A SCALING LIMIT FOR THE DEGREE DISTRIBUTION IN SUBLINEAR PREFERENTIAL ATTACHMENT SCHEMES

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Abstract. We consider a general class of preferential attachment schemes evolving by a reinforcement rule with respect to certain sublinear weights. In these schemes, which grow a random network, the sequence of degree distributions is an object of interest which sheds light on the evolving structures.

In this article, we use a fluid limit approach to prove a functional law of large numbers for the degree structure in this class, starting from a variety of initial conditions. The method appears robust and applies in particular to 'non-tree' evolutions where cycles may develop in the network.

A main part of the argument is to analyze an infinite system of coupled ODEs, corresponding to a rate formulation of the law of large numbers limit, in terms of C_0-semigroup/dynamical systems methods. These results also resolve a question in Chung, Handjani and Jungreis (2003).

1. Introduction

Since the late 90's and early 2000's, much attention has been devoted to 'preferential attachment processes': Networks evolving over time by linking at each time step new nodes to vertices in the existing graph with a probability based on their connectivity. Such schemes relate to 'reinforcement' and other dynamics which have a long history (cf. surveys [27], [34], [40]). Recently, Barabási and Albert (BA) in [6] proposed that versions of these processes may serve as models for growing real-world networks such as the world wide internet web, and types of social structures.

For instance, in a 'friend network', a newcomer may be favorably disposed to link or become friends with an individual with high connectivity, or in other words, one who already has many friends. As observed in [6], when the probability of selecting a vertex is proportional to its degree, the proportions of nodes with degrees $1, 2, \ldots, k, \ldots$ converge as time grows to a power-law distribution $\langle q(k) : k \geq 1\rangle$ where $0 < \lim_{k\uparrow\infty} q(k)k^\theta < \infty$ for some $\theta > 0$. Since the sampled empirical degree structure in many real-world networks also has such a power-law form, such preferential attachment processes, in contrast to Erdös-Rényi graphs where the degree structure decays much more rapidly, have become popular: See [1], [5], [9], [10], [14], [15], [19], [21], [29], [30], [31], and references therein.

At the same time, other versions of preferential attachment, where the selection probability is a nonlinear function of the connectivity have been considered, and interesting effects have been shown: See, among other works, [13], [18], [20], [24],

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[32], [38]. For instance, depending on the scheme and the type of nonlinearity, the degree structure asymptotically may be in the form of a 'stretched exponential' or the graph may evolve into a 'condensed' state in which a single (random) vertex may be linked with almost all the incoming nodes.

To be more specific, consider the following preferential attachment model. Suppose at time $n = 0$, the initial network G_0 is composed of two vertices with a single (undirected) edge between them. The dynamics now is that at time $n = 1$, a new vertex is attached to one of the two vertices in G_0 with probability proportional to a function of its degree to form the new network G_1. This scheme continues: More precisely, at time $n + 1$, a new node is linked to vertex $x \in G_n$ with probability proportional to $w(d_x(n))$, that is chance $w(d_x(n))/\sum_{y\in G_n} w(d_y(n))$, where $d_z(n)$ is the degree at time n of vertex z and $w = w(d) : \mathbb{N} \to \mathbb{R}_+$ is the 'weight' function.

Now, for the moment, to simplify the discussion, let us assume $w(d) = d^\kappa$ for $\kappa > -\infty$. In this way, since the initial graph is a tree, all later networks G_n for $n \geq 0$ are also trees. Let now $\mathcal{Z}_k(n)$ be the number of vertices in G_n with degree k, $\mathcal{Z}_k(n) = \sum_{y\in G_n} 1(d_y(n) = k)$. In [24], a trichotomy of growth behaviors was observed depending on the strength of the exponent κ.

First, when w is linear, that is when $\kappa = 1$, the scheme is the well known Barabasi-Albert model where the degree structure satisfies, for $k \geq 1$,

$$\lim_{n\to\infty} \frac{\mathcal{Z}_k(n)}{n} = \frac{4}{k(k+1)(k+2)} \quad \text{a.s.}$$

This power-law ($\theta = 3$), in mean-value, through an analysis of rates, was found in [6], [24]. In [8], using difference equations/concentration bounds, the limit was proved in probability. Via Pólya urns, another proof was found yielding a.s. convergence, and also central limit theorems [28]. Also, by embedding into continuous time branching processes, the same a.s. limit was proved in [38]; see also [3] where a different type of embedding was used. A form of Stein's method gives rates of convergence in total variation norm [33], [37]. A large deviation approach also obtains the limit [12].

Next, in the strict sublinear case, when $\kappa < 1$, it was shown that

$$\lim_{n\uparrow\infty} \mathcal{Z}_k(n)/n = q(k) \quad \text{a.s.} \tag{1.1}$$

although q is not a power law, but in form where it decays faster than any polynomial [24], [38]: For $k \geq 1$,

$$q(k) = \frac{s^*}{k^\kappa} \prod_{j=1}^{k} \frac{j^\kappa}{s^* + j^\kappa}, \quad \text{and } s^* \text{ is determined by } \quad 1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{j^\kappa}{s^* + j^\kappa}.$$

Asymptotically as $k \uparrow \infty$, when $0 < \kappa < 1$, $\log q(k) \sim -(s^*/(1-\kappa))k^{1-\kappa}$ is in 'stretched exponential' form; when $\kappa < 0$, $\log q(k) \sim \kappa k \log k$; when $\kappa = 0$, the case of uniform attachment when an old vertex is selected uniformly, $s^* = 1$ and q is geometric: $q(k) = 2^{-k}$ for $k \geq 1$.

In the superlinear case, when $\kappa > 1$, 'explosion' or a sort of 'condensation' happens in that in the limiting graph a random single vertex dominates in accumulating connections. In particular, the limiting graph is shown to be a tree

$$\begin{aligned}&\text{where there is a single random vertex with an infinite number of children;}\\&\text{all other vertices have bounded degree, and of these only a finite number}\\&\text{have degree equal or larger than } \lfloor \kappa/(\kappa-1)\rfloor\end{aligned} \tag{1.2}$$

(cf. for a more precise description [32], [24]). Moreover, a corresponding LLN limit, $\lim_{n\uparrow\infty} E\mathcal{Z}_k(n)/n = q(k)$, is proposed where q is degenerate in that $q(1) = 1$ but $q(k) = 0$ for $k \geq 2$ (cf. [24], [21, Chapter 4], [2]).

We now comment on the methods in the papers [38] and [32]. Both use branching process embedding techniques to establish the sublinear and superlinear degree structure results (1.1) and (1.2). More specifically, it seems a tree structure is useful in the proofs, that is the dynamics places no edges between already extant vertices to create cycles.

The purpose of this article, in this context, is to show the LLN for the degree structure in a general class of 'sublinear' preferential attachment models, including the scheme discussed above, starting from various initial conditions, through a new, different 'fluid limit' approach where cycles may develop. We also note that the method taken here seems robust and might be used in other combinatorial schemes (cf. Remark 2.4).

Specifically, we show (Theorem 2.3) a functional LLN for the degree counts in sublinear generalizations of the urn scheme of Chung, Handjani and Jungreis [13] and the graph model of Chung and Lu [14] (cf. Section 2 for model descriptions and assumptions). Moreover, our work solves a question in [13] to show a LLN for the associated degree structure when the weights are sublinear (cf. Remark 2.4).

The 'fluid limit' method is to consider a more complex problem, namely that of the dynamics of paths $\{n^{-1}\mathcal{Z}_k(\lfloor nt\rfloor) : t \in [0,1]\}$ for $k \geq 1$. But, these paths have nice properties and we show their limit points satisfy certain ODEs corresponding to a rate formulation of the degree distribution flow (cf. (2.7)). As all the counts $\{\mathcal{Z}_k(n)\}$ are coupled together in terms of the total 'weight' of the graph $S(n) = \sum_{k\geq 1}\mathcal{Z}_k(n)$ in the selection procedure, the ODE system derived is infinite dimensional and nonlinear, and poses nontrivial difficulties.

The ODEs appear natural and may be of interest in other contexts where there is exchange of proportional flow between chains of components. By a change of variables, the ODEs can be written in terms of a linear 'Kolmogorov' differential equation (cf. [26]) which can be analyzed by C_0-semigroup/dynamical systems arguments. In particular, we show (Theorem 2.1) the ODEs admit a unique solution. Therefore, addressing the original 'fluid limit' taken, all the path limit points are the same and so are uniquely characterized.

There is a large literature on fluid limits in various contexts: See [11], [17], [23], [35], [42], [44]) and references therein. Most of this previous development focuses on finite dimensional spaces. In this respect, the current article considers a nontrivial infinite dimensional fluid limit, whose analysis depends on the type of initial condition, namely 'small' versus 'large' (cf. (LIM) in Section 2), which plays a role in the results Theorems 2.1, 2.3. See also [36] for a different infinite dimensional limit in a type of Erdös-Rényi graph.

In the next Section, we detail the preferential attachment models discussed, and state results. Then, proofs of the main convergence and uniqueness results follow in succeeding Sections.

2. Models and Results

To specify the models considered, let $0 \leq p \leq 1$ be a parameter, and let $w : \{1, 2, \ldots\} \to (0, \infty)$ be a positive function which we will call the 'weight' function.

Graph Model. The following scheme captures the growth of a graph network:

- At time $n = 0$, the initial network G_0 is a finite, possibly disconnected graph.
- At time $n + 1 \geq 1$, form G_{n+1} as follows.
  –With probability $1 - p$, we select independently two old vertices $x, y \in G_n$ with chances $w(|x|)/S(n)$ and $w(|y|)/S(n)$ respectively, and then place an edge connecting x and y to form G_{n+1}.
  –However, with probability p, an edge is placed between a new vertex and an old node $x \in G_n$, chosen with probability $w(|x|)/S(n)$, to form G_{n+1}.

In this model, $|x| \geq 1$ is the degree of the vertex x, and $S(n)$ is the total 'weight' of collection G_n:

$$S(n) \ = \ \sum_{k \geq 1} w(k) Z_k(n)$$

where $Z_k(n)$ is the count of vertices in G_n with degree k for $k \geq 1$.

Note that it may be possible when two vertices $x, y \in G_n$ are selected, they are the same, which means a 'loop' is added to the graph at $x = y$, and our convention here is that the degree of vertex $x = y$ is incremented by 2. In particular, at each time, the total degree of the graph increments by 2. However, as the successive independent choices in actions are random, the total number of vertices at time $n \geq 1$ is $V(G_0)$ plus a sum of n independent Bernoulli(p) variables; here, $V(G_0)$ is the initial number of vertices.

Cases of the above dynamics include the following.

- When $p = 1$ and $w(k) = k^\kappa$, the dynamics matches the preferential attachment graph scheme mentioned in the Introduction, and $Z_k(n) = \mathcal{Z}_k(n)$ for all $k \geq 1$. In this case, for $\kappa \leq 1$, a LLN limit for $Z_k(n)/n$ has been proved, among other results, in [38] as mentioned before.
- When $w(k) = k^\kappa$ and p is arbitrary, the scheme is discussed and many results are proved in [14]. For instance, when $\kappa = 1$, a LLN is proved, $\lim_{n\uparrow\infty} Z_k(n)/n = q(k)$ where q is in 'power law' form with $\theta = 1+2/(2-p)$. Also, in this situation, a central limit theorem for the 'leaves', nodes of degree 1, has been found in [43]. However, when $\kappa < 1$ and $p < 1$, the LLN for $Z_k(n)/n$ has been an open question, now resolved by Theorem 2.3.
- If $p = 0$, the dynamics would always add edges and loops with respect to the initial graph G_0. We will avoid this 'degenerate' growth in what follows.

Urn Model. Consider the following urn dynamics which builds an evolving collection of urns:

- At time $n = 0$, the initial collection G_0 is a finite set of nonempty urns, each containing a finite number of balls.
- At time $n + 1 \geq 1$, G_{n+1} is built as follows.
  –With probability p, a new urn with a single ball is added to the collection to form G_{n+1}.

–However, with probability $1-p$, we select an urn x from G_n with probability $w(|x|)/S(n)$, and place a new ball into it to form G_{n+1}.

Here, $|x| \geq 1$ is the size or number of balls in the urn x, and $S(n)$ is the total 'weight' of collection G_n:

$$S(n) \;=\; \sum_{k\geq 1} w(k) Z_k(n)$$

where $Z_k(n)$ is the number of urns in G_n with exactly k balls for $k \geq 1$.

We note that the total size or number of balls increments by 1 at each time, but as in the graph model, at time $n \geq 1$, the total number of urns is random, namely $U(G_0)$ plus the sum of n independent Bernoulli(p) variables, where $U(G_0)$ is the initial number of urns.

We now remark on some cases of the above dynamics:

- When $w(k) = k^\kappa$, the scheme is discussed in [13], and results on the evolution are given when $\kappa \geq 1$. However, when $\kappa < 1$, a LLN is stated, but the convergence of $Z_k(n)/n$ is left open (cf. Remark 2.4).
- If $p = 1$, the scheme would always add an urn with a single ball to the collection at each time. Also, if $p = 0$, no new urns are added and only the urns in the initial collection grow. Both are 'degenerate' evolutions which we will avoid in assumption (P) below.

We now give assumptions on p with respect to the two models, and on the weight function w under which results are stated.

(P) To avoid 'degeneracies', in the graph model, p is taken $0 < p \leq 1$. However, in the urn scheme, we assume that $0 < p < 1$.

(SUB) We have

$$\lim_{k\uparrow\infty} \frac{w(k)}{k} = 0. \quad \text{Hence, } \sup_{k\geq 1}\big[w(k)/k\big] \leq \mathcal{W} \text{ for some constant } \mathcal{W} < \infty.$$

This large class of weights $w(\cdot)$ includes in particular the well-studied case $w(k) = k^\kappa$ for $\kappa < 1$ discussed in the Introduction.

Let $p_0, q_0 > 0$. In the following, with respect to the two models under (P),

fix in the graph scheme $p_0 = p$ and $q_0 = 2 - p$, and
in the urn scheme $p_0 = p$ and $q_0 = 1 - p$.

Define also the positive function $F_{p_0,q_0} : [0,\infty) \to \mathbb{R} \cup \{\infty\}$ by

$$F_{p_0,q_0}(s) \;=\; \frac{p_0}{q_0} \sum_{k\geq 1} \prod_{j=1}^{k} \frac{q_0 w(j)}{s + q_0 w(j)}.$$

By (SUB), $F_{p_0,q_0}(s) < \infty$ for $s > 0$. Moreover, it is clear in each model that there exists a number $s_0 = s_0(p_0, q_0) > 0$ such that

$$1 \;<\; F_{p_0,q_0}(s_0) \;<\; \infty. \tag{2.1}$$

Given (2.1), the function F_{p_0,q_0} is strictly decreasing on $[s_0, \infty)$ and vanishes at infinity, $\lim_{s\uparrow\infty} F_{p_0,q_0}(s) = 0$. Therefore,

$$\text{there exists a unique } s^* = s^*(p_0, q_0) > s_0 \text{ such that } F_{p_0,q_0}(s^*) = 1. \tag{2.2}$$

We now comment, in [38], which proves the LLN for the degree distribution in the graph model when $p = 1$, the only condition assumed on w is that (2.1) holds, which is more general than (SUB). For instance, a 'linear weights'-type structure where $\liminf_{k\uparrow\infty} w(k)/k > 0$ is not allowed under (SUB) (although see Remark 2.4). However, as remarked in [38], (2.1) itself is only given as a sufficient condition.

A necessary condition might include the requirement, $\sum_{k\geq 1} 1/w(k) = \infty$, implied by (2.1), although this is not pursued here. In this respect, we note Theorem 1.1(ii) in [16] proves that $\sum_{k\geq 1} 1/w(k) = \infty$ is a necessary and sufficient condition for all vertices/urns to have infinite size in the limit network a.s.

The assumption (SUB) is used to enforce control on the tails of the weight sum $S(n)$, so that the limits of $S(n)/n$ and $\{Z_k(n)/n\}$ can be related (cf. Step 2, proof of Theorem 2.3). (SUB) is also useful in the proof of uniqueness of solution to the infinite dimensional ODE system derived (cf. Section 4).

We now derive the evolution scheme of the counts $\{Z_k(n)\}$ for $n \geq 0$. Define sigma-fields $\mathcal{F}_j = \sigma\{\{Z_k(\ell)\} : 0 \leq \ell \leq j\}$ for $j \geq 0$. Note also, in both models, as $w(\cdot) > 0$, $S(n) > 0$ for all $n \geq 0$. For each $k \geq 1$, let

$$Z_k(j+1) - Z_k(j) \ =: \ d_k(j+1) \tag{2.3}$$

where given $\{Z_k(j)\}$, the counts at time j, the difference $d_k(j+1)$ is as follows.

With respect to the urn scheme,

$$d_1(j+1) \ = \ \begin{cases} 1 & \text{with prob. } p \\ -1 & \text{with prob. } (1-p)\frac{w(1)Z_1(j)}{S(j)} \\ 0 & \text{with prob. } (1-p)\big[1 - \frac{w(1)Z_1(j)}{S(j)}\big]. \end{cases}$$

Also, for $k \geq 2$,

$$d_k(j+1) \ = \ \begin{cases} 1 & \text{with prob. } (1-p)\frac{w(k-1)Z_{k-1}(j)}{S(j)} \\ -1 & \text{with prob. } (1-p)\frac{w(k)Z_k(j)}{S(j)} \\ 0 & \text{with prob. } 1-(1-p)\big[\frac{w(k-1)Z_{k-1}(j)}{S(j)} + \frac{w(k)Z_k(j)}{S(j)}\big]. \end{cases}$$

In the graph scheme, $d_k(j+1)$ has a similar, but more involved expression as possible loops need to be considered. These formulas are given in the Appendix.

Consider now an array of counts $\{Z_k^n(\cdot) : k \geq 1\}$ and weights $\{S^n(\cdot)\}$ for $n \geq 1$ where in the nth row the underlying process begins from initial network G_0^n. Define the family of linearly interpolated processes $\{X_k^n(t) : t \geq 0, k \geq 1\}$ for $n \geq 1$, which place the proportion of counts $\{Z_k^n(j)/n\}$ into continuous time trajectories, by

$$X_k^n(t) \ := \ \frac{1}{n}Z_k^n(\lfloor nt \rfloor) + \frac{nt - \lfloor nt \rfloor}{n}\left(Z_k^n(\lceil nt \rceil) - Z_k^n(\lfloor nt \rfloor)\right).$$

The paths $X_k^n : [0,\infty) \to \mathbb{R}_+$, in both schemes as $|d_k(j+1)| \leq 2$, belong to the space of Lipschitz functions with constant at most 2. For $t \geq 0$, let also $\mathcal{S}^n$ be the continuous interpolation of the weights S^n,

$$\mathcal{S}^n(t) \ := \ \frac{1}{n}S^n(\lfloor nt \rfloor) + \frac{nt - \lfloor nt \rfloor}{n}\left(S^n(\lceil nt \rceil) - S^n(\lfloor nt \rfloor)\right). \tag{2.4}$$

Let now, initially, for some constants $c_k^n, c^n, \tilde{c}^n \geq 0$ for $k \geq 1$ that

$$c_k^n \ := \ \frac{1}{n}Z_k^n(0), \quad c^n \ := \ \sum_{k\geq 1} c_k^n, \quad \text{and} \quad \tilde{c}^n := \sum_{k\geq 1} k c_k^n.$$

We will impose the following initial laws of large numbers:

(LIM) For constants $c_k, c, \tilde{c} \geq 0$, we have

$$c_k := \lim_{n\uparrow\infty} c_k^n \quad \text{and} \quad \tilde{c} := \sup_{n\geq 1} \tilde{c}^n < \infty.$$

Hence, $c := \lim_{n\uparrow\infty} c^n = \sum_{k\geq 1} c_k < \infty$ and $\tilde{c} \geq \sum_{k\geq 1} kc_k$.

Here, noting $\sum_{k\geq L} c_k^n \leq L^{-1}\sum_{k\geq 1} kc_k^n \leq \tilde{c}/L$, the c^n-limit follows. The $\tilde{c}$-inequality is Fatou's inequality. Also, as $\sum_{k\geq L} w(k)c_k^n \leq \tilde{c}[\sup_{j\geq L} w(j)/j]$, from (SUB), we may also conclude the initial weights $\sum_{k\geq 1} w(k)c_k^n \to \sum_{k\geq 1} w(k)c_k$.

We will say that the initial configuration is a 'small' configuration if $c_k \equiv 0$, and is a 'large' one if $c_k > 0$ for some $k \geq 1$. In a small configuration, the total degree/size of G_0^n is $o(n)$. In particular, if the total degree/size of G_0^n is uniformly bounded, for example $G_0^n \equiv G_0$ is fixed, the initial configuration is a small one. However, in a large configuration, the initial networks are already developed in that their degree/size is at least εn for some $\varepsilon > 0$. We remark similar initial conditions were used in a different context in [12].

With respect to small initial configurations, we now try to guess the long term behavior of $X_k^n(t)$ for all k. Suppose, as $n \uparrow \infty$, that $X_k^n(t) \to a_k t$ and $\mathcal{S}^n(t) \to bt$ for constants $\{a_k\}$ and $b = \sum_{k\geq 1} w(k)a_k > 0$. Then, since $Z_k(\lfloor nt \rfloor)$ is a type of random walk, considering the drift, if the increment $d_k(j+1)$ were set equal to its conditional expectation given $\mathcal{F}_j$, we would obtain heuristically the following limit equations for $\{a_k\}$ in the two models discussed.

Let $p_0, q_0 > 0$. In the following, under (P), fix $p_0 = p$ and $q_0 = 2 - p$, the parameters correspond to the the graph model. However, in the urn model, set the parameters $p_0 = p$ and $q_0 = 1 - p$.

Then, for $b > 0$, define $a_k = a_k(p_0, q_0, b)$ by

$$\begin{aligned} a_1 &= p_0 - \frac{q_0 w(1) a_1}{b} && (2.5)\\ a_k &= q_0\frac{w(k-1)a_{k-1}}{b} - q_0\frac{w(k)a_k}{b} && \text{for } k \geq 2. \end{aligned}$$

Solving for the $\{a_k\}$, in terms of b, one obtains

$$a_1 = \frac{bp_0}{b + q_0 w(1)}, \text{ and } a_k = \frac{q_0 w(k-1)a_{k-1}}{b + q_0 w(k)} \quad \text{for } k \geq 2.$$

Therefore, for $k \geq 2$,

$$\begin{aligned} a_k &= a_1 \prod_{j=1}^{k-1} \frac{q_0 w(j)}{b + q_0 w(j+1)}\\ &= a_1 \frac{b + q_0 w(1)}{b + q_0 w(k)} \prod_{j=1}^{k-1} \frac{q_0 w(j)}{b + q_0 w(j)}\\ &= a_1 \frac{b + q_0 w(1)}{b}\Big[\prod_{j=1}^{k-1} \frac{q_0 w(j)}{b + q_0 w(j)} - \prod_{j=1}^{k} \frac{q_0 w(j)}{b + q_0 w(j)}\Big]. \end{aligned}$$

Noting $\{a_k\}$ above, under (SUB), the equation $b = \sum w(k)a_k$ takes form

$$b = \frac{bp_0}{q_0} \sum_{k\geq 1} \prod_{j=1}^{k} \frac{q_0 w(k)}{b + q_0 w(k)}$$

or $1 = F_{p_0,q_0}(b)$ (cf. definition near (2.1)).

Hence, the parameter b above is identified from (2.2) as $b = s^*$, which through (2.5) implicitly determines $\{a_k\}$ in terms of $w(\cdot)$, p_0 and q_0. Of course, when $w(k) = k^\kappa$ for $\kappa < 1$, $a_k = q(k)$ for $k \geq 1$ (cf. (1.1)).

In both models, after a calculation, one sees

$$\sum_{k\geq 1} a_k = p_0 \quad \text{and} \quad \sum_{k\geq 1} k a_k = p_0 + q_0. \tag{2.6}$$

At this point, one might also infer an a.s. 'continuous' version of (2.5), a rate formulation for the limit of the functions $\{X_k^n\}$. That is, for each realization in a probability 1 set, we take subsequential limits of X_k^n and conclude that all limit points are nonnegative functions $\varphi_k(\cdot) = \varphi_k(\cdot; p_0, q_0)$ for $k \geq 1$ satisfying the integral form of a coupled system of ODEs:

$$\begin{aligned} \dot{\varphi}_1(t) &= p_0 - \frac{q_0 w(1)\varphi_1(t)}{\sum_{k\geq 1} w(\ell)\varphi_k(t)}, \\ \dot{\varphi}_k(t) &= \frac{q_0}{\sum_{k\geq 1} w(\ell)\varphi_k(t)}\big[w(k-1)\varphi_{k-1}(t) - w(k)\varphi_k(t)\big], \quad \text{for } k \geq 2. \end{aligned} \tag{2.7}$$

with initial condition $\varphi_k(0) = c_k$ for $k \geq 1$. Under small initial configurations ($c_k \equiv 0$), the ODE is singular at $t = 0$, although one can inspect that $\{\varphi_k(t) = a_k(p_0, q_0, s^*)t : k \geq 1\}$ is a solution since $\{a_k(p_0, q_0, s^*) : k \geq 1\}$ verifies (2.5). However, under either large or small initial configurations, it does not seem easy to conclude the ODEs has a unique nonnegative solution.

But, one can think of the ODE system in the following way: Introduce a time-change $t = t(s)$ satisfying $\dot{t} = T(t)$ where $T(t) = \sum_{k\geq 1} w(k)\varphi_k(t)$ and $t(0) = 1$. Then, $\{\psi_k(s) = \varphi_k(t(s)) : k \geq 1\}$ satisfies the integral form of the following autonomous system:

$$\begin{aligned} \dot{\psi}_1(s) &= (p_0 - q_0)w(1)\psi_1(s) + p_0 \sum_{k\geq 2} \psi_k(s), \\ \dot{\psi}_k(s) &= q_0\big[w(k-1)\psi_{k-1}(s) - w(k)\psi_k(s)\big], \qquad \text{for } k \geq 2. \end{aligned} \tag{2.8}$$

It will turn out that one can associate to this system a strongly continuous positive semigroup P_s whose essential growth rate is nonpositive. Such semigroups have nice asymptotics in terms of a 'Perron-Frobenius' eigenvector and eigenvalue. Moreover, it turns out the time-change $t(\cdot)$ can be determined in terms of P_s, which will allow to characterize solutions of (2.7).

Theorem 2.1. *Suppose $p_0, q_0 > 0$ and conditions (SUB) and (LIM) hold, and recall the parameter s^* in* (2.2). *Then, under both small and large initial configurations, there is a unique nonnegative solution $\{\varphi_k(\cdot)\}$ of the integral form of ODEs* (2.7).

Moreover, for $k \geq 1$, under small initial configurations, $\varphi_k(t) = a_k(p_0, q_0, s*)t$. Also, with respect to large initial configurations, $\lim_{t\uparrow\infty} t^{-1}\varphi_k(t) = a_k(p_0, q_0, s^*)$.

Remark 2.2. The proof of Theorem 2.1 is shorter under large initial configurations as there is no time singularity at $t = 0$. In this case, the solution is found implicitly in terms of the semigroup P_s and time-change $t = t(s)$.

However, the full machinery of 'quasi-compact' semigroup asymptotics and the assumption $c = 0$ are used in the small initial configuration case. See the beginning of Section 4 for more remarks on the strategy of the proof.

Finally, we note s^* can be identified in terms of the 'Perron-Frobenius' eigenvalue alluded to earlier (cf. Proposition 4.8).

We now assert that the heuristic derivations (2.5) and (2.7) are correct.

Theorem 2.3. *Suppose conditions (P), (SUB), and (LIM) hold. Let $\{\varphi_k(\cdot, p_0, q_0) : k \geq 1\}$ be the unique nonnegative solution found in Theorem 2.1 to the integral form of ODEs* (2.7)*. Then, with respect to the graph and urn models, for $k \geq 1$, uniformly on compact time sets, we have*

$$\lim_{n\uparrow\infty} X_k^n(t) \;=\; \varphi_k(t; p_0, q_0) \quad \text{a.s.}$$

Remark 2.4. With respect to the urn model, when $w(k) = k^\kappa$ for $\kappa < 1$, the form of $\{a_k\}$ was derived in [13]. However, it was left open in [13] to show that the LLN $\lim_{n\uparrow\infty} Z_k(n)/n = a_k$ holds for $k \geq 1$. In this context, a contribution of Theorem 2.3 is to give a proof of this limit.

The fluid limit argument given seems of potential use in other nonlinear preferential attachment schemes. In particular, the approach should hold for models where at each time only a finite number of vertices/edges or balls/urns are added. In this case, the differences $d_k(j+1)$ are still uniformly bounded in k, j and the paths $X_k^n(t)$ will be Lipschitz, a primary ingredient in the proof.

In addition, although (SUB) excludes the "linear weights" case $w(k) = k + m$ for $k \geq 1$ and $m > -1$, since in this case S_n acts as an affine function of n, and the corresponding ODEs (2.7) can be uniquely integrated (cf. Corollary 1.7 in [12]), a similar fluid limit argument yields yet another proof in this situation.

3. Proof of Theorem 2.3.

We will assume Theorem 2.1, proved in the next Section, and prove Theorem 2.3 in several steps.

Step 1. Since $d_k(j)$ is uniformly bounded, $\|d_k(j)\|_{L^\infty} \leq 2$, we have, for each realization of the evolving scheme, that X_k^n are Lipschitz functions with constant 2 for all $k \geq 1$ and $n \geq 1$. Since $X_k^n(0) = c_k^n$ converges to c_k, by equicontinuity and local compactness of $[0, \infty)$, we may take a diagonal subsequence n_m so that $X_k^{n_m}$ converges uniformly for t in compact subsets of $[0, \infty)$ to a Lipschitz function φ_k with constant 2, for each $k \geq 1$, which may depend on the realization:

$$\lim_{m\uparrow\infty} \sup_{t\in[0,1]} \left| X_k^{n_m}(t) - \varphi_k(t) \right| \;=\; 0.$$

Step 2. With respect to the graph model, as the total degree increments by 2 at each time, we have $\sum_{k\geq 1} k Z_k^n(n) = n\tilde{c}^n + 2n$. On the other hand, in the urn scheme, the total number of balls increases by 1 at each time, and so the total size $\sum_{k\geq 1} k Z_k(n) = n\tilde{c}^n + n$. Hence, in both models, given $\sum_{k\geq 1} k X_k^n(t) \leq \tilde{c}^n + 2t$, we have, for each $L \geq 1$, that

$$\sum_{k\geq L} w(k) X_k^{n_m}(t) \;\leq\; \big[\sup_{k\geq L} w(k)/k\big] \sum_{k\geq 1} k X_k^{n_m}(t) \;\leq\; \big[\sup_{k\geq L} w(k)/k\big](\tilde{c}^n + 2t). \quad (3.1)$$

Therefore,

$$\sum_{k\leq L} w(k) X_k^{n_m}(t) \;\leq\; \mathcal{S}^{n_m}(t) \;\leq\; \sum_{k\leq L} w(k) X_k^{n_m}(t) + (\tilde{c}^n + 2t)[\sup_{k>L} w(k)/k]$$

and also, for $N > 0$, noting (LIM),

$$\lim_{n_m \uparrow \infty} \sup_{t\in[0,N]} \big|\mathcal{S}^{n_m}(t) - \sum_{k\le L} w(k)\varphi_k(t)\big| \ \le\ (\tilde{c}+2N)[\sup_{k>L} w(k)/k].$$

In addition, by Fatou's lemma, from (3.1), we obtain $\sum k\varphi_k(t) \le \tilde{c} + 2t$. Then,

$$\sup_{t\in[0,N]} \sum_{k>L} w(k)\varphi_k(t) \ \le\ \big[\sup_{k>L} w(k)/k\big] \sup_{t\in[0,N]} \sum_{k\ge 1} k\varphi_k(t) \ \le\ (\tilde{c}+2N)\big[\sup_{k>L} w(k)/k\big].$$

Putting together these estimates, we have for each $L \ge 1$ that

$$\lim_{n_m \uparrow \infty} \sup_{t\in[0,N]} \big|\mathcal{S}^{n_m}(t) - \sum_{k\ge 1} w(k)\varphi_k(t)\big| \ \le\ 2(\tilde{c}+2N)[\sup_{k>L} w(k)/k].$$

Therefore, by assumption (SUB), taking $L \uparrow \infty$, we have

$$\mathcal{S} := \lim_{m\uparrow\infty} \mathcal{S}^{n_m} \ = \ \sum_{k\ge1} w(k)\varphi_k$$

converges uniformly for $t \in [0, N]$. Since $\{\mathcal{S}^{n_m}\}$ are continuous functions, we see also that $\mathcal{S}$ is a continuous function.

Step 3. We now derive bounds for the limit function $\mathcal{S}$. Under (SUB) and (LIM), in both models, given the bound $(\tilde{c} + 2)n$ on the total degree/size of the network at time n, we have

$$\mathcal{S}^n(t) \ = \ \sum_{k\ge1} w(k)X^n_k(t) \ \le\ \mathcal{W}\sum_{k\ge1} kX^n_k(t) \ \le\ (\tilde{c}+2t)\mathcal{W}.$$

Also, in both models, for $L \ge 1$, we have

$$\begin{aligned}
\mathcal{S}^n(t) \ &\ge\ \big(\inf_{k\le L} w(k)\big)\sum_{k\le L} X^n_k(t)\\
&\ge\ \big(\inf_{k\le L} w(k)\big)\Big[\sum_{k\ge1} X^n_k(t) - \frac{1}{L+1}\sum_{k\ge1} kX^n_k(t)\Big]\\
&\ge\ \big(\inf_{k\le L} w(k)\big)\Big[\sum_{k\ge1} X^n_k(t) - \frac{\tilde{c}+2t}{L+1}\Big].
\end{aligned}$$

Since, in both models, at time $n \ge 1$, the number of vertices/urns at time n equals nc^n plus the sum of n independent Bernoulli(p) variables, we have $\sum_{k\ge1} X^{n_m}_k(t) \to c + pt$ a.s. Therefore, with $\hat{L}$ such that $\tilde{c}/(\hat{L} + 1) \le c/2$ and $2/(\hat{L} + 1) \le p/2$, we conclude from the above estimates that

$$2^{-1}(c+pt)\big(\inf_{k\le \hat{L}} w(k)\big) \ \le\ \mathcal{S}(t) \ \le\ (\tilde{c}+2t)\mathcal{W}.$$

Step 4. From the martingale decomposition (2.3), we have, for $k \ge 1$, that

$$X^n_k(t) - X^n_k(0) = M^n_k(\lfloor nt\rfloor) + \frac{1}{n}\sum_{j=0}^{\lfloor nt\rfloor -1} E\big[d^n_k(j+1)\big|\mathcal{F}_j\big] + \frac{nt-\lfloor nt\rfloor}{n} d_k(\lfloor nt\rfloor + 1)$$

where

$$M^n_k(\ell) \ = \ \frac{1}{n}\sum_{j=0}^{\ell-1}\Big(d^n_k(j+1) - E\big[d^n_k(j+1)\big|\mathcal{F}_j\big]\Big)$$

is a martingale with respect to $\{\mathcal{F}_\ell : \ell \geq 0\}$, and, for the urn scheme,

$$E\big[d_k^n(j+1)\big|\mathcal{F}_j\big] = \begin{cases} p - (1-p)\frac{w(1)X_1^n(j/n)}{\mathcal{S}^n(j/n)} & \text{for } k=1 \\ \frac{(1-p)w(k-1)X_{k-1}^n(j/n)}{\mathcal{S}^n(j/n)} - \frac{(1-p)w(k)X_k^n(j/n)}{\mathcal{S}^n(j/n)} & \text{for } k \geq 2 \end{cases}$$

and, for the graph model, after calculating with $\{d_k^n\}$ in the Appendix,

$$E\big[d_k^n(j+1)\big|\mathcal{F}_j\big] = \begin{cases} p - (2-p)\frac{w(1)X_1^n(j/n)}{\mathcal{S}^n(j/n)} + \frac{1-p}{n}\Big\{\frac{[w(1)]^2 X_1^n(j/n)}{[\mathcal{S}^n(j/n)]^2}\Big\} & \text{for } k=1 \\ \frac{(2-p)w(1)X_1^n(j/n)}{\mathcal{S}^n(j/n)} - \frac{(2-p)w(2)X_2^n(j/n)}{\mathcal{S}^n(j/n)} + \frac{1-p}{n}\Big\{\frac{-2[w(1)]^2 X_1^n(j/n)}{[\mathcal{S}^n(j/n)]^2} + \frac{[w(2)]^2 X_2^n(j/n)}{[\mathcal{S}^n(j/n)]^2}\Big\} & \text{for } k=2 \\ \frac{(2-p)w(k-1)X_{k-1}^n(j/n)}{\mathcal{S}^n(j/n)} - \frac{(2-p)w(k)X_k^n(j/n)}{\mathcal{S}^n(j/n)} + \frac{1-p}{n}\Big\{\frac{[w(k-2)]^2 X_{k-2}^n(j/n)}{[\mathcal{S}^n(j/n)]^2} + \frac{-2[w(k-1)]^2 X_{k-1}^n(j/n)}{[\mathcal{S}^n(j/n)]^2} + \frac{[w(k)]^2 X_k^n(j/n)}{[\mathcal{S}^n(j/n)]^2}\Big\} & \text{for } k \geq 3, \end{cases}$$

Step 5. Let $\langle M_k^n(j)\rangle$ be the quadratic variation of $M_k^n(j)$. In our context, noting $|d_k^n(j)| \leq 2$ is bounded, we have

$$|\langle M_k^n(\lfloor nt \rfloor)\rangle| \;=\; \frac{1}{n^2}\sum_{j=0}^{\lfloor nt\rfloor -1}\big(d_k^n(j+1) - E[d_k^n(j+1)|\mathcal{F}_j]\big)^2 \;\leq\; Ctn^{-1}.$$

Therefore, for $\epsilon > 0$, by Burkholder-Davis-Gundy inequalities, we have

$$\begin{aligned} P\Big(\sup_{s\in[0,N]} |M_k^n(\lfloor ns \rfloor)| > \epsilon\Big) \;&\leq\; \frac{1}{\epsilon^4}E\Big[\max_{0\leq j\leq \lfloor nN\rfloor} |M_k^n(j)|^4\Big] \\ &\leq\; CE\Big[\langle M_k^n(\lfloor nN \rfloor)\rangle^2\Big] \;\leq CN^2 n^{-2}. \end{aligned}$$

Then, by Borel-Cantelli lemma, $\lim_{n\uparrow\infty}\sup_{t\in[0,N]}|M_k^n(\lfloor nt\rfloor)| = 0$ a.s.

Step 6. To obtain an integral equation, from the development in Step 4, since $X_k^n(0) = c_k^n \to c_k$ by (LIM) and $n^{-1}d_k(\lfloor nt \rfloor + 1)$, $M_k^n(\cdot)$ vanish uniformly a.s., we need only evaluate the limit of

$$\frac{1}{n_m}\sum_{j=0}^{\lfloor n_m t\rfloor -1} E\big[d_k^{n_m}(j+1)\big|\mathcal{F}_j\big] \;=\; \int_0^t E\big[d_k^{n_m}(\lceil n_m s\rceil)\big|\mathcal{F}_{\lfloor n_m s\rfloor}\big]ds.$$

By Steps 2 and 3, given uniform convergence of $X^{n_m}(s) \to \varphi_k(s)$ and $\mathcal{S}^{n_m}(s) \to \mathcal{S}(s)$ for $s \in [0, N]$, and positivity of $\mathcal{S}^{n_m}(s)$ and $\mathcal{S}(s)$ for $s > 0$, in both models, we have for $0 < s \leq N$,

$$\lim_{m\uparrow\infty} E\big[d_k^{n_m}(\lceil n_m s\rceil)\big|\mathcal{F}_{\lfloor n_m s\rfloor}\big] = = \begin{cases} p_0 - q_0\frac{w(1)\varphi_1(s)}{\mathcal{S}(s)} & \text{for } k=1 \\ \frac{q_0}{\mathcal{S}(s)}\big[w(k-1)\varphi_{k-1}(s) - w(k)\varphi_k(s)\big] & \text{for } k\geq 2. \end{cases}$$

Given the pointwise bound $|d_k^{n_m}| \le 2$, by dominated convergence, as $n_m \uparrow \infty$, we conclude $\{\varphi_k : k \ge 1\}$ satisfies the integral equation corresponding to (2.7) with initial condition $\varphi_k(0) = c_k$ for $k \ge 1$.

Finally, as $\{\varphi_k(t)\}$ is nonnegative by construction, by Theorem 2.1, we conclude it is the unique solution to the ODEs. Moreover, under small initial configurations $c_k \equiv 0$, $\varphi_k(t) = a_k(p_0, q_0, s^*)t$ for $k \ge 1$. Hence, as this is the unique limit family $\{\varphi_k : k \ge 1\}$ for each realization in a full probability set, the whole sequence $\{X_k^n : n \ge 1\}$ a.s. must converge uniformly to this solution for $k \ge 1$. □

4. Proof of Theorem 2.1.

After several steps in the form of successive propositions, we complete the proof Theorem 2.1 at the end of the Section. The condition (SUB) will be assumed throughout.

As noted, in Chapter VI in [22], 'semigroups are everywhere', and are useful in many applications. In this vein, our strategy is to exploit the properties of a semigroup associated to the transformed ODEs (2.8). However, our semigroup is neither compact nor 'eventually' compact (see Proposition 6.1 in the Appendix for a proof), properties often useful in the study of population evolutions (cf. Section VI.1 in [22] and references therein).

Also, although the transformed ODEs (2.8) do not fit into the general theoretical framework of linear 'Kolmogorov' differential systems, recently considered in certain host patch/parasite models [25], [26], [7], nevertheless one can use this framework to show the semigroup associated to (2.8) is strongly continuous and to help estimate its growth rate.

Moreover, we show the semigroup is positive and after a shift also 'quasicompact', a statement about its 'essential' growth rate and one allowing certain spectral decompositions and asymptotic analysis (see [4] for a discussion of 'quasicompactness' with respect to ergodic theory). Through semigroup 'Perron-Frobenius' results, we specify further the evolution in terms of a finite-dimensional motion and a part corresponding to spectra in a left-half plane. For large initial configurations, such a decomposition is enough to capture the asymptotic growth of the ODE solutions.

However, under small initial configurations, because of the time singularity at $t = 0$, identification of the solution requires more work. By a time-reversal argument, we show the only part of the evolution, consistent with nonnegativity of the solution and the initial condition, corresponds to motion in terms of a dominant eigenvalue-eigenvector pair, leading to the desired characterization.

With respect to a nonnegative solution of the integral form of the nonlinear ODEs (2.7), define for $t \ge 0$ the nonnegative functions

$$V(t) = \sum_{k\ge1} \varphi_k(t), \quad T(t) = \sum_{k\ge1} w(k)\varphi_k(t), \quad \text{and} \quad D(t) = \sum_{k\ge1} k\varphi_k(t).$$

We now derive properties of the functions V, T and D, representing the scaled vertices/urns, weight, and degree/size of the system respectively. Recall $p_0, q_0 > 0$.

Proposition 4.1. *Suppose (SUB) holds. For $t \ge 0$, $T(\cdot)$ is continuous. Also, $V(t) = c + p_0 t$, $D(t) \le \tilde{c} + (p_0 + q_0)t$, and in addition there is a constant $C_0 > 0$ such that*

$$C_0^{-1}[c + p_0 t] \ \le\ T(t) \ \le\ C_0[\tilde{c} + (p_0 + q_0)t].$$

Proof. We first consider $D(t)$. From the ODEs (2.7), write for $L \geq 1$ that

$$\begin{aligned}\sum_{k=1}^{L} k\varphi_k(t) &= \sum_{k=1}^{L} kc_k + \sum_{k=1}^{L} k\big(\varphi_k(t) - \varphi_k(0)\big) \qquad (4.1)\\ &= \sum_{k=1}^{L} kc_k + \ p_0 t + q_0 \int_0^t \sum_{k=1}^{L-1} \frac{w(k)\varphi_k(u)}{T(u)} du - q_0 L \int_0^t \frac{w(L)\varphi_L(u)}{T(u)} du.\end{aligned}$$

Hence, by (LIM), dropping the last negative term, $D(t) \leq \tilde{c} + (p_0 + q_0)t$ as desired.

Considering $V(t)$ now, write

$$\sum_{k=1}^{L} \varphi_k(t) = \sum_{k=1}^{L} c_k + \ p_0 t - q_0 \int_0^t \frac{w(L)\varphi_L(u)}{T(u)} du.$$

Since $\int_0^t \sum_{k\geq 1} w(k)\varphi_k(u)/T(u)du = t < \infty$, the last integral above vanishes as $L \uparrow \infty$. Hence, by (LIM), $V(t) = c + p_0 t$.

Also, the lower and upper bounds on $T(t)$ follow from the argument as given in Step 3 of the proof of Theorem 2.3 in Section 3. The constant C_0 can be taken as $C_0 = \max\{[(1/2)\inf_{1\leq k\leq \hat{L}} w(k)]^{-1}, \mathcal{W}\}$ where $\hat{L}$ is the smallest integer satisfying $\tilde{c}/(\hat{L}+1) \leq c/2$ and $2/(\hat{L}+1) \leq p_0/2$.

To show T is continuous, write

$$T(t) - T(s) = \sum_{k=1}^{L} w(k)\big(\varphi_k(t) - \varphi_k(s)\big) + \sum_{k>L} w(k)\varphi_k(t) - \sum_{k>L} w(k)\varphi_k(s).$$

The last two terms, for large L, are small by the inequality $\sum_{k\geq L} w(k)\varphi_k(u) \leq [\sup_{j>L} w(j)/j]D(u)$ and (SUB). Now, as $\{\varphi_k\}$ satisfies the integral form of the ODEs (2.7), they are continuous functions. Hence, one sees T is also continuous. □

We now analyze more carefully the time scale $t = t(s)$ and associated system $\{\psi_k(s) = \varphi_k(t(s))\}$ mentioned above the statement of Theorem 2.1. Recall

$$\dot{t}(s) = T(t) \quad \text{and} \quad t(0) = 1.$$

Since T is continuous by Proposition 4.1, a solution $t = t(s)$ exists. Also, given the bounds on T in Proposition 4.1, by comparison estimates, we have for $s \geq 0$ that

$$\begin{aligned}&t(0)e^{C_0^{-1}p_0 s} + \frac{c}{p_0}\big[e^{C_0^{-1}p_0 s} - 1\big] \qquad (4.2)\\ &\quad\leq\ t(s)\ \leq\ t(0)e^{C_0(p_0+q_0)s} + \frac{\tilde{c}}{(p_0+q_0)}\big[e^{C_0(p_0+q_0)s} - 1\big],\\ &t(0)e^{-C_0(p_0+q_0)s} + \frac{\tilde{c}}{(p_0+q_0)}\big[e^{-C_0(p_0+q_0)s} - 1\big]\\ &\quad\leq\ t(-s)\ \leq\ t(0)e^{-C_0^{-1}p_0 s} + \frac{c}{p_0}\big[e^{-C_0^{-1}p_0 s} - 1\big].\end{aligned}$$

In addition, as $T(t) > 0$ for $t > 0$, $t(s)$ is a strictly increasing, invertible function of s. Then, under small initial configurations $c_k \equiv 0$, as $s \downarrow -\infty$, we have $t(s) \downarrow 0$. Under large initial conditions $c > 0$, there is an $-\infty < s_0 < 0$ where $t(s_0) = 0$.

The system $\psi_k(s) = \varphi_k(t(s))$ for $k \geq 1$ obeys the integral form of ODEs (2.8), with boundary conditions, under small initial configurations, $\lim_{s\downarrow-\infty} \psi_k(s) = 0$ and, under large initial configurations, $\psi_k(s_0) = c_k$, for $k \geq 1$. Also, given $\varphi_k(\cdot) \geq 0$, of course, in the corresponding time-ranges $\psi_k(\cdot) \geq 0$ for $k \geq 1$.

With $\Psi = \langle \psi_k : k \geq 1\rangle$,

$$\dot{\Psi} \;=\; A\Psi \tag{4.3}$$

where

$$A \;=\; \begin{pmatrix} (p_0 - q_0)w(1) & p_0 w(2) & p_0 w(3) & \cdots \\ q_0 w(1) & -q_0 w(2) & 0 & \cdots \\ 0 & q_0 w(2) & -q_0 w(3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It will be convenient to write $A = B + K$ where

$$B \;=\; \begin{pmatrix} -q_0 w(1) & 0 & 0 & \cdots \\ q_0 w(1) & -q_0 w(2) & 0 & \cdots \\ 0 & q_0 w(2) & -q_0 w(3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$K \;=\; \begin{pmatrix} p_0 w(1) & p_0 w(2) & p_0 w(3) & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For a vector $x = \langle x_k : k \geq 1\rangle$ where $x_k \in \mathbb{R}$, define the norm $\|x\| = \sum_{k\geq 1} k|x_k|$ and the Banach lattice (cf. Section VI.1b in [22])

$$\Omega \;=\; \big\{x = \langle x_k : k \geq 1\rangle : \|x\| < \infty\big\}.$$

Let ℓ_c be the space of compactly supported vectors, and note $\ell_c \subset \Omega$. The operators A, B and K are well-defined on ℓ_c. Moreover, $A\ell_c \subset \ell_c$, $B\ell_c \subset \ell_c$ and $K\ell_c \subset \ell_c$, and hence A, B, K are densely defined.

Moreover, $K : \Omega \to \Omega$ is a bounded operator: As $w(k) \leq \mathcal{W}k$ for $k \geq 1$ (SUB), the bound $\|K\| \leq p_0\mathcal{W}$ may be computed. Also, since K is a bounded, rank 1 operator on Ω, K is in addition compact.

The operator B with domain ℓ_c is closable: Indeed, let $\{x^n\} \subset \ell_c$ so that $x^n \to 0$ and $Bx^n \to y$ in Ω. Since every row in B is in ℓ_c, we have $(Bx^n)_k \to 0$ for $k \geq 1$. Since projections $\pi_k : \Omega \to \mathbb{R}$ where $\pi_k(x) = x_k$ is continuous, we have $\pi_k(Bx^n) \to \pi_k(y) = 0$ for $k \geq 1$. Hence, $y = 0$.

As a consequence, (A, ℓ_c) is closable. We will denote the closures of A and B by the same names as it will not cause confusion in what follows.

Now, we observe the ODEs associated to B, $\dot{\zeta} = B\zeta$, fall into the framework of the 'Kolmogorov' differential equations considered in [26]. Indeed, given $\sup_k w(k)/k \leq \mathcal{W}$, in the notation of [26], with $\alpha_{k,k} = -q_0 w(k+1)$, $\alpha_{k+1,k} = q_0 w(k+1)$ for $k \geq 0$, $\alpha_{j,k} = 0$ otherwise, $\alpha^\diamond = \sup_k \sum_{j=0}^\infty \alpha_{j,k} = 0$, $c_0 = 2q_0\mathcal{W}$, $c_1 = 2q_0\mathcal{W}$, $\epsilon = 1$ and $\omega = c_1 \vee (\alpha^\diamond + c_0) = c_0$, one inspects $\sum_{j=1}^\infty j\alpha_{j,k} = q_0 w(k+1)$, and the B-ODE system satisfies Assumptions 1, 2 in [26]. We note the full statement of (SUB) is not used in this verification.

Proposition 4.2. *Both A and B generate strongly continuous semigroups $P_t, P_t^B : \Omega \to \Omega$ with bounds $\|P_t\| \leq e^{(2q_0\mathcal{W} + \|K\|)t}$ and $\|P_t^B\| \leq e^{2q_0\mathcal{W}t}$ for $t \geq 0$ respectively.*

Proof. By Theorem 2 in [26] there is a strongly continuous semigroup $P_t^B : \Omega \to \Omega$, generated by the part of B restricted to domain $D(B) = \{x \in \Omega : \sum_k w(k)|x_k| < \infty, Bx \in \Omega\}$, with bound $\|P_t^B\| \leq e^{\omega t}$.

Moreover, by the perturbation Theorem III.1.3 in [22], as K is bounded, $A = B + K$ with domain $D(A) = D(B)$ generates a strongly continuous semigroup $P_t : \Omega \to \Omega$ with bound $\|P_t\| \leq e^{(\omega+\|K\|)t}$. □

Recall that a strongly continuous semigroup $P_t^E : \Omega \to \Omega$ is positive if $(P_t^E x)_k \geq 0$ for $k \geq 1$ when $x \in \Omega$ and $x_k \geq 0$ for all $k \geq 1$ (cf. Section VI.1b in [22]).

Below, in Proposition 4.3, we show the semigroups generated by B and A are positive. In passing, we remark in fact P_t, although not P_t^B, is irreducible, that is $[(\lambda I - A)^{-1}x]_k > 0$ for $k \geq 1$ when $x_k \geq 0$ for all $k \geq 1$ but $x \neq 0$ (cf. Section VI.1b in [22]). Indeed, from the ODEs (4.3) and a calculation left to the reader, $(P_s x)_k = \psi_k(s) > 0$ for $k \geq 1$ and $s > 0$. Then, $(\lambda I - A)^{-1}x = \int_0^\infty e^{-\lambda s} P_s x ds$ is composed of positive entries. We will not need this stronger result in the following.

Proposition 4.3. *The semigroups P_t and P_t^B are both positive.*

Proof. We show P_t is positive; the same argument also proves P_t^B is positve. Since ℓ_c is a core of A, we can calculate $P_t x$ for $x \in \ell_c$ such that $x_k \geq 0$ for $k \geq 1$ by the equation $(d/dt)P_t x = AP_t x$, in other words ODEs (4.3), and initial condition $P_0 x = x$ (cf. Lemma II.1.3 in [22]). As $q_0 > 0$, all off-diagonal entries in A are nonnegative. Hence, inspection of these ODEs reveals that $P_t x \geq 0$. Now, for $x \in \Omega$ and $x_k \geq 0$ for $k \geq 1$, take $x_n \in \ell_c$ so that $(x_n)_k \geq 0$ for $k \geq 1$ and $x_n \to x$ in Ω. Since, for fixed $t \geq 0$, P_t is bounded (cf. Proposition 4.2), $P_t x_n \to P_t x$ in Ω. Hence, $P_t x \geq 0$. □

The growth rate $w_0(E)$ of a semigroup P_t^E is $w_0(E) = \lim_{t\uparrow\infty} t^{-1} \log \|P_t^E\|$ (cf. Proposition IV.2.2 in [22]). Also, the essential growth rate $w_{\rm ess}(E)$ of P_t^E is $w_{\rm ess}(E) = \lim_{t\uparrow\infty} t^{-1} \log \|P_t^E\|_{\rm ess}$ where $\|P_t^E\|_{\rm ess} = \inf\{\|P_t^E - M\| : M \text{ compact}\}$ (cf. Proposition IV.2.10 in [22]). In particular, inputting $M \equiv 0$, we obtain

$$w_{\rm ess}(E) \ \leq \ w_0(E).$$

Proposition 4.4. *We have that $w_0(B) \leq 0$.*

Proof. Again, since the B-ODEs satisfy Assumptions 1,2 in [26], by Theorem 4 in [26], we have

$$w_0(B) \ \leq \ \alpha^\diamond \vee \limsup_{k\to\infty} \sum_{j=1}^\infty j\alpha_{j,k}/k \ = \ 0,$$

recalling $\alpha^\diamond = 0$ and $\limsup_{k\uparrow\infty} \sum_{j=1}^\infty j\alpha_{j,k}/k = \lim_{k\uparrow\infty} q_0 w(k+1)/k = 0$. □

We now show for all small $\varepsilon > 0$ that $e^{-\varepsilon t}P_t$ is a quasi-compact semigroup, that is the essential growth rate $w_{\rm ess}(A - \varepsilon I) < 0$. This is one characterization of being 'quasicompact' (cf. Proposition V.3.5 in [22]). Such semigroups have nice representations which we will leverage later on.

Proposition 4.5. *For all small $\varepsilon > 0$, the semigroup $e^{-\varepsilon t}P_t$ is quasi-compact.*

Proof. We will show that $e^{-\varepsilon t}P_t^B$, the semigroup generated by $B - \varepsilon I$, is quasi-compact. Then, by the perturbation result Proposition V.3.6 in [22], as K is a compact operator, $e^{-\varepsilon t}P_t$ the semigroup generated by $A - \varepsilon I = B + K - \varepsilon I$ is also quasi-compact.

As stated in Proposition V.3.5 in [22], for a strongly continuous semigroup, quasi-compactness is equivalent to the essential growth rate of the semigroup being strictly negative. We will apply this characterization to $B - \varepsilon I$. Since $w_0(B) \leq 0$ by Proposition 4.4, we have $w_{\rm ess}(B - \varepsilon I) \leq w_0(B - \varepsilon I) = w_0(B) - \varepsilon < 0$. □

Now, by the quasi-compact semigroup representation Theorem V.3.7 in [22] applied to $e^{-\varepsilon t}P_t$ for $\varepsilon > 0$, there are only a finite number m of spectral values z, if any, of $A - \varepsilon$, and each of these is a pole of the resolvent $R(\cdot, A-\varepsilon)$ with finite algebraic multiplicity. Moreover, when $m \geq 1$, we may write for $t \geq 0$ that

$$e^{-\varepsilon t}P_t \;=\; \sum_{r=1}^{m} U_r(t) + R(t). \tag{4.4}$$

Here, with respect to the rth pole λ_r with multiplicity k_r and spectral projection Q_r (cf. Proposition IV.1.16 in [22]),

$$U_r(t) \;=\; e^{\lambda_r t}\sum_{j=0}^{k_r-1}\frac{t^j}{j!}(A-(\varepsilon+\lambda_r)I)^j Q_r.$$

Also, Theorem V.3.7 in [22] states $\|R(t)\| \leq Me^{-\beta t}$ for some $\beta > 0$ and $M \geq 1$.

In effect, $e^{-\varepsilon t}P_t$ acts as a finite-dimensional operator on $\mathrm{Range}(Q_r)$ and leaves it invariant for $1 \leq r \leq m$. In particular, $e^{-\varepsilon t}P_t$ and $\{Q_r\}$ commute and

$$e^{-\varepsilon t}P_tQ_r = U_r(t) \;\text{ and }\; R_t = e^{-\varepsilon t}P_t\Big[I - \sum_{r=1}^{m} Q_r\Big]. \tag{4.5}$$

Let now $\sigma(E)$ be the spectrum of a generator E on Ω. The largest real part of the spectrum is denoted $s(E) = \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(E)\}$.

To make use of this representation, we now examine the spectrum of A in a right half plane. A goal in the next propositions is to show that $s(A)$ is positive and a simple eigenvalue. Also, we derive the form of its eigenvector.

Proposition 4.6. *The generator A has only one real eigenvalue in the strict right half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, and it has an eigenvector with all positive entries. As a consequence, $s(A) > 0$.*

Proof. We solve $Ax = \lambda x$ for $\lambda > 0$. We have

$$\begin{aligned}\lambda x_1 &= (p_0 - q_0)x_1 + p_0\sum_{k\geq 2} w(k)x_k\\ \lambda x_k &= q_0\big\{w(k-1)x_{k-1} - w(k)x_k\big\} \quad \text{for } k \geq 2.\end{aligned}$$

This gives, for $k \geq 2$,

$$x_k \;=\; x_1\prod_{r=2}^{k}\frac{q_0 w(r-1)}{\lambda + q_0 w(r)}, \tag{4.6}$$

the same equations for $a_k(p_0, q_0, \lambda)$ (cf. (2.5)).

In particular, by (SUB), a calculation shows that $x \in \Omega$, and

$$\sum_{k\geq 2} w(k)x_k \;=\; x_1 w(1)\sum_{k\geq 2}\prod_{r=2}^{k}\frac{q_0 w(r)}{\lambda + q_0 w(r)}$$

converges for $\lambda > 0$. Hence, plugging into the equation involving x_1 above,

$$\lambda \;=\; p_0 - q_0 + p_0 w(1)\sum_{k\geq 2}\prod_{r=2}^{k}\frac{q_0 w(r)}{\lambda + q_0 w(r)}. \tag{4.7}$$

The left side of the equation (4.7) is strictly increasing in λ, whereas the right-side is strictly decreasing in λ. Also, the right-side of (4.7) diverges to infinity as $\lambda \downarrow 0$.

We conclude therefore there is exactly one $\lambda > 0$ which satisfies (4.7). This λ is the desired unique real eigenvalue, with positive eigenvector x when $x_1 > 0$. $\square$

Proposition 4.7. *For $0 \le \varepsilon < s(A)$, $s(A-\varepsilon I) > 0$ is the only real eigenvalue of $A-\varepsilon I$ in the strict right-half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. All other eigenvalues λ of $A - \varepsilon I$, if they exist, satisfy $\mathrm{Re}(\lambda) < s(A-\varepsilon I)$.*

Proof. First, for $\varepsilon > 0$, as $e^{-\varepsilon t}P_t$ is quasi-compact (Proposition 4.5), as noted above there are only a finite number of spectral values of $A - \varepsilon I$ in the right half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) \ge 0\}$, and these are all eigenvalues. In particular, there are only a finite number of spectral values/eigenvalues of A in the half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) \ge \varepsilon\}$.

Then, as $s(A) > 0$ (Proposition 4.6), and by positivity of P_t (Proposition 4.3) and the 'Perron-Frobenius' type Theorem VI.1.10 in [22], $s(A)$ is an eigenvalue of A and, by Proposition 4.6, the only real one in the strict right-half plane. Moreover, with $\varepsilon = s(A)/2$, as there are only a finite number of eigenvalues z of A with real part $\mathrm{Re}(z) \ge s(A)/2$, by another 'Perron-Frobenius' type Theorem VI.1.12(i) in [22], the boundary spectrum of A must be a singleton. Hence, any other eigenvalue z of A satisfies $\mathrm{Re}(z) < s(A)$.

Then, for all $0 \le \varepsilon < s(A)$, $s(A - \varepsilon I) = s(A) - \varepsilon$ is the only real eigenvalue of $A-\varepsilon I$ in the strict right half-plane, and all other eigenvalues have real part strictly less than $s(A-\varepsilon I)$. $\square$

Define now the dual space

$$\Omega' \;=\; \big\{z : \text{ There exists } C \text{ such that } |z_k| \le Ck \text{ for all } k \ge 1\big\}$$

and $\|z\|_{\Omega'}$ is the smallest such constant C. It will be helpful now to find an eigenvector of

$$A^* \;=\; \begin{pmatrix} (p_0-q_0)w(1) & q_0w(1) & 0 & 0 & \cdots \\ p_0w(2) & -q_0w(2) & q_0w(2) & 0 & \cdots \\ p_0w(3) & 0 & -q_0w(3) & q_0w(3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

with positive entries.

Proposition 4.8. *There exists an eigenvector $x^* \in \Omega'$ of A^*, with all entries positive, corresponding to a real eigenvalue $\lambda^* > 0$. Moreover, λ^* can be taken $\lambda^* = s^*$ where we recall s^* solves $1 = F_{p_0,q_0}(s^*)$ (cf. (2.2)).*

Proof. For a possible eigenpair x^*, λ^*, we obtain equations

$$\begin{aligned} x_1^* &= \frac{q_0w(1)x_2^*}{\lambda^*+q_0w(1)} + \frac{p_0w(1)x_1^*}{\lambda^*+q_0w(1)} \\ x_k^* &= \frac{q_0w(k)x_{k+1}^*}{\lambda^*+q_0w(k)} + \frac{p_0w(k)x_1^*}{\lambda^*+q_0w(k)} \quad \text{for } k \ge 2. \end{aligned} \tag{4.8}$$

Note, by (SUB), the sum

$$\sum_{k\ge 2} \frac{p_0w(k)}{\lambda^*+q_0w(k)} \prod_{r=1}^{k-1} \frac{q_0w(r)}{\lambda^*+q_0w(r)}$$

converges for each $\lambda^* > 0$. Also, consider the equation

$$1 \;=\; \sum_{k\geq 2} \frac{p_0 w(k)}{\lambda^* + q_0 w(k)} \prod_{r=1}^{k-1} \frac{q_0 w(r)}{\lambda^* + q_0 w(r)} + \frac{p_0 w(1)}{\lambda^* + q_0 w(1)}, \tag{4.9}$$

which is the same as $1 = F_{p_0,q_0}(\lambda^*)$ and identifies, as concluded in (2.2), $\lambda^* = s^*$.

Iterating (4.8), we may solve

$$\begin{aligned} x_1^* \;&=\; \lim_{N\uparrow\infty} x_{N+1}^* \prod_{r=1}^{N} \frac{q_0 w(k)}{\lambda + q_0 w(k)} \\ &\quad + x_1^* \sum_{k\geq 2} \frac{p_0 w(k)}{\lambda + q_0 w(k)} \prod_{r=1}^{k-1} \frac{q_0 w(r)}{\lambda + q_0 w(r)} + x_1^* \frac{p_0 w(1)}{\lambda + q_0 w(1)}. \end{aligned}$$

With $\lambda^* = s^*$, necessarily, noting (4.9), $\lim_{N\uparrow\infty} x_{N+1}^* \prod_{r=1}^{N} \frac{q_0 w(k)}{\lambda^* + q_0 w(k)} = 0$.

In this case, for $j \geq 2$, with convention $\prod_{r=j+1}^{j} \cdot = 1$,

$$\begin{aligned} x_j^* \;&=\; x_1^* \sum_{k\geq j} \frac{p_0 w(k+1)}{\lambda + q_0 w(k+1)} \prod_{r=j}^{k} \frac{q_0 w(r)}{\lambda + q_0 w(r)} + x_1^* \frac{p_0 w(j)}{\lambda + q_0 w(j)} \\ &=\; \frac{x_1^* w(j)}{\lambda + q_0 w(j)} \Big[\sum_{k\geq j} \frac{p_0 w(k+1)}{\lambda + q_0 w(k+1)} \prod_{r=j+1}^{k} \frac{q_0 w(r)}{\lambda + q_0 w(r)} + p_0 \Big]. \end{aligned}$$

Again, by (SUB), one sees that $|x_j^*| \leq Cj$ for a uniform constant C for $j \geq 1$. In particular, the eigenvector $x^* \in \Omega'$ and if $x_1^* > 0$, $x_j^* > 0$ for $j \geq 2$. □

Proposition 4.9. *The eigenvalue $s(A)$ of A is simple and moreover $\lambda^* = s(A)$. Also, $x^* \perp w$ for any (generalized) eigenvector w of A other than the one with eigenvalue $s(A)$.*

Proof. Consider the eigenvector x^* with eigenvalue λ^* in Proposition 4.8 consisting of all positive entries when say $x_1^* = 1$. Then, with respect to the positive eigenvector x with eigenvalue $s(A)$ of A in Proposition 4.7, we note $\langle x^*, Ax\rangle = s(A)\langle x^*, x\rangle = \lambda^*\langle x^*, x\rangle$. Since $\langle x^*, x\rangle > 0$, $\lambda^* = s(A)$.

Moreover, suppose there exists a generalized eigenvector w where $(A - s(A)I)w = cx$ for some $c \neq 0$. Then, $\lambda^*\langle x^*, w\rangle = \langle x^*, Aw\rangle = \langle x^*, s(A)w\rangle + c\langle x^*, x\rangle$. Since $\lambda^* = s(A)$ and $\langle x^*, x\rangle > 0$, we must have $c = 0$ which is a contradiction. Hence, $s(A)$ is a simple eigenvalue of A.

Finally, for any eigenvector w of A with eigenvalue $\lambda_w \neq s(A)$, $s(A)\langle x^*, w\rangle = \langle x^*, Aw\rangle = \lambda_w\langle x^*, w\rangle$. Since $\lambda_w \neq s(A)$, we have $x^* \perp w$. If w' is a generalized eigenvector corresponding to eigenvalue λ_w, $(A - \lambda_w)^k w' = cw$ for some power k and constant c. Then, $(s(A) - \lambda_w)^k\langle x^*, w'\rangle = \langle (A^* - \lambda_w)^k x^*, w'\rangle = c\langle x^*, w\rangle = 0$. Again, as $\lambda_w \neq s(A)$, $x^* \perp w'$. □

Consider now, under small initial configurations, the global trajectory $\psi(s) = \langle \psi_k(s) : k \geq 1\rangle$ satisfying the ODEs (4.3) such that $\psi(0) = \phi(1)$ and $\lim_{s\downarrow-\infty} \psi(s) = 0$. To characterize $\psi(s)$, we will need the following estimate.

Lemma 4.10. *Under small initial configurations, for $s < 0$,*

$$Y(s) \;:=\; \|\psi(s)\| \;\leq\; (p_0 + q_0)e^{C_0^{-1}s}$$

where C_0 is the constant in Lemma 4.1.

Proof. From (4.2), applied to small initial configurations ($c = 0$), noting $t(0) = 1$, we have $t(s) \leq e^{C_0^{-1}s}$. Also, note that $D(u) \leq (p_0 + q_0)u$ for $u \geq 0$. Then, $Y(s) = \sum_k k\psi_k(s) = \sum_k k\varphi_k(t(s)) = D(t(s)) \leq (p_0+q_0)t(s) \leq (p_0+q_0)e^{C_0^{-1}s}$. $\square$

Proposition 4.11. *With respect to small initial configurations, we identify $\psi(s) = e^{s(A)s}\psi(0)$ for all $s \in \mathbb{R}$ where $\psi(0)$ is an eigenvector of A with eigenvalue $s(A)$.*

Proof. The argument proceeds in steps.

Step 1. Let $0 < \varepsilon < C_0^{-1}/2$ where C_0 is the constant in Proposition 4.1. Recall the quasi-compact representation of $e^{-\varepsilon t}P_t$ in (4.4). For $u \in \mathbb{R}$, define $\xi(u)$ by the equation

$$e^{-\varepsilon u}\psi(u) \;=\; e^{-\varepsilon u}\sum_{r=1}^m Q_r\psi(u) + \xi(u). \tag{4.10}$$

Then, for $t \geq 0$ and $s \in \mathbb{R}$, on the one hand,

$$e^{-\varepsilon(s+t)}\psi(s+t) \;=\; e^{-\varepsilon(s+t)}\sum_{r=1}^m Q_r\psi(s+t) + \xi(s+t). \tag{4.11}$$

On the other hand, as $R_t = e^{-\varepsilon t}P_t[I - \sum_{r=1}^m Q_r]$ (cf. (4.5)),

$$\begin{aligned} e^{-\varepsilon(s+t)}\psi(s+t) \;&=\; e^{-\varepsilon t}P_t\big(e^{-\varepsilon s}\psi(s)\big) \\ &=\; e^{-\varepsilon t}P_t\big[\sum_{r=1}^m Q_r\big]\big(e^{-\varepsilon s}\psi(s)\big) + R_t\big(e^{-\varepsilon s}\psi(s)\big). \end{aligned}$$

Since $R_t\big(e^{-\varepsilon s}\psi(s)\big) = R_t\xi(s)$, and $e^{-\varepsilon t}P_t$ and $\{Q_r\}$ commute,

$$e^{-\varepsilon(s+t)}\psi(s+t) \;=\; e^{-\varepsilon(s+t)}\sum_{r=1}^m Q_r\psi(s+t) + R_t\xi(s). \tag{4.12}$$

Hence, combining (4.11) and (4.12), we have $\xi(s+t) = R_t\xi(s)$, and with the bound on R_t after (4.4),

$$\|\xi(s+t)\| \;\leq\; \|R(t)\|\|\xi(s)\| \;\leq\; Me^{-\beta t}\|\xi(s)\|.$$

Step 2. We now argue that $\xi(u) = 0$ for all $u \in \mathbb{R}$. First, for $s < 0$, from Lemma 4.10 and $\varepsilon < C_0^{-1}/2$,

$$\|e^{-\varepsilon s}\psi(s)\| \;=\; e^{-\varepsilon s}Y(s) \;\leq\; (p_0+q_0)e^{(C_0^{-1}-\varepsilon)s} \;\leq\; (p_0+q_0)e^{(C_0^{-1}/2)s}. \tag{4.13}$$

Second, from its finite-dimensional form, the operator $e^{-\varepsilon t}P_t|_{\mathrm{Range}(Q_r)}$ is invertible for $t \geq 0$. Denote the inverse on the range of Q_r as

$$e^{\varepsilon t}P_{-t}|_{\mathrm{Range}(Q_r)} \;=\; e^{-\lambda_r t}\sum_{j=0}^{k_r-1}((-t)^j/j!)(A - (\varepsilon+\lambda_r)I)^jQ_r \;=:\; U_r(-t)$$

where U_r is extended to $\mathbb{R}_-$. Then, for $s < 0$, we have

$$e^{\varepsilon s}P_{-s}\sum_{r=1}^m Q_r\big(e^{-\varepsilon s}\psi(s)\big) \;=\; \sum_{r=1}^m Q_r\big(P_{-s}\psi(s)\big) \;=\; \sum_{r=1}^m Q_r\psi(0).$$

Hence, after inverting,

$$e^{-\varepsilon s}\sum_{r=1}^m Q_r\psi(s) \;=\; \sum_{r=1}^m U_r(s)\psi(0). \tag{4.14}$$

Third, with respect to a constant $C = C(\{\lambda_r\}, \{k_r\}, \varphi(1))$, for $s < 0$, from (4.10) and (4.14), and bound (4.13) and $\lambda_r \geq 0$ for $1 \leq r \leq m$,

$$\begin{aligned}\|\xi(s)\| &\leq \|e^{-\varepsilon s}\psi(s)\| + \Big\|\sum_{r=1}^m U_r(s)\psi(0)\Big\| \\ &\leq (p_0 + q_0) + C|s|^{\max_{1\leq r\leq m} k_r - 1}\end{aligned}$$

As a consequence, for fixed $u = s + t$ where $s < 0$ and $t > 0$, as $t \uparrow \infty$, we have

$$\|\xi(u)\| = \|\xi(s+t)\| \leq Me^{-\beta t}\big[(p_0 + q_0) + C|u - t|^{\max k_r - 1}\big] \to 0.$$

Therefore, in equation (4.10), $e^{-\varepsilon s}\psi(u) = \sum_{r=1}^m U_r(u)\psi(0)$ for all $u \in \mathbb{R}$.

Step 3. Recall $\psi(\cdot)$ is assumed nonnegative. We now show that $\psi(u) = e^{s(A)u}\psi(0)$ for all $u \in \mathbb{R}$ where $\lambda = s(A)$ is the simple eigenvalue of A with largest real part. We will also conclude $\psi(0)$ is an eigenvector corresponding to $s(A)$.

Indeed, the eigenvalue λ_r with largest real part is of form $s(A) - \varepsilon > 0$ with a corresponding eigenvector x with all positive entries (cf. Proposition 4.6). Recall x^* the positive eigenvector of A^* with eigenvalue $\lambda^* = s(A)$ and that all (generalized) eigenvectors x^r of $A - \varepsilon I$ corresponding to $\lambda_r \neq s(A) - \varepsilon$ are orthogonal to x^* (cf. Propositions 4.7, 4.8, 4.9).

Let $\{\lambda_r : r \in I_\alpha\}$ be those eigenvalues with the same real part $\mathrm{Re}(\lambda_r) = \alpha$, and I_α the corresponding index set. For $0 \leq j \leq \max_{1\leq r\leq m} k_r - 1$, consider the sum

$$\begin{aligned}A(\alpha, j, s) &:= \sum_{r\in I_\alpha} e^{\lambda_r s}(s^j/j!)(A - (\varepsilon + \lambda_r)I)^j Q_r\psi(0) \\ &= e^{s\alpha}\frac{s^j}{j!}\sum_{r\in I_\alpha} e^{is\mathrm{Im}(\lambda_r)}((A - (\varepsilon + \lambda_r)I)^j Q_r\psi(0).\end{aligned}$$

There are a finite number of nontrivial sums indexed by α, j. Let $\bar\alpha$ be the minimum real part of the eigenvalues $\{\lambda_r\}$ and suppose $\bar\alpha \neq s(A)$, the largest real part. Let $\hat j$ be the maximum of $k_r - 1$ among the eigenvalues λ_r with real part $\bar\alpha$.

Suppose $A(\bar\alpha, \hat j, s) \neq 0$ for some $s \in \mathbb{R}$. We claim we can find integers $\{k_{r,\ell} : r \in I_\alpha\}$ and $n_\ell \geq 1$ where $\lim_{\ell\uparrow\infty} n_\ell = \infty$ and $\max_{r\in I_r} |n_\ell \mathrm{Im}(\lambda_r)/(2\pi) - k_{r,\ell}| \leq n_\ell^{-1/|I_\alpha|}$ for $\ell \geq 1$: Indeed, if $\{\mathrm{Im}(\lambda_r) : r \in I_\alpha\}$ are all rational, this is the case; if one of $\{\mathrm{Im}(\lambda_r) : r \in I_\alpha\}$ is irrational, then Dirichlet's simulataneous Diophantine approximation theorem, Corollary II.1B in [39], implies the claim.

Then, at times $u_\ell = s - n_\ell$ for $\ell \geq 1$, the sum $(e^{u_\ell\bar\alpha}u_\ell{}^{\hat j}/\hat j!)^{-1}A(\bar\alpha, \hat j, u_\ell)$ well approximates $(e^{s\bar\alpha}s^{\hat j}\hat j!)^{-1}A(\bar\alpha, \hat j, s)$ in Ω, and the absolute value $|A(\bar\alpha, \hat j, u_\ell)|$ dominates the magnitudes of all the other sums $A(\alpha, j, u_\ell)$ for $(\alpha, j) \neq (\bar\alpha, \hat j)$ as $\ell \uparrow \infty$.

Therefore,

$$e^{\varepsilon u_\ell}\psi_k(u_\ell) = \Big[\sum_{r=1}^m U_r(u_\ell)\psi(0)\Big]_k \sim A(\bar\alpha, \hat j, u_\ell)_k$$

for $|u_\ell|$ large with respect to components k of $A(\bar\alpha, \hat j, u_\ell)$ which are nonzero. Given $x^* \perp x^r$ for any generalized eigenvector x_r of λ_r, $r \in I_{\bar\alpha}$, and $e^{\varepsilon u_\ell}\psi(u_\ell)$ is real, there must be a component of $A(\bar\alpha, \hat j, u_\ell)$ which is also real and strictly negative. This contradicts the nonnegativity of $e^{\varepsilon u_\ell}\psi(u_\ell)$. Therefore, $A(\bar\alpha, \hat j, s) = 0$ for $s \in \mathbb{R}$.

Similarly, considering the remaining finite number of sums $A(\alpha, j, u)$, strictly ordered according to their growth as $u \downarrow -\infty$, we conclude $A(\alpha, j, s) = 0$ when $\alpha < s(A)$ for $s \in \mathbb{R}$.

Then, for $u \in \mathbb{R}$, $\sum_{r:\lambda_r \neq s(A)} U_r(u)x_0 = 0$ and so $\psi(s) = e^{s(A)u}Q\psi(0)$ where Q is projection onto the eigenvector x of $\lambda = s(A)$ (cf. (4.6)). Finally, $\psi(0)$ is also a corresponding eigenvector since $\psi(0) = Q\psi(0)$. □

We now identify, under small initial configurations, the 'time-change' $t = t(s)$' given in the beginning of the Section.

Lemma 4.12. *With respect to small initial configurations, we have $t(u) = e^{s(A)u}$ for $u \in \mathbb{R}$ and $T(t) = s(A)t$ for $t \geq 0$.*

Proof. Since $\psi(u) = e^{s(A)u}\psi(0)$ from Proposition 4.11, and $\psi(u) = \varphi(t(u))$, we have from Lemma 4.1 that

$$p_0 t(u) \ = \ \sum_k \varphi_k(t(u)) \ = \ e^{s(A)u} \sum_k \psi_k(0).$$

Since $t(0) = 1$, we have $\sum_k \psi_k(0) = p_0$. This shows $t(u) = e^{s(A)u}$ for $u \in \mathbb{R}$. Next, as $s(A)t(u) = \dot{t}(u) = T(t(u))$, and $t = t(u)$ is onto $\mathbb{R}$, $T(t) = s(A)t$ for $t \geq 0$. □

Proof of Theorem 2.1. First, consider large initial configurations and recall the time s_0 defined after (4.2) so that $t(s_0) = 0$, and $\psi(s_0)_k = c_k$ for $k \geq 1$. Then, $\psi(s + s_0) = P_s\psi(s_0)$ for $s \geq 0$ and $\sum_{k\geq 1} \psi_k(s + s_0) = \sum_{k\geq 1} \varphi(t(s + s_0)) = p_0 t(s + s_0) + c$. In particular, $t(\cdot)$ is uniquely specified in terms of $\{\psi_k(\cdot)\}$ and c. Hence, $\{\varphi_k(u) = \psi_k(t^{-1}(u)) : u \geq 0\}$ is uniquely determined.

Moreover, for $\varepsilon > 0$ small and $s > 0$, as $e^{-\varepsilon s}P_s$ satisfies representation (4.4), and the dominant eigenvalue $s(A) - \varepsilon > 0$ is simple (Proposition 4.9), we have $e^{-s(A)s}\psi(s) = e^{-(s(A)-\varepsilon)s}e^{-\varepsilon s}P_s\psi(0)$ converges in Ω to an eigenvector v with eigenvalue $s(A)$ of A as $s \uparrow \infty$ (cf. discussion before Corollary V.3.3 in [22]). Then,

$$p_0 + \frac{c}{t(s)} \ = \ \frac{1}{t(s)} \sum_{k\geq 1} \varphi_k(t(s)) \ = \ \frac{e^{s(A)s}}{t(s)} \cdot e^{-s(A)s} \sum_{k\geq 1} \psi_k(s).$$

Taking $s \uparrow \infty$, as $t(s) \uparrow \infty$ from (4.2), we conclude $p_0 = z\sum_{k\geq 1} v_k$ where $z = \lim_{s\uparrow\infty} e^{s(A)s}/t(s)$, which necessarily converges. By the eigenvector formula (4.6) which $\{a_k(p_0, q_0, s^*)\}$ satisfies, fact $s(A) = s^*$ (Proposition 4.9), and equality $\sum_{k\geq 1} a_k(p_0, q_0, s^*) = p_0$ (cf. (2.6)), we identify $zv_k = a_k(p_0, q_0, s^*)$ for $k \geq 1$. Hence, for $k \geq 1$,

$$\varphi_k(s)/s \ = \ [e^{s(A)t^{-1}(s)}/s][e^{-s(A)t^{-1}(s)}\psi_k(t^{-1}(s))] \ \to \ zv_k \ = \ a_k(p_0, q_0, s^*).$$

Now, consider small initial configurations. By Lemma 4.12, $t(s) = e^{s(A)s}$ is identified and therefore $\{\varphi_k(u) = \psi_k(t^{-1}(u))\}$ is as well uniquely found. However, since $\{a_k(p_0, q_0, s^*)\}$ satisfies (2.5), we conclude $\{a_k(p_0, q_0, s^*)t\}$ solves ODEs (2.7) with $c = 0$. Hence, in this case, $\varphi_k(t) = a_k(p_0, q_0, s^*)t$ for $k \geq 1$. □

5. Appendix: $d_k(j+1)$ in the graph model

As mentioned, formation of loops need to be considered. For $k \geq 3$,

$$d_k(j+1) = \begin{cases} 2 & \text{with prob. } (1-p)\big[\frac{w(k-1)Z_{k-1}(j)}{S(j)}\big]^2 \\ & \quad -(1-p)\frac{[w(k-1)]^2 Z_{k-1}(j)}{[S(j)]^2} \\ 1 & \text{with prob. } p\frac{w(k-1)Z_{k-1}(j)}{S(j)} \\ & \quad +(1-p)\frac{[w(k-2)]^2 Z_{k-2}(j)}{[S(j)]^2} \\ & \quad +2(1-p)\frac{w(k-1)Z_{k-1}(j)}{S(j)}\big[1-\frac{w(k-1)Z_{k-1}(j)}{S(j)}-\frac{w(k)Z_k(j)}{S(j)}\big] \\ 0 & \text{with prob. } p\big[1-\frac{w(k-1)Z_{k-1}(j)}{S(j)}-\frac{w(k)Z_k(j)}{S(j)}\big] \\ & \quad +(1-p)\frac{[w(k-1)]^2 Z_{k-1}(j)}{[S(j)]^2} \\ & \quad +2(1-p)\frac{w(k-1)Z_{k-1}(j)}{S(j)}\frac{w(k)Z_k(j)}{S(j)} \\ & \quad +(1-p)\big[1-\frac{w(k-1)Z_{k-1}(j)}{S(j)}-\frac{w(k)Z_k(j)}{S(j)}\big]^2 \\ & \quad -(1-p)\frac{[w(k-2)]^2 Z_{k-2}(j)}{[S(j)]^2} \\ -1 & \text{with prob. } p\frac{w(k)Z_k(j)}{S(j)} \\ & \quad +(1-p)\frac{[w(k)]^2 Z_k(j)}{[S(j)]^2} \\ & \quad +2(1-p)\frac{w(k)Z_k(j)}{S(j)}\big[1-\frac{w(k-1)Z_{k-1}(j)}{S(j)}-\frac{w(k)Z_k(j)}{S(j)}\big] \\ -2 & \text{with prob. } (1-p)\big[\frac{w(k)Z_k(j)}{S(j)}\big]^2 \\ & \quad -(1-p)\frac{[w(k)]^2 Z_k(j)}{[S(j)]^2}. \end{cases}$$

$$d_1(j+1) = \begin{cases} 1 & \text{with prob. } p\big[1-\frac{w(1)Z_1(j)}{S(j)}\big] \\ 0 & \text{with prob. } p\frac{w(1)Z_1(j)}{S(j)} + (1-p)\big[1-\frac{w(1)Z_1(j)}{S(j)}\big]^2 \\ -1 & \text{with prob. } 2(1-p)\frac{w(1)Z_1(j)}{S(j)}\big[1-\frac{w(1)Z_1(j)}{S(j)}\big] \\ & \quad +(1-p)\frac{[w(1)]^2 Z_1(j)}{[S(j)]^2} \\ -2 & \text{with prob. } (1-p)\big[\frac{w(1)Z_1(j)}{S(j)}\big]^2 \\ & \quad -(1-p)\frac{[w(1)]^2 Z_1(j)}{[S(j)]^2}. \end{cases}$$

$$d_2(j+1) = \begin{cases} 2 & \text{with prob. } (1-p)\big[\frac{w(1)Z_1(j)}{S(j)}\big]^2 \\ & \quad -(1-p)\frac{[w(1)]^2 Z_1(j)}{[S(j)]^2} \\ 1 & \text{with prob. } p\frac{w(1)Z_1(j)}{S(j)} \\ & \quad +2(1-p)\frac{w(1)Z_1(j)}{S(j)}\big[1-\frac{w(1)Z_1(j)}{S(j)}-\frac{w(2)Z_2(j)}{S(j)}\big] \\ 0 & \text{with prob. } p\big[1-\frac{w(1)Z_1(j)}{S(j)}-\frac{w(2)Z_2(j)}{S(j)}\big] \\ & \quad +(1-p)\frac{[w(1)]^2 Z_1(j)}{[S(j)]^2} \\ & \quad +2(1-p)\frac{w(1)Z_1(j)}{S(j)}\frac{w(2)Z_2(j)}{S(j)} \\ & \quad +(1-p)\big[1-\frac{w(1)Z_1(j)}{S(j)}-\frac{w(2)Z_2(j)}{S(j)}\big]^2 \\ -1 & \text{with prob. } p\frac{w(2)Z_2(j)}{S(j)} \\ & \quad +(1-p)\frac{[w(2)]^2 Z_2(j)}{[S(j)]^2} \\ & \quad +2(1-p)\frac{w(2)Z_2(j)}{S(j)}\big[1-\frac{w(1)Z_1(j)}{S(j)}-\frac{w(2)Z_2(j)}{S(j)}\big] \\ -2 & \text{with prob. } (1-p)\big[\frac{w(2)Z_2(j)}{S(j)}\big]^2 \\ & \quad -(1-p)\frac{[w(2)]^2 Z_2(j)}{[S(j)]^2}. \end{cases}$$

6. Appendix: Non-compactness of semigroups

Proposition 6.1. *The semigroups P_t and P_t^B are not compact for any $t \geq 0$.*

Proof. For $x \in \Omega$, let $\zeta(t;x) = P_t^B x$. From the form of B (cf. after (2.8)), we observe that

$$\sum_{k=1}^{L} k\zeta_k(t;x) = \sum_{k=1}^{L} k\zeta_k(0;x) + q_0 \int_0^t \sum_{k=1}^{L-1} w(k)\zeta_k(s;x)ds - q_0 Lw(L) \int_0^t \zeta_L(s;x)ds \tag{6.1}$$

when $x \in \ell_c$ (cf. proof of Proposition 4.1).

Fix now $x \in \ell_c$ positive. Since P_t^B is positive (Proposition 4.3), we have $\zeta_k(\cdot;x) \geq 0$ for $k \geq 1$ and $\sum_{k=1}^L k\zeta_k(t;x) \leq \sum_{k=1}^L k\zeta_k(0;x) + \mathcal{W}q_0 \int_0^t \sum_{k=1}^L k\zeta_k(s;x)ds$. Therefore, the upper bound $\sum_{k\geq 1} k\zeta_k(t;x) \leq e^{\mathcal{W}q_0 t} \sum_{k\geq 1} k\zeta_k(0;x)$.

We now derive a lower bound. In (6.1), by the upper bound, limits of all terms as $L \uparrow \infty$ converge. In particular, by positivity, $\sum_{k\geq 1} w(k) \int_0^t \zeta_k(s;x)ds = \int_0^t \sum_{k\geq 1} w(k)\zeta_k(s;x)ds < \infty$, and so the limit $\lim_L Lw(L) \int_0^t \zeta_L(s;x)ds = 0$. Therefore, from (6.1) and positivity, we get $\sum_{k\geq 1} k\zeta_k(t;x) \geq \sum_{k\geq 1} k\zeta_k(0;x) = \|x\|$.

Let now $t \geq 0$ be fixed. For $n \geq 1$, let $x^n \in \ell_c$ where $x^n_n = n^{-1}$ and $x^n_k = 0$ for $k \neq n$. This sequence is bounded in Ω: $\|x^n\| = \sum_{k\geq 1} k|x^n_k| = 1$.

Then, starting from $n = 1$, let L_1 be an index so that $\sum_{k>L^1} k\zeta_k(t;x^1) \leq 1/2$. For $n \geq 1$, define $L^{n+1} > L^n$ as an index where $\sum_{k>L^{n+1}} k\zeta_k(t;x^{n+1}) \leq 1/2$.

We now show $\|P_t^B x^n - P_t^B x^m\| \geq 1$ for all $1 \leq m < n$. By the form of B, there is no flow 'backwards', that is $\zeta_k(t;x^n) \equiv 0$ for $k < n$. Write $\sum_{k\geq 1} k|\zeta_k(t;x^m) - \zeta_k(t;x^n)| \geq \sum_{k\leq L^m} k|\zeta_k(t;x^m) - \zeta_k(t;x^n)| = \sum_{k\leq L^m} k\zeta_k(t;x^m) \geq \|x^m\| - 1/2 = 1/2$. Hence, P_t^B cannot be a compact operator for any $t \geq 0$.

Similarly, P_t cannot be compact for any $t \geq 0$: Suppose P_{t_0} is compact. Then, as P_u is bounded for each $u \geq 0$, P_{t_0+u} is compact for $u \geq 0$. Because K is compact and $B = A - K$, by say the perturbation result Theorem III.1.14(i) in [22], $P_{t_1}^B$ for some $t_1 \geq 0$ would also be compact, a contradiction. $\square$

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