```python
import numpy as np


def affine_forward(x, w, b):
  """
  Computes the forward pass for an affine (fully-connected) layer.

  The input x has shape (N, d_1, ..., d_k) and contains a minibatch of N
  examples, where each example x[i] has shape (d_1, ..., d_k). We will
  reshape each input into a vector of dimension D = d_1 * ... * d_k, and
  then transform it to an output vector of dimension M.

  Inputs:
  - x: A numpy array containing input data, of shape (N, d_1, ..., d_k)
  - w: A numpy array of weights, of shape (D, M)
  - b: A numpy array of biases, of shape (M,)

  Returns a tuple of:
  - out: output, of shape (N, M)
  - cache: (x, w, b)
  """

  # ================================================================== #
  # YOUR CODE HERE:
  #   Calculate the output of the forward pass.  Notice the dimensions
  #   of w are D x M, which is the transpose of what we did in earlier
  #   assignments.
  # ================================================================== #

  out = x.reshape(x.shape[0], np.prod(x.shape[1:])) @ w + b

  # ================================================================== #
  # END YOUR CODE HERE
  # ================================================================== #

  cache = (x, w, b)
  return out, cache


def affine_backward(dout, cache):
  """
  Computes the backward pass for an affine layer.

  Inputs:
  - dout: Upstream derivative, of shape (N, M)
  - cache: Tuple of:
    - x: A numpy array containing input data, of shape (N, d_1, ..., d_k)
    - w: A numpy array of weights, of shape (D, M)
    - b: A numpy array of biases, of shape (M,)

  Returns a tuple of:
  - dx: Gradient with respect to x, of shape (N, d1, ..., d_k)
  - dw: Gradient with respect to w, of shape (D, M)
  - db: Gradient with respect to b, of shape (M,)
  """
  x, w, b = cache
  dx, dw, db = None, None, None

  # ================================================================== #
  # YOUR CODE HERE:
  #   Calculate the gradients for the backward pass.
  # Notice:
  #   dout is N x M
  #   dx should be N x d1 x ... x dk; it relates to dout through multiplication with w, which is D x M
  #   dw should be D x M; it relates to dout through multiplication with x, which is N x D after reshaping
  #   db should be M; it is just the sum over dout examples
  # ================================================================== #

  dx = (dout @ w.T).reshape(x.shape)
  dw = x.reshape(x.shape[0], np.prod(x.shape[1:])).T @ dout
  db = np.sum(dout, axis=0)

  # ================================================================== #
  # END YOUR CODE HERE
  # ================================================================== #

  return dx, dw, db

def relu_forward(x):
  """
  Computes the forward pass for a layer of rectified linear units (ReLUs).

  Input:
  - x: Inputs, of any shape

  Returns a tuple of:
  - out: Output, of the same shape as x
  - cache: x
  """
  # ================================================================== #
  # YOUR CODE HERE:
  #   Implement the ReLU forward pass.
  # ================================================================== #

  out = np.maximum(0,x)

  # ================================================================== #
  # END YOUR CODE HERE
  # ================================================================== #

  cache = x
  return out, cache


def relu_backward(dout, cache):
  """
  Computes the backward pass for a layer of rectified linear units (ReLUs).

  Input:
  - dout: Upstream derivatives, of any shape
  - cache: Input x, of same shape as dout

  Returns:
  - dx: Gradient with respect to x
  """
  x = cache

  # ================================================================== #
  # YOUR CODE HERE:
  #   Implement the ReLU backward pass
  # ================================================================== #

  dx = (x > 0) * dout

  # ================================================================== #
  # END YOUR CODE HERE
  # ================================================================== #

  return dx

def batchnorm_forward(x, gamma, beta, bn_param):
  """
  Forward pass for batch normalization.

  During training the sample mean and (uncorrected) sample variance are
  computed from minibatch statistics and used to normalize the incoming data.
  During training we also keep an exponentially decaying running mean of the mean
  and variance of each feature, and these averages are used to normalize data
  at test-time.

  At each timestep we update the running averages for mean and variance using
  an exponential decay based on the momentum parameter:

  running_mean = momentum * running_mean + (1 - momentum) * sample_mean
  running_var = momentum * running_var + (1 - momentum) * sample_var

  Note that the batch normalization paper suggests a different test-time
  behavior: they compute sample mean and variance for each feature using a
  large number of training images rather than using a running average. For
  this implementation we have chosen to use running averages instead since
  they do not require an additional estimation step; the torch7 implementation
  of batch normalization also uses running averages.

  Input:
  - x: Data of shape (N, D)
  - gamma: Scale parameter of shape (D,)
  - beta: Shift paremeter of shape (D,)
  - bn_param: Dictionary with the following keys:
    - mode: 'train' or 'test'; required
    - eps: Constant for numeric stability
    - momentum: Constant for running mean / variance.
    - running_mean: Array of shape (D,) giving running mean of features
    - running_var Array of shape (D,) giving running variance of features

  Returns a tuple of:
  - out: of shape (N, D)
  - cache: A tuple of values needed in the backward pass
  """
  mode = bn_param['mode']
  eps = bn_param.get('eps', 1e-5)
  momentum = bn_param.get('momentum', 0.9)

  N, D = x.shape
  running_mean = bn_param.get('running_mean', np.zeros(D, dtype=x.dtype))
  running_var = bn_param.get('running_var', np.zeros(D, dtype=x.dtype))

  out, cache = None, None
  if mode == 'train':

    # ================================================================== #
    # YOUR CODE HERE:
    #   A few steps here:
    #     (1) Calculate the running mean and variance of the minibatch.
    #     (2) Normalize the activations with the running mean and variance.
    #     (3) Scale and shift the normalized activations.  Store this
    #         as the variable 'out'
    #     (4) Store any variables you may need for the backward pass in
    #         the 'cache' variable.
    # ================================================================== #

    batch_mean = x.mean(axis=0)
    batch_var = x.var(axis=0)
    running_mean = momentum * running_mean + (1 - momentum) * batch_mean
    running_var = momentum * running_var + (1 - momentum) * batch_var

    x_norm = (x - batch_mean) / np.sqrt(batch_var + eps)
    out = gamma * x_norm + beta

    cache = {
      'x_norm': x_norm,
      'gamma': gamma,
      'batch_var': batch_var,
      'eps': eps,
      'a': (x - batch_mean),
      'mean': batch_mean,

    }

    # ================================================================== #
    # END YOUR CODE HERE
    # ================================================================== #

  elif mode == 'test':

    # ================================================================== #
    # YOUR CODE HERE:
    #   Calculate the testing time normalized activation.  Normalize using
    #   the running mean and variance, and then scale and shift appropriately.
    #   Store the output as 'out'.
    # ================================================================== #

    x_norm = (x - running_mean) / np.sqrt(running_var + eps)
    out = gamma * x_norm + beta

    # ================================================================== #
    # END YOUR CODE HERE
    # ================================================================== #

  else:
    raise ValueError('Invalid forward batchnorm mode "%s"' % mode)

  # Store the updated running means back into bn_param
  bn_param['running_mean'] = running_mean
  bn_param['running_var'] = running_var

  return out, cache

def batchnorm_backward(dout, cache):
  """
  Backward pass for batch normalization.

  For this implementation, you should write out a computation graph for
  batch normalization on paper and propagate gradients backward through
  intermediate nodes.

  Inputs:
  - dout: Upstream derivatives, of shape (N, D)
  - cache: Variable of intermediates from batchnorm_forward.

  Returns a tuple of:
  - dx: Gradient with respect to inputs x, of shape (N, D)
  - dgamma: Gradient with respect to scale parameter gamma, of shape (D,)
  - dbeta: Gradient with respect to shift parameter beta, of shape (D,)
  """
  dx, dgamma, dbeta = None, None, None

  # ================================================================== #
  # YOUR CODE HERE:
  #   Implement the batchnorm backward pass, calculating dx, dgamma, and dbeta.
  # ================================================================== #

  M = dout.shape[0]

  dbeta = dout.sum(axis=0)
  dgamma = (cache['x_norm'] * dout).sum(axis=0)

  # dx calc
  dx_norm = dout * cache['gamma']
  da = dx_norm / np.sqrt(cache['batch_var'] + cache['eps'])
  dmu = -1 * dx_norm.sum(axis=0) / np.sqrt(cache['batch_var'] + cache['eps'])
  db = cache['a'] * dx_norm
  dc =  -1 * db / (cache['batch_var'] + cache['eps'])
  de = dc * 1/2 * 1/ np.sqrt(cache['batch_var'] + cache['eps'])
  dvar = de.sum(axis=0)

  dx = da + 2/M * cache['a'] * dvar + 1/M * dmu

  # ================================================================== #
  # END YOUR CODE HERE
  # ================================================================== #

  return dx, dgamma, dbeta

def dropout_forward(x, dropout_param):
  """
  Performs the forward pass for (inverted) dropout.

  Inputs:
  - x: Input data, of any shape
  - dropout_param: A dictionary with the following keys:
    - p: Dropout parameter. We drop each neuron output with probability p.
    - mode: 'test' or 'train'. If the mode is train, then perform dropout;
      if the mode is test, then just return the input.
    - seed: Seed for the random number generator. Passing seed makes this
      function deterministic, which is needed for gradient checking but not in
      real networks.

  Outputs:
  - out: Array of the same shape as x.
  - cache: A tuple (dropout_param, mask). In training mode, mask is the dropout
    mask that was used to multiply the input; in test mode, mask is None.
  """
  p, mode = dropout_param['p'], dropout_param['mode']
  if 'seed' in dropout_param:
    np.random.seed(dropout_param['seed'])

  mask = None
  out = None

  if mode == 'train':
    # ================================================================== #
    # YOUR CODE HERE:
    #   Implement the inverted dropout forward pass during training time.
    #   Store the masked and scaled activations in out, and store the
    #   dropout mask as the variable mask.
    # ================================================================== #

    mask = (np.random.random_sample(x.shape) >= p) / (1 - p)
    out = x * mask

    # ================================================================== #
    # END YOUR CODE HERE
    # ================================================================== #

  elif mode == 'test':

    # ================================================================== #
    # YOUR CODE HERE:
    #   Implement the inverted dropout forward pass during test time.
    # ================================================================== #

    out = x

    # ================================================================== #
    # END YOUR CODE HERE
    # ================================================================== #

  cache = (dropout_param, mask)
  out = out.astype(x.dtype, copy=False)

  return out, cache

def dropout_backward(dout, cache):
  """
  Perform the backward pass for (inverted) dropout.

  Inputs:
  - dout: Upstream derivatives, of any shape
  - cache: (dropout_param, mask) from dropout_forward.
  """
  dropout_param, mask = cache
  mode = dropout_param['mode']

  dx = None
  if mode == 'train':
    # ================================================================== #
    # YOUR CODE HERE:
    #   Implement the inverted dropout backward pass during training time.
    # ================================================================== #

    dx = dout * mask

    # ================================================================== #
    # END YOUR CODE HERE
    # ================================================================== #
  elif mode == 'test':
    # ================================================================== #
    # YOUR CODE HERE:
    #   Implement the inverted dropout backward pass during test time.
    # ================================================================== #

    dx = dout

    # ================================================================== #
    # END YOUR CODE HERE
    # ================================================================== #
  return dx

def svm_loss(x, y):
  """
  Computes the loss and gradient using for multiclass SVM classification.

  Inputs:
  - x: Input data, of shape (N, C) where x[i, j] is the score for the jth class
    for the ith input.
  - y: Vector of labels, of shape (N,) where y[i] is the label for x[i] and
    0 <= y[i] < C

  Returns a tuple of:
  - loss: Scalar giving the loss
  - dx: Gradient of the loss with respect to x
  """
  N = x.shape[0]
  correct_class_scores = x[np.arange(N), y]
  margins = np.maximum(0, x - correct_class_scores[:, np.newaxis] + 1.0)
  margins[np.arange(N), y] = 0
  loss = np.sum(margins) / N
  num_pos = np.sum(margins > 0, axis=1)
  dx = np.zeros_like(x)
  dx[margins > 0] = 1
  dx[np.arange(N), y] -= num_pos
  dx /= N
  return loss, dx


def softmax_loss(x, y):
  """
  Computes the loss and gradient for softmax classification.

  Inputs:
  - x: Input data, of shape (N, C) where x[i, j] is the score for the jth class
    for the ith input.
  - y: Vector of labels, of shape (N,) where y[i] is the label for x[i] and
    0 <= y[i] < C

  Returns a tuple of:
  - loss: Scalar giving the loss
  - dx: Gradient of the loss with respect to x
  """

  probs = np.exp(x - np.max(x, axis=1, keepdims=True))
  probs /= np.sum(probs, axis=1, keepdims=True)
  N = x.shape[0]
  loss = -np.sum(np.log(probs[np.arange(N), y])) / N
  dx = probs.copy()
  dx[np.arange(N), y] -= 1
  dx /= N
  return loss, dx
```