

## PART 5: ENTROPY METHOD FOR HYDRODYNAMICS OF ZERO-RANGE PROCESSES

A rigorous proof of hydrodynamics (Theorem 7.1 in Part 4) is given for a class of zero-range models through the ‘entropy’ method of [2]. The main idea is that the drift, which is now an average of a nonlinear function of the occupation variables, may be understood in the scaling limit as a function of the limiting empirical density. This sort of ergodic theorem, which proceeds in two steps, the ‘1-block’ and ‘2-block’ lemmas, is facilitated by estimates on the Dirichlet form with respect to a nonequilibrium density function, and local central limit theorem asymptotics.

Recall the notation and assumptions from Section 7 in Part 4.

### 1. PROOF OF STEP 1

As for simple exclusion, the first step can be divided into the tasks:

- Show that the measures  $Q^N$  on  $D([0, T]; \mathcal{M}_+(\mathbb{T}^d))$  which govern  $\langle \pi_{N^2 t} : t \in [0, T] \rangle$  starting from local equilibrium  $\nu_{\rho_0(\cdot)}$  are tight.
- Show that all limit points of  $\{Q_N\}$  are supported on trajectories with densities  $\pi(t, u)du$ .

**Lemma 1.1.**  $\{Q^N\}$  is tight.

*Proof.* Considering Proposition 2.7 in Part 3, we need only show (1) for each  $t \in [0, T]$  that  $\langle G, \pi_{N^2 t}^N \rangle$  is tight on  $\mathbb{R}$ , and

$$(2') \quad \lim_{\gamma \downarrow 0} \lim_{N \uparrow \infty} P \left( \sup_{|t-s| \leq \gamma} |\langle G, \pi_{N^2 t}^N \rangle - \langle G, \pi_{N^2 s}^N \rangle| > \epsilon \right) = 0.$$

Now, by the basic coupling and Exercise 5.4 in Part 4, we have that  $\nu_{\rho_0(\cdot)} \ll \nu_{\bar{\rho}}$  where  $\bar{\rho} = \|\rho_0\|_{L^\infty}$ . Hence, (1) follows as the absolute first moments are uniformly bounded:

$$\mathbb{E}_{\nu_{\rho_0(\cdot)}} |\langle G, \pi_{N^2 t}^N \rangle| \leq \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} |G(x/N)| \mathbb{E}_{\nu_{\bar{\rho}}} [\eta_{N^2 t}(x)] \leq C(G, \bar{\rho}).$$

As for simple exclusion, to establish (2'), we consider the drift and martingale terms separately. For the drift term, since  $g$  is bounded, we have the uniform bound on  $N$ ,

$$v(N) \int_s^t |L\langle G, \pi_{N^2 u}^N \rangle| du \leq |t-s| \frac{v(N)}{N^{d+2}} \sum_{x,y} (\Delta_{x,x+y}^N G) \|g\|_{L^\infty} \leq \frac{|t-s|}{N} C(g, p),$$

which vanishes as  $N \uparrow \infty$ .

Noting the form of the quadratic variation  $\langle M_{N^2 t}^G \rangle$  in Section 7 in Part 4, the proof is similar than for exclusion processes.  $\square$

**Lemma 1.2.** All limit points of  $\{Q^N\}$  are supported on trajectories with  $L^2$  densities  $\pi(t, u)du$  such that there is a constant  $C = C(\bar{\rho})$  where

$$E_Q \int_0^T \int \pi(t, u)^2 du dt < C.$$

Before beginning the proof, we remark absolute continuity of the limit trajectories means that particles cannot pile up to form point masses. Given the process is ‘attractive’, that is the basic coupling holds, one can bound the probability of large piles in terms of  $\nu_{\bar{\rho}}$  estimates where  $\bar{\rho}$  is an absolute bound on the hydrodynamic density. Without attractiveness, the proof is harder and we refer to [3] for a more general argument.

*Proof.* We show first that the trajectories are absolutely continuous under a limit point  $Q$ . Let  $G : [0, T] \times \mathbb{T}^d \rightarrow \mathbb{R}$  be a smooth function. Note, for  $A > 0$  and for all large  $N$ , that

$$\begin{aligned} & \int_0^T \frac{1}{N} \sum_x G(s, x/N) \eta_{N^2 s}(x) ds \\ & \leq 2(A + E_{\nu_{\bar{\rho}}}[\eta(0)1(\eta(0) > A)]) \|G\|_{L^1([0, T] \times \mathbb{R})} \\ & \quad + \int_0^T \frac{1}{N} \sum_x |G|(s, x/N) (\eta_{N^2 s}(x) 1(\eta_{N^2 s} > A) - E_{\nu_{\bar{\rho}}}[\eta(0)1(\eta(0) > A)]). \end{aligned}$$

The last term is an increasing function of  $\eta_{N^2 s}$ . Let  $\phi(\eta_{N^2 s}(x)) = \eta_{N^2 s}(x) 1(\eta_{N^2 s} > A) - E_{\nu_{\bar{\rho}}}[\eta(0)1(\eta(0) > A)]$ . Therefore, by the basic coupling, Chebychev’s inequality and invariance of  $\nu_{\bar{\rho}}$ , for  $\epsilon > 0$ , we have

$$\begin{aligned} & \mathbb{P}_{\nu_{\rho_0(\cdot)}} \left( \int_0^T \frac{1}{N} \sum_x |G|(s, x/N) \phi(\eta_{N^2 s}(x)) > \epsilon \right) \\ & \leq \mathbb{P}_{\nu_{\bar{\rho}}} \left( \int_0^T \frac{1}{N} \sum_x |G|(s, x/N) \phi(\eta_{N^2 s}(x)) > \epsilon \right) \\ & \leq \frac{T \|G\|_{L^2}}{N \epsilon^2} \text{Var}_{\nu_{\bar{\rho}}}(\eta(0)1(\eta(0) > A)) = O(N^{-1}). \end{aligned}$$

Hence, by simple estimates,

$$Q^N \left( \int_0^T \langle G(s, \cdot), \pi_s \rangle ds \leq 2(A + E_{\nu_{\bar{\rho}}}[\eta(0)1(\eta(0) > A)]) \|G\|_{L^1} + \epsilon \right) \geq 1 - O(N^{-1})$$

and by weak convergence, since the function  $\int_0^T \langle G(s, \cdot), \pi_s \rangle ds$  is continuous in the Skorohod topology and  $Q(F) \geq \lim Q^N(F)$  for closed sets,

$$Q \left( \int_0^T \langle G(s, \cdot), \pi_s \rangle ds \leq 2(A + E_{\nu_{\bar{\rho}}}[\eta(0)1(\eta(0) > A)]) \|G\|_{L^1} + \epsilon \right) = 1.$$

Since  $\epsilon > 0$  is arbitrary, we have a.s. under  $Q$  that all trajectories satisfy

$$\int_0^T \langle G(s, \cdot), \pi_s \rangle ds \leq C \|G\|_{L^1}.$$

Now, note by the tightness estimate (2’) already proven in the previous lemma, all trajectories under  $Q$  are continuous. Then, by choosing  $G$  to approximate  $\delta^{-1}1(t, t + \delta)1(B)$  for  $B \subset \mathbb{T}^d$ , we obtain that  $\pi_t$  is absolutely continuous for each  $t \in [0, T]$ .

To show the display in the lemma, consider the bound

$$\sup_{N \geq 1} \mathbb{E}_{\nu_{\rho_0(\cdot)}} \int_0^T du \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} (\eta_{N^2 s}^{N_\epsilon}(x))^2 \leq T E_{\nu_{\bar{\rho}}}[\eta(0)^2]$$

proved using the basic coupling and Schwarz inequality  $(\eta^{N\epsilon}(x))^2 \leq [(2N\epsilon + 1)^d]^{-1} \sum_{|y-x| \leq N\epsilon} \eta(y)^2$ .

Hence, as  $\bar{Q}$  is a limit point, by Fatou's lemma again, we have

$$E_Q \int_0^T ds \int_{\mathbb{T}^d} dx \left( (2\epsilon)^{-d} \int_{B(x,\epsilon)} \pi(s,u) du \right)^2 \leq C(T, \bar{\rho})$$

where  $B(x, \epsilon)$  is the ball of radius  $\epsilon$  around  $x$ . Taking limit on  $\epsilon \downarrow 0$ , and another application of Fatou's lemma, we finish the proof.  $\square$

## 2. PROOF OF STEP 2 MODULO ERGODIC REPLACEMENT

We now supply, modulo a replacement estimate, the proof of Theorem 7.1 in Part 4. Recall that  $\Psi(\rho) = E_{\nu_\rho}[g(\eta(0))]$  which is a bounded function as  $g$  is bounded. Let  $\tau_x$  be the shift operator,  $(\tau_x \eta)(y) = \eta(x+y)$ .

**Theorem 2.1** (Ergodic Replacement). *We have that*

$$\limsup_{\epsilon \downarrow 0} \limsup_{N \uparrow \infty} \mathbb{E}_{\nu_{\rho_0(\cdot)}} \left[ \int_0^T \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_{N\epsilon}(\eta_{N^2 s}) ds \right] = 0$$

where

$$V_\ell(\eta) = \left| \frac{1}{(2\ell+1)^d} \sum_{|y| \leq \ell} g(\eta(y)) - \Psi(\eta^\ell(0)) \right|.$$

Notice that  $V_\ell$  is a bounded function as  $g$  is bounded.

Let us now see how this replacement allows to finish the proof of hydrodynamics.

**Proof of Theorem 7.1 (Part 4).** First, we establish an equation that the densities  $\pi(t, u)$  under a limit point  $Q$  must satisfy. Let  $G \in C^2([0, T] \times \mathbb{T}^d)$ . Then, following the derivation when  $G$  did not depend on time in Section 7 in Part 4, we obtain

$$\begin{aligned} \langle G(t, \cdot), \pi_{N^2 t}^N \rangle &= \langle G(0, \cdot), \pi_0^N \rangle \\ &+ \int_0^t \frac{N^2}{N^{d+2}} \sum_{x \in \mathbb{T}_N^d} \left[ \frac{\partial}{\partial s} G(s, x/N) \eta_{N^2 s}(x) + \frac{1}{2} \Delta G(s, x/N) g(\eta_{N^2 s}(x)) \right] ds \\ &+ M_{N^2 t}^G + o(1) \end{aligned}$$

where by Doob's inequality,

$$\sup_{t \in [0, T]} \mathbb{E}_{\nu_{\rho_0(\cdot)}} (M_{N^2 t}^G)^2 \leq \mathbb{E}_{\nu_{\rho_0(\cdot)}} (M_{N^2 T}^G)^2 = O(N^2 N^{-d-2})$$

since  $g$  is bounded.

On the other hand, by Theorem 2.1, since  $\Delta G$  is bounded, we have that

$$\lim_{\epsilon \downarrow 0} \limsup_{N \uparrow \infty} \mathbb{E}_{\nu_{\rho_0(\cdot)}} \int_0^T \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \Delta G(s, x/N) |g(\eta_{N^2 s}(x)) - \Psi(\eta^{N\epsilon}(x))| ds = 0$$

in probability.

Putting this together, we have

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \lim_{N \uparrow \infty} \mathbb{E}_{\nu_{\rho_0(\cdot)}} \left| \langle G, \pi_{N^2 T} \rangle - \langle G, \pi_0 \rangle - \int_0^T \langle \frac{\partial}{\partial_s} G, \pi_{N^2 s} \rangle ds \right. \\ \left. - \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \Delta G(s, u) \Psi(\pi_s^{(\epsilon)}(u)) ds \right| = 0 \end{aligned} \quad (2.1)$$

where

$$\pi_s^{(\epsilon)}(u) = \langle (2\epsilon)^{-d} 1_{[-\epsilon, \epsilon]^d}(\cdot - u), \pi_s \rangle.$$

Now, again the quantity in absolute value in (2.1) is a continuous function of  $\pi$  in the Skorohod topology. Hence, any limit point  $Q$ , by Fatou's lemma, satisfies

$$\begin{aligned} \lim_{\epsilon \downarrow 0} E_Q \left| \langle G, \pi_T \rangle - \langle G, \pi_0 \rangle - \int_0^T \langle \frac{\partial}{\partial_s} G, \pi_s \rangle ds \right. \\ \left. - \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \Delta G(s, u) \Psi(\pi_s^{(\epsilon)}(u)) duds \right| = 0. \end{aligned}$$

But, since trajectories have density and are continuous under  $Q$ , and  $\Psi$  is bounded and continuous, by dominated convergence,

$$\lim_{\epsilon \downarrow 0} E_Q \int_0^T \int_{\mathbb{T}^d} |\Psi(\pi_s^{(\epsilon)}(u)) - \Psi(\pi(u, s))| duds = 0.$$

Finally, we obtain then that  $Q$  is supported on trajectories satisfying

$$\langle G, \pi_T \rangle = \langle G, \pi_0 \rangle + \int_0^T \langle G_s, \pi_s \rangle ds + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \Delta G(s, u) \Psi(\pi(s, u)) duds$$

which are weak solutions to

$$\frac{\partial}{\partial_t} \rho = \frac{1}{2} \Delta \Psi(\rho) \quad \rho(0, u) = \rho_0(u).$$

At this point, it is known in PDE that weak solutions with  $L^2$  integrable densities are unique (cf. [3, Appendix 2.4], [4]). Hence, following now the same argument as for simple exclusion, we obtain Theorem 7.1 in Part 4.  $\square$

### 3. ENTROPY PRODUCTION AND CONSEQUENCES

The goal now is to prove Theorem 2.1. Since we start out of an equilibrium, we have to understand the nonequilibrium contribution. Controlling the entropy and Dirichlet form of the associated probability density is the first step. The main vehicles will be Lemmas 2.1 and 3.3 in Part 4.

We remark, although these lemmas and the discussion in Part 4 mostly apply to finite state Markov chains, by decomposing the initial distribution on the number of particles put into the torus  $\mathbb{T}_N^d$ , these results extend to the calculations below.

Let  $\bar{\rho} < \liminf g(k)$  and  $\nu_{\bar{\rho}}$  be the reference measure throughout the rest of the lecture. Let also  $f_t^N = d\nu_{\rho_0(\cdot)} P_{N^2 t} / d\nu_{\bar{\rho}}$  be the density at macroscopic time  $t \geq 0$ . Since the time scaling is  $N^2$ , we have by Lemma 2.1 and Lemma 3.3 in Part 4 that

$$\begin{aligned} \frac{d}{dt} H(\nu_{\rho_0(\cdot)} P_{N^2 t} | \nu_{\bar{\rho}}) &= N^2 E_{\nu_{\bar{\rho}}} [(L f_t^N) \log f_t^N] \\ &\leq -2I(f_t^N). \end{aligned}$$

Let  $H(f_t^N) = H(\nu_{\rho_0(\cdot)} P_{N^2 t} | \nu_{\bar{\rho}})$ . By Exercise 4.1 in Part 4, we know that  $H(f_0^N) \leq CN^d$  for some constant  $C < \infty$ . Hence, by integrating the differential inequality above, we obtain

$$H(f_t^N) + 2N^2 \int_0^t I(f_s^N) ds \leq H(f_0^N) \leq CN^d.$$

Moreover, by convexity of the entropy and Dirichlet form, we have

$$H\left(\frac{1}{t} \int_0^t f_s^N ds\right) \leq CN^d \quad \text{and} \quad I\left(\frac{1}{t} \int_0^t f_s^N ds\right) \leq C(2t)^{-1} N^{d-2}.$$

For later reference, the following estimate will allow to introduce a truncation corresponding to the number of particles in a region.

**Lemma 3.1.** *For  $\ell \geq 1$ , we have*

$$\sup_{x \in \mathbb{T}_N^d, t \in [0, T]} \mathbb{E}_{\nu_{\rho_0(\cdot)}} \left[ \eta_{N^2 t}^\ell(x) 1(\eta_{N^2 t}^\ell(x) > A) \right] \leq E_{\nu_{\bar{\rho}}}[\eta^2(0)] A^{-1}.$$

*Proof.* By Markov's inequality, the basic coupling, and Schwarz inequality, we have the left-side is bounded by

$$A^{-1} \mathbb{E}_{\nu_{\rho_0(\cdot)}} \left[ (\eta_{N^2 t}^\ell(x))^2 \right] \leq A^{-1} E_{\nu_{\bar{\rho}}}[\eta^2(0)].$$

□

#### 4. REDUCTION TO 1-BLOCK AND 2-BLOCK ESTIMATES

Consider the following bound of the integrand in Theorem 2.1, by adding and subtracting a term.

$$\begin{aligned} & \frac{1}{N^d} \sum_x \tau_x V^{N\epsilon}(\eta) \\ & \leq \frac{1}{N^d} \sum_x \left| \frac{1}{(2N\epsilon+1)^d} \sum_{|y-x| \leq N\epsilon} \left\{ g(\eta(y)) - \frac{1}{(2\ell+1)^d} \sum_{|z-y| \leq \ell} g(\eta(z)) \right\} \right| \\ & \quad + \frac{1}{N^d} \sum_x \left| \frac{1}{(2N\epsilon+1)^d} \sum_{|y-x| \leq N\epsilon} \left\{ \frac{1}{(2\ell+1)^d} \sum_{|z-y| \leq \ell} g(\eta(z)) - \Psi(\eta^\ell(y)) \right\} \right| \\ & \quad + \frac{1}{N^d} \sum_x \left| \frac{1}{(2N\epsilon+1)^d} \sum_{|y-x| \leq N\epsilon} \left\{ \Psi(\eta^\ell(y)) - \Psi(\eta^{N\epsilon}(x)) \right\} \right|. \end{aligned}$$

The first term, as  $g$  is bounded, and careful counting, is bounded on order  $O(\ell^d(N\epsilon)^{-1})$ . Note: It would be zero, except for the absolute value; the positive contributions come from the boundaries of the sum.

The contribution of the second term to the estimate in Theorem 2.1, introducing a truncation by Lemma 3.1 and noting  $V_\ell$  is bounded, is estimated as follows, after bringing the absolute value inside the second sum:

**Proposition 4.1** (1-block lemma). *We have for all  $A > 0$  that*

$$\lim_{\ell \uparrow \infty} \lim_{N \uparrow \infty} \sup_{I(f) \leq C_0 N^{d-2}} \mathbb{E}_{\nu_{\bar{\rho}}} \left[ f(\eta) \frac{1}{N^d} \sum_x \tau_x V_\ell(\eta) 1(\eta^\ell(0) \leq A) \right] = 0$$

where the supremum is over densities  $f$ .

The third term is handled by the following development. Since  $\Psi$  is Lipschitz as  $g$  is Lipschitz (cf. Part 4), we have, for fixed  $y$ , that

$$\frac{1}{N^d} \sum_x \left| \Psi(\eta^\ell(x+y)) - \Psi(\eta^{N\epsilon}(x)) \right| \leq \frac{C_g}{N^d} \sum_x \left| \eta^\ell(x+y) - \eta^{N\epsilon}(x) \right|.$$

Now divide the block of width  $2N\epsilon + 1$ , corresponding to  $\eta^{N\epsilon}(x)$ , into cubes of size  $(2\ell+1)^d$  with possibly  $O((N\epsilon)^{d-1})$  leftover blocks of size strictly less than  $(2\ell+1)^d$ . Then, the right-side is bounded above by

$$\frac{C_g}{N^d} \sum_x \frac{1}{(2N\epsilon+1)^d} \sum_{|y| \leq N\epsilon} \frac{(2\ell+1)^d}{(2N\epsilon+1)^d} \sum_j \left| \eta^\ell(x+y) - \eta^\ell(x+z_j) \right| + \frac{C\ell^d}{N\epsilon} \frac{1}{N^d} \sum_x \eta(x)$$

where  $|z_j| \leq N\epsilon$  are the centers of the cubes in the decomposition, and the last term corresponds to nearest-neighbor cubes of size  $O(\ell^d)$  and the leftover blocks. The second term is negligible, with respect to Theorem 2.1, by the basic coupling, and is on order  $O(\ell^d N^{-1})$ . The first term in the last display is less than

$$\begin{aligned} & C_g \frac{1}{(2N\epsilon+1)^d} \sum_{|y| \leq N\epsilon} \frac{(2\ell+1)^d}{(2N\epsilon+1)^d} \sum_j \frac{1}{N^d} \sum_x \left| \eta^\ell(x+y) - \eta^\ell(x+z_j) \right| \\ & \leq C_g \frac{1}{(2N\epsilon+1)^d} \sum_{|y| \leq N\epsilon} \frac{(2\ell+1)^d}{(2N\epsilon+1)^d} \sum_j \frac{1}{N^d} \sum_x \left| \eta^\ell(x) - \eta^\ell(x+z_j-y) \right| \\ & \leq \sup_{2\ell < |w| \leq 2N\epsilon} \frac{C_g}{N^d} \sum_x \left| \eta^\ell(x) - \eta^\ell(x+w) \right|. \end{aligned}$$

Hence, the following estimate, as for the ‘1-block Lemma’, which also introduces truncations (Lemma 3.1), suffices to bound the contribution of the third term.

**Proposition 4.2** (2-block lemma). *We have for all  $A > 0$  that*

$$\lim_{\ell \uparrow \infty} \lim_{\epsilon \downarrow 0} \lim_{N \uparrow \infty} \sup_{I(f) \leq C_0 N^{d-2}} \sup_{2\ell < |y| \leq 2N\epsilon} \mathbb{E}_{\nu_{\bar{\rho}}} \left[ f(\eta) \frac{1}{N^d} \sum_x \left| \eta^\ell(x) - \eta^\ell(x+y) \right| 1(\eta^\ell(x) + \eta^\ell(x+y) \leq A) \right] = 0.$$

## 5. PROOF OF 1-BLOCK LEMMA

To prove Proposition 4.1, the main idea is that the Dirichlet form of the density  $f$  vanishes as  $N \uparrow \infty$ . In some sense, the optimal density  $f$  is almost constant. Things then reduce to a standard ergodic theorem or law of large numbers associated with iid random variables distributed according to  $\kappa_{\bar{\rho}}$ .

To make this strategy precise, define

$$\text{Av}(f) = \frac{1}{N^d} \sum_x \tau_x f(\eta).$$

Then,

$$\begin{aligned} & \mathbb{E}_{\nu_{\bar{\rho}}} \left[ f(\eta) \frac{1}{N^d} \sum_x \tau_x V_\ell(\eta) 1(\eta^\ell(0) \leq A) \right] \\ & = \mathbb{E}_{\nu_{\bar{\rho}}} \left[ \text{Av}(f) \frac{1}{N^d} \sum_x V_\ell(\eta) 1(\eta^\ell(0) \leq A) \right]. \end{aligned} \tag{5.1}$$

Let now

$$\Lambda_\ell = \{-\ell, \dots, \ell\}^d \text{ and } \mathcal{F}_\ell = \sigma\{\eta(x) : x \in \Lambda_\ell\}.$$

Denote  $f_\ell = E_{\nu_{\bar{\rho}}}[\text{Av}(f)|\mathcal{F}_\ell]$ . Since  $V_\ell(\eta)1(\eta^\ell \leq A)$  depends only on coordinate variables  $\eta(x)$  where  $x \in \Lambda_\ell$ , we have the right-side of (5.1) equals

$$E_{\nu_{\bar{\rho}}}[f_\ell(\eta)V_\ell(\eta)1(\eta^\ell \leq A)] = E_{\nu_{\bar{\rho}}^\ell}[f_\ell(\eta)V_\ell(\eta)1(\eta^\ell \leq A)]$$

where  $\nu_{\bar{\rho}}^\ell$  is the product measure over sites in  $\Lambda_\ell$  with marginal  $\kappa_{\bar{\rho}}$ .

Now, recall that the Dirichlet form has a special form. For  $x, y$  such that  $|x-y|=1$ , let

$$I_{x,x+y}(h) = \frac{1}{2}E_{\nu_{\bar{\rho}}}[g(\eta(x))(\sqrt{h(\eta^{x,x+y})} - \sqrt{h(\eta)})^2].$$

Then,  $I(h) = \sum_{x,|y|=1} I_{x,x+y}(h)$ . Define also the Dirichlet form corresponding to the dynamics restricted to  $\Lambda_\ell$  with invariant measure  $\nu_{\bar{\rho}}^\ell$ .

$$I_\ell(h) = \sum_{\substack{x,|y|=1 \\ x,x+y \in \Lambda_\ell}} I_{x,x+y}(h).$$

**Lemma 5.1.** *We have that*

$$\begin{aligned} I_\ell(f_\ell) &\leq I_\ell(\text{Av}(f)) \\ &= (2\ell+1)^{d-1}(2\ell)N^{-d}I(\text{Av}(f)) \\ &\leq (2\ell+1)^{d-1}(2\ell)N^{-d}I(f) \leq C\ell^d N^2. \end{aligned}$$

*Proof.* Note  $f_\ell$  and  $\text{Av}(f)$  are averages— $f_\ell$  is a conditional expectation for instance. Then, the first and second inequalities follow as the Dirichlet form is convex. The equality follows as  $\text{Av}(f)$  and the measure  $\nu_{\bar{\rho}}$  are translation-invariant: Namely,  $I_{x,x+y}(\text{Av}(f)) = I_{z,z+w}(\text{Av}(f))$ . The last line follows from the bound  $I(f) \leq C_0 N^{d-2}$ .  $\square$

Now, taking into account (5.1) and Lemma 5.1, we need to show

$$\lim_{\ell \uparrow \infty} \lim_{N \uparrow \infty} \sup_{I_\ell(f) \leq C\ell^d N^{-2}} E_{\nu_{\bar{\rho}}^\ell}[f(\eta)V_\ell(\eta)1(\eta^\ell \leq A)] = 0 \quad (5.2)$$

where the supremum is over densities  $f$  with respect to  $\nu_{\bar{\rho}}^\ell$ .

In fact, since  $\nu_{\bar{\rho}}^\ell(\eta^\ell \leq A) > 0$ , in the above expression, we may replace the integrating measure  $\nu_{\bar{\rho}}^\ell(\eta)$  by  $\nu_{\bar{\rho}}^\ell(\eta|\eta^\ell \leq A)$  which supports only a finite number of configurations. Also, the supremum now will be over the collection of densities  $f$  with respect to  $\nu_{\bar{\rho}}^\ell(\eta|\eta^\ell \leq A)$ . The corresponding collection of probability measures  $fd\nu_{\bar{\rho}}^\ell(\eta|\eta^\ell \leq A)$ , being on a compact space, is tight and has a converging subsequence by Prokhorov's theorem.

Hence, to evaluate (5.2), for fixed  $\ell$ , we consider a sequence in  $N$  which approaches the limit supremum. Densities  $f^N$  can be found on which the supremum value is well approximated. By tightness, we can find a subsequence where  $f^N$  converges to  $f^*$  for which  $I_\ell(f^*) = 0$ . Hence, it is enough to show

$$\lim_{\ell \uparrow \infty} \sup_{I_\ell(f)=0} E_{\nu_{\bar{\rho}}^\ell}[f(\eta)V_\ell(\eta)1(\eta^\ell \leq A)] = 0.$$

Now, from our prior estimates, given  $I_\ell(f^*) = 0$  and  $\eta^\ell \leq A$ , we know that  $f^*$  is constant on the configurations such that  $\ell^\ell = a$  for  $a \leq A$ . Therefore, we may

decompose  $\nu_{\bar{\rho}}^\ell$  along such ‘hyperplanes’. It is enough now to show that

$$\limsup_{\ell \uparrow \infty} \sup_{a \leq A} E_{\nu_{\bar{\rho}}^\ell} [V_\ell(\eta) | \eta^\ell = a] = 0. \quad (5.3)$$

The left-side is the same as

$$E_{\nu_{\bar{\rho}}} \left[ \left| \frac{1}{(2\ell+1)^d} \sum_{|x| \leq \ell} g(\eta(x)) - E_{\nu_a} [g(\eta(0))] \right| \middle| \eta^\ell = a \right].$$

Here, we removed the restriction of the measure to  $\Lambda_\ell$  since it does not matter. Notice also that the integrating measure is the canonical measure on  $\Lambda_\ell$  with  $a(2\ell+1)^d$  particles. Hence, the density of the underlying grand-canonical measure  $\nu_{\bar{\rho}}$  does not matter, and can be chosen as we like. We may rewrite the above display as

$$\frac{1}{\sqrt{(2\ell+1)^d} \nu_a(\eta^\ell = a)} E_{\nu_a} \left[ \left| \frac{1}{\sqrt{(2\ell+1)^d}} \sum_{|x| \leq \ell} (g(\eta(x)) - E_{\nu_a} [g(\eta(0))]) \right| \middle| 1(\eta^\ell = a) \right]. \quad (5.4)$$

It is an exercise to see that the denominator is bounded away from 0.

**Exercise 5.2.** Use a local central limit theorem to deduce that

$$\lim_{\ell \uparrow \infty} \sqrt{(2\ell+1)^d} \nu_a(\eta^\ell = a) = (2\pi)^{-d/2}.$$

Now, the variables  $g(\eta(x)) - E_{\nu_a}[g]$  in (5.4) are mean-zero and in  $L^2$ . Hence, by an application of Schwarz inequality and Exercise 5.2 again, we have that (5.4) is at most on order  $O(\ell^{-1/4})$  which vanishes.

This concludes the proof of Proposition 4.1.

## 6. PROOF OF 2-BLOCK LEMMA

The argument for the proof of Proposition 4.2 is similar to the proof of the ‘1-block Lemma’ Proposition 4.1. Now, we have to control the differences of averages in separated blocks of width  $2\ell+1$ . To compare these averages, we need again to show that the localized Dirichlet form of the density function is small. Since the jump probabilities are nearest-neighbor, to localize on the two blocks of width  $O(\ell)$ , we will need to extend the Dirichlet form and dynamics to include a long jump from one block to the other. In this way, the localized system is a well-mixing one, and the difference in averages will wash out. There is a cost for this extension which however can be overcome.

As before, with the ‘1-block Lemma’, the first step is to estimate the display in Proposition 4.2 in terms of the averaged density over shifts  $\text{Av}(f)$ . We need to show that

$$\lim_{\ell \uparrow \infty} \lim_{\epsilon \downarrow 0} \lim_{N \uparrow \infty} \sup_{I(f) \leq C_0 N^{d-2}} \sup_{2\ell < |y| \leq 2N\epsilon} E_{\nu_{\bar{\rho}}} \left[ f(\eta) | \eta^\ell(0) - \eta^\ell(y) | 1(|\eta^\ell(0) + \eta^\ell(y)| \leq A) \right] = 0$$

Here the supremum is over translation invariant densities  $f$ .

Next, we localize to the union of the two blocks  $\Lambda_\ell$  and  $\Lambda_\ell^x = x + \Lambda_\ell$ . Let  $\nu_{\bar{\rho}}^{\ell,x}$  be the product measure over  $x \in \Lambda_\ell \cup \Lambda_\ell^x$  with marginal  $\kappa_{\bar{\rho}}$ . Let also  $f^{\ell,x} = E_{\nu_{\bar{\rho}}^{\ell,x}} [f | \mathcal{F}_{\ell,x}]$

where  $\mathcal{F}_{\ell,x} = \sigma\{\eta(x) : x \in \Lambda_\ell \cup \Lambda_\ell^x\}$ . Then, as before, the above display reduces to

$$\lim_{\ell \uparrow \infty} \lim_{\epsilon \downarrow 0} \lim_{N \uparrow \infty} \sup_{I(f) \leq C_0 N^{d-2}} \sup_{2\ell < |y| \leq 2N\epsilon} E_{\nu_{\bar{\rho}}^{\ell,x}} \left[ f^{\ell,x}(\eta) |\eta^\ell(0) - \eta^\ell(y)| 1(|\eta^\ell(0) + \eta^\ell(y) \leq A) \right] = 0.$$

The question now is how to treat the Dirichlet form  $I(f)$ . We would like to localize it so that when the localization would vanish the function would be constant on the hyperplane  $\{\eta_{\Lambda_\ell \cup \Lambda_\ell^x} : \eta^\ell(0) + \eta^\ell(x) = a\}$  for  $a \leq A$  and  $\eta_B = \langle \eta(x) : x \in B \rangle$ . One could just localize on the blocks *separately* as in the last section. Then, configurations would be constant on each block, but these constants may not agree!

The trick is to add a Dirichlet form bond corresponding to a jump say from  $0 \in \Lambda_\ell$  to  $x \in \Lambda_\ell^x$ . Any such jump from a site in  $\Lambda_\ell$  to  $\Lambda_\ell^x$  would work. Then, if the Dirichlet form localized to  $\Lambda_\ell \cup \Lambda_\ell^x$  including this extra bond term vanishes, the function would have to constant on hyperplane configurations comprising both blocks. In terms of dynamics, this analytical picture is means the zero-range process is irreducible on configurations in  $\Omega^{x,\ell} = \{0, 1, \dots\}^{\Lambda_\ell \cup \Lambda_\ell^x}$ , and well-mixing there.

For  $h : \Omega^{x,\ell} \rightarrow \mathbb{R}$ , define

$$I_b(h) = \frac{1}{2} E_{\nu_{\bar{\rho}}} \left[ g(\eta(0)) (\sqrt{h(\eta^{0,x})} - \sqrt{h(\eta)})^2 \right].$$

By itself  $I_b$  is the Dirichlet form for the simple dynamics which moves a particle from 0 to  $x$  and back according to zero-range dynamics, and as such shares all the convexity and other Dirichlet form properties that we have used in the proof of the ‘1-block Lemma’.

Note

$$E_{\nu_{\bar{\rho}}} [g(\eta(0)) F(\eta)] = \alpha(\bar{\rho}) E_{\nu_{\bar{\rho}}} [F(\eta + \delta_0)] \quad (6.1)$$

where  $\eta + \delta_z$  is the configuration where there is an extra particle at  $z$ . Then, it will be helpful to express

$$I_b(h) = \frac{\alpha(\bar{\rho})}{2} E_{\nu_{\bar{\rho}}} \left[ (\sqrt{h(\eta + \delta_x)} - \sqrt{h(\eta + \delta_0)})^2 \right].$$

**Exercise 6.1.** Show (6.1).

Consider now the Schwarz estimate

$$\left( \sum_{j=1}^k q_j \right)^2 \leq k \sum_{j=1}^k q_j^2.$$

Then, by adding and subtracting several terms,

$$\begin{aligned} I_b(h) &= \frac{\alpha(\bar{\rho})}{2} E_{\nu_{\bar{\rho}}} \left[ \left( \sum_{k=1}^{|x|} \sqrt{h(\eta + \delta_{e_k})} - \sqrt{h(\eta + \delta_{q_{k+1}})} \right)^2 \right] \\ &\leq |x| \sum_{j=1}^{|x|} I_{e_j, e_{j+1}}(h) \end{aligned}$$

where  $\{q_k\}$  corresponds to a nearest-neighbor path from 0 to  $x$  in  $|x|$  steps.

We will apply the above estimate to the translation invariant density  $f$ . Since  $I_{e_j, e_{j+1}}(f) \leq N^{-d} I(f)$ , we have that  $I_b(f) \leq |x|^2 N^{-d} I(f)$ . Now, if  $I(f) \leq C_0 N^{d-2}$  and  $|x| \leq C_1 N^2 \epsilon^2$  since the separation between 0 and  $x$  is on order  $O(N\epsilon)$ , then  $I_b(f) \leq C_2 \epsilon^2$ .

Let now  $I^{x,\ell}(h) = I_\ell(h) + I_\ell^x(h) + I_b(h)$  where  $I_\ell^x(h) = \sum_{\substack{z,|w|=1 \\ z,z+w \in \Lambda_\ell^x}} I_{z,z+w}(h)$ . By convexity, we have that

$$I^{x,\ell}(f^{\ell,x}) \leq I^{x,\ell}(f) \leq 2C_0\ell^d N^{-2} + C_2\epsilon^2 \leq C\epsilon^2$$

as  $N \uparrow \infty$ .

Hence, it is enough to show that

$$\lim_{\ell \uparrow \infty} \lim_{\epsilon \downarrow 0} \lim_{N \uparrow \infty} \sup_{I^{x,\ell}(f) \leq C\epsilon^2} \sup_{2\ell < |y| \leq 2N\epsilon} E_{\nu_{\rho}^{\ell,x}} \left[ f(\eta) |\eta^\ell(0) - \eta^\ell(y)| \mathbf{1}(|\eta^\ell(0) + \eta^\ell(y) \leq A) \right] = 0.$$

Here, the supremum is over densities with respect to  $\nu^{\ell,x}$ .

But, at this point, the proof can follow the method as for the ‘1-block Lemma’. We just point out that in the display above that  $\eta^\ell(0)$  and  $\eta^\ell(y)$  share no terms! With respect to  $\nu^{\ell,x}$ , they are then independent random variables which allows the ‘1-block Lemma’ proof to go through. This was the reason for avoiding the nearest-neighbor cubes in the original decomposition of  $\eta^{N\epsilon}(x)$ . This finishes the proof of Proposition 4.2.

## 7. NOTES

An interesting, alternate way of proving hydrodynamics is the ‘Relative Entropy Method’ due to H.T. Yau [6]. The idea behind the ‘relative entropy method’ is to assume that a classical solution to the hydrodynamic PDE exists, and to build a local equilibrium distribution with respect to the solution at time  $v(N)t$ . The method then computes the relative entropy of the distribution at time  $v(N)t$ , started from an initial distribution, to this local equilibrium constructed from the proposed hydrodynamic density. Under conditions on the intial distribution, this relative entropy can be controlled, and hydrodynamics can be proved. More discussion can be found in [3] and [5].

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