Midterm test - Solutions

Closed book. Attempt all questions. Calculators permitted. 13:00-13:50
Please write your name and student identity number on the front page.

(1) Determine the order of convergence of each of the following sequences (if they converge at all).

$$\text{(a)}\ \ x_k = \frac{1}{k!}, \quad \text{(b)}\ \ x_k = 1 + (0.9)^{3^k}, \quad \text{(c)}\ \ x_k = 99^{-k}, \quad \text{(d)}\ \ x_k = 1/k^2$$

[5 marks]

(2) Determine, with justification, which of the following functions is convex ($\ln(x)$ refers to the natural logarithm).

(a) $f(x) = \ln(x)$ for $x > 0$;

(b) $f(x) = \dfrac{1}{x}$ for $x > 0$;

(c) $f(x, y, z) = z^2 - x^2 - y^2$ for $x, y, z \in \mathbb{R}$;

(d) $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 + \|\boldsymbol{x}\|_\infty$.

You may used criteria for convexity seen in the lecture and problem sessions. [5 marks]

(3) Consider the following polyhedron: $x = (x_1, x_2)^T \in \mathbb{R}^2$ such that

$$\begin{aligned} x_1 &\geq 1 \\ x_2 &\geq 1 \\ x_1 + 1.5x_2 &\leq 2 \end{aligned}$$

It is clear that this set is empty, i.e. there exists no $x \in \mathbb{R}^2$ which simultaneously satisfies this set of inequalities. Show this is true using the duality argument we used in class, i.e. find a suitable $\lambda \in \mathbb{R}^3$ satisfying 3 properties.

[5 marks]

(4) Consider the function

$$f(x, y) = \sqrt{1 + x^2 + y^2}$$

By computing the gradient and Hessian, show that this function is convex and determine the unique minimum. Write down the form of one iteration of Newton's method for this function. [5 marks]

Solution (1)

(a) The sequence converges to 0. We have

$$x_{k+1} = \frac{1}{(k+1)!} = \frac{1}{k+1} x_k,$$

so that $\lim_{k\to\infty} \frac{x_{k+1}}{x_k} = 0$. The convergence is *superlinear*.

(b) The sequence converges to 1. We have

$$|x_{k+1} - 1| = (0.9)^{3^{k+1}} = \left((0.9)^{3^k}\right)^3 = |x_k - 1|^3,$$

so that the convergence is cubic.

(c) The sequence converges to 0. Moreover,

$$x_{k+1} = \frac{1}{99^{k+1}} = \frac{1}{99} x_k,$$

so that the sequence converges linearly.

(d) The sequence converges to 0. We have the identity

$$x_{k+1} = \frac{1}{(k+1)^2} = \frac{k^2}{(k+1)^2}\frac{1}{k^2} = \frac{k^2}{(k+1)^2} x_k.$$

For any fixed constant $c < 1$ there is a k such that $1 > k^2/(k+1)^2 > c$, and therefore $x_{k+1} > cx_k$. It follows that the sequence does not converge linearly (or to any higher order).

Solution (2)

(a) The function is not convex. The derivative is $1/x$, which for $x > 0$ is positive. The second derivative is $-1/x^2 < 0$.

(b) The function is convex. The second derivative is $2/x^3 > 0$.

(c) This function is not convex. The Hessian is given by

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

which is not positive definite.

(d) This function is, as the sum of two norms, convex. Precisely, for $\lambda \in [0,1]$,

$$\begin{aligned} \|\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}\|_1 + \|\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}\|_\infty &\leq \lambda\|\boldsymbol{x}\|_1 + (1-\lambda)\|\boldsymbol{y}\|_1 + \lambda\|\boldsymbol{x}\|_\infty + (1-\lambda)\|\boldsymbol{y}\|_\infty \\ &= \lambda(\|\boldsymbol{x}\|_1 + \|\boldsymbol{x}\|_\infty) + (1-\lambda)(\|\boldsymbol{y}\|_1 + \|\boldsymbol{y}\|_\infty). \end{aligned}$$

Solution (3) The matrix and the vectors associated to this problem are

$$\boldsymbol{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1.5 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

The duality argument states that the feasible set $\{x; Ax \leq b\}$ is empty if we can find some $\lambda \in \mathbb{R}^3$ such that (i) $\lambda_i \geq 0$ for $i = 1, \dots, 3$, (ii) $A^T\lambda = 0$, and (iii) $\lambda^T b < 0$. An example is $\lambda = (1, 1.5, 1)^T$. Then $A^T\lambda = 0$ and $\lambda^T b = -0.5 < 0$.

Solution (4) We first compute the gradient and the Hessian of this function.

$$\nabla f(x_1, x_2) = \frac{1}{\sqrt{1 + x^2 + y^2}} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

$$\nabla^2 f(x_1, x_2) = \frac{1}{(1 + x^2 + y^2)^{3/2}} \begin{pmatrix} 1 + y^2 & -xy \\ -xy & 1 + x^2 \end{pmatrix}.$$

We have a stationary point at $(0, 0)$ which is a minimizer, as the function can never fall below $f(0, 0) = 1$. This means that the Hessian is positive definite at $(0, 0)$. There are various ways of verifying that the Hessian is positive definite everywhere, and the function therefore convex. One is direct verification:

$$\boldsymbol{v}^\top \nabla^2 f(x, y) \boldsymbol{v} = v_1^2(1 + y^2) - 2v_1v_2xy + v_2^2(1 + x^2) = v_1^2 + v_2^2 + (v_1y - v_2x)^2 > 0.$$

Newton's method starts with a point $(x_{(0)}, y_{(0)})$, and then for every $k \geq 0$, first solves the system of equations

$$\frac{1}{(1 + x_{(k)}^2 + y_{(k)}^2)^{3/2}} \begin{pmatrix} 1 + y_{(k)}^2 & -x_{(k)}y_{(k)} \\ -x_{(k)}y_{(k)} & 1 + x_{(k)}^2 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \frac{1}{\sqrt{1 + x_{(k)}^2 + y_{(k)}^2}} \begin{pmatrix} x_{(k)} \\ y_{(k)} \end{pmatrix},$$

and then computes

$$(x_{(k+1)}, y_{(k+1)}) = (x_{(k)}, y_{(k)}) - (\Delta x, \Delta y).$$