The University of Jordan

Student's Name: ....

Student's Number: ....

Department of Mathematics

Instructor's Nume: ....

Class Time: ....

Mid-term Exam ◇ Modern Convex Optimization (0301972) ◇ Fall 2017
Note: This exam is composed of 4 questions. You have 75 minutes to finish.

Question 1 [6 points]. Let

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \; q = \begin{bmatrix} -22.0 \\ -14.5 \\ 13.0 \end{bmatrix}, \; \text{and} \; r = 1.$$

Show that $x^\star = (1, 1/2, -1)$ is optimal for the optimization problem

$$\begin{array}{ll} \min & \frac{1}{2}x^\mathsf{T}Px + q^\mathsf{T}x + r \\ \text{s.t.} & -1 \le x_i \le 1, \; i = 1, 2, 3. \end{array}$$

Proof.

Question 2 **[4+4 points].**

(i) Prove Farkas' lemma: Exactly one of the following two holds for a given $A \in \mathbb{R}^{m \times n}$ & $b \in \mathbb{R}^m$:

(1) There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.

(2) There exists $y \in \mathbb{R}^m$ such that $A^\mathsf{T} y \geq 0$ and $y^\mathsf{T} b < 0$.

Proof.

(ii) Use Farkas' lemma to prove the following part of strong duality: **If the dual is infeasible, then the primal is unbounded**, where

$$
\begin{array}{llcll}
 & \underline{\text{Primal}} & & & \underline{\text{Dual}} \\
\max & c^\mathsf{T} x & & \min & b^\mathsf{T} y \\
\text{s.t.} & A^\mathsf{T} x \leq b, & & \text{s.t.} & A^\mathsf{T} y = c, \\
 & & & & y \geq 0.
\end{array}
$$

Proof.

Question 3 [4+4 points]. Let $a \in \mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ and $b \in \mathbb{R}^n$ with $b_k = \frac{1}{a_k}$, and consider the convex optimization problem

$$\begin{array}{ll} \min & -\log(a^\top x) - \log(b^\top x) \\ \text{s.t.} & x \geq 0, \;\; \mathbf{1}^\top x = 1. \end{array}$$

(i) Derive the KKT conditions for this optimization problem.

(ii) Show that the point $x = \left(\frac{1}{2}, 0, \ldots, 0, \frac{1}{2}\right)$ is optimal for this optimization problem.

Solution.

Question 4. [8 points] Consider the optimization problem

$$\begin{array}{lll} \max & x_1 + 2x_2 & \\ \text{s.t.} & x_1 + x_2 & \leq 2, \\ & -x_1 + x_2 & \leq 1, \\ & x_1, x_2 & \geq 0. \end{array}$$

Use the Affine Scaling Algorithm to compute an approximate solution arising by performing one iteration of the algorithm. In your computation, use $\beta = 0.995$ and $x^0 = (0.146, 0.188, 1.666, 0.958)^\top$ (so, x^1 is the required). Show all details.

Solution.