18.06 Problem Set 1 Solutions

Total: 100 points

Section 1.2. Problem 23: The figure shows that $\cos(\alpha) = v_1/\|v\|$ and $\sin(\alpha) = v_2/\|v\|$. Similarly $\cos(\beta)$ is _______ and $\sin(\beta)$ is ______. The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos(\alpha)\cos(\beta) + \sin(\beta)\sin(\alpha)$ for $\cos(\beta - \alpha)$ to find $\cos(\theta) = v \cdot w/\|v\|\|w\|$.

$\boxed{\text{Solution}}$ (4 points)

First blank: $w_1/\|w\|$. Second blank: $w_2/\|w\|$. Substituting into the trigonometry formula yields

$$\cos(\beta - \alpha) = (w_1/\|w\|)(v_1/\|v\|) + (w_2/\|w\|)(v_2/\|v\|) = v \cdot w/\|v\|\|w\|.$$

Section 1.2. Problem 28: Can three vectors in the xy plane have $u \cdot v < 0$ and $v \cdot w < 0$ and $u \cdot w < 0$?

$\boxed{\text{Solution}}$ (12 points)

Yes. For instance take $u = (1, 0)$, $v = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $w = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Notice $u \cdot v = v \cdot w = u \cdot w = -\frac{1}{2}$.

Section 1.3. Problem 4: Find a combination $x_1w_1 + x_2w_2 + x_3w_3$ that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \; w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \; w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are (independent)(dependent). The three vectors lie in a _______. The matrix W with those columns is *not invertible.*

$\boxed{\text{Solution}}$ (4 points)

Observe $w_1 - 2w_2 + w_3 = 0$. The vectors are **dependent**. They lie in a **plane**.

Section 1.3. Problem 13: The very last words say that the 5 by 5 centered difference matrix *is not* invertible. Write down the 5 equations $Cx = b$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero?

$\boxed{\text{Solution}}$ (12 points)
The 5 by 5 centered difference matrix is

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The five equations $Cx = b$ are

$$x_2 = b_1, \; -x_1 + x_3 = b_2, \; -x_2 + x_4 = b_3, \; -x_3 + x_5 = b_4, \; -x_4 = b_5.$$

Observe that the sum of the first, third, and fifth equations is zero. Similarly, $b_1 + b_3 + b_5 = 0$.

Section 2.1. Problem 29: Start with the vector $u_0 = (1, 0)$. Multiply again and again by the same "Markov matrix" $A = [.8\ .3; .2\ .7]$. The next three vectors are u_1, u_2, u_3:

$$u_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \end{bmatrix} \qquad u_2 = Au_1 = \underline{\qquad\quad} \qquad u_3 = Au_2 = \underline{\qquad\quad}.$$

What property do you notice for all four vectors u_0, u_1, u_2, u_3.

$\boxed{\text{Solution}}$ (4 points)
Computing, we get

$$u_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix} \quad u_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}.$$

All four vectors have components that sum to one.

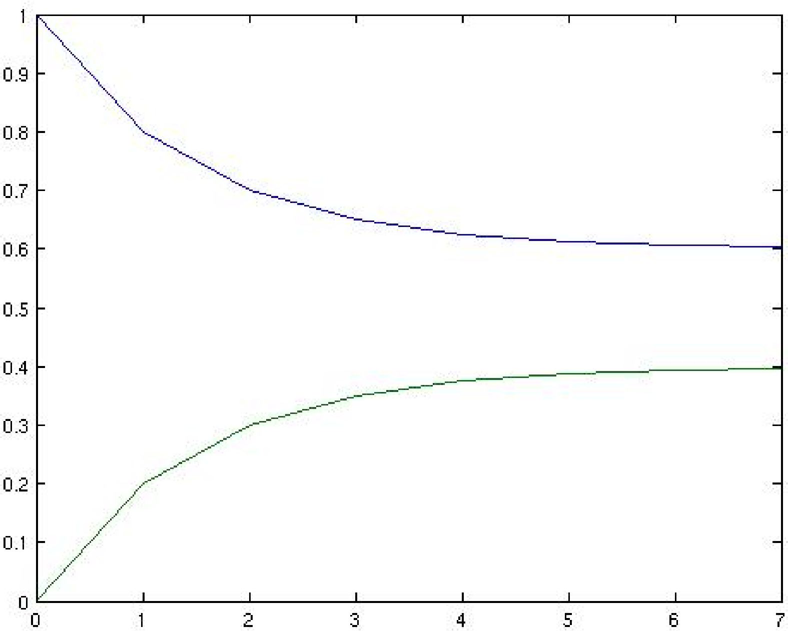
Section 2.1. Problem 30: Continue Problem 29 from $u_0 = (1, 0)$ to u_7, and also from $v_0 = (0, 1)$ to v_7. What do you notice about u_7 and v_7? Here are two MATLAB codes, with while and for. They plot u_0 to u_7 and v_0 to v_7.
The u's and the v's are approaching a steady state vector s. Guess that vector and check that $As = s$. If you start with s, then you stay with s.

$\boxed{\text{Solution}}$ (12 points)

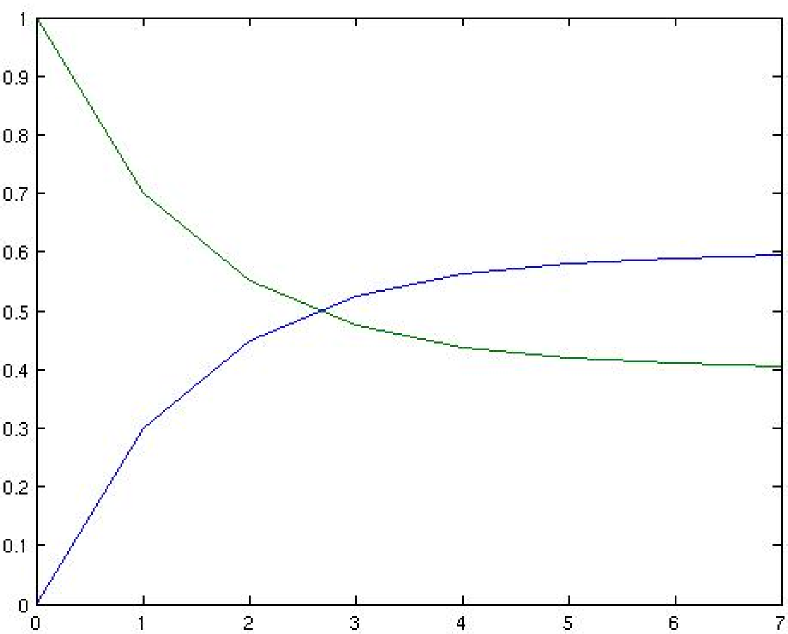
Here is the code entered into MATLAB.

```
u= [1; 0];
A= [.8 .3; .2 .7];
x= u;
k= [0 : 7];
while size (x, 2)<=7
u= A*u;  x= [x, u];
end
plot (k,x)
v= [0;1];
A= [.8 .3; .2 .7];
x= v;
k= [0:7];
for j=1:7
v= A*v;  x=[x, v];
end
plot (k, x)
```

In this graph, we see that the sequence $u_1, u_2, \ldots, u_7$ is approaching $(.6, .4)$.



In this graph, we see that the sequence $v_1, v_2, \ldots, v_7$ is approaching $(.6, .4)$.



From the graphs, we guess that $s = (.6, .4)$ is a steady state vector. We verify this with the computation

$$As = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}.$$

Section 2.2. Problem 20: Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a _________ of the first two rows. Find a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$.

$\boxed{\text{Solution}}$ (4 points)
The system is singular if row 3 of A is a **linear combination** of the first two rows. There are many possible choices of a third equation that cannot be solved together with the ones given. An example is $2x + 5y + 4z = 1$. Note that the left hand side of the third equation is the three times the left hand side of thte first minus the left hand side of the second. However, the right hand side does not satisfy this relation.

Section 2.2. Problem 32: Start with 100 equations $Ax = 0$ for 100 unknowns $x = (x_1, \dots, x_{100})$. Suppose elimination reduces the 100th equation to $0 = 0$, so the system is "singular".
(a) Elimination takes linear combinations of the rows. So this singular system has the singular property: Some linear combination of the 100 **rows** is ________.
(b) Singular systems $Ax = 0$ have infinitely many solutions. This means that some linear combination of the 100 **columns** is _________.
(c) Invent a 100 by 100 singular matrix with no zero entries.
(d) For your matrix, describe in words the row picture and the column picture of $Ax = 0$. Not necessary to draw 100-dimensional space.

$\boxed{\text{Solution}}$ (12 points)
(a) Zero. (b) Zero. (c) There are many possible answers. For instance, the matrix for which every row is $(1\ 2\ 3\ \cdots\ 100)$. (d) The row picture is 100 copies of the hyperplane in 100-space defined by the equation

$$x_1 + 2x_2 + 3x_3 + \cdots + 100x_{100} = 0.$$

The column picture is the 100 vectors proportional to $(1\ 1\ 1\ \cdots\ 1)$ of lengths $10, 20, \dots, 1000$.

Section 2.3. Problem 22: The entries of A and x are a_{ij} and x_j. So the first component of Ax is $\sum a_{1j}x_j = a_{11}x_1 + \cdots + a_{1n}x_n$. If E_{21} subtracts row 1 from row 2, write a formula for
(a) the third component of Ax
(b) the $(2,1)$ entry of $E_{21}A$
(c) the $(2,1)$ entry of $E_{21}(E_{21}A)$
(d) the first component of $E_{21}Ax$.

$\boxed{\text{Solution}}$ (4 points)
(a) $\sum a_{3j}x_j$. (b) $a_{21} - a_{11}$. (c) $a_{21} - 2a_{11}$. (d) $\sum a_{1j}x_j$.

Section 2.3. Problem 29: Find the triangular matrix E that reduces "*Pascal's matrix*" to a smaller Pascal:

$$E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Which matrix M (multiplying several E's) reduces Pascal all the way to I?

$\boxed{\text{Solution}}$ (12 points)

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

One can eliminate the second column with the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and the third column with the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Multiplying these together, we get

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Section 2.4. Problem 32: Suppose you solve $Ax = b$ for three special right sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

$\boxed{\text{Solution}}$ (4 points)
The matrix AX has columns Ax_1, Ax_2, and Ax_3. Therefore, $AX = I$.

Section 2.4. Problem 36: Suppose A is m by n, B is n by p, and C is p by q. Then the multiplication count for $(AB)C$ is $mnp + mpq$. The multiplication count from A times BC with $mnq + npq$ separate multiplications.
(a) If A is 2 by 4, B is 4 by 7, and C is 7 by 10, do you prefer $(AB)C$ or $A(BC)$?
(b) With N-component vectors, would you choose $(u^T v)w^T$ or $u^T(vw^T)$?
(c) Divide by $mnpq$ to show that $(AB)C$ is faster when $n^{-1} + q^{-1} < m^{-1} + p^{-1}$.

$\boxed{\text{Solution}}$ (12 points)
(a) Note that $(AB)C$ has $2 \cdot 4 \cdot 7 + 2 \cdot 7 \cdot 10 = 196$ multiplications and $A(BC)$ has $2 \cdot 4 \cdot 10 + 4 \cdot 7 \cdot 10 = 360$. Hence, we prefer $(AB)C$.
(b) We prefer $(u^T v)w^T$; it requires $2N$ multiplications. On the other hand, the multiplication count for $u^T(vw^T)$ is $2N^2$.
(c) Note $(AB)C$ is faster when $mnp + mpq < mnq + npq$. Dividing by $mnpq$, we get $q^{-1} + n^{-1} < p^{-1} + m^{-1}$.

Section 2.5. Problem 7: If A has row 1 + row 2 = row 3, show that A is not invertible:
(a) Explain why $Ax = (1, 0, 0)$ cannot have a solution.
(b) Which right sides (b_1, b_2, b_3) might allow a solution to $Ax = b$?
(c) What happens to row 3 in elimination?

$\boxed{\text{Solution}}$ (4 points)
(a) Suppose A has row vectors A_1, A_2, A_3, and x is a solution to $Ax = (1, 0, 0)$. Then $A_1 \cdot x = 1$, $A_2 \cdot x = 0$, and $A_3 \cdot x = 0$. But $A_1 + A_2 = A_3$ means that $A_1 \cdot x + A_2 \cdot x = A_3 \cdot x$ and $1 + 0 = 0$, a contradiction.
(b) If $Ax = (b_1, b_2, b_3)$, then $A_1 \cdot x = b_1$, $A_2 \cdot x = b_2$, $A_3 \cdot x = b_3$. Since $A_1 + A_2 = A_3$, we deduce $b_1 + b_2 = b_3$.
(c) In the eliminated matrix, the third row will be zero.

18.06 Linear Algebra
Spring 2010

18.06 Problem Set 2 Solution

Total: 100 points

Section 2.5. Problem 24: Use Gauss-Jordan elimination on $[U\ I]$ to find the upper triangular U^{-1}:

$$UU^{-1} = I \qquad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\boxed{\text{Solution}}$ (4 points): Row reduce $[U\ I]$ to get $[I\ U^{-1}]$ as follows (here R_i stands for the ith row):

$$\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{matrix} (R_1 = R_1 - aR_2) \\ (R_2 = R_2 - cR_2) \\ \end{matrix} \begin{bmatrix} 1 & 0 & b-ac & 1 & -a & 0 \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{matrix} \big(R_1 = R_1 - (b-ac)R_3\big) \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Section 2.5. Problem 40: (Recommended) A is a 4 by 4 matrix with 1's on the diagonal and $-a, -b, -c$ on the diagonal above. Find A^{-1} for this bidiagonal matrix.

$\boxed{\text{Solution}}$ (12 points): Row reduce $[A\ I]$ to get $[I\ A^{-1}]$ as follows (here R_i stands for the ith row):

$$\begin{bmatrix} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{matrix} (R_1 = R_1 + aR_2) \\ (R_2 = R_2 + bR_2) \\ (R_3 = R_3 + cR_4) \\ \end{matrix} \begin{bmatrix} 1 & 0 & -ab & 0 & 1 & a & 0 & 0 \\ 0 & 1 & 0 & -bc & 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{matrix} (R_1 = R_1 + abR_3) \\ (R_2 = R_2 + bcR_4) \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Alternatively, write $A = I - N$. Then N has a, b, c above the main diagonal, and all other entries equal to 0. Hence $A^{-1} = (I - N)^{-1} = I + N + N^2 + N^3$ as $N^4 = 0$.

Section 2.6. Problem 13: (Recommended) Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

$\boxed{\text{Solution}}$ (4 points): Elimination subtracts row 1 from rows 2–4, then row 2 from rows 3–4, and finally row 3 from row 4; the result is U. All the multipliers ℓ_{ij} are equal to 1; so L is the lower triangular matrix with 1's on the diagonal and below it.

$$A \longrightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \longrightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = U, \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The pivots are the nonzero entries on the diagonal of U. So there are four pivots when these four conditions are satisfied: $a \neq 0$, $b \neq a$, $c \neq b$, and $d \neq c$.

Section 2.6. Problem 18: If $A = LDU$ and also $A = L_1D_1U_1$ with all factors invertible, then $L = L_1$ and $D = D_1$ and $U = U_1$. *"The three factors are unique."*

Derive the equation $L_1^{-1}LD = D_1U_1U^{-1}$. Are the two sides triangular or diagonal? Deduce $L = L_1$ and $U = U_1$ (they all have diagonal 1's). Then $D = D_1$.

$\boxed{\text{Solution}}$ (4 points): Notice that $LDU = L_1D_1U_1$. Multiply on the left by L_1^{-1} and on the right by U^{-1}, getting
$$L_1^{-1}LDUU^{-1} = L_1^{-1}L_1D_1U_1U^{-1}.$$
But $UU^{-1} = I$ and $L_1^{-1}L_1 = I$. Thus $L_1^{-1}LD = D_1U_1U^{-1}$, as desired.

The left side $L_1^{-1}LD$ is lower triangular, and the right side $D_1U_1U^{-1}$ is upper triangular. But they're equal. So they're both diagonal. Hence $L_1^{-1}L$ and U_1U^{-1} are diagonal too. But they have diagonal 1's. So they're both equal to I. Thus $L = L_1$ and $U = U_1$. Also $L_1^{-1}LD = D_1U_1U^{-1}$. Thus $D = D_1$.

Section 2.6. Problem 25: For the 6 by 6 second difference constant-diagonal matrix K, put the pivots and multipliers into $K = LU$. (L and U will have only two nonzero diagonals, because K has three.) Find a formula for the i, j entry of L^{-1}, by software like `MATLAB` using `inv(L)` or by looking for a nice pattern.

$$\mathbf{-1,\ 2,\ -1\ matrix} \quad K = \begin{bmatrix} 2 & -1 & & & & \\ -1 & \bullet & \bullet & & & \\ & \bullet & \bullet & \bullet & & \\ & & \bullet & \bullet & \bullet & \\ & & & \bullet & \bullet & -1 \\ & & & & -1 & 2 \end{bmatrix} = \texttt{toeplitz([2 -1 0 0 0 0])}.$$

$\boxed{\text{Solution}}$ (12 points): Here is the transcript of a session with the software `Octave`, which is the open-source GNU clone of `MATLAB`. The decomposition $K = LU$ is found using the teaching code `slu.m`, available from

`http://web.mit.edu/18.06/www/Course-Info/Tcodes.html`

```
octave:1> K=toeplitz([2 -1 0 0 0 0]);
octave:2> [L,U]=slu(K);
octave:3> inv(L)
ans =
   1.00000   0.00000   0.00000   0.00000   0.00000   0.00000
   0.50000   1.00000   0.00000   0.00000   0.00000   0.00000
   0.33333   0.66667   1.00000   0.00000   0.00000   0.00000
   0.25000   0.50000   0.75000   1.00000   0.00000   0.00000
   0.20000   0.40000   0.60000   0.80000   1.00000   0.00000
   0.16667   0.33333   0.50000   0.66667   0.83333   1.00000
```

So the nice pattern is $(L^{-1})_{ij} = j/i$ for $j \le i$ and $(L^{-1})_{ij} = 0$ for $j > i$.

Section 2.6. Problem 26: If you print K^{-1}, it doesn't look good. But if you print $7K^{-1}$ (when K is 6 by 6), that matrix looks wonderful. Write down $7K^{-1}$ by hand, following this pattern:

1 Row 1 and column 1 are (6, 5, 4, 3, 2, 1).
2 On and above the main diagonal, row i is i times row 1.
3 On and below the main diagonal, column j is j times column 1.

Multiply K times that $7K^{-1}$ to produce $7I$. Here is that pattern for $n = 3$:

3 by 3 case
The determinant of this K is 4

$$(K)(4K^{-1}) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix}.$$

$\boxed{\text{Solution}}$ (12 points): For $n = 6$, following the pattern yields this matrix:

$$\begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 10 & 8 & 6 & 4 & 2 \\ 4 & 8 & 12 & 9 & 6 & 3 \\ 3 & 6 & 9 & 12 & 8 & 4 \\ 2 & 4 & 6 & 8 & 10 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}.$$

Here is the transcript of an `Octave` session that multiplies K times that $7K^{-1}$.

```
octave:1> K=toeplitz([2 -1 0 0 0 0]);
octave:2> M=[6 5 4 3 2 1;5 10 8 6 4 2;4 8 12 9 6 3;3 6 9 12 8 4;2 4 6 8 10 5;1 2 3 4 5 6];
octave:3> K*M
ans =
   7   0   0   0   0   0
   0   7   0   0   0   0
   0   0   7   0   0   0
   0   0   0   7   0   0
   0   0   0   0   7   0
   0   0   0   0   0   7
```

Section 2.7. Problem 13: (a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).
(b) Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \ne I$.

$\boxed{\text{Solution}}$ (4 points): (a) Let P move the rows in a cycle: the first to the second, the second to the third, and the third to the first. So

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad P^3 = I.$$

(b) Let $\widehat{P}$ be the block diagonal matrix with 1 and P on the diagonal: $\widehat{P} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & P \end{smallmatrix}\right)$. Since $P^3 = I$, also $\widehat{P}^3 = I$. So $\widehat{P}^4 = \widehat{P} \neq I$.

Section 2.7. Problem 36: A *group* of matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups?
Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P, matrices with $Q^{\mathrm{T}} = Q^{-1}$. ***Invent two more matrix groups.***

$\boxed{\text{Solution}}$ (4 points): Yes, the lower triangular matrices L with 1's on the diagonal form a group. Clearly, the product of two is a third. Further, the Gauss-Jordan method shows that the inverse of one is another.

No, the symmetric matrices do not form a group. For example, here are two symmetric matrices A and B whose product AB is not symmetric.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}.$$

No, the positive matrices do not form a group. For example, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is positive, but its inverse $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ is not.

Yes, clearly, the diagonal invertible matrices form a group.

Yes, clearly, the permutation matrices matrices form a group.

Yes, the matrices with $Q^{\mathrm{T}} = Q^{-1}$ form a group. Indeed, if A and B are two such matrices, then so are AB and A^{-1}, as

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}} = B^{-1} A^{-1} = (AB)^{-1} \quad \text{and} \quad (A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = A^{-1}.$$

There are many more matrix groups. For example, given two, the block matrices $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ form a third as A ranges over the first group and B ranges over the second. Another example is the set of all products cP where c is a nonzero scalar and P is a permutation matrix of given size.

Section 2.7. Problem 40: Suppose Q^{T} *equals* Q^{-1} (transpose equal inverse, so $Q^{\mathrm{T}} Q = I$).

(a) Show that the columns $q_1, \ldots, q_n$ are unit vectors: $\| q_i \|^2 = 1$.

(b) Show that every two distinct columns of Q are perpendicular: $q_i^{\mathrm{T}} q_j = 0$ for $i \neq j$.

(c) Find a 2 by 2 example with first entry $q_{11} = \cos\theta$.

$\boxed{\text{Solution}}$ (12 points): In any case, the ij entry of $Q^{\mathrm{T}} Q$ is $q_i^{\mathrm{T}} q_j$. So $Q^{\mathrm{T}} Q = I$ leads to (a) $q_i^{\mathrm{T}} q_i = 1$ for all i and to (b) $q_i^{\mathrm{T}} q_j = 0$ for $i \neq j$. As for (c), the rotation matrix $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ works.

Section 3.1. Problem 18: True or false (check addition or give a counterexample):

(a) The symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = A$) form a subspace.

(b) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = -A$) form a subspace.

(c) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} \neq A$) form a subspace.

$\boxed{\text{Solution}}$ (4 points): (a) True: $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = B$ lead to $(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}} = A + B$.

(b) True: $A^{\mathrm{T}} = -A$ and $B^{\mathrm{T}} = -B$ lead to $(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}} = -A - B = -(A+B)$.

(c) False: $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Section 3.1. Problem 23: (Recommended) If we add an extra column b to a matrix A, then the column space gets larger unless ____. Give an example where the column space gets larger and an example where it doesn't. Why is $Ax = b$ solvable exactly when the column space *doesn't* get larger—it is the same for A and $[\,A\ b\,]$?

$\boxed{\text{Solution}}$ (4 points): The column space gets larger unless <u>it contains b</u>; that is, b is a linear combination of the columns of A. For example, let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$; then the column space gets larger if $b = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and it doesn't if $b = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. The equation $Ax = b$ is solvable exactly when b is a (nontrivial) linear combination of the columns of A (with the components of x as combining coefficients); so $Ax = b$ is solvable exactly when b lies in the column space, so exactly when the column space doesn't get larger.

Section 3.1. Problem 30: Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$.

(a) **Definition**: The sum $\mathbf{S} + \mathbf{T}$ contains all sums $\mathbf{s} + \mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$. Show that $\mathbf{S} + \mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space.

(b) If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S} + \mathbf{T}$ and $\mathbf{S} \cup \mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and $\mathbf{T}$ or both. Explain this statement: *The span of* $\mathbf{S} \cup \mathbf{T}$ *is* $\mathbf{S} + \mathbf{T}$. (Section 3.5 returns to this word "span.")

$\boxed{\text{Solution}}$ (12 points): (a) Let $\mathbf{s}, \mathbf{s}'$ be vectors in $\mathbf{S}$, let $\mathbf{t}, \mathbf{t}'$ be vectors in $\mathbf{T}$, and let c be a scalar. Then

$$(\mathbf{s} + \mathbf{t}) + (\mathbf{s}' + \mathbf{t}') = (\mathbf{s} + \mathbf{s}') + (\mathbf{t} + \mathbf{t}') \quad \text{and} \quad c(\mathbf{s} + \mathbf{t}) = c\mathbf{s} + c\mathbf{t}.$$

Thus $\mathbf{S} + \mathbf{T}$ is closed under addition and scalar multiplication; in other words, it satisfies the two requirements for a vector space.

(b) If $\mathbf{S}$ and $\mathbf{T}$ are distinct lines, then $\mathbf{S} + \mathbf{T}$ is a plane, whereas $\mathbf{S} \cup \mathbf{T}$ is not even closed under addition. The span of $\mathbf{S} \cup \mathbf{T}$ is the set of all combinations of vectors in this union. In particular, it contains all sums $\mathbf{s} + \mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$, and these sums form $\mathbf{S} + \mathbf{T}$. On the other hand, $\mathbf{S} + \mathbf{T}$ contains both $\mathbf{S}$ and $\mathbf{T}$; so it contains $\mathbf{S} \cup \mathbf{T}$. Further, $\mathbf{S} + \mathbf{T}$ is a vector space. So it contains all combinations of vectors in itself; in particular, it contains the span of $\mathbf{S} \cup \mathbf{T}$. Thus the span of $\mathbf{S} \cup \mathbf{T}$ is $\mathbf{S} + \mathbf{T}$.

Section 3.1. Problem 32: Show that the matrices A and $[\,A\ AB\,]$ (with extra columns) have the same column space. But find a square matrix with $\mathbf{C}(A^2)$ smaller than $\mathbf{C}(A)$. Important point:

An n by n matrix has $\mathbf{C}(A) = \mathbf{R}^n$ exactly when A is an ____ matrix.

$\boxed{\text{Solution}}$ (12 points): Each column of AB is a combination of the columns of A (the combining coefficients are the entries in the corresponding column of B). So any combination of the columns of $[\,A\ AB\,]$ is a combination of the columns of A alone. Thus A and $[\,A\ AB\,]$ have the same column space.

Let $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then $A^2 = 0$, so $\mathbf{C}(A^2) = \mathbf{Z}$. But $\mathbf{C}(A)$ is the line through $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.

An n by n matrix has $\mathbf{C}(A) = \mathbf{R}^n$ exactly when A is an <u>invertible</u> matrix, because $Ax = b$ is solvable for any given b exactly when A is invertible.

18.06 PSET 3 SOLUTIONS

FEBRUARY 22, 2010

Problem 1. (§3.2, #18) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in Problem 17. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

Solution. **(4 points)** The equation $x = 12 + 3y + z$ says it all:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \left(= \begin{bmatrix} 12 + 3y + z \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} \boxed{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \boxed{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \boxed{1} \\ 0 \\ 1 \end{bmatrix}.$$

□

Problem 2. (§3.2, #24) (If possible...) Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$.

Solution. **(4 points)** $\boxed{\text{Not possible}}$: Such a matrix A must be 3×3. Since the nullspace is supposed to contain two independent vectors, A can have at most $3 - 2 = 1$ pivots. Since the column space is supposed to contain two independent vectors, A must have at least 2 pivots. These conditions cannot both be met! □

Problem 3. (§3.2, #36) How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Solution. **(12 points)** $\boxed{\mathbf{N}(C) = \mathbf{N}(A) \cap \mathbf{N}(B)}$ just the intersection: Indeed,

$$C\mathbf{x} = \begin{bmatrix} A\mathbf{x} \\ B\mathbf{x} \end{bmatrix}$$

so that $C\mathbf{x} = 0$ if and only if $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$. (...and as a nitpick, it wouldn't be quite sloppy instead write "if and only if $A\mathbf{x} = B\mathbf{x} = 0$"—those are zero vectors of potentially different length, hardly equal). □

Problem 4. (§3.2, #37) Kirchoff's Law says that *current in = current out* at every node. This network has six currents $y_1, \ldots, y_6$ (the arrows show the positive direction, each y_i could be positive or negative). Find the four equations $A\mathbf{y} = 0$ for Kirchoff's Law at the four nodes. Find three special solutions in the nullspace of A.

Solution. **(12 points)** The four equations are, in order by node,

$$\begin{aligned} y_1 - y_3 + y_4 &= 0 \\ -y_1 + y_2 + y_5 &= 0 \\ -y_2 + y_3 + y_6 &= 0 \\ -y_4 - y_5 - y_6 &= 0 \end{aligned}$$

or in matrix form $A\mathbf{y} = 0$ for

$$\boxed{A = \begin{bmatrix} 1 & 0-1 & 1 & 0 & 0 & \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}}$$

Adding the last three rows to the first eliminates it, and shows that we have three "pivot variables" y_1, y_2, y_4 and three "free variables" y_3, y_5, y_6. We find the special solutions by back-substitution from $(y_3, y_5, y_6) = (1,0,0), (0,1,0), (0,0,1)$:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$$ □

Problem 5. (§3.3, #19) Suppose A and B are n by n matrices, and $AB = I$. Prove from $\text{rank}(AB) \leq \text{rank}(A)$ that the rank of A is n. So A is invertible and B must be its two-sided inverse (Section 2.5). Therefore $BA = I$ (which is not so obvious!).

Solution. **(4 points)** Since A is n by n, $\text{rank}(A) \leq n$ and conversely

$$n = \text{rank}(I_n) = \text{rank}(AB) \leq \text{rank}(A).$$

The rest of the problem statement seems to be "commentary," and not further things to do. □

Problem 6. (§3.3, #25) *Neat fact* **Every m by n matrix of rank r reduces to (m by r) times (r by n):**

$$A = (\text{pivot columns of } A)\,(\text{first } r \text{ rows of } R)) = (\mathbf{COL})\,(\mathbf{ROW})\,.$$

Write the 3 by 4 matrix A in equation (1) at the start of this section as the product of the 3 by 2 matrix from the pivot columns and the 2 by 4 matrix from R.

Solution. **(4 points)**

$$\boxed{A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix}}$$ □

Problem 7. (§3.3, #27) Suppose R is m by n of rank r, with pivot columns first:

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(a) What are the shapes of those four blocks?
(b) Find a *right-inverse* B with $RB = I$ if $r = m$.
(c) Find a *left-inverse* C with $CR = I$ if $r = n$.
(d) What is the reduced row echelon form of R^T (with shapes)?
(e) What is the reduced row echelon form of $R^T R$ (with shapes)?

Prove that $R^T R$ has the same nullspace as R. Later we show that $A^T A$ always has the same nullspace as A (a valuable fact).

Solution. **(12 points)**

(a)

$$\boxed{\begin{bmatrix} r \times r & r \times (n-r) \\ (m-r) \times r & (m-r) \times (n-r) \end{bmatrix}}$$

(b) In this case

$$R = \begin{bmatrix} I & F \end{bmatrix} \qquad \text{so we can take} \qquad \boxed{B = \begin{bmatrix} I_{r\times r} \\ 0_{(n-r)\times r} \end{bmatrix}}$$

(c) In this case

$$R = \begin{bmatrix} I & 0 \end{bmatrix} \qquad \text{so we can take} \qquad \boxed{C = \begin{bmatrix} I_{r\times r} & 0_{r\times(m-r)} \end{bmatrix}}$$

(d) Note that

$$R^T = \begin{bmatrix} I_{r\times r} & 0_{r\times(m-r)} \\ F^T & 0_{(n-r)\times(m-r)} \end{bmatrix} \qquad \text{so that} \qquad rref(R^T) = \boxed{\begin{bmatrix} I_{r\times r} & 0_{r\times(m-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{bmatrix}}$$

(e) Note that

$$R^T R = \begin{bmatrix} I_{r\times r} & F \\ F^T & 0 \end{bmatrix} \qquad \text{so that} \qquad rref(R^T R) = \boxed{\begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}} = R$$

Performing row operations doesn't change the nullspace, so that $\mathbf{N}(A) = \mathbf{N}(rref(A))$ for any matrix A. So, $\mathbf{N}(A) = \mathbf{N}(R^T R)$ by (e). □

Problem 8. (§3.3, #28) Suppose you allow elementary *column* operations on A as well as elementary row operations (which get to R). What is the "row-and-column reduced form" for an m by n matrix of rank r?

Solution. **(12 points)** After getting to R we can use the column operations to get rid of F, and get to

$$\boxed{\begin{pmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix}}$$

□

Problem 9. (§3.3, #17 – Optional)

(a) Suppose column j of B is a combination of previous columns of B. Show that column j of AB is the same combination of previous columnd of AB. Then AB cannot have new pivot columns, so $\text{rank}(AB) \le \text{rank}(B)$.

(b) Find A_1 and A_2 so that $\text{rank}(A_1 B) = 1$ and $\text{rank}(A_2 B) = 0$ for $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution. **(Optional)**

(a) That column j of B is a combination of previous columns of B means precisely that there exist numbers $a_1, \ldots, a_{j-1}$ so that each row vector $\mathbf{x} = (x_i)$ of B satisfies the linear relation

$$x_j = \sum_{i=1}^{j-1} a_i x_i = a_1 x_1 + \cdots + a_{j-1} x_{j-1}$$

The rows of the matrix AB are all linear combinations of the rows of B, and so also satisfy this linear relation. So, column j is the same combination of previous columns of AB, as desired. Since a column is pivot column precisely when it is not a combination of previous columns, this shows that AB cannot have previous columns and the rank inequality.

(b) Take $A_1 = I_2$ and $A_2 = 0_2$ (or for a less trivial example $A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$). □

Problem 10. (§3.4, #13) Explain why these are all false:

(a) The complete solution is any linear combination of $\mathbf{x}_p$ and $\mathbf{x}_n$.
(b) A system $A\mathbf{x} = \mathbf{b}$ has at most one particular solution.
(c) The solution $\mathbf{x}_p$ with all free variables zero is the shortest solution (minimum length $\|x\|$). Find a 2 by 2 counterexample.
(d) If A is invertible there is no solution $\mathbf{x}_n$ in the nullspace.

Solution. **(4 points)**

(a) The coefficient of $\mathbf{x}_p$ must be one.

(b) If $\mathbf{x}_n \in \mathbf{N}(A)$ is in the nullspace of A and $\mathbf{x}_p$ is one particular solution, then $\mathbf{x}_p + \mathbf{x}_n$ is also a particular solution.

(c) Lots of counterexamples are possible. Let's talk about the 2 by 2 case geometrically: If A is a 2 by 2 matrix of rank 1, then the solutions to $A\mathbf{x} = \mathbf{b}$ form a line parallel to the line that is the nullspace. We're asking that this line's closest point to the origin be somewhere not along an axis. The line $x + y = 1$ gives such an example.

Explicitly, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_p = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Then, $\|\mathbf{x}_p\| = 1/\sqrt{2} < 1$ while the particular solutions having some coordinate equal to zero are $(1, 0)$ and $(0, 1)$ and they both have $\|\cdot\| = 1$.

(d) There's always $\mathbf{x}_n = 0$. $\square$

Problem 11. (§3.4, #25) Write down all known relations between r and m and n if $A\mathbf{x} = \mathbf{b}$ has

(a) no solution for some $\mathbf{b}$

(b) infinitely many solutions for every $\mathbf{b}$

(c) exactly one solution for some $\mathbf{b}$, no solution for other $\mathbf{b}$

(d) exactly one solution for every $\mathbf{b}$.

Solution. **(4 points)**

(a) The system has less than full row rank: $r < m$.

(b) The system has full row rank, and less than full column rank: $m = r < n$.

(c) The system has full column rank, and less than full row rank: $n = r < m$.

(d) The system has full row and column rank (i.e., is invertible): $n = r = m$. $\square$

Problem 12. (§3.4, #28) Apply Gauss-Jordan elimination to $U\mathbf{x} = 0$ and $U\mathbf{x} = \mathbf{c}$. Reach $R\mathbf{x} = 0$ and $R\mathbf{x} = \mathbf{d}$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} U & \mathbf{c} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix}.$$

Solve $R\mathbf{x} = 0$ to find $\mathbf{x}_n$ (its free variable is $x_2 = 1$). Solve $R\mathbf{x} = \mathbf{d}$ to find $\mathbf{x}_p$ (its free variable is $x_2 = 0$).

Solution. **(4 points)** Let me just say to whoever's reading: The problem statement is confusing as written!! In any case, I *think* the desired response is:

$$\begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix} \qquad \text{and} \begin{bmatrix} 1 & 2 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$$

so that

$$R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{d} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

and

$$\mathbf{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_p = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

$\square$

Problem 13. (§3.4, #35) Suppose K is the 9 by 9 second difference matrix (2's on the diagonal, -1's on the diagonal above and also below). Solve the equation $K\mathbf{x} = \mathbf{b} = (10, \ldots, 10)$. If you graph $x_1, \ldots, x_9$ above the points $1, \ldots, 9$ on the x axis, I think the nine points fall on a parabola.

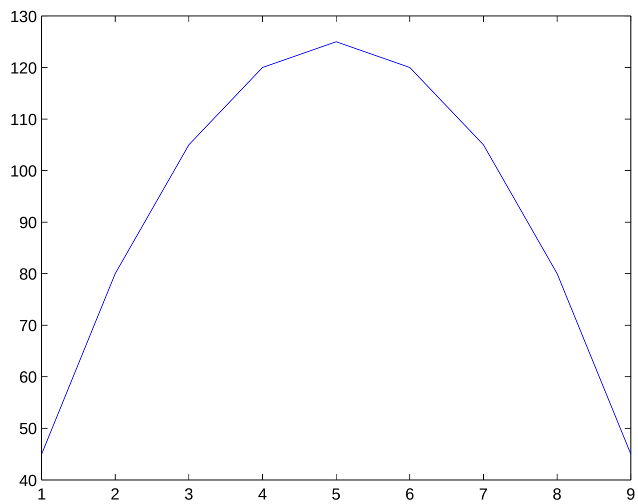
Solution. **(12 points)** Here's some MATLAB code that should do this:

```
K = 2*eye(9)  +  diag(-1*ones(1,8),1)  +  diag(-1*ones(1,8),-1);
b = 10*ones(9,1);
x = K \ b
```

It gives back that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \boxed{\begin{bmatrix} 45 \\ 80 \\ 105 \\ 120 \\ 125 \\ 120 \\ 105 \\ 80 \\ 45 \end{bmatrix}}$$

And for fun, the graph is indeed parabola-like:



$\square$

Problem 14. (§3.4, #36) Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is it true that $A = C$?

Solution. **(12 points)** $\boxed{\text{Yes}}$. In order to check that $A = C$ as matrices, it's enough to check that $A\mathbf{y} = C\mathbf{y}$ for all vectors $\mathbf{y}$ of the correct size (or just for the standard basis vectors, since multiplication by them "picks out the columns"). So let $\mathbf{y}$ be any vector of the correct size, and set $\mathbf{b} = A\mathbf{y}$. Then $\mathbf{y}$ is certainly a solution to $A\mathbf{x} = \mathbf{b}$, and so by our hypothesis must also be a solution to $C\mathbf{x} = \mathbf{b}$; in other words, $C\mathbf{y} = \mathbf{b} = A\mathbf{y}$. $\square$

18.06 Problem Set 4 Solution

Total: 100 points

Section 3.5. Problem 2: (Recommended) Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix})0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_5 = \begin{bmatrix})0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_6 = \begin{bmatrix})0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

$\boxed{\text{Solution}}$ (4 points): Since $v_4 = v_2 - v_1, v_5 = v_3 - v_1$, and $v_6 = v_3 - v_2$, there are at most three independent vectors among these: furthermore, applying row reduction to the matrix $[v_1 v_2 v_3]$ gives three pivots, showing that v_1, v_2, and v_3 are independent.

Section 3.5. Problem 20: Find a basis for the plane $x - 2y + 3z = 0$ in R^3. Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

$\boxed{\text{Solution}}$ (4 points): This plane is the nullspace of the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The special solutions

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

give a basis for the nullspace, and thus for the plane. The intersection of this plane with the xy plane is a line: since the first vector lies in the xy plane, it must lie on the line and thus gives a basis for it. Finally, the vector

$$v_3 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

is obviously perpendicular to both vectors: since the space of vectors perpendicular to a plane in $\mathbb{R}^3$ is one-dimensional, it gives a basis.

Section 3.5. Problem 37: If $AS = SA$ for the shift matrix S, show that A must have this special form:

$$\text{If } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \text{ then } A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & ai \end{bmatrix}$$

"The subspace of matrices that commute with the shift S has dimension __."

$\boxed{\text{Solution}}$ (4 points): Multiplying out both sides gives

$$\begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$$

Equating them gives $d = g = h = 0, e = i = a, f = b$, i.e. the matrix with the form above. Since there are three free variables, the subspace of these matrices has dimension 3.

Section 3.5. Problem 41: Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly indpendent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 = 0$, and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

$\boxed{\text{Solution}}$ (12 points): The other five permutation matrices are

$$P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}, P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}, P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & & 1 \end{bmatrix}$$

Since $P_{21} + P_{31} + P_{32}$ is the all 1s matrix and $P_{32}P_{21} + P_{21}P_{32}$ is the matrix with 0s on the diagonal and 1s elsewhere, $I = P_{21} + P_{31} + P_{32} - P_{32}P_{21} - P_{21}P_{32}$. For the second part, the combination above gives

$$\begin{bmatrix} c_3 & c_1 + c_4 & c_2 + c_5 \\ c_1 + c_5 & c_2 & c_3 + c_4 \\ c_2 + c_4 & c_3 + c_5 & c_1 \end{bmatrix} = 0$$

Setting each element equal to 0 first gives $c_1 = c_2 = c_3 = 0$ along the diagonal, then $c_4 = c_5 = 0$ on the off-diagonal entries.

Section 3.5. Problem 44: (An aside in the text, followed by) *dimension of outputs + dimension of nullspace = dimension of inputs.* For an m by n matrix of rank r, what are those 3 dimensions? Outputs = column space. This question wil be answered in Section 3.6, can you do it now?

$\boxed{\text{Solution}}$ (12 points): You should think about the aside in the text, as well as problem 43: the actual question asked, here, however is quite simple. The dimension of inputs for an m by n matrix is n (the matrix takes n-vectors to m-vectors), while the dimension of the nullspace is $n - r$ and the dimension of outputs = dimension of column space is r. Since $n - r + r = n$, we have the given relation.

Section 3.6. Problem 11: A is an m by n matrix of rank r. Suppose there are right sides $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has *no solution.*

(a) What are all the inequalities ($<$ or $\leq$) that must be true between m, n, and r?

(b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?

$\boxed{\text{Solution}}$ (4 points): (a) The rank of a matrix is always less than or equal to the number of rows and columns, so $r \leq m$ and $r \leq n$. Moreover, by the second statement, the column space is smaller than the space of possible output matrices, i.e. $r < m$.

(b) These solutions make up the left nullspace, which has dimension $m - r > 0$ (that is, there are nonzero vectors in it).

Section 3.6. Problem 24: $A^T\mathbf{y} = \mathbf{d}$ is solvable when $\mathbf{d}$ is in which of the four subspaces? The solution is unique when the __ contains only the zero vector.

$\boxed{\text{Solution}}$ (4 points): It is solvable when $\mathbf{d}$ is in the row space, which consists of all vectors $A^T\mathbf{y}$, and is unique when the left nullspace contains only the zero vector (as any two solutions differ by an element in the left nullspace).

Section 3.6. Problem 28: Find the ranks of the 8 by 8 checkerboard matrix B and the chess matrix C:

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

The numbers r, n, b, k, q, p are all different. Find bases for the rowspace and left nullspace of B and C. Challenge problem: Find a basis for the nullspace of C.

$\boxed{\text{Solution}}$ (4 points): In both cases, elimination kills all but the top two rows, so, if $p \neq 0$, both matrices have rank 2 as well as rowspace bases given by the top two rows (or course, if $p = 0$, C has rank 1 with rowspace generated by the top row). B is symmetric, so its left nullspace is the same as the nullspace, and the special solutions are:

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_6 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Finally, the nullspace of C^T is given by

$$w_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

if $p \neq 0$, and

$$w_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, w_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

if $p = 0$.

Solution (12 points): (Challenge subpart) There are three obvious special solutions of C:

$$u_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

If $p = 0$, the other solutions are similarly straightforward:

$$u_4 = \begin{bmatrix} -\frac{n}{r} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, u_5 = \begin{bmatrix} -\frac{b}{r} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, u_6 = \begin{bmatrix} -\frac{k}{r} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, u_7 = \begin{bmatrix} -\frac{q}{r} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Otherwise, simultaneously solving $c_1 r + c_2 n + b = 0$ and $(c_1 + c_2 + 1)p = 0$ (and similarly for q and k instead of b), we get

$$u_4 = \begin{bmatrix} \frac{n-b}{r-n} \\ \frac{b-r}{r-n} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, u_5 = \begin{bmatrix} \frac{n-q}{r-n} \\ \frac{q-r}{r-n} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, u_6 = \begin{bmatrix} \frac{n-k}{r-n} \\ \frac{k-r}{r-n} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Section 3.6. Problem 30: If $A = \mathbf{u}\mathbf{v}^T$ is a 2 by 2 matrix of rank 1, redraw Figure 3.5 to show clearly the Four Fundamental Subspaces. If B produces those same four subspaces, what is the exact relation of B to A?

Solution (12 points): One draws the same diagram as in the book, but now each space has dimension 1, the column space is the set of multiples of $\mathbf{u}$, the row space is the set of multiples of $\mathbf{v}^T$, the nullspace is the plane perpendicular to $\mathbf{v}$, and the left nullspace is the plane perpendicular to $\mathbf{u}$. If $B = \mathbf{u}'\mathbf{v}'^T$ produces the same four subspaces, $\mathbf{u}'$ is a multiple of $\mathbf{u}$ and $\mathbf{v}'$ is a multiple of $\mathbf{v}$, i.e. B is a multiple of A.

Section 3.6. Problem 31: $\mathbf{M}$ is the space of 3 by 3 matrices. Multiply each matrix X in $\mathbf{M}$ by

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \text{ Notice: } A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Which matrices X lead to $AX = 0$?

(b) Which matrices have the form AX for some matrix X?

(a) finds the "nullspace" of that operation AX and (b) finds the "column space". What are the dimensions of those two subspaces of $\mathbf{M}$? Why do the dimensions add to $(n - r) + r = 9$?

$\boxed{\text{Solution}}$ (12 points): (a) A clearly has rank 2, with nullspace having the basis $[111]^T$. $AX = 0$ precisely when the columns of X are in the nullspace of A, i.e. when they are multiples of the all 1s vector.

$$X = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

(b) On the other hand, the columns of any matrix of the form AX are linear combinations of the columns of A. That is, they are vectors whose components all sum to 0, so a matrix has the form AX if and only if all of its columns individually sum to 0.

$$AX = B \text{ if and only if } B = \begin{bmatrix} a & b & c \\ d & e & f \\ -a-d & -b-e & -c-f \end{bmatrix}$$

The dimension of the "nullspace" is 3, while the dimension of the "column space" is 6. They add up to 9, which is the dimension of the space of "inputs" of this matrix, when treated as a linear map on matrices.

18.06 Problem Set 5 Solution

Total: points

Section 4.1. Problem 7. Every system with no solution is like the one in problem 6. There are numbers $y_1, \ldots, y_m$ that multiply the m equations so they add up to $0 = 1$. This is called Fredholm's Alternative:

Exactly one of these problems has a solution: $A\mathbf{x} = \mathbf{b}$ OR $A^T\mathbf{y} = 0$ with $\mathbf{y}^T\mathbf{b} = 1$.

If $\mathbf{b}$ is not in the column space of A it is not orthogonal to the nullspace of A^T. Multiply the equations $x_1 - x_2 = 1$ and $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2, y_3 chosen so that the equations add up to $0 = 1$.

Solution (4 points) Let $y_1 = 1$, $y_2 = 1$ and $y_3 = -1$. Then the left-hand side of the sum of the equations is

$$(x_1 - x_2) + (x_2 - x_3) - (x_1 - x_3) = x_1 - x_2 + x_2 - x_3 + x_3 - x_1 = 0$$

and the right-hand side is

$$1 + 1 - 1 = 1.$$

Problem 9. If $A^TA\mathbf{x} = 0$ then $A\mathbf{x} = 0$. Reason: $A\mathbf{x}$ is inthe nullspace of A^T and also in the ____________ of A and those spaces are _________. Conclusion: A^TA has the same nullspace as A. This key fact is repeated in the next section.

Solution (4 points) $A\mathbf{x}$ is in the nullspace of A^T and also in the *column space* of A and those spaces are *orthogonal*.

Problem 31. The command `N=null(A)` will produce a basis for the nullspace of A. Then the command `B=null(N')` will produce a basis for the _________ of A.

Solution (12 points) The matrix N will have as its columns a basis for the nullspace of A. Thus if a vector is in the nullspace of N^T it must have dot product 0 with every vector in the basis of $N(A)$, thus it must be in the row space of A. Thus the command `null(N')` will produce a basis for the *row space* of A.

```
>> A = rand(6,12)

A =

  Columns 1 through 6

    0.8147    0.2785    0.9572    0.7922    0.6787    0.7060
    0.9058    0.5469    0.4854    0.9595    0.7577    0.0318
    0.1270    0.9575    0.8003    0.6557    0.7431    0.2769
    0.9134    0.9649    0.1419    0.0357    0.3922    0.0462
    0.6324    0.1576    0.4218    0.8491    0.6555    0.0971
    0.0975    0.9706    0.9157    0.9340    0.1712    0.8235

  Columns 6 through 12

    0.6948    0.7655    0.7094    0.1190    0.7513    0.5472
    0.3171    0.7952    0.7547    0.4984    0.2551    0.1386
    0.9502    0.1869    0.2760    0.9597    0.5060    0.1493
    0.0344    0.4898    0.6797    0.3404    0.6991    0.2575
    0.4387    0.4456    0.6551    0.5853    0.8909    0.8407
    0.3816    0.6463    0.1626    0.2238    0.9593    0.2543
>> N = null(A);
>> B = null(N')

B =

    0.1754    0.3288    0.6264   -0.0997   -0.0309    0.0112
   -0.1000    0.6512   -0.1687    0.5528    0.1346   -0.1632
    0.4380    0.2124   -0.3137   -0.1684    0.0050    0.0507
   -0.0661    0.0094   -0.2313   -0.5029    0.5792   -0.4208
    0.4425   -0.0527    0.2166    0.0525    0.1071   -0.2063
    0.2213    0.3024   -0.3429   -0.2123   -0.1149    0.3415
    0.6724   -0.1323   -0.1677    0.1315   -0.0249    0.0211
   -0.0371    0.4083    0.1074   -0.3610    0.0993   -0.1251
    0.1827    0.1595    0.4375   -0.0581    0.0910    0.0005
    0.1438   -0.1870    0.0025    0.4256    0.4134   -0.3188
   -0.0887    0.1540   -0.0283    0.1444    0.5060    0.5600
    0.0533   -0.2452    0.1759   -0.0477    0.4184    0.4506
>> rref(A) - rref(B')
```

```
ans =

   1.0e-13 *

  Columns 1 through 6

         0         0         0         0         0         0
         0         0         0         0         0         0
         0         0         0         0         0         0
         0         0         0         0         0         0
         0         0         0         0         0         0
         0         0         0         0         0         0

  Columns 6 through 12

   -0.0389    0.0533   -0.0344   -0.0888   -0.2420   -0.2043
    0.0047   -0.0022    0.0044    0.0089    0.0289    0.0200
   -0.1710    0.1843   -0.0999   -0.3020   -0.7816   -0.6395
    0.0153   -0.0089    0.0092    0.0205    0.0444    0.0333
    0.1155   -0.1377    0.0688    0.2132    0.5596    0.4796
    0.1488   -0.1643    0.0888    0.2709    0.6928    0.5684
```

Note that if B has as its columns a basis for the row space of A then the rows of B^T will form a basis for the row space of A. Since the row reduced forms of A and B^T agree (up to 13 decimal places, but the numbers up there are just rounding error) their rows must span the same space, so the columns of B are indeed a basis for the row space of A.

Problem 32. Suppose I give you four nonzero vectors $\mathbf{r}, \mathbf{n}, \mathbf{c}, \mathbf{l}$ in $\mathbf{R}^2$.

a. What are the conditions for those to be bases for the four fundamental subspaces $C(A^T), N(A), C(A), N(A^T)$ of a 2×2 matrix?

b. What is one possible matrix A?

$\boxed{\text{Solution}}$ (12 points)

a. In order for $\mathbf{r}$ and $\mathbf{n}$ to be bases for $N(A)$ and $C(A^T)$ we must have $\mathbf{r} \cdot \mathbf{n} = 0$, as the row space and null space must be orthogonal. Similarly, in order for $\mathbf{c}$ and $\mathbf{l}$ to form bases for $C(A)$ and $N(A^T)$ we need $\mathbf{c} \cdot \mathbf{l} = 0$, as the column space and the left nullspace are orthogonal. In addition, we need $\dim N(A) + \dim C(A^T) = n$ and $\dim N(A^T) + \dim C(A) = m$; however, in this case $n = m = 1$, and as the four vectors we are given are nonzero both of these equations reduce to $1 + 1 = 2$, which is automatically satisfied.

b. One possible such matrix is $A = \mathbf{c}\mathbf{r}^T$. Note that each column of A will be a multiple of $\mathbf{c}$, so it will have the right column space. On the other hand, each row of A will be a multiple of $\mathbf{r}$, so A will have the right row space. The nullspaces don't need to be checked, as any matrix with the correct row and column space will have the right nullspaces (as the nullspaces are just the orthogonal complements of the row and column spaces).

Problem 33. Suppose I give you four nonzero vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{n}_1, \mathbf{n}_2, \mathbf{c}_1, \mathbf{c}_2, \mathbf{l}_1, \mathbf{l}_2$ in $\mathbf{R}^2$.

a. What are the conditions for those to be bases for the four fundamental subspaces $C(A^T), N(A), C(A), N(A^T)$ of a 2×2 matrix?

b. What is one possible matrix A?

$\boxed{\text{Solution}}$ (12 points)

a. Firstly, by the same kind of dimension considerations as before we need the four sets $\{\mathbf{r}_1, \mathbf{r}_2\}$, $\{\mathbf{n}_1, \mathbf{n}_2\}$, $\{\mathbf{c}_1, \mathbf{c}_2\}$ and $\{\mathbf{l}_1, \mathbf{l}_2\}$ to each contain linearly independent vectors. (For example, if $\mathbf{r}_1$ and $\mathbf{r}_2$ are linearly dependent the $\dim C(A^T) = 1$ not 2, and then $\dim C(A^T) + \dim N(A) < 4$ which can't happen.)

Secondly, for $i = 1, 2$ and $j = 1, 2$ we need $\mathbf{r}_i \cdot \mathbf{n}_j = 0$ and $\mathbf{c}_i \cdot \mathbf{l}_j = 0$. This will imply that the specified row space and nullspace are orthogonal, and that the specified column space and left nullpace are also orthogonal. (When we are given subspaces in terms of bases it suffices to check orthogonality on the basis.)

b. One possible such matrix is

$$A = \begin{pmatrix} \mathrm{c}_1 & \mathrm{c}_2 \end{pmatrix} \begin{pmatrix} \mathrm{r}_1 & \mathrm{r}_2 \end{pmatrix}^T.$$

Note that every column of A is a linear combination of c_1 and c_2, so $C(A)$ is at least a subspace of the desired column space. On the other hand, as r_1 and

r_2 are linearly independent we know that $\begin{pmatrix} \mathrm{r}_1 & \mathrm{r}_2 \end{pmatrix}^T$ has full row rank, so A will have rank 2 and thus A has the right column space.

On the other hand,

$$A^T = \begin{pmatrix} \mathrm{r}_1 & \mathrm{r}_2 \end{pmatrix} \begin{pmatrix} \mathrm{c}_1 & \mathrm{c}_2 \end{pmatrix}^T$$

so $C(A^T)$ is spanned by r_1 and r_2, as desired. Thus A has the right row space and column space, and thus will have the right nullspace and left nullspace.

Section 4.2. Problem 13. Suppose A is the 4×4 identity matrix with its last column removed. A is 4×3. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

$\boxed{\text{Solution}}$ (4 points) P will be 4×4 since we take a 4-dimensional vector and project it to another 4-dimensional vector. We will have

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(This can be seen by direct computation, or by simply observing that the column space of A is the wxy-space, so we just need to remove the z coordinate to project.) The projection of $\mathbf{b}$ is $(1, 2, 3, 0)$.

Problem 16. What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\mathbf{b} = (2, 1, 1)$?

$\boxed{\text{Solution}}$ (4 points)

Note that

$$\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1).$$

So this $\mathbf{b}$ is actually in the span of the two given vectors.

Problem 17. If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the ____________.

$\boxed{\text{Solution}}$ (4 points)

$$(I - P)^2 = I^2 - IP - PI + P^2 = I - 2P + P^2 = I - 2P + P = I - P.$$

When P projects onto the column space of A, $I - P$ projects onto the left nullspace of A.

Problem 30.

a. Find the projection matrix P_C ontot he column space of A.

$$\begin{pmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{pmatrix}.$$

b. Find the 3×3 projection matrix P_R onto the row space of A. Multiply $B = P_C A P_R$. Your answer B should be a little suprising — can you explain it?

$\boxed{\text{Solution}}$ (12 points)

a. Note that as A is rank 1 its column space is spanned by the vector $\mathbf{a} = \begin{pmatrix} 3 & 4 \end{pmatrix}^T$. Using this matrix we can compute

$$P_C = \mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}\mathbf{a}^T = \frac{1}{25}\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}.$$

b. The row space of A is spanned by the vector $\mathbf{a} = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix}^T$. Then we compute

$$P_R = \mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}\mathbf{a}^T = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}.$$

Then $B = P_C A P_R = A$. First, note that $P_C A = A$, as for every vector $\mathbf{x}$, $A\mathbf{x} \in C(A)$, so $P_C A\mathbf{x} = A\mathbf{x}$. Analogously, $AP_R = A$, as for every vector $\mathbf{x}$ we can write it uniquely as $\mathbf{x} = \mathbf{n} + \mathbf{r}$ with $\mathbf{n}$ in $N(A)$ and $\mathbf{r}$ in $C(A^T)$. Then $A\mathbf{x} = A\mathbf{n} + A\mathbf{r} = A\mathbf{r}$ by the definition of nullspace. But $P_R\mathbf{x} = P_R\mathbf{n} + P_R\mathbf{r} = P_R\mathbf{r}$, as the nullspace is orthogonal to the row space, so projecting onto the row space kills the nullspace. So $AP_R = A$. Thus $P_C A P_R = (P_C A)P_R = AP_R = A$, as desired.

Problem 31. In $\mathbf{R}^m$, suppose I give you $\mathbf{b}$ and $\mathbf{p}$, and n linearly independent vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$. How would you test to see if $\mathbf{p}$ is the projection of $\mathbf{b}$ onto the subspace spanned by the $\mathbf{a}$'s?

$\boxed{\text{Solution}}$ (12 points)

The projection of $\mathbf{b}$ must lie in the span of the $\mathbf{a}$'s, and must also be the closest vector in this span, meaning that the error will be orthogonal to this span. Thus we need to check (a) that $\mathbf{p}$ is in the span of the $\mathbf{a}$'s, and (b) that $\mathbf{b} - \mathbf{p}$ is orthogonal to $\mathbf{a}_i$ for each $i = 1, \dots, n$. Note that just checking (b) is not enough because if we set $\mathbf{p} = \mathbf{b}$ then (b) will be satisfied but (a) will not be if $\mathbf{b}$ is not in the span of the $\mathbf{a}$'s.

Problem 34. If A has r independent columns and B has r independent rows, AB is invertible.
Proof: When A is m by r with independent columns, we know that A^TA is invertible. If B is r by n with independent rows, show that BB^T is invertible. (Take $A = B^T$.)

Now show that AB has rank r.

$\boxed{\text{Solution}}$ (12 points) Let $A = B^T$. As B has independent rows, A has independent columns, so A^TA is invertible. But $A^TA = (B^T)^TB^T = BB^T$, so BB^T is invertible, as desired.

Note that A^TA is $r \times r$ and is invertible, and BB^T is $r \times r$ and is invertible, so A^TABB^T is $r \times r$ and invertible, so in particular has rank r. Thus we have that $A^T(AB)B^T$ has rank r. We know that multiplying AB by any matrix on the left or on the right cannot increase rank, but can only decrease it. Thus we see that AB has rank at least r. However, AB is $r \times r$, so it has rank r and is therefore invertible.

Section 8.2. Problem 13. With conductances $c_1 = c_2 = 2$ and $c_3 = c_4 = c_5 = 3$, multiply the matrices A^TCA. Find a solution to $A^TCA\mathbf{x} = \mathbf{f} = (1, 0, 0, -1)$. Where these potentials $\mathbf{x}$ and currents $\mathbf{y} = -CA\mathbf{x}$ on the nodes and edges of the square graph.

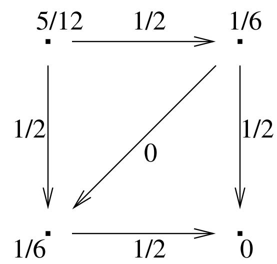
$\boxed{\text{Solution}}$ (4 points) For this graph the incidence matrix is

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

We are told that the conductance matrix has diagonal entries $(2, 2, 3, 3, 3)$. Then

$$A^TCA = \begin{pmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{pmatrix}.$$

A solution to the given equation is $\mathbf{x} = (5/12, 1/6, 1/6, 0)$; then $\mathbf{y} = (1/2, 1/2, 0, 1/2, 1/2)$. The picture associated to this solution is



Problem 17. Suppose A is a 12×9 incidence matrix from a connected (but unknown) graph.

a. How many columns of A are independent?

b. What condition on $\mathbf{f}$ makes it possible to solve $A^T\mathbf{y} = \mathbf{f}$?

c. The diagonal entries of A^TA give the number of edges into each node. What is the sum of those diagonal entries?

$\boxed{\text{Solution}}$ (12 points)

a. Note that as A is 12×9 it is a graph with 9 nodes and 12 edges. As it is connected elimination will produce a tree with 8 edges, so the rank of A is 8, and so it has 8 independent columns.

b. In order to solve $A^T\mathbf{y}$ we need the entries of $\mathbf{f}$ to add up to 0, as $\mathbf{f}$ needs to be in $C(A^T)$, which is orthogonal to $N(A)$ and is generated by $(1, 1, \ldots, 1)$.

c. The sum of the entries of A^TA is the sum of the degrees of all of the nodes. As each edge hits exactly two nodes it will be counted twice, so the sum of the diagonal entries is 24.

18.06 Linear Algebra
Spring 2010

18.06 Problem Set 6 Solutions

Total: 100 points

Section 4.3. Problem 4: Write down $E = \|Ax - b\|^2$ as a sum of four squares– the last one is $(C + 4D - 20)^2$. Find the derivative equations $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$. Divide by 2 to obtain the normal equations $A^T A\widehat{x} = A^T b$.

$\boxed{\text{Solution}}$ (4 points)
Observe

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}, \text{ and define } x = \begin{pmatrix} C \\ D \end{pmatrix}.$$

Then

$$Ax - b = \begin{pmatrix} C \\ C + D - 8 \\ C + 3D - 8 \\ C + 4D - 20 \end{pmatrix},$$

and

$$\|Ax - b\|^2 = C^2 + (C + D - 8)^2 + (C + 3D - 8)^2 + (C + 4D - 20)^2.$$

The partial derivatives are

$$\partial E/\partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 8C + 16D - 72,$$

$$\partial E/\partial D = 2(C + D - 8) + 6(C + 3D - 8) + 8(C + 4D - 20) = 16C + 52D - 224.$$

On the other hand,

$$A^T A = \begin{pmatrix} 4 & 8 \\ 8 & 26 \end{pmatrix}, \; A^T b = \begin{pmatrix} 36 \\ 112 \end{pmatrix}.$$

Thus, $A^T Ax = A^T b$ yields the equations $4C + 8D = 36$, $8C + 26D = 112$. Multiplying by 2 and looking back, we see that these are precisely the equations $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$.

Section 4.3. Problem 7: Find the closest line $b = Dt$, *through the origin*, to the same four points. An exact fit would solve $D \cdot 0 = 0$, $D \cdot 1 = 8$, $D \cdot 3 = 8$, $D \cdot 4 = 20$.

Find the 4 by 1 matrix A and solve $A^T A\widehat{x} = A^T b$. Redraw figure 4.9a showing the best line $b = Dt$ and the e's.

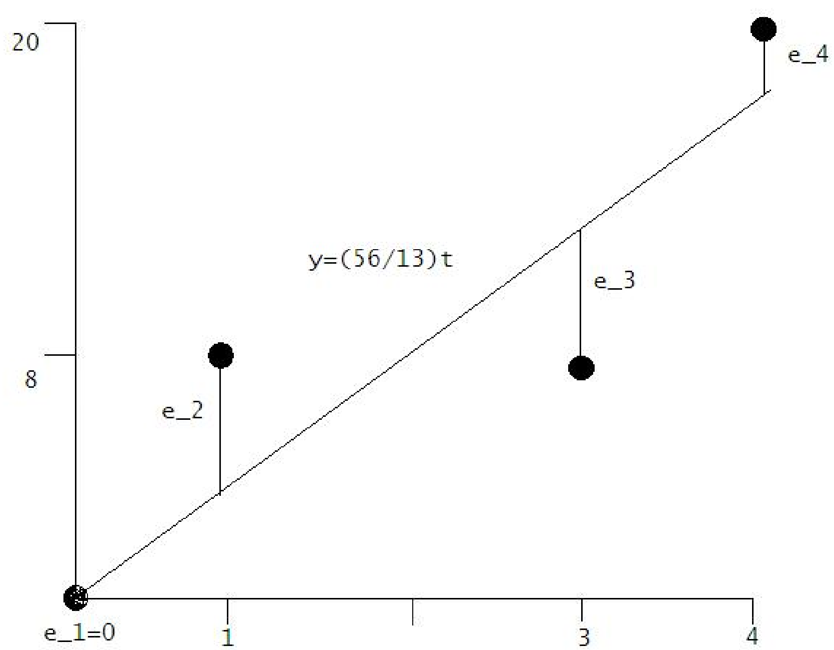
$\boxed{\text{Solution}}$ (4 points) Observe

$$A = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}, \; A^T A = (26), \;\; A^T b = (112).$$

Thus, solving $A^T Ax = A^T b$, we arrive at

$$D = 56/13.$$

Here is the diagram analogous to figure 4.9a.



Section 4.3. Problem 9: Form the closest parabola $b = C + Dt + Et^2$ to the same four points, and write down the unsolvable equations $Ax = b$ in three unknowns

$x = (C, D, E)$. Set up the three normal equations $A^T A\widehat{x} = A^T b$ (solution not required). In figure 4.9a you are now fitting a parabola to 4 points–what is happening in Figure 4.9b?

$\boxed{\text{Solution}}$ (4 points)
Note

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}, \ x = \begin{pmatrix} C \\ D \\ E \end{pmatrix}.$$

Then multiplying out $Ax = b$ yields the equations

$$C = 0, \ C + D + E = 8, \ C + 3D + 9E = 8, \ C + 4D + 16E = 20.$$

Take the sum of the fourth equation and twice the second equation and subtract the sum of the first equation and two times the third equation. One gets $0 = 20$. Hence, these equations are not simultaneously solvable.
Computing, we get

$$A^T A = \begin{pmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{pmatrix}, \ A^T b = \begin{pmatrix} 36 \\ 112 \\ 400 \end{pmatrix}.$$

Thus, solving this problem is the same as solving the system

$$\begin{pmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 36 \\ 112 \\ 400 \end{pmatrix}.$$

The analogue of diagram 4.9(b) in this case would show three vectors $a_1 = (1, 1, 1, 1)$, $a_2 = (0, 1, 3, 4)$, $a_3 = (0, 1, 9, 16)$ spanning a three dimensional vector subspace of $\mathbb{R}^4$. It would also show the vector $b = (0, 8, 8, 20)$, and the projection $p = Ca_1 + Da_2 + Ea_3$ of b into the three dimensional subspace.

Section 4.3. Problem 26: Find the *plane* that gives the best fit to the 4 values $b = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0)$ and $(0, -1)$ of a square. The equations $C + Dx + Ey = b$ at those 4 points are $Ax = b$ with 3 unknowns $x = (C, D, E)$. What is A? At the center $(0, 0)$ of the square, show that $C + Dx + Ey$ is the average of the b's.

$\boxed{\text{Solution}}$ (12 points)

Note

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

To find the best fit plane, we must find x such that $Ax - b$ is in the left nullspace of A. Observe

$$Ax - b = \begin{pmatrix} C + D \\ C + E - 1 \\ C - D - 3 \\ C - E - 4 \end{pmatrix}.$$

Computing, we find that the first entry of $A^T(Ax - b)$ is $4C - 8$. This is zero when $C = 2$, the average of the entries of b. Plugging in the point $(0, 0)$, we get $C + D(0) + E(0) = C = 2$ as desired.

Section 4.3. Problem 29: Usually there will be exactly one hyperplane in $\mathbb{R}^n$ that contains the n given points $x = 0, a_1, \ldots, a_{n-1}$. (Example for n=3: There will be exactly one plane containing $0, a_1, a_2$ unless ________.) What is the test to have exactly one hyperplane in $\mathbb{R}^n$?

$\boxed{\text{Solution}}$ (12 points)

The sentence in paranthesis can be completed a couple of different ways. One could write "There will be exactly one plane containing $0, a_1, a_2$ unless these three points are colinear". Another acceptable answer is "There will be exactly one plane containing $0, a_1, a_2$ unless the vectors a_1 and a_2 are linearly dependent".

In general, $0, a_1, \ldots, a_{n-1}$ will be contained in an unique hyperplane unless all of the points $0, a_1, \ldots, a_{n-1}$ are contained in an $n - 2$ dimensional subspace. Said another way, $0, a_1, \ldots, a_{n-1}$ will be contained in an unique hyperplane unless the vectors $a_1, \ldots, a_{n-1}$ are linearly dependent.

Section 4.4. Problem 10: Orthonormal vectors are automatically linearly independent.
(a) Vector proof: When $c_1q_1 + c_2q_2 + c_3q_3 = 0$, what dot product leads to $c_1 = 0$? Similarly $c_2 = 0$ and $c_3 = 0$. Thus, the q's are independent.
(b) Matrix proof: Show that $Qx = 0$ leads to $x = 0$. Since Q may be rectangular, you can use Q^T but not Q^{-1}.

$\boxed{\text{Solution}}$ (4 points) For part (a): Dotting the expression $c_1q_1 + c_2q_2 + c_3q_3$ with q_1, we get $c_1 = 0$ since $q_1 \cdot q_1 = 1$, $q_1 \cdot q_2 = q_1 \cdot q_3 = 0$. Similarly, dotting the expression with q_2 yields $c_2 = 0$ and dotting the expression with q_3 yields $c_3 = 0$. Thus, $\{q_1, q_2, q_3\}$ is a linearly independent set.

For part (b): Let Q be the matrix whose columns are q_1, q_2, q_3. Since Q has orthonormal columns, Q^TQ is the three by three identity matrix. Now, multiplying the equation $Qx = 0$ on the left by Q^T yields $x = 0$. Thus, the nullspace of Q is the zero vector and its columns are linearly independent.

Section 4.4. Problem 18: Find the orthonormal vectors A, B, C by Gram-Schmidt from a, b, c:

$$a = (1, -1, 0, 0) \quad b = (0, 1, -1, 0) \quad c = (0, 0, 1, -1).$$

Show $\{A, B, C\}$ and $\{a, b, c\}$ are bases for the space of vectors perpendicular to $d = (1, 1, 1, 1)$.

$\boxed{\text{Solution}}$ (4 points) We apply Gram-Schmidt to a, b, c. We have

$$A = \frac{a}{\|a\|} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0, 0\right).$$

Next,

$$B = \frac{b - (b \cdot A)A}{\|b - (b \cdot A)A\|} = \frac{(\frac{1}{2}, \frac{1}{2}, -1, 0)}{\|(\frac{1}{2}, \frac{1}{2}, -1, 0)\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, 0\right).$$

Finally,

$$C = \frac{c - (c \cdot A)A - (c \cdot B)B}{\|c - (c \cdot A)A - (c \cdot B)B\|} = \left(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{\sqrt{3}}{2}\right).$$

Note that $\{a, b, c\}$ is a linearly independent set. Indeed,

$$x_1a + x_2b + x_3c = (x_1, x_2 - x_1, x_3 - x_2, -x_3) = (0, 0, 0, 0)$$

implies that $x_1 = x_2 = x_3 = 0$. We check $a \cdot (1, 1, 1, 1) = b \cdot (1, 1, 1, 1) = c \cdot (1, 1, 1, 1) = 0$. Hence, all three vectors are in the nullspace of $(1, 1, 1, 1)$. Moreover, the dimension of the column space of the transpose and the dimension of the nullspace sum to the dimension of $\mathbb{R}^4$. Thus, the space of vectors perpendicular to $(1, 1, 1, 1)$ is three dimensional. Since $\{a, b, c\}$ is a linearly independent set in this space, it is a basis.

Since $\{A, B, C\}$ is an orthonormal set, it is a linearly independent set by problem 10. Thus, it must also span the space of vectors perpendicular to $(1, 1, 1, 1)$, and it is also a basis of this space.

Section 4.4. Problem 35: Factor $[Q, R] = \mathbf{qr(A)}$ for $A = \mathbf{eye(4)}-\mathbf{diag}([\mathbf{111}], -\mathbf{1})$. You are orthogonalizing the columns $(1, -1, 0, 0)$, $(0, 1, -1, 0)$, $(0, 0, 1, -1)$, and $(0, 0, 0, 1)$ of A. Can you scale the orthogonal columns of Q to get nice integer components?

$\boxed{\text{Solution}}$ (12 points) Here is a copy of the matlab code

```
>> A=eye(4)-diag([1 1 1],-1)

A =

     1     0     0     0
    -1     1     0     0
     0    -1     1     0
     0     0    -1     1

>> [Q,R]=qr(A)

Q =

   -0.7071   -0.4082   -0.2887    0.5000
    0.7071   -0.4082   -0.2887    0.5000
         0    0.8165   -0.2887    0.5000
         0         0    0.8660    0.5000


R =

   -1.4142    0.7071         0         0
         0   -1.2247    0.8165         0
         0         0   -1.1547    0.8660
         0         0         0    0.5000
```

Note that scaling the first column by $\sqrt{2}$, the second column by $\sqrt{6}$, the third column by $2\sqrt{3}$, and the fourth column by 2 yields

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{pmatrix}.$$

Section 4.4. Problem 36: If A is m by n, $\mathbf{qr(A)}$ produces a *square* A and zeroes below R: The factors from MATLAB are $(m$ by $m)(m$ by $n)$

$$A = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}.$$

The n columns of Q_1 are an orthonormal basis for which fundamental subspace? The $m-n$ columns of Q_2 are an orthonormal basis for which fundamental subspace?

$\boxed{\text{Solution}}$ (12 points) The n columns of Q_1 form an orthonormal basis for the column space of A. The $m-n$ columns of Q_2 form an orthonormal basis for the left nullspace of A.

Section 5.1. Problem 10: If the entries in every row of A add to zero, solve $Ax = 0$ to prove $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean $\det A = I$?

$\boxed{\text{Solution}}$ (4 points) If $x = (1, 1, \dots, 1)$, then the components of Ax are the sums of the rows of A. Thus, $Ax = 0$. Since A has non-zero nullspace, it is not invertible and $\det A = 0$. If the entries in every row of A sum to one, then the entries in every row of $A - I$ sum to zero. Hence, $A - I$ has a non-zero nullspace and $\det(A - I) = 0$. This does not mean that $\det A = I$. For example if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the entries of every row of A sum to one. However, $\det A = -1$.

Section 5.1. Problem 18: Use row operations to show that the 3 by 3 "Vandermonde determinant" is

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

$\boxed{\text{Solution}}$ (4 points) Doing elimination, we get

$$\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = \det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} = (b-a)\det \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & c-a & c^2-a^2 \end{pmatrix} =$$

$$= (b-a)\det\begin{pmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 0 & (c-a)(c-b)\end{pmatrix} = (b-a)(c-a)(c-b).$$

Section 5.1. Problem 31: (MATLAB) The Hilbert matrix **hilb(n)** has i, j entry equal to $1/(i+j-1)$. Print the determinants of $\mathbf{hilb(1)}, \mathbf{hilb(2)}, \ldots, \mathbf{hilb(10)}$. Hilbert matrices are hard to work with! What are the pivots of **hilb(5)**?

$\boxed{\text{Solution}}$ (12 points) Here is the relevant matlab code.

```
>> [det(hilb(1)) det(hilb(2)) det(hilb(3)) det(hilb(4))
det(hilb(5)) det(hilb(6)) det(hilb(7)) det(hilb(8))
det(hilb(9)) det(hilb(10))]

ans =

    1.0000    0.0833    0.0005    0.0000    0.0000
    0.0000    0.0000    0.0000    0.0000    0.0000

>> [L,U,P]=lu(hilb(5))

L =

    1.0000         0         0         0         0
    0.3333    1.0000         0         0         0
    0.5000    1.0000    1.0000         0         0
    0.2000    0.8000   -0.9143    1.0000         0
    0.2500    0.9000   -0.6000    0.5000    1.0000


U =

    1.0000    0.5000    0.3333    0.2500    0.2000
         0    0.0833    0.0889    0.0833    0.0762
         0         0   -0.0056   -0.0083   -0.0095
         0         0         0    0.0007    0.0015
         0         0         0         0   -0.0000


P =

     1     0     0     0     0
     0     0     1     0     0
     0     1     0     0     0
     0     0     0     0     1
     0     0     0     1     0
```

Note that the determinants of the 4th through 10th Hilbert matrices differ from zero by less than one ten thousandth. The pivots of the fifth Hilbert matrix are $1, .0833, -.0056, .0007, .0000$ up to four significant figures. Thus, we see that there is even a pivot of the fifth Hilbert matrix that differs from zero by less than one ten thousandth.

Section 5.1. Problem 32: (MATLAB) What is the typical determinant (experimentally) of **rand(n)** and **randn(n)** for $n = 50, 100, 200, 400$? (And what does "Inf" mean in MATLAB?)

$\boxed{\text{Solution}}$ (12 points) This matlab code computes some examples for rand.

```
>> [det(rand(50)) det(rand(50)) det(rand(50)) det(rand(50))
det(rand(50)) det(rand(50))]

ans =

   1.0e+06 *

   -0.5840   -1.1620   -0.0612    0.3953    0.5149   -0.0436

>> [det(rand(100)) det(rand(100)) det(rand(100)) det(rand(100))
det(rand(100)) det(rand(100))]

ans =

   1.0e+26 *

   -0.6288   -0.0001   -0.1463    0.6322    3.5820    0.0929

>> [det(rand(200)) det(rand(200)) det(rand(200)) det(rand(200))
det(rand(200)) det(rand(200))]

ans =

   1.0e+80 *

   -1.2212    0.0246    0.1505    0.0791    8.4722   -4.5166

>> [det(rand(400)) det(rand(400)) det(rand(400)) det(rand(400))
det(rand(400)) det(rand(400))]

ans =

  1.0e+219 *

    0.4479    1.0835    1.8087    5.5787   -0.3650    5.6855
```

As you can see, **rand(50)** is around 10^5, **rand(100)** is around 10^{25}, **rand(200)** is around 10^{79}, and **rand(400)** is around 10^{219}.
This matlab code computes some examples for randn.

```
>> [det(randn(50)) det(randn(50)) det(randn(50)) det(randn(50))
det(randn(50)) det(randn(50))]

ans =

   1.0e+31 *

    1.2894   -0.0421    0.6148   -0.4418    3.0691   -9.5823

>> [det(randn(100)) det(randn(100)) det(randn(100))
det(randn(100)) det(randn(100)) det(randn(100))]

ans =

   1.0e+78 *

   -0.6426    2.7239   -0.6567    2.1435    1.3960   -1.1224

>> [det(randn(200)) det(randn(200)) det(randn(200))
det(randn(200)) det(randn(200)) det(randn(200))]

ans =

  1.0e+187 *

    1.0414    0.0137    0.1884    0.3810   -0.2961   -1.1438

>> [det(randn(400)) det(randn(400)) det(randn(400))
det(randn(400)) det(randn(400)) det(randn(400))]

ans =

   Inf   Inf  -Inf  -Inf   Inf  -Inf
```

Note that **randn(50)** is around 10^{31}, **randn(100)** is around 10^{78}, **randn(200)** is around 10^{186}, and **randn(400)** is just too big for matlab.

18.06 Problem Set 7 Solutions

Total: 100 points

Prob. 16, Sec. 5.2, Pg. 265: F_n is the determinant of the 1, 1, -1 tridiagonal matrix of order n:

$$F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \qquad F_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3 \qquad F_4 = \begin{vmatrix} 1 & -1 & & \\ 1 & 1 & -1 & \\ & 1 & 1 & -1 \\ & & 1 & 1 \end{vmatrix} \neq 4.$$

Expand in cofactors to show that $F_n = F_{n-1} + F_{n-2}$. These determinants are *Fibonacci numbers* 1, 2, 3, 5, 8, 13, ... The sequence usually starts 1, 1, 2, 3 (with two 1's) so our F_n is the usual F_{n+1}.

$\boxed{\text{Solution}}$ (see pg. 535, 4 pts.): The 1, 1 cofactor of the n by n matrix is F_{n-1}. The 1, 2 cofactor has a 1 in column 1, with cofactorF_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the 1, 2 entry to find $F_n = F_{n-1} + F_{n-2}$ (so these determinants are Fibonacci numbers).

Prob. 32, Sec. 5.2, Pg. 268: Cofactors of the 1, 3, 1 matrices in Problem 21 give a recursion $S_n = 3S_{n-1} - S_{n-2}$. Amazingly that recursion produces every second Fibonacci number. Here is the challenge.
Show that S_n is the Fibonacci number F_{2n+2} by proving $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule $F_k = F_{k-1} + F_{k-2}$ starting with $k = 2n + 2$.

$\boxed{\text{Solution}}$ (see pg. 535, 12 pts.): To show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$, keep using Fibonacci's rule:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}.$$

Prob. 33, Sec. 5.2, Pg. 268: The symmetric Pascal matrices have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or cofactors.)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (to explain)}.$$

$\boxed{\text{Solution}}$ (see pg. 535, 12 pts.): The difference from 20 to 19 multiplies its cofactor, which is the determinant of the 3 by 3 Pascal matrix, so equal to 1. Thus the det drops by 1.

Prob. 8, Sec. 5.3, Pg. 279: Find the cofactors of A and multiply AC^{T} to find $\det A$:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and } AC^{\mathrm{T}} = \underline{\qquad}.$$

If you change that 4 to 100, why is $\det A$ unchanged?

$\boxed{\text{Solution}}$ (see pg. 536, 4 pts.): Straightforward computation yields C and $\det A = 3$:

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix} \text{ and } AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

This is $(\det A)I$ and $\det A = 3$.
The 1, 3 cofactor of A is 0.
Multiplying by 4 or by 100: no change.

Prob. 28, Sec. 5.3, Pg. 281: Spherical coordinates ρ, ϕ, θ satisfy $x = \rho \sin\phi\cos\theta$ and $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$. Find the 3 by 3 matrix of partial derivatives: $\partial x/\partial\rho$, $\partial x/\partial\phi$, $\partial x/\partial\theta$ in row 1. Simplify its determinant to $J = \rho^2 \sin\phi$. Then dV in spherical coordinates is $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ the volume of an infinitesimal "coordinate box".

$\boxed{\text{Solution}}$ (4 pts.): The rows are formed by the partials of x, y, z with respect to ρ, ϕ, θ:

$$\begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}.$$

Expanding its determinant J along the bottom row, we get

$$\begin{aligned} J &= \cos\phi(\rho^2\cos\phi\sin\phi)(\cos^2\theta + \sin^2\theta) + \rho^2\sin^3\phi(\cos^2\theta + \sin^2\theta) \\ &= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi. \end{aligned}$$

Prob. 40, Sec. 5.3, Pg. 282: Suppose A is a 5 by 5 matrix. Its entries in row 1 multiply determinants (cofactors) in rows 2–5 to give the determinant. Can you guess a "Jacobi formula" for $\det A$ using 2 by 2 determinants from rows 1–2 *times* 3 by 3 determinants from rows 3–5? Test your formula on the -1, 2, -1 tridiagonal matrix that has determinant 6.

$\boxed{\text{Solution}}$ (12 pts.): A good guess for $\det A$ is the sum, over all pairs i, j with $i < j$, of $(-1)^{i+j+1}$ times the 2 by 2 determinant formed from rows 1–2 and columns i, j times the 3 by 3 determinant formed from rows 3–5 and the complementary columns (this formula is more commonly named after Laplace than Jacobi). There are $\binom{5}{2}$ terms. In the given case, only the first two are nonzero:

$$\det A = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} - \begin{vmatrix} 2 & \\ -1 & -1 \end{vmatrix} \begin{vmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{vmatrix} = (3)(4) - (-2)(-3) = 6.$$

Prob. 41, Sec. 5.3, Pg. 282: The 2 by 2 matrix $AB = (2 \text{ by } 3)(3 \text{ by } 2)$ has a "Cauchy–Binet formula" for $\det AB$:

$$\det AB = \text{ sum of (2 by 2 determinants in } A) \text{ (2 by 2 determinants in } B).$$

(a) Guess which 2 by 2 determinants to use from A and B.
(b) Test your formula when the rows of A are 1, 2, 3 and $1, 4, 7$ with $B = A^{\mathrm{T}}$.

$\boxed{\text{Solution}}$ (12 pts.): (a) A good guess is the sum, over all pairs i, j with $i < j$, of the product of the 2 by 2 determinants formed from columns i, j of A and rows i, j of B.

(b) First, $AA^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 14 & 30 \\ 30 & 66 \end{bmatrix}$. So $\det AA^{\mathrm{T}} = 924 - 900 = 24$.

On the other hand, $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} \begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix} = 4 + 16 + 4 = 24.$

Prob. 19, Sec. 6.1, Pg. 295: A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This is information is enough to find three of these (give the answers where possible):

(a) the rank of B,

(b) the determinant of $B^{\mathrm{T}}B$,

(c) the eigenvalues of $B^{\mathrm{T}}B$,

(d) the eigenvalues of $(B^2 + I)^{-1}$.

Solution (4 pts.): (a) The rank is at most 2 since B is singular as 0 is an eigenvalue. The rank is not 0 since B is not 0 as B has a nonzero eigenvalue. The rank is not 1 since a rank-1 matrix has only one nonzero eigenvalue as every eigenvector lies in the column space. Thus the rank is 2.

(b) We have $\det B^{\mathrm{T}}B = \det B^{\mathrm{T}} \det B = (\det B)^2 = 0 \cdot 1 \cdot 2 = 0$.

(c) There is not enough information to find the eigenvalues of $B^{\mathrm{T}}B$. For example,

$$\text{if } B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix}, \text{ then } B^{\mathrm{T}}B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 4 \end{bmatrix}; \quad \text{if } B = \begin{bmatrix} 0 & 1 & \\ & 1 & \\ & & 2 \end{bmatrix}, \text{ then } B^{\mathrm{T}}B = \begin{bmatrix} 0 & & \\ & 2 & \\ & & 4 \end{bmatrix}.$$

However, the eigenvalues of a triangular matrix are its diagonal entries.

(d) If $Ax = \lambda x$, then $x = \lambda A^{-1}x$; also, any polynomial $p(t)$ yields $p(A)x = p(\lambda)x$. Hence the eigenvalues of $(B^2 + I)^{-1}$ are $1/(0^2 + 1)$ and $1/(1^2 + 1)$ and $1/(2^2 + 1)$, or 1 and 1/2 and 1/5.

Prob. 29, Sec. 6.1, Pg. 296: (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

Solution (4 pts.): Since the eigenvalues of a triangular matrix are its diagonal entries, the eigenvalues of A are $1, 4, 6$. Since the characteristic polynomial of B is

$$\det(B - \lambda I) = (-\lambda)(2 - \lambda)(-\lambda) - 1(2 - \lambda)3 = (2 - \lambda)(\lambda^2 - 3),$$

the eigenvalues of B are $2, \pm\sqrt{3}$. Since C is 6 times the projection onto $(1, 1, 1)$, the eigenvalues of C are $6, 0, 0$.

Prob. 6, Sec. 6.2, Pg. 308: Describe all matrices S that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix},$$

Then describe all matrices that diagonalize A^{-1}.

Solution (see pg. 537, 4 pts.): The columns of S are nonzero multiples of $(2, 1)$ and $(0, 1)$: either order. Same for A^{-1}. Indeed, since the eigenvalues of a triangular matrix are its diagonal entries, the eigenvalues of A are 4, 2. Further, $(2, 1)$ and $(0, 1)$ obviously span the nullspaces of

$$\begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}.$$

Prob. 16, Sec. 6.2, Pg. 309: (Recommended) Find Λ and S to diagonalize A_1 in Problem 15:

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}.$$

What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of the matrix you see the ____ .

$\boxed{\text{Solution}}$ (4 pts.): The columns sum to 1; hence, $A_1 - I$ is singular, and so 1 is an eigenvalue. The two eigenvalues sum to 0.6+0.1; so the other one is -0.3. Further, the nullspaces of

$$\begin{bmatrix} -0.4 & 0.9 \\ 0.4 & -0.9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.9 & 0.9 \\ 0.4 & 0.4 \end{bmatrix}$$

are obviously spanned by $(9,4)$ and $(-1,1)$. Therefore,

$$\Lambda = \begin{bmatrix} 1 & \\ & -0.3 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 9 & -1 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda^k \to \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \quad \text{and}$$
$$S\Lambda^k S^{-1} \to \begin{bmatrix} 9 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \frac{1}{9+4} \begin{bmatrix} 1 & 1 \\ -4 & 9 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 9 & 9 \\ 4 & 4 \end{bmatrix}.$$

In the columns of the last matrix you see the steady state vector.

Prob. 37, Sec. 6.2, Pg. 311: The transpose of $A = S\Lambda S^{-1}$ is $A^{\mathrm{T}} = (S^{-1})^{\mathrm{T}} \Lambda S^{\mathrm{T}}$. The eigenvectors in $A^{\mathrm{T}} y = \lambda y$ are the columns of that matrix $(S^{-1})^{\mathrm{T}}$. They are often called ***left eigenvectors***. How do you multiply matrices to find this formula for A?

$$\textbf{Sum of rank-1 matrices} \quad A = S\Lambda S^{-1} = \lambda_1 x_1 y_1^{\mathrm{T}} + \cdots + \lambda_n x_n y_n^{\mathrm{T}}.$$

$\boxed{\text{Solution}}$ (see pg. 539, 12 pts.): Columns of S times rows of ΛS^{-1} will give r rank-1 matrices ($r =$ rank of A).

Challenge problem: in MATLAB (and in GNU Octave), the command `A=toepliz(v)` produces a symmetric matrix in which each descending diagonal (from left to right) is constant and the first row is v. For instance, if $v = [0\ 1\ 0\ 0\ 0\ 1]$, then `toepliz(v)` is the matrix with 1s on both sides of the main diagonal and on the far corners, and 0s elsewhere. More generally, let $v(n)$ be the vector in $\mathbf{R}^n$ with a 1 in the second and last places and 0s elsewhere, and let `A(n)=toepliz(v(n))`.

(a) Experiment with $n = 5, \ldots, 12$ in MATLAB to see the repeating pattern of $\det A(n)$.

(b) Expand $\det A(n)$ in terms of cofactors of the first row and in terms of cofactors of the first column. Use the known determinant C_n of problem 5.2.13 to recover the pattern found in part (a).

$\boxed{\text{Solution}}$ (12 pts.): (a) The output 2, -4, 2, 0, 2, -4, 2, 0 is returned by this line of code:

```
for n = 5:12; v=zeros(1,n); v(2)=1; v(n)=1; det(toeplitz(v)), endfor.
```

(b) Expand $\det A(n)$ along the first row and then down both first columns to get

$$\det A(n) = -C_{n-2} - (-1)^n + (-1)^{n+1} + (-1)^{n+1}(-1)^n C_{n-2} \quad \text{where } C_n = \begin{cases} 0, & n \text{ odd}; \\ (-1)^{n/2}, & n \text{ even}. \end{cases}$$

Thus $\det A(n) = 2(C_n - (-1)^n)$, which recovers the pattern found in part (a).

18.06 PSET 8 SOLUTIONS

APRIL 15, 2010

Problem 1. (§6.3, #14) The matrix in this question is skew-symmetric ($A^{\mathrm{T}} = -A$):

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \mathbf{u} \qquad \begin{aligned} u_1' &= cu_2 - bu_3 \\ \text{or } u_2' &= au_3 - cu_1 \\ u_3' &= bu_1 - au_2 \end{aligned}$$

(a) The derivative of $\|\mathbf{u}(t)\|^2 = u_1^2 + u_2^2 + u_3^3$ is $2u_1u_1' + 2u_2u_2' + 2u_3u_3'$. Substitute u_1', u_2', u_3' to get *zero*. Then $\|\mathbf{u}(t)\|^2$ stays equal to $\|\mathbf{u}(0)\|^2$.

(b) *When A is skew-symmetric, $Q = e^{At}$ is orthogonal.* Prove $Q^{\mathrm{T}} = e^{-At}$ from the series for $Q = e^{At}$. Then $Q^{\mathrm{T}}Q = I$.

Solution. **(4 points)**

(a)

$$2u_1u_1' + 2u_2u_2' + 2u_3u_3' = 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) = 0.$$

(b) The important points are that $(A^n)^{\mathrm{T}} = (A^{\mathrm{T}})^n = (-A)^n$, and that we can take transpose termwise in a sum:

$$Q^{\mathrm{T}} = \left(\sum_{n=0}^{\infty} A^n \frac{t^n}{n!} \right)^{\mathrm{T}} = \sum_{n=0}^{\infty} (A^n)^{\mathrm{T}} \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-A)^n \frac{t^n}{n!} = e^{-At}.$$

Then,

$$Q^{\mathrm{T}}Q = e^{-At}e^{At} = e^0 = I$$

because A and $-A$ commute (but I don't think the problem intended for you to have to actually check this!). □

Problem 2. (§6.3, #24) Write $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ as $S\Lambda S^{-1}$. Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At}. Check e^{At} and the derivative of e^{At} when $t = 0$.

Solution. **(4 points)**

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

Then,

$$Se^{\Lambda t}S^{-1} = \begin{bmatrix} e^t & \frac{e^{3\,t}}{2} - \frac{e^t}{2} \\ 0 & e^{3\,t} \end{bmatrix}$$

This is the identity matrix when $t = 0$, as it should be.

The derivative matrix is

$$\begin{bmatrix} e^t & 3/2e^{3t} - 1/2e^t \\ 0 & 3e^{3t} \end{bmatrix}$$

which is equal to A when $t = 0$, as it should be. □

Problem 3. (§6.3, #28) Centering $y'' = -y$ in Example 3 will produce $Y_{n+1} - 2Y_n + Y_{n-1} = -(\Delta t)^2 Y_n$. This can be written as a one-step difference equation for $\mathbf{U} = (Y, Z)$:

$$\begin{aligned} Y_{n+1} &= Y_n + \Delta t Z_n \\ Z_{n+1} &= Z_n - \Delta t Y_{n+1} \end{aligned} \qquad \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}.$$

Invert the matrix on the left side to write this as $\mathbf{U}_{n+1} = A\mathbf{U}_n$. Show that $\det A = 1$. Choose the large time step $\Delta t = 1$ and find the eigenvalues λ_1 and $\lambda_2 = \overline{\lambda_1}$ of A:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{has } |\lambda_1| = |\lambda_2| = 1. \text{ Show that } A^6 \text{ is exactly } I.$$

After 6 steps to $t = 6$, $\mathbf{U}_6$ equals $\mathbf{U}_0$. The exact $y = \cos t$ returns to 1 at $t = 2\pi$.

Solution. **(12 points)** We have

$$\begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\Delta t & 1 \end{bmatrix} \qquad \text{and so} \qquad A = \begin{bmatrix} 1 & 0 \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 - (\Delta t)^2 \end{bmatrix}$$

Clearly $\det A = 1$: it is the product of two matrices that are triangular with ones on the diagonal, and so each have determinant 1.

For $\Delta t = 1$, the matrix becomes $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. The eigenvalues are the roots of the polynomial $\lambda^2 - \lambda + 1 = 0$:

$$\lambda_1 = \frac{1 + i\sqrt{3}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1 - i\sqrt{3}}{2} = \overline{\lambda_1}.$$

These numbers are actually pretty special: Since $\lambda^2 = \lambda - 1$, they satisfy $\lambda^3 = \lambda^2 - \lambda = -1$ and so $\lambda^6 = 1$.

Since $\lambda_1 \neq \lambda_2$, there is a basis v_1, v_2 consisting of eigenvectors for A. So to check that $A^6 = I$, it is enough to check this on the basis v_1 and v_2. But, $A^6 v_1 = \lambda_1^6 v_1 = v_1$ and $A^6 v_2 = \lambda_2^6 v_2 = v_2$!

(I don't think there was a question in the last sentence...) □

Problem 4. (§6.3, #29) The centered choice (*leapfrog method*) in Problem 28 is very successful for small time steps Δt. But find the eigenvalues of A for $\Delta t = \sqrt{2}$ and 2:

$$A = \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$$

Both matrices have $|\lambda| = 1$. Compute A^4 in both cases and find the eigenvectors of A. That value $\Delta t = 2$ is at the border of instability. Time steps $\Delta t > 2$ will lead to $|\lambda| > 1$, and the powers in $\mathbf{U}_n = A^n \mathbf{U}_0$ will explode.

Note You might say that nobody would compute with $\Delta t > 2$. But if an atom vibrates with $y'' = -1000000y$, then $\Delta t > .0002$ will give instability. Leapfrog has a very strict stability limit. $Y_{n+1} = Y_n + 3Z_n$ and $Z_{n+1} = Z_n - 3Y_{n+1}$ will explode because $\Delta t = 3$ is too large.

Solution. **(12 points)** For $\Delta t = \sqrt{2}$, the eigenvalues are the roots of $\lambda^2 + 1 = 0$, that is $\boxed{\pm i}$. For $\Delta t = 2$, the eigenvalues are the roots of $\lambda^2 + 2\lambda + 1 = 0$, that is $\boxed{-1}$ (with algebraic multiplicity two).

In the first case, $A^4 = I$ (for the same reason as in the previous problem, or just multiply it out). The eigenvectors of A (for $i, -i$ respectively) are (multiples of)

$$v_1 = \begin{bmatrix} 1 + i \\ -\sqrt{2}i \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} 1 - i \\ \sqrt{2}i \end{bmatrix}.$$

In the second case, we don't get distinct eigenvectors and *have to* multiply it out:

$$A^4 = \begin{bmatrix} -7 & -8 \\ 8 & 9 \end{bmatrix}.$$

The eigenvectors of A for $\lambda = -1$ are (multiples of)

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(Note that the algebraic multiplicity of $\lambda = 0$ is *two*, while the geometric multiplicity is *one*: That is, there is a one-dimensional space of eigenvectors.) ☐

Problem 5. (§6.3, #30) Another good idea for $y'' = -y$ is the trapezoidal method (half forward/half back): This may be the best way to keep (Y_n, Z_n) exactly on a circle.

$$\textbf{Trapezoidal} \quad \begin{bmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}$$

(a) Invert the left matrix to write this equation as $\mathbf{U}_{n+1} = A\mathbf{U}_n$. *Show that A is an orthogonal matrix: $A^{\mathrm{T}}A = I$.* **These points $\mathbf{U}_n$ never leave the circle.** $A = (I-B)^{-1}(I+B)$ is always an orthogonal matrix if $B^{\mathrm{T}} = -B$.

(b) (Optional MATHLAB) Take 32 steps from $\mathbf{U}_0 = (1, 0)$ to $\mathbf{U}_{32}$ with $\Delta t = 2\pi/32$. Is $\mathbf{U}_{32} = \mathbf{U}_0$? I think there is a small error.

Solution. **(12 points)**

(a) I get

$$\begin{bmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{4}{(\Delta t)^2+4} & \frac{2\,\Delta t}{(\Delta t)^2+4} \\ -\frac{2\,\Delta t}{(\Delta t)^2+4} & \frac{4}{(\Delta t)^2+4} \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} \frac{4-(\Delta t)^2}{(\Delta t)^2+4} & \frac{4\,\Delta t}{(\Delta t)^2+4} \\ -\frac{4\,\Delta t}{(\Delta t)^2+4} & \frac{4-(\Delta t)^2}{(\Delta t)^2+4} \end{bmatrix}$$

It's an annoying computation to check directly that $A^{\mathrm{T}}A = I$, but it works.

(b) It's pretty close (approx. $(0.9992, 0.0401)$)…. ☐

Problem 6. (§6.4, #7)

(a) Find a symmetric matrix $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.

(b) How do you know it must have a negative pivot?

(c) How do you know it can't have two negative eigenvalues?

Solution. **(4 points)**

(a) The eigenvalues of that matrix are $1 \pm b$. So take any $b > 1$ (or $b < -1$). In this case, the determinant is $1 - b^2 < 0$.

(b) We saw in the book that the signs of the pivots coincide with the signs of the eigenvalues. (Alternatively, the product of the pivots is the determinant, which is negative in this case. So, precisely one of the two pivots must be negative.)

(c) The product of the eigenvalues equals the determinant, which is negative in this case. Two negative numbers cannot have a negative product! ☐

Problem 7. (§6.4, #10) Here is a quick "proof" that the eigenvalues of all real matrices are real:

$$\textbf{False proof} \qquad A\mathbf{x} = \lambda\mathbf{x} \text{ gives } \mathbf{x}^{\mathrm{T}}A\mathbf{x} = \lambda\mathbf{x}^{\mathrm{T}}\mathbf{x} \text{ so } \lambda = \frac{\mathbf{x}^{\mathrm{T}}A\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \text{ is real.}$$

Find the flaw in this reasoning—a hidden assumption that is not justified. You could test those steps on the 90-degree rotation matrix $[0, -1; 1, 0]$ with $\lambda = i$ and $\mathbf{x} = (i, 1)$.

Solution. **(4 points)** The vector $\mathbf{x}$ doesn't have real components. So, $\mathbf{x}^{\mathrm{T}}\mathbf{x}$ can be zero and neither numerator nor denominator is obviously real… ☐

Problem 8. (§6.4, #23) Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, $Q\Lambda Q^{\mathrm{T}}$?

Solution. **(4 points)** One at a time:

(a) Matrix A is invertible, orthogonal, a permutation matrix, diagonalizable, and Markov! (So everything but a projection...)

Let's see why: A satisfies $A^2 = I$ and $A = A^{\mathrm{T}}$, and so also $AA^{\mathrm{T}} = I$. This means it is invertible, symmetric, and orthogonal. Since it is symmetric, it is diagonalizable (with real eigenvalues!). It is a permutation matrix by just looking at it. It is Markov since the columns add to 1 (just by looking at it), or alternatively because every permutation matrix is. It is not a projection since $A^2 = I \neq A$.

All of the factorizations are possible for it: LU and QR are always possible, $S\Lambda S^{-1}$ is possible since it is diagonalizable, and $Q\Lambda Q^T$ is possible since it is symmetric.

(b) Matrix B is a projection, diagonalizable, and Markov. It is not invertible, not orthogonal, and not a permutation.

Let's see why: B is a projection since $B^2 = B$, it is symmetric and thus diagonalizable, and it's Markov since the columns add to 1. It is not invertible since the columns are visibly linearly dependent, it is not orthogonal since the columns are far from orthonormal, and it's clearly not a permutation.

All the factorizations are possible for it: LU and QR are always possible, $S\Lambda S^{-1}$ is possible since it is diagonalizable, and $Q\Lambda Q^T$ is possible since it is symmetric.

□

Problem 9. (§6.4, #28) For complex matrices, the symmetry $A^{\mathrm{T}} = A$ that produces real eigenvalues changes to $\overline{A}^{\mathrm{T}} = A$. From $\det(A - \lambda I) = 0$, find the eigenvalues of the 2 by 2 "Hermitian" matrix

$$A = \begin{bmatrix} 4 & 2+i \\ 2-i & 0 \end{bmatrix} = \overline{A}^{\mathrm{T}}$$

To see why eigenvalues are real when $\overline{A}^{\mathrm{T}} = A$, adjust equation (1) of the text to $\overline{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$.

Transpose to $\overline{\mathbf{x}}^{\mathrm{T}}\overline{A}^{\mathrm{T}} = \overline{\mathbf{x}}^{\mathrm{T}}\overline{\lambda}$. With $\overline{A}^{\mathrm{T}} = A$, reach equation (2): $\lambda = \overline{\lambda}$.

Solution. **(12 points)** We solve $\lambda^2 - 4\lambda - 5 = 0$ to find $\lambda = -1$ or $\lambda = 5$.

Now let's do the proof:

$$\lambda\overline{\mathbf{x}}^T\mathbf{x} = \left(\overline{\mathbf{x}}^{\mathrm{T}}A\mathbf{x}\right)^T = \mathbf{x}^{\mathrm{T}}A^{\mathrm{T}}\overline{\mathbf{x}} = \mathbf{x}^{\mathrm{T}}\overline{A}\overline{\mathbf{x}} = \overline{\lambda}\mathbf{x}^{\mathrm{T}}\overline{\mathbf{x}}.$$

But now, $\mathbf{x}^{\mathrm{T}}\overline{\mathbf{x}}$ is the complex conjugate of $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}$. Since $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = \sum_i |x_i|^2$ is a non-negative real number, it is its own complex conjugate (and non-zero). Dividing the previous displayed equation by this non-zero number, we get $\lambda = \overline{\lambda}$. □

Problem 10. (§6.4, #30) If λ_{max} is the largest eigenvalue of a symmetric matrix A, no diagonal entry can be larger tha λ_{max}. What is the first entry a_{11} of $A = Q\Lambda Q^T$? Show why $a_{11} \leq \lambda_{max}$.

Solution. **(12 points)** Set $\mathbf{e}_1 = (1, 0, 0, \ldots)^T$ and $\mathbf{v} = Q^{\mathrm{T}}\mathbf{e}_1 = (v_1, \ldots, v_n)$. Then,

$$a_{11} = \mathbf{e}_1^{\mathrm{T}}A\mathbf{e}_1 = \mathbf{e}_1^{\mathrm{T}}Q\Lambda Q^{\mathrm{T}}\mathbf{e}_1 = (Q^{\mathrm{T}}\mathbf{e}_1)^{\mathrm{T}}\Lambda(Q^{\mathrm{T}}\mathbf{e}_1) = \mathbf{v}^{\mathrm{T}}\Lambda\mathbf{v} = \sum_{i=1}^{n}\lambda_i v_i^2.$$

Since Q^{T} is orthogonal,

$$\|\mathbf{v}\| = \|Q^{\mathrm{T}}\mathbf{e}_1\| = \|\mathbf{e}_1\| = 1$$

and so

$$a_{11} \leq \lambda_{max}\sum_{i=1}^{n} v_i^2 = \lambda_{max}\|\mathbf{v}\|^2 = \lambda_{max}.$$

□

Problem 11. (§8.3, #9) Prove that the square of a Markov matrix is also a Markov matrix.

Solution. **(4 points)** A matrix A is matrix precisely if the sum of the components of $A\mathbf{x}$ is equal to the sum of the components of $\mathbf{x}$, i.e. $\sum x_i = \sum (A\mathbf{x})_i$. (In other words, if the "transition probabilities" given by A keep the total probability the same.) But if A doesn't change the sum of the components, then certainly A^2 doesn't either. $\square$

Problem 12. (§8.3, #12) A Markov differential equation is not $d\mathbf{u}/dt = A\mathbf{u}$ but $d\mathbf{u}/dt = (A - I)\mathbf{u}$. The diagonal is negative, the rest of $A - I$ is positive. The coulmns add to zero.

Find the eigenvalues of $B = A - I = \begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix}$. Why does $A - I$ have $\lambda = 0$?

When $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ multiply $\mathbf{x}_1$ and $\mathbf{x}_2$, what is the steady state as $t \to \infty$?

Solution. **(4 points)** The eigenvalues are the roots of $\lambda^2 + 1/2\lambda$, that is $\boxed{0, -1/2}$. This has $\lambda = 0$ as an eigenvalue since A has $\lambda = 1$ as an eigenvalue (since it is Markov).

For $\lambda_1 = 0$, $e^{\lambda_1 t}\mathbf{x}_1 = \mathbf{x}_1$ is already the steady state.

For $\lambda_2 = -1/2$, $e^{\lambda_2 t}\mathbf{x}_2 = e^{-1/2t}\mathbf{x}_2$ goes to the steady state $(0, 0)$ as $t \to \infty$.

$\square$

Problem 13. (§8.3, #16) (Markov again) This matrix has zero determinant. What are its eigenvalues?

$$A = \begin{bmatrix} .4 & .2 & .3 \\ .2 & .4 & .3 \\ .4 & .4 & .4 \end{bmatrix}.$$

Find the limits of $A^k\mathbf{u}_0$ starting from $\mathbf{u}_0 = (1, 0, 0)$ and then $\mathbf{u}_0 = (100, 0, 0)$.

Solution. **(12 points)** The eigenvalues are the roots of $\lambda^3 - 6/5\lambda^2 + 1/5\lambda = 0$, which are $\boxed{0, 1/5, 1}$

We can find corresponding eigenvectors:

- For $\lambda = 0$: $(1, 1, -2)$.
- For $\lambda = 1/5$: $(1, -1, 0)$.
- For $\lambda = 1$: $(3, 3, 4)$ (for $\lambda = 1$).

(And in fact, we only care about the last one since the others have $|\lambda| < 1$)

So, $\lim_{k\to\infty} A^k$ is a (non-orthogonal) projection onto the line spanned by $(3, 3, 4)$. Since A is Markov, $\lim_{k\to\infty} A^k$ is as well and its columns are vectors parallel to $(3, 3, 4)$ whose components sum to 1. This tells us right away what this limit must be:

$$\lim_{k\to\infty} A^k = \begin{bmatrix} .3 & .3 & .3 \\ .4 & .4 & .4 \\ .4 & .4 & .4 \end{bmatrix}$$

The limits we wanted are

$$\lim A^k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .3 \\ .3 \\ .4 \end{bmatrix} \qquad \text{and} \qquad \lim A^K \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \\ 40 \end{bmatrix}$$

Note that we knew ahead of time that the second answer would just be 100 times the first by linearity. I have no idea why the book would ask such a silly thing. $\square$

18.06 Pset 9 Solutions

Problem 6.5, #25: With positive pivots in D, the factorization $A = LDL^{\mathrm{T}}$ becomes $L\sqrt{D}\sqrt{D}L^{\mathrm{T}}$. (Square roots of the pivots give $D = \sqrt{D}\sqrt{D}$.) Then $C = \sqrt{D}L^{\mathrm{T}}$ yields the *Cholesky factorization* $A = C^{\mathrm{T}}C$ which is "symmetrized LU".

$$\text{From} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{find } A. \qquad \text{From} \quad A = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} \quad \text{find } C = \mathbf{chol}(A).$$

$\boxed{\text{Solution}}$ (4 points) From C, we obtain

$$A = C^{\mathrm{T}}C = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}.$$

Conversely, the given A quickly diagonalizes to $\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$ via $L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$: thus

$$C = \mathbf{chol}(A) = \sqrt{D}L^{\mathrm{T}} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}.$$

Problem 6.5, #26: In the Cholesky factorization $A = C^{\mathrm{T}}C$, with $C^{\mathrm{T}} = L\sqrt{D}$, the square roots of the pivots are on the diagonal of C. Find C (upper triangular) for

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}$$

.

$\boxed{\text{Solution}}$ (4 points) For the first matrix A, we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

while for the second matrix we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$$

Problem 6.5, #27: The symmetric factorization $A = LDL^{\mathrm{T}}$ means that $\mathbf{x}^{\mathrm{T}}A\mathbf{x} = \mathbf{x}^{\mathrm{T}}LDL^{\mathrm{T}}\mathbf{x}$:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & (ac-b^2)/a \end{bmatrix}\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The left side is $ax^2 + 2bxy + cy^2$. The right side is $a(x + \frac{b}{a}y)^2 + \_\_\_ y^2$. The second pivot completes the square! Test with $a = 2, b = 4, c = 10$.

$\boxed{\text{Solution}}$ (4 points) Evaluating out the right side gives $ax^2 + 2bxy + cy^2$, so the entry in the space given is $c - \frac{b^2}{a}$, i.e. the second pivot. For the given values, we have $2x^2 + 8xy + 10y^2 = 2(x + 2y)^2 + 2y^2$ as desired.

Problem 6.5, #29: For $F_1(x, y) = x^4/4 + x^2 + x^2y + y^2$ and $F_2(x, y) = x^3 + xy - x$, find the second-derivative matrices H_1 and H_2:

$$H = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix}.$$

H_1 is positive-definite so F_1 is concave up (= convex). Find the minimum point of F_1 and the the saddle point of F_2 (look only where the first derivatives are zero).

$\boxed{\text{Solution}}$ (4 points) For $F_1(x, y)$, we first solve for the stationary point

$$\frac{\partial F_1}{\partial x} = x^3 + 2x + 2xy = 0, \frac{\partial F_1}{\partial y} = x^2 + 2y = 0$$

From (2), we have $y = -x^2/2$. Plug this into (1), we have $2x = 0$ and hence the only critical point is $x = y = 0$. At this point,

$$H_1 = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 3x^3 + 2 + 2y & 2x \\ 2x & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

It is positive definite and hence $(0, 0)$ is a minimal point of $F_1(x, y)$.

REMARK: The problem for $F_1 = x^4/4 + x^2y + y^2$ as originally stated, you get a curve of minima $x^2 + 2y = 0$, and H_1 is only positive semidefinite.

For $F_2(x, y)$, we first solve for the stationary point

$$\frac{\partial F_2}{\partial x} = 3x^2 + y - 1 = 0, \frac{\partial F_2}{\partial y} = x = 0$$

This implies that $y = 1$. At this point $(0, 1)$,

$$H_2 = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of H_2 at $(0,1)$ is the solution to det $(H_2 - \lambda I) = \lambda^2 - 1$, which are $\lambda_1 = 1$ and $\lambda_2 = -1$. They are with opposite signs and hence $(0,1)$ is a saddle point of $F_2(x,y)$.

Problem 6.5, #32: A *group* of nonsingular matrices include AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these are groups (as in 2.7.37)? Invent a "subgroup" of two of these groups (not I by itself = the smallest group).

(a) Positive definite symmetric matrices A.
(b) Orthogonal matrices Q.
(c) All exponentials e^{tA} of a fixed matrix A.
(d) Matrices P with positive eigenvalues.
(e) Matrices D with determinant 1.

Solution (12 points) First, note that all but the first and fourth are groups (assuming we are only referring to square matrices in (b)): on the other hand, $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ are both positive definite and symmetric, but their product is not symmetric. Intersections of these groups give the simplest examples of subgroups (for instance, orthogonal matrices with determinant 1, called the *special orthogonal matrices*), though there are many others.

Problem 6.5, #33: When A and B are symmetric positive definite, AB might not even be symmetric. But its eigenvalues are still positive. Start from $AB\mathbf{x} = \lambda\mathbf{x}$ and take dot products with $B\mathbf{x}$. Then prove $\lambda > 0$.

Solution (12 points) Taking dot products, we get $(AB\mathbf{x})^{\mathrm{T}}B\mathbf{x} = (B\mathbf{x})^{\mathrm{T}}A(B\mathbf{x})$ on the left and $(\lambda\mathbf{x})^{\mathrm{T}}B\mathbf{x} = \lambda\mathbf{x}^{\mathrm{T}}B\mathbf{x}$. Since B is positive definite, $\mathbf{x}^{\mathrm{T}}B\mathbf{x} > 0$, and since A is positive definite, $(B\mathbf{x})^{\mathrm{T}}A(B\mathbf{x})$ is too ($B\mathbf{x}$ is just another vector). Thus, λ must be positive as well.

Problem 6.5, #34: Write down the 5 by 5 sine matrix S from Worked Example 6.5 D, containing the eigenvectors of K when $n = 5$ and $h = 1/6$. Multiply K times S to see the five positive eigenvalues.

Their sum should equal the trace 10. Their product should be det $K = 6$.

$\boxed{\text{Solution}}$ (12 points) S is the matrix

$$S = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 1 & \sqrt{3}/2 & 1/2 \\ \sqrt{3}/2 & \sqrt{3}/2 & 0 & -\sqrt{3}/2 & -\sqrt{3}/2 \\ 1 & 0 & -1 & 0 & 1 \\ \sqrt{3}/2 & -\sqrt{3}/2 & \sqrt{3}/2 & -\sqrt{3}/2 & \sqrt{3}/2 \\ 1/2 & -\sqrt{3}/2 & 1 & -\sqrt{3}/2 & 1/2 \end{bmatrix}$$

The five eigenvalues (corresponding to the columns) are $2-\sqrt{3}, 1, 2, 3$, and $2+\sqrt{3}$, which add up to 10 and multiply to 6 as desired.

Problem 6.5, #35: If A has full column rank, and C is positive-definite, show that $A^{\mathrm{T}}CA$ is positive definite.

$\boxed{\text{Solution}}$ (12 points) Since C is positive-definite, $y^{\mathrm{T}}Cy > 0$ for any $y \neq 0$ in $\mathbb{R}^n$. Now, we need to show that $z^{\mathrm{T}}A^{\mathrm{T}}CAz > 0$ for any $z \neq 0$ in $\mathbb{R}^n$. We can rewrite it as $z^{\mathrm{T}}A^{\mathrm{T}}CAz = (Az)^{\mathrm{T}}C(Az)$. Since A has full column rank, $N(A) = \{0\}$ and in particular, $Az \neq 0$ in $\mathbb{R}^n$. Therefore, we have $(Az)^{\mathrm{T}}C(Az) > 0$. This implies that $A^{\mathrm{T}}CA$ is positive definite.

Problem 8.1, #3: In the free-free case when $A^{\mathrm{T}}CA$ in equation (9) is singular, add the three equations $A^{\mathrm{T}}CA\mathbf{u} = \mathbf{f}$ to show that we need $f_1 + f_2 + f_3 = 0$. Find a solution to $A^{\mathrm{T}}CA\mathbf{u} = \mathbf{f}$ when the forces $\mathbf{f} = (-1, 0, 1)$ balance themselves. Find all solutions!

$\boxed{\text{Solution}}$ (4 points) Dot producting our formula with $(1, 1, 1)$ gives

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$0 = f_1 + f_2 + f_3$$

Substituting $\mathbf{f} = (-1, 0, 1)$ gives the two equations $c_2(u_1 - u_2) = -1, c_3(u_3 - u_2) = 1$ (the middle equation is redundant), with a solution $(-c_2^{-1}, 0, c_3^{-1})$. All other solutions are given by adding multiples of $(1, 1, 1)$, which spans the nullspace.

Problem 8.1, #5: In the fixed-free problem, the matrix A is square and invertible. We can solve $A^{\mathrm{T}}\mathbf{y} = \mathbf{f}$ separately from $A\mathbf{u} = \mathbf{e}$. Do the same for the differential equation:

$$\text{Solve} \quad -\frac{dy}{dx} = f(x) \quad \text{with} \quad y(1) = 0. \quad \text{Graph} \quad y(x) \quad \text{if} \quad f(x) = 1.$$

$\boxed{\text{Solution}}$ (4 points) $y(x) = -\int_1^x f(x)dx$ and if $f(x) = 1$ then $y(x) = 1 - x$. You can graph this.

Problem 8.1, #7: For five springs and four masses with both ends fixed, what are the matrices A and C and K? With $C = I$ solve $K\mathbf{u} = \text{ones}(4)$.

$\boxed{\text{Solution}}$ (4 points) The matrices are

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} c_1 & & & & \\ & c_2 & & & \\ & & c_3 & & \\ & & & c_4 & \\ & & & & c_5 \end{bmatrix},$$

$$K = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & 0 \\ -c_2 & c_2 + c_3 & -c_3 & 0 \\ 0 & -c_3 & c_3 + c_4 & -c_4 \\ 0 & 0 & -c_4 & c_4 + c_5 \end{bmatrix}$$

Inverting K for $c_1 = \cdots = c_5 = 1$ gives

$$K^{-1} = \frac{1}{5} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Multiplying by $(1, 1, 1, 1)$ gives $(2, 3, 3, 2)$.

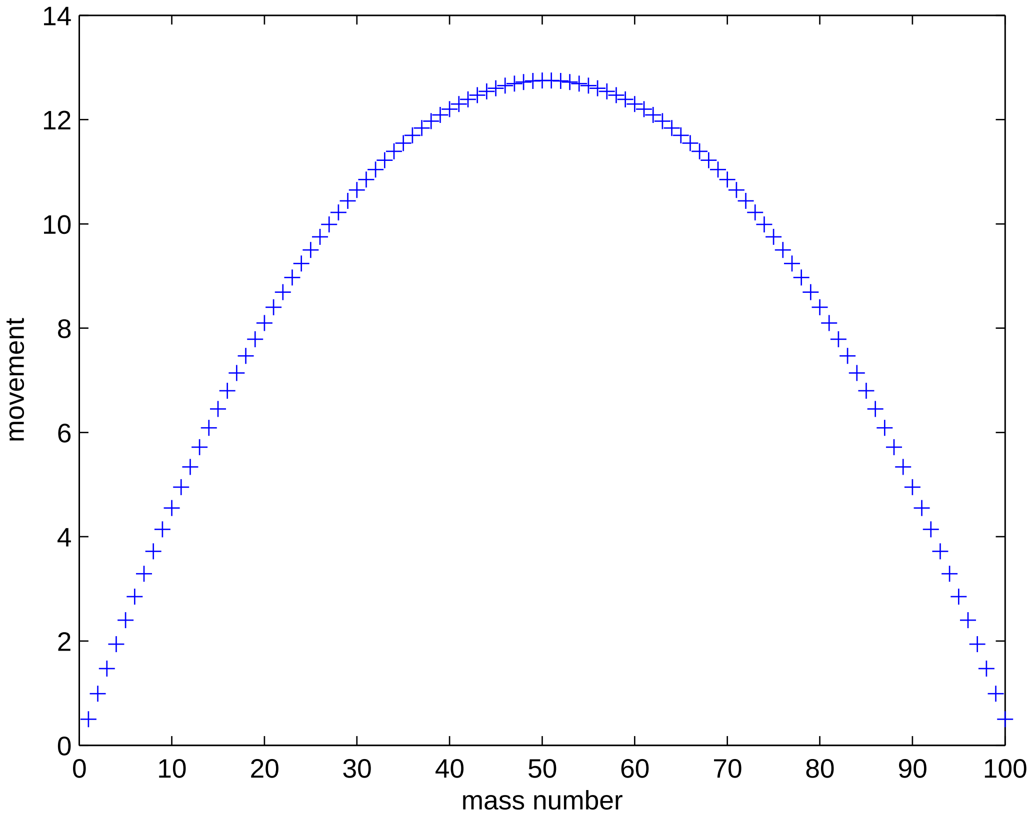
Problem 8.1, #10: (MATLAB) Find the displacements $u(1), \ldots, u(100)$ of 100 masses connected by springs all with $c = 1$. Each force is $f(i) = 0.01$. Print graphs of $\mathbf{u}$ with fixed-fixed and fixed-free ends. Note that $\text{diag}(\text{ones}(n, 1), d)$ is a matrix with n ones along the diagonal d. This print command will graph a vector u:

plot(u,'+'); xlabel('mass number'); ylabel('movement'); print

$\boxed{\text{Solution}}$ (12 points)

```
>> E = diag(ones(99,1),1);
>> K = 2*eye(100)-E-E';
>> f = 0.01*ones(100, 1); u = K\f;
>> plot(u,'+'); xlabel('mass number'); ylabel('movement'); print

>> K(100,100) = 1; u = K\f;
>> plot(u,'+'); xlabel('mass number'); ylabel('movement'); print
```

FIGURE 1. Fixed-fixed



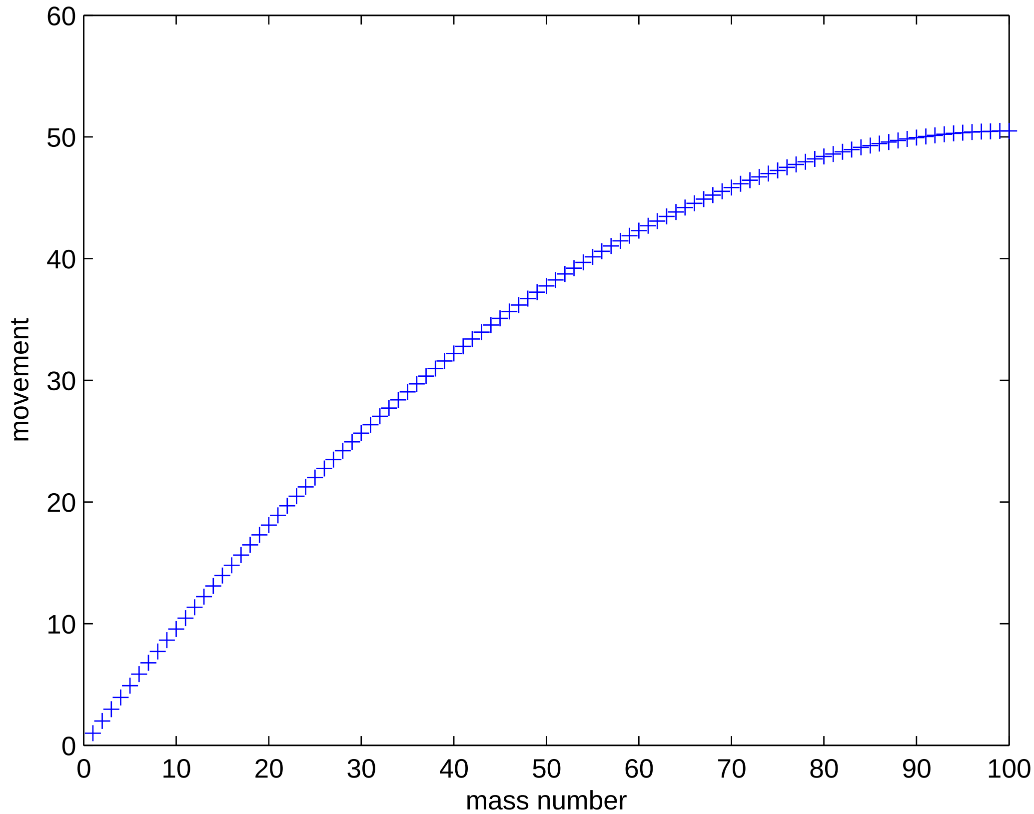
Problem 8.1, #11: (MATLAB) Chemical engineering has a first derivative of du/dx from fluid velocity as well as d^2u/dx^2 from diffusion. Replace du/dx by a *forward* difference, then a *centered* difference, then a *backward* difference, with $\nabla x = \frac{1}{8}$. Graph your numerical solutions of

$$-\frac{d^2u}{dx^2} + 10\frac{du}{dx} = 1 \text{ with } u(0) = u(1) = 0.$$

$\boxed{\text{Solution}}$ (12 points)

```
>> E = diag(ones(6,1),1);
>> K = 64*(2*eye(7) - E - E');
>> D = 80*(-eye(7)+E);
```

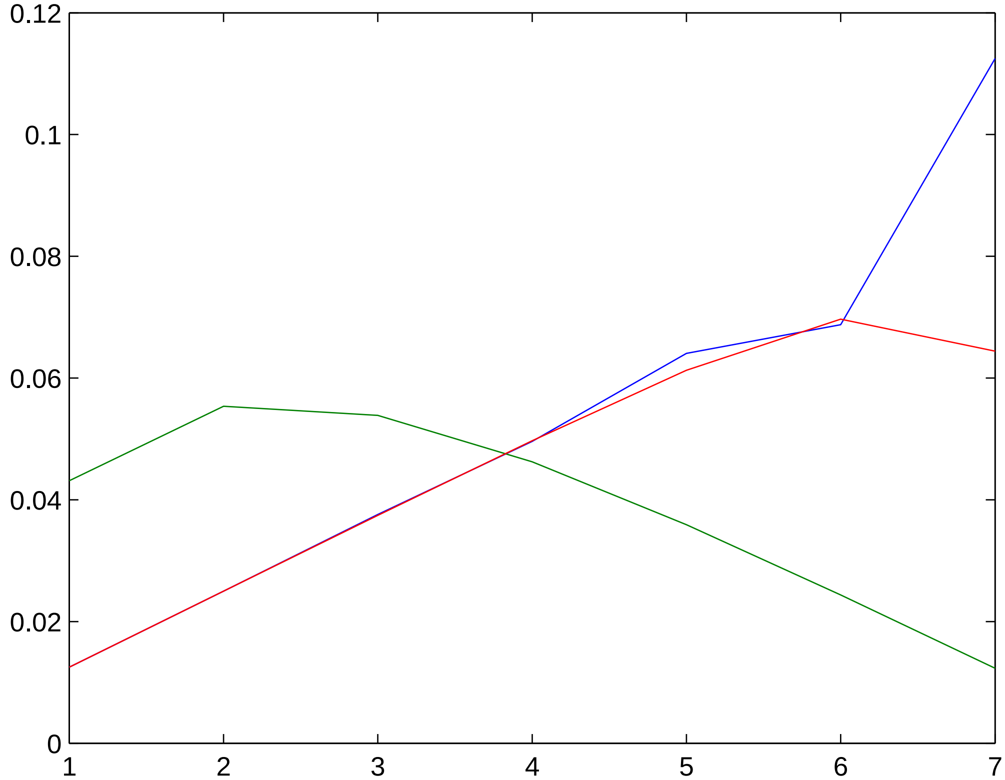
FIGURE 2. Fixed-free




```
>> forward = (K+D)\\ones(7,1)

forward =

    0.0125
    0.0250
    0.0376
    0.0496
    0.0641
    0.0688
    0.1125
```

```
>> backward = (K-D)\\ones(7,1)

backward =

    0.0431
    0.0554
    0.0539
    0.0462
    0.0359
    0.0244
    0.0123

>> centered = (K+D/2-D'/2)\\ones(7,1)

centered =

    0.0125
    0.0250
    0.0374
    0.0497
    0.0613
    0.0697
    0.0644

>> plot(n,forward(n),n,backward(n),n,centered(n))
```

FIGURE 3. Overlayed numerical solutions



18.06 Problem Set 10 Solution

Total: 100 points

Section 6.6. Problem 12. These Jordan matrices have eigenvalues $0, 0, 0, 0$. They have two eigenvectors (one from each block). But the block sizes don't match and they are *not similar*:

$$J = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad K = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

For any matrix M, compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is impossible; *J is not similar to K.*

$\boxed{\text{Solution}}$ (4 points) Let $M = (m_{ij})$. Then

$$JM = \begin{pmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad MK = \begin{pmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{pmatrix}.$$

If $JM = MK$ then

$$m_{21} = m_{22} = m_{24} = m_{41} = m_{42} = m_{44} = 0,$$

which in particular means that the second row is either a multiple of the fourth row, or the fourth row is all 0's. In either of these cases M is not invertible.

Suppose that J were similar to K. Then there would be some invertible matrix M such that $K = M^{-1}JM$, which would mean that $MK = JM$. But we just showed that in this case M is never invertible! Contradiction. Thus J is not similar to K.

Section 6.6. Problem 14. Prove that A^T is *always similar* to A (we know that the λ's are the same):

1. For one Jordan block J_i: find M_i so that $M_i^{-1}J_iM_i = J_i^T$ (see example 3).
2. For any J with blocks J_i: build M_0 from blocks so that $M_0^{-1}JM_0 = J^T$.

3. For any $A = MJM^{-1}$: Show that A^T is similar to J^T and so to J and so to A.

$\boxed{\text{Solution}}$ (4 points)

1. Suppose that we have one Jordan block J_i. Then

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ & \lambda & 1 & \cdots & 0 \\ & & \lambda & \cdots & 0 \\ & & & \ddots & \\ & & & & \lambda \end{pmatrix} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ 0 & 1 & \lambda & & \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

so J is similar to J^T.

2. Suppose that each J_i satisfies $J_i^T = M_i^{-1} J_i M_i$. Let M_0 be the block-diagonal matrix consisting of the M_i's along the diagonal. Then

$$\begin{aligned} M_0^{-1} J M_0 &= \begin{pmatrix} M_1^{-1} & & & \\ & M_2^{-1} & & \\ & & \ddots & \\ & & & M_n^{-1} \end{pmatrix} \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{pmatrix} \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{pmatrix} \\ &= \begin{pmatrix} M_1^{-1} J_1 M_1 & & & \\ & M_2^{-1} J_2 M_2 & & \\ & & \ddots & \\ & & & M_n^{-1} J_n M_n \end{pmatrix} \\ &= \begin{pmatrix} J_1^T & & & \\ & J_2^T & & \\ & & \ddots & \\ & & & J_n^T \end{pmatrix} = J^T \end{aligned}$$

3. $$A^T = (MJM^{-1})^T = (M^{-1})^T J^T M^T = (M^T)^{-1} J^T (M^T).$$

So A^T is similar to J^T, which is similar to J, which is similar to A. Thus any matrix is similar to its transpose.

Section 6.6. Problem 20. Why are these statements all true?

(a) If A is similar to B then A^2 is similar to B^2.

(b) A^2 and B^2 can be similar when A and B are not similar.

(c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$.

(d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is not similar to $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

(e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case $M =$?

$\boxed{\text{Solution}}$ (4 points)

(a) If A is similar to B then we can write $A = M^{-1}BM$ for some M. Then $A^2 = M^{-1}B^2M$, so A^2 is similar to B^2.

(b) Let

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $A^2 = B^2$ (so they are obviously similar) but A is not similar to B because nothing but the zero matrix is similar to the zero matrix.

(c)

$$\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(d) These are not similar because the first matrix has a plane of eigenvectors for the eigenvalue 3, while the second only has a line.

(e) In order to exchange two rows of A we multiply on the left by

$$M = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

In order to exchange two columns we multiply on the right by the same M. As $M = M^{-1}$ we see that the new matrix is similar to the old one, so the eigenvalues stay the same.

Section 6.6. Problem 22. If an $n \times n$ matrix A has all eigenvalues $\lambda = 0$ prove that A^n is the zero matrix.

$\boxed{\text{Solution}}$ (12 points) Suppose that we have a Jordan block of size i with eigenvalue 0. Then notice that J^2 will have a diagonal of 1's two diagonals above the main diagonal and zeroes elsewhere. J^3 will have a diagonal of 1's three diagonals above the main diagonal, and zeroes elsewhere. Therefore $J^i = 0$, since there is no diagonal i diagonals above the main diagonal. If A has all eigenvalues $\lambda = 0$ then then A is similar to some matrix with Jordan blocks $J_1, \ldots, J_k$ with each J_i of size n_i and $\sum_{i=1}^{k} n_k = n$. Each Jordan block will have eigenvalue 0, so we know that $J_i^{n_i} = 0$, and thus $J_i^n = 0$.

As A^n is similar to a block-diagonal matrix with blocks $J_1^n, J_2^n, \ldots, J_k^n$ and each of these is 0 we know that $A^n = 0$.

Another way to see this is to note that if A has all eigenvalues 0 this means that the characteristic polynomial of A must be x^n, as this is the only polynomial of degree n all of whose roots are 0. Thus $A^n = 0$ by the Cayley-Hamilton theorem.

Section 6.6. Problem 23. For the shifted QR method in the Worked Example 6.6 B, show that A_2 is similar to A_1. No change in eigenvalues, and the A's quickly approach a diagonal matrix.

$\boxed{\text{Solution}}$ (12 points) We are asked to show that $A_2 = R_1Q_1 - cs^2I$ is similar to $A_1 = Q_1R_1 - cs^2I$. Note that

$$Q_1A_2Q_1^{-1} = Q_1(R_1Q_1 - cs^2I)Q_1^{-1} = Q_1R_1 - Q_1cs^2IQ_1^{-1} = Q_1R_1 - cs^2I = A_1.$$

Thus A_2 is similar to A_1, and thus their eigenvalues are the same.

Section 6.6. Problem 24. If A is similar to A^{-1}, must all the eigenvalues equal 1 or -1?

$\boxed{\text{Solution}}$ (12 points)

No. Consider:

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is similar to $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$.

Section 6.7. Problem 4. Find the eigenvalues and unit eigenvectors of A^TA and AA^T. Keep each $Av = \sigma u$ for the Fibonacci matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Construct the singular value decomposition and verify that A equal s$U\Sigma V^T$.

$\boxed{\text{Solution}}$ (4 points)

$$A^TA = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that these are the same. (This makes sense, as A is symmetric.) The eigenvalues of this are the roots of $x^2 - 3x + 1$, which are $(3 \pm \sqrt{5})/2$. The unit eigenvectors of this will be

$$\begin{pmatrix} \sqrt{\frac{2}{5-\sqrt{5}}} \\ \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} \\ -\sqrt{\frac{2}{5-\sqrt{5}}} \end{pmatrix}.$$

Then

$$U = \begin{pmatrix} \sqrt{\frac{2}{5-\sqrt{5}}} & -\sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} \\ \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \end{pmatrix} \qquad V = \begin{pmatrix} \sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} \\ \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} & -\sqrt{\frac{2}{5-\sqrt{5}}} \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \\ & \frac{\sqrt{5}-1}{2} \end{pmatrix}.$$

Section 6.7. Problem 11. Suppose A has orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1, \ldots, \sigma_n$. What are U,Σ and V in the SVD?

$\boxed{\text{Solution}}$ (4 points) We will first solve this assuming all of the w_i are nonzero; at the end we will give a modification for the solution in the case that some are 0. As the columns of A are orthogonal we know that A^TA will be a diagonal matrix with diagonal entries $\sigma_1^2, \ldots, \sigma_n^2$. Thus $U = I$ and Σ is the diagonal matrix with entries $\sigma_1, \ldots, \sigma_n$. Then if we define V to be the matrix whose i-th row is the vector w_i/σ_i we will have $A = U\Sigma V^T$, as desired.

Suppose that some of w_i are zero. Take all of the w's that are nonzero and complete them to an orthogonal basis $u_1, \ldots, u_n$ satisfying the conditions that if $w_i \neq 0$ then $u_i = w_i$, and if $w_i = 0$ then $|u_i| = 1$. Then let U, Σ be as above, and V be the matrix whose i-th row is w_i/σ_i if $\sigma_i \neq 0$, and u_i if $\sigma_i = 0$. Then $A = U\Sigma V^T$, as desired.

Section 6.7. Problem 17. The $1, -1$ first difference matrix A has A^TA the second difference matrix. The singular vectors of A are *sine vectors* V and *cosine vectors* u. Then $Av = \sigma u$ is the discrete form of $d/dx(\sin cx) = c(\cos cx)$. This is the best SVD I have seen.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad A^TA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Then the orthogonal sine matrix is

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin \pi/4 & \sin 2\pi/4 & \sin 3\pi/4 \\ \sin 2\pi/4 & \sin 4\pi/4 & \sin 6\pi/4 \\ \sin 3\pi/4 & \sin 6\pi/4 \sin 9\pi/4 & \end{pmatrix}.$$

(a) Put numbers in V: The unit eigenvectors of A^TA are singular vectors of A. Show that the columns of V have $A^TAv = \lambda v$ with $\lambda = 2 - \sqrt{2}, 2, 2 + \sqrt{2}$.

(b) Multiply AV and verify taht its columns are orthogonal. They are $\sigma_1 u_1$ and $\sigma_2 u_2$ and $\sigma_3 u_3$. The first columns of the cosine matrix U are u_1, u_2, u_3.

(c) Since A is 4×3 weneed a fourth orthogonal vector u_4. It comes from the nullspace of A^T. What is u_4?

$\boxed{\text{Solution}}$ (12 points)

(a) We are asked to show that the columns of V are eigenvectors of A^TA. The characteristic polynomial of A^TA is $x^3 - 6x^2 + 10x - 4$, which can be factored as $(x-2)(x^2 - 4x + 2)$. By the quadratic formua the roots of this are exactly the eigenvalues specified.

Note that
$$V = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}.$$

Then note that the three vectors $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$ are scalar multiples of the columns of V, and it is easy to check that they are indeed eigenvectors of A^TA with the right eigenvalues.

(b)

$$AV = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2}-1 & -\sqrt{2} & -\sqrt{2}-1 \\ 1-\sqrt{2} & -\sqrt{2} & 1+\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix}.$$

It is easy to check that these columns are orthogonal.

(c) Note that $A^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$. The nullspace of this is generated by $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Section 8.5. Problem 4. The first three *Legendre polynomials* are $1, x, x^2 - 1/3$. Choose c so that the fourth polynomial $x^3 - cx$ is orthogonal to the first three. All integrals go from -1 to 1.

$\boxed{\text{Solution}}$ (4 points) We compute

$$\int_{-1}^{1} x^3 - cx\,dx = 0 \qquad \int_{-1}^{1} (x^3 - cx)x\,dx = \frac{2}{5} - \frac{2}{3}c \qquad \int_{-1}^{1} (x^3 - cx)(x^2 - \frac{1}{3})\,dx = 0.$$

Thus in order for $x^3 - cx$ to be orthogonal to the other three we need $c = 3/5$.

Section 8.5. Problem 5. For the square wave $f(x)$ in Example 3 show that

$$\int_0^{2\pi} f(x)\cos x\,dx = 0 \qquad \int_0^{2\pi} f(x)\sin x\,dx = 4 \qquad \int_0^{2\pi} f(x)\sin 2x\,dx = 0.$$

Which three Fourier coefficients come from those integrals?

$\boxed{\text{Solution}}$ (4 points) By definition, coefficients that come from these are a_1, b_1, b_2, respectively. We compute

$$\begin{aligned}
\int_0^{2\pi} f(x)\cos x\,dx &= \int_0^{\pi} \cos x\,dx - \int_{\pi}^{2\pi} \cos x\,dx = 0 \\
\int_0^{2\pi} f(x)\sin x\,dx &= \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx = 4 \\
\int_0^{2\pi} f(x)\sin 2x\,dx &= \int_0^{\pi} \sin 2x\,dx - \int_{\pi}^{2\pi} \sin 2x\,dx = 0.
\end{aligned}$$

Section 8.5. Problem 12. The functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ are a basis for a Hilberts space. Write the derivatives of those first five functions as combinations of the same five functions. What is the 5×5 "differentiation matrix" for those functions?

$\boxed{\text{Solution}}$ (12 points)

We know that $1' = 0$, and that

$$(\cos x)' = -\sin x \qquad (\sin x)' = \cos x \qquad (\cos 2x)' = -2\sin 2x \qquad (\sin 2x)' = 2\cos 2x.$$

Thus the "differentiation matrix" is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

Section 8.5. Problem 13. Find the Fourier coefficients a_k and b_k of the square pulse $F(x)$ centered at $x = 0$: $f(x) = 1/h$ for $|x| \leq h/2$ and $F(x) = 0$ for $h/2 < |x| \leq \pi$. As $h \to 0$, this $F(x)$ approaches a delta function. Find the limits of a_k and b_k.

$\boxed{\text{Solution}}$ (12 points) We compute

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{h\pi}\int_{-h/2}^{h/2} 1\,dx = \frac{1}{\pi}. \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos kx\,dx = \frac{1}{\pi h}\int_{-h/2}^{h/2}\cos kx\,dx = \frac{1}{\pi hk}\sin kx\Big|_{-h/2}^{h/2} = \frac{2}{\pi hk}\sin\frac{kh}{2}. \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin kx\,dx = \frac{1}{\pi h}\int_{-h/2}^{h/2}\sin kx\,dx = \frac{1}{\pi k}\cos kx\Big|_{-h/2}^{h/2} = 0. \end{aligned}$$

Thus as $h \to 0$ we see that $a_0 \to 1/\pi$ and $b_k \to 0$. We also compute

$$\lim_{h\to 0} a_k = \lim_{h\to 0}\frac{1}{\pi}\frac{2}{hk}\sin\frac{hk}{2} = \frac{1}{\pi}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{\pi}$$

where we set $x = hk/2$.