Solution to the 1D Heat Equation

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a. Analytical Solution

Given problem	$u_t = u_{xx}$
Domain	$x = [0, 1]$
Boundary condition	$u(0,t) = u(1,t),\ u_x(0,t) = u_x(1,t)$
Initial condition	$u(x,0) = sin(2\pi x)$

The given problem is a well posed homogenous linear PDE and has a unique solution for the given boundary (Neumann) and initial conditions.
The solution is derived as follows by variable separation:

$$\begin{aligned}
\text{Let}\quad & u(x,t) = X(x) * T(t) \\
& u_t = u_{xx} \implies XT_t = X_{xx}T \\
& \tfrac{X_{xx}}{X} = \tfrac{T_t}{T} = \alpha
\end{aligned}$$

(α is a constant since LHS is only a function of x and RHS is a function of t)

$$\begin{aligned}
\text{Case 1:}\quad & \alpha = 0 \\
\implies\quad & X = ax + b,\ T = c \\
\implies\quad & u^{\alpha=0} = a_1x + a_2 \\
\text{Case 2:}\quad & \alpha = \beta^2 > 0 \\
\implies\quad & X = ae^{\beta x} + be^{-\beta x},\ T = ce^{\beta t^2} \\
\implies\quad & u^{\alpha>0} = (b_1e^{\beta x} + b_2e^{-\beta x})e^{\beta t^2}
\end{aligned}$$

where b_1 and b_2 are arbitary constants depending on β

$$\begin{aligned}
\text{Case 3:}\quad & \alpha = -\lambda^2 < 0 \\
\implies\quad & X = acos(\lambda x) + bsin(\lambda x),\ T = ce^{-\lambda t^2} \\
\implies\quad & u^{\alpha<0} = (c_1cos(\lambda x) + c_2sin(\lambda x))e^{-\lambda t^2}
\end{aligned}$$

where c_1 and c_2 are arbitary constants depending on λ

Due to the homogenous and linear nature of the given PDE, the solution is given by

$$\begin{aligned}
u(x,t) \quad = \quad & a_1x + a_2 + \textstyle\sum_{\beta\in S_1}((b_1(\beta)e^{\beta x} + b_2(\beta)e^{-\beta x})e^{\beta t^2}) \\
& + \textstyle\sum_{\lambda\in S_2}((c_1(\lambda)cos(\lambda x) + c_2(\lambda)sin(\lambda x))e^{-\lambda t^2})
\end{aligned}$$

where S_1 and S_2 are countable subsets of Real numbers (since the summation should converge)

Now the solution is finite even for $t \to \infty$ (by Physics of the heat equation with no external energy source). Hence b_1 and b_2 should be 0 for any value of β.

$u(x,t) = a_1 x + a_2 + \sum_{\lambda \in S_2}((c_1(\lambda)cos(\lambda x) + c_2(\lambda)sin(\lambda x))e^{-\lambda t^2})$

Applying condition $u(0,t) = u(1,t)$, we get
$a_2 + \sum_{\lambda \in S_2}((c_1(\lambda))e^{-\lambda t^2}) = a_1 + a_2 + \sum_{\lambda \in S_2}((c_1(\lambda)cos(\lambda) + c_2(\lambda)sin(\lambda))e^{-\lambda t^2})$
$\implies a_1 + \sum_{\lambda \in S_2}([c_1(\lambda)(cos(\lambda) - 1) + c_2(\lambda)sin(\lambda)]e^{-\lambda t^2}) = 0$

Since the above equation is true for any value of t,

$$a_1 = 0 \qquad \text{and}$$

$$c_1(\lambda)(cos(\lambda) - 1) + c_2(\lambda)sin(\lambda) = 0 \qquad \text{for every } \lambda \in S_2 \qquad (1)$$