The following are excerpts from three lessons in Calculus Extended by Gary Taylor and J Michael Shaw

## Lesson 7-4 Power Series, Geometric Power Series, Integration and Differentiation of Power Series

A series with variable terms like $1+x+x^2+x^3+\cdots+x^n+\cdots$ is called a **power series.** Note that this series is a **geometric power series.** If it converges, what must be true about the variable $x$?

For these $x$-values $1+x+x^2+x^3+\cdots+x^n+\cdots=$

This means for these $x$-values, the function $f(x)=\dfrac{1}{1-x}$ can be written as $f(x)=\sum_{n=0}^{\infty}x^n$ .

Examples: Find a power series for each of the following functions. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

1. $f(x)=\dfrac{1}{1+x}$

2. $g(x)=\dfrac{3}{1-2x}$

3. $h(x)=\dfrac{1}{3x}$

**Power Series by Substitution:**

Examples:

4. Using the power series from Example 1, make a new power series for $f\left(x^2\right)$.

5. Using the power series from Example 2, make a new series for $g\left(\sqrt{x}\right)$.

**Power Series by Differentiation:**

Examples:

6. Use the power series for $f(x)=\dfrac{1}{1-x}$ above to write a power series for $f'(x)$.

7. If $j(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}+\cdots$ find $j'(x)$.

Can you identify the function $j(x)$?

**Power Series by integration:**

8. Since $\frac{1}{1+t}=1-t+t^2-t^3+\cdots+(-1)^n t^n+\cdots$ (see Example 1)

$\int_0^x \frac{1}{1+t}\,dt=\int_0^x \left(1-t+t^2-t^3+\cdots+(-1)^n t^n+\cdots\right)dt$ Now integrate both sides.

9. Use the result of Example 8 to write a power series for $f(x)=\ln(x)$.

10. Now write a series for $g(x)=x^2\ln x$.

**Assignment 7-4:**

Find a geometric power series for each of the following functions. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

1. $\frac{1}{1-3x}$
2. $\frac{2}{1-x^3}$
3. $\frac{x}{1+x}$
4. $\frac{1}{1+(-x-3)}$
5. $\frac{3}{4x}$
6. $\frac{1}{2-2x}$

Find a function for each of the following geometric power series. Also give the interval of convergence.

7. $\sum_{n=0}^{\infty}(2x)^n$
8. $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n x^n$
9. $\sum_{n=0}^{\infty}4(x-1)^n$
10. $\sum_{n=1}^{\infty}\left(x^2\right)^n$
11. $\sum_{n=0}^{\infty}(\sin x)^n$

12. Use the result of Example 7 on the previous page to write a power series for $f(x) = e^{x^2}$. Show four terms and the general term.

13. Use the result of Example 7 on the previous page to write a power series for $g(x) = xe^x$. Show four terms and the general term.

14. Find a geometric power series for $g(x) = \frac{1}{1+x}$. Show four terms and the general term.

15. Use the answer to Problem 14 to find a power series for $\frac{1}{1+x^2}$.

16. Use integration to find a power series for $\arctan x$.

Use the function $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ to find the following. Answer using sigma notation.

17. $f(-x)$ 18. $f'(x)$ 19. $\int_0^t f(x)\,dx$

Use the function $g(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$ to find the following.

Show four terms and the general term.

20. $g(\sqrt{x})$ 21. $g'(x)$ 22. $\int_0^x g(t)\,dt$

23. Use the power series for $f(x) = \ln(x)$ in Example 9 on the previous page to find a simplified answer for $f(1)$.

Determine whether the following series converge or diverge. Find the sum when possible.

24. $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$ 25. $\sum_{n=0}^{\infty} \left(\frac{\pi}{e}\right)^n$ 26. $\sum_{n=0}^{\infty} \frac{2n+1}{n+1}$ 27. $\sum_{n=2}^{\infty} \frac{n!}{(n-2)!}$

Determine the convergence/divergence of the following sequences:

28. $\left\{\frac{2n-1}{n+1}\right\}$ 29. $\{e^n\}$

Determine whether the following sequences are monotonic and/or bounded: (assume $n = 1, 2, 3, \cdots$)

30 $\left\{4 + \frac{1}{n}\right\}$ 31. $\left\{\frac{n!}{e^n}\right\}$

32. Evaluate the following improper integral $\int_0^1 \frac{1}{\sqrt{3-x^2-2x}}\,dx$.

Integrate the following:

33. $\int t^2 \tan\left(t^3\right) dt$ 34. $\int \frac{2x}{9+x^2}\,dx$ 35. $\int \frac{2\,dx}{9+x^2}$ 36. $\int \frac{(e^{2y}-1)^2}{e^y}\,dy$

37. $\int \frac{dx}{x\sqrt{\ln x}}$ 38. $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$ 39. $\int_1^2 x\sqrt[3]{x-1}\,dx$

**Selected Answers:**

1. $1+3x+9x^2+27x^3+\cdots+(3x)^n+\cdots=\sum_{n=0}^{\infty}(3x)^n$ ; $\left(-\frac{1}{3},\frac{1}{3}\right)$

2. $2+2x^3+2x^6+2x^9+\cdots+2\left(x^3\right)^n+\cdots=\sum_{n=0}^{\infty}2\left(x^3\right)^n$ ; $(-1,1)$

3. $x-x^2+x^3-x^4+\cdots+(-1)^n x^{n+1}+\cdots=\sum_{n=0}^{\infty}(-1)^n x^{n+1}=\sum_{n=1}^{\infty}(-1)^{n+1}x^n$ ; $(-1,1)$

5. $3+3(1-4x)+3(1-4x)^2+3(1-4x)^3+\cdots+3(1-4x)^n+\cdots=\sum_{n=0}^{\infty}3(1-4x)^n$ ; $\left(0,\frac{1}{2}\right)$

6. $\frac{1}{2}+\frac{1}{2}x+\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots+\frac{1}{2}x^n+\cdots=\sum_{n=0}^{\infty}\frac{1}{2}x^n$ ; $(-1,1)$

7. $\frac{1}{1-2x}$ ; $\left(-\frac{1}{2},\frac{1}{2}\right)$ 8. $\frac{1}{1+\frac{1}{2}x}=\frac{2}{2+x}$ ; $(-2,2)$ 9. $\frac{4}{1-(x-1)}=\frac{4}{2-x}$ ; $(0,2)$

10. $\frac{x^2}{1-x^2}$ ; $(-1,1)$ 12. $e^{x^2}=1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots+\frac{x^{2n}}{n!}+\cdots$

14. $1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots$ 15. $1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+\cdots$

16. $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+\cdots$ 17. $f(-x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}$

18. $f'(x)=\sum_{n=1}^{\infty}x^{n-1}$ 19. $\int_0^t f(x)\,dx=\sum_{n=1}^{\infty}\frac{t^{n+1}}{n(n+1)}$

# **Lesson 7-5** Taylor Series

In this section you will be finding polynomial functions that can be used to approximate transcendental functions. If $P(x)$ is a polynomial function used to approximate some other function $f(x)$, they must contain the same point with some $x$-value $c$. That means $P(c) = f(c)$. To be a better approximation they should have the same slope at that point. This means $P'(c) = f'(c)$. For even greater accuracy, $P''(c) = f''(c)$ and so on. Putting this together gives us the **Taylor Polynomial Expansion:** If $f(x)$ has derivatives of all orders it can be approximated by the polynomial function shown.

$$\boxed{P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n}$$

This is called an $n^{th}$ degree or $n^{th}$ order Taylor Polynomial centered at $c$ or expanded about $c$.

When the center is at $c = 0$ the Taylor polynomial is called a **Maclaurin Polynomial** which can be written as :

$$\boxed{P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n}$$

Example 1. Use the definition of a Maclaurin Polynomial to find the fifth degree Maclaurin Polynomial for $f(x) = e^x$.

| | |
|---|---|
| $f(x) =$ | $f(0) =$ |
| $f'(x) =$ | $f'(0) =$ |
| $f''(x) =$ | $f''(0) =$ |
| $f'''(x) =$ | $f'''(0) =$ |
| $f^{(4)}(x) =$ | $f^{(4)}(0) =$ |
| $f^{(5)}(x) =$ | $f^{(5)}(0) =$ |

$P_5(x) =$

This polynomial is a good approximation for $f(x)=e^x$. By extending the pattern into an infinite series it becomes exactly correct instead of an approximation.

$f(x)=e^x=$

The general form of Taylor and Maclaurin Polynomials can be extended to Taylor and Maclaurin Series.

**Taylor Series** (provided $f(x)$ has derivatives of all orders)

$$\boxed{f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots}$$

**Maclaurin Series**

$$\boxed{f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots}$$

These formulas allow us to form a power series for functions that cannot be written as geometric power series.

Example 2. Use the definition of a Taylor Polynomial to find the fourth order Taylor Polynomial and the Taylor Series for $f(x)=\cos x$ centered at $c=\pi$.

| | |
|---|---|
| $f(x)=$ | $f(\pi)=$ |
| $f'(x)=$ | $f'(\pi)=$ |
| $f''(x)=$ | $f''(\pi)=$ |
| $f'''(x)=$ | $f'''(\pi)=$ |
| $f^{(4)}(x)=$ | $f^{(4)}(\pi)=$ |

$P_4(x)=$

$f(x)=$

Example 3. Use your Taylor Polynomial from example 2 to approximate $\cos 3$.

$\cos 3 \approx$

Example 4. Use the definition of a Maclaurin Series to find the Maclaurin series for $f(x) = \sin x$.

| | |
|---|---|
| $f(x) =$ | $f(0) =$ |
| $f'(x) =$ | $f'(0) =$ |
| $f''(x) =$ | $f''(0) =$ |
| $f'''(x) =$ | $f'''(0) =$ |
| $f^{(4)}(x) =$ | $f^{(4)}(0) =$ |
| $f^{(5)}(x) =$ | $f^{(5)}(0) =$ |

$f(x) = \sin x =$

Example 5. Find a power series for $f(x)$ centered at $c = 1$ if $f(1) = 2$ and $f^{(n)}(1) = n!$.

**Assignment 7-5:**

1. Use the definition to find a fourth degree Maclaurin Polynomial for $f(x) = \dfrac{1}{e^x}$.
2. Use the polynomial from Problem 1 to approximate $\dfrac{1}{\sqrt{e}}$.
3. Use the definition to find a fourth degree Maclaurin Polynomial for $f(x) = e^{3x}$.
4. Use the definition to find a fifth degree Maclaurin Polynomial for $g(x) = xe^x$.
5. Use the definition to find a third degree Taylor Polynomial centered at $c = 1$ for $f(x) = \sqrt[3]{x}$.
6. Use the definition to find a fourth degree Taylor Polynomial centered at $c = 1$ for $h(x) = \ln x$.
7. Use the polynomial from Problem 6 to approximate $\ln 1.3$.

8. Use the definition to find a Taylor Series centered at $c = \frac{\pi}{4}$ for $f(x) = \sin x$.
9. Use the definition to find a Taylor Series centered at $c = 0$ for $f(x) = \cos(2x)$. Show four terms (Zero terms don't count.) and a general term.
10. Use the definition to write a Maclaurin Series for $f(x) = \dfrac{1}{1+x}$. Show four terms and the general term.
11. Write a geometric series expansion for $f(x) = \dfrac{1}{1+x}$. Also give the interval of convergence.
12. Write four terms and the general term of the Taylor series expansion of $f(x) = \dfrac{1}{x-1}$ about $x = 2$.
13. Use the series from Problem 12 to find four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.
14. The Taylor Series of a function about $x = 3$ is given by $f(x) = 1 + \dfrac{3(x-3)}{1!} + \dfrac{5(x-3)^2}{2!} + \dfrac{7(x-3)^3}{3!} + \cdots + \dfrac{(2n+1)(x-3)^n}{n!} + \cdots$ What is the value of $f'''(3)$?
15. Let $f(x)$ be a function such that $f(0) = 2$, $f'(x) = 3f(x)$, and the $n^{th}$ derivative of $f$ is given by $f^{(n)}(x) = 3f^{(n-1)}(x)$.

    (a) Give the first four terms and the general term of the Taylor Series for $f$ centered at $x = 0$.

    (b) Find $f(x)$ by solving the differential equation $f'(x) = 3f(x)$ (that is $y' = 3y$) with the initial condition $f(0) = 2$.
16. Let $f$ be the function defined by the power series $f(x) = 2 + 2x + 2x^2 + 2x^3 + \cdots + 2x^n + \cdots$ If $g'(x) = f(x)$ and $g(0) = 2$, then $g(x) = ?$ Show four terms and the general term.
17. The Taylor series for a function $f$ about $x = 0$ is $2 + \frac{4}{3}x + \frac{8}{9}x^2 + \frac{16}{27}x^3 + \cdots + \dfrac{2^{n+1}x^n}{3^n} + \cdots$ for $-1 < x < 1$. Calculator Allowed.

    (a) Write the first four nonzero terms and the general term for $f'$, the derivative of $f$.

    (b) Using the appropriate second-degree Taylor polynomial approximate $f(0.2)$ and $f'(0.2)$.

    (c) Use the values found in part (b) to approximate the equation of the tangent line to $f$ at $x = 0.2$.
18. The Maclaurin series for $f(x)$ is given by $1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots + \dfrac{x^n}{(n+1)!} + \cdots$

    (a) Find $f'(0)$. (b) Find $f^{(15)}(0)$.
19. Find a geometric power series for $f(x) = \dfrac{x}{1+4x^2}$. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

20. Find a function for the geometric power series $f(x)=\sum_{n=0}^{\infty}4(3x)^n$. Also give the interval of convergence.

Determine whether the following series converge or diverge. Find the sum when possible.

21. $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$
22. $\sum_{n=0}^{\infty}(1.3)^n$
23. $\sum_{n=0}^{\infty}\frac{n^2-1}{n+1}$

24. The derivative of a function is $\frac{dy}{dt}=\frac{-3}{t^2}+1$. If the graph of the function contains the point (3, 10), find the equation of the function.

25. a. Find an equation for the family of functions whose derivative is $y'=3\sqrt{x}$.
b. Find the particular function from the family in Part a. whose curve passes through the point (4, 0).

Evaluate the following definite integrals without a calculator.

26. $\int_1^4\frac{2\sqrt{x}-1}{\sqrt{x}}\,dx$
27. $\int_1^2\frac{dx}{2\sqrt{3x-2}}$
28. $\int_0^3\frac{2x}{\sqrt{x+1}}\,dx$

Differentiate.

29. $h(x)=\sin(\pi x)+\pi x^2$
30. $f(\theta)=\frac{-2\theta}{\sin\theta}$
31. $f(x)=\tan\left(\ln\left(x^2-2x\right)\right)$
32. $h(x)=e^{\sin x\cos x}$

33. Sketch graphs and show shaded areas representing the values of the following definite integrals.

I. $\int_{-2}^{2}\left|x^3+x\right|dx$ II. $\int_{-2}^{2}\left|x^2+5x+6\right|dx$ III. $\int_{-2}^{2}\left|x^2+5x-6\right|dx$ IV. $\int_{-2}^{2}\left|x+1\right|dx$

Match each of the integrals to one of the descriptions below.

a. The integral can be evaluated geometrically using areas of triangles, so that no actual integration is necessary.
b. Absolute value is not even necessary for the given limits of integration.
c. Use of symmetry for the graph allows the problem to be done using only one integral that does not involve absolute value.
d. The integral can only be done by using more than one integral. That is, the problem must be split into two or more integrals to eliminate the absolute value.

34. a. <u>Set up</u> integrals <u>that do not involve absolute value</u> which could be used to integrate the integrals shown in Problem 33 I, II, and III.
b. Evaluate the integral in Problem 33 IV using areas of triangles.

**Selected Answers:**

1. $f(x) \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4$ 2. .60677 3. $f(x) \approx 1 + 3x + \frac{9}{2}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4$

4. $g(x) \approx x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$ 5. $f(x) \approx 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{10}{27 \bullet 3!}(x-1)^3$

7. .261975 8. $f(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}} \bullet \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{\sqrt{2}} \bullet \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{\sqrt{2}} \bullet \frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 + \cdots$

10. $f(x) = 1 - x + x^2 - x^3 + \bullet\bullet\bullet + (-1)^n x^n + \bullet\bullet\bullet$

12. $f(x) = 1 - (x-2) + (x-2)^2 - (x-2)^3 + \bullet\bullet\bullet + (-1)^n (x-2)^n + \bullet\bullet\bullet$

13. $\ln|x-1| = (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \bullet\bullet\bullet + (-1)^n \frac{1}{n+1}(x-2)^{n+1} + \bullet\bullet\bullet$

14. $f'''(3) = 7$ 15a. $f(x) = 2 + 6x + 9x^2 + 9x^3 + \bullet\bullet\bullet + \dfrac{2(3^n)}{n!}x^n + \bullet\bullet\bullet$ b. $f(x) = 2e^{3x}$

16. $g(x) = 2 + \left(2x + x^2 + \frac{2}{3}x^3 + \bullet\bullet\bullet + \frac{2}{n+1}x^{n+1} + \bullet\bullet\bullet\right)$ 18a. $f'(0) = \frac{1}{2}$ b. $f^{(15)}(0) = \frac{1}{16}$

19. $f(x) = x - 4x^3 + 16x^5 - 64x^7 + \bullet\bullet\bullet + (-1)^n 4^n x^{2n+1} + \bullet\bullet\bullet = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n+1}, \quad -\frac{1}{2} < x < \frac{1}{2}$

21. converges to 3 23. diverges 24. $y = \dfrac{3}{t} + t + 6$

25 a. $y = 2x^{\frac{3}{2}} + C$ b. $y = 2x^{\frac{3}{2}} - 16$ 27. $\frac{1}{3}$ 28. $\boxed{\left(\frac{4}{3} \bullet 8 - 8\right) - \left(\frac{4}{3} - 4\right)} = \frac{16}{3}$

29. $h'(x) = \pi \cos(\pi x) + 2\pi x$ 30. $f'(\theta) = \dfrac{-2\sin\theta + 2\theta\cos\theta}{\sin^2\theta}$

31. $f'(x) = \sec^2\left(\ln\left(x^2 - 2x\right)\right)\dfrac{2x-2}{x^2 - 2x}$ 32. $h'(x) = e^{\sin x \cos x}\left(-\sin^2 x + \cos^2 x\right)$

# Lesson 8-1 Error Approximations

**Alternating Series Remainder:**
For a convergent alternating series when approximating the sum of a series by using only the first $n$ terms, the error will be less than or equal to the absolute value of the $n+1^{st}$ term (this is the next term or the first unused term).

Example 1. Approximate the sum of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}$ by using the first 6 terms .

Example 2. Find the upper bound for the remainder for the approximation from Example 1.

Example 3. Find upper and lower bounds for the actual sum of the series in Example 1.

Example 4. Approximate $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}$ with an error of less than .001.

Example 5. Use an elementary series to find the actual value of the series in Example 4.

This remainder can be written as

**Alternating Series Remainder** $\le \left|\frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!}\right|$
First unused term!

If a nonalternating series is approximated, the method is slightly different and slightly harder. It is called the **Lagrange Remainder or Taylor's Theorem Remainder**.

**Lagrange Remainder** $\le \left|\frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}\right|$

**Where $z$ is the $x$-value between $x$ and $c$ inclusive which makes $\left|f^{(n+1)}(z)\right|$ a maximum.**

As in an alternating series remainder the $(n+1)^{st}$ term of the Taylor series is used however, the $(n+1)^{st}$ derivative factor is carefully chosen.

Choose a value of $z$ which makes the $\left|f^{(n+1)}(z)\right|$ factor a maximum. This may be at the center, at the *x*-value where $f$ is to be evaluated, or you may know the maximum value in advance (sine and cosine functions have a maximum value of 1).

Example 6. Estimate $e^2$ using a Maclaurin polynomial of degree 10 for $e^x$.

Example 7. Use the Lagrange form of the remainder (error) to estimate the accuracy of using this partial sum.

Example 8. If $f^{(5)}(x) = 700\sin x$ and if $x = .7$ is in the convergence interval for the power series of $f$ centered at $x = 0$, find an upper limit for the error when the fourth-degree Taylor polynomial is used to approximate $f(.7)$.

Example 9. If $f^{(6)}(x)$ is a positive decreasing function, find the error bound when a 5th degree Taylor polynomial centered at $x = 4$ is used to approximate $f(4.1)$. Assume the series converges for $x = 4.1$.

**Assignment 8-1**

1. Approximate the sum of the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ with an error less than or equal to 0.001.
2. If the first four terms are used to approximate the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3-1}$ find an upper bound for the remainder.
3. Approximate $e^{-1}$ with a sixth degree Maclaurin Polynomial and find an upper limit of the

Alternating Series Remainder.

4. How many terms of a Maclaurin Polynomial are needed to approximate $\sin 1$ with an error of less than 0.001?

5. How many terms of a Maclaurin Polynomial are needed to approximate $\sin 2$ with an error of less than 0.001?

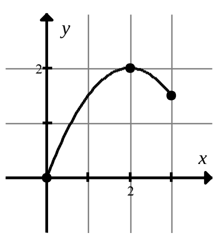
6. If a Taylor Polynomial centered at 1 is used to approximate $\ln 2$ with an error of less than 0.001, how many terms are needed?

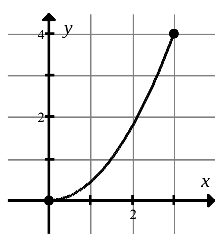
7. If $\left|f^{(4)}(x)\right| \le 4$ find the Lagrange error bound if a third degree Taylor Polynomial centered at $x=1$ is used to approximate $f(2)$. Assume the series converges for $x = 2$.

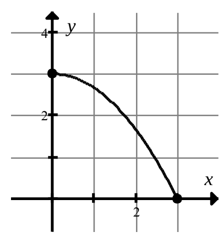
8. If $P_3(2)=5$ for the function from problem 7, find the range of possible values for $f(2)$.

9. If $f^{(6)}(x)=200\sin x$ and $x=.5$ is in the interval of convergence of the power series for $f$, then find the error when a fifth-degree Taylor polynomial, centered at $x=0$ is used to approximate $f(.5)$.

10. If a sixth degree Taylor Polynomial centered at $x=0$ is used to approximate $f(3)$, find the Lagrange error bound for each of the following if the graph shown is a portion of the graph of $f^{(7)}(x)$. Assume the series converges for $x = 3$.

a. 


b. 


c. 


11. Assuming the function from problem 10 is represented by an alternating series, which of the three answers would be the same using an alternating series error bound?

12. The function $f(x)=e^{-2x}$ is approximated by the polynomial $f(x)\approx 1-2x+2x^2-\frac{4}{3}x^3$.
Use the alternating series error bound to determine positive $x$-values for which this approximation
has an error of less than $\frac{27}{8}$ without using a calculator.

13. For $f(x)=\ln x,\ c=1$:

a. Write a Taylor Polynomial $P_4(x)$.

b. Write a power series for $f(x)$ using $\sum$ notation.

c. Approximate $f(1.3)$ using $P_4(1.3)$.

d. Find the actual value of $f(1.3)$.

e. Find the Lagrange error (remainder) bound, $R_4(1.3)$.

f. Find the number of terms from the Taylor Polynomial needed to approximate $f(1.3)$ with an error (remainder) less than .001.

14. Find an upper limit for the error when the Taylor polynomial $T(x) = x - \frac{x^3}{3!}$ is used to approximate $f(x) = \sin x$ at $x = 0.5$.

15. Let $f(x)$ be a function whose Taylor series converges for all $x$. If $\left|f^{(n)}(x)\right| < 1$ what is the minimum number of terms of the Taylor series, centered at $x = 1$, necessary to approximate $f(1.2)$ with an error less than 0.00001? Assume the series has no zero terms.

16. (calculator allowed)

| $x$ | $h(x)$ | $h'(x)$ | $h''(x)$ | $h'''(x)$ | $h^{(4)}(x)$ |
|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 |

Let $h$ be a function having derivatives of all orders for $x > 0$, selected values of $h$ and its first four derivatives are indicated in the table above.

a. Write the first-degree Taylor polynomial for $h$ about $x = 2$ and use it to approximate $h(1.9)$.

b. Write the third-degree Taylor polynomial for $h$ about $x = 2$ and use it to approximate $h(1.9)$.

c. Assuming the fourth derivative of $h$ is a positive increasing function, use the Lagrange error bound to show that the third-degree Taylor polynomial for $h$ about $x = 2$ approximates $h(1.9)$ with error less than $3 \times 10^{-5}$.

17. If $y = \frac{3}{x} - 2\sqrt{x}$, find $\frac{d^2y}{dx^2}$.

18. Find the <u>point(s)</u> where the line(s) tangent to the graph of $f(x) = \frac{1}{3}x^3 - x^2 + x + 3$ is/are parallel to the graph of $y - x = 5$.

Integrate:

19. $\int \frac{3-4t}{t^2+9}\,dt$

20. $\int \frac{\arcsin \frac{x}{2}}{\sqrt{4-x^2}}\,dx$

21. $\int \frac{\sqrt{x-1}}{x}\,dx$ (hint: let $u = \sqrt{x-1}$)

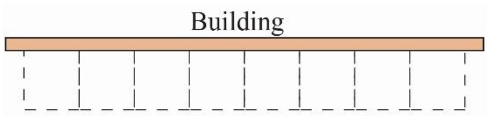
22. Find a general solution of the differential equation $t\frac{dy}{dt} - 2y = \frac{dy}{dt}$. Solve for $y$.

23. The rate of growth of bacteria in a culture is proportional to the number of bacteria present at any time $t$. If there were 2000 bacteria present 3 days after the introduction of bacteria into the culture, and 5000 present 2 days later, find:

a. the growth rate for the bacteria in the culture.

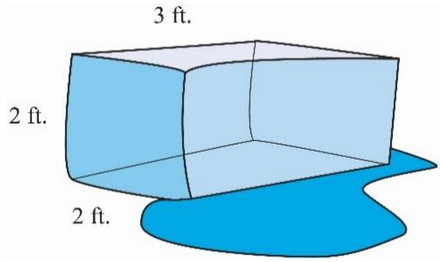
b. the number of bacteria initially introduced into the culture. (Round to the nearest whole number.)
c. the estimated number of bacteria in the culture 12 days after the bacteria were initially introduced. (Round to the nearest hundred.)

24. A small dog kennel with 8 individual rectangular holding pens of equal size is to be constructed using 144 *ft* of chain link fencing material. One side of the kennel is to be placed against a building and requires no fencing, as shown in the figure below.




a. Find the dimensions (for each holding pen) that produce a maximum area for each pen.
b. What is that maximum area for <u>each</u> holding pen?

25. A block of ice is exposed to heat in such a way that the block maintains a similar shape as it melts. The block of ice is initially 2 feet wide, 2 feet high, and 3 feet long, as shown at right. If the change in the width of the ice is $-\frac{1}{3}$ ft/hr, find:




a. the rate of change in the volume of the block of ice when the width is 1 ft.
b. the amount of time it will take for the block of ice to completely melt.

26. Find $\displaystyle\lim_{\theta \to \frac{\pi}{2}} \frac{-2\cos\theta}{e^{\theta-\frac{\pi}{2}}-1}$

**Selected Answers:**

1. .902 2. $R \le \frac{1}{249}$ 3. .368, $R \le \frac{1}{5040}$ 4. three terms 5. five terms

6. 1000 terms 7. $R \le \frac{1}{6}$ 8. $4\frac{5}{6} \le f(2) \le 5\frac{1}{6}$

9. $R \le .0043$ or $R \le .00208$ (better answer)

10a. $R \le .867$ or $.868$ b. $R \le 1.735$ or $1.736$ c. $R \le 1.301$ or $1.302$

11. c 12. $0 < x < \frac{3}{2}$

13a. $P_4 = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$, b. $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$,

c. $f(1.3) \approx P_4(1.3) = .261975$, d. $f(1.3) = \ln(1.3) = .262364$,

e. $R_4(1.3) \le .000486$, f. four terms

14. Alt Series Error $\le .00026$ or Lagrange Error $\le .0026$ 15. five terms

16 (a) $h(x) \approx 3 + 4(x-2)$, $h(1.9) \approx 2.6$,

(b) $h(x) \approx 3 + 4(x-2) + \frac{5}{2}(x-2)^2 + (x-2)^3$, $h(1.9) \approx 2.624$

(c) $R \le .000029 < .00003$

17. $y'' = 6x^{-3} + \frac{1}{2}x^{-\frac{3}{2}}$ 18. $(0,3)$, $\left(2, \frac{11}{3}\right)$ 19. $\arctan\frac{t}{3} - 2\ln\left(t^2+9\right) + C$

20. $\frac{1}{2}\left(\arcsin\frac{x}{2}\right)^2 + C$ 21. $2\sqrt{x-1} - 2\arctan\sqrt{x-1} + C$ 22. $y = C(t-1)^2$