附录 2.成本最小问题(Cost Minimization Problem,CMP)

给定生产函数$q = F(x_1, \dots x_n)$

$$min_{x_1,\dots x_n} w_1 x_1 + \cdots + w_n x_n$$

$$s.t\ F(x_1, \dots x_n) \geq q$$

利用拉格朗日求解：

$$\mathcal{L} = w_1 x_1 + \cdots + w_n x_n + \lambda[F(x_1, \dots x_n) - q]$$

F.O.Cs $\frac{\partial \mathcal{L}}{\partial x_j} = w_j + \lambda \frac{\partial F(x_1,\dots x_n)}{\partial x_j}$*=0,j=1,…n*

由此可得对于$\forall i, j$,$\boldsymbol{\frac{w_i}{w_j} = \frac{\frac{\partial F(x_1,\dots x_n)}{\partial x_i}}{\frac{\partial F(x_1,\dots x_n)}{\partial x_j}}}$

例 1：常替代弹性(the constant elasticity of substitution ,CES)生产函数

$$q = (a_0 + \sum_{j=1}^{n} a_j\, x_j^{\rho})^{\frac{1}{\rho}}$$

由上述推导，可得$\forall i, j$,$\frac{w_i}{w_j} = \frac{a_i x_i^{\rho-1}}{a_j x_j^{\rho-1}}$

因此对于$\forall j \neq i$,可得$x_j = (\frac{w_j}{w_i}\frac{a_i}{a_j})^{\frac{1}{\rho-1}} x_i$

代入$q = (a_0 + \sum_{j=1}^{n} a_j\, x_j^{\rho})^{\frac{1}{\rho}}$中，可得

$$q = [a_0 + \sum_{j=1}^{n} a_j\, (\frac{w_j}{w_i}\frac{a_i}{a_j})^{\frac{\rho}{\rho-1}} x_i^{\rho}]^{\frac{1}{\rho}} ==> x_i = [\frac{q^{\rho} - a_0}{\sum_{j=1}^{n} a_j (\frac{w_j a_i}{w_i a_j})^{\frac{\rho}{\rho-1}}}]^{\frac{1}{\rho}}$$

成本函数为$C(q) = \sum_{i=1}^{n} w_i \left[\frac{q^{\rho}-a_0}{\sum_{j=1}^{n} a_j (\frac{w_j a_i}{w_i a_j})^{\frac{\rho}{\rho-1}}}\right]^{\frac{1}{\rho}}$

例 2:Cobb-Douglus 生产函数

$$q = K^{\alpha} L^{\beta}$$

要素价格分别为r, w

$$\frac{r}{w} = \frac{\alpha K^{\alpha-1} L^{\beta}}{\beta K^{\alpha} L^{\beta-1}} = \frac{\alpha L}{\beta K} ==> L = \frac{r}{w}\frac{\beta}{\alpha} K$$

代入$q = K^{\alpha} L^{\beta}$中，可得$K^{\alpha}(\frac{w_1}{w_2}\frac{\beta}{\alpha}K)^{\beta} = q ==> K = q^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{w}{r}\right)^{\frac{\beta}{\alpha+\beta}}$

进而可得$L = \frac{r}{w}\frac{\beta}{\alpha} q^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{w}{r}\right)^{\frac{\beta}{\alpha+\beta}} = q^{\frac{1}{\alpha+\beta}} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{r}{w}\right)^{\frac{\alpha}{\alpha+\beta}}$

$$C(q) = r q^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{w}{r}\right)^{\frac{\beta}{\alpha+\beta}} + r q^{\frac{1}{\alpha+\beta}} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{r}{w}\right)^{\frac{\alpha}{\alpha+\beta}}$$

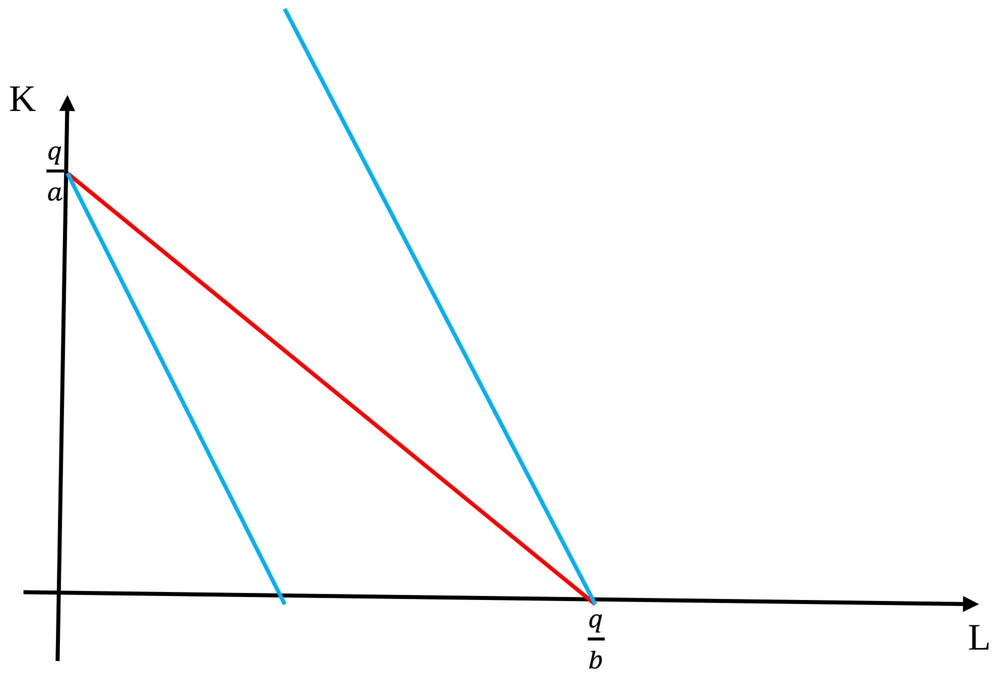
$$= q^{\frac{1}{\alpha+\beta}} r^{\frac{\alpha}{\alpha+\beta}} w^{\frac{\beta}{\alpha+\beta}} [\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}]$$

<u>不能使用拉格朗日方法求解的问题：</u>

1.完全替代的生产函数

$$F(K, L) = aK + bL$$

若$\frac{w}{r} > \frac{b}{a}$



是成本最小的要素组合为$K^* = \frac{q}{a}, L^* = 0$,成本函数为$C(q) = r\frac{q}{a}$

解释：$\frac{w}{b} > \frac{r}{a}$,生产 1 单位产出只使用劳动力的成本>生产 1 单位产出只使用资本的成本。

同理，$\frac{w}{r} < \frac{b}{a}, K^* = 0, L^* = \frac{q}{b}$,成本函数为$C(q) = w\frac{q}{b}$

因此成本函数为$C(q) = \min\{\frac{w}{b}, \frac{r}{a}\}q$

2.完全互补生产函数

$$F(K, L) = \min\{aK，bL\}$$

为了生产 q 单位，所选取的最优要素组合满足$aK^* = bL^* = q$

成本函数为$C(q) = r\frac{q}{a} + w\frac{q}{b} = (\frac{w}{b} + \frac{r}{a})q$

解释：为什么要花钱在不会带来产出的要素投入上？

<u>短期成本函数与长期成本函数</u>：

例 1.给定生产函数$q = F(K, L) = K^{\alpha}L^{\beta}$,短期内资本数量固定为$K = K_0$

则短期内成本最小化问题如下：

$$min_{L\geq 0} wL + rK_0$$

$$s.t\ K_0^{\alpha} L^{\beta} \geq q$$

因此可得最优选择为$L_S^* = \frac{q^{\frac{1}{\beta}}}{K_0^{\frac{\alpha}{\beta}}}$,短期成本函数为$C_S(q) = rK_0 + w\frac{q^{\frac{1}{\beta}}}{K_0^{\frac{\alpha}{\beta}}} = rK_0 + Bq^{\frac{1}{\beta}}$

长期成本函数:

$$C_L(q) = rq^{\frac{1}{\alpha+\beta}}\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\left(\frac{w}{r}\right)^{\frac{\beta}{\alpha+\beta}} + rq^{\frac{1}{\alpha+\beta}}\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{r}{w}\right)^{\frac{\alpha}{\alpha+\beta}}$$

$$= q^{\frac{1}{\alpha+\beta}} r^{\frac{\alpha}{\alpha+\beta}} w^{\frac{\beta}{\alpha+\beta}}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] = Aq^{\frac{1}{\alpha+\beta}}$$

$$C_L(q) - C_S(q) = Aq^{\frac{1}{\alpha+\beta}} - rK_0 - Bq^{\frac{1}{\beta}}$$

$$C_L'(q) - C_S'(q) = q^{\frac{1}{\alpha+\beta}-1}\left(A\frac{1}{\alpha+\beta} - \frac{B}{\beta}q^{\frac{1}{\beta}-\frac{1}{\alpha+\beta}}\right)$$

当$A\frac{1}{\alpha+\beta} - \frac{B}{\beta}q^{\frac{1}{\beta}-\frac{1}{\alpha+\beta}} = 0$时，可知$C_L(q) - C_S(q)$取最大值 0，因此$C_L(q) - C_S(q) \leq 0$.

思考：给定生产函数$q = F(K, L, M) = K^{\alpha}L^{\beta}M^{\gamma}$,短期内资本数量固定为$K = K_0$,求解长期成本函数和短期成本函数。