SSY345 – Sensor Fusion and Non-Linear Filtering
Home Assignment 1 - Analysis

Basic information

This home assignment is related to the material in lecture 1 and 2. A large part of the assignment focuses on understanding of the basic concepts that we will rely on later in the course.

In the analysis part we want you to use the toolbox that you have developed and apply it to a practical scenario. Associated with each scenario is a set of tasks that we would like you to perform.

The result of the tasks should in general be visualised and compiled together in a report (pdf-file). A template for the report can also be found on course homepage. Note that, it is sufficient to write short concise answers to the questions but they should be clearly motivate by what can be seen in the figures. Only properly referenced or captioned figures such that it is understandable what will result in POE. Also, all the technical terms and central concepts to explain an answer should be used without altering their actual meaning. The report should be uploaded on the course homepage before the deadline.

1 Transformation of Gaussian random variables

The purpose of this scenario is to get familiar with a few common types of transformations and to practice how to calculate mean and covariance of transformed variables.

Suppose that we are interested in the position of an object in Cartesian xy-coordinates, denoted $\mathbf{x}$. However, we are not able to measure $\mathbf{x}$ directly, but rather some related entity $\mathbf{y} = h(\mathbf{x})$, where $h(\cdot)$ describes the relationship between the position of the object and what we observe. Further, assume that we have prior knowledge of the position of the object modelled as,

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.3 & 0 \\ 0 & 8 \end{bmatrix} \right), \tag{1}$$

As we will see later in the course, it is often of interest to describe what we already know (our prior on $\mathbf{x}$) in the same coordinate system as our observed entity $\mathbf{y}$. That is, we are interested in the transformed density $p(\mathbf{y}) = p(h(\mathbf{x}))$.

Task: Investigate how the properties of $p(h(\mathbf{x}))$ depend on the properties of the transforming function $h(\cdot)$ (also know as the measurement model).

a) Suppose we have a sensor which can measure the sum and the difference between the x- and y-position of the object. That is,

$$\mathbf{y} = h(\mathbf{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}. \tag{2}$$

Use both the `approxGaussianTransform` and the `affineGaussianTransform` to calculate/approximate the mean and covariance of the transformed density $p(\mathbf{y}) = p(h(\mathbf{x}))$. Plot the transformed samples (from `approxGaussianTransform`) together with the mean and 3σ-ellipses from both functions. Do the ellipses match the sample points well? Why? How does the approximated ellipse's fit to the analytically calculated ellipse change with number of samples?

b) Now, suppose that we instead have a radar sensor which is capable of observing the angle, ϕ, in radians, and distance, ρ, in m to the object. In this case,

$$\mathbf{y} = \begin{bmatrix} \rho \\ \phi \end{bmatrix} = h(\mathbf{x}) = \begin{bmatrix} ||\mathbf{x}|| \\ \pi - \text{atan2}(y, x) \end{bmatrix}, \tag{3}$$

where $\text{atan2}(\cdot, \cdot)$ is the angle in radians between the object and the positive x-axis[1]. Draw samples from (1) and approximate the mean and covariance of the transformed density $p(h(\mathbf{x}))$ by using `approxGaussianTransform`. Plot the transformed samples together with the approximated mean and 3σ-ellipse. Does the ellipse match the sample points well? Why? How does the approximated ellipse's fit to true distribution change with number of samples?

[1] There is a Matlab command with the same name and the same functionality

c) Repeat the task a) and b) for the prior knowledge of the position model

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.3 & 0 \\ 0 & 0.1 \end{bmatrix} \right). \tag{4}$$

What has changed? In what way does this change your conclusions from a) and b) and why?

2 Vacation in Turkey

In this problem you will practice using conditional densities, and investigate their relation to joint densities. A second purpose is that you should understand the machinery behind the Bayesian measurement updates for Gaussian densities in detail.

Anders in Sweden wants to have great vacation at summer and chooses between going to Antalya or Izmir, the best locations to have a great vacation in Turkey. He checked the forecast website yesterday and Antalya's water temperature supposedly should be $15C^\circ$, and Izmir's water temperature should have $13C^\circ$. He also knows that these forecasts are not very reliable; in fact he considers the forecasts to be a Gaussian function, $p(x) = \mathcal{N}(x; \mu, 2.5^2)$, where x is the true temperature and μ is what the forecast reported.

Since Anders does not like to take any chances when he is going to have a vacation, he calls his friends who are already in these areas. His friend Anna in Antalya says the water temperature there is $17C^\circ$, while his friend Oguz in Izmir says the water temperature there is $12C^\circ$. Now he is very careful and also considers his friends skills to determine the temperature of his friends, which he models as $y = x+r$, where $r \sim \mathcal{N}(0, 1.7^2)$ for Anna, and $r \sim \mathcal{N}(0, 5.2^2)$ for Oguz.

Hint: You may find the command *normpdf* useful when plotting 1-D Gaussian densities, but remember to check if the functions you use take the standard deviation or the variance as input.

Task:

a) Use the code you developed for `jointGaussian` and plot the 3σ-ellipsoid of $p(x, y)$ for the water temperatures at Antalya and Izmir respectively. What can you say about the dependency between x and y by examining the plots? E.g. what does the slope of the major axis of the ellipse tell you?

b) Use `posteriorGaussian` and plot the two densities of the water temperatures at Antalya and Izmir, $p(x_A|y_A)$ and $p(x_I|y_O)$, given the information Anders gained from his friends and from the forecast. How are these 1-dimensional densities related to the plot of $p(x, y)$ in task (a)?

c) Anders likes to have a vacation where the water temperature is between $15C^\circ - 17C^\circ$. If he bases his decision on where to go by maximizing the probability that temperature is between $15C^\circ - 17C^\circ$, where should he go?

d) Finally assume that the Anders's friends can determine the temperature without any error (e.g. y=x). What will happen to posterior density? Motivate! (You may discuss this without plotting the results!)

3 MMSE and MAP estimates for Gaussian mixture posteriors

The purpose of this scenario is to understand what the MMSE estimator is and which fundamental properties it has, specifically in comparison to the MAP estimator.

Task: Plot all three posterior densities along with the MMSE estimates obtained in each case. Also mark, with a different symbol, roughly where the MAP estimate is. How does the MMSE and MAP estimator differ? If the MAP estimate is ambiguous, it is enough to mark one MAP estimate.

a) $p(\theta|y) = 0.2\mathcal{N}(\theta;\ 1, 0.5) + 0.8\mathcal{N}(\theta;\ 1, 9)$.

b) $p(\theta|y) = 0.49\mathcal{N}(\theta;\ 6, 2) + 0.51\mathcal{N}(\theta;\ -6, 2)$.

c) $p(\theta|y) = 0.4\mathcal{N}(\theta;\ 1.5, 2.5) + 0.6\mathcal{N}(\theta;\ 3, 1.4)$.