MEAN CURVATURE INTERFACE LIMIT FROM GLAUBER+ZERO-RANGE INTERACTING PARTICLES

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Abstract. We derive a continuum mean-curvature flow as a certain hydrodynamic scaling limit of a class of Glauber+Zero-range particle systems. The Zero-range part moves particles while preserving particle numbers, and the Glauber part governs the creation and annihilation of particles and is set to favor two levels of particle density. When the two parts are simultaneously seen in certain different time-scales, the Zero-range part being diffusively scaled while the Glauber part is speeded up at a lesser rate, a mean-curvature interface flow emerges, with a homogenized 'surface tension-mobility' parameter reflecting microscopic rates, between the two levels of particle density. We use relative entropy methods, along with a suitable 'Boltzmann-Gibbs' principle, to show that the random microscopic system may be approximated by a 'discretized' Allen-Cahn PDE with nonlinear diffusion. In turn, we show the behavior of this 'discretized' PDE is close to that of a continuum Allen-Cahn equation, whose generation and propagation interface properties we also derive.

Keywords: interacting, particle system, zero-range, Glauber, relative entropy, motion by mean curvature, Allen-Cahn equation, nonlinear diffusion, surface tension.

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Contents

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1. Introduction

We study the emergence of continuum mean curvature interface flow from a class of microscopic interacting particle systems. Such a concern in the context of phase separating interface evolution is a long standing one in statistical physics; see Spohn [44] for a discussion. The aim of this paper is to understand the formation of a continuum mean curvature interface flow, with a homogenized 'surface tension-mobility' parameter reflecting microscopic rates, as a scaling limit in a general class of reaction-diffusion interacting particle systems. We focus on so-called Glauber+Zero-range processes on discrete tori $\mathbb{T}_N^d = (\mathbb{Z} \setminus N\mathbb{Z})^d$ for dimensions $d \geq 2$ and scaling parameter N, where the Glauber part governs reaction rates favoring two levels of mass density, and the Zero-range part controls nonlinear rates of exploration.

A 'two step' approach to derive the continuum interface flow would consider scaling the Zero-range part of the dynamics, but not speeding up the Glauber rates. The first step would be to obtain the space-time mass hydrodynamic limit in terms of an Allen-Cahn reaction-diffusion PDE. The second step would be to scale the reaction term in this Allen-Cahn PDE and to obtain mean-curvature interface flow in this limit.

However, in a nutshell, our purpose is to obtain 'directly' the mean curvature interface flow, up to the time of singularity, by scaling *both* the Glauber and Zero-range parts simultaneously. The Zero-range part is diffusively scaled while the Glauber part is scaled at a lesser level. By means of a probabilistic relative entropy method, and a new 'Boltzmann-Gibbs' principle, we show that the microscopic system may be approximated by a 'discretized' Allen-Cahn equation whose reaction term is being speeded up. To show convergence of the solutions of 'discretized' Allen-Cahn equation, we consider its approximation to the continuum version. With respect to this continuum Allen-Cahn PDE, which features a nonlinear diffusion term, as the reaction term coefficient diverges, we show that an interface moving by mean-curvature, with a nontrivial 'surface tension-mobility' parameter arising as a consequence of this nonlinearity, is generated and propagated.

In the continuum, motion by mean curvature is a time evolution of $(d-1)$-dimensional hypersurface Γ_t in $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d = [0,1)^d$ with periodic boundary conditions, or in $\mathbb{R}^d$ defined by

$$V = \kappa,$$

where V is a normal velocity and κ is the mean curvature of Γ_t multiplied by $d-1$. Such a flow is of course a well-studied geometric object (cf. book Bellettini [5]).

Mean curvature flow is known to arise from Allen-Cahn equations, which are reaction-diffusion equations of the form

$$\partial_t u = \Delta u + \frac{1}{\varepsilon^2} f(u), \quad t > 0,\ x \in D, \tag{1.1}$$

in terms of a 'sharp interface limit' as $\varepsilon \downarrow 0$. Here, $D = \mathbb{T}^d$ or a domain in $\mathbb{R}^d$, for $d \geq 2$, with Neumann boundary conditions at ∂D, $\varepsilon > 0$ is a small parameter and f is a bistable function with stable points $\alpha_\pm$ and unstable point $\alpha_* \in (\alpha_-, \alpha_+)$ satisfying the balance condition:

$$\int_{\alpha_-}^{\alpha_+} f(u)\, du \,\big(= F(\alpha_-) - F(\alpha_+)\big) = 0,$$

where F is the potential associated with f such that $f = -F'$. The sharp interface limit is as follows: The solution $u = u^\varepsilon$ of the Allen-Cahn equation satisfies

$$u^\varepsilon(t,x) \underset{\varepsilon\downarrow 0}{\longrightarrow} \chi_{\Gamma_t}(x) := \begin{cases} \alpha_+, & \text{on one side of } \Gamma_t, \\ \alpha_-, & \text{on the other side of } \Gamma_t, \end{cases}$$

where Γ_t moves according to the motion by mean curvature, and the sides are determined from Γ_0. This limit has a long history; among other works, see Alfaro et al. [2], Bellettini

[5], Chen et al. [11], Funaki [25], Chapter 4 of Funaki [26] and references therein. Although we do not consider the case $d = 1$, we remark the phenomenon in dimension $d = 1$ is much different given that the 'interface' consists of points; see Carr et al. [8].

Informally, the Zero-range process follows a collection of continuous time random walks on $\mathbb{T}_N^d$ such that each particle interacts infinitesimally only with the particles at its location: At a site x, one of the particles there jumps with rate given by a function of the occupation number η_x at x, say $g(\eta_x)$, and then displaces by y with rate $p(y)$. We will consider the case that jumps occur only to neighboring sites with equal rate, that is $p(y) = 1(|y| = 1)$. It is known that, under the diffusive scaling in space and time, namely when space squeezed by N while time speeded up by N^2, in the limit as $N \to \infty$, the evolution of the macroscopic mass density profile of the microscopic particles, namely the 'hydrodynamics', follows a nonlinear PDE (cf. [35])

$$\partial_t u = \Delta\varphi(u),$$

where φ can be seen as a homogenization of the microscopic rate g. We remark when $g(k) \equiv k$, and so $\varphi(u) \equiv u$, the associated Zero-range process is the system of independent particles.

We may add the effect of Glauber dynamics to the Zero-range process. Namely, we allow now creation and annihilation of particles at a location with rates which depend on occupation numbers nearby. This mechanism is also speeded up by a factor $K = K(N) \nearrow \infty$ as $N \to \infty$. We will impose that K grows much slower than the time scale N^2 for the Zero-range part, in fact we will take that $K = O(\sqrt{\log\log N})$ (see below for some discussion).

If K were kept constant with respect to N, the associated hydrodynamic mass density solves a nonlinear reaction-diffusion equation, a type of nonlinear Allen-Cahn equation, in the diffusive scaling limit:

$$\partial_t u = \Delta\varphi(u) + Kf(u)$$

where f reflects a homogenization of the Glauber creation and annihilation rates (cf. [39]; see also [14] and [19] in which related Glauber+Kawasaki dynamics was studied).

As mentioned above, with notation $1/\varepsilon^2$ instead of K, in the PDE literature, taking the limit of solutions $u = u^{(K)}$, as $K \uparrow \infty$, in these Allen-Cahn equations, when say $\varphi(u) \equiv u$ and f is bistable, that is $f(u) = -F'(u)$ with F being a 'balanced' double-well potential, is called the sharp interface limit. This scaling limit leads to a continuum motion by mean curvature of an interface separating two phases, here say two levels of mass density.

In our stochastic setting, by properly choosing the rates of creation and annihilation of particles in Glauber part, we observe, in the microscopic system itself, the whole domain $\mathbb{T}_N^d$ separates in a short time into 'denser' and 'sparser' regions of particles with an interface region of width $O(K^{-1/2})$ between (cf. Theorems 9.1 and 9.2). In particular, our paper derives as a main result, as $N \uparrow \infty$, motion of a continuum interface by mean curvature directly from these microscopic particle systems as a combination of the ideas of the hydrodynamic limit and the sharp interface limit (cf. Theorem 2.1).

In the probabilistic part, for the hydrodynamic limit, we apply the so-called relative entropy method originally due to Yau [45]. As a consequence of the method, we show that the microscopic configurations are not far from the solution to a deterministic discrete approximation to the nonlinear Allen-Cahn equation (cf. Theorem 2.2). To control the errors in this approximation, we will need a new 'quantified' replacement estimate, which can be seen as a type of 'Boltzmann-Gibbs' principle (cf. Theorem 4.4). In the PDE part, we compare the discretized Allen-Cahn equation with its continuous counterpart (cf. Theorem 2.3) by constructing super and sub solutions in terms of those for the continuum PDE. We show a sharp interface limit, with respect to the Allen-Cahn equation, now with nonlinear diffusion term $\Delta\varphi(u)$ (cf. Theorems 2.4 and 2.5). Such a derivation is obtained by keeping a 'corrector' term in the expansion, or second order term in $\varepsilon = K^{-1/2}$, of the solutions $u = u^{(K)}$ in variables depending on the distance to a certain level set. It seems this sharp interface limit for the nonlinear Allen-Cahn equation is unknown even in the continuum setting.

Previous work on such problems in particles systems with creation and annihilation rates concentrates on Glauber+Kawasaki dynamics (where the Zero-range part is replaced by Kawasaki dynamics) [7], [18], [33], [30], and [29]. In these papers, the Kawasaki part is a simple exclusion process. For K fixed with respect to N, the macroscopic mass hydrodynamic equation is a more standard Allen-Cahn PDE with linear diffusion Δu (instead of $\Delta\varphi(u)$),

$$\partial_t u = \Delta u + K f(u).$$

See also related work on Glauber dynamics with Kac type long range mean field interaction [6], [16], [17], [34], on fast-reaction limit for two-component Kawasaki dynamics [15], and on spatial coalescent models of population genetics [22].

Phenomenologically, when there is a nonlinear Laplacian, say $\Delta\varphi(u)$, as in our case of the Glauber +Zero-range process, this nonlinearity affects the limit motion of the hypersurface interface. When now f satisfies a modified balance condition due to the nonlinearity (cf. condition (BS)), we obtain in the limit a mean curvature motion speeded up by a nontrivial in general 'surface tension-mobility' speed λ_0 reflecting a homogenization of the Glauber and Zero-range microscopic rates,

$$V = \lambda_0 \kappa$$

(cf. flow (P^0) (2.13)). We derive two formulas for λ_0, one of them below, and the other found in (6.11), from which λ_0 is seen as the 'surface tension' multiplied by the 'mobility' of the interface. We remark, in the case of Glauber+Kawasaki dynamics, or for independent particles, the speed $\lambda_0 = 1$ is not affected by the microscopic rates.

The discretized hydrodynamic equation, or discretized Allen-Cahn PDE,

$$\partial_t u^N = \Delta^N \varphi(u^N) + K f(u^N),$$

with discrete Laplacian Δ^N, plays a role to cancel the first order terms in the occupation numbers in the computation of the time derivative of the relative entropy of the law of the microscopic configuration at time t with respect to a local equilibrium measure with average profile given by u^N. But, in the present situation, the problem is more complex than say in the application to Glauber+Kawasaki dynamics since we need to handle nonlinear functions of occupation numbers, which do not appear in the Glauber+Kawasaki process, by replacing them by linear ones. Once this is done, in a quantified way, the relative entropy can be suitably estimated, yielding that the microscopic configuration on $\mathbb{T}_N^d$ is 'near' the values u^N.

The replacement scheme, a type of 'quantified' second-order estimate or 'Boltzmann-Gibbs principle, takes on here an important role. This estimate, in comparison with a related bound for Kawasaki+Glauber systems in [29], seems to hold in more generality, and its proof is quite different. In particular, the technique used in [29] does not seem to apply for Glauber+Zero-range processes, relying on the structure of the Kawasaki generator. Moreover, as a byproduct of the 'quantified' second order estimate here, the form of the discretized hydrodynamic equation found turns out to satisfy a comparison theorem without any additional assumptions, such as the assumption (A3) for the creation and annihilation rates in [29]. This is another advantage of our Boltzmann-Gibbs principle, beyond its more general validity (cf. Remark 2.1). We remark, in passing, different 'quantitative' replacement estimates, in other settings, have been recently considered [32], [20].

The outline of the paper is as follows: In Section 2, we introduce Glauber+Zero-range process in detail. In particular, we describe a class of invariant measures ν_ρ (cf. (2.2)), and a spectral gap assumption (SP) for the Zero-range part, and then specify a proper choice of the creation and annihilation rates for the Glauber part, favoring two levels of mass density (cf. (2.11) and (2.12)), so that the corresponding macroscopic reaction function f satisfies a form of balanced bistability, matched to the nonlinear diffusion term $\Delta\varphi(u)$ obtained from the Zero-range part (cf. condition (BS)).

Our main result on the direct passage from the microscopic system to the continuum interface flow is formulated in Theorem 2.1. Its proof, given in Section 3, relies on two theorems: Theorem 2.2, which is probabilistic, stating that the microscopic system is close to that of a discretized reaction-diffusion equation, and Theorem 2.3, which is more PDE related, stating that the discrete PDE evolution is close to the continuum interface flow. Theorem 2.2 follows as a combination of the relative entropy method developed in Section 5 and a Boltzmann-Gibbs principle stated in Subsection 4.4 and proved in Section 10. On the other hand, Theorem 2.3 is shown via PDE arguments for the sharp interface limit continuous Allen-Cahn equation with nonlinear diffusion operator, in terms of 'generation' and 'propagation' of the interface phenomena, in Section 9.

In Section 4, we develop, in addition to stating the Boltzmann-Gibbs principle, some preliminary results for the discrete PDE, namely a comparison theorem, a priori energy estimates, and L^∞-bounds on discrete derivatives. Some proofs of these last bounds are deferred to Section 11.

In Section 5, we prove the probabilistic part, Theorem 2.2, by implementing the method of relative entropy: We compute the time derivative of the relative entropy of our dynamics μ_t^N at time t with respect to the local equilibrium state ν_t^N constructed from the solution of the discretized hydrodynamic equation (2.18). As remarked earlier, in the case of Kawasaki dynamics instead of the Zero-range process, the first order terms appearing in these computations are all written already in occupation numbers η_x or its normalized variables, see [29]. In our case, in contrast, nonlinear functions of η_x appear, that is, the jump rate $g(\eta_x)$ of the Zero-range part, as well as reaction rates $c_x^{\pm}(\eta)$ of the Glauber part. We mention, in [29], the relative entropy method of Jara and Menezes [32], a variant of [45], was applied. This method does not seem to apply for Glauber+Zero-range processes. However, because of our Boltzmann-Gibbs principle, the original method of Yau [45] turns out to be enough.

The Boltzmann-Gibbs principle with a quantified error is essential in our work to replace nonlinear functions of η_x, for instance $g(\eta_x)$ and those arising from the Glauber part, by linearizations in terms of the occupation numbers η_x. Its proof is given in Section 10. The argument makes use of time averaging and mixing properties of the Zero-range process in the form of a spectral gap condition (SP), verified for a wide variety of rates g.

Nonlinear functions, such as $g(\eta_x)$, are estimated by their conditional expectation given local average densities $\eta_x^\ell = \ell^{-d}\sum_{|y-x|\le\ell}\eta_y$. In the standard 'one-block' estimate of Guo-Papanicolaou-Varadhan (cf. [35]), which gives errors of order $o(1)$ without quantification, ℓ is of the order N, and so η_x^ℓ is close to the local macroscopic density. Here, errors multiplied by diverging functions of K need to be controlled, because of the form of certain terms in the discrete hydrodynamic equation. The idea then is to consider $\ell = N^\alpha$ where $\alpha > 0$ is small, and so η_x^ℓ is a type of 'mesoscopic' average. The spectral gap condition (SP) is also an ingredient used to quantify the errors suitably.

The growth of K of order $O(\sqrt{\log\log N})$ that we impose is due to energy estimates for the discrete hydrodynamic equation that we derive in Subsection 4.3 and show in Section 11. In the case of the Glauber+Kawasaki model, a growth order of $O(\sqrt{\log N})$ was obtained in [29], afforded by the linear diffusion term in its discrete hydrodynamic equation, as opposed to the nonlinear one $\Delta^N\varphi(u^N)$ which seems not as well behaved. We remark that, in the work of [7] and [33], for Glauber+Kawasaki processes, K can be of order $O(N^\beta)$ for a small $\beta > 0$, the difference being that the method of correlation functions was used instead of relative entropy. This method, relying on the structure of the Kawasaki model, does not seem to generalize to the systems considered here.

In Section 6 we discuss informally our derivation of the sharp interface limit from Allen-Cahn with a nonlinear diffusion term. To study the limit as $K \uparrow \infty$, it is essential to consider the asymptotic expansion of the solution up to the second order term in K. This plays a role of the corrector in the homogenization theory and, by the averaging effect for the nonlinear diffusion operator, a constant speed λ_0 arises in the motion by mean curvature,

$$\lambda_0 = \frac{\displaystyle\int_{\alpha_-}^{\alpha_+} \varphi'(u)\sqrt{W(u)}du}{\displaystyle\int_{\alpha_-}^{\alpha_+} \sqrt{W(u)}du},$$

where $W(u) = \int_u^{\alpha_+} f(s)\varphi'(s)ds$ (cf. (2.14) and (2.15)). We refer also to (6.11) for the other formula for λ_0 in terms of surface tension and mobility of the interface.

These arguments are made precise in the next two sections. In Section 7, we prove the 'generation' of interface, or 'initial layer' result in Theorem 2.4. In Section 8, we argue the 'propagation' of interface result, or motion by mean curvature with a homogenized 'surface tension-mobility' speed, for the continuum Allen-Cahn equation with nonlinear diffusion operator, in Theorem 2.5.

In Section 9, we extend the 'generation' and 'propagation' of the interface results to the discrete PDE as $N \uparrow \infty$ and $K = K(N) \uparrow \infty$, in Theorems 9.1 and 9.2, by employing certain comparisons. Finally, as a consequence, the proof of Theorem 2.3 is completed in Subsection 9.3.

2. Models and main results

We now introduce the Glauber+Zero-range model in detail in Subsection 2.1, and state our main results in Subsections 2.2 and 2.3.

2.1. **Glauber+Zero-range processes.**

Let $\mathbb{T}_N^d := (\mathbb{Z}/N\mathbb{Z})^d = \{1, 2, \dots, N\}^d$ be the d-dimensional square lattice of size N with periodic boundary condition. We consider the Glauber+Zero-range processes on $\mathbb{T}_N^d$. The configuration space is $\mathcal{X}_N = \{0, 1, 2, \dots\}^{\mathbb{T}_N^d} \equiv$

$\mathbb{Z}_+^{\mathbb{T}_N^d}$ and its element is denoted by $\eta = \{\eta_x\}_{x\in\mathbb{T}_N^d}$, where η_x represents the number of particles at the site x. The generator of our process is of the form $L_N = N^2 L_{ZR} + K L_G$, where L_{ZR} and L_G are Zero-range and Glauber operators, respectively, defined as follows. Here, K is a parameter, which will later depend on the scaling parameter N.

Zero-range specification

To define the Zero-range part, let the jump rate $g = \{g(k) \geq 0\}_{k\in\mathbb{Z}_+}$ be given such that $g(k) = 0$ if and only if $k = 0$. Consider the symmetric simple zero-range process with generator

$$L_{ZR} f(\eta) = \sum_{x\in\mathbb{T}_N^d} \sum_{e\in\mathbb{Z}^d:|e|=1} g(\eta_x)\{f(\eta^{x,x+e}) - f(\eta)\}, \tag{2.1}$$

where $\eta = \{\eta_x\}_{x\in\mathbb{T}_N^d} \in \mathcal{X}_N$, $|e| = \sum_{i=1}^d |e_i|$ for $e = (e_i)_{i=1}^d \in \mathbb{Z}^d$ and $\eta^{x,y} \in \mathcal{X}_N$ for $x, y \in \mathbb{T}_N^d$ is defined from η satisfying $\eta_x \geq 1$ by

$$(\eta^{x,y})_z = \begin{cases} \eta_x - 1 & \text{when } z = x \\ \eta_y + 1 & \text{when } z = y \\ \eta_z & \text{otherwise,} \end{cases}$$

for $z \in \mathbb{T}_N^d$; $\eta^{x,y}$ describes the configuration after one particle at x in η jumps to y.

We remark the case $g(k) \equiv k$ corresponding to the motion of independent particles, however when g is not linear, the infinitesimal interaction is nontrivial.

The invariant measures of the Zero-range process are translation-invariant product measures $\{\bar\nu_\varphi : 0 \leq \varphi < \varphi^* := \liminf_{k\to\infty} g(k)\}$ on $\mathcal{X}_N$ with one site marginal given by

$$\bar\nu_\varphi(k) \equiv \bar\nu_\varphi(\eta_x = k) = \frac{1}{Z_\varphi}\frac{\varphi^k}{g(k)!}, \quad Z_\varphi = \sum_{k=0}^\infty \frac{\varphi^k}{g(k)!}. \tag{2.2}$$

Here, $g(k)! = g(1)\cdots g(k)$ for $k \geq 1$ and $g(0)! = 1$, see Section 2.3 of [35].

(De) We assume that $\rho(\varphi) = \sum_{k=0}^\infty k\bar\nu_\varphi(k)$ diverges as $\varphi \uparrow \varphi^*$, meaning that all densities $0 \leq \rho < \infty$ are possible in the system.

We denote, for $\rho \geq 0$, that

$$\nu_\rho := \bar\nu_{\varphi(\rho)}$$

by changing the parameter so that the mean of the marginal is ρ. In fact, ρ and $\varphi = \varphi(\rho)$ is related by

$$\rho = \varphi(\log Z_\varphi)' \left(= \frac{1}{Z_\varphi}\sum_{k=0}^\infty k\frac{\varphi^k}{g(k)!} =: \langle k\rangle_{\bar\nu_\varphi} \right).$$

Also, note that

$$\varphi = \langle g(k)\rangle_{\bar\nu_\varphi} \left(:= \frac{1}{Z_\varphi}\sum_{k=1}^\infty \frac{\varphi^k}{g(k-1)!} \right).$$

Moreover, one can compute that $\varphi'(\rho) = \frac{1}{\varphi(\rho)} E_{\nu_\rho}\big[(\eta_0 - \rho)^2\big] > 0$, and so $\varphi = \varphi(\rho)$ is a strictly increasing function.

We observe when $g(k) \equiv k$ that the marginals of ν_ρ are Poisson distributions with mean ρ. When $ak \leq g(k) \leq bk$ for all $k \geq 0$ with $0 < a < b < \infty$, we have $a\rho \leq \varphi(\rho) \leq b\rho$ for $\rho \geq 0$. When $g(k) = 1(k \geq 1)$, i.e., $g(k) = 1$ for $k \geq 1$ and 0 for $k = 0$, we have $\varphi(\rho) = \rho/(1+\rho)$ for $\rho \geq 0$.

We will need the following condition to use and prove the 'Boltzmann-Gibbs principle'(cf. proofs of Theorem 2.2 and Theorem 4.4).

(LG) We assume $g(k) \le Ck$ for all $k \ge 0$ with some $C > 0$.

Later, we also consider $\bar{\nu}_\varphi$ and ν_ρ as the product measures on the configuration space $\mathcal{X} = \mathbb{Z}_+^{\mathbb{Z}^d}$ on an infinite lattice $\mathbb{Z}^d$ instead of $\mathbb{T}_N^d$.

Let $u : \mathbb{T}_N^d \to [0, \infty)$ be a function. We define the (inhomogeneous) product measure on $\mathcal{X}_N$ by

$$\nu_{u(\cdot)}(\eta) = \prod_{x \in \mathbb{T}_N^d} \nu_{u(x)}(\eta_x), \quad \eta = \{\eta_x\}_{x \in \mathbb{T}_N^d}, \tag{2.3}$$

with means $u(\cdot) = \{u(x)\}_{x \in \mathbb{T}_N^d}$ over sites in $\mathbb{T}_N^d$.

In the sequel, we will assume a certain 'spectral gap' bound on the Zero-range operator: Let $\Lambda_k = \{x \in \mathbb{T}_N^d : |x| \le k\}$ for $k \ge 1$ with N large enough. Let $L_{ZR,k}$ be the restriction of L_{ZR} to Λ_k, that is

$$L_{ZR,k} f(\eta) = \sum_{\substack{|x-y|=1 \\ x,y \in \Lambda_k}} g(\eta_x) \big\{ f(\eta^{x,y}) - f(\eta) \big\}.$$

When there are $j \ge 0$ particles on Λ_k, the process generated by $L_{ZR,k}$ is an irreducible continuous-time Markov chain. The operator $L_{ZR,k}$ is self-adjoint with respect to the unique canonical invariant measure $\nu_{k,j} = \nu_\beta\big\{ \cdot \,|\, \sum_{x \in \Lambda_k} \eta_x = j \big\}$; here $\nu_{k,j}$ does not depend on $\beta > 0$. For the operator $-L_{ZR,k}$, the value 0 is the bottom of the spectrum. Let $gap(k, j)$ denote the value of the next smallest eigenvalue.

(SP) There exists $C > 0$ so that $gap(k, j)^{-1} \le Ck^2(1 + j/|\Lambda_k|)^2$ for all $k \ge 2$ and $j \ge 0$.

Such bounds have been shown for Zero-range processes with different jump rates g:

- Suppose there is $C, r_1 > 0$ and $r_2 \ge 1$ such that $g(k) \le Ck$ and $g(k+r_2) \ge g(k)+r_1$ for all $k \ge 0$. Then, there is a constant $C' > 0$ such that $gap(k, j)^{-1} \le C'k^2$ independent of j [36].
- Suppose $g(k) = k^\gamma$ for $0 < \gamma < 1$. Then, there is a $C > 0$ such that $gap(k, j)^{-1} \le Ck^2(1 + j/|\Lambda_k|)^{1-\gamma}$ [40].
- Suppose $g(k) = 1(k \ge 1)$. Then, there is a $C > 0$ such that $gap(k, j)^{-1} \le Ck^2(1 + j/|\Lambda_k|)^2$ [38], [36].

We remark that all of these g's satisfy (De) and (LG).

Glauber specification

For Glauber part, we consider the creation and annihilation of a single particle when a change happens, though it is possible to consider the case that several particles are created or annihilated at once. Let τ_x be the shift acting on $\mathcal{X}_N$ so that $\tau_x \eta = \eta_{\cdot + x}$ for $\eta \in \mathcal{X}_N$ and $\tau_x f(\eta) = f(\tau_x \eta)$ for functions f on $\mathcal{X}_N$.

The generator of the Glauber part is given by

$$L_G f(\eta) = \sum_{x \in \mathbb{T}_N^d} \Big[c_x^+(\eta)\{f(\eta^{x,+}) - f(\eta)\} + c_x^-(\eta) 1(\eta_x \ge 1)\{f(\eta^{x,-}) - f(\eta)\} \Big] \tag{2.4}$$

where $\eta^{x,\pm} \in \mathcal{X}_N$ are determined from $\eta \in \mathcal{X}_N$ by $(\eta^{x,\pm})_z = \eta_x \pm 1$ when $z = x$ and $(\eta^{x,\pm})_z = \eta_z$ when $z \ne x$, note that $\eta^{x,-}$ is defined only for $\eta \in \mathcal{X}_N$ satisfying $\eta_x \ge 1$.

Here, $c_x^\pm(\eta) = \tau_x c^\pm(\eta)$ and $c^\pm(\eta)$ are nonnegative local functions on $\mathcal{X}$, that is, those depending on finitely many $\{\eta_x\}$ so that these can be viewed as functions on $\mathcal{X}_N$ for N large enough. We assume that $c^\pm(\eta)$ are written in form

$$c^\pm(\eta) = \hat{c}^\pm(\eta)\hat{c}^{0,\pm}(\eta_0), \tag{2.5}$$

where $\hat{c}^\pm$ are functions of $\{\eta_y\}_{y\neq 0}$ and $\hat{c}^{0,\pm}$ are functions of η_0 only. Moreover, since the rate of annihilation at an empty site vanishes, namely $c^-(\eta) = c^-(\eta)1(\eta_0 \geq 1)$, we may take $\hat{c}^{0,-}(0) = 0$ so that $\hat{c}^{0,-}(\eta_0) = \hat{c}^{0,-}(\eta_0)1(\eta_0 \geq 1)$ and $c^-(\eta) = c^-(\eta)1(\eta_0 \geq 1)$. In particular, we may drop $1(\eta_x \geq 1)$ in (2.4), since it is now included in $c_x^-(\eta)$ by the specification that $\hat{c}^{0,-}(0) = 0$.

As an example, we may choose

$$\hat{c}^{0,+}(\eta_0) = \frac{1}{g(\eta_0+1)} \quad \text{and} \quad \hat{c}^{0,-}(\eta_0) = 1(\eta_0 \geq 1) \tag{2.6}$$

and therefore

$$c_x^+(\eta) = \frac{\hat{c}_x^+(\eta)}{g(\eta_x+1)} \quad \text{and} \quad c_x^-(\eta) = \hat{c}_x^-(\eta)1(\eta_x \geq 1) \tag{2.7}$$

with $\hat{c}_x^\pm(\eta) = \tau_x \hat{c}^\pm(\eta)$; see (2.11) and (2.12) below with further choices of $\hat{c}^\pm(\eta)$.

Glauber+Zero-range specification.

Let now $\eta^N(t) = \{\eta_x(t)\}_{x\in\mathbb{T}_N^d}$ be the Markov process on $\mathcal{X}_N$ corresponding to the Glauber+Zero-range generator $L_N = N^2 L_{ZR} + KL_G$. The macroscopically scaled empirical measure on $\mathbb{T}^d(= [0,1)^d$ with the periodic boundary) associated with $\eta \in \mathcal{X}_N$ is defined by

$$\alpha^N(dv;\eta) = \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} \eta_x \delta_{\frac{x}{N}}(dv), \quad v \in \mathbb{T}^d,$$

and we denote

$$\alpha^N(t,dv) = \alpha^N(dv;\eta^N(t)), \quad t \geq 0.$$

Define $\langle \alpha, \phi\rangle$ to be the integral $\int \phi d\alpha$ with respect to test functions ϕ and measure α on $\mathbb{T}^d$. Sometimes, when α has a density, $\alpha = rdv$, we will write $\langle r, \phi\rangle = \int \phi r dv$ when the context is clear.

When K is a fixed parameter, one may deduce that a hydrodynamic limit can be shown: The empirical measure $\langle \alpha^N(t,dv), \phi\rangle$ with ϕ converges to $\langle \rho(t,v)dv, \phi\rangle$ as $N \to \infty$ in probability if initially this limit holds at $t = 0$, where $\rho(t,v)$ is a unique weak solution of the reaction-diffusion or 'nonlinear' Allen-Cahn equation,

$$\partial_t \rho = \Delta\varphi(\rho) + Kf(\rho), \quad v \in \mathbb{T}^d, \tag{2.8}$$

with an initial value $\rho_0(x) = \rho(0,x)$. Here, functions φ and f are defined by

$$\varphi(\rho) \equiv \tilde{g}(\rho) = E_{\nu_\rho}[g(\eta_0)], \tag{2.9}$$

$$f(\rho) \equiv \widetilde{c^+}(\rho) - \widetilde{c^-}(\rho) = E_{\nu_\rho}[c^+(\eta)] - E_{\nu_\rho}[c^-(\eta)], \tag{2.10}$$

respectively, where E_{ν_ρ} is expectation with respect to ν_ρ. As noted earlier, φ is an increasing function since $\varphi'(\rho) = \varphi(\rho)/E_{\nu_\rho}[(\eta_0-\rho)^2] > 0$.

More generally, we denote the ensemble averages of local functions $h = h(\eta)$ on $\mathcal{X}$ under ν_ρ by

$$\tilde{h}(\rho) \equiv \langle h \rangle_{\nu_\rho} := E_{\nu_\rho}[h], \quad \rho \geq 0.$$

It is known that $\tilde{h}$ is C^∞-smooth, and so in particular both $\varphi, f \in C^\infty$.

Such hydrodynamic limits, and our later results do not depend on knowledge of the invariant measures of the Glauber+Zero-range process. Indeed, when the process rates are irreducible, there is a unique invariant measure, but it is not explicit. See [21] for some discussion in infinite volume about these measures.

We now impose the following assumptions on the rates $c^\pm$:

(P) $c^\pm(\eta) \geq 0$.

(BR) $\|c^+(\eta)g(\eta_0+1)\|_{L^\infty} < \infty$ and $\|c^-(\eta^{0,+})g^{-1}(\eta_0+1)\|_{L^\infty} < \infty$.

(BS) f is a 'bistable' function with three zeros at $\alpha_-, \alpha_*, \alpha_+$ such that $0 < \alpha_- < \alpha_* < \alpha_+$, $f'(\alpha_-) < 0$, $f'(\alpha_*) > 0$ and $f'(\alpha_+) < 0$. Also, the 'φ-balance' condition $\int_{\alpha_-}^{\alpha_+} f(\rho)\varphi'(\rho)d\rho = 0$ holds.

The first assumption (P) was already mentioned. We mention, under the choice (2.6), if we further impose that $g(k) \geq C_0 > 0$ for $k \geq 1$, (BR) is implied by

$$\|\hat{c}^\pm(\eta)\|_{L^\infty} < \infty.$$

Note also that $\varphi(\rho) = \rho$ for the linear Laplacian so that $\varphi'(\rho) = 1$, in which case the 'φ-balance' condition is the more familiar 'balance' condition $\int_{\alpha_-}^{\alpha_+} f(\rho)d\rho = 0$.

An example of the rates $c^\pm(\eta)$ and the corresponding reaction term $f(\rho)$ determined by (2.10) is the following. Define, with respect to (2.6) and (2.7), that

$$c^+(\eta) = \frac{C}{g(\eta_0+1)}\big\{(a_- + a_* + a_+)1(\eta_{e_1} \geq 1)1(\eta_{e_2} \geq 1) + a_-a_*a_+\big\}, \tag{2.11}$$

$$c^-(\eta) = \frac{C}{g(\eta_{e_3}+1)}\big\{1(\eta_{e_1} \geq 1)1(\eta_{e_2} \geq 1) + (a_-a_* + a_-a_+ + a_*a_+)\big\}1(\eta_0 \geq 1), \tag{2.12}$$

where $C > 0$ and $a_-, a_+, a_* > 0$. Here, $e_1, e_2, e_3 \in \mathbb{Z}^d$ are distinct points not equal to $0 \in \mathbb{Z}^d$. In this case, setting $r(\rho) = E_{\nu_\rho}[1(\eta_0 \geq 1)]$ and $v(\rho) = E_{\nu_\rho}[g(\eta_0+1)^{-1}] = r(\rho)/\varphi(\rho)$, we have

$$f(\rho) = -Cv(\rho)(r(\rho) - a_-)(r(\rho) - a_*)(r(\rho) - a_+),$$

which has three zeros since $r(\rho)$ is strictly increasing from 0 to 1 as ρ increases from 0 to ∞.

One can find $0 < a_- < a_* < a_+ < 1$ so that $\int_{\alpha_-}^{\alpha_+} f(\rho)\varphi'(\rho)d\rho = 0$, where $\alpha_\pm = r^{-1}(a_\pm)$. Indeed, take $0 < a_- < a_+ < 1$ arbitrarily and observe that this integral is negative if $a_* \in (a_-, a_+)$ is close to a_+, while it is positive if a_* is close to a_-. When also $\inf_{k\geq 1} g(k) > 0$ say, the rates $c^\pm$ satisfy conditions (P), (BR) and (BS).

2.2. **Results on Glauber+Zero-range particle systems.** Let now μ_0^N be the initial distribution of $\eta^N(0)$ on $\mathcal{X}_N$. Let $\{u^N(0,x)\}_{x\in\mathbb{T}_N^d}$ be a collection of nonnegative values and consider the inhomogeneous product measure $\nu_0^N := \nu_{u^N(0,\cdot)}$ defined by (2.3).

We make the following assumptions on $\{u^N(0,x)\}_{x\in\mathbb{T}_N^d}$:

(BIP1) $u_- \leq u^N(0,x) \leq u_+$ for some $0 < u_- < u_+$.

(BIP2) $u^N(0,x) = u_0(\frac{x}{N}), x \in \mathbb{T}_N^d$ with some $u_0 \in C^3(\mathbb{T}^d)$.

Further, $\Gamma_0 := \{v \in \mathbb{T}^d; u_0(v) = \alpha_*\}$ is a $(d-1)$-dimensional $C^{4+\theta}$, $\theta > 0$, hypersurface in $\mathbb{T}^d$ without boundary such that ∇u_0 is non-degenerate to the normal direction to Γ_0 at every point $v \in \Gamma_0$.

Also, $u_0 > \alpha_*$ in D_0^+ and $u_- < u_0 < \alpha_*$ in D_0^- where $D_0^\pm$ are the regions separated by Γ_0.

Consider a family of closed smooth $C^{4+\theta}$, $\theta > 0$, hypersurfaces $\{\Gamma_t\}_{t\in[0,T]}$ in $\mathbb{T}^d$, without boundary, whose evolution is governed by a 'homogenized' mean curvature motion:

$$(P^0) \quad \begin{cases} V = \lambda_0 \kappa & \text{on } \Gamma_t \\ \Gamma_t\big|_{t=0} = \Gamma_0\,, \end{cases} \tag{2.13}$$

where V is the normal velocity of Γ_t from the α_--side to the α_+-side defined below, κ is the mean curvature at each point of Γ_t multiplied by $d-1$, the constant $\lambda_0 = \lambda_0(\varphi, f)$ is given by

$$\lambda_0 = \frac{\displaystyle\int_{\alpha_-}^{\alpha_+} \varphi'(u)\sqrt{W(u)}du}{\displaystyle\int_{\alpha_-}^{\alpha_+} \sqrt{W(u)}du} \tag{2.14}$$

and the potential W is defined by

$$W(u) = \int_u^{\alpha_+} f(s)\varphi'(s)ds\,. \tag{2.15}$$

We note that we also derive another expression for λ_0 (cf. (6.11)), from which it is interpreted as the 'surface tension' multiplied by the 'mobility' of the interface.

In the linear case of independent particles, that is when $g(k) \equiv k$ and so $\varphi(u) \equiv u$, we recover the value $\lambda_0 = 1$. Here, $T > 0$ is the time such that the Γ_t is smooth for $t \le T$. If Γ_0 is smooth, such a $T > 0$ always exists; see Section 6.

We denote

$$\chi_{\Gamma_t}(v) = \begin{cases} \alpha_- & \text{for } v \text{ on one side of } \Gamma_t \\ \alpha_+ & \text{for } v \text{ on the other side of } \Gamma_t. \end{cases} \tag{2.16}$$

These sides are determined by how u_0 is arranged with respect to Γ_0, and then continuously kept in time for Γ_t.

We will also denote by $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ the process measure and expectation with respect to $\eta^N(\cdot)$ starting from initial measure μ. When $\mu = \mu_0^N$, we will call $\mathbb{P}_{\mu_0^N} = \mathbb{P}_N$ and $\mathbb{E}_{\mu_0^N} = \mathbb{E}_N$. Let also E_μ denote expectation with respect to measure μ.

Recall that the relative entropy between two probability measures μ and ν on $\mathcal{X}_N$ is given as

$$H(\mu|\nu) := \int_{\mathcal{X}_N} \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} d\nu.$$

The main result of this article is now formulated as follows.

Theorem 2.1. *Suppose $d \ge 2$ and the assumptions (De), (LG), (SP), (P), (BR), (BS) stated in Subsection 2.1 and (BIP1), (BIP2). Suppose also that the relative entropy at $t = 0$ behaves as*

$$H(\mu_0^N|\nu_0^N) = O(N^{d-\epsilon})$$

as $N \uparrow \infty$, where $\epsilon > 0$. Suppose further that $K = K(N) \uparrow \infty$ as $N \uparrow \infty$ and satisfies $1 \le K(N) \le \sqrt{\delta_1 \log(\delta_2 \log N)}$, with respect to small $\delta_1 = \delta_1(\epsilon, T)$, $\delta_2 = \delta_2(\epsilon, T)$.

Then, for $0 < t \le T$, $\varepsilon > 0$ and $\phi \in C^\infty(\mathbb{T}^d)$, we have that

$$\lim_{N\to\infty} \mathbb{P}_N \Big(\big| \langle \alpha^N(t), \phi \rangle - \langle \chi_{\Gamma_t}, \phi \rangle \big| > \varepsilon \Big) = 0. \tag{2.17}$$

As we will see in Theorem 9.2, the macroscopic width of the interface Γ_t is $O(K^{-1/2})$. Our result (2.17) shows that, apart from this area, the local particle density, that is the local empirical average of particles' number, is close to either α_- or α_+. In other words, the whole domain is separated into sparse or dense regions of particles and the interface Γ_t separating these two regions move macroscopically according to the motion by mean curvature (P^0).

Remark 2.1. In [7] and [33], the growth condition for K was $K = O(N^\beta)$ for a small power $\beta > 0$, whereas in [29], the growth condition was $K \le \delta_0 \sqrt{\log N}$. The condition here on K is worse primarily due to the nonlinearity of the Zero-range rates.

The proof of Theorem 2.1 is in two main parts. The first part establishes that the microscopic evolution is close to a discrete PDE motion through use of the relative entropy method and the Boltzmann-Gibbs principle, Theorem 2.2. The second part shows that the discrete PDE evolution converges to that of the 'homogenized' mean curvature flow desired, Theorem 2.3.

To state Theorem 2.2, let $u^N(t,\cdot) = \{u^N(t,x)\}_{x\in\mathbb{T}_N^d}$ be the nonnegative solution of the discretized hydrodynamic equation:

$$\partial_t u^N(t,x) = \sum_{i=1}^d \Delta_i^N \{\varphi(u^N(t,x))\} + K f(u^N(t,x)), \tag{2.18}$$

with initial values $u^N(0,\cdot) = \{u^N(0,x)\}_{x\in\mathbb{T}_N^d}$, where

$$\Delta_i^N \varphi(u(x)) := N^2 \left(\varphi(u(x+e_i)) + \varphi(u(x-e_i)) - 2\varphi(u(x)) \right), \tag{2.19}$$

where $u(\cdot) = \{u(x)\}_{x\in\mathbb{T}_N^d}$ and $\{e_i\}_{i=1}^d$ are standard unit basis vectors of $\mathbb{Z}^d$. Recall also that φ and f are functions given by (2.9) and (2.10), respectively. We will later denote

$$\Delta^N = \sum_{i=1}^d \Delta_i^N. \tag{2.20}$$

Let $\nu_t^N = \nu_{u^N(t,\cdot)}$ be the inhomogeneous product measure with Zero-range marginals defined by (2.3) from $u^N(t,\cdot)$ for $t \ge 0$.

The next theorem shows that the 'microscopic motion is close to the discretized hydrodynamic equation'. We note this result holds in all $d \ge 1$.

Theorem 2.2. *Suppose $d \ge 1$ and let μ_t^N be the distribution of $\eta^N(t)$ on $\mathcal{X}_N$. Suppose all conditions in Subsection 2.1 and that (BIP1) holds with respect to $u^N(0)$ and the initial measure μ_0^N is such that*

$$H(\mu_0^N | \nu_0^N) = O(N^{d-\epsilon})$$

as $N \to \infty$ for some $\epsilon > 0$. Then, when $K = K(N)$ is a sequence as in the statement of Theorem 2.1, we have, for an $0 < \epsilon_1 = \epsilon_1(\epsilon, d)$, that

$$H(\mu_t^N | \nu_t^N) = O(N^{d-\epsilon_1})$$

for $t \in [0,T]$ *as* $N \to \infty$.

We comment that ϵ_1 can be taken as $\epsilon_1 = (\varepsilon_0 \wedge \epsilon)/2$ where $\varepsilon_0 = 2d/(9d+2)$.

We now capture the behavior of $u^N(t)$ as $N \uparrow \infty$ in terms of the motion by mean curvature (P^0) when $d \geq 2$. Define the step function

$$u^N(t,v) = \sum_{x \in \mathbb{T}_N^d} u^N(t,x) 1_{B(\frac{x}{N},\frac{1}{N})}(v), \quad v \in \mathbb{T}^d, \tag{2.21}$$

where $B(\frac{x}{N},\frac{1}{N}) = \prod_{i=1}^d [\frac{x_i}{N} - \frac{1}{2N}, \frac{x_i}{N} + \frac{1}{2N})$ is a box with center $\frac{x}{N}$, $x = (x_i)_{i=1}^d$, and side length $\frac{1}{N}$.

Theorem 2.3. *Let* $d \geq 2$ *and assume (BS), (BIP1) and (BIP2). Then, for* $v \notin \Gamma_t$ *and* $t \in (0,T]$*, we have that*

$$\lim_{N\to\infty} u^N(t,v) = \chi_{\Gamma_t}(v).$$

The proof of Theorem 2.3 can be viewed as a corollary, for the discrete stochastic interacting Zero-range + Glauber particle system, of the results in Subsection 2.3 for continuum nonlinear Allen-Cahn partial differential equations, of their own interest.

2.3. **Results on Allen-Cahn equation with nonlinear diffusion.** We will state below generation and propagation of interface properties for an Allen-Cahn equation with nonlinear diffusion. More precisely, we study the problem

$$(P^\varepsilon) \begin{cases} \partial_t u = \Delta\varphi(u) + \dfrac{1}{\varepsilon^2} f(u) & \text{in } [0,\infty) \times \mathbb{T}^d \\ u(0,v) = u_0(v) & \text{for } v \in \mathbb{T}^d \end{cases} \tag{2.22}$$

where the unknown function u denotes say 'mass density', $d \geq 2$, and $\varepsilon > 0$ is a small parameter. We remark the parameter ε can be viewed in terms of K, which we use to describe the microscopic Glauber+Zero-range dynamics, as $\varepsilon = K^{-1/2}$ or $\varepsilon^{-2} = K$.

In the following, $\mathbb{R}_+$ stands for $\mathbb{R}_+ = [0,\infty)$.

The nonlinear functions φ and f satisfy the following properties: In line with the previous specification of the microscopic dynamics, we assume (minimally) that $f \in C^2(\mathbb{R}_+)$ has three zeros $f(\alpha_-) = f(\alpha_+) = f(\alpha_*) = 0$, where $0 < \alpha_- < \alpha_* < \alpha_+$, and

$$f'(\alpha_-) < 0, f'(\alpha_+) < 0, f'(\alpha_*) > 0. \tag{2.23}$$

Also, $f(0) > 0$ so that the later evolution starting positive stays positive.

In addition, we assume that $\varphi \in C^4(\mathbb{R}_+)$ and

$$\varphi'(u) \geq C(\varphi, u_-, u_+) \ \text{ for } \ u_- \leq u \leq u_+ \tag{2.24}$$

for some positive constant $C(\varphi, u_-, u_+)$. We give one more assumption on f and φ, namely

$$\int_{\alpha_-}^{\alpha_+} \varphi'(s) f(s) ds = 0. \tag{2.25}$$

We note in the particle system context that $\varphi, f \in C^\infty(\mathbb{R}_+)$ and $\varphi'(u) > 0$ for $u > 0$, and so $\varphi'(u)$ is bounded away from 0 and ∞ for $u \in [u_-, u_+]$.

As for the initial condition u_0, following (BIP1) and (BIP2), we assume $u_0 \in C^3(\mathbb{T}^d)$ and $0 < u_- \le u_0 \le u_+$. As a consequence, $u(t, \cdot)$ is also bounded between u_- and u_+. We define C_0 as follows,

$$C_0 := \|u_0\|_{C^0(\mathbb{T}^d)} + \|\nabla u_0\|_{C^0(\mathbb{T}^d)} + \|\Delta u_0\|_{C^0(\mathbb{T}^d)}. \tag{2.26}$$

Furthermore we define Γ_0 by

$$\Gamma_0 := \{v \in \mathbb{T}^d : u_0(v) = \alpha_*\}. \tag{2.27}$$

In addition, recalling assumption (BIP2), we suppose Γ_0 is a $C^{4+\theta}$, $0 < \theta < 1$, hypersurface without boundary such that

$$\nabla u_0(v) \cdot n(v) \neq 0 \text{ if } v \in \Gamma_0 \tag{2.28}$$

$$u_0 > \alpha_* \text{ in } D_0^+,\ u_0 < \alpha_* \text{ in } D_0^- \tag{2.29}$$

where $D_0^\pm$ denote the regions separated by Γ_0 and n is the outward normal vector to D_0^+. It is standard that Problem (P^ε) possesses a unique classical solution u^ε.

We now study the singular limit of u^ε as $\varepsilon \downarrow 0$. We first present our generation of interface result. We will use below the following notation:

$$\gamma = f'(\alpha_*),\ t^\varepsilon = \gamma^{-1}\varepsilon^2|\log\varepsilon|,\ \delta_0 := \min(\alpha_* - \alpha_-, \alpha_+ - \alpha_*). \tag{2.30}$$

Theorem 2.4. *Let u^ε be the solution of the problem (P^ε), δ be an arbitrary constant satisfying $0 < \delta < \delta_0$. Then there exist positive constants ε_0 and M_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have the following:*

(1) *For all $v \in \mathbb{T}^d$,*

$$\alpha_- - \delta \le u^\varepsilon(t^\varepsilon, v) \le \alpha_+ + \delta. \tag{2.31}$$

(2) *If $u_0(v) \ge \alpha_* + M_0\varepsilon$, then*

$$u^\varepsilon(t^\varepsilon, v) \ge \alpha_+ - \delta. \tag{2.32}$$

(3) *If $u_0(v) \le \alpha_* - M_0\varepsilon$, then*

$$u^\varepsilon(t^\varepsilon, v) \le \alpha_- + \delta. \tag{2.33}$$

To understand more this statement, we remark that the assumption (2.28) implies that $u_0(v)$ is away from α_* when v is away from Γ_0.

After the interface has been generated, the diffusion term has the same order as the reaction term. As a result the interface starts to propagate slowly. Later we will prove that the interface moves according to the motion equation (P^0) (cf. (2.13)). It is well known that Problem (P^0) possesses locally in time a unique smooth solution. Let $T > 0$ be the maximal time interval for the existence of the smooth solution of (P^0) and denote this solution by $\Gamma = \cup_{0 \le t < T}(\{t\} \times \Gamma_t)$. Moreover we deduce from [9] that the regularity of the interface exactly follows the regularity of the initial interface, so that $\Gamma \in C^{\frac{4+\theta}{2}, 4+\theta}$.

Let D_t^+ denote the region 'enclosed' by the interface Γ_t, continuously determined from D_0^+, and set $D_t^- := \mathbb{T}^d \setminus \overline{D_t^+}$. Let $\overline{d}(t, v)$ be the signed distance function to Γ_t defined by

$$\overline{d}(t, v) := \begin{cases} \mathrm{dist}(v, \Gamma_t) & \text{for } v \in \overline{D_t^-} \\ -\mathrm{dist}(v, \Gamma_t) & \text{for } v \in D_t^+. \end{cases}$$

The second main theorem deals with the propagation of the interface.

Theorem 2.5. *Under the conditions given in Theorem 2.4 and those mentioned above, for any given* $0 < \delta < \delta_0$ *there exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that*

$$u^\varepsilon(t,v) \in \begin{cases} [\alpha_- - \delta, \alpha_+ + \delta] & \text{for } v \in \mathbb{T}^d \\ [\alpha_+ - \delta, \alpha_+ + \delta] & \text{if } \overline{d}(t,v) \leq -\varepsilon C \\ [\alpha_- - \delta, \alpha_- + \delta] & \text{if } \overline{d}(t,v) \geq \varepsilon C \end{cases} \tag{2.34}$$

for all $\varepsilon \in (0, \varepsilon_0)$ *and for all* $t \in (t^\varepsilon, T]$.

3. Mean curvature interface limit from Glauber+Zero-range systems: Proof of Theorem 2.1

As we mentioned, Theorem 2.1 is shown mainly as a combination of Theorems 2.2 and 2.3. To make this precise, define, for $\varepsilon > 0$ and a test function $\phi \in C^\infty(\mathbb{T}^d)$, the event

$$\mathcal{A}^\varepsilon_{N,t} = \{\eta \in \mathcal{X}_N; |\langle \alpha^N, \phi\rangle - \langle u^N(t,\cdot), \phi\rangle| > \varepsilon\}.$$

Proposition 3.1. *There exists* $C = C(\varepsilon) > 0$ *such that*

$$\nu^N_t(\mathcal{A}^\varepsilon_{N,t}) \leq e^{-CN^d}.$$

Proof. Write

$$\langle \alpha^N, \phi\rangle - \langle u^N(t,\cdot), \phi\rangle = \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} (\eta_x - u^N(t,x))\phi(x/N) + o(1).$$

Under ν^N_t, the variable η_x has mean $u^N(t,x)$ and a variance $\sigma^2_{x,t}$ in terms of $u^N(t,x)$. Under the condition (BIP1), by the comparison Lemma 4.1, we have that $u^N(t,\cdot)$, and so also $\sigma^2_{x,t}$, is uniformly bounded away from 0 and ∞.

The desired bound, since ϕ is uniformly bounded, follows from a standard application of exponential Markov inequalities. □

Now note that the entropy inequality, for an event A, gives

$$\mu^N_t(A) \leq \frac{\log 2 + H(\mu^N_t|\nu^N_t)}{\log\{1 + 1/\nu^N_t(A)\}}.$$

Combined with Proposition 3.1 and the relative entropy Theorem 2.2, we have that

$$\lim_{N\to\infty} \mu^N_t(\mathcal{A}^\varepsilon_{N,t}) = 0.$$

However, the PDE convergence Theorem 2.3 shows that $\langle u^N(t,\cdot), \phi\rangle \to \langle \chi_{\Gamma_t}, \phi\rangle$ as $N \uparrow \infty$, finishing the proof of Theorem 2.1.

4. Comparison, a priori estimates, and a 'Boltzmann-Gibbs' principle

Let $u^N(t,\cdot) = \{u^N(t,x)\}_{x\in\mathbb{T}^d_N}$ be the nonnegative solution of the discretized hydrodynamic equation (2.18) with given sequence $1 \leq K = K(N)$. In this section, we do not impose a growth condition on $K = K(N)$, stating results in terms of K.

4.1. **Comparison theorem.** The equation (2.18) satisfies a comparison theorem. We will say that profiles $u(\cdot) = (u_x)_{x\in\mathbb{T}_N^d}$ and $v(\cdot) = (v_x)_{x\in\mathbb{T}_N^d}$ are ordered $u(\cdot) \geq v(\cdot)$ when $u_y \geq v_y$ for all $y \in \mathbb{T}_N^d$.

We say that $u^+(t,\cdot)$ and $u^-(t,\cdot)$ are super and sub solutions of (2.18), if u^+ and u^- satisfy (2.18) with "$\geq$ " and "$\leq$ " instead of "$=$ " respectively.

Lemma 4.1. *Suppose initial conditions $u^-(0,\cdot) \leq u^+(0,\cdot)$. Then, the corresponding super and sub solutions $u^+(t,\cdot)$ and $u^-(t,\cdot)$ to the discrete PDE* (2.18)*, for all $t \geq 0$, satisfy*

$$u^-(t,\cdot) \leq u^+(t,\cdot).$$

Furthermore, suppose (BIP1) holds: $u_- \leq u^N(0,x) \leq u_+$ for some $0 < u_- < u_+ < \infty$. Then, for $t \geq 0$ and $x \in \mathbb{T}_N^d$, we have

$$u_- \wedge \alpha_- \leq u^N(t,x) \leq u_+ \vee \alpha_+.$$

Proof. Assume that $u^+(t,\cdot) \geq u^-(t,\cdot)$ and $u^-(t,x) = u^+(t,x)$ holds at some space-time point (t,x). Then, since the reaction term f cancels, and φ is an increasing function, we have

$$\begin{aligned}
\partial_t(u^+ - u^-)(t,x) &\geq \Delta^N\{\varphi(u^+) - \varphi(u^-)\}(t,x) + K\big(f(u^+(t,x)) - f(u^-(t,x))\big) \\
&= N^2 \sum_{\pm e_i} \big\{(\varphi(u^+) - \varphi(u^-))(t,x\pm e_i) - (\varphi(u^+) - \varphi(u^-))(t,x)\big\} \\
&= N^2 \sum_{\pm e_i} \{\varphi(u^+) - \varphi(u^-)\}(t,x\pm e_i) \geq 0.
\end{aligned}$$

This implies $\partial_t(u^+ - u^-)(t,x) \geq 0$ and shows that $u^-(t)$ can not exceed $u^+(t)$ for all $t > 0$.

In particular, if we take $u^+(0,x) \equiv u_+ \vee \alpha_+$, then by the condition (BS), the solution $u^+(t,\cdot)$ with this initial datum is decreasing in t so that we obtain $u^N(t,\cdot) \leq u^+(t,\cdot) \leq u_+ \vee \alpha_+$. We can similarly show $u^N(t,\cdot) \geq u_- \wedge \alpha_-$. $\square$

4.2. **A priori estimates.** Define for $\{u_x = u(x)\}_{x\in\mathbb{T}^d}$ and $1 \leq i \leq d$,

$$\begin{aligned}
&\nabla_i^N u(x) = N\big(u(x+e_i) - u(x)\big), \quad \text{and} \\
&\nabla^N u(x) = \big(\nabla_i^N u(x)\big)_{i=1}^d.
\end{aligned}$$

Lemma 4.2. *Suppose bounds (BIP1) hold for $u^N(0,\cdot)$. Then, for a constant $C > 0$, we have*

$$\frac{1}{2}\sum_{x\in\mathbb{T}_N^d} u^N(t,x)^2 + c_0\int_0^T \sum_{x\in\mathbb{T}_N^d} |\nabla^N u^N(t,x)|^2 dt \leq \frac{1}{2}\sum_{x\in\mathbb{T}_N^d} u^N(0,x)^2 + CKTN^d,$$

where $c_0 := \inf_{\rho>0}\varphi'(\rho) > 0$ (see [35] *p.30), and as a consequence*

$$\frac{N^2}{\ell^2}\frac{1}{N^d}\int_0^T \sum_{x\in\mathbb{T}_N^d} \Big(\frac{1}{(2\ell+1)^d}\sum_{|z-x|\leq\ell} u^N(t,z) - u^N(t,x)\Big)^2 dt \leq \frac{CKT}{c_0}, \tag{4.1}$$

where $|x| = \sum_{i=1}^d |x_i|$ for $x = (x_i)_{i=1}^d \in \mathbb{Z}^d$.

Proof. Recall $u^N(t,\cdot)$ is the solution of (2.18). By Lemma 4.1, we have that $u^N(t,\cdot)$ is between $u_-^* = u_- \wedge \alpha_-$ and $u_+^* = u_+ \vee \alpha_+$ uniformly in time. Since $\varphi'(u) \geq c_0 > 0$ and $f(u)$ is bounded for u between u_-^* and u_+^*, we have by the mean-value theorem that

$$\begin{aligned}
\tfrac{1}{2}\partial_t \sum_{x\in\mathbb{T}_N^d} u^N(t,x)^2 &= \sum_{x\in\mathbb{T}_N^d} u^N(t,x)\left(\Delta^N\varphi(u^N(t,x)) + Kf(u^N(t,x))\right)\\
&= -\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^d \nabla_i^N u^N(t,x)\nabla_i^N\varphi(u^N(t,x)) + K\sum_{x\in\mathbb{T}_N^d} u^N(t,x)f(u^N(t,x))\\
&\leq -c_0\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^d |\nabla_i^N u(t,x)|^2 + CKN^d.
\end{aligned}$$

Integrating in time gives the first inequality in the lemma. The second inequality now follows from the first, utilizing Jensen's inequality and the relation $(a_1+\cdots+a_j)^2 \leq j(a_1^2+\cdots+a_j^2)$. $\square$

4.3. **L^∞-estimates on discrete derivatives.** We next state the following L^∞-estimates for the (macroscopic) discrete derivatives of the solution $u^N(t,x)$ of (2.18).

Theorem 4.3. *Suppose $|\nabla_i^N u^N(0,x)| \leq C_0K$ for all $x\in\mathbb{T}_N^d$, $1\leq i\leq d$, and the condition (BIP1): $0 < u_- \leq u^N(0,x) \leq u_+ < \infty$ for all $x\in\mathbb{T}_N^d$. Then, we have*

$$|\nabla_i^N u^N(t,x)| \leq K(C_0 + C\sqrt{t}), \tag{4.2}$$

$$|\nabla_{i_2}^N\nabla_{i_1}^N u^N(t,x)| \leq C(\|\nabla^N\nabla^N u^N(0)\|_{L^\infty} + K^3)e^{CK^2t}, \tag{4.3}$$

for all $t\in[0,T]$, $x\in\mathbb{T}_N^d$, $1\leq i_1,i_2\leq d$ and some $C>0$.

In particular, if $\|\nabla_\cdot^N\nabla_\cdot^N u^N(0,\cdot)\|_{L^\infty} \leq CK^3$ holds, we have

$$|\nabla_{i_2}^N\nabla_{i_1}^N u^N(t,x)| \leq CK^3e^{CK^2T} \quad\text{and}\quad |\Delta^N\varphi(u^N(t,x))| \leq CK^3e^{CK^2T}. \tag{4.4}$$

We note a sufficient condition for

$$\|\nabla^N u^N(0,\cdot)\|_{L^\infty} \leq CK \quad\text{and}\quad \|\nabla_\cdot^N\nabla_\cdot^N u^N(0,\cdot)\|_{L^\infty} \leq CK^2$$

to hold is condition (BIP2): $u^N(0,x) = u_0(x/N)$ and $u_0\in C^2(\mathbb{T}^d)$.

The proof of Theorem 4.3 will be given in Section 11.

4.4. **A 'Boltzmann-Gibbs' principle.** For a local function $h = h(\eta)$, with support in a finite square box $\Lambda_h\subset\mathbb{T}_N^d$, and parameter $\beta\geq 0$, let

$$\tilde{h}(\beta) = E_{\nu_\beta}[h].$$

In this section, we suppose that the function h satisfies, in terms of constants C_1, C_2, the bound

$$|h(\eta)| \leq C_1\sum_{y\in\Lambda_h}|\eta_y| + C_2. \tag{4.5}$$

With respect to an evolution $\{u^N(t,x)\}_{x\in\mathbb{T}_N^d}$ satisfying the discrete PDE (2.18), let

$$f_x(\eta) = \tau_x h(\eta) - \tilde{h}(u^N(t,x)) - \tilde{h}'(u^N(t,x))\big(\eta_x - u^N(t,x)\big). \tag{4.6}$$

Recall that $\mathbb{P}_N$ is the underlying process measure governing $\eta^N(\cdot)$ starting from μ_0^N and μ_t^N is the distribution of $\eta^N(t)$ for $t \geq 0$. Recall $K = K(N) \geq 1$ for $N \geq 1$ is speed of the Glauber jumps in the process $\eta^N(\cdot)$ with generator L_N. We will not impose a growth condition here on K but state results in terms of K. With respect to the evolution $u^N(t,\cdot)$, define $\nu_t^N = \nu_{u^N(t,\cdot)}$ as the inhomogeneous Zero-range product measure with stationary marginal indexed over $x \in \mathbb{T}_N^d$ with density $u^N(t,x)$.

We now state a so-called 'Boltzmann-Gibbs' principle, under the relative entropy assumption $H(\mu_0^N|\nu_0^N) = O(N^d)$, weaker than the one assumed for Theorem 2.2. It is a 'second-order' estimate valid in $d \geq 1$ with a remainder given in terms of a relative entropy term and a certain error.

Theorem 4.4. *Suppose bounds (BIP1) hold for the initial values $\{u^N(0,x)\}_{x\in\mathbb{T}_N^d}$, and the initial relative entropy $H(\mu_0^N|\nu_0^N) = O(N^d)$. Suppose $\{a_{t,x} : x \in \mathbb{T}_N^d, t \geq 0\}$ are non-random coefficients with uniform bound*

$$\sup_{x\in\mathbb{T}_N^d, t\geq 0} |a_{t,x}| \leq M. \tag{4.7}$$

Then, there exist $\epsilon_0, C > 0$ such that

$$\mathbb{E}_N\left|\int_0^T \sum_{x\in\mathbb{T}_N^d} a_{t,x} f_x dt\right| \leq O(MKN^{d-\epsilon_0}) + CM\int_0^T H(\mu_t^N|\nu_t^N)\, dt. \tag{4.8}$$

Moreover, we may take $\epsilon_0 = 2d/(9d+2)$.

The proof of Theorem 4.4 is given in Section 10.

Remark 4.1. We remark that this proof relies on the form of the discrete PDE (2.18) only in that u^N satisfies the statements in Lemmas 4.1 and 4.2. So, this Boltzmann-Gibbs principle would hold also for other evolutions u^N for which these bounds are valid.

5. Microscopic motion is close to the 'discrete PDE': Proof of Theorem 2.2

Recall the Glauber+Zero-range process $\eta^N(t)$ generated by $L_N = N^2 L_{ZR} + K(N)L_G$, where $K = K(N)$. For a function f on $\mathcal{X}_N$ and a measure ν on $\mathcal{X}_N$, set

$$\mathcal{D}_N(f;\nu) = 2N^2\mathcal{D}_{ZR}(f;\nu) + K\mathcal{D}_G(f;\nu),$$

where

$$\mathcal{D}_{ZR}(f;\nu) = \frac{1}{4}\sum_{\substack{|x-y|=1\\ x,y\in\mathbb{T}_N^d}} \int_{\mathcal{X}_N} g(\eta_x)\{f(\eta^{x,y}) - f(\eta)\}^2 d\nu, \tag{5.1}$$

$$\mathcal{D}_G(f;\nu) = \sum_{x\in\mathbb{T}_N^d} \int_{\mathcal{X}_N} c_x^+(\eta)\{f(\eta^{x,+}) - f(\eta)\}^2 + c_x^-(\eta)\{f(\eta^{x,-}) - f(\eta)\}^2 d\nu,$$

and recall $c_x^-(\eta) = 0$ when $\eta_x = 0$.

Recall μ_t^N is the law of $\eta^N(t)$ on $\mathcal{X}_N$ and $\nu_t^N = \nu_{u^N(t,\cdot)}$. Let m be a reference measure on $\mathcal{X}_N$ with full support in $\mathcal{X}_N$. Define

$$\psi_t^N := \frac{d\nu_t^N}{dm}.$$

In general, we denote the adjoint of an operator L on $L^2(\nu_t^N)$ by L^{*,ν_t^N}.

We now state an estimate for the derivative of relative entropy. Such estimates go back to the work of Guo-Papanicolaou-Varadhan (cf. [35]) and Yau [45]. A more recent bound is the following; see [27], [29] and [32] for a proof.

Proposition 5.1.

$$\frac{d}{dt}H(\mu_t^N|\nu_t^N) \le -\mathcal{D}_N\left(\sqrt{\frac{d\mu_t^N}{d\nu_t^N}};\nu_t^N\right) + \int_{\mathcal{X}_N}(L_N^{*,\nu_t^N}1 - \partial_t \log \psi_t^N)d\mu_t^N.$$

We remark that in our later development we need only the inequality, originally derived in [45], where the Dirichlet form term is dropped:

$$\frac{d}{dt}H(\mu_t^N|\nu_t^N) \le \int_{\mathcal{X}_N}(L_N^{*,\nu_t^N}1 - \partial_t \log \psi_t^N)d\mu_t^N. \tag{5.2}$$

To control the relative entropy $H(\mu_t^N|\nu_t^N)$ we will develop a bound of the right-hand side of (5.2) in the following subsection. With the aid of these bounds, which use a 'Boltzmann-Gibbs' estimate shown in Section 10, we later give a proof of Theorem 2.2 in Subsection 5.2.

5.1. **Computation of $L_N^{*,\nu_t^N}1 - \partial_t \log \psi_t^N(\eta)$.** We first formulate a few lemmas in the abstract. Let $\{u(x)\ge 0\}_{x\in\mathbb{T}_N^d}$ be given and let $\nu = \nu_{u(\cdot)}$ be the product measure given as in (2.3). Recall that Δ_i^N and Δ^N are defined in (2.19) and (2.20), respectively.

Lemma 5.2. *We have*

$$\begin{aligned} L_{ZR}^{*,\nu}1 &= \sum_{x\in\mathbb{T}_N^d}\frac{N^{-2}(\Delta^N\varphi)(u(x))}{\varphi(u(x))}g(\eta_x) \\ &= \sum_{x\in\mathbb{T}_N^d}\frac{N^{-2}(\Delta^N\varphi)(u(x))}{\varphi(u(x))}\{g(\eta_x)-\varphi(u(x))\}. \end{aligned}$$

Proof. Similar computations results are found in [35], pp.120–121. Take any $f = f(\eta)$ on $\mathcal{X}_N$ as a test function and compute

$$\begin{aligned} \int L_{ZR}^{*,\nu}1\cdot f d\nu &= \int L_{ZR} f d\nu \\ &= \sum_{x\in\mathbb{T}_N^d}\sum_{|e|=1}\sum_{\eta\in\mathcal{X}_N} g(\eta_x)\{f(\eta^{x,x+e}) - f(\eta)\}\nu(\eta). \end{aligned}$$

Then, by fixing x, e and making change of variables $\zeta = \eta^{x,x+e}$, we have

$$\sum_\eta g(\eta_x)f(\eta^{x,x+e})\nu(\eta) = \sum_\zeta g(\zeta_x+1)f(\zeta)\nu(\zeta^{x+e,x}).$$

However, since

$$\begin{aligned} \nu(\zeta^{x+e,x}) &= \frac{\bar\nu_{u(x+e)}(\zeta_{x+e}-1)}{\bar\nu_{u(x+e)}(\zeta_{x+e})}\frac{\bar\nu_{u(x)}(\zeta_x+1)}{\bar\nu_{u(x)}(\zeta_x)}\nu(\zeta) \\ &= \frac{g(\zeta_{x+e})}{\varphi(u(x+e))}\frac{\varphi(u(x))}{g(\zeta_x+1)}\nu(\zeta), \end{aligned}$$

we obtain

$$
\begin{aligned}
L_{ZR}^{*,\nu}1 &= \sum_{x,e}\left\{\frac{\varphi(u(x))}{\varphi(u(x+e))}g(\eta_{x+e})-g(\eta_x)\right\}\\
&= \sum_{x,e}\left\{\frac{\varphi(u(x-e))}{\varphi(u(x))}-1\right\}g(\eta_x)=\sum_{x}\frac{N^{-2}(\Delta^N\varphi)(u(x))}{\varphi(u(x))}g(\eta_x).
\end{aligned}
$$

The last equality follows by noting that $\sum_x(\Delta^N\varphi)(u(x))=0$. □

Lemma 5.3. *We have*

$$
L_G^{*,\nu}1=\sum_{x\in\mathbb{T}_N^d}\left\{c_x^+(\eta^{x,-})\frac{g(\eta_x)}{\varphi(u(x))}+c_x^-(\eta^{x,+})\frac{\varphi(u(x))}{g(\eta_x+1)}-c_x^+(\eta)-c_x^-(\eta)\right\}.
$$

Proof. Taking any $f=f(\eta)$ on $\mathcal{X}_N$, we have

$$
\begin{aligned}
&\int L_G^{*,\nu}1\cdot f d\nu=\int L_G f d\nu\\
&=\sum_{x\in\mathbb{T}_N^d}\sum_{\eta\in\mathcal{X}_N}\Big\{c^+(\eta)\{f(\eta^{x,+})-f(\eta)\}+c^-(\eta)1(\eta_x\geq 1)\{f(\eta^{x,-})-f(\eta)\}\Big\}\nu(\eta)
\end{aligned}
$$

Then, by making change of variables $\zeta=\eta^{x,\pm}$, we have

$$
\begin{aligned}
\sum_{\eta}c^+(\eta)f(\eta^{x,+})\nu(\eta)&=\sum_{\zeta}c^+(\zeta^{x,-})1(\zeta_x\geq 1)f(\zeta)\nu(\zeta^{x,-}),\\
\sum_{\eta}c^-(\eta)1(\eta_x\geq 1)f(\eta^{x,-})\nu(\eta)&=\sum_{\zeta}c^-(\zeta^{x,+})f(\zeta)\nu(\zeta^{x,+}).
\end{aligned}
$$

However, since

$$
\begin{aligned}
\nu(\zeta^{x,-})1(\zeta_x\geq 1)&=1(\zeta_x\geq 1)\frac{\nu_{u(x)}(\zeta_x-1)}{\nu_{u(x)}(\zeta_x)}\nu(\zeta)=\frac{g(\zeta_x)}{\varphi(u(x))}\nu(\zeta),\\
\nu(\zeta^{x,+})&=\frac{\nu_{u(x)}(\zeta_x+1)}{\nu_{u(x)}(\zeta_x)}\nu(\zeta)=\frac{\varphi(u(x))}{g(\zeta_x+1)}\nu(\zeta),
\end{aligned}
$$

we obtain

$$
L_G^{*,\nu}1=\sum_{x}\left\{c_x^+(\eta^{x,-})\frac{g(\eta_x)}{\varphi(u(x))}+c_x^-(\eta^{x,+})\frac{\varphi(u(x))}{g(\eta_x+1)}-c_x^+(\eta)-c_x^-(\eta)1(\eta_x\geq 1)\right\}.
$$

Finally, by our convention with respect to c_x^-, we have that $c_x^-(\eta)1(\eta_x\geq 1)=c_x^-(\eta)$. □

Example 5.1. *If we choose $c_x^{\pm}(\eta)$ as in (2.7), noting that $\hat{c}_x^{\pm}(\eta)$ do not depend on η_x, we have $L_G^{\nu,*}1$ equals*

$$
\sum_{x\in\mathbb{T}_N^d}\hat{c}_x^+(\eta)\left(\frac{1(\eta_x\geq 1)}{\varphi(u(x))}-\frac{1}{g(\eta_x+1)}\right)+\sum_{x\in\mathbb{T}_N^d}\hat{c}_x^-(\eta)\left(\frac{\varphi(u(x))}{g(\eta_x+1)}-1(\eta_x\geq 1)\right).
$$

Lemma 5.4. *Now we take $u(\cdot)=\{u^N(t,x)\}_{x\in\mathbb{T}_N^d}$. Then, we have*

$$
\partial_t\log\psi_t^N(\eta)=\sum_{x\in\mathbb{T}_N^d}\frac{\partial_t\varphi(u^N(t,x))}{\varphi(u^N(t,x))}(\eta_x-u^N(t,x)).
$$

Proof. Since

$$\psi_t^N(\eta) = \frac{\nu_{u^N(t,\cdot)}(\eta)}{m(\eta)} = \frac{\prod_x \nu_{u^N(t,x)}(\eta_x)}{m(\eta)},$$

we have

$$\partial_t \log \psi_t^N(\eta) = \sum_{x\in\mathbb{T}_N^d} \frac{\partial_t \nu_{u^N(t,x)}(\eta_x)}{\nu_{u^N(t,x)}(\eta_x)}.$$

Here,

$$\begin{aligned}
\partial_t \nu_{u^N(t,x)}(k) &= \partial_t \left(\frac{1}{Z_{\varphi(u^N(t,x))}} \frac{\varphi(u^N(t,x))^k}{g(k)!} \right) \\
&= \frac{1}{Z_{\varphi(u^N(t,x))}} \frac{k\varphi(u^N(t,x))^{k-1}}{g(k)!} \partial_t \varphi(u^N(t,x)) - \frac{Z'_{\varphi(u^N(t,x))}\partial_t\varphi(u^N(t,x))}{Z^2_{\varphi(u^N(t,x))}} \frac{\varphi(u^N(t,x))^k}{g(k)!} \\
&= \nu_{u^N(t,x)}(k)\partial_t\varphi(u^N(t,x)) \frac{1}{\varphi(u^N(t,x))}(k - u^N(t,x)),
\end{aligned}$$

where we have used the formula $\frac{\partial}{\partial\varphi}\log Z_\varphi = \rho/\varphi$. This shows the conclusion. $\square$

These three lemmas, combined with the comparison estimates, discrete derivative bounds, and Boltzmann-Gibbs principle in Section 4, are the main ingredients for the following theorem.

Theorem 5.5. *Suppose $u^N(t,x)$ satisfies (2.18), with $K \geq 1$. Then, there are $\varepsilon_0, C > 0$ such that*

$$\begin{aligned}
&\int_0^T \int_{\mathcal{X}_N} \Big\{ L_N^{*,\nu_t^N} 1 - \partial_t \log \psi_t^N \Big\} d\mu_t^N dt \\
&\qquad \leq CK^3 e^{CTK^2} \int_0^T H(\mu_t^N|\nu_t^N)dt + O\big(K^4 e^{CTK^2} N^{d-\varepsilon_0}\big).
\end{aligned}$$

Proof. By Lemmas 5.2, 5.3 and 5.4, we have $L_N^{*,\nu_t^N} 1 - \partial_t \log\psi_t^N$ equals

(5.3)

$$\begin{aligned}
&\sum_x \frac{(\Delta^N\varphi)(u^N(t,x))}{\varphi(u^N(t,x))} \{g(\eta_x) - \varphi(u^N(t,x))\} \\
&\quad + K \sum_{x\in\mathbb{T}_N^d} \left\{ c_x^+(\eta^{x,-}) \frac{g(\eta_x)}{\varphi(u^N(t,x))} - c_x^+(\eta) + c_x^-(\eta^{x,+}) \frac{\varphi(u^N(t,x))}{g(\eta_x+1)} - c_x^-(\eta)1(\eta_x \geq 1) \right\} \\
&\quad - \sum_{x\in\mathbb{T}_N^d} \frac{\partial_t\varphi(u^N(t,x))}{\varphi(u^N(t,x))}(\eta_x - u_x^N(t)).
\end{aligned}$$

To analyze further, we will apply the Boltzmann-Gibbs principle, along with comparison estimates and bounds for the discrete derivatives of the discrete PDE, with respect to the first two lines in the above display (5.3).

First, let $h(\eta) = g(\eta_x) - \varphi(u^N(t,x))$. By the assumption (LG), h satisfies the bound in (4.5). Observe that $\tilde{h}(\beta) \equiv E_{\nu_\beta}[h] = \varphi(\beta) - \varphi(u^N(t,x))$ as $\tilde{g}(\beta) \equiv E_{\nu_\beta}[g] = \varphi(\beta)$ for $\beta \geq 0$. This implies $\tilde{h}(u^N(t,x)) = 0$ and $\tilde{h}'(u^N(t,x)) = \varphi'(u^N(t,x))$.

Let now $a_{t,x} = \Delta^N \varphi(u^N(t,x)/\varphi(u^N(t,x))$. Since u^N is bounded between $u_- \wedge \alpha_-$ and $u_+ \vee \alpha_+$ according to Lemma 4.1, $\varphi(u^N(t,x))$ is uniformly bounded away from 0. Also, by Theorem 4.3, we have the estimate $\|\Delta^N \varphi(u^N(t,\cdot))\|_{L^\infty} = O(K^3 e^{CTK^2})$. Then, we conclude that $\|a(t,\cdot)\|_{L^\infty} = O(K^3 e^{CTK^2})$.

Therefore, by the Boltzmann-Gibbs principle (Theorem 4.4), we obtain that

$$\begin{aligned}
&\mathbb{E}_N \Big| \int_0^T \sum_{x\in\mathbb{T}_N^d} \frac{(\Delta^N\varphi)(u^N(t,x))}{\varphi(u^N(t,x))} \big(g(\eta_x(t)) - \varphi(u^N(t,x))\big)\, dt \\
&\quad - \int_0^T \sum_{x\in\mathbb{T}_N^d} \frac{(\Delta^N\varphi)(u^N(t,x))}{\varphi(u^N(t,x))} \varphi'(u^N(t,x))\big(\eta_x(t) - u^N(t,x)\big) dt \Big| \\
&\quad \le CK^3 e^{CTK^2} \int_0^T H(\mu_t^N|\nu_t^N) dt + O(K^4 e^{CTK^2} N^{d-\varepsilon_0}).
\end{aligned}$$

Secondly, observe that

$$\widetilde{\Big(\frac{c_x^-(\eta^{x,+})}{g(\eta_x+1)}\Big)}(\beta) \equiv E_{\nu_\beta}\left[\frac{c_x^-(\eta^{x,+})}{g(\eta_x+1)}\right] = \frac{1}{\varphi(\beta)} E_{\nu_\beta}[c_x^-(\eta) 1(\eta_x \ge 1)]. \tag{5.4}$$

Indeed, recall $c_x^-(\eta) = \hat{c}_x^-(\eta)\hat{c}_x^{0,-}(\eta_x)$ where $\hat{c}_x^-$ does not depend on η_x. Then,

$$E_{\nu_\beta}[c_x^-(\eta^{x,+})g(\eta_x+1)^{-1}] = E_{\nu_\beta}[\hat{c}_x^-(\eta)] E_{\nu_\beta}[\hat{c}_x^{0,-}(\eta_x+1)g(\eta_x+1)^{-1}].$$

The factor $E_{\nu_\beta}[\hat{c}_x^{0,-}(\eta_x+1)g(\eta_x+1)^{-1}]$ is rewritten as

$$\begin{aligned}
\frac{1}{Z_\varphi}\sum_{k=0}^\infty \frac{\hat{c}_x^{0,-}(k+1)}{g(k+1)}\frac{\varphi^k}{g(k)!} &= \frac{1}{Z_\varphi}\varphi^{-1}\sum_{k=0}^\infty \frac{\varphi^{k+1}}{g(k+1)!}\hat{c}_x^{0,-}(k+1) \\
&= \varphi^{-1}\frac{1}{Z_\varphi}\sum_{k=0}^\infty \hat{c}_x^{0,-}(k)1(k\ge 1)\frac{\varphi^k}{g(k)!} = \frac{1}{\varphi}E_{\nu_\beta}[c_x^{0,-}(\eta_x)1(\eta_x \ge 1)],
\end{aligned}$$

where $\varphi = \varphi(\beta)$. This shows (5.4) by noting the independence of $\hat{c}_x^-(\eta)$ and functions of η_x under ν_β.

Let now

$$h(\eta) = c_x^-(\eta^{x,+})\frac{\varphi(u^N(t,x))}{g(\eta_x+1)} - c_x^-(\eta_x)1(\eta_x \ge 1).$$

Since h is seen to be uniformly bounded by assumption (BR), condition (4.5) holds. Moreover, from (5.4), we see

$$\tilde{h}(\beta) = E_{\nu_\beta}[\hat{c}_x^-(\eta)]\frac{E_{\nu_\beta}[\hat{c}_x^{0,-}(\eta_x)1(\eta_x\ge 1)]}{\varphi(\beta)}\big(\varphi(u^N(t,x)) - \varphi(\beta)\big).$$

Then, in particular $\tilde{h}(u^N(t,x)) = 0$ and

$$\tilde{h}'(u^N(t,x)) = -E_{\nu_{u^N(t,x)}}[\hat{c}_x^-(\eta)]\frac{E_{\nu_{u^N(t,x)}}[\hat{c}_x^{0,-}(\eta_x)1(\eta_x\ge 1)]}{\varphi(u^N(t,x))}\varphi'(u^N(t,x)).$$

Since $\hat{c}_x^{0,-}(0) = 0$ by our convention, we see that $E_{\nu_\beta}[\hat{c}_x^-(\eta)]E_{\nu_\beta}[\hat{c}_x^{0,-}(\eta_x)1(\eta_x \ge 1)] = E_{\nu_\beta}[c_x^-(\eta)]$.

Let now $a_{t,x} \equiv K$. By the Boltzmann-Gibbs principle, Theorem 4.4, we conclude that

$$\begin{aligned}
&\mathbb{E}_N\Big|K\int_0^T \sum_{x\in\mathbb{T}_N^d}\Big\{c_x^-(\eta^{x,+}(t))\frac{\varphi(u^N(t,x))}{g(\eta_x(t)+1)} - c_x^-(\eta_x(t))1(\eta_x(t)\geq 1)\Big\}\,dt \\
&\qquad + K\int_0^T \sum_{x\in\mathbb{T}_N^d} E_{\nu_{u^N(t,x)}}[c^-(\eta)]\frac{\varphi'(u^N(t,x))}{\varphi(u^N(t,x))}\big(\eta_x(t)-u^N(t,x)\big)dt\Big| \\
&\qquad \leq CK\int_0^T H(\mu_t^N|\nu_t^N)dt + O(K^2N^{d-\varepsilon_0}).
\end{aligned}$$

Thirdly, we consider

$$h(\eta) = c_x^+(\eta^{x,-})\frac{g(\eta_x)}{\varphi(u^N(t,x))} - c_x^+(\eta).$$

Again, by the assumption (BR), h is uniformly bounded and so satisfies (4.5). Also, from a calculation similar to (5.4), we see

$$\tilde{h}(\beta) = \frac{E_{\nu_\beta}[c^+(\eta)]}{\varphi(u^N(t,x))}(\varphi(\beta)-\varphi(u^N(t,x))).$$

Therefore, for this choice $\tilde{h}(u^N(t,x)) = 0$ and

$$\tilde{h}'(u^N(t,x)) = \frac{E_{\nu_{u^N(t,x)}}[c^+(\eta)]}{\varphi(u^N(t,x))}\varphi'(u^N(t,x)).$$

Here, also let $a_{t,x} \equiv K$. Again, by the Boltzmann-Gibbs principle, Theorem 4.4, we have that

$$\begin{aligned}
&\mathbb{E}_N\Big|K\int_0^T \sum_{x\in\mathbb{T}_N^d}\Big\{c_x^+(\eta^{x,-}(t))\frac{g(\eta_x(t))}{\varphi(u^N(t,x))} - c_x^+(\eta(t))\Big\}\,dt \\
&\qquad - K\int_0^T \sum_{x\in\mathbb{T}_N^d} E_{\nu_{u^N(t,x)}}[c^+(\eta)]\frac{\varphi'(u^N(t,x))}{\varphi(u^N(t,x))}\big(\eta_x(t)-u^N(t,x)\big)dt\Big| \\
&\qquad \leq CK\int_0^T H(\mu_t^N|\nu_t^N)dt + O(K^2N^{d-\varepsilon_0}).
\end{aligned}$$

Finally, we note, with respect to the third line of (5.3), that

$$\partial_t\varphi(u^N(t,x)) = \varphi'(u^N(t,x))\partial_t u^N(t,x).$$

Then, combining these observations, $\int_0^T \big(L_N^{*,\nu_t^N}1 - \partial_t \log\psi_t^N\big)dt$ is approximated in $L^1(\mathbb{P}_N)$ by

$$\begin{aligned}
(5.5)\qquad &\int_0^T \sum_x \Big[\frac{(\Delta^N\varphi)(u^N(t,x))}{\varphi(u^N(t,x))}\varphi'(u^N(t,x))\{\eta_x(t)-u^N(t,x)\} \\
&\qquad + K\sum_x \frac{\varphi'(u^N(t,x))}{\varphi(u^N(t,x))}E_{\nu_{u^N(t,x)}}\big[c^+(\eta)-c^-(\eta)\big]\{\eta_x(t)-u^N(t,x)\}
\end{aligned}$$

$$- \sum_{x \in \mathbb{T}_N^d} \frac{\varphi'(u^N(t,x))}{\varphi(u^N(t,x))} \partial_t u^N(t,x) \{\eta_x(t) - u^N(t,x)\} \Bigg] dt$$

with error $CK^3 e^{CTK^2} \int_0^T H(\mu_t^N | \nu_t^N) dt + O(K^4 e^{CTK^2} N^{d-\varepsilon_0})$. Since $u^N(t,x)$ satisfies the discretized equation (2.18), the display (5.5) vanishes. Hence, $\int_0^T \left(L_N^{*,\nu_t^N} 1 - \partial_t \log \psi_t^N \right) dt$ is within the L^1 error bound desired. □

5.2. **Proof of Theorem 2.2.** From (5.2) and Theorem 5.5, we have, for $t \in [0.T]$, that

$$H(\mu_t^N | \nu_t^N) \leq H(\mu_0^N | \nu_0^N) + CK^3 e^{CTK^2} \int_0^t H(\mu_s^N | \nu_s^N) ds + O(K^4 e^{CTK^2} N^{d-\varepsilon_0}),$$

where $\varepsilon_0 = 2d/(9d+2)$. Then, by Gronwall's estimate, we obtain, for $t \in [0,T]$, that

$$H(\mu_t^N | \nu_t^N) \leq \left\{ H(\mu_0^N | \nu_0^N) + O(K^4 e^{CTK^2} N^{d-\varepsilon_0}) \right\} \exp \left\{ CTK^3 e^{CTK^2} \right\}.$$

Suppose now that

$$K^2(N) \leq \delta_1 \log \left(\delta_2 \log N \right)$$

for $\delta_1, \delta_2 > 0$ such that $2CT\delta_1 \leq 1$ and $\delta_2 < (\varepsilon_0 \wedge \varepsilon)/2$. Since the initial entropy $H(\mu_0^N | \nu_0^N) = O(N^{d-\epsilon})$, we will have for $t \leq T$ that

$$H(\mu_t^N | \nu_t^N) = o(N^{d-(\varepsilon_0 \wedge \epsilon)/2}).$$

This finishes the proof. □

6. Formal derivation of the interface motion from nonlinear Allen-Cahn equations

In this section, we derive the interface motion equation corresponding to Problem (P^ε) (cf. (2.22)) by using formal asymptotic expansions. This equation is determined by the two first terms of the asymptotic expansion. We refer to [41], [1], [3] for a similar formal analysis for other equations with a bistable nonlinear reaction term. Let us also mention some other papers [4], [23] and [43] involving the method of matched asymptotic expansions for related phase transition problems.

Problem (P^ε) possesses a unique solution u^ε. As $\varepsilon \to 0$, the qualitative behavior of this solution is the following. In the very early stage, the nonlinear diffusion term is negligible compared with the reaction term $\varepsilon^{-2} f(u)$. Hence, rescaling time by $\tau = t/\varepsilon^2$, the equation is well approximated by the ordinary differential equation $u_\tau = f(u)$ where $u_\tau = \partial_\tau u$. In view of the bistable nature of f, u^ε quickly approaches the values α_- or α_+, the stable equilibria of the ordinary differential equation, and an interface is formed between the regions $\{u^\varepsilon \approx \alpha_-\}$ and $\{u^\varepsilon \approx \alpha_+\}$. Once such an interface is developed, the nonlinear diffusion term becomes large near the interface, and comes to balance with the reaction term so that the interface starts to propagate, on a much slower time scale.

To study such interfacial behavior, it is useful to consider a formal asymptotic limit of (P^ε) as $\varepsilon \to 0$. Then, the limit solution will be a step function taking the value α_- on one side of the interface, and α_+ on the other side. This sharp interface, which we will denote by Γ_t, obeys a certain law of motion, which is expressed as (P^0) (cf. (2.13))

It follows from the standard local existence theory for parabolic equations that Problem (P^0) possesses locally in time a unique smooth solution. In fact, by using an appropriate parametrization, one can express Γ_t as a graph over a $N-1$ manifold without boundary and transfer the motion equation (P^0) into a parabolic equation on the manifold, at least locally in time. Let $0 \leq t < T^{max}$, $T^{max} \in (0, \infty]$ be the maximal time interval for the existence of the solution of (P^0) and denote this solution by $\Gamma = \cup_{0 \leq t < T^{max}}(\{t\} \times \Gamma_t)$. Hereafter, we fix T such that $0 < T < T^{max}$ and work on $[0, T]$. Since Γ_0 is a $C^{4+\vartheta}$ hypersurface, we also see that Γ is of class $C^{\frac{4+\vartheta}{2}, 4+\vartheta}$. For more details concerning problems related to (P^0), we refer to Chen [9], [10] or Chen and Reitich [12].

Remark 6.1. From another viewpoint, we may construct the time evolution of Γ_t embedded continuously in a large flat box $(0, N)^d$ with periodic boundary conditions by solving the corresponding nonlinear PDE. It is known that such an interface Γ_t stays in $(0, N)^d$ at all times until it disappears.

We set

$$Q_T := (0, T) \times \mathbb{T}^d\,,$$

and, for each $t \in [0, T]$, we denote by $\Omega_t^{(1)}$ the region of one side of the hypersurface Γ_t, and by $\Omega_t^{(2)}$ the region of the other side of Γ_t. We define a step function $\tilde{u}(t, v)$ by

$$\tilde{u}(t, v) = \begin{cases} \alpha_- & \text{in } \Omega_t^{(1)} \\ \alpha_+ & \text{in } \Omega_t^{(2)} \end{cases} \quad \text{for } t \in [0, T]\,, \tag{6.1}$$

which represents the formal asymptotic limit of u^ε (or the *sharp interface limit*) as $\varepsilon \to 0$.

More specifically, we define Γ_t^ε using the solution u^ε of (P^ε). Denote Γ_t^ε as follows;

$$\Gamma_t^\varepsilon := \{v \in \mathbb{T}^d : u^\varepsilon(t, v) = \alpha_*\}.$$

Assume that, for some $T > 0$, Γ_t^ε is a smooth hypersurface without boundary for each $t \in [0, T], \varepsilon > 0$. Define the signed distance function to Γ_t^ε as follows;

$$\overline{d}^\varepsilon(t, v) := \begin{cases} \operatorname{dist}(v, \Gamma_t^\varepsilon) & \text{for } v \in \overline{D_t^{\varepsilon,-}} \\ -\operatorname{dist}(v, \Gamma_t^\varepsilon) & \text{for } v \in D_t^{\varepsilon,+} \end{cases}$$

where $D_t^{\varepsilon,-}$ is the region 'enclosed' by Γ_t^ε and $D_t^{\varepsilon,+} := \mathbb{T}^d \setminus \{D_t^{\varepsilon,-} \cup \Gamma_t^\varepsilon\}$. Note that $\overline{d}^\varepsilon = 0$ on Γ_t^ε and $|\nabla \overline{d}^\varepsilon| = 1$ near Γ_t^ε. Suppose further that $\overline{d}^\varepsilon$ is expanded in the form

$$\overline{d}^\varepsilon(t, v) = \overline{d}_0(t, v) + \varepsilon \overline{d}_1(t, v) + \varepsilon^2 \overline{d}_2(t, v) + \cdots\,.$$

Define

$$\begin{aligned} \Gamma_t &:= \{v \in \mathbb{T}^d : \overline{d}_0(t, v) = 0\}, \\ \Gamma &:= \cup_{0 \leq t \leq T}(\{t\} \times \Gamma_t), \\ D_t^- &:= \{v \in \mathbb{T}^d : \overline{d}_0(t, v) > 0\}, \\ D_t^+ &:= \{v \in \mathbb{T}^d : \overline{d}_0(t, v) < 0\}. \end{aligned}$$

As we will see later, the values of u^ε are close to $\alpha_\pm$ on the domains $D_t^\pm$, which is consistent with $D_0^\pm$ in (BIP2) and (2.29).

Assume that u^ε has the expansions

$$u^\varepsilon(t, v) = \alpha_\pm + \varepsilon u_1^\pm(t, v) + \varepsilon^2 u_2^\pm(t, v) + \cdots$$

away from the interface Γ and

$$u^\varepsilon(t, v) = U_0(t, v, \xi) + \varepsilon U_1(t, v, \xi) + \varepsilon^2 U_2(t, v, \xi) + \cdots \tag{6.2}$$

near Γ, where $\xi = \dfrac{\overline{d}_0}{\varepsilon}$. Here the variable ξ was given to describe the rapid transition between the regions $\{u^\varepsilon \simeq \alpha_+\}$ and $\{u^\varepsilon \simeq \alpha_-\}$. In addition, we normalize U_0 and U_k in a way that

$$U_0(t,v,0) = \alpha_* \tag{6.3}$$

$$U_k(t,v,0) = 0. \tag{6.4}$$

To match the inner and outer expansions, we require that

$$U_0(t,v,\pm\infty) = \alpha_\mp,\ U_k(t,v,\pm\infty) = u_k^\mp(t,v) \tag{6.5}$$

for all $k \geq 1$.

After substituting the expansion (6.2) into (P^ε) we consider collecting the ε^{-2} terms, which yields the following equation

$$\varphi(U_0)_{zz} + f(U_0) = 0.$$

Since the equation only depends on the variable z, we may assume that U_0 is only a function of the variable z. Thus we may assume $U_0(t,v,z) = U_0(z)$. In view of the conditions (6.3) and (6.5), we find that U_0 is the unique solution of the following problem

$$\begin{cases} (\varphi(U_0))_{zz} + f(U_0) = 0, \\ U_0(-\infty) = \alpha_+, U_0(0) = \alpha_*, U_0(\infty) = \alpha_-. \end{cases} \tag{6.6}$$

To understand this more clearly, for $u \geq 0$, we set

$$b(u) := f(\varphi^{-1}(u)),$$

where φ^{-1} is the inverse function of $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and define $V_0(z) := \varphi(U_0(z))$; note that such transformation is possible by the condition (2.24). The condition (BS) on f implies that $b(u)$ has exactly three zeros $\varphi(\alpha_-)$, $\varphi(\alpha_*)$ and $\varphi(\alpha_+)$ where

$$b'(\varphi(\alpha_-)) < 0,\ b'(\varphi(\alpha_*)) > 0, \quad \text{and} \quad b'(\varphi(\alpha_+)) < 0.$$

Substituting V_0 into equation (6.6) yields

$$\begin{cases} V_{0zz} + b(V_0) = 0, \\ V_0(-\infty) = \varphi(\alpha_+), V_0(0) = \varphi(\alpha_*), V_0(\infty) = \varphi(\alpha_-). \end{cases} \tag{6.7}$$

Condition (2.25) then implies

$$\int_{\varphi(\alpha_-)}^{\varphi(\alpha_+)} b(u)du = 0,$$

which gives the existence and uniqueness up to translations of the solution of (6.7), and especially in our case that the speed of the traveling wave solution V_0 vanishes.

Next, we consider the collection of ε^{-1} terms in the asymptotic expansion. In view of the definition of $U_0(z)$ and the condition (6.3), for each (t,v), this yields the following problem

$$\begin{cases} (\varphi'(U_0)\overline{U_1})_{zz} + f'(U_0)\overline{U_1} = U_{0z}\partial_t\overline{d}_0 - (\varphi(U_0))_z\Delta\overline{d}_0, \\ \overline{U_1}(t,v,0) = 0,\ \varphi'(U_0)\overline{U_1} \in L^\infty(\mathbb{R}). \end{cases} \tag{6.8}$$

To see the existence of the solution of (6.8) we perform the change of unknown function $\overline{V_1} = \varphi'(U_0)\overline{U_1}$, which yields the problem

$$\begin{cases} \overline{V_1}_{zz} + b'(V_0)\overline{V_1} = \dfrac{V_{0z}}{\varphi'(\varphi^{-1}(V_0))}\partial_t \overline{d}_0 - V_{0z}\Delta \overline{d}_0, \\ \overline{V_1}(t,v,0) = 0, \ \overline{V_1} \in L^\infty(\mathbb{R}). \end{cases} \tag{6.9}$$

Lemma 2.2 of [3] implies the existence of V_1 provided that

$$\int_{\mathbb{R}} \left(\frac{1}{\varphi'(\varphi^{-1}(V_0))}\partial_t \overline{d}_0 - \Delta \overline{d}_0 \right) V_{0z}^2 dz = 0.$$

Substituting $V_0 = \varphi(U_0)$ and $V_{0z} = \varphi'(U_0)U_{0z}$ in the above equation yields

$$\partial_t \overline{d}_0 = \frac{\int_{\mathbb{R}} V_{0z}^2 dz}{\int_{\mathbb{R}} \frac{V_{0z}^2}{\varphi'(\varphi^{-1}(V_0))} dz}\Delta \overline{d}_0 = \frac{\int_{\mathbb{R}} (\varphi'(U_0)U_{0z})^2 dz}{\int_{\mathbb{R}} \varphi'(U_0)U_{0z}^2 dz}\Delta \overline{d}_0. \tag{6.10}$$

It is well known that $\partial_t \overline{d}_0$ is equal to the normal velocity V of the interface Γ_t, and $\Delta \overline{d}_0$ is equal to κ where κ is the mean curvature of Γ_t multiplied by $d-1$. Thus, we obtain the interface motion equation on Γ_t:

$$V = \lambda_0 \kappa,$$

where

$$\lambda_0 = \frac{\int_{\mathbb{R}} (\varphi'(U_0)U_{0z})^2 dz}{\int_{\mathbb{R}} \varphi'(U_0)U_{0z}^2 dz}. \tag{6.11}$$

This speed λ_0 is interpreted as the 'surface tension' multiplied by the 'mobility' of the interface; see equations (4.7), (4.8) and (4.9) in [44].

Finally, we derive the explicit form (2.14) of λ_0. To that purpose, we multiply the equation (6.6) by $\varphi(U_0)_z$; this yields

$$\varphi(U_0)_{zz}\varphi(U_0)_z + f(U_0)\varphi(U_0)_z = 0\,,$$

which we integrate from $-\infty$ to z to obtain

$$\frac{1}{2}\left[\varphi(U_0)_z\right]^2(z) + \int_{-\infty}^{z} f(U_0)\varphi(U_0)_z dz = 0$$

or else

$$\frac{1}{2}\left[\varphi(U_0)_z\right]^2(z) + \int_{\alpha_+}^{U_0(z)} f(s)\varphi'(s) ds = 0\,,$$

which in turn implies

$$\varphi(U_0)_z(z) = -\sqrt{2}\sqrt{W(U_0(z))}\,, \tag{6.12}$$

where W is given by

$$W(u) = \int_u^{\alpha_+} f(s)\varphi'(s) ds.$$

It follows that

$$\int_{\mathbb{R}} \varphi(U_0)_z U_{0z}(z) dz = -\sqrt{2}\int_{\mathbb{R}} \sqrt{W(U_0(z))}U_{0z}(z) dz$$

so that also

$$\int_{\mathbb{R}} \varphi'(U_0)U_{0z}^2(z) dz = \sqrt{2}\int_{\alpha_-}^{\alpha_+} \sqrt{W(u)} du.$$

Similarly, since

$$\int_{\mathbb{R}} (\varphi'(U_0)U_{0z})^2 dz = -\sqrt{2}\int_{\mathbb{R}} \varphi'(U_0)\sqrt{W(U_0(z))}U_{0z}dz$$

it follows that

$$\int_{\mathbb{R}} (\varphi'(U_0)U_{0z})^2 dz = \sqrt{2}\int_{\alpha_-}^{\alpha_+} \varphi'(u)\sqrt{W(u)}du$$

so that we finally obtain the formula (2.14).

7. Generation of the interface: Proof of Theorem 2.4

In this section we prove a generation of interface property in Theorem 2.4.

The main idea of the proof is based on the comparison principle. Thus, we need to construct appropriate sub and super solutions for the problem (P^ε). In this first stage, we expect that the solution behaves as that of the corresponding ordinary differential equation and we construct sub and super solutions as solutions of the following initial value problem ordinary differential equation;

$$\begin{cases} \partial_\tau Y(\tau,\zeta) = f(Y(\tau,\zeta)), & \tau > 0, \\ Y(0,\zeta) = \zeta, & \zeta \in \mathbb{R}_+. \end{cases} \tag{7.1}$$

Recall C_0 defined in (2.26), $\gamma = f'(\alpha_*)$, t^ϵ, δ_0 defined in (2.30), and set

$$-\bar{\gamma} = \min_{\zeta\in[u_-\wedge\alpha_-,\, u_+\vee\alpha_+]} f'(\zeta);$$

note that $\gamma, \bar{\gamma} > 0$.

Lemma 7.1. *Let $\delta \in (0,\delta_0)$ be arbitrary. Then,*

(1) *There exists a constant $C_1 = C_1(\delta) > 0$ such that*

$$0 < e^{-\bar{\gamma}\tau} < Y_\zeta(\tau,\zeta) \le C_1 e^{\gamma\tau}$$

for all $\zeta \in [u_-, u_+]$ and $\tau \ge 0$.

(2) *There exists a constant $C_2 = C_2(\delta) > 0$ such that, for all $\tau > 0$ and all $\zeta \in (0, 2C_0)$,*

$$\left|\frac{Y_{\zeta\zeta}(\tau,\zeta)}{Y_\zeta(\tau,\zeta)}\right| \le C_2(e^{\gamma\tau}-1), \quad |Y_{\zeta\zeta}(\tau,\zeta)| \le C_2(e^{\gamma\tau}-1)e^{\gamma\tau}, \quad \text{and}$$

$$|Y_{\zeta\zeta\zeta}(\tau,\zeta)| \le 2C_2(e^{2\gamma\tau}-1)e^{\gamma\tau}. \tag{7.2}$$

(3) *There exist constants $\varepsilon_0, C_3 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$:*

(a) *For all $\zeta \in (0, 2C_0)$, in terms of a constant $C_0 > 0$,*

$$\alpha_- - \delta \le Y(\gamma^{-1}|\log\varepsilon|,\zeta) \le \alpha_+ + \delta. \tag{7.3}$$

(b) *If $\zeta \ge \alpha_* + C_3\varepsilon$, then*

$$Y(\gamma^{-1}|\log\varepsilon|,\zeta) \ge \alpha_+ - \delta. \tag{7.4}$$

(c) *If $\zeta \le \alpha_* - C_3\varepsilon$, then*

$$Y(\gamma^{-1}|\log\varepsilon|,\zeta) \le \alpha_- + \delta. \tag{7.5}$$

Proof. We deduce from [3] that the right-hand sides of (1), (2) and (3) hold except (7.2).

To show the left-hand side of (1), set $Z = Z(\tau, \zeta) := Y_\zeta(\tau, \zeta)$. Then, Z satisfies $\partial_\tau Z = f'(Y)Z$ and $Z(0, \zeta) = 1$ so that, by solving this ordinary differential equation, we have

$$Z(\tau, \zeta) = \exp\Big\{ \int_0^\tau f'(Y(s, \zeta))ds \Big\}.$$

However, we see $Y(s, \zeta) \in [u_- \wedge \alpha_-, u_+ \vee \alpha_+]$ for every $\zeta \in [u_-, u_+]$ and $s \geq 0$. Therefore, we have $f'(Y(s, \zeta)) \geq -\bar{\gamma}$ and this implies the lower bound in (1).

To show (7.2), we use

$$\begin{aligned} &Y_{\zeta\zeta}(\tau, \zeta) = A(\tau, \zeta) Y_\zeta(\tau, \zeta), \quad A(\tau, \zeta) = \int_0^\tau f''(Y(r, \zeta)) Y_\zeta(r, \zeta) dr, \\ &|A(\tau, \zeta)| \leq C_A(e^{\gamma\tau} - 1), \end{aligned}$$

given in Lemmas 3.3 and 3.4 of [3] where $C_A > 0$ is some constant. Indeed, we have

$$Y_{\zeta\zeta\zeta}(\tau, \zeta) = A_\zeta(\tau, \zeta) Y_\zeta(\tau, \zeta) + A(\tau, \zeta) Y_{\zeta\zeta}(\tau, \zeta).$$

Thus there exists $C' > 0$ such that A_ζ in the first term can estimated as

$$\begin{aligned} |A_\zeta(\tau, \zeta)| &= \left| \int_0^\tau \left\{ f'''(Y(r, \zeta)) Y_\zeta^2(r, \zeta) + f''(Y(r, \zeta)) Y_{\zeta\zeta}(r, \zeta) \right\} dr \right| \\ &\leq C' \int_0^\tau e^{2\gamma r} dr \leq C'(e^{2\gamma\tau} - 1). \end{aligned}$$

Thus, by choosing C_2 bigger if necessary, we obtain

$$|Y_{\zeta\zeta\zeta}(\tau, \zeta)| \leq C_2(e^{2\gamma\tau} - 1)e^{\gamma\tau} + C_2(e^{\gamma\tau} - 1)^2 e^{\gamma\tau} \leq 2C_2(e^{2\gamma\tau} - 1)e^{\gamma\tau}.$$

□

7.1. **Construction of sub and super solutions.** We now construct sub and super solutions on $\mathbb{T}^d$ for the proof of Theorem 2.4. Define sub and super solutions as follows

$$w_\varepsilon^\pm(t, v) = Y\left(\frac{t}{\varepsilon^2}, u_0(v) \pm P(t) \right), \tag{7.6}$$

where

$$P(t) = \varepsilon^2 C_4 \left(e^{\gamma t/\varepsilon^2} - 1 \right),$$

for some constant $C_4 > 0$. Note that $P(t) \leq \varepsilon^2 C_4(\varepsilon^{-1} - 1) \leq \varepsilon C_4$ for $t \leq t^\varepsilon$, where t^ε is defined in (2.30). In particular, since $u_0(v) \geq u_- > 0$, we have $u_0(v) - P(t) > 0$ for sufficiently small $\varepsilon > 0$. Given we work on the torus $\mathbb{T}^d$, or on $\mathbb{R}^d$ with periodic u_0, the constructed sub and super solutions $w_\varepsilon^\pm(t, v)$ are periodic for all $t \in [0, t^\varepsilon]$.

Denote also the operator $\mathcal{L}$ by

$$\mathcal{L}u = \partial_t u - \Delta\varphi(u) - \frac{1}{\varepsilon^2} f(u).$$

We set also, noting $\varphi(u), \varphi'(u) > 0$,

$$C_\varphi := \max \varphi(u) + \max \varphi'(u) + \max |\varphi''(u)|,$$

where 'max' is maximum over $u \in [0, (2C_0) \vee \alpha_+]$.

Lemma 7.2. *There exist constants $\varepsilon_0, C_4 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, $w_\varepsilon^\pm$ is a pair of sub and super solutions of (P^ε) in the domain $[0, t^\varepsilon] \times \mathbb{T}^d$.*

In particular, in terms of a constant $C_5 > 0$, we have

$$\mathcal{L}w_\varepsilon^+ \geq C_5 e^{-\bar{\gamma}\tau} \quad \text{and} \quad \mathcal{L}w_\varepsilon^- \leq -C_5 e^{-\bar{\gamma}\tau}, \ (\tau, v) \in [0, t^\varepsilon] \times \mathbb{T}^d. \tag{7.7}$$

Proof. We only prove w_ε^+ is the desired super solution; the bound for $\mathcal{L}w_\varepsilon^-$ can be shown in a similar way.

Then, direct computation gives

$$\begin{aligned}
\mathcal{L}w_\varepsilon^+ &= \frac{1}{\varepsilon^2} Y_\tau + P'(t) Y_\zeta \\
&\qquad - \Big(\varphi''(w_\varepsilon^+)|\nabla u_0|^2 (Y_\zeta)^2 + \varphi'(w_\varepsilon^+)\Delta u_0 Y_\zeta + \varphi'(w_\varepsilon^+)|\nabla u_0|^2 Y_{\zeta\zeta} + \frac{1}{\varepsilon^2} f(Y)\Big) \\
&= \frac{1}{\varepsilon^2}(Y_\tau - f(Y)) \\
&\qquad + Y_\zeta \Big(P'(t) - \big(\varphi''(w_\varepsilon^+)|\nabla u_0|^2 Y_\zeta + \varphi'(w_\varepsilon^+)\Delta u_0 + \varphi'(w_\varepsilon^+)|\nabla u_0|^2 \frac{Y_{\zeta\zeta}}{Y_\zeta}\big)\Big).
\end{aligned}$$

By the definition of Y, the first term on the right side vanishes. By choosing $\varepsilon_0 > 0$ sufficiently small, for $0 \leq t \leq t_\varepsilon$ we have

$$P(t) \leq P(t^\varepsilon) = \varepsilon^2 C_4 (e^{\gamma t^\varepsilon/\varepsilon^2} - 1) \leq \varepsilon^2 C_4 (\varepsilon^{-1} - 1) < C_0,$$

which implies $0 < u_0 + P(t) < 2C_0$. Applying Lemma 7.1, noting that $0 < w_\varepsilon^+ < (2C_0) \vee \alpha_+$, gives

$$\begin{aligned}
\mathcal{L}w_\varepsilon^+ &\geq Y_\zeta \Big(C_4 \gamma e^{\gamma t/\varepsilon^2} - (C_0^2 C_\varphi C_1 e^{\gamma t/\varepsilon^2} + C_0 C_\varphi + C_0^2 C_\varphi C_2 (e^{\gamma t/\varepsilon^2} - 1))\Big) \\
&= Y_\zeta \Big((C_4\gamma - C_0^2 C_\varphi C_1 - C_0^2 C_\varphi C_2) e^{\gamma t/\varepsilon^2} + C_0^2 C_\varphi C_2 - C_0 C_\varphi\Big) \\
&\geq Y_\zeta (C_4\gamma - C_0^2 C_\varphi C_1 - C_0^2 C_\varphi C_2 + C_0^2 C_\varphi C_2 - C_0 C_\varphi) \\
&\geq Y_\zeta (C_4\gamma - C_0^2 C_\varphi C_1 - C_0 C_\varphi).
\end{aligned}$$

Since $Y_\zeta > e^{-\bar{\gamma}\tau}$ from part (1) of Lemma 7.1, for C_4 large enough we have, for a $C_5 > 0$ that

$$\mathcal{L}w_\varepsilon^+ \geq C_5 e^{-\bar{\gamma}\tau}.$$

Thus, the estimate (7.7) for w_ε^+ is shown and w_ε^+ is a super solution for Problem (P^ε). $\square$

Remark 7.1. It follows from $\mathcal{L}w_\varepsilon^- \leq 0 \leq \mathcal{L}w_\varepsilon^+$ that $w_\varepsilon^\pm$ are sub and super solutions. However, the stronger estimate (7.7) will be useful in the proof of Theorem 9.1 in the discrete setting.

7.2. **Generation of the interface: Proof of Theorem 2.4.** From the construction of the sub and super solutions, we obtain

$$w_\varepsilon^-(t^\varepsilon, v) \leq u^\varepsilon(t^\varepsilon, v) \leq w_\varepsilon^+(t^\varepsilon, v). \tag{7.8}$$

By the definition of C_0 in (2.26), we have, for ε_0 small enough, and $\varepsilon \in (0, \varepsilon_0)$, that

$$0 < u_0(v) - P(t) \leq u_0(v) + P(t) \leq u_0(v) + \varepsilon C_4 < 2C_0, \quad \text{for } v \in \mathbb{T}^d, t \leq t^\varepsilon.$$

Thus, the assertion (2.31) is a direct consequence of (7.3) and (7.8).

For (2.32), first we choose M_0 large enough so that $M_0\varepsilon - C_4\varepsilon \geq C_3\varepsilon$. Then, for any $t \leq t^\varepsilon$ and $v \in \mathbb{T}^d$ such that $u_0(v) \geq \alpha_* + M_0\varepsilon$ we have

$$u_0(v) - P(t) \geq \alpha_* + M_0\varepsilon - C_4\varepsilon \geq \alpha_* + C_3\varepsilon.$$

Thus, with (7.4) and (7.8), we see that

$$u^\varepsilon(t^\varepsilon, v) \geq \alpha_+ - \delta$$

for any $v \in \mathbb{T}^d$ such that $u_0(v) \geq \alpha_* + M_0\varepsilon$, which implies (2.32). Note that (2.33) can be shown in the same way. This completes the proof of Theorem 2.4. $\square$

8. Propagation of the interface: Proof of Theorem 2.5

We now argue the propagation of the interface given in Theorem 2.5. Again, we will need to construct appropriate sub and super solutions, but now in terms of the first two terms U_0 and U_1 in the expansion (6.2).

8.1. **A modified signed distance function.** For future usage, we introduce a cut-off signed distance function $d = d(t, v)$ as follows. Choose $d_0 > 0$ small enough so that the signed distance function $\overline{d} = \overline{d}(t, v)$ from the interface Γ_t evolving under (P^0) is smooth in the set

$$\{(t, v) \in [0, T] \times \mathbb{T}^d, |\overline{d}(t, v)| < 3d_0\}.$$

Let $h(s)$ be a smooth non-decreasing function on $\mathbb{R}$ such that

$$h(s) = \begin{cases} s & \text{if } |s| \leq d_0 \\ -2d_0 & \text{if } s \leq -2d_0 \\ 2d_0 & \text{if } s \geq 2d_0. \end{cases}$$

We then define the cut-off signed distance function d by

$$d(t, v) = h(\overline{d}(t, v)),\ (t, v) \in [0, T] \times \mathbb{T}^d.$$

Note that, as d coincides with $\overline{d}$ in the region

$$\{(t, v) \in [0, T] \times \mathbb{T}^d : |d(t, v)| < d_0\},$$

we have

$$\partial_t d = \lambda_0 \Delta d \text{ on } \Gamma_t.$$

Moreover, d is constant far away from Γ_t and the following properties hold. Recall $\Gamma = \cup_{0\leq t\leq T}\big(\{t\} \times \Gamma_t\big)$ is in $C^{\frac{4+\theta}{2}, 4+\theta}$, $\theta > 0$.

Lemma 8.1. *There exists a constant $C_d > 0$ such that*

(1) $|\partial_t d| + |\nabla d| + |\Delta d| + |\nabla \Delta d| + |\Delta\Delta d| \leq C_d$,
(2) $|\partial_t d - \lambda_0 \Delta d| \leq C_d |d|$

in $[0, T] \times \mathbb{T}^d$.

8.2. **Estimates for the functions U_0 and U_1.** Here we give estimates for the functions which will be used in constructing the sub and super solution. Recall that $U_0 = U_0(z)$, $z \in \mathbb{R}$ (cf. (6.6)) is a solution of the equation

$$(\varphi(U_0))_{zz} + f(U_0) = 0.$$

We have the following lemma.

Lemma 8.2. *There exist constants $\hat{C}_0, \lambda_1 > 0$ such that for all $z \in \mathbb{R}$*

(1) $|U_0| \le \hat{C}_0$,
(2) $|U_{0z}|,\ |U_{0zz}| \le \hat{C}_0 \exp(-\lambda_1 |z|)$.

Proof. Recall that $V_0 = \varphi(U_0)$ satisfies the equation (6.7). Lemma 2.1 of [3] implies that there exist some positive constants $\overline{C}_0$ and λ_1 such that for all $z \in \mathbb{R}$

$$\begin{aligned}&|V_0| \le \overline{C}_0,\\ &|V_{0z}|,\ |V_{0zz}| \le \overline{C}_0 \exp(-\lambda_1|z|).\end{aligned}$$

Since $\varphi \in C^4(\mathbb{R}_+)$, we have the desired results. □

With respect to the cut-off signed distance function d, for each (t, v), we define $U_1 : \mathbb{R} \to \mathbb{R}$ as the solution of the following equation:

$$\begin{cases}(\varphi'(U_0)U_1)_{zz} + f'(U_0)U_1 = (\lambda_0 U_{0z} - (\varphi(U_0))_z)\Delta d\\ U_1(t, v, 0) = 0,\ \varphi'(U_0)U_1 \in L^\infty(\mathbb{R}).\end{cases} \tag{8.1}$$

Existence of the solution U_1 can be shown in the same way as proving the existence of $\bar{U}_1$ in (6.8).

Finally let us give the following estimates for $U_1 = U_1(t, v, z)$.

Lemma 8.3. *There exist constants $\hat{C}_1, \lambda_1 > 0$ such that for all $z \in \mathbb{R}$*

(1) $|U_1|,\ |\nabla U_1|,\ |\Delta U_1|,\ |U_{1t}| \le \hat{C}_1$,
(2) $|U_{1z}|,\ |U_{1zz}|,\ |\nabla U_{1z}| \le \hat{C}_1 \exp(-\lambda_1|z|)$.

Here, the operators ∇ and Δ act on the variable $v \in \mathbb{T}^d$.

Proof. Define $V_1(t, v, z) := \varphi'(U_0(z))U_1(t, v, z)$. From (8.1) we see that V_1 satisfies (6.9) with $\partial_t d$ replaced by $\lambda_0 \Delta d$:

$$\begin{cases}V_{1zz} + b'(V_0)V_1 = \Big[\lambda_0 \dfrac{V_{0z}}{\varphi'(\varphi^{-1}(V_0))} - V_{0z}\Big]\Delta d,\\ V_1(t, v, 0) = 0,\ V_1 \in L^\infty(\mathbb{R}).\end{cases} \tag{8.2}$$

Applying Lemmas 2.2, 2.3 of [3] and Lemma 8.1 to (8.2) gives the boundedness of V_1 and the exponential decay of $|V_{1z}|, |V_{1zz}|$. Moreover, as we assume d to be smooth enough on $[0, T] \times \mathbb{T}^d$, we can apply Lemma 2.2 of [3] to obtain the boundedness of $\nabla V_1, \Delta V_1, V_{1t}$. Using the same method as in the proof of Lemma 8.2 gives the desired estimates for the function U_1. □

8.3. **Construction of sub and super solutions.** We construct the sub and super solutions as follows: Given $0 < \varepsilon < 1$, we define

$$u^{\pm}(t,v) \equiv u_{\varepsilon}^{\pm}(t,v) = U_0\left(\frac{d(t,v) \pm \varepsilon p(t)}{\varepsilon}\right) + \varepsilon U_1\left(t, v, \frac{d(t,v) \pm \varepsilon p(t)}{\varepsilon}\right) \pm q(t), \tag{8.3}$$

where

$$p(t) = e^{-\beta t/\varepsilon^2} - e^{Lt} - \hat{L},$$
$$q(t) = \sigma\left(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}\right).$$

Here $\beta, \sigma, L, \hat{L} > 0$ are constants which will be determined later (cf. (8.4), (8.6) and Lemma 8.7). Although we work on $\mathbb{T}^d$, if we take the viewpoint of working on $\mathbb{R}^d$, we may regard the signed distance function d as periodic with period 1 so that $u^{\pm}(t,v)$ are periodic as well for all $t \in [0,T]$.

We first give a lemma on the uniform negativity of $f'(U_0(z)) + (\varphi'(U_0(z))_{zz}$.

Lemma 8.4. *There exist $b > 0$ and $C_b > 0$ such that $f'(U_0(z)) + (\varphi'(U_0))_{zz} < -C_b$ on $\{z : U_0(z) \in [\alpha_-,\ \alpha_- + b] \cup [\alpha_+ - b,\ \alpha_+]\}$.*

Proof. We can choose $b_1, C_b > 0$ such that

$$f'(U_0(z)) < -2C_b$$

on $\{z : U_0(z) \in [\alpha_-,\ \alpha_- + b_1] \cup [\alpha_+ - b_1,\ \alpha_+]\}$.

Note that $(\varphi'(U_0))_{zz} = \varphi'''(U_0)U_{0z}^2 + \varphi''(U_0)U_{0zz}$. From Lemma 8.2, we can choose $b_2 > 0$ small enough so that

$$|(\varphi'(U_0))_{zz}| < C_b$$

on $\{z : U_0(z) \in [\alpha_-,\ \alpha_- + b_2] \cup [\alpha_+ - b_2,\ \alpha_+]\}$. Define $b := \min\{b_1, b_2\} > 0$. Then we have

$$f'(U_0(z)) + (\varphi'(U_0))_{zz} < C_b - 2C_b = -C_b < 0,$$

on the desired domain. $\square$

Fix $b > 0$ which satisfies the result of Lemma 8.4. Denote $J_1 := \{z : U_0(z) \in [\alpha_-,\ \alpha_- + b] \cup [\alpha_+ - b,\ \alpha_+]\}$, $J_2 = \{z : U_0(z) \in [\alpha_- + b,\ \alpha_+ - b]\}$. Let

$$\beta := -\sup\left\{\frac{f'(U_0(z)) + (\varphi'(U_0(z)))_{zz}}{3} : z \in J_1\right\} > 0. \tag{8.4}$$

Then, we have the following result which plays an important role in computing the sub and super solutions.

Lemma 8.5. *There exists a constant $\sigma_0 > 0$ small enough such that for every $0 < \sigma < \sigma_0$, we have*

$$U_{0z} + \sigma(f'(U_0) + (\varphi'(U_0))_{zz}) \le -3\sigma\beta.$$

Proof. To show the assertion, it is sufficient to show that there exists σ_0 such that for all $0 < \sigma < \sigma_0$

$$\frac{U_{0z}}{\sigma} + f'(U_0) + (\varphi'(U_0))_{zz} \le -3\beta.$$

We prove the result on each the sets J_1, J_2.

On the set J_1: Note that $U_{0z} < 0$ on $\mathbb{R}$. If $z \in J_1$, for any $\sigma > 0$ we have

$$\frac{U_{0z}}{\sigma} + f'(U_0) + (\varphi'(U_0))_{zz} \le \sup_{z \in J_1}(f'(U_0) + (\varphi'(U_0))_{zz}) = -3\beta.$$

On the set J_2: Note that the set J_2 is compact set in $\mathbb{R}$. Thus there exist $c_1, c_2 > 0$ such that $\sup_{z\in J_2} U_{0z}(z) \le -c_1$ and $\sup_{z\in J_2}(f'(U_0)+(\varphi'(U_0))_{zz}) \le c_2$. Therefore, we have

$$\begin{aligned}
&\limsup_{\sigma\downarrow 0}\sup_{z\in J_2}\left(\frac{U_{0z}}{\sigma} + f'(U_0) + (\varphi'(U_0))_{zz}\right)\\
&\qquad \le \limsup_{\sigma\downarrow 0}\sup_{z\in J_2}\left(\frac{U_{0z}}{\sigma}\right) + \sup_{z\in J_2}(f'(U_0) + (\varphi'(U_0))_{zz}) = -\infty
\end{aligned}$$

which implies the assertion. □

Before rigorously proving that $u^\pm$ are sub and super solutions, we first give some preliminary computations needed in the sequel. First, note for u^+ in (8.3) that

$$\begin{aligned}
\varphi(u^+) &= \varphi(U_0) + (\varepsilon U_1 + q)\varphi'(U_0) + (\varepsilon U_1 + q)^2 \int_0^1 (1-s)\varphi''(U_0 + (\varepsilon U_1 + q)s)ds\\
f(u^+) &= f(U_0) + (\varepsilon U_1 + q) f'(U_0) + \frac{(\varepsilon U_1 + q)^2}{2} f''(\theta(t,v))
\end{aligned}$$

where $\theta(t,v)$ is a function taking values between $U_0(t,v)$ and $U_0(t,v) + \varepsilon U_1(t,v) + q(t)$. Straightforward computations yield

$$\partial_t u^+ = U_{0z}\cdot\left(\frac{\partial_t d + \varepsilon\partial_t p}{\varepsilon}\right) + \varepsilon U_{1t} + U_{1z}\cdot(\partial_t d + \varepsilon \partial_t p) + \partial_t q$$

and with the help of above identity for $\varphi(u^+)$,

$$\begin{aligned}
&\Delta\varphi(u^+)\\
&= \nabla\cdot\Big((\varphi(U_0))_z \frac{\nabla d}{\varepsilon} + U_{1z}\varphi'(U_0)\nabla d\\
&\quad + \varepsilon\nabla U_1 \varphi'(U_0) + (\varepsilon U_1 + q)(\varphi'(U_0))_z\frac{\nabla d}{\varepsilon} + \nabla R\Big)\\
&= (\varphi(U_0))_{zz}\frac{|\nabla d|^2}{\varepsilon^2} + (\varphi(U_0))_z\frac{\Delta d}{\varepsilon}\\
&\quad + (U_{1z}\varphi'(U_0))_z\frac{|\nabla d|^2}{\varepsilon} + U_{1z}\varphi'(U_0)\Delta d\\
&\quad + 2\nabla U_{1z}\varphi'(U_0)\cdot\nabla d + \nabla U_1(\varphi'(U_0))_z\cdot\nabla d + \varepsilon\Delta U_1\varphi'(U_0)\\
&\quad + (U_1\varphi'(U_0)_z)_z\frac{|\nabla d|^2}{\varepsilon} + q(\varphi'(U_0))_{zz}\frac{|\nabla d|^2}{\varepsilon^2} + \nabla U_1(\varphi'(U_0))_z\cdot\nabla d\\
&\quad + (\varepsilon U_1 + q)(\varphi'(U_0))_z\frac{\Delta d}{\varepsilon} + \Delta R,
\end{aligned}$$

where $R(t,v) = (\varepsilon U_1 + q)^2 \int_0^1 (1-s)\varphi''(U_0 + (\varepsilon U_1 + q)s)ds$.

The next task is to compute ΔR. To do this, define

$$r(t,v) = \int_0^1 (1-s)\varphi''(U_0 + (\varepsilon U_1 + q)s)ds.$$

Then, we have

$$\begin{aligned}
\Delta R(t,v) &= \nabla\cdot\nabla\Big[\Big((\varepsilon U_1)^2+2\varepsilon q U_1+q^2\Big)r\Big]\\
&= \nabla\cdot\Big[\Big(2\varepsilon U_1\left(U_{1z}\nabla d+\varepsilon\nabla U_1\right)+2q\left(U_{1z}\nabla d+\varepsilon\nabla U_1\right)\Big]r(t,v)\\
&\quad+\Big((\varepsilon U_1)^2+2\varepsilon q U_1+q^2\Big)\nabla r(t,v)\Big]\\
&= \Big[2\left(U_{1z}\nabla d+\varepsilon\nabla U_1\right)^2\\
&\quad+2\varepsilon U_1\left(U_{1zz}\frac{|\nabla d|^2}{\varepsilon}+U_{1z}\Delta d+2\nabla U_{1z}\cdot\nabla d+\varepsilon\Delta U_1\right)\Big]r(t,v)\\
&\quad+2q\left(U_{1zz}\frac{|\nabla d|^2}{\varepsilon}+U_{1z}\Delta d+2\nabla U_{1z}\cdot\nabla d+\varepsilon\Delta U_1\right)r(t,v)\\
&\quad+2\Big[2\varepsilon U_1\left(U_{1z}\nabla d+\varepsilon\nabla U_1\right)+2q\left(U_{1z}\nabla d+\varepsilon\nabla U_1\right)\Big]\nabla r(t,v)\\
&\quad+\Big((\varepsilon U_1)^2+2\varepsilon q U_1+q^2\Big)\Delta r(t,v),
\end{aligned}$$

where

$$\begin{aligned}
\nabla r(t,v) &= \int_0^1(1-s)\varphi'''(U_0+(\varepsilon U_1+q)s)\left((U_0+\varepsilon U_1 s)_z\frac{\nabla d}{\varepsilon}+\varepsilon\nabla U_1 s\right)ds,\\
\Delta r(t,v) &= \int_0^1(1-s)\varphi'''(U_0+(\varepsilon U_1+q)s)\times\\
&\qquad\left((U_0+\varepsilon U_1 s)_z\frac{\Delta d}{\varepsilon}+(U_0+\varepsilon U_1 s)_{zz}\frac{|\nabla d|^2}{\varepsilon^2}+(2\nabla U_{1z}\cdot\nabla d+\varepsilon\Delta U_1)s\right)ds\\
&\quad+\int_0^1(1-s)\varphi^{(4)}(U_0+(\varepsilon U_1+q)s)\left((U_0+\varepsilon U_1 s)_z\frac{\nabla d}{\varepsilon}+\varepsilon\nabla U_1 s\right)^2ds.
\end{aligned}$$

Define $l(t,v), r_i(t,v), i=1,2,3$ as follows:

$$\begin{aligned}
l(t,v) &= U_{1zz}\frac{|\nabla d|^2}{\varepsilon}+U_{1z}\Delta d+2\nabla U_{1z}\cdot\nabla d+\varepsilon\Delta U_1,\\
r_1(t,v) &= \Big[2\left(U_{1z}\nabla d+\varepsilon\nabla U_1\right)^2+2\varepsilon U_1 l(t,v)\Big]r(t,v)\\
&\quad+4\varepsilon U_1\left(U_{1z}\nabla d+\varepsilon\nabla U_1\right)\nabla r(t,v)+(\varepsilon U_1)^2\Delta r(t,v),\\
r_2(t,v) &= 2ql(t,v)r(t,v)+4q\left(U_{1z}\nabla d+\varepsilon\nabla U_1\right)\nabla r(t,v)+2\varepsilon q U_1\Delta r(t,v),\\
r_3(t,v) &= q^2\Delta r(t,v).
\end{aligned}$$

Thus, $\Delta R=r_1+r_2+r_3$. Then we have the following estimates for r_i, $i=1,2,3$.

Lemma 8.6. *There exists $C_r>0$ such that*

$$|r_1|\le C_r,\ |r_2|\le\frac{q}{\varepsilon}C_r,\ |r_3|\le\frac{q^2}{\varepsilon^2}C_r, \tag{8.5}$$

for all $0<\varepsilon<1$, $t\in[0,T]$ and $v\in\mathbb{T}^d$.

Proof. Note that, by Lemmas 8.2, 8.3, and recalling that σ, β, L are constants determined later, the term $U_a:=U_0+(\varepsilon U_1+q)s,\ s\in[0,1]$, is uniformly bounded, thus the terms

$\varphi''(U_a), \varphi'''(U_a), \varphi^{(4)}(U_a)$ are uniformly bounded. Moreover, by Lemmas 8.2, 8.3 again,

$$\nabla d, \Delta d, \nabla U_1, \Delta U_1, U_{0z}, U_{1z}, U_{0zz}, U_{1zz}, \nabla U_{1z}$$

are all bounded. It follows then that there exist some constants $c, c_\nabla, c_\Delta > 0$ such that

$$|r| \le c, \ |\nabla r| \le \frac{c_\nabla}{\varepsilon}, \ |\Delta r| \le \frac{c_\Delta}{\varepsilon^2}.$$

In particular, by Lemmas 8.2, 8.3, there exists a positive constant c_l such that

$$|l(t,v)| \le \frac{c_l}{\varepsilon}.$$

Combining these estimates yields (8.5). $\square$

We now start to prove that $u^\pm = u^\pm_\varepsilon(t,v)$ in (8.3) are sub and super solutions for the problem (P^ε) for $t \in [0,T]$. Recall $0 < \varepsilon < 1$. The constant $\beta > 0$ in $p(t), q(t)$ defined below (8.3) has been determined via (8.4). Then, the constant $\sigma > 0$ is taken such that

$$0 < \sigma \le \min\{\sigma_0, \sigma_1, \sigma_2\}, \tag{8.6}$$

where $\sigma_0 > 0$ is the constant defined in Lemma 8.5, and $\sigma_1, \sigma_2 > 0$ are given as follows:

$$\sigma_1 = \frac{1}{2(\beta+1)}, \ \sigma_2 = \frac{\beta}{(F + C_r)(\beta+1)}, \ F = \|f''\|_{L^\infty([0,C_u])}$$

where C_r is the constant defined in Lemma 8.6 and

$$C_u = \sup_{t\in[0,T], v\in\mathbb{T}^d, 0<\varepsilon<1, \pm} u^\pm(t,v)$$

which is finite (depending on L) as we saw in the proof of Lemma 8.6.

We note that the constants $L, \hat{L} > 0$ in definition of $p(t), q(t)$ will be determined in the next lemma.

Lemma 8.7. *Let β, σ be given by (8.4), (8.6). Then for each $\hat{L} > 1$ there exist $L > 0$ large enough and $\varepsilon_0 > 0$ small enough such that for a constant $C > 0$ we have*

$$\mathcal{L}u^- \le -C < C \le \mathcal{L}u^+ \ \text{in } [0,T] \times \mathbb{T}^d \tag{8.7}$$

for every $\varepsilon \in (0, \varepsilon_0)$. Hence, $u^\pm$ are sub and super solutions for the problem (P^ε).

Proof. We only show that u^+ is a super solution; one can show that u^- is a sub solution in a similar way.

Combining the computations above, we obtain

$$\begin{aligned}\mathcal{L}u^+ &= \partial_t u^+ - \Delta(\varphi(u^+)) - \frac{1}{\varepsilon^2} f(u^+) \\ &= E_1 + E_2 + E_3 + E_4 + E_5 + E_6,\end{aligned}$$

where

$$\begin{aligned}E_1 &= -\frac{1}{\varepsilon^2}\left((\varphi(U_0))_{zz}|\nabla d|^2 + f(U_0)\right) - \frac{|\nabla d|^2 - 1}{\varepsilon^2} q(\varphi'(U_0))_{zz} \\ &\quad - \frac{|\nabla d|^2 - 1}{\varepsilon}(U_1\varphi'(U_0))_{zz}, \\ E_2 &= \frac{1}{\varepsilon} U_{0z}\partial_t d - \frac{1}{\varepsilon}\left((\varphi(U_0))_z \Delta d + (U_1\varphi'(U_0))_{zz} + U_1 f'(U_0)\right), \\ E_3 &= [U_{0z}\partial_t p + \partial_t q] - \frac{1}{\varepsilon^2}\left[q f'(U_0) + q(\varphi'(U_0))_{zz} + \frac{q^2}{2} f''(\theta)\right] - r_3(t,v),\end{aligned}$$

$$
\begin{aligned}
E_4 &= \varepsilon U_{1z}\partial_t p - \frac{q}{\varepsilon}\Big[(\varphi'(U_0))_z \Delta d + U_1 f''(\theta)\Big] - r_2(t,v),\\
E_5 &= \varepsilon U_{1t} - \varepsilon \Delta U_1 \varphi'(U_0),\\
E_6 &= U_{1z}\partial_t d - 2\nabla U_{1z}\varphi'(U_0)\cdot\nabla d - 2\nabla U_1(\varphi'(U_0))_z\cdot\nabla d\\
&\quad - (U_1\varphi'(U_0))_z\Delta d - r_1(t,v) - \frac{(U_1)^2}{2} f''(\theta),
\end{aligned}
$$

and $\theta = \theta(t,v)$ is the function which appeared when we expanded $f(u^+)$.

Estimates for the term E_1: Using (6.6) for $f(U_0)$, we write E_1 in the form

$$
E_1 = -\frac{|\nabla d|^2 - 1}{\varepsilon^2}((\varphi(U_0))_{zz} + q(\varphi'(U_0))_{zz}) - \frac{|\nabla d|^2-1}{\varepsilon}(U_1\varphi'(U_0))_{zz}.
$$

We only consider the second term $E_{1,2} := \dfrac{|\nabla d|^2-1}{\varepsilon}(U_1\varphi'(U_0))_{zz}$; the first term can be bounded similarly. In the region where $|d| \leq d_0$, we have $|\nabla d| = 1$ so that $E_{1,2} = 0$. If $|d| > d_0$, by Lemma 8.3, there exists some constant $\tilde{C} > 0$ such that

$$
\frac{|(U_1\varphi'(U_0))_{zz}|}{\varepsilon} \leq \frac{\tilde{C}}{\varepsilon} e^{-\lambda_1\left|\frac{d}{\varepsilon}+p(t)\right|} \leq \frac{\tilde{C}}{\varepsilon} e^{-\lambda_1\left[\frac{d_0}{\varepsilon}-|p(t)|\right]} \leq \frac{\tilde{C}}{\varepsilon} e^{-\lambda_1\left[\frac{d_0}{\varepsilon}-(1+e^{LT}+\hat{L})\right]}.
$$

Choosing $\varepsilon_0 > 0$ small enough, compared with $L, \hat{L}$, such that

$$
\frac{d_0}{2\varepsilon_0} - \left(1 + e^{LT} + \hat{L}\right) \geq 0,
$$

we deduce that

$$
\frac{|(U_1\varphi'(U_0))_{zz}|}{\varepsilon} \leq \frac{\tilde{C}}{\varepsilon} e^{-\lambda_1 \frac{d_0}{2\varepsilon}} \to 0 \text{ as } \varepsilon \downarrow 0.
$$

Thus, $\dfrac{|(U_1\varphi'(U_0))_{zz}|}{\varepsilon}$ is uniformly bounded, so that there exists $\hat{C}_2$ independent of ε, L such that

$$
|E_{1,2}| \leq \hat{C}_2.
$$

Applying the same method, there exists $\tilde{C}_1$ independent of ε, L such that

$$
|E_1| \leq \tilde{C}_1. \tag{8.8}
$$

Estimate for the term E_2: Using (8.1) we write E_2 in the form

$$
E_2 = \frac{1}{\varepsilon}U_{0z}\partial_t d - \frac{1}{\varepsilon}\lambda_0 U_{0z}\Delta d = \frac{U_{0z}}{\varepsilon}(\partial_t d - \lambda_0\Delta d).
$$

Applying Lemmas 8.1 and 8.2 gives

$$
|E_2| \leq C_d\hat{C}_0\frac{|d|}{\varepsilon}e^{-\lambda_1\left|\frac{d}{\varepsilon}+p\right|} \leq C_d\hat{C}_0 \max_{\xi\in\mathbb{R}} |\xi| e^{-\lambda_1|\xi+p|}.
$$

Note that $\max_{\xi\in\mathbb{R}} |\xi|e^{-\lambda_1|\xi+p|} \leq |p| + \frac{1}{\lambda_1}$ [31]. Thus, there exists $\tilde{C}_2 > 0$ such that

$$
|E_2| \leq \tilde{C}_2(1 + e^{Lt} + \hat{L}). \tag{8.9}
$$

Estimate for the term E_3: Substituting $\partial_t p = -\dfrac{q}{\varepsilon^2\sigma}$ and then replacing q by its explicit form gives

$$
E_3 = \frac{q}{\varepsilon^2\sigma}\left[-U_{0z} - \sigma(f'(U_0) + (\varphi'(U_0))_z) - \sigma q\left(\frac{1}{2}f''(\theta) + \frac{\varepsilon^2}{q^2}r_3\right)\right] + \partial_t q
$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon^2}\left(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}\right)\Big[- U_{0z} - \sigma(f'(U_0) + (\varphi'(U_0))_z) \\
&\qquad - \sigma^2(\beta e^{-\frac{\beta t}{\varepsilon^2}} + L\varepsilon^2 e^{Lt})\left(\frac{1}{2}f''(\theta) + \frac{\varepsilon^2}{q^2} r_3\right)\Big] - \frac{1}{\varepsilon^2}\sigma\beta^2 e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2\sigma L^2 e^{Lt} \\
&= \frac{1}{\varepsilon^2}\beta e^{-\frac{\beta t}{\varepsilon^2}}(I - \sigma\beta) + Le^{Lt}[I + \varepsilon^2\sigma L],
\end{aligned}
$$

where

$$
I := -U_{0z} - \sigma(f'(U_0) + (\varphi'(U_0))_z) - \sigma^2(\beta e^{-\frac{\beta t}{\varepsilon^2}} + L\varepsilon^2 e^{Lt})\left(\frac{1}{2}f''(\theta) + \frac{\varepsilon^2}{q^2} r_3\right).
$$

We now choose $\varepsilon_0 > 0$ small enough, by making smaller if necessary, such that

$$
\varepsilon_0^2 L e^{LT} \leq 1.
$$

Applying Lemma 8.5, and using the fact $\sigma \leq \sigma_2$ yields

$$
\begin{aligned}
I &\geq 3\sigma\beta - \sigma\sigma_2\left(\beta + L\varepsilon^2 e^{Lt}\right)\left(|f''(\theta)| + \frac{\varepsilon^2}{q^2} r_3\right) \\
&\geq 3\sigma\beta - \sigma\sigma_2\left(\beta + 1\right)\left(|f''(\theta)| + \frac{\varepsilon^2}{q^2} r_3\right) \\
&\geq 2\sigma\beta,
\end{aligned}
$$

for every $\varepsilon \in (0, \varepsilon_0)$, so that

$$
E_3 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\frac{\beta t}{\varepsilon^2}} + 2\sigma\beta L e^{Lt}, \ \varepsilon \in (0, \varepsilon_0). \tag{8.10}
$$

Estimate for the term E_4: Substituting $\partial_t p = -\dfrac{q}{\varepsilon^2\sigma}$ and then replacing q by its explicit form gives

$$
\begin{aligned}
E_4 &= \frac{q}{\varepsilon\sigma}\left(-U_{1z} - \sigma((\varphi'(U_0))_z \Delta d + U_1 f''(\theta)) - \sigma\frac{\varepsilon}{q} r_2\right) \\
&= \frac{1}{\varepsilon}\left(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}\right)\left(-U_{1z} - \sigma((\varphi'(U_0))_z \Delta d + U_1 f''(\theta)) - \sigma\frac{\varepsilon}{q} r_2\right).
\end{aligned}
$$

Applying Lemmas 8.1, 8.2, 8.3 and 8.6 gives the uniform boundedness of the last term of the above equation. Thus there exists a constant $\tilde{C}_4 > 0$ such that

$$
|E_4| \leq \tilde{C}_4 \frac{1}{\varepsilon}\left(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}\right). \tag{8.11}
$$

Estimates for the terms E_5 and E_6: Applying Lemmas 8.1, 8.2 and 8.3, it follows that there exists $\tilde{C}_5 > 0$ such that

$$
|E_5| + |E_6| \leq \tilde{C}_5. \tag{8.12}
$$

Combining estimates: Collecting the estimates (8.8),(8.9),(8.10),(8.11),(8.12), we obtain

$$
\begin{aligned}
\mathcal{L}u^+ &\geq \left[\frac{\sigma\beta^2}{\varepsilon^2} - \tilde{C}_4\frac{\beta}{\varepsilon}\right] e^{-\frac{\beta t}{\varepsilon^2}} + \left[2\sigma\beta L - \varepsilon\tilde{C}_4 L - \tilde{C}_2\right] e^{Lt} - \tilde{C}_1 - \tilde{C}_2 - \tilde{C}_5 - \tilde{C}_2\hat{L} \\
&\geq \left[\frac{\sigma\beta^2}{\varepsilon^2} - \tilde{C}_4\frac{\beta}{\varepsilon}\right] e^{-\frac{\beta t}{\varepsilon^2}} + \left[\frac{2\sigma\beta L}{3} - \varepsilon\tilde{C}_4 L\right] e^{Lt} \\
&\quad + \left[\frac{2\sigma\beta L}{3} - \tilde{C}_2\right] e^{Lt} + \left[\frac{2\sigma\beta L}{3} - \tilde{C}_6\right],
\end{aligned}
$$

for every $\varepsilon \in (0, \varepsilon_0)$, where $\tilde{C}_6 = \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_5 + \tilde{C}_2 \hat{L} > 0$. Choosing L large and $\varepsilon_0 > 0$ small enough again, we obtain, for a constant $C > 0$, that $\mathcal{L}u^+ \geq C$. □

Remark 8.1. To show that $u^\pm$ are sub and super solutions, it would have been enough in the above proof to show that $\mathcal{L}u^- \leq 0 \leq \mathcal{L}u^+$. However, the stronger estimate (8.7) found will be useful in the proof of Theorem 9.2 in the discrete setting.

8.4. **Proof of Theorem 2.5.** The proof of Theorem 2.5 is divided in two steps: (i) For large enough $J > 0$, we prove that $u^-(t,v) \leq u^\varepsilon(t+t^\varepsilon, v) \leq u^+(t,v)$ for $t \in [0, T-t^\varepsilon], v \in \mathbb{T}^d$ and (ii) prove the desired result.

Step 1. Fix $\sigma, \beta > 0$ as in (8.4), (8.6). Without loss of generality, we may assume that

$$0 < \delta < \min\{\delta_0, \sigma\beta\}$$

be arbitrary. Theorem 2.4 implies the existence of constants ε_0 and M_0 which satisfy (2.31)-(2.33). Conditions (2.28) and (2.29) imply that there exists a constant $M_1 > 0$ such that

$$\begin{aligned} &\text{if } \mathrm{dist}(v) \geq M_1\varepsilon \ \text{ then } u_0(v) \leq \alpha_* - M_0\varepsilon, \\ &\text{if } \mathrm{dist}(v) \leq -M_1\varepsilon \ \text{ then } u_0(v) \geq \alpha_* + M_0\varepsilon, \end{aligned}$$

where $\mathrm{dist}(v) = \bar{d}(0,v)$ denotes the signed distance function associated with Γ_0. From this we deduce, by applying (2.31), (2.33), that

$$u^\varepsilon(t^\varepsilon, v) \leq H^+(v) := \begin{cases} \alpha_+ + \frac{\delta}{4} & \text{if } \mathrm{dist}(v) \leq M_1\varepsilon, \\ \alpha_- + \frac{\delta}{4} & \text{if } \mathrm{dist}(v) > M_1\varepsilon, \end{cases}$$

and, by applying (2.31), (2.32), that

$$u^\varepsilon(t^\varepsilon, v) \geq H^-(v) := \begin{cases} \alpha_+ - \frac{\delta}{4} & \text{if } \mathrm{dist}(v) < -M_1\varepsilon, \\ \alpha_- - \frac{\delta}{4} & \text{if } \mathrm{dist}(v) \geq -M_1\varepsilon. \end{cases}$$

Next, we fix a sufficient large constant $\hat{L} > 0$ such that

$$U_0(M_1 - \hat{L}) \geq \alpha_+ - \frac{\delta}{4} \text{ and } U_0(-M_1 + \hat{L}) \leq \alpha_- + \frac{\delta}{4}$$

For such a constant $\hat{L}$, Lemma 8.7 implies the existence of constants ε_0 and $L > 0$ such that the inequalities in (8.7) holds. We claim that

$$u^+(0,v) \geq H^+(v), \ u^-(0,v) \leq H^-(v). \tag{8.13}$$

We only prove the former inequality; the latter inequality can be proved similarly. By Lemma 8.3, we have $|U_1| \leq \hat{C}_1$. Thus, we can choose $\varepsilon_0 > 0$ small enough, by making it smaller if necessary, so that $\varepsilon_0 \hat{C}_1 \leq \frac{\sigma\beta}{4}$ and the following inequality holds:

$$\begin{aligned} u^+(0,v) &\geq U_0\left(\frac{\mathrm{dist}(v) + \varepsilon p(0)}{\varepsilon}\right) - \varepsilon\hat{C}_1 + \sigma\beta + \varepsilon^2\sigma L \\ &> U_0\left(\frac{\mathrm{dist}(v)}{\varepsilon} - \hat{L}\right) + \frac{3}{4}\delta, \end{aligned}$$

for $\varepsilon \in (0, \varepsilon_0)$ if $|\mathrm{dist}(v)| \leq d_0$, recalling $\delta < \sigma\beta$. Therefore, on the set $\{v \in \mathbb{T}^d : \mathrm{dist}(v) \leq M_1\varepsilon\}$, from the inequalities above, the fact that U_0 is a decreasing function implies

$$u^+(0,v) > U_0(M_1 - \hat{L}) + \frac{3}{4}\delta \geq \alpha_+ + \frac{\delta}{2} > H^+(v),$$

by choosing $\hat{L} > 0$ large enough. On the other hand, on the set $\{v \in \mathbb{T}^d : \mathrm{dist}(v) > M_1\varepsilon\}$, since $U_0 \geq \alpha_-$ we have

$$u^+(0,v) > \alpha_- + \frac{3}{4}\delta > H^+(v).$$

Thus we proved the claim (8.13) above. In particular, we have shown that $u^-(0,v) \leq u^\varepsilon(t^\varepsilon, v) \leq u^+(0,v)$.

The claim (8.13) and Lemma 8.7 give a sufficient condition to apply the comparison principle. Thus, we have

$$u^-(t,v) \leq u^\varepsilon(t+t^\varepsilon, v) \leq u^+(t,v) \quad \text{for} \quad t \in [0, T-t^\varepsilon], v \in \mathbb{T}^d. \tag{8.14}$$

Step 2. Let $0 < \delta < \delta_0$ be given as in Theorem 2.5 and let $\varepsilon_0, L > 0$ be as in Lemma 8.7. We choose $\sigma > 0$ as in (8.7) and also satisfying $\sigma(\beta + \varepsilon_0^2 L e^{LT}) \leq \delta/3$. With this σ, we construct sub and super solutions $u^\pm(t,v)$. Then, $u^\pm(t,v)$ satisfy (8.14).

Choose now $C > 0$ (different from the C in Lemma 8.7 so that

$$U_0(-C - e^{LT} - \hat{L}) \geq \alpha_+ - \delta/3 \text{ and } U_0(C + e^{LT} + \hat{L}) \leq \alpha_- + \delta/3$$

Then, from (8.14) and noting that $0 < q(t) < \delta/3$, $t \in [0,T]$ and $|U_1|$ is bounded by Lemma 8.3, by choosing $\varepsilon_0 > 0$ smaller if necessary, we have

$$\begin{aligned} u^\varepsilon(t+t^\varepsilon, v) &\geq \alpha_+ - \delta \quad \text{if } d(t,v) \leq -\varepsilon C, \\ u^\varepsilon(t+t^\varepsilon, v) &\leq \alpha_- + \delta \quad \text{if } d(t,v) \geq \varepsilon C, \end{aligned}$$

and

$$u^\varepsilon(t+t^\varepsilon, v) \in [\alpha_- - \delta, \alpha_+ + \delta],$$

for every $\varepsilon \in (0, \varepsilon_0)$.

This finishes the proof of Theorem 2.5. □

9. Generation and propagation of the interface for the 'discrete PDE': Proof of Theorem 2.3

Recall that the initial data $\{u^N(0,x)\}_{x\in\mathbb{T}_N^d}$ of the discrete PDE (2.18) satisfy (BIP1) and (BIP2). Previously, in Subsections 7.1 and 8.3 (cf. (7.6) and (8.3)), we have constructed super and sub solutions

$$w_\varepsilon^\pm(t,v) \equiv w_K^\pm(t,v) \quad \text{and} \quad u_\varepsilon^\pm(t,v) \equiv u_K^\pm(t,v), t \geq 0, v \in \mathbb{T}^d,$$

of the problem (P^ε) with $\varepsilon = K^{-1/2}$.

We will show that these functions, $w_K^\pm(t,v)$ and $u_K^\pm(t,v)$, restricted to the discrete torus $\frac{1}{N}\mathbb{T}_N^d$ actually play the role of super and sub solutions of the discretized hydrodynamic equation (2.18). The proof relies on the comparison argument.

More precisely, we show

$$\mathcal{L}^{N,K} w_K^+ \geq 0 \geq \mathcal{L}^{N,K} w_K^- \quad \text{and} \quad \mathcal{L}^{N,K} u_K^+ \geq 0 \geq \mathcal{L}^{N,K} u_K^-,$$

where $\mathcal{L}^{N,K}$ is the operator associated with (2.18). These estimates will follow from estimates shown in the continuum setting, namely $\mathcal{L}w_\varepsilon^+ \geq C_5 e^{-\bar{\gamma}\tau} > -C_5 e^{-\bar{\gamma}\tau} \geq \mathcal{L}w_\varepsilon^-$ (cf. (7.7)), and $\mathcal{L}u_\varepsilon^+ \geq C > -C \geq \mathcal{L}u_\varepsilon^-$ (cf. (8.7)), in combination with the error estimates on $(\mathcal{L} - \mathcal{L}^{N,K})w_K^\pm$ and $(\mathcal{L} - \mathcal{L}^{N,K})u_K^\pm$.

9.1. **Generation of a discrete interface.** Recall $Y(\tau) = Y(\tau, \zeta)$ for $\tau \geq 0, \zeta \in \mathbb{R}_+$, is the solution of the ordinary differential equation (7.1), with the initial value $Y(0) = \zeta$.

Theorem 9.1. *Let $u^N(t,\cdot)$ be the solution of the discrete PDE (2.18) with initial value $u^N(0,\cdot)$. Let also $\delta \in (0, \delta_0)$ where $\delta_0 = \min\{\alpha_* - \alpha_-, \alpha_+ - \alpha_*\}$, and $t^N = \frac{1}{2\gamma K} \log K$. Suppose that $K \equiv K(N) = o(N^{2\gamma/(3\gamma+\bar{\gamma})})$. Then, there exist $N_0, M_0 > 0$ such that the following hold for every $N \geq N_0$:*
(1) For all $x \in \mathbb{T}_N^d$,

$$\alpha_- - \delta \leq u^N(t^N, x) \leq \alpha_+ + \delta.$$

(2) If $u_0(\frac{x}{N}) \geq \alpha_* + M_0 K^{-1/2}$, then

$$u^N(t^N, x) \geq \alpha_+ - \delta.$$

(3) If $u_0(\frac{x}{N}) \leq \alpha_* - M_0 K^{-1/2}$, then

$$u^N(t^N, x) \leq \alpha_- + \delta.$$

Proof. Using $Y(\tau, \zeta)$ and $u_0 = u_0(x)$, we define sub and super solutions of the continuous system as

$$w_K^{\pm}(t, v) = Y(Kt, u_0(v) \pm P(t)), \quad v \in \mathbb{T}^d,$$

where $P(t) = C_4(e^{K\gamma t} - 1)/K$. Define the operators $\mathcal{L}^K$ and $\mathcal{L}^{N,K}$ by

$$\mathcal{L}^K u = \partial_t u - \Delta\varphi(u) - Kf(u), \quad v \in \mathbb{T}^d,$$

with respect to the continuous Laplacian Δ on $\mathbb{T}^d$ and also continuous functions $u = \{u(t,v)\}_{v \in \mathbb{T}^d}$, and

$$\mathcal{L}^{N,K} u = \partial_t u - \Delta^N \varphi(u) - Kf(u), \quad x \in \mathbb{T}_N^d,$$

for discrete functions $u = \{u(t,x)\}_{x \in \mathbb{T}_N^d}$, respectively.

We now make use of an estimate in the proof of Theorem 2.5: In Lemma 7.2, it is shown that

$$\mathcal{L}^K w_K^+ \geq C_5 e^{-\bar{\gamma} K t^N} = C_5 K^{-\bar{\gamma}/2\gamma} > 0$$

holds for some $C_5 > 0$ and large enough K. However,

$$\mathcal{L}^{N,K} w_K^+ = \mathcal{L}^K w_K^+ + (\Delta\varphi(w_K^+) - \Delta^N \varphi(w_K^+)),$$

and, by Taylor's formula, the second term is bounded by

$$\frac{C_2}{N} \sup_{v \in \mathbb{T}^d} \left| D_v^3 \{\varphi(w_K^+(t,v))\} \right|,$$

where $|D_v^3\{\,\cdot\,\}|$ means the sum of the absolute values of all third derivatives in v.

Since $u_0 \in C^3(\mathbb{T}^d)$ and $\varphi \in C^3(\mathbb{R}_+)$ (note that $w_K^{\pm}$ takes only bounded values so that $\varphi \in C_b^3([0,M])$), from (1)-(3) of Lemma 7.1 and noting $e^{3\gamma K t^N} = K^{3/2}$, we obtain

$$\sup_{0 \leq t \leq t^N, x \in \mathbb{T}_N^d} |\Delta\varphi(w_K^+(t, \tfrac{x}{N})) - \Delta^N \varphi(w_K^+(t, \tfrac{x}{N})))| \leq \frac{C_3}{N} K^{3/2}.$$

Thus, this term is absorbed by $C_5 K^{-\bar{\gamma}/2\gamma}$ if $K = o(N^{2\gamma/(3\gamma+\bar{\gamma})})$ and N is large enough.

Therefore, we obtain $\mathcal{L}^{N,K} w_K^+ \geq 0$ for $N \geq N_0$ with some $N_0 > 0$. By Lemma 4.1, we see $u^N(t,x) \leq w_K^+(t, \frac{x}{N})$. Similarly, one can show $w_K^-(t, x/N) \leq u^N(t,x)$. Thus, the proof of the theorem is concluded similarly to the proof of Theorem 2.4 in Subsection 7.2. $\square$

9.2. **Propagation of a discrete interface.** Recall the interface flow Γ_t, and the two functions $u^\pm(t,v) \equiv u_K^\pm(t,v)$ defined by (8.3), namely

$$u^\pm(t,v) = U_0\left(K^{1/2}d(t,v) \pm p(t)\right) + K^{-1/2}U_1\left(t,v,K^{1/2}d(t,v) \pm p(t)\right) \pm q(t),$$

and $u^N(t,v)$ defined in (2.21) from the discretized hydrodynamic equation (2.18).

Theorem 9.2. *Assume that the following inequality* (9.1) *holds at* $t=0$ *and* $K = o(N^{2/3})$ *for* $K = K(N) \uparrow \infty$. *Then, taking* $\beta, \sigma, L, \hat{L} > 0$ *in* $p(t)$ *and* $q(t)$ *as in Lemma 8.7, there exists* $N_0 \in \mathbb{N}$ *such that*

$$u^-(t,v) \le u^N(t,v) \le u^+(t,v), \tag{9.1}$$

holds for every $t \in [0,T]$, $v = x/N, x \in \mathbb{T}_N^d$ *and* $N \ge N_0$.

Proof. The upper bound in (9.1) follows from Lemma 4.1, once we can show that

$$\mathcal{L}^{N,K}u^+ = \partial_t u^+ - \Delta^N\varphi(u^+) - Kf(u^+) \ge 0, \quad x \in \mathbb{T}_N^d, \tag{9.2}$$

for every $N \ge N_0$ with some $N_0 \in \mathbb{N}$. As in the proof of Theorem 9.1,

we decompose

$$\mathcal{L}^{N,K}u^+ = \mathcal{L}^K u^+ + (\Delta\varphi(u^+) - \Delta^N\varphi(u^+)), \tag{9.3}$$

where $\mathcal{L}^K u^+ = \partial_t u^+ - \Delta\varphi(u^+) - Kf(u^+)$.

We now make use of an estimate derived in the proof of Theorem 2.4: By Lemma 8.7, the first term $\mathcal{L}^K u^+$ in (9.3) is bounded on $[0,T] \times \mathbb{T}^d$ as

$$\mathcal{L}^K u^+ \ge C > 0, \tag{9.4}$$

if we choose parameters $\beta, \sigma, L, \hat{L} > 0$ there properly.

For the second term in (9.3), since $d(t,v)$ and $U(z)$ are smooth so that $u^\pm$ are smooth in v, we have

$$\left|\Delta\varphi(u^+(t,\tfrac{x}{N})) - \Delta^N\varphi(u^+(t,\tfrac{x}{N}))\right| \le C_1\tfrac{K^{3/2}}{N}.$$

Indeed, this follows from Taylor expansion for $\Delta^N\varphi(u^+)$ up to the third order term, noting that $\varphi \in C^3(\mathbb{R}_+)$ and $u^+(t,v)$ is bounded. Therefore, if $K = o(N^{2/3})$, this term is absorbed by the positive constant C in (9.4) for $\mathcal{L}^K u^+$. This proves (9.2).

The lower bound by $u^-(t,v)$ is shown similarly. □

9.3. **Proof of Theorem 2.3.** The proof of Theorem 2.3 follows from Theorems 9.1 and 9.2. By the assumption (BIP2), $\nabla u_0(v)\cdot n(v) \ne 0$ for $v \in \Gamma_0$. Hence, for $v \notin \Gamma_0$, we have that $u_0(v) \ne \alpha_*$. Then, for N large enough, we would have $|u_0(v) - \alpha_*| \ge \epsilon_v > M_0K^{-1/2}$, where M_0 is the constant in Theorem 9.1.

Recall $u^N(t,v)$ in (2.21). By Theorem 9.1, at time $t^N = (2\gamma K)^{-1}\log K$, either $u^N(t^N,v) \ge \alpha_+ - \delta$ or $u^N(t,v) \le \alpha_- + \delta$ for a small $\delta > 0$.

Since for large N, we have $u^-(0,v) \le u^N(t^N,v) \le u^+(0,v)$, thinking of $u^N(t^N,\cdot)$ as an initial condition, by Theorem 9.2, we can 'propagate' and obtain $u^-(t-t^N,v) \le u^N(t,v) \le u^+(t-t^N,v)$ for $t^N \le t \le T$. As $N \uparrow \infty$, we obtain, for each $0 < t \le T$ and $v \notin \Gamma_t$ that $u^N(t,v) \to \chi_{\Gamma_t}(v)$, concluding the proof. □

10. A 'Boltzmann-Gibbs' principle: Proof of Theorem 4.4

We give now an outline of the proof of Theorem 4.4, referring to statements proved in the following subsections. The constant C will change from line to line.

We have, by Lemmas 10.4 and 10.5, that

$$
\begin{aligned}
&\mathbb{E}_N\left|\int_0^T \sum_{x\in\mathbb{T}_N^d} a_{t,x}f_x dt\right| \\
&\leq \mathbb{E}_N\left|\int_0^T \sum_{x\in\mathbb{T}_N^d} a_{t,x}f_x 1(\sum_{y\in\Lambda_h}\eta_{y+x}\leq A)dt\right| + \int_0^T CMH(\mu_t^N|\nu_t^N)dt + \frac{CMTN^d}{A} \\
&\leq \mathbb{E}_N\left|\int_0^T \sum_{x\in\mathbb{T}_N^d} a_{t,x}f_x 1(\sum_{y\in\Lambda_h}\eta_{y+x}\leq A)1(\eta_x^\ell\leq B)dt\right| \\
&\qquad + 2\int_0^T CMH(\mu_t^N|\nu_t^N)dt + \frac{CMTN^dA}{B} + \frac{CMTN^d}{A},
\end{aligned}
\tag{10.1}
$$

if A, B satisfy $(A+1)/B \leq 1$.

The expectation in the right-side of (10.1) is bounded by

$$
\mathbb{E}_N\left|\int_0^T \sum_{x\in\mathbb{T}_N^d} a_{t,x}m_x dt\right| + \mathbb{E}_N\left|\int_0^T E_{\nu_\beta}\Big[a_{t,x}f_x 1(\sum_{y\in\Lambda_h}\eta_{y+x}\leq A|\eta_x^\ell\Big]1(\eta_x^\ell\leq B)dt\right|
$$

where m_x is defined in (10.7). By Lemma 10.6, the first term is bounded, and by Lemmas 10.9 and 10.10, the second term is estimated. Adding these bounds, the display is bounded by

$$
\begin{aligned}
&\frac{CTMKN^d}{G} + \frac{CTMG\ell^{d+2}A^2B^2N^d}{N^2} + \frac{CMN^d}{G} \\
&\quad + CM\int_0^T H(\mu_t^N|\nu_t^N)dt + \frac{CTMN^d}{\ell^d} + \frac{CTMKN^d\ell^2B}{N^2} + \frac{CTMN^d}{A}.
\end{aligned}
$$

Here, A, B, G, ℓ are in form $A = N^{\alpha_A}$, $B = N^{\alpha_B}$, $G = N^{\alpha_G}$ and $\ell = N^{\alpha_\ell}$ for parameters $\alpha_A, \alpha_B, \alpha_G, \alpha_\ell > 0$. By the assumptions of Lemmas 10.5 and 10.6, we assume that $\alpha_B = 2\alpha_A$ and

$$
\alpha_A + \alpha_G + (d+2)\alpha_\ell + 2\alpha_B - 2 = 5\alpha_A + \alpha_G + (d+2)\alpha_\ell - 2 < 0. \tag{10.2}
$$

Combining the estimates, as $A/B = 1/A$ and $K \geq 1$, the left-hand side of (4.8) is bounded by

$$
C\int_0^T MH(\mu_t^N|\nu_t^N))dt + CTMN^d\Big(\frac{1}{A} + \frac{K}{G} + \frac{G\ell^{d+2}A^2B^2}{N^2} + \frac{1}{\ell^d} + \frac{KB\ell^2}{N^2}\Big).
$$

We need to fix α_A, α_G, $d\alpha_\ell$ such that $2 - [\alpha_G + [(d+2)/d]d\alpha_\ell + 2\alpha_A + 2\alpha_B] > 0$, noting that the constraint (10.2) would also hold. A convenient choice is $\varepsilon_0 = \alpha_A = \alpha_G = d\alpha_\ell = 2 - (7 + (d+2)/d)\varepsilon_0$, or when $\varepsilon_0 = 2d/(9d+2)$, from which the right-hand side of (4.8) follows, and the proof of Theorem 4.4 is completed. □

We now turn to the estimates used in the proof of Theorem 4.4. We will assume throughout this section the assumptions (BIP1) and $H(\mu_0^N|\nu_0^N) = O(N^d)$.

10.1. **Preliminary estimates.** Recall the 'entropy inequality' following from the variational form of the relative entropy between two probability measures μ and ν:

$$E_\mu[F] \le H(\mu|\nu) + \log E_\nu\big[e^F\big].$$

Lemma 10.1. *We have, for a small $\gamma > 0$, that*

$$E_{\mu_t^N}\big[\sum_{x\in\mathbb{T}_N^d}\eta_x\big] \le \frac{H(\mu_t^N|\nu_t^N)}{\gamma} + O(N^d).$$

Proof. Write

$$\begin{aligned} E_{\mu_t^N}\big[\sum_{x\in\mathbb{T}_N^d}\eta_x\big] &\le \frac{H(\mu_t^N|\nu_t^N)}{\gamma} + \frac{1}{\gamma}\log E_{\nu_t^N} e^{\gamma\sum_{x\in\mathbb{T}_N^d}\eta_x} \\ &\le \frac{H(\mu_t^N|\nu_t^N)}{\gamma} + \frac{N^d}{\gamma}\max_x E_{\nu_t^N} e^{\gamma\eta_x} \\ &\le \frac{H(\mu_t^N|\nu_t^N)}{\gamma} + O(N^d) \end{aligned}$$

as $\max_{x\in\mathbb{T}_N^d} E_{\nu_t^N} e^{\gamma\eta_x} < \infty$ for a $\gamma > 0$ small, noting the uniform estimate on u^N in Lemma 4.1. □

Lemma 10.2. *For $\beta > 0$ and the γ in Lemma 10.1, we have*

$$H(\mu_t^N|\nu_\beta) \le (1+\gamma^{-1})H(\mu_t^N|\nu_t^N) + O(N^d).$$

In particular, when $H(\mu_0^N|\nu_0^N) = O(N^d)$, we have $H(\mu_0^N|\nu_\beta) = O(N^d)$.

Proof. Write

$$H(\mu_t^N|\nu_\beta) = \int \log\frac{d\mu_t^N}{d\nu_\beta}d\mu_t^N = H(\mu_t^N|\nu_t^N) + \int \log\frac{d\nu_t^N}{d\nu_\beta}d\mu_t^N \tag{10.3}$$

and

$$\frac{d\nu_t^N}{d\nu_\beta} = \prod_x \frac{d\nu_t^N}{d\nu_\beta}(\eta_x).$$

From Lemma 4.1, we have that u^N is uniformly bounded between $c_- = u_- \wedge \alpha_-$ and $c_+ = u_+ \vee \alpha_+$. Since $Z_{u^N(t,x)} = \sum \varphi(u^N(t,x))^k/g(k)!$ and φ is an increasing function, we have $Z_{u^N(t,x)} \ge Z_{c_-}$. Also, $\varphi(u^N(t,x)) \le \varphi(c_+)$. Then,

$$\frac{d\nu_{u^N(t,x)}}{d\nu_\beta}(k) = \frac{Z_{u^N(t,x)}^{-1}\frac{\varphi(u^N(t,x))^k}{g(k)!}}{Z_\beta^{-1}\frac{\varphi(\beta)^k}{g(k)!}} = \frac{Z_\beta}{Z_{u^N(t,x)}}\frac{\varphi(u^N(t,x))^k}{\varphi(\beta)^k} \le \frac{Z_\beta}{Z_{c_-}}\left(\frac{\varphi(c_+)}{\varphi(\beta)}\right)^k.$$

Therefore, (10.3) is bounded by

$$H(\mu_t^N|\nu_t^N) + N^d\log\frac{Z_\beta}{Z_{c_-}} + \frac{\varphi(c_+)}{\varphi(\beta)}E_\mu\big[\sum_{x\in\mathbb{T}_N^d}\eta_x\big].$$

Noting that $E_{\mu_t^N}\big[\sum_{x\in\mathbb{T}_N^d}\eta_x\big] \le \gamma^{-1}H(\mu_t^N|\nu_t^N) + O(N^d)$ by Lemma 10.1, the proof is complete. □

We first give an estimate to be used several times in the sequel. Let $\Lambda_k = \{x \in \mathbb{T}_N^d : |x| \le k\}$ be a cube of width $2k+1$. Let $q = q(\eta)$ be a function supported in Λ_k. Denote $q_x = \tau_x q$ for $x \in \mathbb{T}_N^d$. Consider the collection of $|\Lambda_k|$ regular sublattices $\mathbb{T}_{N,z,k}^d \subset \mathbb{T}_N^d$, where $z \in \Lambda_k$ and neighboring points in the grid are separated by $2k+1$.

Lemma 10.3. *We have*

$$\log E_{\nu_t^N}\big[e^{\sum_{x\in\mathbb{T}_N^d} q_x}\big] \le \frac{1}{|\Lambda_k|}\sum_{z\in\Lambda_k} \log E_{\nu_t^N}\big[e^{|\Lambda_k|\sum_{w\in\mathbb{T}_{N,z,k}^d} q_w}\big] \tag{10.4}$$
$$= \frac{1}{|\Lambda_k|}\sum_{x\in\mathbb{T}_N^d} \log E_{\nu_t^N}\big[e^{|\Lambda_k| q_x}\big].$$

Proof. One can write $\sum_{x\in\mathbb{T}_N^d} q_x = \sum_{z\in\Lambda_k}\sum_{w\in\mathbb{T}_{N,z,k}^d} q_w$. The inequality in (10.4) results from a Hölder's inequality. The last equality follows since elements $\{q_w : w \in \mathbb{T}_{N,z,k}^d\}_{z\in\Lambda_k}$ are independent under ν_t^N. □

10.2. **Truncation estimates.** With respect to the left-hand side of (4.8), we now develop useful truncation estimates, since under the Glauber+Zero-range dynamics, there is no a priori bound on the number of particles at a site $x \in \mathbb{T}_N^d$. If one were working with Glauber+Kawasaki dynamics, such estimates would already be in place given the maximum occupancy of a site is 1.

The first limits the particle numbers in $\tau_x\Lambda_h$, where we recall that Λ_h is the box including the support of the function h through which f_x is defined in (4.6).

Lemma 10.4. *Let $A = A_N = N^{\alpha_A}$ for $0 < \alpha_A$ small. Then, with respect to constants C, C', we have*

$$ME_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d} |f_x| 1\big(\sum_{y\in\Lambda_h}\eta_{y+x} > A\big)\Big] \le CMH(\mu_t^N|\nu_t^N) + \frac{C'MN^d}{A}.$$

Proof. Write, through the entropy inequality and Lemma 10.3, with respect to a $\gamma_1 > 0$, that

$$\begin{aligned}
&ME_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d} |f_x| 1\big(\sum_{y\in\Lambda_h}\eta_{y+x} > A\big)\Big]\\
&\le \frac{MH(\mu_t^N|\nu_t^n)}{\gamma_1} + \frac{M}{\gamma_1}\log E_{\nu_t^N}\Big[e^{\gamma_1\sum_{x\in\mathbb{T}_N^d}|f_x|1(\sum_{y\in\Lambda_h}\eta_{y+x}>A)}\Big]\\
&\le \frac{MH(\mu_t^N|\nu_t^N)}{\gamma_1} + \frac{M}{\gamma_1|\Lambda_h|}\sum_{x\in\mathbb{T}_N^d}\log E_{\nu_t^N}\Big[e^{\gamma_1|\Lambda_h||f_x|1(\sum_{y\in\Lambda_h}\eta_{y+x}>A)}\Big]\\
&= \frac{MH(\mu_t^N|\nu_t^N)}{\gamma_1}\\
&\quad + \frac{M}{\gamma_1|\Lambda_h|}\sum_{x\in\mathbb{T}_N^d}\log\Big\{1 - P_{\nu_t^N}\big(\sum_{y\in\Lambda_h}\eta_{y+x} > A\big) + E_{\nu_t^N}\Big[1\big(\sum_{y\in\Lambda_h}\eta_{y+x} > A\big)e^{\gamma_1|\Lambda_h||f_x|}\Big]\Big\}.
\end{aligned}$$

The last line is further estimated with the inequality $\log(1+x) \le x$ for $x \ge 0$, and then Markov's inequality:

$$\begin{aligned}
&\frac{MH(\mu_t^N|\nu_t^N)}{\gamma_1} + \frac{M}{\gamma_1|\Lambda_h|}\sum_{x\in\mathbb{T}_N^d} E_{\nu_t}\Big[1\big(\sum_{y\in\Lambda_h}\eta_{y+x} > A\big)\big(e^{\gamma_1|\Lambda_h||f_x|}-1\big)\Big]\\
&\le \frac{MH(\mu_t^N|\nu_t^N)}{\gamma_1} + \frac{M}{\gamma_1|\Lambda_h|A}\sum_{x\in\mathbb{T}_N^d} E_{\nu_t^N}\big[\sum_{y\in\Lambda_h}\eta_{y+x}e^{\gamma_1|\Lambda_h||f_x|}\big]. \qquad (10.5)
\end{aligned}$$

We first note that $f_x(\eta) \le C\sum_{y\in\Lambda_h}\eta_{x+y} + C$ through the bounds (4.5). Then, by the uniform estimate Lemma 4.1, we may choose γ_1 small enough so that

$$\sup_{x\in\mathbb{T}_N^d} E_{\nu_t^N}\big[\sum_{y\in\Lambda_h}\eta_{y+x}e^{\gamma_1|\Lambda_h||f_x|}\big] < \infty$$

The display (10.5) is then bounded by $CMH(\mu_t^N|\nu_t^N) + C'MN^d/A$, as desired. ☐

We now truncate the average number of particles in a block of width ℓ around x. Define

$$\eta_x^\ell = \frac{1}{(2\ell+1)^d}\sum_{|z-x|\le\ell}\eta_z.$$

Lemma 10.5. *Let $B = B_N = A_N^2 = N^{2\alpha_A}$, and $\ell \ge 1$. Then, with respect to constants C, C', for large N, we have*

$$\begin{aligned}
&ME_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d}|f_x|1\big(\sum_{y\in\Lambda_h}\eta_{y+x}\le A\big)1(\eta_x^\ell > B)\Big]\\
&\qquad\le \frac{CM(A+1)}{B}H(\mu_t^N|\nu_t^N) + \frac{C'M(A+1)}{B}N^d.
\end{aligned}$$

Proof. Since $f_x(\eta) \le C\sum_{y\in\Lambda_h}\eta_{x+y} + C$ by the bounds (4.5), and $\sum_{x\in\mathbb{T}_N^d}\eta_x^\ell = \sum_{x\in\mathbb{T}_N^d}\eta_x$, then

$$\begin{aligned}
E_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d}|f_x|1\big(\sum_{y\in\Lambda_h}\eta_{y+x}\le A\big)1(\eta_x^\ell > B)\Big] &\le \frac{\max(C_1,C_2)(A+1)}{B}E_{\mu_t^N}\big[\sum_{x\in\mathbb{T}_N^d}\eta_x^\ell\big]\\
&= \frac{\max(C_1,C_2)(A+1)}{B}E_{\mu_t^N}\big[\sum_{x\in\mathbb{T}_N^d}\eta_x\big]. \qquad (10.6)
\end{aligned}$$

Now, by Lemma 10.1, $E_{\mu_t^N}\big[\sum_{x\in\mathbb{T}_N^d}\eta_x\big] \le CH(\mu_t^N|\nu_t^N) + O(N^d)$. Then, the display (10.6) is bounded, for large N, by

$$\frac{CM(A+1)}{B}H(\mu_t^N|\nu_t^N) + \frac{C'M(A+1)}{B}N^d,$$

finishing the estimate. We note this bound does not depend on the size of $\ell \ge 1$. ☐

10.3. **Main estimates.** We now estimate the remaining portions of $\sum_{x\in\mathbb{T}_N^d} f_x$. In Subsection 10.3.1, we show that f_x is in a sense close to its conditional mean given the local density of particles. In Subsection 10.3.2, we estimate this conditional mean.

10.3.1. *Bound on 'concentration' around conditional mean.* For $x \in \mathbb{T}_N^d$, let

$$m_x = \Big(f_x 1(\sum_{y\in\Lambda_h} \eta_{y+x} \le A) - E_{\nu_\beta}\big[f_x 1(\sum_{y\in\Lambda_h} \eta_{y+x} \le A)|\eta_x^\ell\big]\Big)1(\eta_x^\ell \le B). \tag{10.7}$$

Lemma 10.6. *Let $\ell = \ell_N = N^{\alpha_\ell}$ and $G = N^{\alpha_G}$ for $0 < \alpha_\ell, \alpha_G$ small. Suppose $\alpha_A + \alpha_G + (d+2)\alpha_\ell + 2\alpha_B - 2 < 0$. Then, for constants C, C', we have*

$$\mathbb{E}_N\Big|\int_0^T \sum_{x\in\mathbb{T}_N^d} a_{t,x} m_x dt\Big| \le \frac{C(T+1)MKN^d}{G} + \frac{C'TMG\ell^{d+2}A^2B^2N^d}{N^2}.$$

Remark 10.1. The estimate on the spectral gap in (SP) is used in the proof of Lemma 10.6. For the specific cases when a sharper estimate holds, the right-hand side bound in Lemma 10.6 may be improved.

Proof of Lemma 10.6. We apply the entropy inequality, with respect to a Zero-range invariant measure ν_β, to obtain

$$\mathbb{E}_N\Big|\int_0^T \sum_{x\in\mathbb{T}_N^d} a_{t,x} m_x dt\Big| \le \frac{H(\mu_0^N|\nu_\beta)}{\gamma} + \frac{1}{\gamma}\log E_{\nu_\beta}\Big[e^{\gamma|\int_0^T \sum_{x\in\mathbb{T}_N^d} a_{t,x}m_x dt|}\Big],$$

for every $\gamma > 0$. The second term, on the right-hand side of the display, noting $e^{|z|} \le e^z + e^{-z}$, is bounded by the Feynman-Kac formula in Appendix 1.7 (whose proof does not require ν_β to be an invariant measure of L_N) in [35].

Then, considering $\gamma = G/M$, we have

$$\begin{aligned}\mathbb{E}_{\mu^N}&\Big|\int_0^T \sum_{x\in\mathbb{T}_N^d} a_{t,x} m_x dt\Big| \\ &\le \frac{MH(\mu_0^N|\nu_\beta)}{G} + 2M\int_0^T \sup_h \Big\{\langle M^{-1}\sum_{x\in\mathbb{T}_N^d} a_{t,x}m_x, h\rangle + \frac{1}{G}D_N(\sqrt{h})\Big\}dt\end{aligned}$$

where h is a density with respect to ν_β, and $D^N(f) = E_{\nu_\beta}[f(S_N f)]$ is the quadratic form given in terms of $S_N = (L_N + L_N^*)/2$ and the $L^2(\nu_\beta)$ adjoint L_N^*.

By Lemma 10.2 and our initial assumption, $H(\mu_0^N|\nu_\beta) \le O\big(H(\mu_0^N|\nu_0^N)\big) + O(N^d) = O(N^d)$.

To estimate the supremum, write $D_N(f) = -N^2 D_{ZR}(f) + KQ_G(f)$. By (5.1),

$$\begin{aligned}D_{ZR}(f) &= E_{\nu_\beta}[f(-L_{ZR}f)] = \mathcal{D}_{ZR}(f;\nu_\beta)\\ &= \frac{1}{4}\sum_{\substack{|x-y|=1\\ x,y\in\mathbb{T}_N^d}} E_{\nu_\beta}\big[g(\eta_x)\big(f(\eta^{x,y}) - f(\eta)\big)^2\big].\end{aligned}$$

Also, $Q_G(f) = E_{\nu_\beta}[(L_G f)f]$ is explicit following calculations say in Lemma 5.3 as

$$\begin{aligned}Q_G(f) = &-\sum_{x\in\mathbb{T}_N^d} E_{\nu_\beta}\Big[c_x^+(\eta)\big(f(\eta^{x,+}) - f(\eta)\big)^2\Big]\\ &-\sum_{x\in\mathbb{T}_N^d} E_{\nu_\beta}\Big[c^-(\eta)1(\eta_x \ge 1)\big(f(\eta^{x,-}) - f(\eta)\big)^2\Big]\end{aligned}$$

$$- \sum_{x\in\mathbb{T}_N^d} E_{\nu_\beta}\Big[f(\eta)f(\eta^{x,+})c_x^+(\eta) + f(\eta)f(\eta^{x,-})c_x^-(\eta)1(\eta_x \geq 1)\Big]$$
$$+ \sum_{x\in\mathbb{T}_N^d} E_{\nu_\beta}\Big[f^2(\eta)\Big(c_x^+(\eta^{x,-})\frac{g(\eta_x)}{\varphi(\beta)} + c_x^-(\eta^{x,+})\frac{\varphi(\beta)}{g(\eta_x+1)}\Big)\Big].$$

As the rates $c^\pm \geq 0$, only the last line in the display for $Q_G(f)$ is nonnegative. By our assumption (BR), however, we have that $c_x^+(\eta^{x,-})g(\eta_x)$ and $c_x^-(\eta^{x,+})/g(\eta_x+1)$ are bounded. When f is a nonnegative function such that f^2 is a density with respect to ν_β, that is $E_{\nu_\beta}[f^2(\eta)] = 1$, we have the upper bound

$$\frac{1}{G}D_N(f) = -\frac{N^2}{G}D_{ZR}(f) + \frac{K}{G}Q_G(f) \leq -\frac{N^2}{G}D_{ZR}(f) + \frac{CKN^d}{G}$$

and therefore

$$\begin{aligned}
&\mathbb{E}_{\mu^N}\Big|\int_0^T \sum_{x\in\mathbb{T}_N^d} a_{t,x}m_x dt\Big| \leq \frac{CMN^d}{G}\\
&\quad + 2M\int_0^T \sup_h \Big\{M^{-1}\langle \sum_{x\in\mathbb{T}_N^d} a_{t,x}m_x, h\rangle - \frac{N^2}{G}D_{ZR}(\sqrt{h})\Big\}dt + \frac{CTMKN^d}{G}.
\end{aligned} \tag{10.8}$$

To analyze further, define

$$D_{\ell,x}(f) = E_{\nu_\beta}[f(-L_{\ell,x}f)] = \frac{1}{4}\sum_{\substack{|w-z|=1\\ w,z\in\Lambda_{\ell,x}}} E_{\nu_\beta}\Big[g(\eta_w)\big(f(\eta^{w,z}) - f(\eta)\big)^2\Big]$$

where $L_{\ell,x}$ is the Zero-range generator restricted to sites $\Lambda_{\ell,x} = \{y + x : |y| \leq \ell\}$.

Define also the associated canonical process on $\Lambda_{\ell,x}$ where the number of particles $\sum_{y\in\Lambda_{\ell,x}}\eta_y = j$ is fixed for $j \geq 0$. Let $L_{\ell,x,j}$ denote its generator and let $\nu_{\ell,x,j} = \nu_\beta(\cdot|\sum_{y\in\Lambda_{\ell,x,j}}\eta_y = j)$ be its canonical invariant measure on the configuration space $\big\{\{\eta_z\}_{z\in\Lambda_{\ell,x}} : \sum_{y\in\Lambda_{\ell,x,j}}\eta_y = j\big\}$. By translation-invariance, $\nu_{\ell,x,j}$ does not depend on x.

Then, counting the overlaps, we have

$$\sum_{x\in\mathbb{T}_N^d} D_{\ell,x}(\sqrt{h}) = (2\ell+1)^d D_{ZR}(\sqrt{h}).$$

The supremum on the right-hand side of (10.8) is less than

$$\sum_{x\in\mathbb{T}_N^d} \sup_h \Big\{E_{\nu_\beta}[(a_{t,x}/M)m_x h] - \frac{N^2}{G\ell_*^d}D_{\ell,x}(\sqrt{h})\Big\} \tag{10.9}$$

where $\ell_* = 2\ell + 1$.

Recall that $G = N^{\alpha_G}$ for a small $\alpha_G > 0$, and that m_x vanishes unless the density of particles in the ℓ-block is bounded, $\eta^\ell(x) \leq B$. By conditioning on the number of particles on $\Lambda_{\ell,x}$, and dividing and multiplying by $E_{\nu_\beta}[h|\sum_{z\in\Lambda_{\ell,x}}\eta_z = j]$, we have for each x that

$$\sup_h \Big\{E_{\nu_\beta}[(a_{t,x}/M)m_x h] - \frac{N^2}{G\ell_*^d}D_{\ell,x}(\sqrt{h})\Big\}$$

$$\leq \sup_{j\leq B(2\ell+1)^d} \sup_h \Big\{ E_{\nu_{\ell,x,j}}[(a_{t,x}/M)m_x h] - \frac{N^2}{G\ell_*^d} D_{\ell,x,j}(\sqrt{h}) \Big\}$$

where h is a density with respect to $\nu_{\ell,x,j}$.

Now, by the Rayleigh estimate in [35] p. 375, Theorem 1.1, in terms of the spectral gap of the canonical process $gap(\ell, j)$, which does not depend on x by translation-invariance, the last display is bounded by

$$\frac{G\ell_*^d}{N^2} \frac{E_{\nu_{\ell,x,j}}[(a_{t,x}/M)m_x\{(-L_{\ell,x,j})^{-1}(a_{t,x}/M)m_x\}]}{1 - 2\|(a_{t,x}/M)m_x\|_{L^\infty} \frac{G\ell_*^d}{N^2} gap(\ell, j)^{-1}}. \tag{10.10}$$

By the bounds on f_x via (4.5) and those on $\{a_{t,x}\}$, we have $\|(a_{t,x}/M)m_x\|_{L^\infty} = O(A)$. Since m_x is mean-zero with respect to $\nu_{\ell,x,j}$, we have

$$E_{\nu_{\ell,x,j}}\big[(a_{t,x}/M)m_x\{(-L_{\ell,x,j})^{-1}(a_{t,x}/M)m_x\}\big] \leq gap(\ell, j)^{-1}\|(a_{t,x}/M)m_x\|^2_{L^\infty}.$$

Recall the spectral gap assumption (SP) that $gap(\ell, j)^{-1} \leq C\ell^2(j/\ell^d)^2$. Since $j/\ell^d \leq CB$, we have that $gap(\ell, j)^{-1} \leq C\ell^2 B^2$. Choosing $\alpha_A + \alpha_G + (d+2)\alpha_\ell + 2\alpha_B - 2 < 0$, we have

$$AG\ell^d gap(\ell, j)^{-1}/N^2 \leq CAG\ell^{d+2}B^2N^{-2} = o(1)$$

and so the denominator in (10.10) is bounded below.

Hence, (10.9) is bounded above by

$$\frac{CG\ell^d}{N^2}\|m_x\|^2_{L^\infty}\ell^2 \leq \frac{CG\ell^d A^2\ell^2 B^2}{N^2},$$

and the desired estimate follows by inserting back into (10.8). $\square$

10.3.2. *Bound on conditional mean.* To treat the conditional expectation

$$E_{\nu_\beta}[a_{t,x}f_x 1(\sum_{y\in\Lambda_h} \eta_{y+x} \leq A)|\eta_x^\ell]1(\eta_x^\ell \leq B), \tag{10.11}$$

we will need two preliminary estimates (Lemmas 10.7 and 10.8).

Since f_x is mean-zero with respect to $\nu_{u^N(t,x)}$ (cf. (4.6)), the next bound as seen by its argument is a type of bound on the tail $f_x 1(\sum_{y\in\Lambda_h}\eta_{y+x} > A)$.

Lemma 10.7. *For $x \in \mathbb{T}_N^d$, let $y_x = \eta_x^\ell - u^N(t, x)$. Fix also $\delta > 0$. We have that*

$$\Big|E_{\nu_\beta}\big[(a_{t,x}/M)f_x 1(\sum_{y\in\Lambda_h}\eta_{y+x} \leq A)|\eta_x^\ell\big]\Big|1(|y_x| \leq \delta) \leq Cy_x^2 1(|y_x| \leq \delta) + \frac{C}{\ell^d} + \frac{C}{A}.$$

Proof. The argument is by a local central limit theorem, or in other words equivalence of ensembles. Let $b_x = (a_{t,x}/M)f_x 1(\sum_{y\in\Lambda_h}\eta_{y+x} \leq A)$. Recall $\|a_{t,x}\|_\infty/M \leq 1$. By Corollary 1.7 in Appendix 2 of [35], when $|y_x| \leq \delta$, we have that

$$\Big|E_{\nu_\beta}[b_x|\eta_x^\ell]\Big| \leq \big|E_{\nu_{u^N(t,x)+y_x}}[b_x]\big| + \frac{C}{\ell^d}.$$

We now expand $E_{\nu_{u^N(t,x)+y_x}}[b_x]$ in terms of y_x around 0. Choose $\lambda = \lambda(y_x)$ so that

$$\frac{E_{\nu_{u^N(t,x)}}[\eta_x e^{\lambda(\eta_x - u^N(t,x))}]}{E_{\nu_{u^N(t,x)}}[e^{\lambda(\eta_x - u^N(t,x))}]} = u^N(t, x) + y_x. \tag{10.12}$$

Note from (10.12) that $\lambda(0) = 0$ and $\lambda'(0) := \frac{d}{dy_x}\lambda(0) = E_{\nu_{u^N(t,x)}}[(\eta_x - u^N(t,x))^2]^{-1}$, and so

$$\begin{aligned}\frac{d}{dy_x} E_{\nu_{u^N(t,x)+y_x}}[b_x]\big|_{y_x=0} &= E_{\nu_{u^N(t,x)}}\Big[b_x e^{\lambda \sum_{y\in\Lambda_h}(\eta_{y+x} - u^N(t,x))}\Big] \\ &= \lambda'(0) E_{\nu_{u^N(t,x)}}\Big[b_x\big(\sum_{y\in\Lambda_h}(\eta_{y+x} - u^N(t,x))\big)\Big].\end{aligned}$$

Since u^N is uniformly bounded (Lemma 4.1), $\lambda'(0)$ is bounded; also, from (10.12), one can see that $\lambda''(a) = \frac{d^2}{dy_x^2}\lambda(a)$ for $|a| \le \delta$ is also bounded say by $C(\delta)$ for $|a| \le \delta$. Then,

$$E_{\nu_{u^N(t,x)+y_x}}[b_x] = E_{\nu_{u^N(t,x)}}[b_x] + \Big[\frac{d}{dy_x} E_{\nu_{u^N(t,x)+y_x}}[b_x]\big|_{y_x=0}\Big] y_x + r_x \tag{10.13}$$

where $|r_x| \le (C(\delta)/2) y_x^2$.

We now estimate that the first two terms on the right-hand side of (10.13) are of order A^{-1} to finish the argument. Indeed,

$$\begin{aligned}&|E_{\nu_{u^N(t,x)}}[b_x]| = \big|E_{\nu_{u^N(t,x)}}[(a_{t,x}/M) f_x] - E_{\nu_{u^N(t,x)}}[(a_{t,x}/M) f_x 1(\sum_{y\in\Lambda_h}\eta_{y+x} > A)]\big| \\ &\le \frac{1}{A} E_{\nu_{u^N(t,x)}}\big[|f_x| \sum_{y\in\Lambda_h}\eta_{y+x}\big] \le \frac{C}{A}\end{aligned}$$

as f_x is mean-zero with respect to $\nu_{u^N(t,x)}$ and $E_{\nu_{u^N(t,x)}}[|f_x|\sum_{y\in\Lambda_h}\eta_{y+x}]$ is uniformly bounded as $u^N(t,\cdot)$ is uniformly bounded in Lemma 4.1.

The other term is similar:

$$\begin{aligned}&\big|\frac{d}{dy_x} E_{\nu_{u^N(t,x)+y_x}}[b_x]|_{y_x=0}\big| \\ &\le \big|\frac{d}{dy_x} E_{\nu_{u^N(t,x)+y_x}}[(a_{t,x}/M) f_x]|_{y_x=0}\big| \\ &\qquad + \big|\frac{d}{dy_x} E_{\nu_{u^N(t,x)+y_x}}[(a_{t,x}/M) f_x(\sum_{y\in\Lambda_h}\eta_{y+x} > A)]|_{y_x=0}\big| \\ &\le \frac{\lambda'(0)}{A}\big|E_{\nu_{u^N(t,x)}}[(a_{t,x}/M) f_x(\sum_{y\in\Lambda_h}\eta_{y+x})(\sum_{y\in\Lambda_h}(\eta_{y+x} - u^N(t,x)))]\big| \le \frac{C}{A},\end{aligned}$$

since first $a_{t,x}$ is non-random and f_x satisfies $0 = \frac{d}{dy_x} E_{\nu_{u^N(t,x)+y_x}}[f_x]|_{y_x=0}$, and second

$$E_{\nu_{u^N(t,x)}}[(a_{t,x}/M) f_x(\sum_{y\in\Lambda_h}\eta_{y+x})(\sum_{y\in\Lambda_h}(\eta_{y+x} - u^N(t,x)))]$$

is uniformly bounded as $u^N(t,\cdot)$ is uniformly bounded (Lemma 4.1). □

Let now

$$\tilde{y}_x = \frac{1}{(2\ell+1)^d} \sum_{|z-x|\le \ell^d} \big(\eta_z - u^N(t,z)\big) \quad \text{and} \quad \ell_* = 2\ell + 1.$$

We will need that the following exponential moment is uniformly bounded.

Lemma 10.8. *For $\gamma, \delta > 0$ small, we have*

$$\sup_{\ell} E_{\nu_t^N}\Big[e^{\gamma \ell_*^d \tilde{y}_x^2} 1(|\tilde{y}_x| \le \delta)\Big] < \infty.$$

To get a feel for this estimate, consider the case that the variables are i.i.d. Poisson with parameter κ. Then, $\tilde{y}_x$ has the distribution of ℓ_*^{-d} times a centered Poisson $\ell_*^d\kappa$ random variable. In this case, the expectation in this lemma equals

$$\sum_{|k-\ell_*^d\kappa| \le \ell_*^d\delta} e^{\gamma \ell_*^{-d}(k-\ell_*^d\kappa)^2} e^{-\ell_*^d\kappa} \frac{(\ell_*^d\kappa)^k}{k!}.$$

A typical summand, say with $k \sim (\lambda + \delta)\ell_*^d$ is estimated by Stirling's formula as

$$e^{\gamma \ell_*^d \delta^2} e^{-\ell_*^d\kappa} \frac{(\ell_*^d\kappa)^{\ell_*^d(\kappa+\delta)}}{(\ell_*^d(\kappa+\delta))!} \sim e^{-c\ell_*^d},$$

with $c > 0$, when $\gamma\delta^2 < \lambda$. Since there are only ℓ_*^d order summands, Lemma 10.8 holds in this setting.

We now give an argument for the general case through use of a local central limit theorem. Let κ denote $\kappa = \ell_*^{-d} \sum_{|z| \le \ell^d} u^N(t, x+z)$.

Proof of Lemma 10.8. Write the expectation in (10.8) as

$$\begin{aligned}
(10.14) \qquad & \sum_{|k-\ell_*^d\kappa| < \ell_*^d\delta} e^{\gamma \ell_*^{-d}(k-\ell_*^d\kappa)^2} \nu_t^N\Big(\sum_{|z|\le\ell} \eta_{z+x} = k\Big) \\
&= \sum_{k=\lceil \ell_*^d\kappa\rceil}^{\lfloor \ell_*^d(\kappa+\delta)\rfloor} e^{\gamma \ell_*^{-d}(k-\ell_*^d\kappa)^2} \nu_t^N\Big(\sum_{|z|\le\ell} \eta_{z+x} = k\Big) \\
&\quad + \sum_{k=\lceil \ell_*^d(\kappa-\delta)\rceil}^{\lceil \ell_*^d\kappa\rceil - 1} e^{\gamma \ell_*^{-d}(k-\ell_*^d\kappa)^2} \nu_t^N\Big(\sum_{|z|\le\ell} \eta_{z+x} = k\Big).
\end{aligned}$$

We now bound uniformly the first sum, and discuss the second sum afterwards.

Write the first sum on the right-hand side of (10.14), where a is a positive constant, as

$$\begin{aligned}
(10.15) \qquad & \sum_{k=\lceil \ell_*^d\kappa\rceil}^{\lceil \ell_*^d\kappa\rceil + a\lceil \ell_*^{d/2}\rceil} e^{\gamma \ell_*^{-d}(k-\ell_*^d\kappa)^2} \nu_t^N\Big(\sum_{|z|\le\ell} \eta_{z+x} = k\Big) \\
&\quad + \sum_{k=\lceil \ell_*^d\kappa\rceil + a\lceil \ell_*^{d/2}\rceil + 1}^{\lfloor \ell_*^d(\kappa+\delta)\rfloor} e^{\gamma \ell_*^{-d}(k-\ell_*^d\kappa)^2} \nu_t^N\Big(\sum_{|z|\le\ell} \eta_{z+x} = k\Big).
\end{aligned}$$

The first object, since $0 \le k - \ell_*^d\kappa \le a\ell_*^{d/2}$, is bounded by $e^{a^2\gamma}$.

To estimate the second object in (10.15), we write the probability in the sum as a difference of $1 - F(\ell_*^{-d/2}(k-1-\ell_*^d\kappa))$ and $1 - F(\ell_*^{d/2}(k-\ell_*^d\kappa))$, where F is the distribution function of $\ell_*^{d/2}\tilde{y}_x$, and then rewrite the second object in (10.15), summing by parts as

$$\sum_{k=\lceil \ell_*^d\kappa\rceil + a\lceil \ell_*^{d/2}\rceil+1}^{\lfloor \ell_*^d(\kappa+\delta)\rfloor -1} \left[e^{\gamma\ell_*^{-d}(k+1-\ell_*^d\kappa)^2} - e^{\gamma\ell_*^{-d}(k-\ell_*^d\kappa)^2}\right]\left[1-F(\ell_*^{-d/2}(k-\ell_*^d\kappa))\right] \tag{10.16}$$
$$+ e^{\gamma\ell_*^{-d}(k-\ell_*^d\kappa)^2}\left[1-F(\ell_*^{-d/2}(k-1-\ell_*^d\kappa))\right]\big|_{k=\lceil \ell_*^d\kappa\rceil + a\lceil \ell_*^{d/2}\rceil+1}$$
$$- e^{\gamma\ell_*^{-d}(k-\ell_*^d\kappa)^2}\left[1-F(\ell_*^{-d/2}(k-\ell_*^d\kappa))\right]\big|_{k=\lfloor \ell_*^d(\kappa+\delta)\rfloor}.$$

In Theorem 10 of Chapter 8 in [42] (page 230), subject to assumptions, namely that equations (2.3), (2.4) and (2.5) in Chapter 8 [42] hold, a uniform estimate on the tail of the distribution function is given. These assumptions hold when there is an $H > 0$ small where $R_{z,t}(u) = \log E_{\nu_t^N} e^{u\eta_z}$ is uniformly bounded in z and t for $|u| \le H$, and also when $\sigma^2_{z,t} = E_{\nu_t^N}[(\eta_z - u^N(t,z))^2]$ is uniformly bounded away from 0 in z and t. These specifications follow straightforwardly from the uniform bounds on u^N (Lemma 4.1). Then, $v_\ell := \sqrt{\ell_*^{-d}\sum_{|z|\le\ell}\sigma^2_{z,t}}$ is uniformly bounded away from 0 and ∞.

Therefore, by Theorem 10 in Chapter 8 [42], there is a constant τ such that for $0 \le x \le \tau\ell_*^{d/2}$ we have

$$1-F(x) \le C(\tau)\big(1-\Phi(x/v_\ell)\big)\exp\Big\{\frac{x^3}{v_\ell^3\ell_*^{d/2}}\kappa^1(x/(v_\ell\ell_*^{d/2}))\Big\} \ \text{ and}$$
$$F(-x) \le C(\tau)\Phi(x/v_\ell)\exp\Big\{-\frac{x^3}{v_\ell^3\ell_*^{d/2}}\kappa^1(-x/(v_\ell\ell_*^{d/2}))\Big\}, \tag{10.17}$$

where $\kappa^1(\cdot)$ is uniformly bounded for small arguments, and Φ is the Normal$(0,1)$ distribution function.

Note that

$$e^{\gamma\ell_*^{-d}(k+1-\ell_*^d\kappa)^2} - e^{\gamma\ell_*^{-d}(k-\ell_*^d\kappa)^2} = e^{\gamma\ell_*^{-d}(k-\ell_*^d\kappa)^2}\Big(e^{2\gamma\ell_*^{-d}(k-\ell_*^d\kappa)+\gamma\ell_*^{-d}} - 1\Big).$$

Also, when $x/v_\ell = \ell_*^{-d/2}(k-\ell_*^d\kappa)/v_\ell \ge 1$, which is the case when $k \ge \ell_*^d\kappa + a\ell_*^{d/2}$ and a is fixed large enough, we have that

$$\Big\{1-\Phi(\ell_*^{-d/2}(k-\ell_*^d\kappa)/v_\ell)\Big\}e^{\kappa^1(x/(v_\ell\ell_*^{d/2}))v_\ell^{-3}\ell_*^{-2d}(k-\ell_*^d\kappa)^3}$$
$$\le \frac{1}{\sqrt{2\pi}}e^{-\ell_*^{-d}(k-\ell_*^d\kappa)^2/2v_\ell^2}e^{\kappa^1(x/(v_\ell\ell_*^{d/2}))v_\ell^{-3}\ell_*^{-2d}(k-\ell_*^d\kappa)^3}.$$

With the aid of these observations, we deduce now that (10.16) is uniformly bounded in ℓ. Indeed, to see that the sum in (10.16) is bounded, observe since $a\ell_*^{d/2} \le k-\ell_*^d\kappa \le \delta\ell_*^d$ that

$$e^{2\gamma\ell_*^{-d}(k-\ell_*^d\kappa)+\gamma\ell_*^{-d}} - 1 \le 2\gamma\ell_*^{-d}(k-\ell_*^d\kappa) + \gamma\ell_*^{-d},$$

and $\kappa^1(x/(v_\ell\ell_*^{d/2})) \le \bar\kappa$ where $\bar\kappa$ is a constant, when δ small. Then, each summand is bounded by

$$e^{(\gamma+\delta\bar\kappa v_\ell^{-3}-2^{-1}v_\ell^{-2})\ell_*^{-d}(k-\ell_*^d\kappa)^2} \le 2\gamma\ell_*^{-d}(k-\ell_*^d\kappa+1)e^{-c(\gamma,\delta)\ell_*^{-d}(k-\ell_*^d\kappa)^2}$$

for γ,δ chosen small and $c(\gamma,\delta) > 0$. Hence, the sum may be bounded uniformly in ℓ in terms of the integral $C(\gamma)\int_a^\infty ze^{-c(\gamma,\delta)z^2}dz$, for some constant $C(\gamma)$.

The other two terms in (10.16) are bounded using similar ideas.

Finally, the second sum in (10.14) is bounded uniformly in ℓ analogously, using the left tail estimate in (10.17). $\square$

With these preliminary bounds in place, we resume the argument and consider the conditional expectation (10.11) when $|y_x| \le \delta$.

Lemma 10.9. *For $\delta > 0$ small, we have*

$$\int_0^T E_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d} E_{\nu_\beta}[a_{t,x}[f_x 1(\sum_{y\in\Lambda_h}\eta_{y+x}\le A)|\eta_x^\ell]1(\eta_x^\ell\le B)1(|y_x|\le\delta)\Big]dt$$
$$\le CM\int_0^T H(\mu_t^N|\nu_t^N)dt + \frac{CTMN^d}{\ell^d} + \frac{2CTMKN^d\ell^2}{N^2} + \frac{CTMN^d}{A}.$$

Proof. We first divide and multiply the left-hand side of the display by M. By Lemma 10.7, we first bound the term

$$\Big|E_{\nu_\beta}[(a_{t,x}/M)f_x 1(\sum_{y\in\Lambda_h}\eta_{y+x}\le A)|\eta_x^\ell]1(\eta_x^\ell\le B)1(|y_x|\le\delta)\Big|$$
$$\le\ Cy_x^2 1\big(|y_x|\le\delta\big) + \frac{C}{\ell^d} + \frac{C}{A}.$$

The last two terms when multiplied by M and summed over $x\in\mathbb{T}_N^d$ give rise to terms $CMN^d/\ell^d + CMN^d/A$ present in the right-hand side of the display of Lemma 10.9.

We now concentrate on the terms $y_x^2 1(|y_x|\le\delta)$. Recall $\tilde y_x = \ell_*^{-d}\sum_{|z-x|\le\ell}(\eta_z - u(t,z))$. Consider the following estimate of $1(|y_x|\le\delta)$ in terms of $1(|\tilde y_x|\le 2\delta)$:

$$\begin{aligned}1(|y_x|\le\delta) &= 1(|y_x|\le\delta)[1(|\tilde y_x|\le 2\delta)+1(|\tilde y_x|>2\delta)]\\ &\le 1(|\tilde y_x|\le 2\delta) + 1(|y_x|\le\delta)1(|\tilde y_x|>2\delta)\\ &\le 1(|\tilde y_x|\le 2\delta) + 1(|y_x|\le\delta)\big[1(|y_x|>\delta)+1(|y_x-\tilde y_x|>\delta)\big]\\ &\le 1(|\tilde y_x|\le 2\delta) + 1(|y_x|\le\delta)1(|y_x-\tilde y_x|\ge\delta).\end{aligned}$$

Hence,

$$M\int_0^T E_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d} y_x^2 1(|y_x|\le\delta)\Big]dt\ \le\ M\int_0^T E_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d} y_x^2 1(|\tilde y_x|\le 2\delta)\Big]dt$$
$$+ M\int_0^T E_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d} y_x^2 1(|y_x|\le\delta)1(|y_x-\tilde y_x|\ge\delta)\Big]dt. \tag{10.18}$$

To bound the second term in (10.18), since $|y_x-\tilde y_x| = \big|\ell_*^{-d}\sum_{|z|\le\ell}u^N(t,z)-u^N(t,x)\big|$, we have by Markov's inequality and Lemma 4.2 that

$$M\int_0^T E_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d} y_x^2 1(|y_x|\le\delta)1(|y_x-\tilde y_x|\ge\delta)\Big]dt \le \delta^2\frac{CTMKN^d\ell^2}{\delta^2 N^2}. \tag{10.19}$$

To bound the first term in (10.18), write

$$y_x^2 \le 2\tilde y_x^2 + 2\Big(\frac{1}{\ell_*^d}\sum_{|z-x|\le\ell}(u^N(t,z)-u^N(t,x))\Big)^2. \tag{10.20}$$

By Lemma 4.2 again,

$$M\int_0^T E_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d} y_x^2 1(|y_x|\le\delta)\Big]dt \tag{10.21}$$
$$\le \frac{CTMKN^d\ell^2}{N^2} + 2M\int_0^T E_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d}\tilde{y}_x^2 1(|y_x|\le\delta)\Big]dt.$$

The sum of the first terms on the right-hand sides of (10.19) and (10.21) gives the third term in Lemma 10.9.

To address the remaining second term in (10.21), write

$$ME_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d}\tilde{y}_x^2 1(|\tilde{y}_x|\le 2\delta)\Big] \le \frac{MH(\mu_t^N|\nu_t^N)}{\gamma_2} + \frac{M}{\gamma_2}\log E_{\nu_t^N}\Big[e^{\gamma_2\sum_{x\in\mathbb{T}_N^d}\tilde{y}_x^2 1(|\tilde{y}_x|\le 2\delta)}\Big]$$
$$\le \frac{MH(\mu_t^N|\nu_t^N)}{\gamma_2} + \frac{M}{\gamma_2\ell^d}\sum_{x\in\mathbb{T}_N^d}\log E_{\nu_t^N}\Big[e^{\gamma_2\ell^d\tilde{y}_x^2 1(|\tilde{y}_x|\le 2\delta)}\Big], \tag{10.22}$$

using Lemma 10.3 where the grid spacing is $2\ell+1$. By Lemma 10.8, we have that

$$\log E_{\nu_t^N}\Big[e^{\gamma_2\ell^d\tilde{y}_x^2 1(|\tilde{y}_x|\le 2\delta)}\Big] \le \log\Big\{1 + E_{\nu_t^N}\Big[e^{\gamma_2\ell^d\tilde{y}_x^2}1(|\tilde{y}_x|\le 2\delta)\Big]\Big\}$$
$$\le E_{\nu_t^N}\Big[e^{\gamma_2\ell^d\tilde{y}_x^2}1(|\tilde{y}_x|\le 2\delta)\Big]$$

is uniformly bounded in ℓ for small γ_2,δ. Hence, the right-hand side of (10.22) is bounded $MH(\mu_t^N|\nu_t)/\gamma_2 + CMN^d/(\gamma_2\ell^d)$, finishing the argument. □

Finally, our last estimate bounds the conditional expectation in (10.11) when $|y_x|>\delta$.

Lemma 10.10. *We have, for small γ_3,δ and constant $c_1 = c_1(\delta,\gamma_3)>0$, that*

$$M\int_0^T \mathbb{E}_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d} E_{\nu_\beta}[|f_x||y_x|]1(\eta_x^\ell\le B)1(|y_x|>\delta)\Big]dt \tag{10.23}$$
$$\le \frac{M}{\gamma_3}\int_0^T H(\mu_t^N|\nu_t^N)dt + \frac{CTMKBN^d\ell^2}{\delta^2N^2} + \frac{CTMN^d}{\gamma_3\ell^d}e^{-c_1\ell^d}.$$

Proof. First, by our assumptions on f_x (cf (4.5)), using exchangeability of the canonical measure and the uniform bounds on u^N in Lemma 4.1, we have that

$$E[|f_x||y_x] \le C(|\Lambda_h|)\Big\{\eta_x^\ell + C\Big\}$$
$$= C(|\Lambda_h|)\Big\{\tilde{y}_x + \frac{1}{\ell_*^d}\sum_{|z-x|\le\ell}u^N(t,z) + C\Big\}$$
$$\le C\tilde{y}_x + C'.$$

Hence, we need only bound $ME_{\mu_t^N}\big[\sum_{x\in\mathbb{T}_N^d}\big(C\tilde{y}_x + C'\big)1(\eta_x^\ell\le B)1(|y_x|>\delta)\big]$. Since

$$1(|y_x|>\delta) \le 1\Big(\Big|\frac{1}{\ell_*^d}\sum_{|z-x|\le\ell}\big(u^N(t,x)-u^N(t,z)\big)\Big|>\delta/2\Big) + 1(|\tilde{y}_x|>\delta/2)$$

and $|\tilde{y}_x| 1(\eta_x^\ell \le B) \le B + \sup_x \|u^N(t,x)\|_{L^\infty} \le 2B$ say, by Lemma 4.1, for large N, in turn, we need only bound

$$(10.24) \qquad \big(1+\delta^{-1}\big)\int_0^T ME_{\mu_t^N}\Big[\sum_{x\in\mathbb{T}_N^d} |\tilde{y}_x| 1(|\tilde{y}_x|>\delta/2)\Big]dt$$
$$+\int_0^T \frac{8MB}{\delta^2}\sum_{x\in\mathbb{T}_N^d}\Big(\frac{1}{\ell_*^d}\sum_{|z-x|\le\ell}\big(u^N(t,x)-u^N(t,z)\big)\Big)^2 dt.$$

The second term in (10.24) is bounded by $CTMKBN^d\ell^2/\big(\delta^2N^2\big)$ via Lemma 4.2.

However, the integrand of the first expression in (10.24) is bounded, by Lemma 10.3 with grid spacing $2\ell+1$, by

$$\frac{MH(\mu_t^N|\nu_t^N)}{\gamma_3}+\frac{M}{\gamma_3\ell_*^d}\sum_{x\in\mathbb{T}_N^d}\log\Big(1-\nu_t^N(|y_x|>\delta/2)+E_{\nu_t^N}\Big[e^{\gamma_3\ell_*^d|\tilde{y}_x|}1(|\tilde{y}_x|>\delta/2)\Big]\Big).$$

By Schwarz inequality, we have

$$E_{\nu_t^N}\Big[e^{\gamma_3\ell_*^d|\tilde{y}_x|}1(|\tilde{y}_x|>\delta/2)\Big]\le\Big\{E_{\nu_t^N}\Big[e^{2\gamma_3\ell_*^d|\tilde{y}_x|}\Big]\cdot\nu_t^N(|\tilde{y}_x|>\delta/2)\Big\}^{1/2}.$$

Now, for $s>0$,

$$\begin{aligned}\nu_t^N(|\tilde{y}_x|>\delta) &\le E_{\nu_t^N}\big[e^{s\tilde{y}_x\ell^d}\big]e^{-s\ell^d\delta}+E_{\nu_t^N}\big[e^{-s\tilde{y}_x\ell^d}\big]e^{-s\ell^d\delta}\\ &\le \prod_{|z|\le\ell}E_{\nu_t^N}\big[e^{s(\eta_{x+z}-u^N(t,x+z))}\big]e^{-s\delta}+\prod_{|z|\le\ell}E_{\nu_t^N}\big[e^{-s(\eta_{x+z}-u^N(t,x+z))}\big]e^{-s\delta}.\end{aligned}$$

Moreover, recalling $\sigma_{z,t}^2=E_{\nu_t^N}[(\eta_z-u^N(t,z))^2]$, we have

$$\log E_{\nu_t^N}\big[e^{\pm s(\eta_y-u^N(t,y)}\big]=s^2\sigma_{y,t}^2/2+o(s^2).$$

Hence, with $s=\varepsilon\delta$ and $\varepsilon>0$ small, noting that $\sigma_{x+z,t}^2$ is uniformly bounded away from 0 and infinity by Lemma 4.1, we have

$$\nu_t^N\big(|\tilde{y}_x|>\delta\big)\le 2\prod_{|z|\le\ell}e^{-\delta^2(\varepsilon-\varepsilon^2\sigma_{x+z,t}^2/2)}\le e^{-c(\delta,\varepsilon)\ell^d}$$

for a small constant $c(\delta,\varepsilon)>0$

At the same time, for $\gamma_3>0$ small, as the means $u^N(t,\cdot)$ are uniformly bounded via Lemma 4.1 again, we have, in terms of $0\le\gamma_3'\le\gamma_3$, that

$$\begin{aligned}&\log E_{\nu_t^N}\big[e^{2\gamma_3\ell^d|\tilde{y}_x|}\big]\\ &\le \log\Big[\prod_{|z-x|\le\ell}E_{\nu_t^N}\big[e^{2\gamma_3(\eta_z-u^N(t,z))}\big]+\prod_{|z-x|\le\ell}E_{\nu_t^N}\big[e^{-2\gamma_3(\eta_z-u^N(t,z))}\big]\Big]\\ &\le C\gamma_3^2\sum_{|z-x|\le\ell}E_{\nu_t^N}\big[(\eta_z-u^N(t,z))^2e^{2\gamma_3'|\eta_z-u^N(t,z)|}\big]=O(\gamma_3^2\ell^d).\end{aligned}$$

Hence, we bound (10.24), taking $\gamma_3>0$ small enough compared to $c(\delta,\varepsilon)$, by

$$\frac{M}{\gamma_3}H(\mu_t^N|\nu_t^N)+\frac{MN^d}{\gamma_3\ell^d}e^{-c_1(\delta,\varepsilon,\gamma_3)\ell^d},$$

finishing the proof. □

11. L^∞ estimates for the discrete derivatives of the 'discrete PDE': Proof of Theorem 4.3

After some preliminary estimates, we prove the statements (4.2), (4.3) and (4.4) in Lemma 4.3 in succession.

To begin, we first summarize the definitions and simple properties of discrete derivatives. For $f = f(x)$, define

$$\nabla_i f(x) := f(x+e_i) - f(x), \quad \tau_y f(x) := f(x+y), \quad \Delta := -\sum_{j=1}^d \nabla_j \nabla_j^*,$$

where ∇_i^* is the dual operator of ∇_i with respect to the inner product

$$\langle f, g\rangle = \sum_{x\in\mathbb{T}_N^d} f(x)g(x)$$

given by

$$\nabla_i^* f(x) := f(x-e_i) - f(x).$$

Then, we have

$$\nabla_i^* = -\tau_{-e_i}\nabla_i, \quad [\tau_y, \nabla_i] \equiv \tau_y\nabla_i - \nabla_i\tau_y = 0, \quad [\nabla_i, \nabla_j] = 0, \quad [\nabla_i, \Delta] = 0,$$

and

$$\nabla_i(fg)(x) = (\tau_{e_i} g \cdot \nabla_i f + f\cdot \nabla_i g)(x)$$
$$\Big(= \tfrac{1}{2}\{(g+\tau_{e_i}g)\cdot\nabla_i f + (f+\tau_{e_i} f)\cdot \nabla_i g\}(x)\Big).$$

Further, let

$$\nabla_i^N := N\nabla_i, \quad \nabla_i^{N,*} := N\nabla_i^*,$$
$$\Delta^N := N^2\Delta = -\sum_{j=1}^d \nabla_j^N\nabla_j^{N,*} = \sum_{j=1}^d \nabla_j^N \tau_{-e_j}\nabla_j^N.$$

In the next lemma, we first rewrite the nonlinear Laplacian $\Delta^N\varphi(u)$ into a divergence form $L_a^N u$ with coefficients $a(x,e_j) = \varphi'(x,e_j;u)$ as in (11.1). Then, we compute the commutator $[\nabla_i^N, L_a^N]$ which has a gradient form given as in (11.3). Once we regard u in the coefficients of L_a^N as already given, L_a^N is a linear difference operator of second order.

Lemma 11.1. *We have*

$$\Delta^N\varphi(u)(x) = L_a^N u(x) := \sum_{j=1}^d \nabla_j^N\left\{\varphi'(x,e_j;u)\tau_{-e_j}\nabla_j^N u(x)\right\}, \tag{11.1}$$

where

$$\varphi'(x,e;u) := \frac{\varphi(u(x)) - \varphi(u(x-e))}{u(x)-u(x-e)}. \tag{11.2}$$

We also have

$$\nabla_i^N\Delta^N\varphi(u)(x) = L_a^N\nabla_i^N u(x) + \sum_{j=1}^d \nabla_j^N\left\{\nabla_i^N\varphi'(x,e_j;u)\tau_{e_i-e_j}\nabla_i^N u(x)\right\}, \tag{11.3}$$

and

$$|\nabla_i^N \varphi'(x,e_j;u)| \le C\{|\nabla_i^N u(x)| + |\tau_{e_i-e_j}\nabla_j^N u(x)| + |\tau_{-e_j}\nabla_j^N u(x)|\}. \tag{11.4}$$

Proof. Recalling the definition of $\varphi'(x,e_j;u)$, the first identity (11.1) is shown as

$$\begin{aligned}
\Delta^N \varphi(u)(x) &= \sum_{j=1}^d \nabla_j^N \cdot N\left\{\varphi(u(x)) - \varphi(u(x-e_j))\right\} \\
&= \sum_{j=1}^d \nabla_j^N \left\{\varphi'(x,e_j;u) N(u(x)-u(x-e_j))\right\} \\
&= \sum_{j=1}^d \nabla_j^N \left\{\varphi'(x,e_j;u)\tau_{-e_j}\nabla_j^N u(x)\right\}.
\end{aligned}$$

The second identity (11.3) is shown as

$$\begin{aligned}
&\nabla_i^N \Delta^N \varphi(u)(x) = \sum_{j=1}^d \nabla_i^N \nabla_j^N \left\{\varphi'(x,e_j;u)\tau_{-e_j}\nabla_j^N u(x)\right\} \\
&= \sum_{j=1}^d \nabla_j^N \nabla_i^N \left\{\varphi'(x,e_j;u)\tau_{-e_j}\nabla_j^N u(x)\right\} \\
&= \sum_{j=1}^d \nabla_j^N \left\{\varphi'(x,e_j;u)\nabla_i^N\tau_{-e_j}\nabla_j^N u(x) + \tau_{e_i}\tau_{-e_j}\nabla_j^N u(x)\cdot\nabla_i^N\varphi'(x,e_j;u)\right\} \\
&= L_a^N \nabla_i^N u(x) + \sum_{j=1}^d \tau_{e_i-e_j}\nabla_j^N u(x)\cdot \nabla_i^N \varphi'(x,e_j;u).
\end{aligned}$$

Finally, to show the bound (11.4), by Taylor's formula, one can find u_1^*, u_2^*, u_3^* such that

$$\begin{aligned}
\nabla_i^N \varphi'(x,e_j;u) &= N\left\{\frac{\varphi(u(x+e_i)) - \varphi(u(x+e_i-e_j))}{u(x+e_i)-u(x+e_i-e_j)} - \frac{\varphi(u(x))-\varphi(u(x-e_j))}{u(x)-u(x-e_j)}\right\} \\
&= N\Big\{\varphi'(u(x+e_i)) + \tfrac12\varphi''(u_1^*)(u(x+e_i-e_j)-u(x+e_i)) \\
&\qquad - \varphi'(u(x)) - \tfrac12\varphi''(u_2^*)(u(x-e_j)-u(x))\Big\} \\
&= \varphi''(u_3^*)\nabla_i^N u(x) + \tfrac12\varphi''(u_1^*)\nabla_j^{N,*}\tau_{e_i}u(x) - \tfrac12\varphi''(u_2^*)\nabla_j^{N,*}u(x).
\end{aligned}$$

Noting that $|\varphi''(u)| \le C$ for $u \in [u_-,u_+]$, we obtain the bound (11.4). □

Remark 11.1. The lines (11.1) and (11.3) in Lemma 11.1 correspond to the following simple continuum setting identities

$$\begin{aligned}
&\Delta\varphi(u)(x) = \operatorname{div}\big(\varphi'(u(x))\nabla u(x)\big), \\
&\partial_{x_i}\Delta\varphi(u)(x) = \operatorname{div}\big(\varphi'(u(x))\nabla\partial_{x_i}u(x)\big) + \operatorname{div}\big(\varphi''(u(x))\partial_{x_i}u(x)\nabla u(x)\big), x\in\mathbb{R}^d.
\end{aligned}$$

From (11.1), we may regard the nonlinear Laplacian $\Delta^N\varphi(u)$ as a linear Laplacian $L_{a(t)}^N u$ with coefficients $a(t,x,e) = \varphi'(x,e;u^N(t))$:

$$\Delta^N\varphi(u)(x) = L_a^N u(x) = N^2 \sum_{e:|e|=1} a(x,e;u)\{u(x+e)-u(x)\}.$$

The coefficient $a(x,e;u) \equiv \varphi'(x+e,e;u)$ is defined in (11.2) and determined by the mean value theorem as

$$\varphi(u(x+e)) - \varphi(u(x)) = \varphi'(u_*)(u(x+e) - u(x)),$$

with some $u_* > 0$ taken between $u(x+e)$ and $u(x)$.

This coefficient a is symmetric: $a(x,e;u) = a(x+e,-e;u)$, positive, and bounded:

$$0 < \varphi'(u_- \wedge \alpha_-) \le a \le \varphi'(u_+ \vee \alpha_+) < \infty.$$

In particular, L_a^N is symmetric, $(L_a^N)^* = L_a^N$.

Thus, by Theorem 1.1 of [13] (with non-random but spatially and temporally inhomogeneous coefficients), $L^N_{a(\cdot,\cdot;u^N(\cdot))}$ has a fundamental solution

$$p^N(s,y,t,x) \equiv p_{a(\cdot,\cdot;u^N(\cdot))}(s,y,t,x),$$

satisfying the bound:

$$\big|\nabla_x^N p^N(0,y,t,x)\big|, \big|\nabla_y^N p^N(0,y,t,x)\big| \le \frac{C}{\sqrt{t}}\,\bar p^N(ct, x-y), \quad t \in (0,T], \tag{11.5}$$

where $C = C_T > 0$ and $\bar p^N(t,x-y) = p^N_{a^*}(0,x,t,y) (= p^N_{a^*}(0,y,t,x))$ with $a^* = 1_{|e|=1}$; see also Lemma 4.2 of [29]. This bound for $\nabla_y^N p^N$ is derived by thinking of $p^N(s,y,t,x)$ as the fundamental solution of the symmetric operator in (s,y) in the backward sense.

From Duhamel's formula, with respect to the discrete PDE (2.18), we have

$$u^N(t,x) = \sum_{y\in\mathbb{T}_N^d} u^N(0,y)p^N(0,y,t,x) + K\int_0^t ds \sum_{y\in\mathbb{T}_N^d} f(u^N(s,y))p^N(s,y,t,x). \tag{11.6}$$

We denote the first and the second terms in the right hand side of (11.6) by $I_1^N(t,x)$ and $I_2^N(t,x)$, respectively.

By noting $f(u^N(s,y))$ is bounded from Lemma 4.1 and (BIP1) and applying (11.5), we obtain

$$|\nabla_i^N I_1^N(t,x)| \le \frac{C}{\sqrt{t}}, \quad t \in (0,T], \tag{11.7}$$

and also, by applying (11.5) by shifting by s, we have

$$|\nabla_i^N I_2^N(t,x)| \le KC\sqrt{t}, \quad t \in [0,T], \tag{11.8}$$

with $C = C_T$.

11.1. **Proof of** (4.2)**.** To improve (11.7) and obtain (4.2), we consider the equation for $\nabla_i^N u^N(t,x)$ by applying the discrete derivative ∇_i^N to (2.18) with $f \equiv 0$–in other words, we are concerned only with the first term $\nabla_i^N I_1^N(t,x)$ acted on by ∇_i^N.

Indeed, to derive a better estimate for $\nabla_i^N I_1^N(t,x)$ than (11.7) for small $t > 0$, we use (11.3) in Lemma 11.1 for the right hnad side of the equation $\partial_t \nabla_i^N u^N(t,x) = \nabla_i^N \Delta^N \varphi(u^N(t,x))$, Duhamel's formula and summation by parts in y to obtain

$$\nabla_i^N u^N(t,x) = \sum_{y\in\mathbb{T}_N^d} \nabla_i^N u^N(0,y)p^N(0,y,t,x) \tag{11.9}$$

$$+\sum_{j=1}^{d}\int_0^t ds \sum_{y\in\mathbb{T}_N^d} \nabla_i^N \varphi'(y,e_j;u^N(s))\tau_{e_i-e_j}\nabla_i^N u(s,y)\cdot \nabla_{j,y}^{N,*} p^N(s,y,t,x).$$

Let $v(t,x) := |\nabla^N u^N(t,x)|$. Applying estimates (11.4) and (11.5) show that

$$\|v(t)\|_{L^\infty} \le C\|v(0)\|_{L^\infty} + C\int_0^t \frac{\|v(s)\|_{L^\infty}^2}{\sqrt{t-s}} ds. \tag{11.10}$$

Let $m(t) := \sup_{0\le s\le t} \|v(s)\|_{L^\infty}$. By (11.10) and the bound $\|v(0)\|_{L^\infty} \le C_0 K$ assumed in Theorem 4.3, we have

$$0 \le m(t) \le C_T C_0 K + C_T m(t)^2 2\sqrt{t}, \quad t\in[0,T].$$

This proves

$$m(t) \le \frac{1-\sqrt{1-8\sqrt{t}C_T^2 C_0 K}}{4C_T\sqrt{t}}, \quad \text{for } \sqrt{t}\le\sqrt{t_*} := \frac{1}{8C_T^2 C_0 K}.$$

The right-side is increasing in $\sqrt{t}$ so that it is bounded by the value at $t=t_*$. Thus, we obtain the bound

$$m(t) \le 2C_T C_0 K, \quad t\le t_*.$$

From (11.7), we also have

$$m(t) \le \frac{C_T}{\sqrt{t}}, \quad t\in(0,T]. \tag{11.11}$$

From these two bounds, especially using (11.11) for $t>t_*$, we obtain

$$m(t) \le CK, \quad t\in[0,T],$$

which, when combined with (11.8), proves (4.2). □

11.2. **Proof of** (4.3)**.** With respect to the discrete PDE (2.18), using (11.3), we have for $u^N = u^N(t,x)$ that

$$\begin{aligned}\partial_t \nabla_{i_2}^N\nabla_{i_1}^N u^N &= \nabla_{i_2}^N\nabla_{i_1}^N \Delta^N \varphi(u^N) + K\nabla_{i_2}^N\nabla_{i_1}^N f(u^N)\\ &= \nabla_{i_2}^N L_a \nabla_{i_1}^N u^N + \sum_j \nabla_{i_2}^N\nabla_j^N\{\nabla_{i_1}^N\varphi'(x,e_j;u^N)\tau_{e_{i_1}-e_j}\nabla_{i_1}^N u^N\}\\ &\quad + K\nabla_{i_2}^N\{f'(x,e_{i_1};u^N)\nabla_{i_1}^N u^N\}.\end{aligned} \tag{11.12}$$

Here, $f'(x,e;u)$ is defined in a similar way as $\varphi'(x,e;u)$ in (11.2). Consider the first term in the right-side of (11.12). To exchange $\nabla_{i_2}^N$ and L_a, we compute the commutator $[\nabla_i^N, L_a]$:

$$\begin{aligned}\nabla_i^N L_a u(x) &= \sum_{j=1}^d \nabla_j^N \nabla_i^N\{\varphi'(x,e_j;u)\tau_{-e_j}\nabla_j^N u(x)\}\\ &= \sum_{j=1}^d \nabla_j^N\{\tau_{e_i-e_j}\nabla_j^N u(x)\cdot\nabla_i^N\varphi'(x,e_j;u) + \varphi'(x,e_j;u)\nabla_i^N\tau_{-e_j}\nabla_j^N u(x)\}\\ &= L_a\nabla_i^N u(x) + \sum_{j=1}^d \nabla_j^N\{\tau_{e_i-e_j}\nabla_j^N u(x)\cdot\nabla_i^N\varphi'(x,e_j;u)\}.\end{aligned} \tag{11.13}$$

For the second term in the right-side of (11.12), we expand

$$\tag{11.14} \begin{aligned} &\nabla_{i_2}^N\{\nabla_{i_1}^N\varphi'(x,e_j;u^N)\tau_{e_{i_1}-e_j}\nabla_{i_1}^N u^N\} \\ &= \nabla_{i_2}^N\nabla_{i_1}^N\varphi'(x,e_j;u^N)\tau_{e_{i_1}-e_j+e_{i_2}}\nabla_{i_1}^N u^N + \nabla_{i_1}^N\varphi'(x,e_j;u^N)\tau_{e_{i_1}-e_j}\nabla_{i_2}^N\nabla_{i_1}^N u^N. \end{aligned}$$

Here, similar to the proof of (11.4), we now claim

$$\tag{11.15} \begin{aligned} &\nabla_{i_2}^N\nabla_{i_1}^N\varphi'(x,e_j;u) \\ &\qquad = \tfrac{3}{2}\varphi''(u(x))\nabla_{i_2}^N\nabla_{i_1}^N u(x) - \tfrac{1}{2}\varphi''(u(x))\nabla_{i_2}^N\nabla_{i_1}^N u(x-e_j) + R^N(x). \end{aligned}$$

where $R^N(x)$ is a quadratic function of $\{\nabla_i^N u^N(z); i=i_1,i_2,j, z=x,x+e_{i_1},x+e_{i_2},x-e_j,x+e_{i_1}-e_j,x+e_{i_2}-e_j,x+e_{i_1}+e_{i_2}-e_j\}$, which is explicitly given below.

Indeed, the left hand side of (11.15) is given as

$$\begin{aligned} &N^2\Big\{\frac{\varphi(u(x+e_{i_2}+e_{i_1}))-\varphi(u(x+e_{i_2}+e_{i_1}-e_j))}{u(x+e_{i_2}+e_{i_1})-u(x+e_{i_2}+e_{i_1}-e_j)} \\ &\quad -\frac{\varphi(u(x+e_{i_1}))-\varphi(u(x+e_{i_1}-e_j))}{u(x+e_{i_1})-u(x+e_{i_1}-e_j)} \\ &\quad -\frac{\varphi(u(x+e_{i_2}))-\varphi(u(x+e_{i_2}-e_j))}{u(x+e_{i_2})-u(x+e_{i_2}-e_j)} + \frac{\varphi(u(x))-\varphi(u(x-e_j))}{u(x)-u(x-e_j)}\Big\} \\ &= N^2\Big\{\varphi'(u(x+e_{i_2}+e_{i_1})) \\ &\qquad + \tfrac{1}{2}\varphi''(u(x+e_{i_2}+e_{i_1}))(u(x+e_{i_2}+e_{i_1})-u(x+e_{i_2}+e_{i_1}-e_j)) \\ &\qquad + \tfrac{1}{6}\varphi'''(u_1^*)(u(x+e_{i_2}+e_{i_1})-u(x+e_{i_2}+e_{i_1}-e_j))^2 \\ &\quad - \varphi'(u(x+e_{i_1})) - \tfrac{1}{2}\varphi''(u(x+e_{i_1}))(u(x+e_{i_1})-u(x+e_{i_1}-e_j)) \\ &\qquad - \tfrac{1}{6}\varphi'''(u_2^*)(u(x+e_{i_1})-u(x+e_{i_1}-e_j))^2 \\ &\quad - \varphi'(u(x+e_{i_2})) - \tfrac{1}{2}\varphi''(u(x+e_{i_2}))(u(x+e_{i_2})-u(x+e_{i_2}-e_j)) \\ &\qquad - \tfrac{1}{6}\varphi'''(u_3^*)(u(x+e_{i_2})-u(x+e_{i_2}-e_j))^2 \\ &\quad + \varphi'(u(x)) + \tfrac{1}{2}\varphi''(u(x))(u(x)-u(x-e_j)) - \tfrac{1}{6}\varphi'''(u_4^*)(u(x)-u(x-e_j))^2\Big\}. \end{aligned}$$

The terms containing $\frac{1}{6}\varphi'''$ are summarized as

$$\begin{aligned} R_1^N(x) :=& \tfrac{1}{6}\varphi'''(u_1^*)(\nabla_j^{N,*}u(x+e_{i_2}+e_{i_1}))^2 - \tfrac{1}{6}\varphi'''(u_2^*)(\nabla_j^{N,*}u(x+e_{i_1}))^2 \\ &- \tfrac{1}{6}\varphi'''(u_3^*)(\nabla_j^{N,*}u(x+e_{i_2}))^2 + \tfrac{1}{6}\varphi'''(u_4^*)(\nabla_j^{N,*}u(x))^2, \end{aligned}$$

and this is a quadratic function of $\{\nabla_j^N u^N(z); z=x-e_j,x+e_{i_1}-e_j,x+e_{i_2}-e_j,x+e_{i_1}+e_{i_2}-e_j\}$.

The terms containing φ' are summarized as

$$\begin{aligned} &N^2\{\varphi'(u(x+e_{i_2}+e_{i_1}))-\varphi'(u(x+e_{i_1}))-\varphi'(u(x+e_{i_2}))+\varphi'(u(x))\} \\ &= N^2\Big\{\varphi'(u(x))+\varphi''(u(x))(u(x+e_{i_2}+e_{i_1})-u(x)) \\ &\quad + \tfrac{1}{2}\varphi'''(u_5^*)(u(x+e_{i_2}+e_{i_1})-u(x))^2 \\ &\quad - \varphi'(u(x))-\varphi''(u(x))(u(x+e_{i_1})-u(x)) + \tfrac{1}{2}\varphi'''(u_6^*)(u(x+e_{i_1})-u(x))^2 \\ &\quad - \varphi'(u(x))-\varphi''(u(x))(u(x+e_{i_2})-u(x)) \end{aligned}$$

$$
\begin{aligned}
&\qquad + \tfrac{1}{2}\varphi'''(u_7^*)(u(x+e_{i_2})-u(x))^2 + \varphi'(u(x))\Big\} \\
&= N^2\varphi''(u(x))\big(u(x+e_{i_2}+e_{i_1})-u(x+e_{i_1})-u(x+e_{i_2})+u(x)\big) + R_2^N(x) \\
&= \varphi''(u(x))\nabla_{i_2}^N\nabla_{i_1}^N u(x) + R_2^N(x),
\end{aligned}
$$

where $R_2^N(x)$ is a quadratic function of $\{\nabla_{i_1}^N u^N(x), \nabla_{i_2}^N u^N(x), \nabla_{i_1}^N u^N(x+e_{i_2})\}$.

The terms containing $\frac{1}{2}\varphi''$ are summarized as

$$
\begin{aligned}
&\tfrac{1}{2}N^2\Big\{\varphi''(u(x+e_{i_2}+e_{i_1}))(u(x+e_{i_2}+e_{i_1})-u(x+e_{i_2}+e_{i_1}-e_j)) \\
&\quad - \varphi''(u(x+e_{i_1}))(u(x+e_{i_1})-u(x+e_{i_1}-e_j)) \\
&\quad - \varphi''(u(x+e_{i_2}))(u(x+e_{i_2})-u(x+e_{i_2}-e_j)) + \varphi''(u(x))(u(x)-u(x-e_j))\Big\} \\
&= \tfrac{1}{2}N^2\varphi''(u(x))\Big\{(u(x+e_{i_2}+e_{i_1})-u(x+e_{i_2}+e_{i_1}-e_j)) \\
&\qquad - (u(x+e_{i_1})-u(x+e_{i_1}-e_j)) \\
&\qquad - (u(x+e_{i_2})-u(x+e_{i_2}-e_j)) + (u(x)-u(x-e_j))\Big\} + R_3^N(x) \\
&= \tfrac{1}{2}\varphi''(u(x))\big\{\nabla_{i_2}^N\nabla_{i_1}^N u(x) - \nabla_{i_2}^N\nabla_{i_1}^N u(x-e_j)\big\} + R_3^N(x),
\end{aligned}
$$

where $R_3^N(x)$ is a quadratic function of

$$
\begin{aligned}
&\{\nabla_{i_1}^N u^N(x), \nabla_{i_2}^N u^N(x), \nabla_{i_1}^N u^N(x+e_{i_2}), \nabla_j^N u^N(z) \\
&\qquad : z = x+e_{i_1}-e_j, x+e_{i_2}-e_j, x+e_{i_1}+e_{i_2}-e_j\}.
\end{aligned}
$$

This completes the proof of (11.15) with $R^N(x) = R_1^N(x) + R_2^N(x) + R_3^N(x)$.

Now we come back to the proof of (4.3). Insert (11.13) taking $i = i_2$ and $u = \nabla_{i_1}^N u^N$ into the first term of (11.12). Then, we see

$$
\partial_t \nabla_{i_2}^N\nabla_{i_1}^N u^N = L_a \nabla_{i_2}^N\nabla_{i_1}^N u^N + Q^N(x),
$$

where the remainder term $Q^N(x)$ is the sum of the second and third terms in the right side of (11.12) and the last term in (11.13).

Then, by Duhamel's formula, we obtain for $v_{i_1,i_2}(t,x) := \nabla_{i_2}^N\nabla_{i_1}^N u^N(t,x)$,

$$
\begin{aligned}
v_{i_1,i_2}(t,x) &= \sum_y v_{i_1,i_2}(0,y)p^N(0,y,t,x) \\
&+ \int_0^t ds \sum_y \sum_{j=1}^d \{\tau_{e_i-e_j}\nabla_j^N\nabla_i^N u^N(s,y)\cdot\nabla_i^N\varphi'(y,e_j;u^N(s))\}\nabla_j^{N,*}p^N(s,y,t,x) \\
&+ \int_0^t ds \sum_y \nabla_{i_2}^N\{\nabla_{i_1}^N\varphi'(y,e_j;u^N(s))\tau_{e_i-e_j}\nabla_{i_1}^N u^N\}(s,y)\nabla_j^{N,*}p^N(s,y,t,x) \\
&+ K\int_0^t ds \sum_y f'(y,e_{i_1};u^N(s))\nabla_{i_1}^N u^N(s,y)\nabla_{i_2}^{N,*}p^N(s,y,t,x).
\end{aligned}
$$

Let $v(t,x) = \sum_{i_1,i_2}|v_{i_1,i_2}(t,x)|$ and note that $|\nabla^N u^N(s,y)| \le CK$ was shown in (4.2). We have from (11.4), (11.5), (11.14) and (11.15) that

$$
\|v(t)\|_{L^\infty} \le C\|v(0)\|_{L^\infty} + CK\int_0^t \frac{ds}{\sqrt{t-s}}\|v(s)\|_{L^\infty} + CK^3. \tag{11.16}
$$

This implies, by the argument in [24], p. 144, that

$$\|v(t)\|_{L^\infty} \le C(\|v(0)\|_{L^\infty} + K^3)e^{CK^2 t}, \tag{11.17}$$

concluding the proof of (4.3). □

Remark 11.2. Instead of using the argument in [24], we may give a simple method to derive (11.17) with $e^{CK^p t}, p > 2$ in place of $e^{CK^2 t}$. We apply Hölder's inequality to (11.16), taking $p > 2, 1 < q < 2$ so that $\frac{1}{p} + \frac{1}{q} = 1$, to get

$$\|v(t)\|_{L^\infty}^p \le C(\|v(0)\|_{L^\infty} + K^3)^p + CK^p \left(\int_0^t (t-s)^{-\frac{q}{2}} ds \right)^{p/q} \int_0^t \|v(s)\|_{L^\infty}^p ds.$$

Gronwall's inequality now shows the desired estimate for $\|v(t)\|_{L^\infty}$ with $e^{CK^p t}$ in place of $e^{CK^2 t}$ in (11.17).

Remark 11.3. For the linear Laplacian (i.e., $\varphi(u) = cu$), we have $\nabla_x^N p^N = \nabla_y^N p^N$ due to $[\nabla, \Delta] = 0$ or $p^N = p^N(t-s, x-y)$ so that the computations made in Lemma 11.1 are unnecessary. In Lemma 11.1, especially (11.3), the second term, which is the error term obtained by computing $[\nabla_i^N, L_a^N]$, is of the form of a gradient. This is important to make the summation by parts in y and move the discrete derivative ∇_j^N to p.

11.3. **Proof of** (4.4)**.** Equation (4.4) is now a straightforward consequence of (4.2), (4.3), (11.1) and (11.4). □

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