmatrix} = F.$ $\qquad F^{-1} = \begin{bmatrix}e^{2t}&-e^{2t}\\4e^{-3t}&e^{-3t}\end{bmatrix}\frac{1}{5e^{-t}}$

$$v' = F^{-1}\begin{bmatrix}-5t+2\\-8t-8\end{bmatrix} = \begin{bmatrix}\frac{e^{3t}}{5}(-5t+2) - \frac{e^{3t}}{5}(-8t-8)\\ \frac{4}{5}e^{-2t}(-5t+2) + \frac{e^{-2t}}{5}(-8t-8)\end{bmatrix} = \begin{bmatrix}\frac{3e^{3t}}{5}t + 2e^{3t}\\ -\frac{28}{5}e^{-2t}t\end{bmatrix}$$

$$\therefore v = \begin{bmatrix}\frac{te^{3t}}{5}+\frac{3}{5}e^{3t}\\ \frac{14}{5}e^{-2t}t+\frac{7}{5}e^{-2t}\end{bmatrix}$$

$$\vec{x}_p = F\vec{v} = \begin{bmatrix}\frac{t}{5}+\frac{3}{5}+\frac{14}{5}t+\frac{7}{5}\\ -\frac{4t}{5}-\frac{12}{5}+\frac{14}{5}t+\frac{7}{5}\end{bmatrix} = \begin{bmatrix}3t+2\\2t-1\end{bmatrix} \text{ Ans.}$$

4I-2

a) Using the work from above:

$$v' = \frac{1}{5}\begin{bmatrix}e^{3t}&-e^{3t}\\4e^{-2t}&e^{-2t}\end{bmatrix}\begin{bmatrix}e^{-2t}\\-2e^{t}\end{bmatrix} = \frac{1}{5}\begin{bmatrix}e^t+2e^{4t}\\4e^{-4t}-2e^{-t}\end{bmatrix}$$

$$v = \frac{1}{5}\begin{bmatrix}e^t+\frac{e^{4t}}{2}\\-e^{-4t}+2e^{-t}\end{bmatrix} \qquad \vec{x} = Fv = \frac{1}{5}\begin{bmatrix}e^{-2t}+e^t/2-e^{-2t}+2e^t\\-4e^{-2t}-2e^t-e^{-2t}+2e^t\end{bmatrix}$$

$$\therefore \vec{x}_p = \frac{1}{5}\begin{bmatrix}5/2\,e^t\\-5e^{-2t}\end{bmatrix} = \begin{bmatrix}e^t/2\\-e^{-2t}\end{bmatrix}$$

Add to $\vec{x}_p$ the $\vec{x}_h = c_1\begin{bmatrix}1\\-4\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{2t}$ +

$\vec{x}_p = \vec{c}e^{-2t}+\vec{d}e^t$ Substitute in to the equation:

$$-2\vec{c}e^{-2t}+\vec{d}e^t = \begin{bmatrix}1&1\\4&-2\end{bmatrix}\vec{c}e^{-2t} + \begin{bmatrix}1&1\\4&-2\end{bmatrix}\vec{d}e^t + \begin{bmatrix}1\\0\end{bmatrix}e^{-2t} + \begin{bmatrix}0\\-2\end{bmatrix}e^t$$

$$\therefore -2\vec{c} = \begin{bmatrix}1&1\\4&-2\end{bmatrix}\vec{c} + \begin{bmatrix}1\\0\end{bmatrix}$$

and $$\vec{d} = \begin{bmatrix}1&1\\4&-2\end{bmatrix}\vec{d} + \begin{bmatrix}0\\-2\end{bmatrix}$$

Writing the left side of the 1st system as $-2\begin{bmatrix}1&0\\0&1\end{bmatrix}\vec{c}$, it becomes (I'm just being cute – you could just write it all out + hack away) on subtracting $\begin{bmatrix}1&1\\4&-2\end{bmatrix}\vec{c}$ from both sides

$$\begin{bmatrix}-3&-1\\-4&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \quad \text{or: } \begin{aligned}-3c_1-c_2&=1\\-4c_1&=0\end{aligned} \quad \therefore \begin{aligned}c_1&=0\\c_2&=-1\end{aligned}$$

Similarly for the other system:

$$\begin{bmatrix}0&-1\\-4&3\end{bmatrix}\begin{bmatrix}d_1\\d_2\end{bmatrix} = \begin{bmatrix}0\\-2\end{bmatrix} \qquad \begin{aligned}-d_2&=0\\-4d_1+3d_2&=-2\end{aligned} \qquad \therefore \begin{aligned}d_2&=0\\d_1&=1/2\end{aligned}$$

Thus $$\vec{x}_p = \begin{bmatrix}0\\-1\end{bmatrix}e^{-2t} + \begin{bmatrix}1/2\\0\end{bmatrix}e^t = \begin{bmatrix}e^t/2\\-e^{-2t}\end{bmatrix}$$ same as before, ta-da.

$\boxed{4I-4}$

Solve reduced equation first: $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{x}$

char eqn: $m^2 - 1 = 0$

$\underline{m=1}$: $\alpha_1 - \alpha_2 = 0$ $\qquad$ $\underline{m=-1}$: $3\alpha_1 - \alpha_2 = 0$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ soln. $\qquad$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$ soln.

To find particular sol'n, since $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ is a soln of reduced equation, we have to use as the trial soln not just $\vec{c}e^t$ but

$$\vec{x}_p = \vec{c}e^t + \vec{d}te^t$$

Substituting into the ODE's:

$$\vec{c}e^t + \vec{d}e^t + \vec{d}te^t = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}(\vec{c}e^t + \vec{d}te^t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

$$\therefore \quad \vec{c} + \vec{d} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{c} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{d} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{d} \ldots$$

Solving second system: $\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \quad \vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} k$

(as done in prob (2b))

solving first system: $\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1-k \\ -1-k \end{bmatrix}$

Subtract 3 × first row from second: $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1-k \\ -4+2k \end{bmatrix} \quad \therefore k = 2$

get: $-c_1 + c_2 = -1$ $\qquad$ so take $\vec{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (other $\vec{c}$ are possible)

Soln: $\vec{x}_g = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t + c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$

to this could be added

$\boxed{4I-5}$

$\vec{x}' = A\vec{x} + \vec{X}_0$. Try $\vec{x}_p = \vec{c}$. Substituting:

$A\vec{c} + \vec{X}_0 = 0. \quad \therefore \quad \vec{x}_p = -A^{-1}\vec{X}_0$ if A is nonsingular!

[If A is singular, you only get soln $\vec{x}_p = \vec{c}$ if $A\vec{c} = -\vec{X}_0$ is consistent. In general, if rank $A = n - r$, you use $\vec{x}_p = \vec{c}_0 + \vec{c}_1 t + \cdots + \vec{c}_{r-1} t^{r-1}$

18.03 Differential Equations
Spring 2010

$\boxed{5A-1}$ (a) Critical points occur where
$$x^2 - y^2 = 0 \quad \text{and} \quad x - xy = 0$$
Now $x^2 - y^2 = 0 \Rightarrow x = \pm y$

Also $x - xy = 0 \Rightarrow x(1-y) = 0$
$$\Rightarrow x = 0 \text{ or } y = 1$$
$\therefore$ $x = 0$ and $y = 0$

OR $y = 1$ and $x = 1$

OR $y = 1$ and $x = -1$

$\therefore$ $(0,0)$, $(1,1)$ and $(-1,1)$ are the critical points

(b) Critical points occur where
$$1 - x + y = 0 \quad \text{and} \quad y + 2x^2 = 0$$
i.e. $y = x - 1$

Then $0 = x - 1 + 2x^2$

i.e. $x = \frac{1}{2}$ or $x = -1$

But $x = \frac{1}{2} \Rightarrow y = -\frac{1}{2}$

and $x = -1 \Rightarrow y = -2$

$\therefore$ $(\frac{1}{2}, -\frac{1}{2})$ and $(-1,-2)$ are the critical points.

$\boxed{5A-2}$ (a) Let $y = x'$

Then $y' = x'' = -\mu(x^2-1)x' - x$

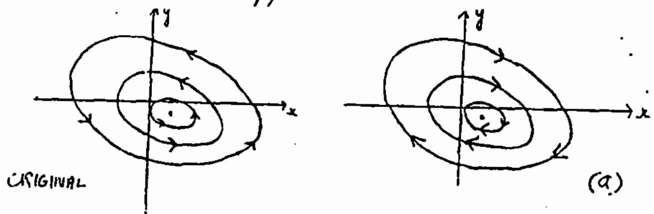
The autonomous equations are then
$$\begin{cases} x' = y \\ y' = -\mu(x^2-1)y - x \end{cases}$$
Critical points occur at
$$y = 0$$
$$-\mu(x^2-1)y - x = 0 \quad \text{i.e. at } (0,0)$$

(b) Let $y = x'$

Then $y' = x'' = x' - 1 + x^2$

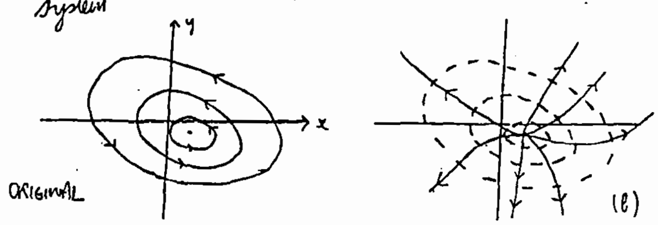
The autonomous equations are then
$$\begin{cases} x' = y \\ y' = y - 1 + x^2 \end{cases}$$
Critical points occur at
$$y = 0$$
$$y - 1 + x^2 = 0 \quad \therefore x^2 = 1 \quad \therefore x = \pm 1$$
So the critical points occur at $(1,0)$ and $(-1,0)$

$\boxed{5A-3}$ (a) For this system the tangent vector $(-f(x,y), -g(x,y))$ to the trajectories is equal in magnitude and opposite in direction to the tangent vector $(f(x,y), g(x,y))$ to the original system. So the trajectories are the same but are traversed in the opposite direction



The critical points occur at
$$\left.\begin{array}{l} f(x,y) = 0 \\ g(x,y) = 0 \end{array}\right\} \text{ i.e. the same for both systems}$$

$\boxed{5A-3}$ (b) For this system the tangent vector $(g(x,y), -f(x,y))$ to the trajectories is perpendicular to the tangent vector $(f(x,y), g(x,y))$ to the original system. So (b) represents the orthogonal trajectories of the original system



The critical points of (b) occur at
$$\left.\begin{array}{l} g(x,y) = 0 \\ -f(x,y) = 0 \end{array}\right\} \text{ i.e. the same as for the original system}$$

5A-4 (a) Let $u = t - t_0$, let $\bar{x}(t) = x_1(t-t_0)$.

Then $x_1(t-t_0) = x_1(u)$ as a function of u

$= \bar{x}(t)$ as a function of t

[As an example: if $x_1 = t^2$, then $x_1(u) = u^2$;
and $\bar{x}(t) = t^2 - 2t_0 t + t_0^2$]

By hypothesis:

$$\frac{dx_1(t)}{dt} = f(x_1(t), y_1(t)) \qquad \frac{dy_1(t)}{dt} = g(x_1(t), y_1(t))$$

$\therefore$ changing letters formally:

$$\frac{dx_1(u)}{du} = f(x_1(u), y_1(u)) \qquad \frac{dy_1(u)}{du} = f(x_1(u), y_1(u)) \quad (*)$$

But $\frac{d\bar{x}(t)}{dt} = \frac{dx_1(u)}{du}\cdot\frac{du}{dt} = \frac{dx_1(u)}{du}$; similarly $\frac{d\bar{y}(t)}{dt} = \frac{dy_1(u)}{du}$

Therefore, from $(*)$ we get

$$\frac{d\bar{x}(t)}{dt} = f(\bar{x}(t), \bar{y}(t)) \qquad \frac{d\bar{y}(t)}{dt} = g(\bar{x}(t), \bar{y}(t)),$$

which shows that $\bar{x}(t), \bar{y}(t)$ is also a solution.

$\begin{cases}\bar{x}(t) \\ \bar{y}(t)\end{cases} = \begin{cases}x_1(t-t_0) \\ y_1(t-t_0)\end{cases}$ represent the same motion as $\begin{cases}x_1(t) \\ y_1(t)\end{cases}$, but occurring t_0 time-units later.

That is, $\begin{cases}\bar{x}(t_1+t_0) \\ \bar{y}(t_1+t_0)\end{cases} = \begin{cases}x_1(t_1) \\ y_1(t_1)\end{cases}$ so wherever $\begin{cases}x_1 \\ y_1\end{cases}$ is at time t_1, $\begin{cases}\bar{x} \\ \bar{y}\end{cases}$ is there at time $t_1 + t_0$.

[This is the essential property of an autonomous system — the vector field does not change with time, so if we start at a given point t_0 seconds later, we follow the same trajectory as before, but delayed by t_0 seconds.]

(b) Let $\begin{pmatrix}x_1(t)\\ y_1(t)\end{pmatrix}$ $\begin{pmatrix}x_2(t)\\ y_2(t)\end{pmatrix}$ be two trajectories which intersect at (a,b)

i.e. $\begin{pmatrix}x_1(t_0)\\ y_1(t_0)\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}x_2(t_1)\\ y_2(t_1)\end{pmatrix}$ some t_0, t_1.

By part (a) $\begin{pmatrix}\bar{x}_2(t)\\ \bar{y}_2(t)\end{pmatrix} \equiv \begin{pmatrix}x_2(t-t_0+t_1)\\ y_2(t-t_0+t_1)\end{pmatrix}$

is also a solution to the ODE

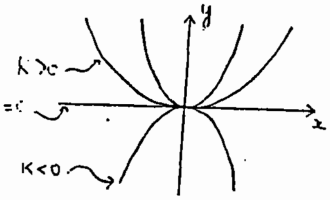
But $\begin{pmatrix}\bar{x}_2(t_0)\\ \bar{y}_2(t_0)\end{pmatrix} = \begin{pmatrix}x_2(t_1)\\ y_2(t_1)\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix}$

Thus by the uniqueness theorem

$$\begin{pmatrix}x_1(t)\\ y_1(t)\end{pmatrix} = \begin{pmatrix}\bar{x}_2(t)\\ \bar{y}_2(t)\end{pmatrix} = \begin{pmatrix}x_2(t-t_0+t_1)\\ y_2(t-t_0+t_1)\end{pmatrix} \text{ for all } t$$

i.e. $\begin{pmatrix}x_1(t)\\ y_1(t)\end{pmatrix}$ and $\begin{pmatrix}x_2(t)\\ y_2(t)\end{pmatrix}$ are the same trajectory and differ at most by a change in parameter.

5B-1 (a) $\frac{y'}{x'} = \frac{dy}{dx} = \frac{-2y}{-x}$



$\therefore \quad \frac{dy}{y} = 2\frac{dx}{x}$

$\therefore \quad y = Kx^2$

(b) Let $\vec{x}(t) = \begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$ and $M = \begin{pmatrix}-1 & 0\\ 0 & -2\end{pmatrix}$

Then $\vec{x}'(t) = M\vec{x}(t)$. This has solution

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$$

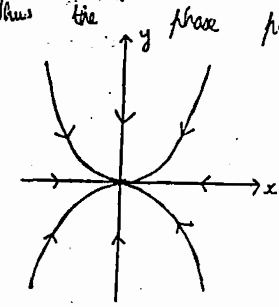
where λ_1 and λ_2 are the (distinct) eigenvalues of M with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$

Here $\lambda_1 = -1$, $\lambda_2 = -2$

$\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$

Then $\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}c_1 e^{-t}\\ c_2 e^{-2t}\end{pmatrix}$ All trajectories $\to (0,0)$ as $t \to +\infty$

Thus the phase picture is:



The new trajectories are $\begin{cases} x = 0 \\ y = c_2 e^{-2t} \end{cases}$ ($c > 0, < 0, = 0$)

i.e. the positive and negative y-axis, and the trivial trajectory $\vec{x}(t) = 0$ (the origin)

c) As the picture shows, 3 trajectories are need to cover a typical solution curve from part (a): $\searrow$, $\swarrow$, and $\bullet$ (the origin).

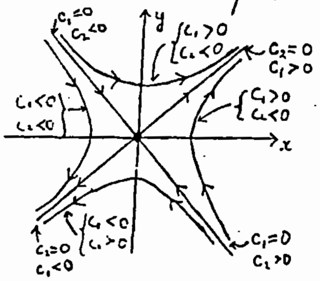
(d) This system may be obtained from the original by replacing t by $-t$. Thus we have the same trajectories but with the direction of the arrows reversed.

5B-2

a) $\frac{dy/dt}{dx/dt} = \frac{x}{y}$ $\therefore \frac{dy}{dx} = \frac{x}{y}$ soln: $y^2 - x^2 = c$ hyperbolas shown

b) $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

Thus the phase picture is.



The only "new" trajectory is the trivial one $\vec{x}(t) = 0$ (the origin)

c) In general, each solution curve (part (a)) is covered by one trajectory. However, the two lines \ and / each require 3 trajectories to cover them.

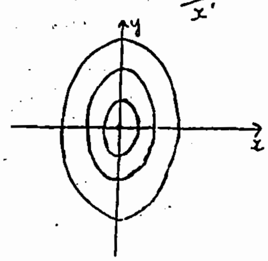
(d) The system $\begin{cases} x' = -y \\ y' = -x \end{cases}$ has the same trajectories as the original system except the arrows are reversed.

5B-3

(a) $\frac{y'}{x'} = \frac{dy}{dx} = \frac{-2x}{y}$

$\therefore \; y\,dy = -2x\,dx$

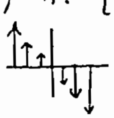
$\therefore \; \frac{y^2}{2} + x^2 = C$



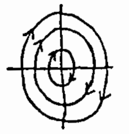
These are ellipses (for $C > 0$) and the point $(0,0)$ (for $C = 0$)

(b) For example, along the x-axis $(y=0)$, the tangent vectors ~~are~~ at $(x_0, 0)$ is: $\begin{cases} x' = 0 \\ y' = -2x_0 \end{cases}$, i.e., $(0, -2x_0)$

Thus the field is



So the direction of motion along the ellipses is clockwise.



5B-4

(a) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$

Thus $\vec{x}'(t) = M\vec{x}(t)$

M has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The system has a critical point at $(0,0)$ which is a saddle point

The general solution is

$$\vec{x}(t) = C_1 \vec{v}_1 e^{\lambda_1 t} + C_2 \vec{v}_2 e^{\lambda_2 t}$$

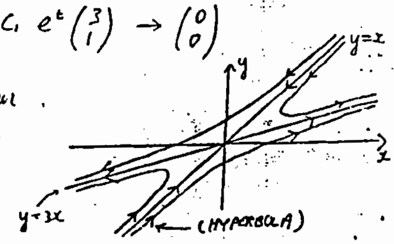
For $C_1 = 0$ and as $t \to \infty$

$$\vec{x}(t) = C_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Also for $C_2 = 0$ and $t \to -\infty$

$$\vec{x}(t) = C_1 e^{t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus the behaviour near the saddle point looks like



(b) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$

Then $\vec{x}'(t) = M\vec{x}(t)$.

M has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The system has an unstable node at $(0,0)$

The general solution is

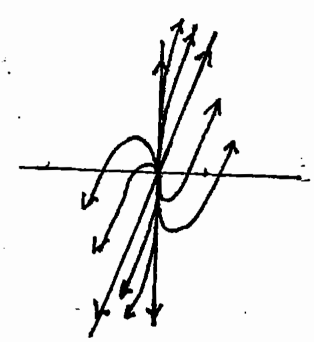
$$\vec{x}(t) = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$$

So as $t \to -\infty$ all trajectories $\to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus the behaviour near the node looks like:

For $t \approx -\infty$, $C_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$ is dominant term, $\therefore$ solns are near the y-axis

For $t \approx \infty$, $C_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$ dominates so solns are parallel to $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$



5B-4

(c) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix}$

Then $\vec{x}'(t) = M\vec{x}(t)$

M has eigenvalues $\lambda_1 = -4$, $\lambda_2 = -1$

with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

The system has an asympt. stable node at $(0,0)$

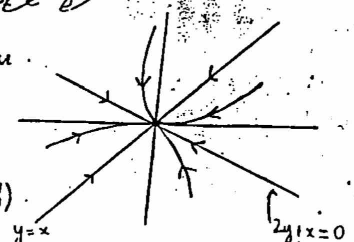
The general solution is

$\vec{x}(t) = c_1 \vec{v}_1 e^{-4t} + c_2 \vec{v}_2 e^{-t}$

As $t \to \infty$ all trajectories $\to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

~~$x(t) = c_1 e^{-4t} + 2c_2 e^{-t}$, $y(t) = c_1 e^{-4t} - c_2 e^{-t}$ non-trivial curve!~~

The behaviour near the node looks like.



For $t \approx -\infty$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}$ dominates so sol'ns are parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $t \approx \infty$, $\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t}$ dominates, so solns are $\underbrace{\text{close to}}_{\text{"like"}}$ $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.)

(d) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

Then $\vec{x}'(t) = M\vec{x}(t)$

M has eigenvalues $\lambda_1 = 1 + i\sqrt{2}$, $\lambda_2 = 1 - i\sqrt{2}$

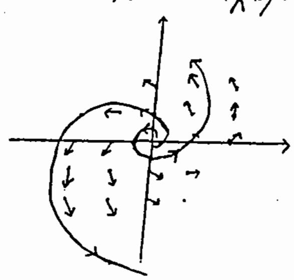
The system then has an unstable spiral around $(0,0)$.

When $y = 0$

$x' = x$

$\therefore$ x is increasing where the spiral cuts the +ve x-axis

As we e^t behaviour the spiral is outwards from the origin



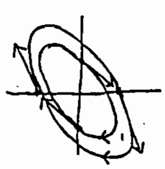
e) $\vec{x}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \vec{x}$

Eigenvalues are $\pm i$ (pure imaginary), so the system is a stable center.

(The curves are ellipses, since $\frac{dy}{dx} = \frac{-2x-y}{x+y}$ which integrates easily after cross-multiplying to $2x^2 + 2xy + y^2 = c$.)

Direction of motion:

For example, at $(1,0)$, the vector field is $x' = 1$, $y' = -2$. So motion is counterclockwise.



(a few other vectors are shown, inaccurately drawn...)

5B-5

(a) Let $y = x'$

Then, assuming $m \neq 0$,

$y' = x'' = -\frac{c}{m}x' - \frac{k}{m}x$

$\therefore$ The system is thus $\begin{cases} x' = y \\ y' = -\frac{k}{m}x - \frac{c}{m}y \end{cases}$

(b) The eigenvalues of $M = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}$

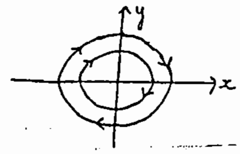
are $\lambda_\pm = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m}$

(i) $c = 0 \Rightarrow \lambda_\pm = \pm i\sqrt{\frac{k}{m}}$

Thus there is a stable centre at $(0,0)$.

Physically we'd expect this as putting $c = 0$ $(m, k > 0)$ in the ODE gives the SHM equation. Then x and x' are periodic with period $2\pi\sqrt{\frac{m}{k}}$

Thus we expect periodic trajectories in phase space



Here $c^2 - 4km < 0$

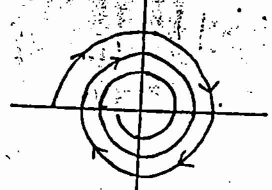
(ii) $\sqrt{c^2 - 4km} = 2i\sqrt{km}\left(1 - \frac{c^2}{4km}\right)^{\frac{1}{2}}$

or, neglecting c, $\approx 2i\sqrt{km}$

Then $\lambda_\pm = -\frac{c}{m} \pm i\sqrt{\frac{k}{m}}$ (eigenvalues)

The behaviour near $(0,0)$ is that of an asymptotically stable spiral (since $-\frac{c}{m} < 0$)

The "radius" of the spiral decays as $t \to \infty$ like $e^{-\frac{c}{m}t}$ ie very slowly indeed!



Physically we have lightly damped harmonic motion eg. a particle at the end of a spring oscillating in air. The motion is almost simple harmonic but the amplitude of oscillation decays slowly with time.

(iii) No!

When $c^2 - 4km \geq 0$, then as $k, m \geq 0$

we see $\sqrt{c^2 - 4km} \leq |c|$

Thus adding or subtracting $\sqrt{c^2 - 4km}$ to $-c$ cannot change its sign.

i.e. when the λ's are real, ~~either~~ they're both ~~positive or~~ ~~both~~ negative (since $c \geq 0$ always).

$\boxed{5C\text{-}5}$

$\boxed{5C\text{-}1}$

$x' = x - y + xy$
$y' = 3x - 2y - xy$

linearization is (at $(0,0)$): $x' = x - y$, $y' = 3x - 2y$

$\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$ char eqn: $m^2 + m + 1 = 0$, $m = \dfrac{-1 \pm \sqrt{-3}}{2}$ $\therefore$ asymp. stable spiral

$\boxed{5C\text{-}2}$

$x' = x + 2x^2 - y^2$
$y' = x - 2y + x^3$

lin'zn: $x' = x$, $y' = x - 2y$ $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$

eigenvalues are $1, -2$ (since mx. is Δular) $\therefore$ unstable saddle

$\boxed{5C\text{-}3}$

$x' = 2x + y + xy^3$
$y' = x - 2y - xy$

lin'zn: $x' = 2x + y$, $y' = x - 2y$ $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$m^2 - 5 = 0$
$m = \pm\sqrt{5}$ unstable saddle.

$\boxed{5C\text{-}4}$

$x' = 1 - y$
$y' = x^2 - y^2$

critical pts: $1 - y = 0$ $\therefore y = 1$; $x^2 - y^2 = 0$ $\therefore x = \pm 1$ — $(1,1)$ and $(-1,1)$.

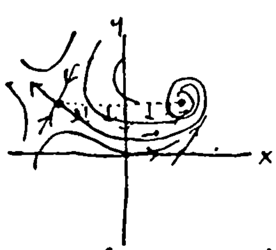
At $(1,1)$: since in general the Jac. matrix (of partial derivs) is $\begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix}$, the lin'zn is $\begin{bmatrix} x_1' \\ y_1' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$m^2 + 2m + 2 = 0$
$m = -1 \pm \dfrac{\sqrt{-4}}{2} = -1 \pm i$ $\therefore$ asym. stable spiral.

At $(-1,1)$: lin'zn is (again using Jacobian): $\begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$ $\therefore m^2 + 2m - 2 = 0$, $m = -1 \pm \sqrt{3} = -2.73, +.73$

$\therefore$ unstable saddle. Eigenvectors: $-m\alpha_1 - \alpha_2 = 0$ $\therefore \begin{bmatrix} 1 \\ -m \end{bmatrix}$ $\therefore \begin{bmatrix} 1 \\ -.73 \end{bmatrix}, \begin{bmatrix} 1 \\ 2.73 \end{bmatrix}$

Using this info: (Along dotted line, $y = 1$, a few dir. field vectors are drawn, using the original system: $x' = 0$, $y' = x^2 - 1$



A few other vectors are drawn in to help the sketch

This one is work, but instructive: think of x, y as 2 population which mutually destroy each other: $x - x^2$, $3y - 2y^2$ represent their "natural" growth laws, the $-xy$ terms their mutual destruction. [Like two hostile tribes, non-cannibalistic].

$x' = x - x^2 - xy$
$y' = 3y - 2y^2 - xy$

Critical points: $x(1 - x - y) = 0$
$y(3 - 2y - x) = 0$

From equation 1, either $x = 0$, or $1 - x - y = 0$.

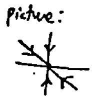
If $x = 0$, eqn 2 says: $y = 0$ or $y = 3/2$

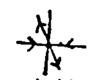
If $1 - x - y = 0$, eqn 2 says:
either $y = 0$ (in which case $1 - x = 0$, $x = 1$)
or $3 - 2y - x = 0$ (in which case we solve the 2 eqns: $1 - x - y = 0$, $3 - 2y - x = 0$ getting $y = 2$, $x = -1$)

Summary: critical points are
$(0,0)$, $(0, 3/2)$, $(1, 0)$, $(-1, 2)$.

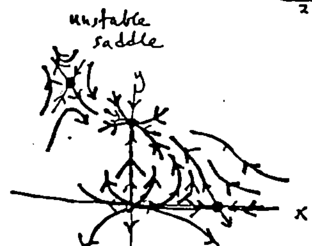
Now we determine their types: Jacobian matrix: $\begin{bmatrix} 1-2x-y & -x \\ -y & -x+3-4y \end{bmatrix}$

$(0,0)$: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$  unstable node.

$(0, 3/2)$: $\begin{bmatrix} -1/2 & 0 \\ -3/2 & -3 \end{bmatrix}$ eigenvs: $-1/2, -3$ vectors: $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ asymp. stable node picture: 

$(1, 0)$ $\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$ eigenvs: $-1, 2$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}$  unstable saddle

$(-1, 2)$ $\begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}$ $m^2 + 3m - 2 = 0$, $m = \dfrac{-3 \pm \sqrt{17}}{2}$, $m \approx 1/2$, $m \approx -7/2$: $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 5/2 \end{bmatrix}$ unstable saddle



the fat lines are impressionistic pieces of solution curves. Note there is no mutual coexistence! The tribe y always wins, (unless there is none of it to start with), essentially because of its stronger growth rate.

-NL

$\boxed{5D-1}$

a) Putting right-side of equations in (2) $= 0$ gives (assume $x \neq 0, y \neq 0$)

$-\frac{x}{y} = 1 - x^2 - y^2 = \frac{y}{x}$ $\therefore -x^2 = y^2$

so $x^2 + y^2 = 0$ $\therefore x = 0, y = 0$ (contradiction)

b) $(\cos t, \sin t)$ satisfies the system (just substitute); trajectory is the unit circle.

c) Equation (3) shows that if $R > 1$, the direction field points in towards the unit $\odot$, and (along circle of radius R) if $R < 1$, it points out towards the unit circle. Thus every solution curve is always getting closer to the unit $\odot$.

$\boxed{5D-2}$

a) Bendixson criterion:

$\text{div}(f,g) = (1+3x^2) + (1+3y^2) > 0$

$\therefore$ no limit cycle in xy-plane

b) System has no critical points, since $x^2+y^2 = 0 \Rightarrow x=0, y=0$, and this does not make $1+x-y = 0$.

$\therefore$ no limit cycles.

c) System has no critical points if $x < -1$, $\therefore$ no limit cycles in this region.

[To see this: $x^2 - y^2 = 0 \Rightarrow y = \pm x$

$2x + x^2 + y^2 = 0$ and $y = \pm x$ $\Rightarrow 2x + 2x^2 = 0$ $\therefore x = 0, -1$

thus critical pts. are $(0,0), (-1,1), (-1,-1)$.]

d) Bendixson's criterion:

$$\begin{aligned}\text{div}(f,g) &= a + 2bx - 2cy \\ &\quad + 2cy - 2bx \\ &= a\end{aligned}$$

$\therefore$ no limit cycles in xy-plane if $a \neq 0$.

$\boxed{5D-3}$

The system (7) is

$x' = y$

$y' = -v(x) - u(x)y$

a) By Bendixson's criterion,

$\text{div}(f,g) = 0 - u(x) < 0$ for all x, y if $u(x) > 0$.

$\therefore$ no periodic solution.

b) $v(x) > 0 \Rightarrow$ system has no critical point [at a critical point, $y = 0$, $\therefore v(x) = 0$]

$\therefore$ no periodic solution.

$\boxed{5D-5}$ (like 5D-1)

$\boxed{5E-1}$ a) linearization is

$$\begin{pmatrix}x\\y\end{pmatrix}' = \begin{bmatrix}1 & -4\\ 2 & -1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} \quad \text{at } (0,0).$$

Char. eqn: $\lambda^2 + 7 = 0$

$(0,0)$ is a center.

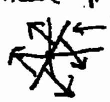
For non-lin. system, $(0,0)$ could be a center; or, unstable or asymptotically stable spiral.

b) linearization is

$$\begin{pmatrix}x\\y\end{pmatrix}' = \begin{pmatrix}3 & -1\\ -6 & +2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \quad \text{at } (0,0)$$

char. eqn: $\lambda^2 - 5\lambda = 0$, $\lambda = 0, 5$

$\therefore$ $(0,0)$ is not isolated – it is one of a line of critical points, all unstable:



← line of crit pts

For non-linear system, picture could stay like this; or turn into an unstable node or saddle.

$\boxed{5E-2}$ a) $x' = y$, $y' = x(1-x)$ $\qquad J = \begin{bmatrix}0 & 1\\ 1-2x & 0\end{bmatrix}$

Crit. pts: $(0,0)$, $(1,0)$

At $\underline{(0,0)}$, $J = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $\lambda^2 - 1 = 0$

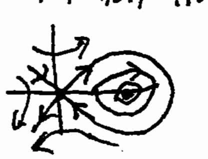
$\lambda = 1$, $\vec{\alpha} = \begin{pmatrix}1\\1\end{pmatrix}$; $\lambda = -1$, $\vec{\alpha} = \begin{pmatrix}1\\-1\end{pmatrix}$.

This is an <u>unstable saddle</u>.

At $(1,0)$, $J = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $\lambda = \pm i$

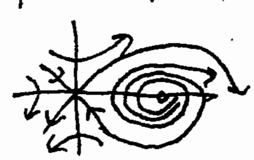
This is a <u>center</u>, <u>clockwise motion</u>.

For non-linear system, three possibilities:



(1,0) center

asymp. stable spiral



unstable spiral

$\boxed{5E-2}$ b) $x' = x^2 - x + y$, $y' = -yx^2 - y$

Crit. pts: $\begin{cases} x^2 - x - y = 0 \\ -y(x^2+1) = 0 \end{cases}$ $\quad \therefore\ y = 0,\ x = 0, 1$

Two crit. pts: $(0,0)$, $(1,0)$.

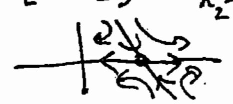
$$J = \begin{bmatrix}2x-1 & 1\\ -2xy & -x^2-1\end{bmatrix}$$

At $(0,0)$: $\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}$ $\quad \lambda = -1$, $\vec{\alpha} = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad$ repeated incomplete eigenvalue asy. stable node

Picture:

At $(1,0)$: $\begin{bmatrix}1 & 1\\ 0 & -2\end{bmatrix}$ $\quad \lambda_1 = 1$, $\vec{\alpha}_1 = \begin{pmatrix}1\\0\end{pmatrix}$; $\lambda_2 = -2$, $\vec{\alpha}_2 = \begin{pmatrix}1\\-3\end{pmatrix}$

Picture:



unstable saddle.

For non-linear system, two possibilities

(0,0) becomes an asym. stable node

(0,0) becomes an asym. stable spiral

$\boxed{5E-3}$ The new system is

$$x' = \tfrac{5}{4}ax - pxy,$$
$$y' = -by + qxy$$

whose critical pt is $\left(\frac{b}{q}, \frac{5a/4}{p}\right)$.

Crit. pt. for the orig. system is: $\left(\frac{b}{q}, \frac{a}{p}\right)$.

So the effect is to leave the flower population the same, but to increase the <u>borer</u> population by 25%.

Section 6 Solutions

$\boxed{6A-1}$ All of these use the ratio test:
if $\lim\limits_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = L$, $\sum_0^\infty b_n$ converges if $L<1$, diverges if $L>1$.

a) $\left|\frac{(n+1)x^{n+1}}{n x^n}\right| = \left(\frac{n+1}{n}\right)|x| \longrightarrow |x|$ as $n\to\infty$

$\therefore$ converges if $|x|<1$, so $R=1$

b) $\left|\frac{x^{2(n+1)}}{(n+1)2^{n+1}}\cdot\frac{n\cdot 2^n}{x^{2n}}\right| = \frac{n}{(n+1)2}\cdot|x|^2$

$\xrightarrow[\text{as } n\to\infty]{} \frac{1}{2}|x|^2$, and $\frac{|x|^2}{2}<1$ if $|x|<\sqrt{2}$

$\therefore$ converges if $|x|<\sqrt{2}$, so $R=\sqrt{2}$

c) $\left|\frac{(n+1)!\,x^{n+1}}{n!\ \ x^n}\right| = (n+1)|x| \longrightarrow \infty$ as $n\to\infty$ (if $x\neq 0$).

$\therefore$ converges only when $x=0$; $R=0$.

d) $\left|\frac{[2(n+1)]!}{(n+1)!^2}\cdot x^{n+1}\cdot\frac{(n!)^2}{(2n)!\,x^n}\right|$

$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)}\,|x| \longrightarrow 4|x|$ as $n\to\infty$

$\therefore$ converges if $4|x|<1$, i.e., $|x|<\frac{1}{4}$, so $R = 1/4$

$\boxed{6A-2}$ a) $\frac{1}{1-x} = \sum_0^\infty x^n$

$\therefore \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_1^\infty n x^{n-1}$

$= \sum_0^\infty (n+1)x^n$

(replacing n by $n+1$).

b) $e^x = \sum_0^\infty \frac{x^n}{n!}$, $\therefore e^{-x^2} = \sum_0^\infty \frac{(-1)^n x^{2n}}{n!}$

$x e^{-x^2} = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{n!}$

$\boxed{6A-2}$c) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \sum_0^\infty (-1)^n x^{2n}$

Integrating:
$\tan^{-1}x = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{2n+1} + \overset{0}{c}$

($c=0$: substitute $x=0$ on both sides to see that $c=0$)

d) $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \sum_0^\infty (-1)^n x^n$

Integrating:
$\ln(1+x) = \sum_0^\infty \frac{(-1)^n x^{n+1}}{n+1} + c^{=0}$

(see that $c=0$ by substituting $x=0$ on both sides)

[series could also be written $\sum_1^\infty (-1)^{n-1}\frac{x^n}{n}$ (putting n for $n+1$)]

$\boxed{6A-3}$a) $y = \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$

$y' = \sum_0^\infty \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_0^\infty \frac{x^{2n}}{(2n)!}$

$y'' = \sum_1^\infty \frac{2n\,x^{2n-1}}{(2n)!} = \sum_1^\infty \frac{x^{2n-1}}{(2n-1)!}$

the 0 term disappears

$= \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$ (changing $n\to n+1$)

This shows $y''=y$, or $y''-y=0$.

b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$

$\therefore \frac{e^x - e^{-x}}{2} = \frac{2x}{2} + \frac{2x^3}{2\,3!} + \frac{2}{2}\frac{x^5}{5!} + \ldots$

$= \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$.

4a) $\sum_0^\infty x^{3n+2} = x^2\sum_0^\infty x^{3n}$

$= x^2\cdot\frac{1}{1-x^3}$.

(since $\sum_0^\infty x^{3n} = \sum_0^\infty (x^3)^n = \frac{1}{1-(x^3)}$).

6A-4b) Start with $\sum_0^\infty x^n = \frac{1}{1-x}$

Integrate both sides:
$$\sum_0^\infty \frac{x^{n+1}}{n+1} = -\ln(1-x) + c \overset{=0}{} \quad \text{(substitute } x=0\text{)}$$
$$\therefore \sum_0^\infty \frac{x^n}{n+1} = -\frac{\ln(1-x)}{x}.$$

4c) Start with $\sum_0^\infty x^n = \frac{1}{1-x}$

Differentiating, $\sum_1^\infty n x^{n-1} = \frac{1}{(1-x)^2}$
$$\therefore \sum_1^\infty n x^n = \frac{x}{(1-x)^2}$$
note $\to$ 1 or 0 (makes no difference)

6B-1

a) Since $y(0)=1$,
$$y = 1 + a_1x + a_2x^2 + a_3x^3$$
$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$$
$$y^2 = (1+a_1x+a_2x^2+\dots)(1+a_1x+a_2x^2+\dots)$$
$$= 1 + 2a_1x + (2a_2+a_1^2)x^2 + (2a_3+2a_2a_1)x^3 + \dots$$
(this is far enough to get a_3.)

$y' = x + y^2$ says that
$$a_1 + 2a_2x + 3a_3x^2 + \dots = 1 + (2a_1+1)x + (2a_2+a_1^2)x^2 + \dots$$
$\therefore$ equating coefficients of like powers of x gives us:
$$a_1 = 1,\ 2a_2 = 2a_1 + 1 = 3, \therefore a_2 = 3/2$$
$$3a_3 = 2a_2 + a_1^2 = 4, \therefore a_3 = 4/3$$

So: $\boxed{y = 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \dots}$

b) Using Taylor's formula: $y(0)=1$

$y' = x + y^2$ $\qquad\therefore y'(0) = 0 + 1^2 = 1$

$\therefore y'' = 1 + y'\cdot 2y$ $\qquad y''(0) = 1 + 1\cdot(2\cdot 1) = 3$

$y''' = y''\cdot 2y + y'\cdot 2y'$ $\qquad y'''(0) = 3\cdot 2 + 1\cdot 2 = 8$

By Taylor's formula,
$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$
$$\therefore y = 1 + x + \frac{3}{2}x^2 + \frac{8}{6}x^3 + \dots$$
just as in part (a).

6B-2

a) $y = \sum_0^\infty a_nx^n$
$$y' = \sum_{0\,(\text{or }1)}^\infty n a_n x^{n-1} \rightsquigarrow \sum_0^\infty (n+1)a_{n+1}x^n$$
$y' - y = x$ says that
$$(n+1)a_{n+1} - a_n = 0 \text{ if } n \neq 1$$
$$= 1 \text{ if } n = 1,$$
that is, (since $y(0)=0$):
$$a_0 = 0,\quad a_{n+1} = \frac{a_n}{n+1} \text{ if } n \neq 1$$
$$\text{and } 2a_2 - a_1 = 1.$$
This gives:
$$a_0 = 0,\ a_1 = 0,\ a_2 = 1/2,\ a_3 = \frac{1}{3}\cdot\frac{1}{2},$$
$$a_4 = \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}, \text{ etc}\dots$$
so $y = \sum_{n=2}^\infty \frac{x^n}{n!} = e^x - 1 - x$

b) $y = \sum_0^\infty a_nx^n \xrightarrow{\cdot x} -xy = \sum_0^\infty a_n x^{n+1}$
$$y' = \sum_{1\,(\text{or }0)}^\infty n a_n x^{n-1} \xrightarrow{n\to n+2} \sum_{-1} (n+2)a_{n+2}x^{n+1}$$
note

$y' = -xy \Rightarrow$
$$(n+2)a_{n+2} = -a_n \qquad n = 0,1,2,\dots$$
$$a_1 = 0 \qquad (\text{corresponding to } n=-1)$$
$$a_0 = 1 \qquad (\text{since } y(0)=1)$$
$$\therefore a_{n+2} = \frac{-a_n}{n+2} \qquad n = 0,1,2,\dots$$
so
$$a_0 = 1,\ a_2 = \frac{-1}{2},\ a_4 = \frac{1}{4}\cdot\frac{1}{2},\ a_6 = \frac{-1}{6\cdot 4\cdot 2}$$
$$a_1 = a_3 = a_5 = \dots = 0.$$
So $y = \sum_0^\infty \frac{x^{2n}(-1)^n}{2^n\cdot n!} = e^{-x^2/2}$

6B-2

c) $y = \sum_0^\infty a_n x^n$

$y' = \sum_{0 \,(\text{or } 1)}^\infty n a_n x^{n-1} \rightsquigarrow \sum_0^\infty (n+1) a_{n+1} x^n$

$x y' = \sum_0^\infty n a_n x^n$

$\therefore (1-x)y' - y = 0 \implies$ (equating the coeff of x^n to 0)

$$(n+1)a_{n+1} - n a_n - a_n = 0$$

or $a_{n+1} = \frac{(n+1)a_n}{n+1} = a_n$

$y(0) = 1 \implies a_0 = 1$

$\therefore y = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$

6C-1

a) $\sum_1^\infty a_n x^{n+3} \underset{n \to n-3}{=} \sum_4 a_{n-3} x^n$

this starts with x^4, so this must also

b) $\sum_0^\infty n(n-1) a_n x^{n-2} \underset{n \to n+2}{=} \sum_0^\infty (n+2)(n+1) a_{n+2} x^n$

starts with x^0, so this must also

c) $\sum_1^\infty (n+1) a_n x^{n-1} \underset{n \to n+1}{=} \sum_0^\infty (n+2) a_{n+1} x^n$

starts with x^0, so this must also

6C-2

$y = \sum_0^\infty a_n x^n \xrightarrow[\cdot 4]{} 4y = \sum_0^\infty 4 a_n x^n$

$y'' = \sum_0^\infty a_n \cdot n(n-1) x^{n-2} \xrightarrow[n \to n+2]{} \sum_0^\infty a_{n+2}(n+2)(n+1) x^n$

$y'' - 4y = 0 \implies a_{n+2}(n+2)(n+1) - 4a_n = 0$

or $\boxed{a_{n+2} = \frac{4\, a_n}{(n+2)(n+1)}}$ Recursion formula

$\therefore a_2 = \frac{4a_0}{2 \cdot 1}, \quad a_4 = \frac{4}{4 \cdot 3} \cdot \frac{4}{2 \cdot 1} a_0 = \frac{4^2}{4!} a_0$

$a_3 = \frac{4a_1}{3 \cdot 2} \qquad a_5 = \frac{4}{5 \cdot 4} \cdot \frac{4}{3 \cdot 2} \cdot a_1 = \frac{4^2 a_1}{5!}$

continued above

6C-2

(continued)

get one series by taking $a_0 = 1, a_1 = 0$:

$y_0 = 1 + \frac{4}{2!}x^2 + \frac{4^2}{4!}x^4 + \frac{4^3}{6!}x^6 + \dots$

other series: take $a_0 = 0$, $a_1 = 1$

$y_1 = x + \frac{4x^3}{3!} + \frac{4^2 x^5}{5!} + \dots$

In summation notation:

$y_0 = \sum_0^\infty \frac{4^n x^{2n}}{n!}, \quad y_1 = \sum_0^\infty \frac{4^n x^{2n+1}}{(2n+1)!}$

(can also write numerator as $(2x)^{2n}$)

6C-3

Not solved.

6C-4

$\boxed{y'' - 2xy' + ky = 0}$; $\boxed{k = 2m}$.

$y = \sum_0^\infty a_n x^n \xrightarrow[\cdot 2m]{} \sum_0^\infty 2m a_n x^n$

$y' = \sum_0^\infty n a_n x^{n-1} \xrightarrow[\cdot -2x]{} \sum_0^\infty -2n a_n x^n$

$y'' = \sum_0^\infty n(n-1) a_n x^{n-2} \xrightarrow[n \to n+2]{} \sum_0^\infty (n+2)(n+1) a_{n+2} x^n$

Since $y'' - 2xy' + ky = 0$, this gives

$(n+2)(n+1)a_{n+2} - 2n a_n + 2m a_n = 0$

or $\boxed{a_{n+2} = \frac{2(n-m)}{(n+2)(n+1)} a_n}$

If $n = m$, then $a_{m+2} = 0$, etc.

So: if m is odd,

take $a_0 = 0$, $a_1 = 1$; then

all $a_0 = a_2 = a_4 = \dots = 0$

and all $a_{m+2} = a_{m+4} = 0 = \dots$

so $y_1 = a_1 x + a_3 x^3 + \dots + a_m x^m$

If m is even, take $a_1 = 0$.

Then similarly, (so $a_3 = 0, a_5 = 0 \dots$)

$y_0 = a_0 + a_1 x^2 + \dots + a_m x^m$

6C-5

$$\boxed{y'' = xy}$$

$$y = \sum_0^\infty a_n x^n \xrightarrow[\cdot x]{xy:} \sum_0^\infty a_n x^{n+1} = \sum_1^\infty a_{n-1} x^n$$

$$y'' = \sum_0^\infty n(n-1) a_n x^{n-2} \underset{n\to n+2}{\rightsquigarrow} \sum_0^\infty (n+2)(n+1) a_{n+2} x^n$$

Equating coeff's of like powers of x (since $y''=xy$) gives

$$(n+2)(n+1)a_{n+2} = a_{n-1} \quad (n \ge 1) \rightsquigarrow \therefore \boxed{a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}}, \quad n \ge 1.$$
$$= 0 \quad (n=0)$$

Recursion formula

$$a_2 = 0$$

$\therefore a_0, a_1$ are arbitrary, $2a_2 = 0$ (so $a_2=0$), $(\therefore a_5 = a_8 = a_{11} = \ldots = 0$ by the recursion formula)

and other terms are: $a_3 = \dfrac{a_0}{3\cdot 2}$, $a_6 = \dfrac{a_0}{6\cdot5\cdot3\cdot2} \ldots$

$$a_4 = \frac{a_1}{4\cdot 3}, \quad a_7 = \frac{a_1}{7\cdot6\cdot4\cdot3} \ldots$$

Taking $a_0 = 1$, $a_1 = 0$ gives $y_0 = 1 + \dfrac{x^3}{3\cdot2} + \dfrac{x^6}{6\cdot5\cdot3\cdot2} + \cdots + \dfrac{x^{3n}}{3n\cdot(3n-1)(3n-3)\cdots3\cdot2} + \cdots$

taking $a_0 = 0$, $a_1 = 1$ gives $y_1 = x + \dfrac{x^4}{4\cdot3} + \dfrac{x^7}{7\cdot6\cdot4\cdot3} + \cdots + \dfrac{x^{3n+1}}{(3n+1)\cdot3n\cdot(3n-2)\cdots4\cdot3} + \cdots$

6C-6

$$y = \sum_0^\infty a_n x^n \rightsquigarrow 6y = \sum_0^\infty 6a_n x^n$$

$$y' = \sum_0^\infty n a_n x^{n-1} \rightsquigarrow -2xy' = \sum_0^\infty -2n a_n x^n$$

$$y'' = \sum_0^\infty n(n-1)a_n x^{n-2} \rightsquigarrow y'' = \sum_0^\infty (n+2)(n+1)a_{n+2} x^n$$

$$\rightsquigarrow -x^2y'' = -\sum_0^\infty n(n-1)a_n x^n$$

$$\boxed{y'' - x^2y'' - 2xy' + 6y = 0}$$

Equating coeff of x^n to 0 gives:

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 6a_n = 0$$

or $a_{n+2} = a_n \dfrac{[n(n-1) + 2n - 6]}{(n+2)(n+1)}$

or $\boxed{a_{n+2} = \dfrac{(n+3)(n-2)}{(n+2)(n+1)} a_n}$

RECURSION FORMULA.

This gives solutions

$y_0 = 1 - 3x^2 \quad (a_0 = 1, a_1 = 0 = a_3 = a_5 = \ldots)$

$y_1 = x - \frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{4}{35}x^7 - \cdots \quad \Leftarrow$

Radius of convergence for y_1 is determined by ratio test: $\left|\dfrac{a_{n+2}x^{n+2}}{a_n x^n}\right| = \dfrac{(n+3)(n-2)}{(n+2)(n+1)}|x|^2 \xrightarrow[\text{as } n\to\infty]{} x^2$, $|x|^2 < 1$ if $|x| < 1$

$\therefore R = 1$. This is expected, since in standard form, ODE is $y'' - \frac{2x}{1-x^2}y' + \frac{6}{1-x^2}y = 0$, and coefficients become infinite at $|x| = 1$.

6C-7

$$y = \sum_0^\infty a_n x^n, \rightsquigarrow xy = \sum_1^\infty a_{n-1} x^n$$

$$y' = \sum_0^\infty n a_n x^{n-1} \rightsquigarrow 2y' = 2\sum_0^\infty (n+1) a_{n+1} x^n$$

$$y'' = \sum_0^\infty (n+2)(n+1) a_{n+2} x^n$$

$\therefore y'' + 2y' + (x-1)y = 0$ leads to the recursion:

$$(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + a_{n-1} - a_n = 0$$

leading to two sol'ns: $y_0 = 1 + x^2/2 - x^3/2 + \ldots \quad \binom{a_0=1}{a_1=0}$

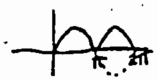
$y_1 = x - x^2 + 5/6\, x^3 + \ldots \quad (a_0 = 0, a_1 = 1)$

FOURIER SERIES

7A-1

a) For $\sin kt$, $\cos kt$ the frequency is k, and $\boxed{(\text{frequency})(\text{period}) = 2\pi}$.

$\therefore \quad \frac{\pi}{3} \cdot P = 2\pi, \quad P = \boxed{6}$

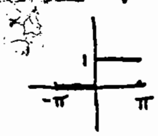
b)  Period is $\boxed{\pi}$: $|\sin(t+\pi)| = |-\sin t| = |\sin t|$

c) $\cos 3t$ has period $= \frac{2\pi}{3}$ (see problem 4)

$\cos^2 3t$ has period $\frac{1}{2} \cdot \frac{2\pi}{3}$ (as in prob. 9):

$\left(\cos 3(t+\frac{\pi}{3})\right)^2 = (\cos(3t+\pi))^2 = (-\cos(3t))^2 = (\cos(3t))^2$

7A-2 a)



$$a_n = \frac{1}{\pi}\int_0^\pi \cos nt\,dt = \frac{\sin nt}{\pi n}\Big]_0^\pi = 0$$

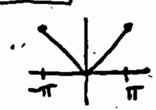
$$\left(a_0 = \frac{1}{\pi}\int_0^\pi dt = 1\right)$$

$$b_n = \frac{1}{\pi}\int_0^\pi \sin nt\,dt = \frac{-\cos nt}{n\pi}\Big]_0^\pi = \frac{-(-1)^n-(-1)}{n\pi}$$

$$= \frac{1-(-1)^n}{n\pi} = \begin{cases} 0, & n \text{ even} \\ \frac{2}{n\pi}, & n \text{ odd} \end{cases}$$

$$\therefore f(t) \sim \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \dots\right)$$

7A-2 b



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|t|\,dt = \frac{2}{\pi}\int_0^\pi t\,dt = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \boxed{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|t|\cos nt}_{\text{even function}}\,dt = \frac{2}{\pi}\int_0^\pi t\cos nt\,dt$$

$$= \frac{2}{\pi}\left[t\,\frac{\sin nt}{n} - \int \frac{\sin nt}{n}\,dt\right]_0^\pi$$

$$= \frac{2}{\pi}\left(0 + \left[\frac{\cos nt}{n^2}\right]_0^\pi\right) = \frac{2}{\pi}\left[\frac{(-1)^n-1}{n^2}\right]$$

$$= \begin{cases} 0, & n \text{ even} \\ -4/\pi n^2, & n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|t|\sin nt}_{\text{odd function}}\,dt = 0$$

$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \dots\right)$$

7A-3

$$\int_{-\pi}^{\pi}\cos mt\cos nt\,dt = \frac{1}{2}\int_{-\pi}^{\pi}(\cos(m+n)t + \cos(m-n)t)\,dt$$

$$\begin{cases} = \frac{1}{2}\left[\frac{\sin(m+n)t}{m+n} + \frac{\sin(m-n)t}{m-n}\right]_{-\pi}^{\pi} = 0 \text{ if } m \neq n \\ = \frac{1}{2}\left[\frac{\sin 2mt}{2m} + t\right]_{-\pi}^{\pi} = \frac{\pi-(-\pi)}{2} = \pi, \text{ if } m = n \end{cases}$$

7A-4

a)
$$\int_P^{a+P} f(t)\,dt = \int_0^a f(u+P)\,du = \int_0^a f(u)\,du$$

$u = t-P$, so $t = u+P$ (since $f(u+P) = f(u)$)

Then: (b)

$$\int_a^{a+P} f(t)\,dt = \int_a^P f(t)\,dt + \int_P^{a+P} f(t)\,dt$$

$$= \int_a^P f(t)\,dt + \int_0^a f(t)\,dt \quad \text{by the first part}$$

$$= \int_0^P f(t)\,dt.$$

7B-1. a)

$a_0 = 2\int_0^1 (1-t)\,dt = 2t - t^2\Big]_0^1 = 1$

$a_n = 2\int_0^1 (1-t)\cos n\pi t\,dt$ Integ. by parts:

$= 2\left[(1-t)\frac{\sin n\pi t}{n\pi} - \int (-1)\frac{\sin n\pi t}{n\pi}\,dt\right]_0^1$

$= 2\left[(1-t)\frac{\sin n\pi t}{n\pi} + -\frac{\cos n\pi t}{(n\pi)^2}\right]_0^1$

$= \frac{-2}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ even} \\ 4/n^2\pi^2, & n \text{ odd}\end{cases}$

$f(t) \sim \frac{1}{2} + \frac{4}{\pi^2}\left(\cos \pi t + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^2} + \ldots\right)$

Fourier cosine series (picture below)

b) $b_n = 2\int_0^1 (1-t)\sin n\pi t\,dt$ Integ. by parts

$= 2\left[(1-t)\left(\frac{-\cos n\pi t}{n\pi}\right) - \int (-1)\left(\frac{-\cos n\pi t}{n\pi}\right)\right]_0^1$

(this part is 0)

$= 2\left[0 + \frac{1}{n\pi}\right]$

$\therefore f(t) \sim \frac{2}{\pi}\left[\sin \pi t + \frac{\sin 2\pi t}{2} + \frac{\sin 3\pi t}{3} + \ldots\right]$

Fourier sine series (picture below)

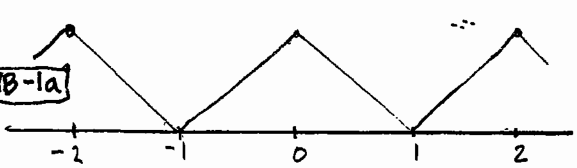
$\boxed{\text{7B-3}}$ a) $\int_{-a}^{0} f(t)\,dt = \int_a^0 f(-u)(-du) = \int_0^a f(u)\,du$

f even   (put $t = -u$) $(f(-u) = f(u))$

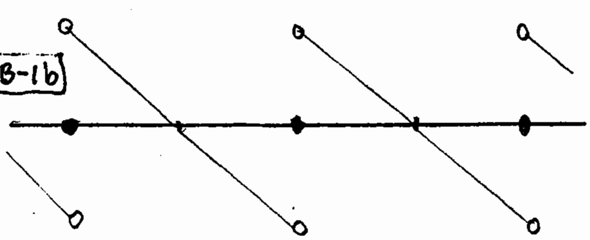
b) $\int_{-a}^{0} f(t)\,dt = \int_a^0 -f(u)(-du) = -\int_0^a f(u)\,du$

f odd $t = -u$, $f(-u) = -f(u)$

$\boxed{\text{7B-1a}}$

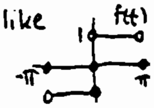


$\boxed{\text{7B-1b}}$



$\boxed{\text{7B-2a}}$ $x'' + 2x = 1$, $x(0) = x(\pi) = 0$

) First expand 1 in a Fourier sine series. This means the (odd) periodic extension (f(t)) looks like



. We can then get a F. sine series for x(t), & it will fit the bdry. conditions.

By (21), 8.1,

$f(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \ldots\right)$ (*)

) Look for a series $x(t) = \sum b_n \sin nt$ (this satisfies $x(0) = x(\pi) = 0$).

$x'' = \sum -b_n \cdot n^2 \sin nt$

$+\ 2x = \sum 2b_n \sin nt$ Adding

$f(x) = \sum b_n(2-n^2)\sin nt$

$= \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \ldots\right)$

$\therefore b_n = 0$, n even

$b_n = \frac{4}{\pi}\cdot\frac{1}{2-n^2}\cdot\frac{1}{n}$, if n is odd

$= \frac{-4}{n(n^2-2)\pi}$, n odd.

$\therefore x(t) = \frac{-4}{\pi}\sum_{n\,odd} \frac{\sin nt}{n(n^2-2)}$, $0 \le t \le \pi$

$\boxed{\text{7B-2b}}$

$x'' + 2x = t$, $x'(0) = x'(\pi) = 0$

a) Expand t in a Fourier cosine series; (we will then get a F. cosine series for x(t), & it will satisfy the 2 endpoint conditions).

Get $a_n = \frac{2}{\pi}\int_0^\pi t\cos nt\,dt$ Integ by parts

$= \frac{2}{\pi}\left[t\frac{\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^\pi = \frac{2}{\pi}\cdot\frac{(-1)^n - 1}{n^2}$

$a_n = \begin{cases} = \frac{-4}{n^2\pi} & \text{if } n \text{ odd} \\ = 0 & \text{if } n \text{ even.}\end{cases}$ $a_0 = \frac{2}{\pi}\int_0^\pi t\,dt = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$

$\therefore t \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \ldots\right)$

b) $x = \frac{A_0}{2} + \sum A_n \cos nt$ $(\times 2$

$x'' = -\sum n^2 A_n \cos nt$ Adding,

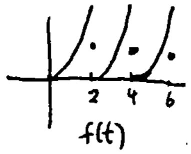
$t = A_0 + \sum A_n(2-n^2)\cos nt$

$\therefore A_0 = \frac{\pi}{2}$, $A_n = 0$ if n even $A_n = -4$

$A_n = -\frac{4}{\pi}\cdot\frac{1}{n^2(2-n^2)}$ if n odd

7B-4

$$f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_1^\infty \frac{\cos n\pi t}{n^2} - \frac{4}{\pi}\sum_1^\infty \frac{\sin n\pi t}{n}$$



$$f'(t) \overset{?}{=} -\frac{4}{\pi^2}\sum \frac{\sin n\pi t}{n} - \frac{4}{\pi}\sum_1^\infty \cos n\pi t$$

This series doesn't converge (the cosine terms don't add up — for example, when $t=0$). So it certainly can't converge to $f'(t)$

7C-1

Preliminary remarks

$$mx'' + kx = F(t)$$

The natural frequency of the spring-mass system is $\omega_0 = \sqrt{k/m}$

The typical term of the Fourier expansion of $F(t)$ is $\cos \frac{n\pi}{L}t$, $\sin \frac{n\pi}{L}t$; thus we get pure resonance if and only if the Fourier series has a $\cos\frac{n\pi}{L}t$ or $\sin\frac{n\pi}{L}t$ term where $\boxed{\frac{n\pi}{L} = \omega_0}$

a) $\omega_0 = \sqrt{5}$ for spring-mass system

$L = 1$

Fourier series is $\sum b_n \sin n\pi t$

$n\pi \neq \sqrt{5}$ $\quad\therefore$ no resonance

b) $\omega_0 = 2\pi$ $\quad L = 1$

Fourier series is $\sum b_n \sin n\pi t$, and $n\pi = 2\pi$ if $n=2$

Example 1, 8.4 shows that this term actually occurs in the Fourier series for $2t$ (just change scale). $\therefore$ get resonance

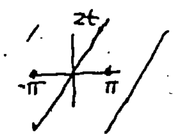
c) $\omega_0 = 3$ $\quad$ Fourier series is a sine series ($F(t)$ is odd):

$F(t) = \sum b_n \sin nt$ $\quad$ all odd n occur (see Problem 8.3/11, or ex. 1, 8.1)

$\therefore$ $n=3$ occurs, so we get resonance.

7C-2

Fourier series for $F(t)$



will be same (up to factor 2) as the Fourier sine series in Example 1, 8.3 $\quad (L=\pi)$

$$F(t) = 4\left(\sin t - \tfrac{1}{2}\sin 2t + \tfrac{1}{3}\sin 3t - \dots\right)$$

$$\begin{aligned} x' &= \sum B_n \sin nt \quad | \times 3 \\ x'' &= \sum -B_n \cdot n^2 \sin nt \quad \text{Adding:} \\ \hline F(t) &= \sum B_n(3-n^2)\sin nt \end{aligned}$$

$$\therefore B_n = (-1)^{n+1}\cdot\frac{4}{n}\cdot\frac{1}{(3-n^2)} = \frac{(-1)^n\cdot 4}{n(n^2-3)}$$

7C-3a

The natural frequency of the undamped spring is $\omega_0 = \sqrt{18/2} = 3$

This frequency occurs in the Fourier series for $F(t)$ (see problem 3). Thus the $n=3$ term should dominate. (The actual series is

$$x_{sp}(t) \approx .25\sin(t-.0063) - .20\sin(2t-.02) + 4.44\sin(3t-1.5708) - .07\sin(4t-3.1130)\dots$$

(steady periodic soln – no transients)

7C-3b

The natural frequency of the undamped spring is $\sqrt{30/3} = \sqrt{10}$

Expanding the force in a Fourier series, since $L = 1$ (half-period), & $F(t)$ is odd, it will be $F(t) = \sum b_n \sin n\pi t$

It's virtually certain all terms will occur (since $F(t)$ looks so messy) — (check soln to 8.4/5 in back of book)

$\therefore$ since $\sqrt{10} \approx \pi$, $b_1 \sin \pi t$ should be the dominant term in the $x_{sp}(t)$ series (this checks with answer given in back of book)

[NOTE: Edwards & Penney 4th edn: 8.4 (16), p. 590 has a sign error in denominators — cf. (13), which is correct.]

18.03 Hour Exam I Solutions: February 24, 2010

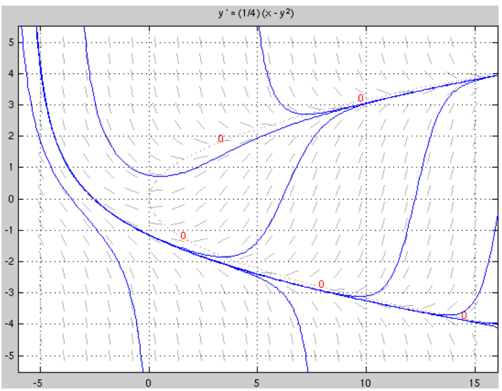
1. (a) $x(t)$ = number of rats at time t; t measured in years. $\dot{x} = kx$. So $x(t) = x(0)e^{kt}$. $ex(0) = x(1) = x(0)e^{kt}$ implies $k = 1$.

(b) $\dot{x} = k\left(1 - \dfrac{x}{R}\right)x = \left(1 - \dfrac{x}{1000}\right)x$.

(c) $\dot{x} = \left(1 - \dfrac{x}{R}\right)x - a$. The pest control people hope for an equilibrium at $x = \frac{3}{4}R$. $\dot{x} = 0$ at equilibrium, so $a = (1 - \frac{3}{4})\frac{3}{4}R = \frac{3}{16}R = 375$.

2. (a) The phase line shows unstable critical points at $x = -2$ and $x = 1$ and a stable critical point at $x = 0$. The arrows of time are directed up above 1 and between -2 and 0, and down between 0 and 1 and below -2.

(b) There are seven basic solution types: three equilibria; a solution rising above $x = 1$, a solution falling from 1 towards 0, a solution rising from -2 towards 0, and a solution falling away from -2.



(f) True. After the solution crosses the nullcline, it is "inside" the parabola and its derivative is positive. If it were to cross the nullcline again it would have to cross the upper branch, from below. But the slope of the nullcline is positive, while at the moment of crossing the slope of the solution would have to be zero. So it does not cross again; it stays below the upper branch of the nullcline, which has equation $y = \sqrt{x}$.

3. (a)

k	x_k	y_k	$m_k = x_k + y_k$	hm_k
0	0	1	1	1/2
1	1/2	3/2	4	1
2	1	5/2	7/2	7/4
3	3/2	17/4		

Ans: 17/4.

(b) The equation is $\frac{d}{dt}(tx) = \cos t$, so $tx = \sin t + c$ and $x = \dfrac{c + \sin t}{t}$. $1 = x(\pi) = \frac{c}{\pi}$ so $c = \pi$ and $x = \dfrac{\pi + \sin t}{t}$.

4. (a) $\frac{1}{3+2i} = \frac{3-2i}{3^2+2^2}$: $a = \frac{3}{13}$, $b = -\frac{2}{13}$.

(b) $r = |1 - i| = \sqrt{2}$. $\theta = \mathrm{Arg}(1 - i) = -\frac{\pi}{4}$.

(c) $|1 - i| = \sqrt{2}$ and $\mathrm{Arg}(1 - i) = \frac{\pi}{4}$, so $|(1 - i)^8| = (\sqrt{2})^8 = 16$ and $\mathrm{Arg}((1 - i)^8) = 8\frac{\pi}{4} = 2\pi$, so $(1 - i)^8 = 16$: $a = 16, b = 0$.

(d) If $(a+bi)^3 = -1$ then $|a+bi|^3 = |(a+bi)^3| = |-1| = 1$ so $|a+bi| = 1$, and $3\mathrm{Arg}(a+bi) = \mathrm{Arg}(-1) = \pi$ (or 3π or 5π) so $\mathrm{Arg}(a + bi) = \frac{\pi}{3}$ or π or $\frac{5\pi}{3}$. The first is the one with positive imaginary part, so $a = \cos\frac{\pi}{3} = \frac{1}{2}$, $b = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

(e) $e^{\ln 2 + i\pi} = e^{\ln 2}e^{i\pi} = 2(-1) = -2$: $a = -2$, $b = 0$.

(f) A, ϕ are the polar coordinates of $(a, b) = (2, -2)$: $A = 2\sqrt{2}$, $\phi = -\frac{\pi}{4}$

5. (a) Try $x = Ae^{2t}$, so that $\dot{x} = A2e^{2t}$ and $e^{2t} = A2e^{2t} + 3Ae^{2t} = 5Ae^{2t}$ so $A = \frac{1}{5}$: $x_p = \frac{1}{5}e^{2t}$. The transient is ce^{-3t}, so $x = \frac{1}{5}e^{2t} + ce^{-3t}$ is a valid solution for any c as well.

(b) $1 = x(0) = \frac{1}{5} + c$ implies $c = \frac{4}{5}$: this particular solution is $x = \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}$

(c) $\dot{z} + 3z = e^{2it}$.

(d) Try $z = Ae^{2it}$: $\dot{z} = A2ie^{2it}$, so $e^{2it} = \dot{z} + 3z = A(3+2i)e^{2it}$. This gives $A = \frac{1}{3+2i}$ and solution $z_p = \frac{1}{3+2i}e^{2it} = \frac{3-2i}{13}(\cos(2t) + i\sin(2t))$, which has real part $x_p = \frac{3}{13}\cos(2t) + \frac{2}{13}\sin(2t)$.

18.03 Hour Exam II Solutions: March 17, 2010

1. (a) The characteristic polynomial is $p(s) = s^2 + s + k = \left(s + \frac{1}{2}\right)^2 + \left(k - \frac{1}{4}\right)$. This has a repeated root when $k = \frac{1}{4}$.

(b) If k is larger, the contents of the square root become negative and the roots become non-real: so underdamped. (Note that this does not require the solution to **(a)**.)

(c) Vanishing twice implies underdamped. The pseudoperiod is 2 (since a damped sinusoid vanishes twice for each period), so $\omega_d = \frac{2\pi}{2} = \pi$. From $p(s) = s^2 + s + k = \left(s + \frac{1}{2}\right)^2 + \left(k - \frac{1}{4}\right)$ we find $\omega_d = \sqrt{k - \frac{1}{4}}$, so $k = \pi^2 + \frac{1}{4}$.

2. (a) Variation of parameters: $x = ue^{2t}$. $\dot{x} = (\dot{u} + 2u)e^{2t}$, $\ddot{x} = (\ddot{u} + 4\dot{u} + 4u)e^{2t}$, so $\ddot{x} + x = (\ddot{u} + 4\dot{u} + 5u)e^{2t}$, and u must satisfy $\ddot{u} + 4\dot{u} + 5u = 5t$. Undetermined coefficients: $u_p = at + b$, $\dot{u}_p = a$, $\ddot{u}_p = 0$, so $4a + 5(at + b) = 5t$, $a = 1$, $b = -\frac{4}{5}$: $u_p = t - \frac{4}{5}$, $x_p = (t - \frac{4}{5})e^{2t}$.

(b) The homogeneous equation has general solution $a\cos t + b\sin t$, so the general solution of $\ddot{x} + x = 5te^{2t}$ is $x = y + a\cos t + b\sin t$. $3 = x(0) = y(0) + a = 1 + a$ so $a = 2$. $5 = \dot{x}(0) = \dot{y}(0) + b = 2 + b$ so $b = 3$: $x = y + 2\cos(t) + 3\sin(t)$.

3. (a) The complex replacement $\ddot{z} + b\dot{z} + kz = e^{i\omega t}$ has exponential solution $z_p = \dfrac{e^{i\omega t}}{p(i\omega)}$. The amplitude of $\mathrm{Re}(z_p)$ is $\dfrac{1}{|p(i\omega)|}$, so we find what value of k minimizes $|p(i\omega)|$. $p(i\omega) = (k - \omega^2) + bi\omega$, so $k = \omega^2$ minimizes the absolute value. [This is interesting; the spring constant resulting in largest gain is the one resulting in a system whose natural frequency matches the driving frequency, independent of the damping constant.]

(b) $p(s) = s^3 - s = s(s-1)(s+1)$, so the modes are $e^{0t} = 1$, e^t, and e^{-t}. The general solution is $ae^{-t} + b + ce^t$.

4. (a) By time invariance and linearity we can suppose the input signal is $\cos(\omega t)$. The complex input is $y_{\mathrm{cx}} = e^{i\omega t}$, and $\ddot{z} + \dot{z} + 6z = 6e^{i\omega t}$ has exponential solution $z_p = \dfrac{6}{p(i\omega)}e^{i\omega t} = \dfrac{6}{p(i\omega)}y_{\mathrm{cx}}$, so the complex gain is $H(\omega) = \dfrac{6}{p(i\omega)} = \dfrac{6}{(6 - \omega^2) + i\omega}$.

(b) $H(2) = \frac{6}{(6-4)+2i} = \frac{3}{1+i}$, so $g(2) = |H(2)| = \frac{3}{\sqrt{2}}$.

(c) $\phi = -\mathrm{Arg}(H)(\omega) = \mathrm{Arg}(1 + i) = \frac{\pi}{4}$.

5. (a) If we write $q(t) = 4\cos(2t)$, the new input signal is $4\cos(2t - 1) = q(t - \frac{1}{2})$, so by time-invariance, $x = \frac{1}{2}(t - \frac{1}{2})\sin(2(t - \frac{1}{2}))$ solves the new equation.

(b) By linearity, $x = t\sin(2t)$.

(c) The form of the solution indicates resonance: so $\pm 2i$ are roots of the characteristic polynomial, which must thus be $p(s) = m(s - 2i)(s + 2i) = m(s^2 + 4)$. Thus $b = 0$ and $k = 4m$. By the Exponential Response Formula with resonance, $m\ddot{z} + kz = 4e^{2it}$ has solution $\dfrac{4t}{p'(2i)}e^{2it} = \dfrac{4t}{4mi}e^{2it} = \dfrac{t}{mi}e^{2it}$, so the original equation has solution $\frac{1}{m}t\sin(2t)$. Thus $m = 2$, $b = 0$, $k = 8$.

18.03 Hour Exam III Solutions: April 23, 2010

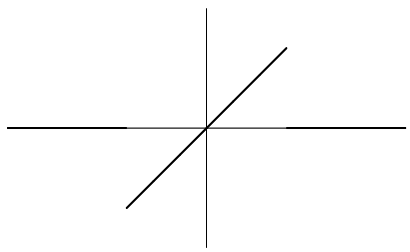
1. (a) The miminal period is 2.

(b) $f(t)$ is even.

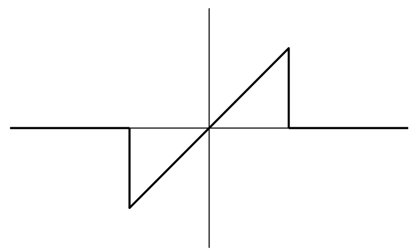
(c) $x_p(t) = \dfrac{1}{\omega_n^2} + \dfrac{\cos(\pi t)}{2(\omega_n^2 - \pi^2)} + \dfrac{\cos(2\pi t)}{4(\omega_n^2 - 4\pi^2)} + \dfrac{\cos(3\pi t)}{8(\omega_n^2 - 9\pi^2)} + \cdots$

(d) There is no periodic solution when $\omega_n = 0, \pi, 2\pi, 3\pi, \ldots$.

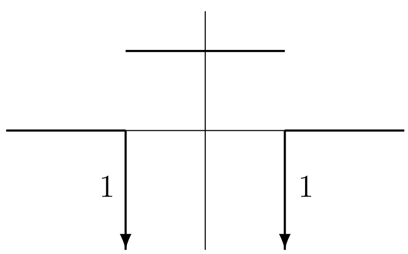
2. (a)



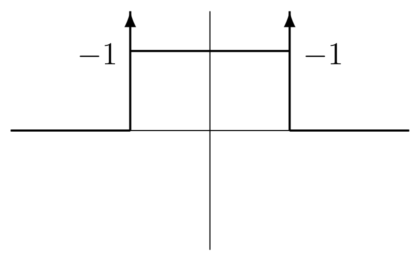
or



(b)



or



(c) $f'(t) = (u(t+1) - u(t-1)) - \delta(t+1) - \delta(t-1)$; $f_r'(t) = u(t+1) - u(t-1)$, $f_s'(t) = -\delta(t+1) - \delta(t-1)$.

3. (a) $v(t) = w(t) * u(t) = \displaystyle\int_0^t w(t-\tau)u(\tau)\, d\tau = \int_0^t (e^{-(t-\tau)} - e^{-3(t-\tau)})\, d\tau$

$= e^{-t}\, e^{\tau}\big|_0^t - e^{-3t}\, \dfrac{e^{3\tau}}{3}\Big|_0^t = (1 - e^{-t}) - \dfrac{1 - e^{-3t}}{3} = \dfrac{2}{3} - e^{-t} + \dfrac{e^{-3t}}{3}.$

(b) $W(s) = \mathcal{L}[w(t)] = \dfrac{1}{s+1} - \dfrac{1}{s+3}.$

(c) $W(s) = \dfrac{(s+3)-(s+1)}{(s+1)(s+3)} = \dfrac{2}{s^2+4s+3}$, so $p(s) = \frac{1}{2}(s^2+4s+3)$.

4. (a) $\dfrac{s-1}{s} = 1 - \dfrac{1}{s} \rightsquigarrow \delta(t) - u(t)$, so $\dfrac{e^{-s}(s-1)}{s} \rightsquigarrow \delta(t-1) - u(t-1)$.

(b) $F(s) = \dfrac{s+10}{s^3+2s^2+10s} = \dfrac{a}{s} + \dfrac{b(s+1)+c}{(s+1)^2+9}$. By coverup, $a = \frac{10}{10} = 1$. By complex coverup (multiply through by $(s+1)^2+9$ and set s to be a root, say $-1+3i$), $b(3i)+c = \frac{9+3i}{-1+3i} = -3i$, so $b = -1$, $c = 0$, and $F(s) = \dfrac{1}{s} - \dfrac{s+1}{(s+1)^2+9}$, which is the Laplace transform of $1 - e^{-t}\cos(3t)$.

5. (a) $\{0, -1+3i, -1-3i\}$.

(b) $X(s) = W(s)F(s)$. $F(s) = \dfrac{2}{s^2+4}$, so $X(s) = \left(\dfrac{s+10}{s^3+2s^2+10s}\right)\left(\dfrac{2}{s^2+4}\right)$.

Solutions of Spring 2008 Final Exam

1. (a) The isocline for slope 0 is the pair of straight lines $y = \pm x$. The direction field along these lines is flat.

   The isocline for slope 2 is the hyperbola on the left and right of the straight lines. The direction field along this hyperbola has slope 2.

   The isocline for slope -2 is the hyperbola above and below the straight lines. The direction field along this hyperbola has slope -2.

   (b) The sketch should have the following features:

   The curve passes through $(-2, 0)$. The slope at $(-2, 0)$ is $(-2)^2 - (0)^2 = 4$.

   Going backward from $(-2, 0)$, the curve goes down $(dy/dx > 0)$, crosses the left branch of the hyperbola $x^2 - y^2 = 2$ with slope 2, and gets closer and closer to the line $y = x$ but never touches it.

   Going forward from $(-2, 0)$, the curve first goes up, crosses the left branch of the hyperbola $x^2 - y^2 = 2$ with slope 2, and becomes flat when it intersects with $y = -x$. Then the curve goes down and stays between $y = -x$ and the upper branch of the hyperbola $x^2 - y^2 = -2$, until it becomes flat as it crosses $y = x$. Finally, the curve goes up again and stays between $y = x$ and the right branch of the hyperbola $x^2 - y^2 = 2$ until it leaves the box.

   (c) $f(100) \approx 100$

   (d) It follows from the picture in (b) that $f(x)$ reaches a local maximum on the line $y = -x$. Therefore $f(a) = -a$.

   (e) Since we know $f(-2) = 0$, to estimate $f(-1)$ with two steps, the step size is 0.5. At each step, we calculate

   $$x_n = x_{n-1} + 0.5, \qquad y_n = y_{n-1} + 0.5(x_{n-1}^2 - y_{n-1}^2)$$

   The calculation is displayed in the following table.

n	x_n	y_n	$0.5(x_n^2 - y_n^2)$
0	-2	0	2
1	-1.5	2	-0.875
2	-1	1.125	

   The estimate of $f(-1)$ is $y_2 = 1.125$.

2. (a) The equation is $\dot{x} = x(x-1)(x-2)$. The phase line has three equilibria $x = 0, 1, 2$.

   For $x < 0$, the arrow points down.

   For $0 < x < 1$, the arrow points up.

   For $1 < x < 2$, the arrow points down.

   For $x > 2$, the arrow points up.

   (b) The horizontal axis is t and the vertical axis is x. There are three constant solutions $x(t) \equiv 0, 1, 2$. Their graphs are horizontal.

   Below $x = 0$, all solutions are decreasing and they tend to $-\infty$.

Between $x = 0$ and $x = 1$, all solutions are increasing and they approach $x = 1$.
Between $x = 1$ and $x = 2$, all solutions are decreasing and they approach $x = 1$.
Above $x = 2$, all solutions are increasing and they tend to $+\infty$.

(c) A point of inflection $(a, x(a))$ is where $\ddot{x}$ changes sign. In particular, $\ddot{x}(a)$ must be zero. Differentiating the given equation with respect to t, we have

$$\ddot{x} = 2\dot{x} - 6x\dot{x} + 3x^2\dot{x} = \dot{x}(2 - 6x + 3x^2)$$

If $x(t)$ is not a constant solution, $\dot{x}(a) \neq 0$ so that $x(a)$ must satisfy

$$2 - 6x(a) + 3x(a)^2 = 0 \qquad \Leftrightarrow \qquad x(a) = 1 \pm \frac{1}{\sqrt{3}}.$$

(d) Typo in the original version: The material being added into the reactor should be Bo instead of Ct.
Let $x(t)$ be the number of moles of Bo in the reactor at time t. The rate of loading is 2 moles per year. Hence $x(t)$ satisfies $\dot{x} = -kx + 2$, where k is the decay rate of Bo. Since the half life of Bo is 2 years, $e^{-k\cdot 2} = 1/2$ so that $k = (\ln 2)/2$. Therefore we have

$$\dot{x} = -\frac{\ln 2}{2}\,x + 2.$$

The initial condition is $x(0) = 0$.

(e) The differential equation is linear. Since we have

$$y' + \left(\frac{3}{x}\right) y = x$$

an integrating factor is given by

$$\exp\left(\int \frac{3}{x}\,dx\right) = \exp(3\ln x) = x^3.$$

Multiply the above equation by x^3 and integrate:

$$(x^3y)' = x^3y' + 3x^2y = x^4 \qquad \Rightarrow \qquad x^3y = \frac{1}{5}\,x^5 + c$$

Since $y(1) = 1$, we have $c = 4/5$ and

$$y = \frac{1}{5}\,x^2 + \frac{4}{5}\,x^{-3}.$$

3. (a) Express all complex numbers in polar form:

$$\frac{ie^{2it}}{1+i} = \frac{e^{i\pi/2}e^{2it}}{\sqrt{2}e^{i\pi/4}} = \frac{1}{\sqrt{2}}\,e^{i(2t+\pi/2-\pi/4)} = \frac{1}{\sqrt{2}}\,e^{i(2t+\pi/4)}$$

The real part is

$$\mathrm{Re}\left(\frac{ie^{2it}}{1+i}\right) = \frac{1}{\sqrt{2}}\cos\left(2t + \frac{\pi}{4}\right).$$

(b) The trajectory is an outgoing, clockwise spiral that passes through 1.

(c) The polar form of $8i$ is $8e^{i\pi/2}$. Its three cubic roots are

$$\begin{aligned}
2e^{i\pi/6} &= 2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6} = \sqrt{3} + i, \\
2e^{i(\pi/6+2\pi/3)} &= 2\cos\frac{5\pi}{6} + 2i\sin\frac{5\pi}{6} = -\sqrt{3} + i, \\
2e^{i(\pi/6+4\pi/3)} &= 2e^{3i\pi/2} = -2i.
\end{aligned}$$

4. (a) Let $x_p(t) = at^2 + bt + c$. Plug it into the left hand side of the equation

$$\begin{aligned}
\ddot{x} + 2\dot{x} + 2x &= (2a) + 2(2at + b) + 2(at^2 + bt + c) \\
&= 2at^2 + (4a + 2b)t + (2a + 2b + 2c)
\end{aligned}$$

and compare coefficients

$$2a = 1, \qquad 4a + 2b = 0, \qquad 2a + 2b + 2c = 1.$$

The solution is $a = 1/2$, $b = -1$, $c = 1$. Therefore $x_p(t) = \dfrac{1}{2}\,t^2 - t + 1$.

(b) The characteristic polynomial is $p(s) = s^2 + 2s + 2$. Using the ERF and linearity,

$$x_p(t) = \frac{e^{-2t}}{p(-2)} + \frac{1}{p(0)} = \frac{e^{-2t}}{2} + \frac{1}{2}$$

(c) Consider the complex equation

$$\ddot{z} + 2\dot{z} + 2z = e^{it}.$$

For any solution z_p, its imaginary part $x_p = \operatorname{Im} z_p$ satisfies the real equation

$$\ddot{x} + 2\dot{x} + 2x = \sin t.$$

The ERF provides a particular solution of the complex equation

$$z_p(t) = \frac{e^{it}}{p(i)} = \frac{e^{it}}{1 + 2i} = \frac{e^{it}}{\sqrt{5}\,e^{i\phi}} = \frac{1}{\sqrt{5}}\,e^{i(t-\phi)}$$

where ϕ is the polar angle of $1 + 2i$. Take the imaginary part of z_p

$$x_p(t) = \operatorname{Im} z_p(t) = \frac{1}{\sqrt{5}}\,\sin(t - \phi)$$

This is a sinusoidal solution of the real equation. Its amplitude is $1/\sqrt{5}$.

(d) If $x(t) = t^3$ is a solution, then $q(t) = \ddot{x} + 2\dot{x} + 2x = 6t + 6t^2 + t^3$.

(e) The general solution is $x(t) = t^3 + x_h(t)$, where $x_h(t)$ is a solution of the associated homogeneous equation. Since the characteristic polynomial $s^2 + 2s + 2$ has roots $-1 \pm i$,

$$x(t) = t^3 + x_h(t) = t^3 + c_1 e^{-t}\cos t + c_2 e^{-t}\sin t.$$

5. (a) See the formula sheet for the definition of sq(t). The graph of $f(t)$ is a square wave of period 2π. It has a horizontal line segment of height 1 in the range $-\pi/2 < t < \pi/2$ and a horizontal line segment of height -1 in the range $\pi/2 < t < 3\pi/2$.

(b) Replace t by $t + \pi/2$ in the definition of sq(t)

$$\begin{aligned} f(t) = \mathrm{sq}\left(t + \frac{\pi}{2}\right) &= \frac{4}{\pi}\left[\sin\left(t + \frac{\pi}{2}\right) + \frac{1}{3}\sin\left(3t + \frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(5t + \frac{5\pi}{2}\right) + \dots\right] \\ &= \frac{4}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t + \dots\right) \end{aligned}$$

(c) First consider the complex equation

$$\ddot{z} + z = e^{int} \qquad \text{for a positive integer } n.$$

The characteristic polynomial is $p(s) = s^2 + 1$. One of the ERFs provides a particular solution of the complex equation

$$\begin{aligned} z_p(t) &= \frac{e^{int}}{p(in)} = \frac{e^{int}}{1-n^2}\,, \qquad n \neq 1 \\ z_p(t) &= \frac{te^{it}}{p'(i)} = \frac{te^{int}}{2i}\,, \qquad n = 1 \end{aligned}$$

The imaginary parts of these functions

$$\begin{aligned} u_p(t) &= \mathrm{Im}\left(\frac{e^{int}}{1-n^2}\right) = \frac{\sin nt}{1-n^2}\,, \qquad n \neq 1 \\ u_p(t) &= \mathrm{Im}\left(\frac{te^{it}}{2i}\right) = -\frac{1}{2}t\cos t\,, \qquad n = 1 \end{aligned}$$

satisfy the imaginary part of the above complex equation, namely

$$\ddot{u} + u = \sin nt.$$

By linearity, a solution of $\ddot{x} + x = \mathrm{sq}(t)$ is given by

$$x_p(t) = \frac{4}{\pi}\left(-\frac{1}{2}t\cos t + \frac{1}{3}\cdot\frac{\sin 3t}{1-3^2} + \frac{1}{5}\cdot\frac{\sin 5t}{1-5^2} + \dots\right).$$

6. (a) For $t < 0$, the graph is flat on t-axis.
For $0 < t < 1$, the graph is flat at 1 unit above t-axis.
For $1 < t < 3$, the graph is flat at 1 unit below t-axis.
For $3 < t < 4$, the graph is flat at 1 unit above t-axis.
For $t > 4$, the graph is flat on t-axis.

(b)
$$\begin{aligned} v(t) &= [u(t) - u(t-1)] - [u(t-1) - u(t-3)] + [u(t-3) - u(t-4)] \\ &= u(t) - 2u(t-1) + 2u(t-3) - u(t-4) \end{aligned}$$

(c) The graph coincides with t-axis for all t, except for two upward spikes at $t = 0, 3$ and two downward spikes at $t = 1, 4$.

(d) $\dot{v}(t) = \delta(t) - 2\delta(t-1) + 2\delta(t-3) - \delta(t-4)$

(e) By the fundamental solution theorem (a.k.a. Green's formula),

$$x(t) = (q * w)(t) = \int_0^t q(t-\tau)w(\tau)\,d\tau = \int_{a(t)}^{b(t)} w(\tau)\,d\tau.$$

Now $q(t-\tau) = 1$ only for $0 < t - \tau < 1$, or $t - 1 < \tau < t$, and it is zero elsewhere. Therefore the upper limit $b(t)$ equals t. The lower limit $a(t)$ is $t-1$ if $t-1 > 0$, or 0 if $t - 1 < 0$. In other words, $a(t) = (t-1)u(t-1)$.

7. (a) The transfer function is $W(s) = \dfrac{1}{p(s)} = \dfrac{1}{2s^2 + 8s + 16}$.

(b) The unit impulse response $w(t)$ is the inverse Laplace transform of $W(s)$. In other words,

$$\begin{aligned} \mathcal{L}(w(t)) &= \frac{1}{2s^2+8s+16} = \frac{1}{2[(s+2)^2+4]} \\ \Rightarrow \quad \mathcal{L}(e^{2t}w(t)) &= \frac{1}{2(s^2+4)} = \frac{1}{4}\,\mathcal{L}(\sin 2t) \end{aligned}$$

Therefore $e^{2t}w(t) = \dfrac{1}{4}\sin 2t$, and $w(t) = \dfrac{1}{4}e^{-2t}\sin 2t$.

(c) Take the Laplace transform of

$$p(D)x = 2\ddot{x}(t) + 8\dot{x}(t) + 16x(t) = \sin t$$

with the initial conditions $x(0+) = 1$, $\dot{x}(0+) = 2$. This yields

$$\begin{aligned} 2[s^2X(s) - s - 2] + 8[sX(s) - 1] + 16X(s) &= \frac{1}{s^2+1} \\ \Rightarrow \quad X(s) &= \frac{1}{2s^2+8s+16}\left(\frac{1}{s^2+1} + 2s + 12\right) \end{aligned}$$

8. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 12 \\ 3 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 36 = (\lambda - 8)(\lambda + 4).$$

Therefore the eigenvalues are $\lambda = 8, -4$.

(b) For $\lambda = 8$, solve $(A - 8I)\mathbf{v} = \mathbf{0}$. Since $A - 8I = \begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix}$, a solution is $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

For $\lambda = -4$, solve $(A + 4I)\mathbf{v} = \mathbf{0}$. Since $A + 4I = \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix}$, a solution is $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(c) The following is a fundamental matrix for $\dot{\mathbf{u}} = B\mathbf{u}$

$$F(t) = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$

Then e^{tB} can be computed as $F(t)F(0)^{-1}$.

$$F(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad F(0)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$e^{tB} = F(t)F(0)^{-1} = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t + e^{2t} & e^t - e^{2t} \\ e^t - e^{2t} & e^t + e^{2t} \end{bmatrix}$$

(d) The general solution of $\dot{\mathbf{u}} = B\mathbf{u}$ is

$$\mathbf{u}(t) = c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -e^{2t} \\ e^{2t} \end{bmatrix} = F(t)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The given initial condition implies

$$\begin{aligned} \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= F(0)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= F(0)^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \end{aligned}$$

Therefore the solution of the initial value problem is $\mathbf{u}(t) = \dfrac{1}{2}\begin{bmatrix} 3e^t + e^{2t} \\ 3e^t - e^{2t} \end{bmatrix}$.

9. (a) The phase portrait has the following features:
   - All trajectories start at $(0,0)$ and run off to infinity.
   - There are straight line trajectories along the lines $y = \pm x$.
   - All other trajectories are tangent to $y = x$ at $(0,0)$.
   - No two trajectories cross each other.

   (b) $\operatorname{Tr} A = a + 1, \quad \det A = a + 4, \quad \Delta = (\operatorname{Tr} A)^2 - 4(\det A) = (a-5)(a+3)$

   (i) $\det A < 0 \quad \Leftrightarrow \quad a < -4$
   (ii) not for any a
   (iii) $\Delta > 0$, $\operatorname{Tr} A < 0$ and $\det A > 0 \quad \Leftrightarrow \quad -4 < a < -3$
   (iv) $\Delta < 0$ and $\operatorname{Tr} A < 0 \quad \Leftrightarrow \quad -3 < a < -1$; counterclockwise
   (v) $\Delta < 0$ and $\operatorname{Tr} A > 0 \quad \Leftrightarrow \quad -1 < a < 5$
   (vi) $\Delta = 0$ and $\operatorname{Tr} A > 0 \quad \Leftrightarrow \quad a = 5$

10. (a) The equilibria are the solutions of

$$\dot{x} = x^2 - y^2 = 0, \qquad \dot{y} = x^2 + y^2 - 8 = 0.$$

   This implies $(x^2, y^2) = (4,4)$, so that $(x,y) = (2,2), (2,-2), (-2,2), (-2,-2)$.

   (b) The Jacobian is $J(x,y) = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix}$. In particular, $J(-2,-2) = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix}$.

(c) The linearization of the nonlinear system at $(-2,-2)$ is the linear system $\dot{\mathbf{u}} = J(-2,-2)\mathbf{u}$. A computation shows that the eigenvalues of $J(-2,-2)$ are $-4 \pm 4i$. The first component of $\mathbf{u}(t)$ is of the form

$$c_1 e^{-4t} \cos 4t + c_2 e^{-4t} \sin 4t = A e^{-4t} \cos(4t - \phi).$$

This means $x(t) \approx -2 + Ae^{-4t}\cos(4t - \phi)$ near $(-2,-2)$.

(d) Let $f(x) = 2x - 3x^2 + x^3$. The phase line in problem 2(a) shows that $\dot{x} = f(x)$ has a stable equilibrium at $x = 1$.

The linearization of the nonlinear equation at $x = 1$ is the linear equation $\dot{u} = f'(1)u = -u$. Its solutions are $u(t) = Ae^{-t}$. This means $x(t) \approx 1 + Ae^{-t}$ near $x = 1$.