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INVARIANT MEASURES FOR THE ZERO RANGE PROCESS[1]

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On a countable set of sites S, the zero range process is constructed when the stochastic matrix $p(x, y)$ determining the one particle motion satisfies a mild assumption. The set of invariant measures for this process is described in the following two cases:

a) The system is attractive and $p(x, y)$ is recurrent.
b) The system is attractive, $p(x, y)$ corresponds to a simple random walk on the integers and the rate at which particles leave any site is bounded.

1. Introduction. In [20] Spitzer introduced several infinite particle systems, among them the zero range and the simple exclusion processes. The former describes the behavior of infinitely many undistinguishable particles moving on a countable set of sites S, according to the following laws: if a site x is occupied by k particles, the rate at which a particle will leave x is $g(k)$; once a particle leaves x it goes to y with probability $p(x, y)$. In particular, if $g(k) = k$ the particles perform independent movements and if $g(k) = 1$, one can consider that each site has an exponential clock with parameter one and when the clock at site x rings, a particle chosen at random among the ones at x moves. In the simple exclusion process, each site is occupied by at most one particle; the number of sites is again countable. The particles attempt to move at independent exponential times with parameter one. Once a particle at x attempts to move, it will try to go to y with probability $p(x, y)$. The interaction among particles is the following: if y is occupied, the particle stays at x.

One of the interesting problems concerning these systems is to describe the set of invariant measures. This was done by Liggett and Spitzer ([10], [11], [21]) for the simple exclusion process when $p(x, y)$ is symmetric, and by Liggett ([12], [13]) when $p(x, y)$ is either positive recurrent and reversible or corresponds to a random walk with mean 0 in $\mathbb{Z}$ or to an asymmetric simple random walk on $\mathbb{Z}$. However, for the zero range process only, the cases where $g(k) = k$ or $p(x, y)$ is positive recurrent and $g(k)$ is nondecreasing are understood ([14], [23]). In the first case the particles move independently and in the second the invariant measures concentrate on configurations with a finite number of particles. Since these properties play an important role in the proofs and do not hold in general, other methods have to be used to consider different cases. In this paper we use coupling techniques similar to the ones employed in [13] for the simple exclusion process.

We should point out that in [20] Spitzer proved that if S is finite, $g(k) > 0$ for all $k > 0$ and π: $S \to \mathbb{R}_+$ satisfies $\sum_x \pi(x)p(x, y) = \pi(y)$ and $\pi(x) < \liminf g(k)$ for all $x \in S$, then the product measure μ_π in $(\mathbb{Z}_+)^S$ given by the following marginals:

$$\mu_\pi\{\eta \in (\mathbb{Z}_+)^S : \nu(x) = k\} = \begin{cases} \gamma_x \dfrac{(\pi(x))^k}{g(1) \cdots g(k)}, & \text{if } k > 0 \\ \gamma_x, & \text{if } k = 0 \end{cases}$$

is invariant for the zero range process (γ_x is a normalizing constant). The primary aim of this paper is to prove that even when S is infinite these measures are invariant and, in some cases, that every invariant measures is a mixture of these. To avoid uninteresting complications, we will assume throughout that $p(x, y)$ is irreducible and that $\lim_{x\to\infty} p(x, y) = 0$ for all y in S.

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The first difficulty in studying the zero range process is to show the existence of a mathematical object corresponding to its description. This was done by Holley ([5]), under the constraints: $\lim g(k) < \infty$, $S = \mathbb{Z}$, $p(x, y) = p(0, y - x)$ and $\sum_x |x| p(0, x) < \infty$. In [9] Liggett considers an arbitrary S and proves the existence of the process when g satisfies:

$$\sup_k |g(k+1) - g(k)| = K < \infty. \tag{1.1}$$

In this case, not all elements of $(\mathbb{Z}_+)^S$ are allowed as initial configurations for the process. This is necessary to avoid infinitely many particles coming to the same site in finite time.

In Section 2 we will give another construction using the method developed by Liggett and Spitzer in [16]. This construction uses slightly weaker assumptions that the ones in [9] and has the advantage of making the proofs easier of some properties of the process. The configurations of particles that will be allowed are the η's in $(\mathbb{Z}_+)^S$ satisfying:

$$\| \eta \| = \sum_{x \in S} \eta(x) \alpha(x) < \infty. \tag{1.2}$$

Here, α is a strictly positive real valued function on S such that

$$\sum_{y \in S} p(x, y) \alpha(y) \leq M \alpha(x) \tag{1.3}$$

for all $x \in S$ and some constant M. One can construct such an α in the following way: fix $x_0 \in S$ and let $\alpha(x) = \sum_{n=0}^{\infty} (1/M)^n p^n(x, x_0)$ for some $M > 1$. A simple computation shows that α satisfies (1.3) and we will later see that this α has some desirable properties. For that reason we are going to assume in Sections 3 to 8 that the construction given in Section 2 has been carried out with this α.

The subset of elements of $\mathbb{Z}_+^S$ satisfying (1.2) will be denoted by ε. $\mathscr{L}$ will be the set of real valued functions f on ε such that $|f(\eta) - f(\xi)| \leq c \| \eta - \xi \|$ for all $\eta, \xi \in \varepsilon$ and some constant c. $L(f)$ will be the smallest constant satisfying this inequality.

The zero range process is well defined for any initial configuration with a finite number of particles, since it is a countable state Markov chain and the transition rates are bounded. We will denote by η_t the random configuration obtained by time t when the initial configuration was η. The following theorem allows us to extend the process to ε.

THEOREM 1.4. *If $\sup_k |g(k+1) - g(k)| = K < \infty$, then there exists a semigroup $S(t)$ of operators on $\mathscr{L}$ such that $S(t)f(\eta) = E^\eta f(\eta_t)$ for all $f \in \mathscr{L}$ and finite η. This semigroup satisfies*:

$$|S(t)f(\eta) - S(t)f(\xi)| \leq L(f) e^{K(M+2)t} \| \eta - \xi \| \tag{1.5}$$

for $\eta, \xi \in \varepsilon$ and $f \in \mathscr{L}$.

As in [16], this defines the joint distribution of $\{\eta_t(x) : x \in S\}$ for all $\eta \in \varepsilon$. In this way we get a probability measure $P^\eta[\eta_t \in A]$ on the Borel σ algebra of $(\mathbb{Z}_+)^S$ such that

$$S(t)f(\eta) = \int f(\gamma) p^\eta[\eta_t \in d\gamma] \tag{1.6}$$

for $f \in \mathscr{L}$ that depends on a finite number of coordinates. By (1.5) $E_\eta \| \eta_t \| \leq e^{K(M+2)t} \| \eta \|$. This shows that $p^\eta[\eta_t \in \varepsilon] = 1$ if $\eta \in \varepsilon$. Now (1.6) allows us to define $S(t)f$ for any $f \in \mathscr{L}$ or $f \geq 0$.

It will also be shown in Section 2 that the natural generator of the process defined by

$$(\Omega f)(\eta) = \sum_{x \in S} g(\eta(x)) \sum_{y \in S} p(x, y)(f(\eta_{xy}) - f(\eta))$$

where

$$\eta_{xy}(z) = \begin{cases} \eta(z) - 1 \text{ if } z = x, & \eta(x) > 0 \text{ and } x \neq y \\ \eta(z) + 1 \text{ if } z = y, & \eta(x) > 0 \text{ and } x \neq y \\ \eta(z) \quad \text{otherwise} & \end{cases}$$

has the following property: $S(t)f(\eta) = f(\eta) + \int_0^t \Omega S(s) f(\eta)\, ds$ for $f \in \mathscr{L}$ and $\eta \in \varepsilon$. Note that $\Omega f(\eta)$ is well defined for $f \in \mathscr{L}$, $\eta \in \varepsilon$ and $|\Omega f(\eta)| \le \sum_x g(\eta(x)) \sum_y p(x, y) |f(\eta_{xy}) - f(\eta)| \le L(f) \sum_x g(\eta(x)) \sum_y p(x, y)(\alpha(x) + \alpha(y))$, this implies:

(1.7) $$|\Omega f(\eta)| \le L(f)(M + 1) ||| \eta |||$$

where $$||| \eta ||| = \sum_{x\in S} g(\eta(x))\alpha(x).$$

To give a formal definition of an invariant measure, we introduce $\mathscr{L}'$, the set of bounded functions in $\mathscr{L}$. Now, given a probability measure μ on ε, we let $\mu S(t)$ be the measure satisfying $\int f d\mu S(t) = \int S(t) f d\mu$ for all $f \in \mathscr{L}'$. We will say that μ is invariant if $\mu S(t) = \mu$ for all $t \ge 0$. This is equivalent to: $\int S(t) f d\mu = \int f d\mu$ for all $f \in \mathscr{L}'$, $t \ge 0$. From now on $\mathscr{I}$ will denote the set of invariant measures.

Through Sections 3 to 8 to avoid pathologies we assume that $g(k) > 0$ for all $k > 0$ and for notational purposes we assume that $g(0) = 0$.

In Section 3 we prove the following.

THEOREM 1.8. *If* $\pi: S \to R_+$ $\pi \not\equiv 0$ *satisfies*: $\sum_{x\in S} \pi(x)p(x, y) = \pi(y)$ *for all* $y \in S$, $\pi(x) < \liminf g(k)$ *for all* $x \in S$ *and* $\mu_\pi(\varepsilon) = 1$ *then* $\mu_\pi \in \mathscr{I}$.

The condition $\mu_\pi(\varepsilon) = 1$ is satisfied in many cases of interest, among others when π is constant or when $g(k) > \delta k$ for some positive δ and all $k > 0$. Unfortunately there are cases where the measure μ_π does not satisfy $\mu_\pi(\varepsilon) = 1$. This problem can be solved by allowing configuration η satisfying $||| \eta ||| < \infty$. This restriction is more natural since it involves the rate at which particles leave the different sites and not the number of particles at each site. In particular if $\sup_y \sum_x p(x, y) = C < \infty$ and $\sup g(k) < \infty$ we can allow all configurations taking $\alpha(x) = \sum_n (1/M^n)p^n(x, x_0)$ for some fixed $x_0 \in S$ and $M > C$. (In this case $||| \eta ||| = \sum_x g(\eta(x))\alpha(x) \le \sup_k g(k) \cdot \sum yx\, \alpha(x) = \sup_k g(k) \cdot \sum_n 1/M^n\, p^n(x, x_0) \le \sup g(k)$.

$\sum_n \cdot C^n/M^n < \infty$ for all $\eta \in (\mathbb{Z}+)^S$). The construction of the process allowing all η's such that $||| \eta ||| < \infty$ can be found in the appendix of [0]. It has the advantage that the condition $\mu_\pi(\varepsilon) = 1$ can be dropped from Theorem 1.8 since it is implied by the other hypothesis of the theorem. We omit, however, this construction in this paper because it requires many technical lemmas and long computations.

It should be mentioned that (1.1) is maintained in both constructions; at the end of Section 2, we give an example showing that something like (1.1) is necessary in order to avoid influence from ∞ in a sense similar to the one described by Griffeath in [4].

Now we introduce a partial order in the set ε: $\eta \le \xi$ will mean $\eta(x) \le \xi(x)$ for all $x \in S$. The main results of this paper require the coupling of two copies of the zero range process in such a way that if $\eta \le \xi$ then $p^{(\eta,\xi)}(\eta_t \le \xi_t) = 1$ for all $t \ge 0$. For this to be possible, we need $g(k) \le g(k + 1)$ for $k = 1, 2, \cdots,$; this condition will be assumed from Section 4 to Section 8. The properties of the coupled process that we need, including the one stated above, are proved in Section 4.

In Section 5 we consider the case in which $S = \mathbb{Z}^d$. We denote by $\mathscr{S}$ the set of translation invariant probability measures on ε.

When $p(x, y)$ is translation invariant, $\pi(x) \equiv \rho$ satisfies $\sum \pi(x)p(x, y) = \pi(y)$; for these π's we will write μ_ρ instead of μ_π. Theorem 1.8 says that $\mu_\rho \in \mathscr{I}$ if $\rho \in [0, \lim g(k))$. Using the results of Section 4, we prove the following.

THEOREM 1.9. *If* $S = \mathbb{Z}^d$, $p(x, y) = p(0, y - x)$ *and* $g(k) \le g(k + 1)$ $\quad k = 0, 1, 2 \cdots$ *then* $\mathscr{I} \cap \mathscr{S} = \{\int \mu_\rho d\lambda(\rho)$: λ *is a probability measure on* $[0, \lim g(k))\}$.

The proof of this theorem is similar to the proof of Theorem 1.1 in [13]. However, some complications arise due to the fact that $\int \eta(x) d\mu(\eta)$ is infinite for some probability measures μ on ε.

In Section 6 we consider the case in which $p(x, y)$ is null recurrent. For $p(x, y)$ positive recurrent we refer the reader to Waymire [23]. It is proved there that, in this case, all invariant measures concentrate on finite configuration. The proof is done for $g(k) \equiv 1$, but using the construction we give in Section 2, it is possible to extend it to all g's satisfying $g(k) \le g(k+1)$.

When $p(x, y)$ is recurrent, all positive solutions of $\sum_x \pi(x)p(x, y) = \pi(y)$ are multiples of a given one. To simplify notation in the following theorem, we fix one solution π and denote by μ_ρ the measure $\mu_{\rho\pi}$ and by A the set $\{\rho : \rho \ge 0$ and $\rho\pi(x) < \sup g(k)$ for all $x \in S\}$. (A is an interval containing 0 but might be open or closed on its right).

THEOREM 1.10. *If $p(x, y)$ is null recurrent, $g(k) \le g(k+1)$ $k = 0, 1, \cdots$ and $\mu_\rho(\varepsilon) = 1$ for all $\rho \in A$ then*:

$$\mathscr{I} = \{\textstyle\int \mu_\rho \, d\lambda(\rho) : \lambda \text{ is a probability measure on } A\}.$$

As in Theorem 1.8, the assumption $\mu_\rho(\varepsilon) = 1$ can be dropped by using the construction given in the appendix of [0].

In Section 7 we consider a nearest neighbor random walk in $\mathbb{Z}$ and, letting μ_ρ be as in Theorem 1.9, we prove the following.

THEOREM 1.11. *If $S = \mathbb{Z}$ $(p(x, x+1) = p, p(x, x-1) = q) p + q = 1, g(k) \le g(k+1)$ $k = 0, 1, \cdots$ and $\sup g(k) < \infty$ then*

$$\mathscr{I} = \{\textstyle\int \mu_\rho d\lambda(\rho) : \lambda \text{ is a probability measure on } [0, \sup g(k))\}.$$

The conclusion of this theorem is false is we do not require $g(k)$ to be bounded. For instance if $g(k) = k$ and $\frac{1}{2} < p < 1$, $\pi(x) = (p/q)^x$ gives us an invariant measure μ_π which is not translation invariant, (μ_π will be an invariant measure for any $g(k) \uparrow \infty$ if we use the construction given in the appendix of [0]). Note that in the corresponding simple exclusion process there are also invariant measures which are not translation invariant; see Theorem 1.4 in [13].

Since the measure μ_π mentioned above concentrates on configurations "growing" with x and with finitely many particles to the left of zero, the following question becomes natural: What happens to these configurations when $g(k)$ is bounded? This is answered by the following theorem proved in Section 8.

THEOREM 1.12. *Suppose that $\pi : S \to \mathbb{R}_+$ satisfies $\sum_{x \in S} \pi(x)p(x, y) = \pi(y)$ and the set $\{\pi(x) : x \in S\}$ has a sequence of isolated points diverging to ∞. Suppose, also, that $g(k) \le g(k+1) \cdots$ and $\sup g(k) < \infty$. Then if η is any configuration such that $\sum_{\{x : \pi(x) < L\}} \eta(x) < \infty$ for all $L \in \mathbb{R}$ then $P(\eta_t(x) > 0) \to 0$ as $t \to \infty$ for all $x \in S$.*

In particular, if $S = \mathbb{Z}_+$, $g(k) \le g(k+1)$ $k = 1, 2, \cdots \sup g(k) < \infty$ and $p(x, x+1) = p > q = p(x, x-1)$ for all $x \ge 1$, $p(0, 0) = q$, $p(0, 1) = p$, then the only invariant measure is μ_0, the point mass at the vacant configuration.

In Sections 7 and 8, $g(k) \le g(k+1)$ and $\sup g(k) < \infty$. Then under the restriction $\sup_y \sum_x p(x, y) < \infty$ it is possible to construct the process in the whole space $(\mathbb{Z}+)^S$ with the techniques of [8]. The auxiliary processes used in Section 8 can also be constructed with these techniques. In this new context, Theorem 1.11 remains valid and, assuming $\sup_y \sum_x p(x, y) < \infty$. Theorem 1.12 can be proved without the unnatural condition on the divergent sequence contained in $\{\pi(x) : x \in S\}$; i.e. there is no need to require that the elements of that sequence are isolated in $\{\pi(x) : x \in S\}$. We have not used this approach, however, because it forces us to prove again technical lemmas like the ones in Section 2 and 4.

2. The construction. Since the proof of the existence of the process is very similar to the one used in [16] for other processes, we will omit many details. However, some estimates are more complicated because the speed at which individual particles move is not independent of the presence of other particles. We will prove these estimates assuming that (1.1) holds.

LEMMA 2.1. *Suppose S is finite; then*

$$E^{\eta}(\eta_t(y)) \le \sum_{x\in S} \eta(x) \sum_{\ell=0}^{\infty} \frac{(Kt)^{\ell}}{\ell!} p^{(\ell)}(x,y).$$

PROOF. Consider a process η_t^* with the following properties: each particle generates a new particle at independent exponential times of parameter K; if the original particle is at x the new particle is placed at y with probability $p(x,y)$. The generator of this process is given by:

$$(\Omega^* f)(\eta) = \sum_{x,y\in S} K\eta(x) p(x,y)(f(\eta_y) - f(\eta))$$

where

$$\eta_y(x) = \begin{cases} \eta(x) & \text{if } x \ne y \\ \eta(x) + 1 & \text{if } x = y. \end{cases}$$

A simple coupling argument shows that $E(\eta_t(y)) \le E(\eta_t^*(y))$ for all $y \in S$.

η_t^* is a multi-type branching process; from Section V—7.2 of [1] we get:

$$E(\eta_t^*(y)) = \sum_{x\in S} \eta(x) \sum_{\ell=0}^{\infty} \frac{(Kt)^{\ell}}{\ell!} p^{(\ell)}(x,y)$$

and this proves the lemma.

Given a transition probability matrix $p(x,y)$ on S, we define the generator of the zero range process Ω in the following way:

$$\Omega f(\eta) = \sum_{x\in S} g(\eta(x)) \sum_{y\in S} p(x,y)(f(\eta_{xy}) - f(\eta))$$

for all $f \in \mathscr{L}$, $\eta \in \varepsilon$. When S is finite, this generates a semi-group of operators $S(t)$ on $\mathscr{L}$ having the following property:

$$S(t)f(\eta) = E^{\eta}(f(\eta_t)).$$

LEMMA 2.2. *Suppose S is finite and $f \in \mathscr{L}$, then $S(t)f \in \mathscr{L}$ and $L(S(t)f) \le L(f)e^{K(M+1)t}$*

PROOF. Let $\bar{S}(t)$ be the semi-group of a process with state space $\mathbb{Z}_+^S \times \mathbb{Z}_+^S$ and generator:

$$\begin{aligned}(\bar{\Omega} f)(\eta,\zeta) &= \sum_{x: g(\eta(x)) \ge g(\zeta(x))} (g(\eta(x)) - g(\zeta(x))) \sum_{y\in S} p(x,y)(f(\eta_{xy},\zeta) - f(\eta,\zeta)) \\ &\quad + \sum_{x: g(\eta(x)) \le g(\zeta(x))} (g(\zeta(x) - g(\eta(x)))) \sum_{y\in S} p(x,y)(f(\eta,\zeta_{xy}) - f(\eta,\zeta)) \\ &\quad + \sum_{x\in S} \min(\{g(\eta(x)), g(\zeta(x))\}) \sum_{y\in S} p(x,y)(f(\eta_{xy},\zeta_{xy}) - f(\eta,\zeta)).\end{aligned}$$

This process is well defined since, for any initial configuration (η,ζ), it is a finite state Markov chain. Note that both marginals of this process are the zero range process.

Suppose that $f \in \mathscr{L}$ and let $g, h: (\mathbb{Z}_+)^S \times (\mathbb{Z}_+)^S \to \mathbb{R}$ be given by $g(\eta,\zeta) = f(\eta) - f(\zeta)$ and $h(\eta,\zeta) = \|\eta - \zeta\|$.

Now $|S(t)f(\eta) - S(t)f(\xi)| = |(\bar{S}(t)g)(\eta,\xi)| \le L(f)(\bar{S}(t)h)(\eta,\xi)$. But $|(\bar{\Omega}h)(\eta,\xi)| \le K\sum_x |\eta(x) - \xi(x)| \sum_y p(x,y)(\alpha(x) + \alpha(y)) < K(M+1)h(\eta,\xi)$. Therefore, $|\bar{S}(t)h(\eta,\xi)| \le e^{K(M+1)t}h(\eta,\xi)$ and this implies the lemma.

The following two lemmas are obvious:

LEMMA 2.3. *If S is finite and $f \in \mathscr{L}$ satisfies $|f(\eta)| \le c\|\eta\|$ for some constant c and all η then $|S(t)f(\eta)| \le ce^{K(M+1)t}\|\eta\|$.*

LEMMA 2.4. *If S is finite, p_1 and p_2 are two transition probability matrices on S, Ω_1 and Ω_2 the corresponding generators of the zero range process and $f \in \mathscr{L}$ then*

$$|(\Omega_1 - \Omega_2)f(\eta)| \le L(f)K\sum_{x,y} \eta(x)|p_1(x,y) - p_2(x,y)|(\alpha(x) + \alpha(y)).$$

Given Ω_1 and Ω_2 as in Lemma 2.4, we denote by $S_1(t)$ and $S_2(t)$ the semigroups corresponding to Ω_1 and Ω_2. Now suppose that p_1 and p_2 satisfy (1.3); then from the identity

$$S_1(t)f(\eta) - S_2(t)f(\eta) = \int_0^t S_1(s)(\Omega_1 - \Omega_2)S_2(t-s)f(\eta)\,ds$$

and Lemmas 2.2, 2.4 and 2.1 it follows that:

$$|S_1(t)f(\eta) - S_2(t)f(\eta)| \le KL(f)\int_0^t e^{K(M+1)(t-s)} \textstyle\sum_{x,y,z}$$

$$\eta(z)\left(\sum_{\ell=0}^{\infty}\frac{(Ks)^\ell}{\ell!}p_1^{(\ell)}(z,x)\right)|p_1(x,y) - p_2(x,y)|(\alpha(x)+\alpha(y))\,ds. \tag{2.5}$$

We consider now a countable set of sites S. Let S_n be an increasing sequence of finite subsets of S such that $S = \cup_1^\infty S_n$ and define

$$p_n(x,y) = \begin{cases} p(x,y) & \text{if } x, y \in S_n\ x \neq y \\ 1 & \text{if } x = y \notin S_n \\ p(x,x) + \sum_{z \notin S_n} p(x,z) & \text{if } x = y \in S_n \end{cases}$$

where $p(x,y)$ is a transition probability on S satisfying (1.3). Since $p_n(x,y) \le p(x,y) + \delta xy$, it follows that p_n satisfies 1.3 with M replaced by $M+1$.

Let $S_n(t)$ be the semigroup of operators on $\mathscr{L}$ corresponding to $p_n(x,y)$. This is well defined since p_n has no transitions off the finite set S_n.

LEMMA 2.6. *$S_n(t)f(\eta)$ converges as $n \to \infty$ for all $f \in \mathscr{L}$ and $\eta \in \varepsilon$.*

PROOF. Since the integrand of (2.5) is bounded by

$$e^{K(M+1)(t-s)}\sum_{z,x\in S}\eta(z)\sum_{\ell=0}^{\infty}\frac{(Ks)^\ell}{\ell!}p_1^{(\ell)}(z,x)2(M+1)\alpha(x)$$

$$\le e^{K(M+1)(t-s)}2(M+1)\sum_z \eta(z)\sum_{\ell=0}^{\infty}\frac{(KsM)^\ell}{\ell!}\alpha(z)$$

$$\le 2e^{K(M+1)t}(M+1)\|\eta\|$$

the lemma follows from $p_n(x,y) \to p(x,y)$ and dominated convergence.

$S(t)f(\eta)$ is now defined as the limit of $S_n(t)f(\eta)$. Lemma 2.2 implies that $|S(t)f(\eta) - S(t)f(\xi)| \le L(f)e^{K(M+2)t}\cdot\|\eta - \xi\|$ for $\eta, \xi \in \varepsilon$.

Following the comments below Lemma 2.10 and the proofs of Lemmas 2.12 and 2.16 of [16], one proves that $S(t)$ satisfies the following equalities and inequalities for all $f \in \mathscr{L}$ and all $\eta \in \varepsilon$.

LEMMA 2.7.

a) $S(t_1 + t_2) = S(t_1)S(t_2)$, $S(0) = I$.

b) $S(t)f(\eta) = f(\eta) + \int_0^t \Omega S(s)f(\eta)\,ds$.

c) $|S(t)f(\eta) - f(\eta)| \le \|\eta\|L(f)(e^{K(M+2)t} - 1)$

d) $S(s)f(\eta)$ is continuous in s.

e) $\Omega S(s)f(\eta)$ is continuous in s.

f) $\lim_{t\downarrow 0}\dfrac{S(t)f(\eta) - f(\eta)}{t} = \Omega f(\eta)$.

g) $\Omega S(t)f(\eta) = S(t)\Omega f(\eta)$.

The purpose of the following two lemmas is to characterize invariant measures.

LEMMA 2.8. *$(S(t) \,|||\ \ |||\,)(\eta) \le e^{K(M+1)t} \,|||\,\eta\,|||$ for all $\eta \in \varepsilon$.*

PROOF. It suffices to show this for the operators S_n and for these it follows from:

$$|\,\Omega_n \,|||\ \ |||\,(\eta)\,| \le \textstyle\sum_{x,y\in S_n, y\ne x} g(\eta(x))p_n(x,y)\cdot(\,|||\,\eta_{xy}\,|||\,-\,|||\,\eta\,|||\,)$$
$$\le \textstyle\sum_{x\in S_n} g(\eta(x))K\sum_{y\in S_n,y\ne x} p_n(x,y)(\alpha(x)+\alpha(y)) \le K(M+1)\,|||\,\eta\,|||$$

LEMMA 2.9. *If μ is a probability measure on ε satisfying $\int |||\,\eta\,|||\,d\mu < \infty$, the following properties are equivalent*:

(i) $\displaystyle\int \Omega f\,d\mu = 0$ *for all* $f \in \mathscr{L}$.

(ii) $\mu \in \mathscr{I}$.

PROOF. Note that for any $f \in \mathscr{L}$ and $\eta \in \varepsilon$

$$\begin{aligned}|S(s)\Omega f(\eta)| = |\Omega S(s)f(\eta)| &\le \textstyle\sum_{x\in S} g(\eta(x))\sum_{y\in S} p(x,y)(|S(s)f(\eta_{xy}) - S(s)f(\eta)|)\\ &\le L(f)(M+1)e^{K(M+2)s}\,|||\,\eta\,|||\,,\end{aligned} \tag{2.10}$$

where the equality comes from 2.7 g) and the last inequality is implied by the comments following the definition of $S(t)f(\eta)$.

Suppose that i) holds and $f \in \mathscr{L}'$, then:

$$\int (S(t)f - f)\,d\mu = \int\left(\int_0^t (\Omega S(s)f)\,ds\right)d\mu.$$

By (2.10), we may use Fubini and the right hand side becomes

$$\int_0^t\left(\int \Omega S(s)f\,d\mu\right)ds = 0$$

by i) and the fact that $S(s)f \in \mathscr{L}$. Hence $\int S(t)f\,d\mu = \int f\,d\mu$ and $\mu \in \mathscr{I}$.

For the converse let

$$f_N(\eta) = \begin{cases} f(\eta) & \text{if } |f(\eta)| \le N\\ N & \text{if } f(\eta) \ \ > N\\ -N & \text{if } f(\eta) \ \ < -N\end{cases}$$

by ii) $0 = \int (S(t)f_N - f_N)\,d\mu = \int (\int_0^t S(s)\Omega f_N\,ds)\,d\mu = \int_0^t (\int S(s)\Omega f_N\,d\mu)\,ds$. The second equality follows from 2.7 b) and g) and the third from 2.10 and Fubini's theorem.

By 2.7 e) and g) and 2.10, $\int S(s)\Omega f_N\,d\mu$ is a continuous function of s, hence it must be identically 0; in particular $\int \Omega f_N\,d\mu = 0$.

But $|\Omega f_N|$ and $|\Omega f|$ are bounded by

$$\textstyle\sum_x g(\eta(x))\sum_y p(x,y)L(f)(\alpha(x)+\alpha(y)) \le (M+1)L(f)\,|||\,\eta\,|||\, \in L^1(d\mu);$$

since $\Omega f_N(\eta) \to \Omega f(\eta)$ for all $\eta \in \varepsilon$, i) follows from dominated convergence.

In Section 4, we will need Lemma 2.11. Before stating it we introduce some notation. Let

$$\|\eta\|_\ell = \textstyle\sum_{x\notin S_\ell} \eta(x)\alpha(x) \quad\text{and}\quad L'(f) = L(f) + 2\sup|f|.$$

Note that $L'(f) < \infty$ when $f \in \mathscr{L}'$.

LEMMA 2.11. *Suppose μ_n is a sequence of probability measures on ε converging weakly to μ_0 and there exists a sequence of real numbers $c_\ell \to 0$ and satisfying $\int (\|\eta\|_\ell \wedge 1)\,d\mu_n \le c_\ell$ for all n., Then $\int f\,d\mu_n \to \int f\,d\mu_0$ for all $f \in \mathscr{L}'$.*

PROOF. Let

$$\eta^{\ell}(x) = \begin{cases} \eta(x) & \text{if } x \in S_\ell \\ 0 & \text{otherwise} \end{cases}$$

and $h_\ell(\eta) = f(\eta) - f(\eta^\ell)$. Since

$$\mu_n \to_w \mu_0 \int f(\eta^\ell)\, d\mu_n(\eta) \to \int f(\eta^\ell)\, d\mu_0(\eta) \quad \text{and} \quad \int |h_\ell(\eta)|\, d\mu_n(\eta)$$
$$\leq L'(f) \int (\|\eta\|_\ell \wedge 1)\, d\mu_n(\eta) \leq L'(f) c_\ell,$$

we must have:

$$\limsup_n \int f(\eta)\, d\mu_n(\eta) \leq \int f(\eta^\ell)\, d\mu_0(\eta) + L'(f) c_\ell.$$

Letting ℓ go to ∞ we get:

$$\limsup_n \int f(\eta)\, d\mu_n(\eta) \leq \int f(\eta)\, d\mu_0(\eta).$$

Similarly one proves that $\liminf_n \int f(\eta)\, d\mu_n(\eta) \geq \int f(\eta)\, d\mu_0(\eta)$.

As mentioned in the introduction, we give an example that shows the importance of (1.1). Let $S = \{0, 1, 2, \cdots\}$,

$$g(k) = \begin{cases} k & \text{if } k = n^3 + 1 \text{ for some } n \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$p(0, n) = \frac{1}{2^n}\ n \geq 1\ p(n, n-1) = 1\ n \geq 1.$$

The α given in Section 1 satisfies $\alpha(n) \leq 1/2^{n-1}$; therefore the configuration η having x^3 particles at site x is allowable and $P(\eta_t(0) > 0) = 0$ for all t since the particles do not move. However if we start the process from η_y where

$$\eta_y(x) = \begin{cases} x^3 & \text{if } x \neq y \\ x^3 + 1 & \text{if } x = y \end{cases}$$

we have $\lim_{y\to\infty} P(\eta_{yt}(0) > 0) > 0$.

Although in this example we have $g(k) = 0$ for some positive k's it is possible to modify this g(k)'s to positive numbers and still have:

$$\liminf_{y\to\infty} P(\eta_{yt}(0) > 0) > P(\eta_t(0) > 0).$$

3. The invariance of μ_π. To prove Theorem 1.8 we will need the following two lemmas.

LEMMA 3.1. *If $\pi : S \to \mathbb{R}_+$ satisfies $\sum_x \pi(x)p(x, y) = \pi(y)$ for all $y \in S$, then* $\sum \pi(x)\alpha(x) < \infty$.

PROOF. $\sum_x \pi(x)\alpha(x) = \sum_{n=0}^{\infty} (1/M)^n \sum_{x\in S} \pi(x) P^{(n)}(x, x_0) = (M/(M-1))\, \pi(x_0) < \infty$.

LEMMA 3.2. *If $\pi : S \to \mathbb{R}_+$ satisfies $\sum yx\, \pi(x)p(x, y) = \pi(y)$ for all $y \in S$ and $\pi(x) <$ $\liminf g(k)$ for all $x \in S$ then $\int |||\, \eta\, |||\, d\mu_\pi < \infty$.*

PROOF. $\int |||\, \eta\, |||\, d\mu_\pi = \int \sum_{x\in S} g(\eta(x))\alpha(x)\, d\mu_\pi$

$$= \sum_{k=1}^{\infty} \sum_{x\in S} g(k)\alpha(x)\gamma_x \frac{(\pi(x))^k}{g(1)\cdots g(k)}$$
$$= \sum_{x\in S} \pi(x)\alpha(x)\left(\gamma_x + \sum_{k=2}^{\infty} \gamma_x \frac{(\pi(x))^{k-1}}{g(1)\cdots g(k-1)}\right) = \sum_{x\in S} \pi(x)\alpha(x) < \infty$$

by Lemma 3.1.

PROOF OF THEOREM 1.8. Let S_n be an increasing sequence of finite subsets of S such that $\cup S_n = S$ and define

$$p_n(x, y) = \begin{cases} 1 & \text{if } x = y \notin S_n \\ p(x, y) + Q_n^{-1}[\sum_{z \notin S_n} p(x, z)][\sum_{z \notin S_n} \pi(z) p(z, y)] & \text{if } x, y \in S_n \\ 0 & \text{otherwise} \end{cases}$$

where $Q_n = \sum_{z \in S_n, y \notin S_n} \pi(z) p(z, y)$. Note that

$$Q_n = \sum_{z \in S_n} \pi(z) - \sum_{z \in S_n} \pi(z) \sum_{y \in S_n} p(z, y)$$
$$= \sum_{y \in S_n} \pi(y) - \sum_{y, z \in S_n} \pi(z) p(z, y) = \sum_{y \in S_n, z \notin S_n} \pi(z) p(z, y).$$

Now a simple computation shows that

$$\sum_{y \in S} p_n(x, y) = 1 \quad \text{for all} \quad x \in S,$$

$p_n(x, y) \to p(x, y)$ for all $x, y \in S$ and

$$\sum_{x \in S} \pi(x) p_n(x, y) = \pi(y) \quad \text{for all} \quad y \in S.$$

$S_n(t)$ and Ω_n will be the semigroup and generator respectively of the zero range process corresponding to $p_n(x, y)$. This process has no transitions off S_n and, therefore, can be considered as having a finite number of sites. These approximations of p have been used by Liggett in [15] to study a different process and by Waymire in [22] to prove Theorem 1.8 under the additional assumptions $\sum \alpha(x) < \infty$ and $\sup_{x \in S} \pi(x) < \infty$.

In view of Lemma 2.9 to prove Theorem 1.8, it suffices to show that $\int \Omega f \, d\mu_\pi = 0$ for all $f \in \mathscr{L}$. But $\int \Omega_n f \, d\mu_\pi = 0$ since Theorem 1.8 is proved in [20] for finite S. Hence Theorem 1.8 follows if $\int |\Omega_n f - \Omega f| \, d\mu_\pi \to 0$ as $n \to \infty$. But

$$\int |(\Omega_n - \Omega) f| \, d\mu_\pi \le L(f) \int \sum_{x \in S} g(\eta(x)) \sum_{y \ne x} |p(x, y) - p_n(x, y)| (\alpha(x) + \alpha(y)) \, d\mu_\pi$$
$$(3.3) \qquad = L(f) \int \sum_{x \in S} g(\eta(x)) \sum_{y \ne x} |p(x, y) - p_n(x, y)| \alpha(x) \, d\mu_\pi$$
$$+ L(f) \int \sum_{x \in S} g(\eta(x)) \sum_{y \ne x} |p(x, y) - p_n(x, y)| \alpha(y) \, d\mu_\pi.$$

Now let $f_n(x) = \sum_{y \in S} |p(x, y) - p_n(x, y)| \le 2$. With this notation, the first integrand is bounded by:

$$\sum_{x \in S} g(\eta(x)) f_n(x) \alpha(x) \le 2 \, |||\eta||| \in L^1(d\mu_\pi)$$

by Lemma 3.2.

Suppose that $x \in S_n$; then

$$f_n(x) = \sum_{y \in S_n} |p(x, y) - p_n(x, y)| + \sum_{y \notin S_n} p(x, y) = 2 \sum_{y \notin S_n} p(x, y)$$

and converges to 0 as n goes to ∞. Therefore, by dominated convergence, the first integral of (3.3) goes to 0.

The second integrand of (3.3) is bounded by:

$$\sum_{x \notin S_n} g(\eta(x)) \sum_{y \in S, y \ne x} |p(x, y) - p_n(x, y)| \alpha(y)$$
$$+ \sum_{x \in S_n} g(\eta(x)) \sum_{y \notin S_n} |p(x, y) - p_n(x, y)| \alpha(y)$$
$$+ \sum_{x \in S_n} g(\eta(x)) \sum_{y \in S_n} |p(x, y) - p_n(x, y)| \alpha(y)$$
$$\le \sum_{x \notin S_n} g(\eta(x)) \sum_{y \in S} p(x, y) \alpha(y) + \sum_{x \in S_n} g(\eta(x)) \sum_{y \notin S_n} p(x, y) \alpha(y)$$
$$+ \sum_{x, y \in S_n} g(\eta(x)) |p(x, y) - p_n(x, y)| \alpha(y).$$

The first and second terms of this last expression are bounded by $M\ |||\eta|||$, hence their integrals go to 0 by Lemma 3.2 and dominated convergence. For the third term write:

$$\int \sum_{x,y\in S_n} g(\eta(x))\,|p(x,y)-p_n(x,y)|\,\alpha(y)\,d\mu_\pi$$

$$= \sum_{k=1}^{\infty} \sum_{x,y\in S_n} g(k) Q_n^{-1}[\sum_{z\notin S_n} p(x,z)][\sum_{z\notin S_n} \pi(z)p(z,y)]\alpha(y)\gamma_x \frac{(\pi(x))^k}{g(1)\cdots g(k)}$$

$$= Q_n^{-1}[\sum_{x\in S_n}\sum_{z\notin S_n} \pi(x)p(x,z)][\sum_{y\in S_n}\sum_{z\notin S_n} \pi(z)p(z,y)\alpha(y)]$$

$$\le M \sum_{z\notin S_n} \pi(z)\alpha(z) \to 0 \quad \text{as } n\to\infty$$

by Lemma 3.1.

4. The coupled process. In this section, and the ones that follow it, we assume that $g(k) \le g(k+1)$ $k = 0, 1, \cdots$. As we shall see, this is crucial for the use of coupling techniques.

We now introduce a Markov process, called the coupled process, on $\varepsilon \times \varepsilon = \bar{\varepsilon}$. This process can be described by

i) The marginal processes have $S(t)$ as their semigroup.
ii) Particles of different marginals occupying the same site move together as much as possible.

More formally, we let $\bar{\mathscr{L}}$ be the set of functions $f: \varepsilon \times \varepsilon \to \mathbb{R}$ such that

$$|f(\eta_1, \xi_1) - f(\eta_2, \xi_2)| \le L(f)(\|\eta_1 - \eta_2\| + \|\xi_1 - \xi_2\|)$$

for some constant $L(F)$. $\bar{\mathscr{L}}'$ will be the subset of bounded functions of $\bar{\mathscr{L}}$.

The semigroup of the coupled process $\bar{S}(t)$ is constructed from the following formal generator:

$$(\bar{\Omega}f)(\eta,\xi) = \sum_{x:\eta(x)\ge\xi(x)} (g(\eta(x)) - g(\xi(x))) \sum_{y\in S} p(x,y)(f(\eta_{xy},\xi) - f(\eta,\xi))$$
$$+ \sum_{x:\eta(x)\le\xi(x)} (g(\xi(x)) - g(\eta(x))) \sum_{y\in S} p(x,y)(f(\eta,\xi_{xy}) - f(\eta,\xi))$$
$$+ \sum_{x\in S} \min(g(\eta(x))g(\xi(x))) \sum_{y\in S} p(x,y)(f(\eta_{xy},\xi_{xy}) - f(\eta,\xi))$$

for all $f \in \bar{\mathscr{L}}$.

$\bar{S}(t)$ and $\bar{\Omega}$ will satisfy Lemmas 2.7, 2.8 and 2.9 where $\|(\eta,\xi)\|$, $|||\eta,\xi|||$ and $\|(\eta,\xi)\|_{\ell}$ are defined as $\|\eta\| + \|\xi\|$, $|||\eta||| + |||\xi|||$ and $\|\eta\|_{\ell} + \|\xi\|_{\ell}$ respectively.

The proofs concerning the construction and these lemmas are the same as the ones for the single process.

We now prove the properties of the coupled process that will be needed later. In the following lemma, the condition $g(k) \le g(k+1)$ is essential.

LEMMA 4.1. *If we start the coupled process from (η, ξ) and $\eta \le \xi$ then $p^{(\eta,\xi)}(\eta_t \le \xi_t) = 1$ for all $t \ge 0$. A similar statement holds for $\eta \ge \xi$ and for $\eta = \xi$.*

PROOF. Let

$$f(\eta,\xi) = \begin{cases} 0 & \text{if } \eta \le \xi \\ \sup\{\alpha(x) : x\in S \text{ and } \eta(x) > \xi(x)\} & \text{otherwise} \end{cases}$$

and let $\bar{S}_n(t)$ be the sequence of semigroups approximating $\bar{S}(t)$, as in Section 2. Then if $\eta \le \xi$ $\bar{S}_n(t)f(\eta,\xi) = 0$ and passing to the limit we get $\bar{S}(t)f(\eta,\xi) = 0$, this implies the lemma.

We will denote by $\bar{\mathscr{I}}$ the set of invariant measures for the coupled process; i.e., the probability measures ν on $\bar{\varepsilon}$ such that $\nu\bar{S}(t) = \nu$. In particular if $\nu \in \bar{\mathscr{I}}$, its marginals are in $\mathscr{I}$. $\mathscr{I}_e$ and $\bar{\mathscr{I}}_e$ will be the sets of extremal points in $\mathscr{I}$ and $\bar{\mathscr{I}}$ respectively.

LEMMA 4.2. *If μ_1 and μ_2 are in $\mathscr{I}$ then there exists $\nu_0 \in \bar{\mathscr{I}}$ with marginals μ_1 and μ_2.*

PROOF. Let $\nu = \mu_1 \times \mu_2$. Since the marginals of $\nu\bar{S}(t)$ are independent of t, the collection of measures

$$\frac{1}{T}\int_0^T \nu\bar{S}(t)\,dt, \quad T \in [1, \infty),$$

is tight. Let $T_n \to \infty$ in such a way that $\nu_n = (1/T_n)\int_0^{T_n} \nu\bar{S}(t)\,dt$ converges weakly to ν_0. The marginals of ν_0 are μ_1 and μ_2; hence $\nu_0(\bar{\varepsilon}) = 1$.

Since the marginals of ν_n are independent of n, Lemma 2.11 applies. Let $f \in \bar{\mathscr{L}}'$. Then:

$$\int \bar{S}(t_0)f\,d\nu_0 = \lim_{n\to\infty} \int \bar{S}(t_0)f\left(\frac{1}{T_n}\int_0^{T_n} \nu\bar{S}(t)\,dt\right) = \lim_{n\to\infty}\frac{1}{T_n}\int_{0_T}^{T_n}\left(\int \bar{S}(t+t_0)f\,d\nu\right)dt$$

$$= \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\left(\int \bar{S}(t)f\,d\nu\right)dt = \lim_{n\to\infty}\frac{1}{T_n}\int f\left(\int_0^{T_n}\nu\bar{S}(t)\,dt\right) = \int f\,d\nu_0$$

and $\nu_0 \xi \bar{\mathscr{I}}$.

LEMMA 4.3. *If μ_1 and μ_2 are in $\mathscr{I}_e$, then there exists a $\nu \in \bar{\mathscr{I}}_e$ with marginals μ_1 and μ_2.*

PROOF. Let $\mathscr{A}$ be the subset of $\bar{\mathscr{I}}$ formed by the measures with marginals μ_1 and μ_2. By Lemma 4.2. $\mathscr{A} \neq \emptyset$ and since $\mathscr{A}$ is compact in the weak topology, $\mathscr{A}_e \neq \emptyset$ by the Krein-Millman theorem. Take $\nu \in \mathscr{A}_e$ and suppose $\nu = \lambda\nu_1 + (1-\lambda)\nu_2$ where $\nu_1, \nu_2 \in \bar{\mathscr{I}}$ and $0 < \lambda < 1$. Since the marginals of ν are extremal, ν_1 and ν_2 must have the same marginals as ν, therefore $\nu_1, \nu_2 \in \mathscr{A}$ and this implies that $\nu_1 = \nu_2 = \nu$ because ν is extremal in $\mathscr{A}$. Hence $\nu \in \bar{\mathscr{I}}_e$.

When $S = \mathbb{Z}^d$ we denote by $\mathscr{S}$ and $\bar{\mathscr{S}}$ the translation invariant probability measures on ε and $\bar{\varepsilon}$ respectively.

LEMMA 4.4. *Lemmas 4.2 and 4.3 hold if we replace $\mathscr{I}$ by $\mathscr{I} \cap \mathscr{S}$ and $\bar{\mathscr{I}}$ by $\bar{\mathscr{I}} \cap \bar{\mathscr{S}}$.*

The proof is the same as in Lemmas 4.2 and 4.3.

LEMMA 4.5. *If $\nu \in \bar{\mathscr{I}}_e$ (or $(\bar{\mathscr{I}} \cap \bar{\mathscr{S}})_e$) and $\nu\{(\eta, \varepsilon): \eta \geq \xi$ or $\xi \geq \eta\} = 1$ then*

$$\nu\{(\eta, \xi): \eta \geq \xi\} = 1 \quad \text{or} \quad \nu\{\eta, \xi): \xi \geq \eta\} = 1.$$

PROOF. Let $A = \{(\eta, \xi): \eta \geq \xi\}$. We assume, without loss of generality, that $\nu(A) = \lambda > 0$. Then there exist two probability measures ν_1 and ν_2 such that $\nu_1(A) = 1$, $\nu_2(\bar{\varepsilon} - A) = 1$ and $\nu = \lambda\nu_1 + (1-\lambda)\nu_2$; then $\nu = \nu\bar{S}(t) = \lambda\nu_1\bar{S}(t) + (1-\lambda)\nu_2\bar{S}(t)$. By Lemma 4.1, $\nu_1\bar{S}(t)(A) = 1$; therefore $\nu_1\bar{S}(t) = \nu_1$ and, since ν is extremal, λ must be one.

The purpose of the next two lemmas is to give sufficient conditions under which $\nu \in \bar{\mathscr{I}}$ satisfies $\nu\{(\eta, \xi): \eta \geq \xi$ or $\xi \geq \eta\} = 1$.

LEMMA 4.6. *If $\nu \in \bar{\mathscr{I}}$ and $x, y \in S$ satisfy*

$$\nu\{(\eta, \xi): \eta(x) > \xi(x) = 0, \eta(y) < \xi(y)\} = 0$$

then

$$\nu\{(\eta, \xi): \eta(x) > \xi(x), \eta(y) < \xi(y)\} = 0.$$

PROOF. We prove $\nu\{(\eta, \xi): \eta(x) > \xi(x) = k, \eta(y) < \xi(y)\} = 0$ by induction on k. For

$k = 0$, it is part of the hypothesis; suppose it is true for $k - 1$ and let $f(\eta, \xi)$ be the indicator function of $\{(\eta, \xi) : \eta(x) > \xi(x) = k - 1, \eta(y) < \xi(y)\}$.

We would like to use Lemma 2.9 to conclude $\int \bar{\Omega} f \, d\nu = 0$. Unfortunately we might have $\int |||(\eta, \xi)||| \, d\nu = \infty$. For that reason we consider ν_k the restriction of ν to the set $\{(\eta, \xi) : |||\eta, \xi||| \leq K\}$. ν_K is a subprobability measure and since $f \geq 0$ and $\int \bar{S}(t) f \, d\nu = \int f \, d\nu = 0$ we must have: $\int \bar{S}(t) f \, d\nu_K = \int f \, d\nu_K = 0$. Now the argument used in the proof of Lemma 2.9 shows that $\int \bar{\Omega} f \, d\nu_K = 0$. But $\bar{\Omega} f(\eta, \xi) < 0$ only if $f(\eta, \xi) = 1$ and this occurs only on a set of ν_K measure 0, hence

$$0 = \int \bar{\Omega} f \, d\nu_K \geq g(k) \nu_K \{(\eta, \xi) : \eta(x) > \xi(x) = k, \eta(y) < \xi(y)\} \cdot \left(\textstyle\sum_{z \neq x} p(x, z)\right) \geq 0.$$

Since this is true for all K, we can complete our induction by letting K go to ∞.

Lemma 4.7. *If $\nu \in \bar{\mathscr{I}}$ and $\nu\{(\eta, \xi) : \eta(x) > \xi(x), \eta(y) < \xi(y)\} = 0$ whenever $p(x, y) > 0$ then $\nu\{(\eta, \xi) : \eta(x) > \xi(x), \eta(y) < \xi(y)\} = 0$ for all x and y in S.*

Proof. Since $p(x, y)$ is irreducible, there exists a finite sequence $x_0 = x, x_1, x_2, \cdots, x_n = y$ such that $p(x_i, x_{i+1}) > 0$, $i = 0, 1, \cdots, n - 1$. We prove the lemma using induction on n. Again the case $n = 1$ is trivial. We assume that it holds for $n - 1$.

Let $f(\eta, \xi)$ be the indicator function of the set $\{(\eta, \xi) : \eta(x_1) > \xi(x_1), \eta(y) < \xi(y)\}$ and ν_K be as in the previous lemma. We now have

$$\begin{aligned} 0 = \int \bar{\Omega} f \, d\nu_K &\geq \textstyle\sum_{k=1}^{\infty} g(k) p(x, x_1) \nu_K \{(\eta, \xi) : \eta(x) = k, \\ &\qquad \xi(x) = 0, \eta(x_1) \geq \xi(x_1), \eta(y) < \xi(y)\} \\ &= \textstyle\sum_{k=1}^{\infty} g(k) p(x, x_1) \nu_K \{(\eta, \xi) : \eta(x) = k, \xi(x) = 0, \eta(y) < \xi(y)\} \geq 0. \end{aligned}$$

Therefore, letting K go to ∞, we get:

$$\nu\{(\eta, \xi) : \eta(x) = k, \xi(x) = 0, \eta(y) < \xi(y)\} = 0.$$

Since this is true for all k, Lemma 4.7 follows from Lemma 4.6.

Corollary 4.8. *If $\nu \in \bar{\mathscr{I}}$ satisfies*

$$\nu\{(\eta, \xi) : \eta(x) > 0, \xi(x) = 0, \eta(y) < \xi(y)\} = 0$$

whenever $p(x, y) > 0$ then

$$\nu\{(\eta, \xi) : \eta \geq \xi \quad \text{or} \quad \xi \geq \eta\} = 1.$$

Definition 4.9. If μ_1 and μ_2 are probability measures on ε we will say that $\mu_1 \leq \mu_2$ if there exists a probability measures ν on $\bar{\varepsilon}$ with first marginal μ_1, second marginal μ_2 and such that $\nu(\{(\eta, \xi) : \eta \leq \xi\}) = 1$.

Lemma 4.10. *If $\mu_1 \leq \mu_2$ then for all $(x_1, \cdots, x_n) \in S^n$ and all $(k_1, \cdots, k_n) \in (Z_+)^n$ we have*:

$$\begin{aligned} \mu_1(\{\eta : \eta(x_i) \geq k_i, 1 \leq i \leq n\}) &= \nu(\{(\eta, \xi) : \eta(x_i) \geq k_i, 1 \leq i \leq n\}) \\ \leq \nu(\{(\eta, \xi) : \xi(x_i) \geq k_i, 1 \leq i \leq n\}) &= \mu_2(\{\xi : \xi(x_i) \geq k_i, 1 \leq i \leq n\}). \end{aligned}$$

Proposition 4.11. *The relation $\leq$ is a partial order on the set of probability measures on ε.*

Proof. If $\mu_1 \leq \mu_2$ and $\mu_2 \leq \mu_1$ then by lemma 4.10, μ_1 and μ_2 have the same finite dimensional distributions and, therefore, are equal.

If μ is a probability measures on ε we can define a measure ν on $\bar{\varepsilon}$ with the following equality: $\nu(A) = \mu(D^{-1}(A))$ where D is the diagonal embedding of ε on $\bar{\varepsilon}$ ($D(\eta) = (\eta, \eta)$) and A is a measurable set in $\bar{\varepsilon}$. Both marginals of ν are μ and $\nu(\{(\eta, \eta)\}) = 1$, hence $\mu \leq \mu$.

Finally, suppose $\mu_1 \leq \mu_2$, $\mu_2 \leq \mu_3$ and let $\nu_{1,2}$ and $\nu_{2,3}$ be the corresponding measures on $\bar{\varepsilon}$. Given S_0, a finite subset of S, denote by $\eta \mid S_0$ the restriction of $\eta \in \varepsilon$ to S_0. Now we construct a probability measure ν_0 on $(Z_+)^{S_0}X(Z_+)^{S_0}$in this way: given $(\eta_0, \xi_0) \in (Z_+)^{S_0}X(Z+)^{S_0}$ we let

$$\nu_0(\{\eta_0, \xi_0\}) = \sum_{\delta_0\in(Z_+)^{S}0,\eta_0\leq\delta_0\leq\xi_0,\mu_2(\{\delta:\delta\mid S_0=\delta_0\})>0}$$

$$\nu_{2,3}(\{(\delta, \xi) : \delta \mid S_0 = \delta_0, \xi \mid S_0 = \xi_0\}) \cdot \frac{1}{\mu_2(\{\delta : \delta \mid S_0 = \delta_0\})}.$$

A simple computation shows that the marginals of ν_0 are equal to the restriction of μ_1 and μ_3 to $(Z_+)^{S_0}$. Since $\nu_0(\{(\eta, \xi) : \eta \leq \xi\}) = 1$, the Kolmogorof extension theorem provides a measure ν on $\bar{\varepsilon}$ which shows that $\mu_1 \leq \mu_3$.

Lemma 4.12. *If π_1 and π_2 are strictly positive real valued function on S, $\pi_1 \leq \pi_2$, and μ_{π_i} is a probability measure on ε ($i = 1, 2$) then $\mu_{\pi_1} \leq \mu_{\pi_2}$*

Proof. Since μ_{π_1} and μ_{π_2} are product measures, it suffices to show that $\mu_{\pi_1}(\{\eta : \eta(x) \geq k\}) \leq \mu_{\pi_2}(\{\eta : \eta(x) \geq k\})$ for all $x \in S$ and all k in the set of positive integers. This inequality is equivalent to:

$$\frac{\mu_{\pi_1}(\{\eta : \eta(x) \geq k\})}{\mu_{\pi_1}(\{\eta : \eta(x) < k\})} \leq \frac{\mu_{\pi_2}(\{\eta : \eta(x) \geq k\})}{\mu_{\pi_2}(\{\eta : \eta(x) < k\})}$$

and this is implied by:

$$\frac{\mu_{\pi_1}(\{\eta : \eta(x) = \ell\})}{\mu_{\pi_1}(\{\eta : \eta(x) = j\})} \leq \frac{\mu_{\pi_2}(\{\eta : \eta(x) = \ell\})}{\mu_{\pi_2}(\{\eta : \eta(x) = j\})} \quad \text{for all} \quad j < \ell.$$

A straightforward computation shows that this inequality, and hence the lemma, hold.

5. The translation invariant case. Throughout this section, S will be the d-dimensional integer lattice $\mathbb{Z}^d$ and p will satisfy: $p(x, y) = p(0, y - x)$ for all $x, y \in \mathbb{Z}^d$.

Proposition 5.1. *If $\nu \in \bar{\mathscr{I}} \cap \bar{\mathscr{S}}$ and $\int |||\xi||| \, d\nu\,(\eta, \xi) < \infty$, then*

$$\nu\{(\eta, \xi) : \eta \geq \xi \quad \text{or} \quad \xi \leq \eta\} = 1.$$

Proof. By Corollary 4.8 it suffices to show that

$$\nu\{(\eta, \xi) : \eta(x) > \xi(x) = 0, \eta(y) < \xi(y)\}p(x, y) = 0$$

for all $x, y \in \mathbb{Z}^d$. Fix $x, y \in \mathbb{Z}^d$ $x \neq y$ and consider the map h: $\bar{\varepsilon} \to \bar{\varepsilon}$ defined by $h(\eta, \xi) = (\eta', \xi)$ where

$$\eta'(z) = \begin{cases} \xi(z) \wedge \eta(z) & \text{if} \quad z \neq x \\ (\eta(x) \wedge 1) \vee (\eta(x) \wedge \xi(x)) & \text{if} \quad z = x. \end{cases}$$

Then $\nu'(A) = \nu(h^{-1}(A))$ defines a measure on $\bar{\varepsilon}$.

Let $f(\eta, \xi) = |\xi(y) - \eta(y)|^+$ and consider a coupling of 3 copies of the process such that the projection over any pair of coordinates coincides with the coupled process of Section 4. Starting this process from the configuration (η', η, ξ) where $h(\eta, \xi) = (\eta', \xi)$ and noting that $\eta' \leq \eta$, we deduce:

$$(\bar{S}(t)f)\; h(\eta, \xi) \geq (\bar{S}(t)f)(\eta, \xi).$$

Since $\int (\bar{S}(t)f)(\eta, \xi)\; d\nu'(\eta, \xi) = \int (\bar{S}(t)f) \,|\, h(\eta, \xi) \,|\, d\nu(\eta, \xi)$ and $\int \bar{S}(t)f \, d\nu = \int f \, d\nu'$, it follows that $\int \bar{S}(t)f \, d\nu' \geq \int f \, d\nu'$ and since $\int |||\eta, \xi||| \; d\nu' < \infty$, an argument similar to the proof of Lemma 2.9 shows that $\int \bar{\Omega} f\, d\nu' \le 0$. But among the negative terms of $\int \bar{\Omega} f\, d\nu'$ we have:

$$\sum_{z\neq y} p(y,z) \sum_{k>\ell}(g(k)-g(\ell))\nu'\{\eta(y)=\ell, \xi(y)=k\}$$
$$= \Big(\sum_{z\neq y} p(y,z)\Big) \sum_{k>\ell}(g(k)-g(\ell))\nu\{\eta(y)=\ell, \xi(y)=k\} \tag{5.2}$$

and

$$g(1)\nu'\{\eta(x)=1, \xi(x)=0, \eta(y)<\xi(y)\}p(x,y)$$
$$= g(1)\nu\{\eta(x)\ge 1, \xi(x)=0, \eta(y)<\xi(y)\}p(x,y) \tag{5.3}$$

and the positive terms are bounded by:

$$\sum_{z\neq y} p(z,y) \sum_{k>\ell}(g(k)-g(\ell))\nu'\{\xi(z)=k, \eta(z)=\ell\}$$
$$= \sum_{z\neq y} p(z,y) \sum_{k>\ell}(g(k)-g(\ell))\nu\{\xi(z)=k, \eta(z)=\ell\}.$$

Since p and ν are translation invariant, this last expression is equal to 5.2; hence 5.3 is 0 and the proposition follows from Corollary 4.8.

COROLLARY 5.4. *If* $\mu_1, \mu_2 \in (\mathscr{I}\cap\mathscr{S})e$ *and* $\int |||\eta|||\, d\mu_i < \infty$ *for* $i = 1$ *or* 2 *then* $\mu_1 \le \mu_2$ *or* $\mu_2 \le \mu_1$.

PROOF. This follows from Proposition 5.1 and Lemmas 4.4 and 4.5.

Since $\pi(x) \equiv \rho$ satisfies $\sum \pi(x)p(x,y) = \pi(y)$, it follows from Lemmas 3.1 and 3.2 that $\sum \alpha(x) < \infty$ and $\int |||\eta|||\, d\mu\rho < \infty$ for all $0 < \rho < \sup g(k)$. A simple estimate also shows that $\int \eta(x) d\mu\rho(\eta) < \infty$ for all $0 \le \rho < \sup g(k)$, hence by Theorem 1.8 $\mu\rho \in \mathscr{I}$ for these ρ's.

PROPOSITION 5.5. *If* $\mu \in (\mathscr{I}\cap\mathscr{S})_e$ *then* $\mu = \mu\rho_0$ *for some* $0 \le \rho_0 < \sup g(k)$.

PROOF. Since $\mu\rho$ is ergodic, $\mu\rho \in \mathscr{S}e$ and since it is invariant, $\mu\rho \in (\mathscr{I}\cap\mathscr{S})e$. By Corollary 5.4, $\mu \ge \mu\rho$ or $\mu \le \mu\rho$ for all $0 < \rho < \sup g(k)$. But $\mu \ge \mu_\rho$ cannot hold for all $0 < \rho < \sup g(k)$ because $\lim_{\rho\uparrow\sup g(k)} \mu_\rho\{\eta : \eta(x) \ge k\} = 1$ for all $k \in Z_+$. Hence by Proposition 4.11 and Lemma 4.12, there exists a ρ_0 such that $\mu \ge \mu_\rho$ for all $\rho < \rho_0$ and $\mu \le \mu\rho$ for all $\rho > \rho_0$; this forces μ to be equal to $\mu\rho_0$ on sets of the form $\{\eta : \eta(x_1) \le k_1, \cdots, \eta(x_n) \le k_n\}$ because of the continuity and monotonicity in ρ of the $\mu\rho$ measures of these sets. Hence $\mu = \mu\rho_0$.

PROOF OF THEOREM 1.9. To simplify the notation, we denote again by S the set of sites. Let $\bar{\mathbb{Z}}_+$ be the one point compactification of $\mathbb{Z}_+$ and consider $\mathbb{Z}_+^S$ embedded in $(\bar{\mathbb{Z}}_+)^S$, a compact metric space. $\mathscr{I}\cap\mathscr{S}$ becomes a set of probability measures on $(\bar{\mathbb{Z}}_+)^S$ and $C(\mathscr{I}\cap\mathscr{S})$ will denote its weak closure. $C(\mathscr{I}\cap\mathscr{S})$ is convex, compact and metrizable in the weak topology. See Theorem 1.11 on [19]. By Choquet's Theorem (page 19–20 in [17]) for any $\mu \in \mathscr{I}\cap\mathscr{S}$ there is probability measure λ on the set of extreme points of $C(\mathscr{I}\cap\mathscr{S})$ such that:

$$\mu = \int \beta\, d\lambda(\beta).$$

Since $1 = \mu(\varepsilon) = \int \beta(\varepsilon)\, d\lambda(\beta)$, λ must concentrate on β's such that $\beta(\varepsilon) = 1$. Hence Theorem 1.9 is implied by Proposition 5.5 and the following lemma.

LEMMA 5.6. *If* β *is an extreme point of* $C(\mathscr{I}\cap\mathscr{S})$ *and* $\beta(\varepsilon) = 1$ *then* $\beta \in \mathscr{I}\cap\mathscr{S}$.

To prove this lemma we first establish the convention $\Delta > n$ for all $n \in \mathbb{Z}_+$ where $\{\Delta\} = \bar{\mathbb{Z}}_+ - \mathbb{Z}_+$. The definition of $\eta \ge \xi$, $\eta, \xi \in (\mathbb{Z}_+)^S$ is now extended to elements of $(\bar{\mathbb{Z}}_+)^S$ and this allows us to extend definition 4.9 to probability measure on $(\bar{\mathbb{Z}}_+)^S$. Lemma 5.6 follows from the next lemma.

LEMMA 5.7. *If β is an extreme point of $C(\mathcal{I} \cap \mathcal{S})$ and $0 \le \rho < \sup g(k)$ then $\beta \ge \mu\rho$ or $\beta \le \mu\rho$.*

PROOF. Fix $0 < \rho < \sup g(k)$ and let β_n be a sequence in $\mathcal{I} \cap \mathcal{S}$ converging to β and $\nu_{n,\rho}$ elements of $\bar{\mathcal{I}} \cap \bar{\mathcal{S}}$ with marginals β_n and $\mu\rho$. By Proposition 5.1, $\nu_{n,\rho}\{\eta \ge \xi \text{ or } \xi \ge \eta\} = 1$. Let $\lambda_n = \nu_{n,\rho}\{\eta \ge \xi\}$. If there is a subsequence where $\lambda_n = 0$ or 1, the lemma is obvious; we can, therefore, assume that $0 < \lambda_n < 1$. Take $\nu^1_{n,\rho}$ and $\nu^2_{n,\rho}$ to be probability measures on $\bar{\varepsilon}$ such that $\nu^1_{n,\rho}\{\eta \ge \xi\} = 1$, $\nu^2_{n,\rho}\{\xi \ge \eta\} = 1$ and $\nu_{n,\rho} = \lambda_n \nu^1_{n,\rho} + (1 - \lambda_n)\nu^2_{n,\rho}$. The proof of Lemma 4.5 shows that $\nu^1_{n,\rho} \in \bar{\mathcal{I}}, \cap \bar{\mathcal{S}}$, hence $\nu^2_{n,\rho} \in \bar{\mathcal{I}} \cap \bar{\mathcal{S}}$. Since $\mu\rho$ is ergodic, it is in $(\mathcal{I} \cap \mathcal{S})e$; therefore the second marginal of $\nu^i_{n,\rho}$ is $\mu\rho$, $i = 1, 2$. Taking an appropriate subsequence we get $\nu\rho = \lambda\nu^1_\rho + (1 - \lambda)\nu^2_\rho$ where $\nu\rho$, ν^1_ρ and ν^2_ρ are probability measures on $(\bar{\mathbb{Z}}_+)^S \times (\bar{\mathbb{Z}}_+)^S$ such that the first marginal of $\nu\rho$ is β, the first marginal of ν^i_ρ is in $C(\mathcal{I} \cap \mathcal{S})$, $i = 1, 2$, the second marginals of $\nu\rho$, ν^1_ρ and ν^2_ρ are all $\mu\rho$, $\nu^1_\rho\{\eta \ge \xi\} = 1$ and $\nu^2_\rho\{\xi \ge \eta\} = 1$. If $\lambda = 0$ or 1 the lemma is proved; if $0 < \lambda < 1$, since β is extremal, the first marginal of ν^i_ρ is β $i = 1, 2$ and the lemma follows.

6. The null-recurrent case. In this section $p(x, y)$ will be null recurrent, hence all positive solutions of $\sum_x \pi(x)p(x, y) = \pi(y)$ are multiples of a given one. To simplify notation, we fix a solution π and denote by $\mu\rho$ the measure $\mu\rho_\pi$.

Let $A = \{\rho \ge 0 : \rho\pi(x) < \sup g(k) \text{ for all } x \in S\}$. Throughout this section we assume that $\mu\rho(\varepsilon) = 1$ for all $\rho \in A$.

LEMMA 6.1. *Let $h(x) = \sum_{k=1}^\infty g(k)\mu\{\eta(x) = k\}$ where $\mu \in \mathcal{I}$ and $\int |||\ \eta\ |||\ d\mu < \infty$, then $h(x) = \sum_{y\in S} h(y)p(y, x)$.*

PROOF. We apply Lemma 2.9 to $g(\eta) = \eta = (x) : \int \Omega g\, d\mu = 0$. Computing the positive and negative terms we get $h(x) \sum_{z\ne x} p(x, z) = \sum_{y\ne x} h(y)p(y, x)$. Adding $h(x)p(x, x)$ to both sides of this equality proves the lemma.

LEMMA 6.2. *Suppose f is a non-negative function on S satisfying $f(x) \le \sum_y f(y)p(y,x)$ for all $x \in S$ and $f(x) \le \rho\pi(x)$ for some positive ρ and all $x \in S$; then $f(x) = \sum_y f(y)p(y, x)$ for all $x \in S$.*

PROOF. Let $g(x) = \rho\pi(x) - f(x) \ge 0$; *then* $\sum_x g(x)p(x, y) \le g(y)$. By Proposition 6.4 in [7], $\sum_x g(x)p(x, y) = g(y)$; since $\pi(x)$ also satisfies this equality, so does $f(x)$.

PROPOSITION 6.3. *If $\nu \in \bar{\mathcal{I}}$ and $\int |||\ \xi\ |||\ d\nu(\eta, \xi) < \infty$ then $\nu\{(\eta\ \xi) : \eta \ge \xi \text{ or } \xi \ge \eta\} =$* 1.

PROOF. As in the proof of Proposition 5.1 we get

$$\sum_{z\ne y} p(y, z) \sum_{k>\ell}(g(k) - g(\ell))\nu\{\eta(y) = \ell, \xi(y) = k\}$$
$$+ g(1)\nu\{\eta(x) \ge 1, \xi(x) = 0, \eta(y) < \xi(y)\}p(x, y)$$
$$\le \sum_{z\ne y} p(z, y)\Big[\sum_{k>\ell}(g(k) - g(\ell))\nu\{\xi(z) = k, \eta(z) = \ell\}\Big].$$

Adding to both sides $p(y, y) \sum_{k>\ell}(g(k) - g(\ell))\nu\{\xi(y) = k, \eta(y) = \ell\}$, we conclude that $f(x) = \sum_{k>\ell}(g(k) - g(\ell))\nu\{\xi(x) = k, \eta(x) = \ell\}$ satisfies $f(x) \le \sum_y f(y)p(y, x)$. Since $f(x) \le \sum_{k=1}^\infty g(k)\nu\{\xi(x) = k\} = h(x)$ and, by Lemma 6.1, $h(x) = \sum_y h(y)p(y, x)$ we can apply Lemma 6.2 to conclude that $f(x) = \sum_y f(y)p(y, x)$. This implies that $g(1)\nu\{\xi(x) = 0, \eta(x) \ge 1, \eta(y) < \xi(y)\}p(x, y) = 0$ and the Proposition follows from Corollary 4.8.

Proposition 6.4. *If $\mu \in \mathscr{I}e$ then $\exists \rho_0 \in A$ such that $\mu \geq \mu_\rho$ for all $\rho < \rho_0$ and $\mu \leq \mu_\rho$ for all $\rho > \rho_0 (\rho \in A)$.*

Proof. It is not possible to follow the argument of Section 5 because we cannot prove directly that $\mu_\rho \in \mathscr{I}e$.

Let $\rho \in A$, by Lemma 4.2 there exists $\nu_\rho \in \bar{\mathscr{I}}$ with marginals μ and μ_ρ. By Proposition 6.3, $\nu_\rho\{\eta \geq \xi \text{ or } \xi \geq \eta\} = 1$. Let $\lambda\rho = \nu\rho\{\eta \geq \xi\}$ and suppose $0 < \lambda_0 < 1$; then the proof of Lemma 4.5 shows that $\nu\rho = \lambda_\rho \nu_\rho^1 + (1 - \lambda_\rho)\nu_\rho^2$ where $\nu_\rho^i \in \bar{\mathscr{I}}$, $i = 1, 2$, $\nu_\rho^1\{\eta \geq \xi\} = 1$ and $\nu_\rho^2\{\eta \leq \xi\} = 1$. Since μ is extremal, the first marginal of ν_ρ^i is μ, $i = 1, 2$. The second marginals μ_ρ^1 and μ_ρ^2 are absolutely continuous with respect to μ_ρ and satisfy $\mu_\rho^1 \leq \mu$, $\mu_\rho^2 \geq \mu$.

Since $\sum \pi(x) = \infty$ we can write an array of elements $x_{ij} \in S$ $i = 1, 2, \cdots, 1 \leq j \leq n_i$ such that $x_{ij} \neq x_{hk}$ if $i \neq h$ or $j \neq k$ and

$$\textstyle\sum_{j=1}^{n_i} \pi(x_{ij}) \geq 1 \quad \text{for} \quad i = 1, 2, \cdots.$$

Now we estimate the first two moments of

$$f_i(\xi) = \frac{\sum_{j=1}^{n_i} g(\xi(x_{ij}))}{\sum_{j=1}^{n_i} \pi(x_{ij})}$$

where the distribution of ξ is given by μ_ρ:

$$\int g(\eta(x))\, d\mu_\rho(\eta) = \textstyle\sum_{k=1}^{\infty} g(k)\mu_\rho\{\eta(x) = k\} = \sum_{k=1}^{\infty} g(k)\lambda_x \dfrac{(\rho\pi(x))^k}{g(1)\cdots g(k)}$$

$$= \rho\pi(x)\left(\gamma_x + \textstyle\sum_{k=1}^{\infty} \gamma_x \dfrac{(\rho\pi(x))^k}{g(1)\cdots g(k)}\right) = \rho\pi(x).$$

Hence

$$\int f_i(\xi)\, d\mu\rho(\xi) = \rho \tag{6.5}$$

$$\int [g(\eta(x))]^2\, d\mu_\rho(\eta) = \textstyle\sum_{k=1}^{\infty} g^2(k)\mu_\rho\{\eta(x) = k\}$$

$$\leq C\left[\gamma_x\rho\pi(x) + \textstyle\sum_{k=2}^{\infty} g(k)g(k-1)\gamma_x \dfrac{(\rho\pi(x))^k}{g(1)\cdots g(k)}\right]$$

where

$$C = \sup\left\{g(1), \frac{g(2)}{g(1)}, \frac{g(3)}{g(2)} \cdots\right\}.$$

But our last expression is bounded by $C\rho\pi(x) + C(\rho\pi(x))^2$ since $\gamma_x \leq 1$. Using this estimate and the independence of $\eta(x)$ and $\eta(y)$ for $x \neq y$ under μ_ρ we get:

$$\begin{aligned}\int f_i^2(\eta)\, d\mu_\rho(\eta) &\leq \left(\textstyle\sum_{j=1}^{n_i} C\rho^2\pi^2(x_{ij}) + \sum_{j=1}^{n_i} C\rho\pi(x_{ij})\right). \dfrac{1}{(\sum_{j=1}^{n_i}\pi(x_{ij}))^2} \\ &+ \dfrac{2\rho^2}{(\sum_{j=1}^{n_i}\pi(x_{ij}))^2}\left(\textstyle\sum_{1\leq j_1 \leq j_2 \leq n_i} \pi(x_{ij_1})\pi(x_{ij_2})\right) \leq C\rho^2 + C\rho + \rho^2.\end{aligned} \tag{6.6}$$

Theorem 5.1.2 in [2], (6.5) and (6.6) show that $(1/n)\sum_{i=1}^{n} f_i(\eta) \to \rho$. a.s. $(d\mu_\rho)$. Since μ^1 and μ^2 are absolutely continuous with respect to $\mu\rho$ and $\mu_\rho^1 \leq \mu$ and $\mu_\rho^2 \geq \mu$, we must have $(1/n)\sum_{i=1}^{n} f_i(\eta) \to \rho$ a.s. $(d\mu)$. This can happen for at most one value of ρ. For all others, our assumption $0 < \lambda_\rho < 1$ must be false; hence $\mu \geq \mu_\rho$ or $\mu \leq \mu_\rho$ for all values of ρ except, at most, one. This, together with 4.11 and 4.12, prove the proposition if A is a bounded closed interval. If $A = [0, \infty)$ then $\lim_{\rho\uparrow\infty} \mu_\rho\{\eta : \eta(x) \geq k\} = 1$ for all $k \in Z_+$ and $\mu > \mu_\rho$ cannot be satisfied for all $\rho \in A$; hence, in this case, the proposition also holds. Finally if

$A = [0, a)$, $a < \infty$ then there exists an element z in S such that $\pi(z) = \sup \pi(y)$, $y \in S$ and $\pi(z).a = \sup g(k)$. Then $\lim_{\rho \uparrow a} \mu_\rho\{\eta : \eta(z) \geq k\} = 1$ for all $k \in Z_+$ and again $\mu \geq \mu_\rho$ cannot be satisfied for all $\rho \in A$ and the proposition holds in this case too.

PROPOSITION 6.7. *If $\mu \in \mathscr{I}e$ then $\mu = \mu_{\rho_0}$ for some $\rho_0 \in A$.*

PROOF. Let $\rho_0 \in A$ be as in the conclusion of Proposition 6.4. Two cases have to be considered:

a) $\rho_0 < \sup A$

b) $\rho_0 = \sup A$

In case a) the proof of 6.7 is the same as the proof of Proposition 5.5. In case b) we need the next two lemmas.

LEMMA 6.8. *If $\sup_k g(k) < \infty$ and $\mu \in \mathscr{I}$ then $f(x) = \sum_k (g(k))(\mu\{\eta(x) = k\})$ satisfies $f(x) = \sum_y f(y)p(y, x)$.*

PROOF. Let $h_N(\eta) = \eta(x) \wedge N$, then

$$S(t)h_N(\eta) - h_N(\eta) = \int_0^t S(s)\Omega h_N(\eta)\, ds = \int_0^t S(s)([\Omega h_N(\eta)]^+ - [\Omega h_N(\eta)]^-)\, ds$$

where the first equality comes from Lemma 2.7.b).

Since $[\Omega h_N(\eta)]^- \leq \sup g(k) < \infty$, we can separate the integral in two parts and get:

$$S(t)h_N(\eta) - h_N(\eta) = \int_0^t S(s)[\Omega h_N(\eta)]^+\, ds - \int_0^t S(s)[\Omega h_N(\eta)]^-\, ds.$$

Now, integrating both sides against μ, the left hand side becomes 0 and because the second term of the right hand side is bounded by $t \sup g(k) < \infty$, we can write this side as $\int(\int_0^t S(s)[\Omega h_N(\eta)]^+\, ds)\, d\mu - \int(\int_0^t S(s)[\Omega h_N(\eta)]^-\, \mathrm{ds})\, \mathrm{d}\mu$. Hence

$$\int_0^t \left(\int S(s)[\Omega h_N(\eta)]^+\, d\mu\right) ds = \int_0^t \left(\int S(s)[\Omega h_N(\eta)]^-\, d\mu\right) ds.$$

Since $\mu \in \mathscr{I}$, this implies $\int[\Omega \mathrm{h}_N(\eta)]^+\, d\mu = \int[\Omega h_N(\eta)]^-\, d\mu$. Computing both sides of this last equality proves the Lemma.

LEMMA 6.9. *If $\mu \geq \mu_{\rho_0}$, $\mu \neq \mu_{\rho_0}$ and $\mu \in \mathscr{I}$ then $\sum_k g(k)\mu\{\eta(x) = k\} > \sum_k g(k)\mu_{\rho_0}\{\eta(x) = k\}$ for some $x \in S$.*

PROOF. Take $\nu_{\rho_0} \in \bar{\mathscr{I}}$ with marginals μ and μ_{ρ_0} and such that $\nu\rho_0\{\eta \geq \xi\} = 1$. Since $\mu \neq \mu_{\rho_0}$ there exists an $x \in S$ and $k < \ell$ satisfying $\nu\rho_0\{\eta(\mathrm{x}) = \ell, \xi(\mathrm{x}) = \mathrm{k}\} > 0$. Using the same inductive argument as in the proof of Lemma 4.6, one shows that $\nu\rho_0\{\eta(\mathrm{x}) = \ell - \mathrm{k}, \xi(\mathrm{x}) = 0\} > 0$. This implies the lemma.

To finish the proof of Proposition 6.7 note first that case b) can only occur if $\sup g(k) < \infty$. Given that $\rho_0 = \sup A$, we must also have $\sup_x \rho_0 \pi(x) = \sup_k g(k)$. ρ_0 being as in the conclusion of Proposition 6.4, we must have $\mu \geq \mu_\rho$ for all $\rho < \rho_0$, hence $\mu \geq \mu_{\rho_0}$. We argue by contradiction: suppose $\mu \neq \mu_{\rho_0}$, then by Lemma 6.9 for some $x \in S$, $\sum g(k)\mu\{\eta(x) = k\} > \sum g(k)\mu_{\rho_0}\{\eta(x) = k\}$. By Lemma 6.8 and the recurrence of $p(x, y)$, $f_1(x) = \sum g(k)\mu\{\eta(x) = k\}$ is a multiple of $f_2(x) = \sum g(k)\mu_{\rho_0}\{\eta(x) = k\}$. Hence $\sup_x f_1(x) > \sup_x f_2(x) = \sup_x \sum g(k)\mu_{\rho_0}\{\eta(x) = k\} =$

$$\sup_x \sum_{k=1}^\infty g(k)\gamma_x \frac{(\rho_0\pi(x))^k}{g(1)\cdots g(k)} = \sup_x \rho_0\pi(x) = \sup_k g(k).$$

This is not possible since $f_1(x)$ is bounded by $\sup_k g(k)$. Therefore $\mu = \mu_{\rho_0}$ and Proposition 6.7 is proved.

The special case of Theorem 12.2 in [3], obtained by considering the measurable function ℓ as identically one, says that any $\mu \in \mathscr{I}$ can be written as $\int \beta \, d\lambda(\beta)$ where λ is a probability measure on $\mathscr{I}e$. This and Proposition 6.7 prove Theorem 1.10.

7. The asymmetric simple random walk on $\mathbb{Z}$. In this section the set of sites will be $\mathbb{Z}$, $g(k) \le g(k+1)$ $k = 1, 2, \cdots$, $\sup g(k) = K_0 < \infty$ and p will correspond to an asymmetric simple random walk, i.e. $p(x, x+1) = p$, $p(x, x-1) = q$ for all $x \in \mathbb{Z}$ and $p + q = 1$. We will assume that $0 < p < \frac{1}{2}$. The case $\frac{1}{2} < p < 1$ is similar.

Since in this case $\sum \alpha(x) < \infty$, the condition $\sup g(k) < \infty$ guarantees $\int |||\eta||| \, d\mu < \infty$ for any probability measure μ on ε. As in Section 5, we write μ_ρ instead of μ_π if $\pi(x) \equiv \rho$. By Theorem 1.8, $\mu_\rho \in \mathscr{I}$ if $0 < \rho < K_0$. Informally, the property of the process we will use to prove that all extremal invariant measures are the μ_ρ's is that the number of change of signs of $\eta_t(x) - \xi_t(x)$ does not increase with t when η_t and ξ_t are coupled in the natural way.

LEMMA 7.1. *Let $\mu \in \mathscr{I}$, $\nu_\rho \in \bar{\mathscr{I}}$ with marginal μ and μ_ρ, $0 \le \rho < K_0$, $f_n(\eta, \xi)$ be the number of changes of sign of $\eta(-n) - \xi(-n), \cdots, \eta(n) - \xi(n)$ $n = 1, 2, \cdots$ and $A_n = \int (f_{n+1} - f_n) \, d\nu_\rho$. Then there exists a subsequence of A_n converging to 0.*

PROOF. We argue by contradiction: suppose $A_n > \varepsilon > 0$ for all $n \ge n_0$. Then there exists a k such that $\int (f_{n+k} - f_n) \, d\nu_\rho > 4$ for all n; since $|f_{n+k} - f_n| \le 2k$ this implies that there exists $\varepsilon_1 > 0$ such that $\nu_\rho\{f_{n+k} - f_n \ge 3\} > \varepsilon_1$ for all n. Hence there is an $\varepsilon_2 > 0$ such that for all n there exists a and b satisfying:

$$\nu_\rho\{\eta(a) > \xi(a), \eta(a+1) = \xi(a+1), \cdots, \eta(b-1) = \xi(b-1), \eta(b) < \xi(b)\}$$
$$+ \nu_\rho\{\eta(a) < \xi(a), \eta(a+1) = \xi(a+1), \cdots, \eta(b-1) = \xi(b-1), \eta(b) > \xi(b)\} > \varepsilon_2$$

and either $n < a < b \le n + k$ or $-n - k \le a < b < -n$. Since the second marginal of ν_ρ is μ_ρ there is an $\varepsilon_3 > 0$ and $L \in \mathbb{Z}_+$ such that for all n there exist a and b satisfying:

$$\nu_\rho\{\eta(a) > \xi(a), \eta(a+1) = \xi(a+1), \cdots, \eta(b-1) = \xi(b-1), \eta(b) < \xi(b),$$
$$\xi(i) \le L \;\; i = a, a+1, \cdots, b\}$$
$$+ \nu_\rho\{\eta(a) \le \xi(a), \eta(a+1) = \xi(a+1), \cdots, \eta(b-1) = \xi(b-1), \eta(b) > \xi(b),$$
$$\xi(i) \le L \;\; i = a, a+1, \cdots, b\} > \varepsilon_3$$

and either $n < a < b \le n + k$ or $-n - k \le a < b < -n$. Now a long and tedious induction shows that there is an $\varepsilon_4 > 0$ such that for all n there exists an a satisfying:

$$\nu_\rho\{\eta(a) > \xi(a), 0 = \eta(a+1) < \xi(a+1)\}$$
$$+ \nu_\rho\{0 = \eta(a) < \xi(a), \eta(a+1) > \xi(a+1)\} > \varepsilon_4.$$

and either $n < a \le n + k$ or $-n - k \le a < -n$. This induction is similar to the one used in Lemmas 4.6 and 4.7. One has to note, however, that if f is the indicator function of a cylinder set depending only on coordinates $a, a+1, \cdots, b$ where $b - a \le k$, then the negative terms of $\int \bar{\Omega} f \, d\nu_\rho$ are bounded by the product of $\int f \, d\nu_\nu$ and $K_0(k+2)$. Also note that $\int \bar{\Omega} f \, d\nu_\rho = 0$, since $\int |||\eta, \xi||| \, d\nu_\rho < \infty$, and there is no need here to truncate the measure ν_ρ as we did in the proofs on Lemmas 4.6 and 4.7.

Let $g_M(\eta, \xi) = \sum_{x \in [-M, M]} |\eta(x) - \xi(x)|$, by Lemma 2.9 $\int \bar{\Omega} g_M \, d\nu_\rho = 0$. Take $M = k\ell$, then the negative terms of $\int \bar{\Omega} g_M \, d\nu_\rho$ are in absolute value greater than or equal to $2 \min\{p, q\} g(1) \varepsilon_4 \ell$ but the positive terms are bounded by K_0; since ℓ is arbitrary we have a contradiction.

LEMMA 7.2. *Let ν_ρ and f_n be as in Lemma 7.1 then*

$$\nu_\rho\{(\eta, \xi) : f_n(\eta, \xi) < 3\} = 1 \quad \textit{for all} \quad n \ge 1.$$

PROOF. Again we argue by contradiction: suppose there exist a, b, c, d such that $a < b < c < d$ and

$$\nu_\rho\{\eta(a) < \xi(a), \eta(b) > \xi(b), \eta(c) < \xi(c), \eta(d) > \xi(d), \eta(i) = \xi(i)$$
$$i = a+1, \cdots, b-1, b+1, \cdots, c-1, c+1, \cdots, d-1\} > 0.$$

Then an inductive argument shows that:

$$\nu_\rho\{\eta(a) < \xi(a), \eta(a+1) > \xi(a+1), \eta(a+2) < \xi(a+2), \eta(a+3) > \xi(a+3)\} > 0$$

but this implies:

$$\begin{aligned}&\nu_\rho\{\eta(a) < \xi(a), 1 = \eta(a+1), \xi(a+1) = 0, 0 = \eta(a+2) < \xi(a+2), \eta(a+3) > \xi(a+3)\}\\ &\quad + \nu_\rho\{\eta(a) < \xi(a), \eta(a+1) > \xi(a+1) = 0,\\ &\qquad 0 = \eta(a+2), 1 = \xi(a+2), \eta(a+3) > \xi(a+3)\} > 0.\end{aligned} \tag{7.3}$$

Now take $A_{n_\ell} \to 0$. By Lemma 2.9 $\int \bar{\Omega} f_{n_\ell} d\nu_\rho = 0$. The positive terms of this integral are bounded by $A_{n_\ell} K_0$ and, therefore, go to 0 as ℓ goes to ∞, but according to 7.3 the absolute value of the negative terms is bounded below by a positive constant for $n_\ell > |a| + 3$; a contradiction.

DEFINITION. Given η, $\xi \in \varepsilon$ we will say that $\eta \circledge \xi$ if there exists $x_0 \in \mathbb{Z}$ such that $\eta(x) \geq \xi(x)$ for all $x \geq x_0$.

LEMMA 7.4. *If* $\eta \circledge \xi$ *then* $P^{\eta,\xi}(\eta_t \circledge \xi_t) = 1$ *for all* $t \geq 0$.

PROOF. Let $\eta \circledge \xi$

$$g_{m_1,m_2}(\eta, \xi) = \begin{cases} 1 & \text{if} \quad \eta(x) < \xi(x) \quad \text{for some} \quad m_1 \leq x \leq m_2 \\ 0 & \text{otherwise.}\end{cases}$$

and $\bar{S}_n(t)$ be the sequence of semigroups that approximate $\bar{S}(t)$ as in Section 2.

Since the Markov processes, starting at (η, ξ), corresponding to $\bar{S}_n(t)$ have a finite number of states and there is a bound, independent of n, for the rate at which particles move we must have:

$$\lim_{m_1 < m_2, m_1 \to \infty} \sup_n \bar{S}_n(t) g_{m_1,m_2}(\eta, \xi) = 0 \quad \text{for all} \quad t \geq 0.$$

This shows that

$$\lim_{m_1 < m_2, m_1 \to \infty} \bar{S}(t) g_{m_1,m_2}(\eta, \xi) = 0 \quad \text{for all} \quad t \geq 0.$$

Hence, given $\varepsilon > 0$ there exists m_1 such that for all m_2 $P^{(\eta,\xi)}(\eta_t(x) < \xi_t(x)$ for some $m_1 \leq x \leq m_2) < \varepsilon$. Letting m_2 go to ∞ we have

$$P^{(\eta,\xi)}(\eta_t(x) < \xi_t(x) \quad \text{for some} \quad x \geq m_1) < \varepsilon.$$

Since ε is arbitrary, the lemma follows.

DEFINITION. Given two probability measures μ_1 and μ_2 on ε, we will say that $\mu_1 \circledleq \mu_2$ if there exists a probability measure ν on $\bar{\varepsilon}$ with marginals μ_1 and μ_2 and such that $\nu\{(\eta, \xi) : \eta \circledleq \xi\} = 1$.

PROPOSITION 7.5. *If* $\mu \in \mathscr{I}e$ *then* $\mu = \mu_\rho$ *for some* $0 \leq \rho < K_0$.

PROOF. By Lemma 7.2 $\nu_\rho\{\eta \circledge \xi$ or $\xi \circledge \eta\} = 1$. Let ν_ρ^1 and ν_ρ^2 be the restrictions of ν_ρ to $\{\eta \circledge \xi\}$ and its complement respectively. Let $\lambda = \nu_\rho^1\{\bar{\varepsilon}\}$ and suppose $0 < \lambda < 1$.

Using Lemma 7.4 and the same argument as in Lemma 4.5 we can show that $\nu_\rho^1 \bar{S}(t) = \nu_\rho^1$ and $\nu_\rho^2 \bar{S}(t) = \nu_\rho^2$. Hence the first marginals of ν_ρ^1 and ν_ρ^2 are in $\mathscr{I}$, and since μ is extremal they must both be μ. This shows that there exist measures μ_{ρ_1} and μ_{ρ_2} absolutely continuous with respect to μ_ρ and such that $\mu \circledcirc{\geq} \mu_{\rho_1}$ and $\mu \circledcirc{\leq} \mu_{\rho_2}$. This implies that $\mu\{\eta \mid \lim_{n\to\infty} \sum_{x=1}^{n} (\eta(x)/n) = h(\rho)\} = 1$ where $h(\rho) = \sum_1^\infty \gamma_\rho k(\rho^k/g(1) \cdots g(k))$. Since $h(\rho)$ is a strictly increasing function of p, this can happen for at most one ρ. For all the others, λ must be 0 or 1. Therefore:

$$\nu_\rho\{\eta \circledcirc{\geq} \xi\} = 1 \quad \text{or} \quad \nu_\rho\{\eta \circledcirc{\leq} \xi\} = 1 \tag{7.6}$$

except for at most one ρ. Let

$$f_1(x) = \textstyle\sum_{k>\ell} (g(k) - g(\ell))\nu_\rho\{\eta(x) = k, \xi(x) = \ell\}$$

and

$$f_2(x) = \textstyle\sum_{k>\ell} (g(k) - g(\ell))\nu_\rho\{\eta(x) = \ell, \xi(x) = k\}.$$

By Lemma 2.9 $\int \bar{\Omega}(\eta(x) - \xi(x))\, d\nu_\rho = 0$. Computing the positive and negative terms, we get:

$$f_1(x) - f_2(x) = \textstyle\sum_y p(y, x)(f_1(y) - f_2(y)) \quad \text{for all} \quad x \in \mathbb{Z}.$$

Similarly, $\int \bar{\Omega} \mid \eta(x) - \xi(x) \mid d\nu_\rho = 0$ and this gives rise to:

$$f_1(x) + f_2(x) \leq \textstyle\sum_y p(y, x)(f_1(y) + f_2(y)) \quad \text{for all} \quad x \in \mathbb{Z}.$$

Therefore,

$$f_i(x) \leq \textstyle\sum_y p(y, x)(f_i(y)) \quad \text{for all} \quad x \in \mathbb{Z} \quad i = 1, 2 \tag{7.7}$$

since $f_i(x) \leq K_0 g_i(x) = K_0 - f_i(x)$ is greater than or equal to 0 and satisfies $g_i(x) \geq \sum_y p(y, x) g_i(y)$ for all x $i = 1, 2$. By Proposition 3.4 in Chapter 2 of [18] and Theorem 1.3 in Chapter 5 of [18], $g_i(x)$ is a decreasing function of x. (Here we use that our random walk has a drift to the left). Hence $f_i(x)$ increases with x. This and 7.6 shows that either $f_1(x) \equiv 0$ or $f_2(x) \equiv 0$, therefore $\nu_\rho\{\eta \geq \xi\} = 1$ or $\nu_\rho\{\xi \geq \eta\} = 1$. Hence, except for at most one ρ, $\mu \geq \mu_\rho$ or $\mu \leq \mu_\rho$, by reasoning as in Proposition 5.5 we can prove that $\mu = \mu_{\rho_0}$ for some $0 < \rho_0 < K_0$.

Theorem 1.11 follows from this Proposition and Theorem 12.2 in [3].

8. Convergence to the vacant configuration. In this section we consider the case in which $\sup g(k) = K_0 < \infty$, $K = \sup |g(k+1) - g(k)|$, $g(k) \leq g(k+1)$ and there exists an unbounded $\pi: S \to \mathbb{R}_+$ satisfying $\sum \pi(x)p(x, y) = \pi(y)$ for all $y \in S$. For technical reasons we also assume that there exists a sequence of isolated points of the set $\{\pi(x) : x \in S\}$ that diverges to ∞.

First we introduce the following notation: L will be an isolated point in $\{\pi(x) : x \in S\}$, $T = \{x \in S : \pi(x) < L\}$, $T' = S - T$ and $\rho = \inf\{\lambda \geq 0 : \pi(x) \geq K_0$ for all $x \in T'\}$. Now we construct a process on a subset of $(\mathbb{Z}_+)^T$ with formal generator:

$$\bar{\Omega} f(\eta) = \textstyle\sum_{x,y\in T} g(\eta(x))p(x, y)(f(\eta_{xy}) - f(\eta)) + \sum_{x\in T} g(\eta(x)) \sum_{y\in T'} p(x, y)(f(\eta_{x^*}) - f(\eta))$$
$$+ \textstyle\sum_{x\in T'} \rho\pi(x) \sum_{y\in T} p(x, y)(f(\eta_y) - f(\eta)),$$

where

$$\eta_{x*}(y) = \begin{cases} \eta(y) & \text{if} \quad y \neq x \quad \text{or} \quad \eta(x) = 0 \\ \eta(x) - 1 & \text{if} \quad y = x \quad \text{and} \quad \eta(x) > 0. \end{cases}$$

The intuitive meaning of this generator is: when a particle attempts to move to T' it disappears but the sites in T' act as sources of particles. Note that when S is finite, this defines a nonexplosive countable state Markov chain.

Proceeding as in Section 2, we construct this process on

$$\tilde{\varepsilon} = \{\eta \in (\mathbb{Z}_+)^T : \textstyle\sum_{x\in T} \eta(x)\alpha(x) < \infty\}$$

where α is as in the previous sections, i.e. $\alpha(x) = \sum_n (1/M^n)p^n(x, x_0)$, $M > 1$. Again for η, $\xi \in \varepsilon$, we define $\|\eta\| = \sum_{x\in T} \eta(x)\alpha(x)$ and $\|\eta - \xi\| = \sum_{x\in T} |\eta(x) - \xi(x)|\alpha(x)$. Our process will be given by a semigroup $\tilde{S}(t)$ of operators on $\tilde{\mathscr{L}}$ where $\tilde{\mathscr{L}} = \{f: \tilde{\varepsilon} \to \mathbb{R} : |f(\eta) - f(\xi)| \le c\|\eta - \xi\|$ for all $\eta, \xi \in \tilde{\varepsilon}\}$. Given $f \in \tilde{\mathscr{L}}$, $L(f)$ will be the smallest number satisfying $|f(\eta) - f(\xi)| \le L(f)\|\eta - \xi\|$ for all $\eta, \xi \in \tilde{\varepsilon}$. This construction is similar to the one given in Section 2; for this reason details are omitted.

We denote by $\tilde{\mathscr{I}}$ the set of probability measures on $\tilde{\varepsilon}$ invariant for this process. As in Section 2, one shows that a measure μ on $\tilde{\varepsilon}$ such that $\int \|\eta\|\, d\mu < \infty$ is in $\tilde{\mathscr{I}}$ if and only if $\int \tilde{\Omega} f\, d\mu = 0$ for all $f \in \tilde{\mathscr{L}}$

Let $\mu_{\rho\pi}$ be the product measure on $(\mathbb{Z}_+)^T$ having marginals given by.

$$\mu_{\rho_\pi}\{\eta : \eta(x) = k\} = \begin{cases} \gamma_x \dfrac{(\rho\pi(x))^k}{g(1)\cdots g(k)} & \text{if} \quad k > 0 \\ \gamma_x & \text{if} \quad k = 0 \end{cases}$$

where γ_x is a normalizing constant.

The following lemma is a special case of Theorem 4.5 in [6] and can be checked by means of a routine computation.

Lemma 8.1. *If S is finite then $\mu_{\rho\pi} \in \tilde{\mathscr{I}}$.*

Proposition 8.2. *Let S be countable; then $\mu_{\rho\pi}(\tilde{\varepsilon}) = 1$ and $\mu_{\rho\pi} \in \tilde{\mathscr{I}}$.*

Proof. The first statement follows from our condition on L (i.e., L is an isolated point of $\{\pi(x) : x \in S\}$) and the invariance of $\mu_{\rho\pi}$ can be proved by proceeding as in Section 3. Details are omitted again.

Given $\eta \in \varepsilon$ and $\xi \in \tilde{\varepsilon}$, we say that $\eta \le \xi$ if $\eta(x) \le \xi(x)$ for all $x \in T$, and given μ and λ probability measures on ε and $\tilde{\varepsilon}$ respectively we say that $\mu \le \lambda$ if there exists a probability measure ν on $(\mathbb{Z}_+)^S \times (\mathbb{Z}_+)^T$ with marginals μ and λ and such that $\nu\{(\eta, \xi) : \eta \le \xi\} = 1$. We also define a partial order for $\eta, \xi \in \tilde{\varepsilon}$ and for μ and λ probability measures on $\tilde{\varepsilon}$ in the same way.

Proposition 8.3. *If $\eta \in \varepsilon$ is such that $\sum_{x\in T} \eta(x) < \infty$, then*

$$\limsup_{t\to\infty} p^\eta(\eta_t(x) > 0) \le \mu_{\rho_\pi}\{\eta : \eta(x) > 0\}$$

for all $x \in T$.

Proof. Let η_T be the restriction of η to T and δ_η, δ_{η_T} the point mass measures at η and η_T respectively.

A simple coupling arguments shows that for all $t \ge 0$

$$\delta_\eta S(t) \le S_{\eta_T}\tilde{S}(t). \tag{8.4}$$

Let $\mu_{\rho\pi,\eta_T}$ be the product probability measures on $(\mathbb{Z}_+)^T$ whose marginals are: point mass at $\eta(x)$ if $\eta(x) > 0$ and the same marginal as $\mu_{\rho\pi}$ if $\eta(x) = 0$. Another simple coupling shows that for all

$$t \ge 0,\ \delta_{\eta_T}\tilde{S}(t) \le \mu_{\rho\pi,\eta_T}\tilde{S}(t). \tag{8.5}$$

Since $\mu_{\rho\pi} \in \tilde{\mathscr{I}}$ and $\mu_{\rho\pi} = \lambda\mu_{\rho\pi,\eta_T} + (1 - \lambda)\mu$ for some $\lambda > 0$ and some probability measure μ on $\tilde{\varepsilon}$, the collection of measures $\mu_{\rho\pi,\eta_T}\tilde{S}(t)$ $t \ge 0$ is tight. In view of this, (8.4) and (8.5) to prove the proposition, it suffices to show that if μ_0 is a weak limit of $\mu_{\rho\pi,\eta_T}\tilde{S}(t)$ then $\mu_0 \le \mu_{\rho\pi}$.

To prove this last statement, consider a probability measure ν on $(\mathbb{Z}_+)^T \times (Z_+)^T$ with marginals $\mu_{\rho\pi,\eta_T}$ and $\mu_{\rho\pi}$ and such that $\nu\{(\eta, \xi) : \eta(x) = \xi(x)\} = 1$ for all $x \in T$ such that $\eta_T(x) = 0$. Denote by $\tilde{\tilde{S}}(t)$ the semigroup corresponding to the natural coupling of two versions of $\tilde{S}(t)$.

Consider the measures $\nu_N = (1/N) \int_0^N \nu\bar{\bar{S}}(t)\, dt\ N \in [1, \infty)$. Due to our previous remark on $\mu_{\rho\pi,\eta_T}\tilde{S}(t)$ and the fact that $\mu_{\rho\pi} \in \tilde{\mathscr{I}}$, this set of measures is tight. Taking a subsequence N_n, we get as limit an invariant measure ν_0.

Let $A = \sum_{x \in T} \eta(x)$. Approximating $\bar{\bar{S}}(t)$ as it was done for $S(t)$ in Section 2, one shows that

$$\nu\bar{\bar{S}}(t)\{(\eta, \xi) : \textstyle\sum_{x\in T} [\eta(x) - \xi(x)]^+ > A\} = 0 \quad \text{for all} \quad t \geq 0.$$

Hence
$$\nu_0\{(\eta, \xi) : \textstyle\sum_{x\in T}[\eta(x) - \xi(x)]^+ > A\} = 0;$$

since $\nu_0\bar{\bar{S}}(t) = \nu_0$ and p is irreducible, this and an inductive argument show that $\nu_0\{(\eta, \xi) : \sum_{x\in T}[\eta(x) - \xi(x)]^+ > 0\} = 0$.

Again using the approximations of $\bar{\bar{S}}(t)$ one shows that $\int \sum_{x\in T}[\eta(x) - \xi(x)]^+\, d[\nu\, \bar{\bar{S}}(t)(\eta, \xi)]$ is non-increasing in t. This together with our result on ν_0 show that:

$$\lim_{t\to\infty} \int \textstyle\sum_{x\in T}[\eta(x) - \xi(x)]^+\, d[\nu\bar{\bar{S}}(t)(\eta, \xi)] = 0$$

and this proves our statement on the weak limits of $\mu_{\rho\pi,\eta_T}\tilde{S}(t)$ and, therefore, the proposition.

PROOF OF THEOREM 1.12. Let L_n be an increasing sequence of isolated points of $\{\pi(x) : x \in S\}$ going to ∞,

$$T_n = \{x \in S : \pi(x) < L_n\},\ \rho_n = \inf\{\lambda \geq 0 : \lambda\ \pi(x) \geq K_0 \quad \text{for all} \quad x \in S - T_n\}$$

and $\eta \in \varepsilon$ such that $\sum_{x\in T} \eta(x) < \infty$ for all $n \geq 1$.

Fix $x \in S$ and let n_0 be large enough for $x \in T_{n_0}$. Then by Proposition 8.3 for all $n \geq n_0$, $\limsup_{t\to\infty} P^\eta(\eta_t(x) > 0) \leq \mu_{\rho_n\pi}\{\eta : \eta(x) > 0\}$ and, letting n go to ∞, Theorem 1.12 follows since $\rho_n \to 0$.

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REFERENCES

[0] ANDJEL E. D. (1980). Ph.D. thesis. University of California, Los Angeles.

[1] ATHREYA, K. B. and NEY, P. E. (1972). *Branching Processes.* Springer, New York.

[2] CHUNG, K. L. (1974). *A Course in Probability Theory*, second edition. Academic, New York.

[3] DYNKIN, E. B. (1978). Sufficient statistics and extreme points. *Ann. Probability* **6** 705–730.

[4] GRIFFEATH, D. (1979). *Additive and Cancellative Interacting Particle Systems.* Springer, Berlin.

[5] HOLLEY, R. (1970). A class of interaction in an infinite particle system. *Adv. in Math.* **5** 291–309.

[6] JACKSON, J. R. (1963). Jobshop-like queueing systems. *Management Science* **10** 131–142.

[7] KEMENY, J. G., SNELL, J. L. and KNAPP, A. W. (1976). *Denumerable Markov Chains*, second edition. Springer, New York.

[8] LIGGETT, T. M. (1972). Existence theorems for infinite particle systems. *Trans. Amer. Math. Soc.* **165** 471–481.

[9] LIGGETT, T. M. (1973). An infinite particle system with zero range interactions. *Ann. Probability* **1** 240–253.

[10] LIGGETT, T. M. (1973). A characterization of the invariant measures for an infinite particle system with interactions. *Trans. Amer. Math. Soc.* **179** 433–453.

[11] LIGGETT, T. M. (1974). A characterization of the invariant measures for an infinite particle system with interaction, II. *Trans. Amer. Math. Soc.* **198** 201–213.

[12] LIGGETT, T. M. (1974). Convergence to total occupancy in an infinite particle system with interaction. *Ann. Probability* **2** 989–998.

[13] LIGGETT, T. M. (1976). Coupling the simple exclusion process. *Ann. Probability* **4** 339–356.

[14] LIGGETT, T. M. (1978). Random invariant measures for Markov chains and independent particle systems. *Z. Wahrsch. verw. Gebiete* **45** 297–313.

[15] LIGGETT, T. M. (1980). Long range exclusion processes. *Ann. Probability* **8** 861–889.

[16] LIGGETT, T. M. and SPITZER, F. (1981). Ergodic theorems for coupled random walks and other

systems with locally interacting components. *Z. Wahrsch. verw. Gebiete* **56** 443–468.
[17] PHELPS, R. R. (1965). *Lectures on Choquet's Theorem.* Van Nostrand, Princeton.
[18] REVUZ, D. (1975). *Markov Chains.* North Holland, Amsterdam.
[19] PROKHOROV, YU. V. (1956). Convergence of random processes and limit theorems in probability theory. *Theor. Probability Appl.* **1** 157–214.
[20] SPITZER, F. (1970). Interaction of Markov processes. *Adv. in Math.* **5** 246–290.
[21] SPITZER, F. (1974). Recurrent random walk of an infinite particle system. *Trans. Amer. Math. Soc.* **198** 191–199.
[22] WAYMIRE, E. (1976). Contributions to the theory of interacting particle systems. Ph.D. thesis, University of Arizona.
[23] WAYMIRE, E. (1980). Zero range interaction at Bose-Einstein speeds under a positive recurrent single particle law. *Ann. Probability* **8** 441–450.

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