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# On Hydrodynamic Limits of Young Diagrams

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**Abstract**

We consider a family of stochastic models of evolving two-dimensional Young diagrams, given in terms of certain energies, with Gibbs invariant measures. 'Static' scaling limits of the shape functions, under these Gibbs measures, have been shown in the literature. The purpose of this article is to study corresponding, but less understood, 'dynamical' limits. We show that the hydrodynamic scaling limits of the diagram shape functions may be described by different types of parabolic PDEs, depending on the energy structure.

**Keywords:** Young diagram; Gibbs measure; interacting particle system; zero-range; weakly; hydrodynamic; shape; dynamic.
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## 1 Introduction

Young diagrams or tableaux, originally introduced in the context of combinatorics and representation theory (cf. [11], [29]), have proved to be useful in a variety of disciplines ranging from mathematical physics to genetics. In particular, language involving Young diagrams and their shape functions may be used to describe phenomena such as Bose-Einstein condensation [8], polymerization and molecular assembly [5], [17], and random partitions in coagulation-fragmentation processes [2], [23], and references therein, among others.

In this paper, we present a family of stochastic evolutions of two-dimensional Young diagrams, with invariant Gibbs measures given in terms of certain energy structures, and show that the hydrodynamic scaling limits of the associated shape functions obey different types of parabolic PDEs, reflecting the type of the energy formulation. Previously, there seems to be only a small literature studying dynamical Young diagrams, for instance [7], [13] and [14], which treat processes where there is birth and death evolution of squares in the diagrams. See also the monograph [12] which reviews some of this work. The purpose of this article is to analyze a natural, but different class of

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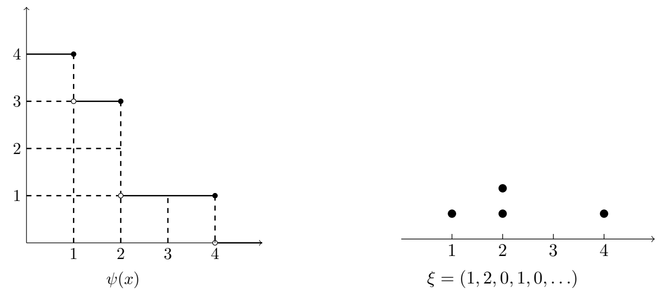


Figure 1: The Young diagram and particle description associated with the partition $(4, 2, 2, 1)$.

models, through new techniques which might be of use in other settings. Later, we give a brief comparison with the results in [12], [13], and as well as those in [7], [14], the former pair however closest to ours in spirit.

To describe our results, we first discuss certain 'static' limits, which set the stage. Let $\varphi = (p_1, p_2, \ldots, p_n)$ with $p_m \geq p_{m+1}$ be a partition of the integer $M(\varphi) := \sum_{m=1}^{n} p_m$. For example, $\varphi = (4, 2, 2, 1)$ corresponds to $9 = 4 + 2 + 2 + 1$. We call $\xi = (\xi(k; \varphi))_{k \in \mathbb{N}}$, where $\xi(k; \varphi) = \#\{m : p_m = k\}$, the size density of the partition $\varphi$. Vice versa, given $\xi$, one can reconstruct $\varphi$, and so in a sense they are interchangeable. In terms of $\xi$, $M(\varphi) = \sum_{k \geq 1} k\xi(k; \varphi)$. Denote by $\psi(x)$ the associated shape (height) function:

$$\psi(x) = \sum_{k \geq x} \xi(k; \varphi).$$

The graph of $\psi$ is the Young diagram of $\varphi$. Since $\xi(k; \varphi) = \psi(k) - \psi(k+1)$, the numbers $\xi$ can be viewed as the gradient particle description of the associated partition $\varphi$. See Figure 1.

Let $\mathscr{P}_M$ be the uniform probability measure on all partitions of an integer $M$. A classical result of A. Vershik [26] states that in the limit as $M \to \infty$, the rescaled shape functions $\psi_M(x) := \psi(x\sqrt{M})/\sqrt{M}$ converge in probability with respect to the canonical measure $\mathscr{P}_M$ to the curve

$$\psi(x) = -\frac{\sqrt{6}}{\pi} \ln\left(1 - e^{-\pi x/\sqrt{6}}\right). \tag{1.1}$$

Namely, for every $\epsilon > 0$ and $a > 0$,

$$\lim_{M \to \infty} \mathscr{P}_M\left(\sup_{x \geq a} \left|\psi_M(x) - \psi(x)\right| > \epsilon\right) = 0.$$

Such results have a long history, and limits and phenomena different than the one above may appear if other ensembles, such as those with respect to Haar statistics, the Plancherel measure or Ewens measure are employed: see [3], [6], [9], [22], [18], [24], [25], [26], [27], [28], and references therein.

In this article, we will consider grand canonical ensembles of sizes $\{\xi(k) : k \geq 1\}$, including those prescribed in [9]:

$$\mathscr{P}_{\beta,N}(\xi) = \frac{1}{Z_{\beta,N}} e^{-\beta \sum_{k \geq 1} \xi(k)\mathcal{E}_k - N^{-1}M}$$

where $\mathcal{E}_k \geq 0$ is the energy of a summand of size $k$, total size $M = \sum_{k \geq 1} k\xi(k)$, inverse temperature $\beta \geq 0$, $Z_{\beta,N}$ is the normalizing factor, and $N$ is a scaling parameter. When

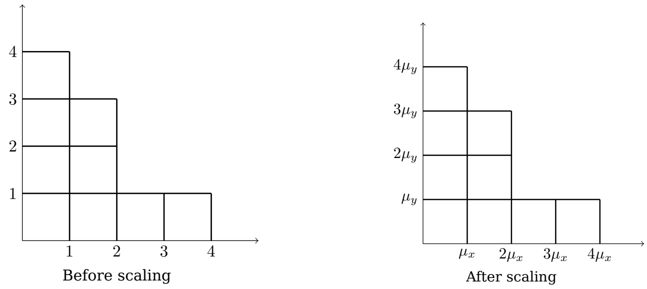


Figure 2: Young diagrams before and after rescaling. $\mu_x = 1/N$, $\mu_y = N/R_{\beta,N}(M)$ in the rescaling from $\psi$ to $\psi_{\beta,N}$.

$\beta = 0$, the canonical, or conditional measures, with size $M$, are of course $\mathscr{P}_M$. As discussed in [9], such ensembles may be understood to govern the size distributions of structures in polymeric melts. In this 'melt', a 'polymer' of size $k$ has an energy $\mathcal{E}_k$.

The scaled shape function $\psi_{\beta,N}(x) := N\psi(Nx)/R_{\beta,N}$, where $R_{\beta,N} = N^2 e^{-\beta \mathcal{E}_N}$, as shown in [9], is of the order of the expected value of $M = \sum_{k\geq 1} k\xi(k) = \int_0^\infty \psi(x)dx$ with respect to $\mathscr{P}_{\beta,N}$. This scaling is such that the expected area of the rescaled Young diagrams, $\mathbb{E}_{\mathscr{P}_{\beta,N}}\left[\int_0^\infty \psi_{\beta,N}(x)dx\right]$ is of order 1; see Figure 2. As $N \to \infty$, $\psi_{\beta,N}(x)$ will converge with respect to $\mathscr{P}_{\beta,N}$ to different limits, depending on the choice of the energy $\mathcal{E}_k$.

Following [9], we assume that the energy function $\mathcal{E}_k$ is in form $\mathcal{E}_k = u(\ln k)$, where $u$ is a positive function diverging at infinity. In particular, we consider two cases in this work: (1) $u'(x) \to 1$, and (2) $u'(x) \to 0$. We refer to these cases as $\mathcal{E}_k \sim \ln k$, and $1 \ll \mathcal{E}_k \ll \ln k$ respectively. The precise specification later given in Condition 2.1 provides a large, varied class of energies, amenable to the scaling limits that we will take.

We remark, if $\mathcal{E}_k$ is not in this form, for instance the case $\mathcal{E}_k \gg \ln k$, there will be a finite number of particles, uniform over $N$, in the system (cf. Proposition 2.1 in [9]), and so the associated scaling limits will be trivial. Also, if $\mathcal{E}_k$ is constant, the situation is tantamount to taking $\beta = 0$, and so we do not distinguish this case. Furthermore, when $\mathcal{E}_k \sim \ln k$ and $\beta > 1$, the variance of the scaled shape function $\psi_{\beta,N}$ diverges, and does not vanish for $\beta = 1$ (cf. Proposition 2.4 in [9]). There are also other interesting 'boundary' energy scenarios discussed in [9], including condensation regimes, which we do not pursue here.

The following convergences follow from Propositions 2.1 and 2.2 of [9]: For $\epsilon > 0$,

1. $\beta = 0$: $\mathscr{P}_{\beta,N}\left(\left|\psi_{\beta,N} - \ln(1-e^{-x})\right| > \epsilon\right) \to 0$;
2. $\mathcal{E}_k \sim \ln k$, $0 < \beta < 1$: $\mathscr{P}_{\beta,N}\left(\left|\psi_{\beta,N} - \int_x^\infty u^{-\beta}e^{-u}du\right| > \epsilon\right) \to 0$;
3. $1 \ll \mathcal{E}_k \ll \ln k$, $\beta > 0$: $\mathscr{P}_{\beta,N}\left(\left|\psi_{\beta,N} - e^{-x}\right| > \epsilon\right) \to 0$.

We remark, the limit when $\beta = 0$, is similar to Vershik's result, and in some sense, a reflection of the equivalence of ensembles between the canonical measures $\mathscr{P}_M$ and $\mathscr{P}_{0,N}$ as $M$ and $N$ diverge.

With this background, the purpose of the article is to consider a natural dynamics of these varied shapes and to understand their hydrodynamic limits. Consider the gradient particle system associated with the Young diagrams with generator

$$Lf(\xi) = \sum_{k=1}^{\infty} \Big\{ \lambda_k \left[ f\left(\xi^{k,k+1}\right) - f(\xi)\right] \chi_{\{\xi(k)>0\}} + \left[ f\left(\xi^{k,k-1}\right) - f(\xi)\right] \chi_{\{\xi(k)>0, k>1\}} \Big\}$$

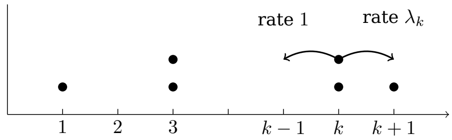


Figure 3: Gradient particle system: Particles at sites $k \geq 2$ move to the left with rate $1$, to the right with rate $\lambda_k$; particles at $k=1$, move only to the right with rate $\lambda_1$.

where $\lambda_k = e^{-\beta(\mathcal{E}_{k+1}-\mathcal{E}_k)-1/N}$ (cf. Figure 3). Here, $\xi^{k,k\pm1}$ is the configuration obtained by moving a particle from $k$ to $k \pm 1$.

The interpretation of this dynamics, which preserves particle mass, in the 'language of polymers' is as follows: A monomer is added to a polymer of size $k$ with rate $\lambda_k$ and removed with rate $1$. In this dynamics, the gradients $\xi$ qualitatively tend to states of lower energy $\mathcal{E}_\cdot$. This dynamics is spatially inhomogeneous when $\beta > 0$ in that $\lambda_k \neq \lambda_{k+1}$, and is not translation-invariant in general, being limited to $\mathbb{Z}^+$, rather than $\mathbb{Z}$. An important feature is that the grand canonical measures $\mathscr{P}_{\beta,N}$ are invariant under $L$. In fact, as $\lambda_k$ and the generator $L$ depend only on the energy difference $\beta\mathcal{E}_{k+1} - \beta\mathcal{E}_k$, all grand canonical ensembles characterized by energies which differ from $\beta\mathcal{E}_\cdot$ by a constant are invariant under the same dynamics. Thus, for a given $L$, there will be a family of invariant measures indexed in terms of these constants (cf. Section 2.1).

Moreover, in terms of the associated Young diagrams, an 'empty' lower left corner, adjacent to three squares, with vertex at $(k,\cdot)$ is filled with a square with rate $\lambda_k$, and a square, with an upper right corner not adjacent to any other square, is removed with rate $1$; for instance, in Figure 1, turning the empty corner at $(1,3)$ into a square corresponds with the particle at $k=1$ moving to location $k=2$, and removing the square with corner $(2,3)$ means a particle at $k=2$ moves to $k=1$.

From a more applied view, motivating our study are various polymeric melts which exhibit aggregation, or condensation behaviors. In particular, rod assembly in chromonic liquid crystals has attracted significant attention in recent years. In these systems disc-shaped monomers with flat hydrophobic faces form stacks (rods) to minimize the total (free) energy penalty. Much of the physics literature, for instance [4], [5], [10], [17], describes models of varying degrees of complexity, however, most of them are kinetic or thermodynamic in nature, that is they describe evolution of various averaged quantities and do not deal with specific microscopic statistical ensembles. In this view, the dynamics generated by $L$ provides a microscopic description grounded on Gibbs grand canonical ensembles $\mathscr{P}_{\beta,N}$ and specific monomer association and dissociation rates. In particular, we consider simple linear polymers (or aggregates) which may absorb or release monomers from or into solution at certain rates related to the energies of aggregates, which remain essentially constant (or growing slowly, that is logarithmically or sub-logarithmically) with size, reflecting that the principal contribution into the aggregate energy comes from the hydrophobic edge length of a linear structure (cf. Figure 3 in [9]). In this context, our work provides a rigorous mathematical connection between the microscopic model and the macroscopic equations for a class of evolutions of the aggregate sizes in the form of a hydrodynamic limit.

Let $\xi_t$ denote the associated Markov process. We will be interested in the process $\eta_t = \xi_{N^2 t}$ seen in diffusive scale, where time is speeded up by $N^2$ and space by $N$. Since $\eta_t$ is viewed as the negative gradient of its corresponding height function $\psi$, the scaling from $\psi$ to $\psi_{\beta,N}$ (cf. Figure 2) motivates the following definition of the empirical measure

$$\pi_t^N(dx) = \frac{N_\beta}{N} \sum_{k=1}^{\infty} \eta_t(k) \delta_{k/N}(dx).$$

Here, $N_\beta = e^{\beta\mathcal{E}_N}$ is a choice so that the total mass of $\pi_0^N$ under $\mathscr{P}_{\beta,N}$ is of $O(1)$.

We will show (Theorems 2.4, 2.5, and 2.6), under diffusive scalings, for a large class of initial conditions supported on configurations with $O(NN_\beta^{-1})$ expected number of particles at level $N$, that the empirical measures $\pi_t^N$ converge weakly to a delta mass supported on the unique weak solution of a macroscopic equation, depending on the structure of the energy $\mathcal{E}_\cdot$, as $N\to\infty$:

1. $\beta = 0$: $\partial_t\rho = \partial_x^2\dfrac{\rho}{\rho+1} + \partial_x\dfrac{\rho}{\rho+1}$;
2. $0<\beta<1$, $\mathcal{E}_k \sim \ln k$: $\partial_t\rho = \partial_x^2\rho + \partial_x\left(\dfrac{\beta+x}{x}\rho\right)$;
3. $\beta>0$, $1 \ll \mathcal{E}_k \ll \ln k$: $\partial_t\rho = \partial_x^2\rho + \partial_x\rho$.

Since the particle density is related to the shape function by $\psi(x) = \int_x^\infty \rho(u)du$, we obtain (Corollary 2.7) the macroscopic equations for $\psi$:

(1') $\beta=0$: $\partial_t\psi = \partial_x\left(\dfrac{\partial_x\psi}{1-\partial_x\psi}\right) + \dfrac{\partial_x\psi}{1-\partial_x\psi}$;

(2') $0<\beta<1$, $\mathcal{E}_k\sim\ln k$: $\partial_t\psi = \partial_x^2\psi + \dfrac{\beta+x}{x}\partial_x\psi$;

(3') $\beta>0$, $1\ll\mathcal{E}_k\ll\ln k$: $\partial_t\psi = \partial_x^2\psi + \partial_x\psi$.

To shed light on these limits, the drift $N(\lambda_k - 1)$ is quite informative. When $\beta=0$, or when $1\ll\mathcal{E}_k\ll\ln k$, this drift tends to $-1$, but when $\mathcal{E}_k\sim\ln k$, it converges to a function of the scaled position. The function $\rho/(1+\rho)$ is in a sense the macroscopic average value of $\chi_{\{\eta_t(k)>0\}}$ with respect to the grand canonical ensemble. When $\beta=0$, the scaling limit recovers this form. But, when $\beta>0$, as there is an additional scaling factor involved to obtain a nontrivial limit, what needs to be replaced is $N_\beta\chi_{\{\eta_t(k)>0\}}$, which is close to the linearization of $\rho/(1+\rho)$, namely $\rho$; see Step 1 of Section 5 for a more technical discussion. From a physical perspective, the linear PDE limits reflect an effective transport of mass, which was a surprise to us.

The proof strategy is to consider the evolution of the empirical measure $\pi_t^N$ acting on test functions through Itô's formula with respect to the zero-range process $\eta_t$. In calculating the generator action, nonlinear functions of $\eta_\cdot$ emerge. However, because of non translation-invariance and inhomogeneity, standard methods such as 'entropy' or 'relative entropy' do not apply immediately to replace these terms with averaged expressions in terms of $\pi_t^N$. We leave open the possibility to adapt these methods with non-trivial modifications to serve our purposes in this model.

The idea we adopt here is to formulate certain 'local' hydrodynamic $1$ and $2$-block replacement estimates suitable to the current setting. These not so well-known 'local' replacements were originally introduced in [15] to study the 'tagged' particles. In particular, the replacements combine spectral gap estimates that we provide and Feynman-Kac and Rayleigh formulas for certain eigenvalues. Interestingly, only when $\beta=0$, does one need both 'local' $1$ and $2$-block replacements. Otherwise, when $\beta>0$, a 'local' $1$-block replacement suffices. In the proof of the $1$ and $2$-block estimates, we use that the process is 'attractive', a feature which allows a certain coupling to be employed, facilitating truncation and other estimates. Then, with tightness of the empirical measures, and uniqueness of weak solutions in appropriate classes that we provide and define, the limits follow. See Sections 4, 5, and 6 for more detailed proof outlines and remarks.

Previously, in [13], Funaki and Sasada studied an evolutional model of the Young diagrams, with respect to the 'uniform' grand ensembles $\mathscr{P}_{0,N}$, as well as certain 'restricted' uniform ensembles when $\beta=0$, providing a dynamical interpretation with respect to the Vershik curve $\psi$ (1.1). We note, although equations $(1), (1')$ when $\beta=0$

match that in [13], up to a constant in front of the first order derivative term, our results are different in several ways. Here, the dynamics that we work with is weakly asymmetric zero-range process (WAZRP) on $\mathbb{Z}^+$, which is in general spatially inhomogeneous, and one whose evolution preserves the total number of particles. However, the model in [13] is a different WAZRP on $\mathbb{Z}^+$, one which does not conserve particle mass, with a weakly asymmetric reservoir at site $0$. Importantly, the proof in [13] relies on the presence of this reservoir. Also, [13] considers initial profiles $\psi(0,x)$ where $\lim_{x\to 0}\psi(0,x)=\infty$ and obtain scaling limits $\psi(t,x)$ such that also $\lim_{x\to 0}\psi(t,x)=\infty$ and the hydrodynamic equation when $\beta=0$ holds. However, the initial conditions are different in our case: We consider initial profiles, finite at time $0$ and for all later times $t$, that is $\psi(t,0)=\psi(0,0)<\infty$, by conservation of particles in the dynamics. Moreover, it seems such profiles are not admissible with respect to the proof in [13], nor it seems are diverging profiles $\psi(0,x)$ amenable to our arguments, which make use that there are a finite number of particles at each level $N$.

From a broader point of view, random growth of Young diagrams also relates with the much studied corner growth model in which only the addition of squares to the diagram is allowed. Formally, in the study of hydrodynamic limits of the corner growth model, the problem is often converted, by considering gradients, to a totally asymmetric simple exclusion process, and the scaling is Euler, that is time and space are scaled at the same order. See [7] which discusses such and other dynamics. In contrast, our model of evolutional Young diagrams is studied via their gradient systems which is a WAZRP. Our analysis is also directly on this WAZRP on $\mathbb{Z}^+$ and no further transformation to simple exclusion processes is employed.

We also remark that, in a different vein, in [14], a dynamical model of Young diagrams connected with group theoretical ensembles, which keeps the Plancherel measure invariant, is studied and a hydrodynamic limit is shown there in terms of free probability notions.

**Organization of the article.** The precise description of the model and results are given in Section 2. Then, after preliminary definitions and estimates with respect to basic martingales in Section 3, we give the proof outlines of Theorems 2.4, 2.5, and 2.6, and Corollary 2.7 in Sections 4, 5, and 6 respectively. Main inputs into the proof are tightness and other estimates of the underlying measures given in Section 7. In Section 8, the important $1$ and $2$-block estimates are shown. Useful properties of the initial measures are given in Section 9. Uniqueness of weak solution to the hydrodynamic equations is proved in Section 10. Finally, in the appendix, some remarks about boundary phenomena of invariant measures are made.

## 2 Model description and results

We first specify certain Gibbs measures and their 'static' limits, which inform and motivate next our dynamical model that we introduce. Then, after prescribing the initial conditions considered, we give the hydrodynamic limit results.

### 2.1 Grand canonical ensembles and 'static' limits

Let $\mathbb{N}=\{1,2,\ldots\}$ be the natural numbers, and $\Omega=\{0,1,2,\ldots\}^{\mathbb{N}}$ be the space of particle configurations. A configuration $\xi=(\xi(k))_{k\in\mathbb{N}}\in\Omega$ specifies that there are $\xi(k)$ particles at sites $k\geq 1$.

Suppose that each particle at site $k$ carries energy $\mathcal{E}_k$, with respect to a function $\mathcal{E}_\cdot:\{0,1,2,\ldots\}\mapsto\mathbb{R}^+:=[0,\infty)$. Following [9], we will assume that the energy function $\mathcal{E}_k$ has the following structure. Let $\mathbb{R}^+_\circ:=(0,\infty)$.

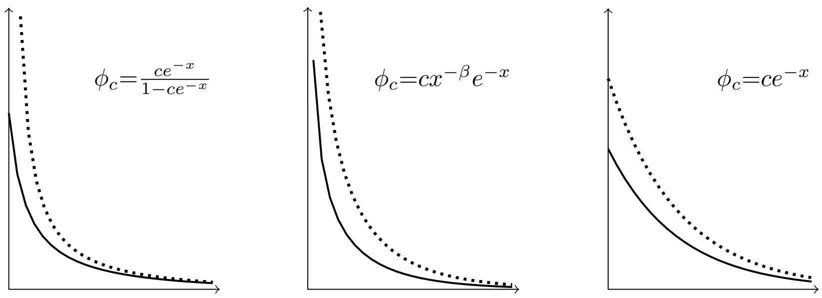


Figure 4: Examples of $\phi_c$ in all the three regimes. The dotted curves represent $c = c_0$ and solid curves are for general $c$'s which are strictly less than $c_0$.

**Condition 2.1.** *$\mathcal{E}_k = u(\ln k)$ where $u(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+_\circ$ is differentiable and $u'(\cdot)$ is bounded, $\lim_{x\to\infty} u(x) = \infty$, and $\lim_{x\to\infty} u'(x) = 0$ or $1$. We will say*

- *'$\mathcal{E}_k \sim \ln k$' denotes the case $\lim_{x\to\infty} u'(x) = 1$ and*
- *'$1 \ll \mathcal{E}_k \ll \ln k$' stands for the case $\lim_{x\to\infty} u'(x) = 0$.*

In passing, we note the constant $1$ in the limit when $\mathcal{E}_k \sim \ln k$ is chosen to be definite, although it could be specified as another positive constant. Also, as the derivative $u'$ is bounded, that the infimum $\inf \mathcal{E}_k/k = 0$ is achieved as $k \uparrow \infty$, a specification important in [9]. In addition, the condition allows a comparison, $\mathcal{E}_k - \mathcal{E}_l = u'(\ln y)\ln(k/l)$, where $y$ is between $k$ and $l$, afforded by the mean value theorem, which will be useful in some later estimates.

For fixed $\beta \geq 0$, we now introduce $\{\mathscr{R}_{c,N}\}$, a family of probability measures on $\Omega$, which will be seen as invariant measures for the dynamics specified in the next subsection. Let

$$c_0 = \min_{k\in\mathbb{N}} e^{\beta\mathcal{E}_k}.$$

Trivially $c_0 = 1$ when $\beta = 0$ and $c_0 \geq 1$ otherwise. For fixed $\beta$ and $0 \leq c \leq c_0$, we define the product measures on $\Omega$,

$$\mathscr{R}_{c,N}(\xi) = \prod_{k\in\mathbb{N}} \mathscr{R}_{\beta,c,N,k}(\xi(k)).$$

Here, the marginal $\mathscr{R}_{\beta,c,N,k}$ is the Geometric distribution with parameter

$$\theta_{k,c} = ce^{-\beta\mathcal{E}_k - k/N},$$

that is, for $n \geq 0$,

$$\mathscr{R}_{\beta,c,N,k}(n) = (1-\theta_{k,c})\theta_{k,c}^n.$$

Notice that $\theta_{k,c} < 1$ as $0 \leq c \leq c_0$ and $k \in \mathbb{N}$, For each site $k$, the marginal has mean

$$\rho_{k,c} = \frac{\theta_{k,c}}{1-\theta_{k,c}} = \frac{ce^{-\beta\mathcal{E}_k - k/N}}{1-ce^{-\beta\mathcal{E}_k - k/N}}. \tag{2.1}$$

The strength of the parameter $c$ reflects the density of the sizes $\{\xi(k)\}$ in the system. For example, the case $c = 0$ is trivial: $\mathscr{R}_{0,N}$ puts no particles anywhere.

When $0 < c \leq c_0$, $\mathscr{R}_{c,N}$ can also be written in an exponential form which illuminates its connection to the grand canonical ensemble $\mathscr{P}_{\beta,N}$ introduced in Section 1:

$$\mathscr{R}_{c,N}(\xi) = \frac{1}{Z_{\beta,c,N}} e^{-\beta\sum_{k\geq 1}\xi(k)\mathcal{E}_k + \sum_{k\geq 1}\ln(c)\xi(k) - N^{-1}\sum_{k\geq 1}k\xi(k)}$$

where $Z_{\beta,c,N}$ is the normalizing constant. When $c = 1$, $\mathscr{R}_{c,N}$ is exactly $\mathscr{P}_{\beta,N}$. Moreover, $\mathscr{R}_{c,N}$ is the grand canonical ensemble associated with the shifted energy $\beta\mathcal{E}_k - \ln c$.

In the rest of this subsection, we present the associated 'static' limits of $\mathscr{R}_{c,N}$. Recall

$$N_\beta = e^{\beta\mathcal{E}_N}. \tag{2.2}$$

We distinguish three regimes depending on the form of $\mathcal{E}_k$ and $\beta$:

(1) $\beta = 0$: $N_\beta = 1$,
(2) $\mathcal{E}_k \sim \ln k$ and $0 < \beta < 1$: $N_\beta = o(N)$ and $\lim_{N\uparrow\infty} N_\beta = \infty$,
(3) $1 \ll \mathcal{E}_k \ll \ln k$ and $\beta > 0$: $N_\beta = o(N)$ and $\lim_{N\uparrow\infty} N_\beta = \infty$.

When $c < c_0$, in Lemma 9.4, we show the following mean $E_{\mathscr{R}_{c,N}}$ and variance $\mathrm{Var}_{\mathscr{R}_{c,N}}$ estimates, under $\mathscr{R}_{c,N}$, for the number of particles in the system:

$$E_{\mathscr{R}_{c,N}} \sum_{k=1}^{\infty} \xi(k) = O(NN_\beta^{-1}), \quad \text{and} \quad \mathrm{Var}_{\mathscr{R}_{c,N}} \sum_{k=1}^{\infty} \xi(k) = o(N^2 N_\beta^{-2}). \tag{2.3}$$

However, when $c = c_0$, we show in Lemma A.1 in the Appendix that the orders of the expected value and variance are strictly greater. In a sense, the case $c = c_0$ represents a boundary, avoided for the most part in the sequel, so that we may unify statements and techniques.

In the three cases above, we now associate certain profiles $\phi_c$:

(1) $\phi_c = \dfrac{ce^{-x}}{1 - ce^{-x}}$ when $\beta = 0$,
(2) $\phi_c = cx^{-\beta}e^{-x}$ when $\mathcal{E}_k \sim \ln k$ and $0 < \beta < 1$,
(3) $\phi_c = ce^{-x}$ when $1 \ll \mathcal{E}_k \ll \ln k$ and $\beta > 0$,

cf. Figure 4. When $0 \le c < c_0$, we observe that $\phi_c \in L^1(\mathbb{R}^+)$. These profiles are the 'static' limits of the gradients under the measures $\mathscr{R}_{c,N}$.

**Proposition 2.2.** *Suppose $\mathcal{E}$ and $\beta$ satisfy the conditions of regimes (1), (2) or (3) above. Fix $0 \le c < c_0$. Then, for any test function $G \in C_c^\infty(\mathbb{R}_\circ^+)$ and $\delta > 0$*

$$\lim_{N\to\infty} \mathscr{R}_{c,N}\left[\left|\frac{N_\beta}{N}\sum_{k=1}^{\infty} G(k/N)\xi(k) - \int_0^\infty G(x)\phi_c(x)dx\right| > \delta\right] = 0 \tag{2.4}$$

*where $\phi_c$ takes the appropriate form in each regime (1), (2) or (3).*

In passing, we remark, when $c = c_0$, the above limit still holds. See Lemma A.2 in the Appendix for an argument.

We will state later in Subsection 2.3 that this proposition is a corollary of (2.9), which is proved in Proposition 9.10.

### 2.2 Dynamics

We now define the gradient evolutions of the Young diagrams. Let $\theta_k = \theta_{k,1} = e^{-\beta\mathcal{E}_k - k/N}$. Informally, particles at site $k$ jump to its right site $k+1$ with rate $\lambda_k := \dfrac{\theta_{k+1}}{\theta_k}$ and to its left site $k-1$ with rate 1. Particles at site 1 jump only to site 2.

For each $N \ge 1$, the evolution is a type of zero-range Markov process, $\xi_t = (\xi_t(k))_{k\ge1} \in \Omega$, on $\mathbb{Z}^+$ and generator

$$Lf(\xi) = \sum_{k=1}^{\infty} \left\{\lambda_k \left[f\left(\xi^{k,k+1}\right) - f(\xi)\right] \chi_{\{\xi(k)>0\}} + \left[f\left(\xi^{k,k-1}\right) - f(\xi)\right] \chi_{\{\xi(k)>0,k>1\}}\right\}$$

where

$$\lambda_k = \frac{\theta_{k+1}}{\theta_k} = e^{-\beta(\mathcal{E}_{k+1}-\mathcal{E}_k)-1/N}. \tag{2.5}$$

Here, $\xi^{x,y}(k) = \xi(k) - 1$, $\xi(k) + 1$, and $\xi(k)$ when respectively $k = x$, $k = y$, and $k \neq x, y$. We note when $\beta > 0$, the process has spatially inhomogeneous rates in that $\lambda_k$ is not constant in $k$. See [1] for more discussion about zero-range processes.

Under the initial measures we use, there will be a large, but finite number of particles, of order $O(NN_\beta^{-1})$, at all times in the system, and so in fact the process can be seen as a countable state space chain.

In Lemma 9.1, we verify that $\mathscr{R}_{c,N}$ is a reversible measure with respect to $L$. Therefore, the family of measures $\{\mathscr{R}_{c,N}\}$ is invariant under the dynamics generated by $L$.

We will observe the evolution speeded up by $N^2$, and consider in the sequel the process $\eta_t := \xi_{N^2 t}$, generated by $N^2 L$, for times $0 \le t \le T$, where $T > 0$ refers to a fixed time horizon.

We will access the space-time structure of the process through the scaled mass empirical measure,

$$\pi_t^N(dx) := \frac{N_\beta}{N} \sum_{k=1}^{\infty} \eta_t(k)\delta_{k/N}(dx).$$

Clearly $\pi_t^N$ is a locally finite measure on $\mathbb{R}_\circ^+$. Let $\mathcal{M}$ be the space of locally finite measures on $\mathbb{R}_\circ^+ = (0,\infty)$, and observe that $\pi_t^N \in \mathcal{M}$. Let also $C_c(\mathbb{R}_\circ^+)$ be the space of compactly supported continuous functions on $\mathbb{R}_\circ^+$, endowed with the topology of uniform convergence on compact sets. For $\{f_k\}_{k\in\mathbb{N}}$ a countable dense set in $C_c(\mathbb{R}_\circ^+)$, we equip $\mathcal{M}$ with the distance

$$d(\mu,\nu) = \sum_{k=1}^{\infty} 2^{-k} \frac{\left|\int f_k(d\mu - d\nu)\right|}{1 + \left|\int f_k(d\mu - d\nu)\right|}.$$

Then, $(\mathcal{M}, d)$ is a complete separable metric space and, for a sequence of measures in $\mathcal{M}$, convergence in the metric $d$ is equivalent to convergence in the vague topology. Here, the trajectories $\{\pi_t^N : 0 \le t \le T\}$ are elements of the Skorokhod space $D([0,T],\mathcal{M})$, endowed with the associated Skorokhod topology.

In the following, for $G \in C_c(\mathbb{R}_\circ^+)$ and $\pi \in \mathcal{M}$, denote $\langle G, \pi\rangle = \int_0^\infty G(u)d\pi(u)$. Also, for a given measure $\mu$, we denote expectation and variance with respect to $\mu$ by $E_\mu$ and $\mathrm{Var}_\mu$. Also, the process measure and associated expectation governing $\eta_\cdot$ starting from $\mu$ will be denoted by $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$.

#### 2.2.1 Attractiveness of the dynamics

Since $\chi_{\{\xi(k)>0\}}$ is an increasing function in $\xi$, the dynamics generated by $L$ is 'attractive', a fact that allows use of the 'basic coupling' in our proofs (cf. [1], Chapter II in [21]): Let $\mu$, $\nu$ be two probability measures on $\Omega$. We say that $\mu \le \nu$, that is $\mu$ is stochastically dominated by $\nu$, if for all $f : \Omega \to \mathbb{R}$ coordinately increasing, we have $E_\mu(f) \le E_\nu(f)$. Attractiveness asserts that if $\mu \le \nu$, then we have $\mathbb{E}_\mu(f(\xi_t)) \le \mathbb{E}_\nu(f(\xi_t))$ for all $t \ge 0$.

### 2.3 Initial conditions

We first specify a set of natural initial conditions, which will be a case of a more general class of initial conditions given later. Consider an initial density profile $\rho_0 : \mathbb{R}_\circ^+ \to \mathbb{R}^+$ such that $\rho_0 \in L^1(\mathbb{R}_\circ^+)$. For all $N, k \in \mathbb{N}$, let

$$\overline{\rho}_{N,k} = N \int_{(k-1)/N}^{k/N} \rho_0(x)dx.$$

Define a sequence of 'local equilibrium' measures $\left\{\mu^N\right\}_{N\in\mathbb{N}}$ corresponding to $\rho_0$:

1. For all $N \in \mathbb{N}$ and $\eta \in \Omega$, $\mu^N(\eta) = \prod_{k=1} \mu_k^N(\eta(k))$ with $\mu_k^N$ Geometric distributions with parameter $\theta_{N,k}$.
2. $\lim_{N\to\infty} \dfrac{1}{N} \sum_{k=1}^{\infty} |N_\beta \rho_{N,k} - \overline{\rho}_{N,k}| = 0$ where $\rho_{N,k} = \dfrac{\theta_{N,k}}{1-\theta_{N,k}}$ is the mean of $\mu_k^N$.
3. $\mu^N$ is stochastically bounded by $\mathscr{R}_{c,N}$ for some $0 \le c < c_0$.

We note that the last condition, given that the marginals of $\mu^N$ are Geometric, is equivalent to $\theta_{N,k} \le \theta_{k,c} = c\theta_k = ce^{-\beta\mathcal{E}_k - k/N}$.

As might be suspected, given the family of profiles $\{\phi_c\}$ are the static limits when the process is started from $\{\mathscr{R}_{c,N}\}$ (Proposition 2.2), we show in Lemma 9.3, that the invariant measures $\mathscr{R}_{c,N}$, for $0 \le c < c_0$, are local equilibrium measures with $\theta_{N,k} \equiv \theta_{k,c}$ and $\rho_0 = \phi_c$.

We now specify a more general class of initial measures $\nu^N$, namely those which satisfy the following condition. In Proposition 9.5, we verify that the local equilibria $\mu^N$ are in fact explicit members of this class.

**Condition 2.3.** *For $N \in \mathbb{N}$, let $\nu^N$ be a sequence of probability measures on $\Omega$.*

1. *Suppose $\rho_0 \in L^1(\mathbb{R}^+)$, and for each $N \in \mathbb{N}$, $\nu^N$ is a product measure, $\nu^N(\eta) = \prod_{k=1} \nu_k^N(\eta(k))$ such that marginals $\nu_k^N$ have mean $m_{N,k}$ where*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{\infty} |N_\beta m_{N,k} - \overline{\rho}_{N,k}| = 0.$$

2. *We have $\nu^N$ is stochastically bounded by $\mathscr{R}_{c,N}$ for a $0 \le c < c_0$.*
3. *The relative entropy of $\nu^N$ with respect to $\mathscr{R}_{c,N}$ is of order $NN_\beta^{-1}$: Let $f_0 = d\nu^N/d\mathscr{R}_{c,N}$. Then, $H(\nu^N|\mathscr{R}_{c,N}) := \int f_0 \ln f_0 d\mathscr{R}_{c,N} = O(NN_\beta^{-1})$.*

When the process starts from $\{\nu^N\}_{N\in\mathbb{N}}$, in the class satisfying Condition 2.3, we will denote by $\mathbb{P}_N := \mathbb{P}_{\nu^N}$ and $\mathbb{E}_N := \mathbb{E}_{\nu^N}$, the associated process measure and expectation. Members of this class have the following properties, useful in later arguments:

- Total bound on the number of particles (Lemma 9.7): For $0 \le t \le T$,

$$\mathbb{E}_N \sum_{k=1}^{\infty} \eta_t(k) = O(NN_\beta^{-1}). \tag{2.6}$$

- Variance bound (Lemma 9.8): For $0 \le t \le T$,

$$\sum_{k=1}^{\infty} \mathrm{Var}_{\mathbb{P}_N}(\eta_t(k)) = o\big(N^2 N_\beta^{-2}\big). \tag{2.7}$$

- Site particle bound (Lemma 9.9): For $0 < a < b$ and $0 \le t \le T$,

$$\sup_N \sup_{aN \le k \le bN} \sup_{0 \le t \le T} N_\beta \mathbb{E}_N\big[\eta_t(k)\big] < \infty. \tag{2.8}$$

- Initial convergence (Proposition 9.10): For any $G \in C_c^\infty(\mathbb{R}_\circ^+)$, and $\delta > 0$,

$$\lim_{N\to\infty} \nu^N\Big[\Big|\frac{N_\beta}{N}\sum_{k=1}^{\infty} G(k/N)\eta(k) - \int_0^\infty G(x)\rho_0(x)dx\Big| > \delta\Big] = 0. \tag{2.9}$$

By the discussion of attractiveness in Subsection 2.2.1, and that $\nu^N \le \mathscr{R}_{c,N}$ and $\mathscr{R}_{c,N}$ is an invariant measure, we have

$$\mathbb{E}_N\left[f(\eta_t)\right] \le \mathbb{E}_{\mathscr{R}_{c,N}}\left[f(\eta_t)\right] = E_{\mathscr{R}_{c,N}}\left[f(\eta)\right], \tag{2.10}$$

for all functions $f$ increasing coordinatewise, and all $t \ge 0$.

In addition, we see that Proposition 2.2 is a corollary of (2.9), since the invariant measures $\mathscr{R}_{c,N}$, for $c < c_0$, are local equilibrium measures, and in fact satisfy Condition 2.3.

We note, as a consequence of the attractiveness and (2.9), that $\int_0^\infty G(x)\rho_0(x)dx \le \int_0^\infty G(x)\phi_c(x)dx$ for nonnegative $G$, and so necessarily $\rho_0 \le \phi_c$.

### 2.4 Results

Following on the discussion of 'static' limits, we now arrive at our main results on the evolution of macroscopic density. These separate into three limits depending on which of the three regimes are in force.

Let $\mathscr{C}$ be the space of functions $\rho : [0,T] \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that the map $t \in [0,T] \mapsto \rho(t,x)dx \in \mathcal{M}$ is vaguely continuous; that is, for each $G \in C_c^\infty(\mathbb{R}_\circ^+)$, the map $t \in [0,T] \mapsto \int_0^\infty G(x)\rho(t,x)dx$ is continuous.

A standing assumption in the sequel is that the process $\eta_\cdot$ begins from initial measures $\{\nu^N\}_{N\in\mathbb{N}}$ satisfying Condition 2.3.

**Theorem 2.4.** *Suppose $\beta = 0$ and $\rho_0 \in L^1(\mathbb{R}^+)$. Then, for any $t \ge 0$, test function $G \in C_c^\infty(\mathbb{R}_\circ^+)$, and $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}_N\left[\left|\langle G, \pi_t^N\rangle - \int_0^\infty G(x)\rho(t,x)dx\right| > \delta\right] = 0,$$

*where $\rho(t,x)$ is the unique weak solution in the class $\mathscr{C}$ of the equation*

$$\begin{cases} \partial_t\rho = \partial_x^2 \dfrac{\rho}{\rho+1} + \partial_x \dfrac{\rho}{\rho+1} \\ \rho(0,\cdot) = \rho_0(\cdot), \quad \displaystyle\int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx \\ \rho(t,\cdot) \le \phi_c(\cdot) \in L^1(\mathbb{R}^+) \text{ for all } t \in [0,T]. \end{cases} \tag{2.11}$$

**Theorem 2.5.** *Suppose $\mathcal{E}_k \sim \ln k$, $0 < \beta < 1$ and $\rho_0 \in L^1(\mathbb{R}^+)$. Then, for any $t \ge 0$, test function $G \in C_c^\infty(\mathbb{R}_\circ^+)$, and $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}_N\left[\left|\langle G, \pi_t^N\rangle - \int_0^\infty G(x)\rho(t,x)dx\right| > \delta\right] = 0,$$

*where $\rho(t,x)$ is the unique weak solution in the class $\mathscr{C}$ of the equation*

$$\begin{cases} \partial_t\rho = \partial_x^2\rho + \partial_x\Big(\dfrac{\beta+x}{x}\rho\Big) \\ \rho(0,\cdot) = \rho_0(\cdot), \quad \displaystyle\int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx \\ \rho(t,\cdot) \le \phi_c(\cdot) \in L^1(\mathbb{R}^+) \text{ for all } t \in [0,T]. \end{cases} \tag{2.12}$$

**Theorem 2.6.** *Suppose $1 \ll \mathcal{E}_k \ll \ln k$, $\beta > 0$ and $\rho_0 \in L^1(\mathbb{R}^+)$. Then, for any $t \ge 0$, test function $G \in C_c^\infty(\mathbb{R}_\circ^+)$, and $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}_N\left[\left|\langle G, \pi_t^N\rangle - \int_0^\infty G(x)\rho(t,x)dx\right| > \delta\right] = 0,$$

*where $\rho(t,x)$ is the unique weak solution in the class $\mathscr{C}$ of the equation*

$$\begin{cases} \partial_t \rho = \partial_x^2 \rho + \partial_x \rho \\ \rho(0,\cdot) = \rho_0(\cdot), \quad \int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx \\ \rho(t,\cdot) \le \phi_c(\cdot) \in L^1(\mathbb{R}^+) \text{ for all } t \in [0,T]. \end{cases} \tag{2.13}$$

We now go back to the Young diagrams and explain the results in this context. For each particle configuration $\eta_t$, the corresponding shape function of the diagram is

$$\psi_N(t,x) = \frac{N_\beta}{N} \sum_{k \ge xN} \eta_t(k). \tag{2.14}$$

The hydrodynamic limits for the diagrams will follow from the hydrodynamic limits of the density profiles.

Let $\mathcal{W}$ be the class of continuous functions $\psi : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that, for each $t \in [0,T]$, $\psi(t,\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is absolutely continuous.

**Corollary 2.7.** *With respect to the shape functions, the following limits hold.*

1. *Consider the assumptions of Theorem 2.4. Then, for any $t \ge 0$, test function $G \in C_c^\infty(\mathbb{R}_\circ^+)$, and $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}_N \Big[ \Big| \int_0^\infty G(x)\psi_N(t,x)dx - \int_0^\infty G(x)\psi(t,x)dx \Big| > \delta \Big] = 0, \tag{2.15}$$

*where $\psi(t,x)$ is the unique weak solution in the class $\mathcal{W}$ of the equation*

$$\begin{cases} \partial_t \psi = \partial_x \Big( \dfrac{\partial_x \psi}{1 - \partial_x \psi} \Big) + \dfrac{\partial_x \psi}{1 - \partial_x \psi} \\ \psi(0,x) = \int_x^\infty \rho_0(u)du, \quad \lim_{x\to\infty} \psi(t,x) = 0 \\ \psi(t,0) = \psi(0,0), \quad 0 \le -\partial_x \psi(t,\cdot) \le \phi_c(\cdot) \text{ for all } t \in [0,T]. \end{cases} \tag{2.16}$$

2. *Consider the assumptions of Theorem 2.5. Then, for any $t \ge 0$, test function $G \in C_c^\infty(\mathbb{R}_\circ^+)$, and $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}_N \Big[ \Big| \int_0^\infty G(x)\psi_N(t,x)dx - \int_0^\infty G(x)\psi(t,x)dx \Big| > \delta \Big] = 0,$$

*where $\psi(t,x)$ is the unique weak solution in the class $\mathcal{W}$ of the equation*

$$\begin{cases} \partial_t \psi = \partial_x^2 \psi + \dfrac{\beta + x}{x} \partial_x \psi \\ \psi(0,x) = \int_x^\infty \rho_0(u)du, \quad \lim_{x\to\infty} \psi(t,x) = 0 \\ \psi(t,0) = \psi(0,0), \quad 0 \le -\partial_x \psi(t,\cdot) \le \phi_c(\cdot) \text{ for all } t \in [0,T]. \end{cases} \tag{2.17}$$

3. *Consider the assumptions of Theorem 2.6. Then, for any $t \ge 0$, test function $G \in C_c^\infty(\mathbb{R}_\circ^+)$, and $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}_N \Big[ \Big| \int_0^\infty G(x)\psi_N(t,x)dx - \int_0^\infty G(x)\psi(t,x)dx \Big| > \delta \Big] = 0,$$

*where $\psi(t,x)$ is the unique weak solution in the class $\mathcal{W}$ of the equation*

$$\begin{cases} \partial_t \psi = \partial_x^2 \psi + \partial_x \psi \\ \psi(0,x) = \int_x^\infty \rho_0(u)du, \quad \lim_{x\to\infty} \psi(t,x) = 0 \\ \psi(t,0) = \psi(0,0), \quad 0 \le -\partial_x \psi(t,\cdot) \le \phi_c(\cdot) \text{ for all } t \in [0,T]. \end{cases} \tag{2.18}$$

## 3 Martingale framework

The proofs of the main results make use of the stochastic differential of $\langle G, \pi_t^N \rangle$, written in terms of certain martingales. Let $G$ be a compactly supported smooth function on $[0,T] \times \mathbb{R}_\circ^+$, and let us write $G_t(x) := G(t,x)$, for $t \geq 0$. Consider the mean zero martingale,

$$M_t^{N,G} = \left\langle G_t, \pi_t^N \right\rangle - \left\langle G_0, \pi_0^N \right\rangle - \int_0^t \Big\{ \left\langle \partial_s G_s, \pi_s^N \right\rangle + N^2 L \left\langle G_s, \pi_s^N \right\rangle \Big\} ds.$$

Define the discrete Laplacian $\Delta_N$ and discrete gradient $\nabla_N$ as

$$\begin{aligned} \Delta_N G\Big(\frac{k}{N}\Big) :=& N^2 \Big( G\Big(\frac{k+1}{N}\Big) + G\Big(\frac{k-1}{N}\Big) - 2G\Big(\frac{k}{N}\Big)\Big), \\ \nabla_N G\Big(\frac{k}{N}\Big) :=& N\Big( G\Big(\frac{k+1}{N}\Big) - G\Big(\frac{k}{N}\Big)\Big). \end{aligned}$$

Then, we may compute

$$\begin{aligned} N^2 L \left\langle G_s, \pi_s^N \right\rangle = \frac{1}{N} \sum_{k=2}^{\infty} \left( \Delta_N G_s\Big(\frac{k}{N}\Big) + \frac{\lambda_k - 1}{1/N} \nabla_N G_s\Big(\frac{k}{N}\Big)\right) N_\beta \chi_{\{\eta_s(k)>0\}} \\ + N\lambda_1 \nabla_N G_s\Big(\frac{1}{N}\Big) N_\beta \chi_{\{\eta_s(1)>0\}}. \end{aligned} \tag{3.1}$$

Since $G_s$ is compactly supported on $\mathbb{R}_\circ^+$, we note that the last term vanishes for all $N$ large.

For later reference, we will call

$$D_{N,k}^{G,s} := \Delta_N G_s\Big(\frac{k}{N}\Big) + \frac{\lambda_k - 1}{1/N} \nabla_N G_s\Big(\frac{k}{N}\Big). \tag{3.2}$$

Define also

$$\alpha(x,\beta) := \lim_{\substack{N\to\infty \\ k/N \to x}} \frac{\lambda_k - 1}{1/N}.$$

Observing

$$\lambda_k = e^{-\beta(\mathcal{E}_{k+1} - \mathcal{E}_k) - 1/N} = e^{-\beta(u(\ln k+1) - u(\ln k)) - 1/N},$$

we have for all $x > 0$ that

$$\alpha(x,\beta) = \begin{cases} -1 & \text{when } \beta = 0 \text{ or } 1 \ll \mathcal{E}_k \ll \ln k \\ -\frac{\beta + x}{x} & \text{when } \mathcal{E}_k \sim \ln k. \end{cases} \tag{3.3}$$

Moreover, for $0 < a < b < \infty$, $N$ large, and $aN \leq k \leq bN$, we conclude

$$\left| D_{N,k}^{G,s} \right| \leq \|\Delta G\|_\infty + \frac{\beta + b}{a} \|\nabla G\|_\infty. \tag{3.4}$$

The quadratic variation of $M_t^{N,G}$ is given by

$$\langle M^{N,G} \rangle_t = \int_0^t \left\{ N^2 L \left( \left\langle G_s, \pi_s^N \right\rangle^2 \right) - 2 \left\langle G_s, \pi_s^N \right\rangle N^2 L \left\langle G_s, \pi_s^N \right\rangle \right\} ds.$$

Straightforward calculation shows that

$$\begin{aligned} \langle M^{N,G} \rangle_t =& \frac{N_\beta}{N} \int_0^t \left\{ \frac{1}{N} \sum_{k=1}^{\infty} \lambda_k \left( \nabla_N G_s\left(k/N\right)\right)^2 N_\beta \chi_{\{\eta_s(k)>0\}} \right. \\ & \left. + \frac{1}{N} \sum_{k=2}^{\infty} \left( \nabla_N G_s\left(k/N\right)\right)^2 N_\beta \chi_{\{\eta_s(k)>0\}} \right\} ds. \end{aligned}$$

A useful bound on this variation is as follows. Recall the estimates on $N_\beta$ (cf. (2.2)).

**Lemma 3.1.** *For smooth $G$ with compact support in $[0,T]\times\mathbb{R}_\circ^+$, there is a constant $C_G$ such that for large $N$,*

$$\sup_{0\le t\le T}\mathbb{E}_N\langle M^{N,G}\rangle_t \le C_G T N_\beta N^{-1}.$$

*Proof.* Suppose that $G_t$ is supported on $[a,b]$ with $0<a<b<\infty$ for all $t$. For $N$ large, we have

$$\begin{aligned}\mathbb{E}_N\langle M^{N,G}\rangle_t =& N_\beta N^{-1}\mathbb{E}_N\Big[\int_0^t \frac{1}{N}\sum_{k=aN}^{bN}\widehat{D}^{G,s}_{N,k}N_\beta\chi_{\{\eta_s(k)>0\}}ds\Big]\\ \le& C^1_G N_\beta N^{-1}\mathbb{E}_N\Big[\int_0^t\frac{1}{N}\sum_{k=aN}^{bN}N_\beta\chi_{\{\eta_s(k)>0\}}ds\Big],\end{aligned}$$

where $\widehat{D}^{G,s}_{N,k}=\lambda_k\left(\nabla_N G_s\left(k/N\right)\right)^2+\left(\nabla_N G_s\left(k/N\right)\right)^2$ and $|\widehat{D}^{G,s}_{N,k}|\le C^1_G$.

For the case $\beta=0$, since $N_\beta=1$, we bound $\chi_{\{\eta(k)>0\}}$ by $1$. Then, $\mathbb{E}_N\langle M^{N,G}\rangle_t\le C^1_G N^{-1}(b-a)t$, from which the lemma follows.

For the other two cases of $\beta>0$, we bound $\chi_{\{\eta(k)>0\}}$ by $\eta(k)$. Then,

$$\begin{aligned}\mathbb{E}_N\langle M^{N,G}\rangle_t &\le C^1_G N_\beta N^{-1}\mathbb{E}_N\Big[\int_0^t\frac{1}{N}\sum_{k=1}^{\infty}N_\beta\eta_s(k)ds\Big]\\ &= C^1_G N_\beta N^{-1}t\,\mathbb{E}_N\Big[\frac{1}{N}\sum_{k=1}^{\infty}N_\beta\eta_0(k)\Big].\end{aligned}$$

We have used that total number of particles is conserved in the last equality. Then, by (2.6), we obtain $\sup_N\mathbb{E}_N\Big[\dfrac{1}{N}\sum_{k=1}^\infty N_\beta\eta_0(k)\Big]<\infty$, thereby finishing the argument. ☐

## 4 Proof outline: Hydrodynamic limits when $\beta=0$

We give the proof of Theorem 2.4 in outline form, referring to estimates proved in later sections. Since $N_\beta=1$ for $\beta=0$, we have

$$\pi^N_t(dx)=\frac{1}{N}\sum_{k=1}^\infty\eta_t(k)\delta_{k/N}(dx).$$

We denote by $Q^N$ the probability measure on the trajectory space $D([0,T],\mathcal{M})$ governing $\pi^N_\cdot$ when the process starts from $\nu^N$. By Lemma 7.1 the family of measures $\{Q^N\}_{N\in\mathbb{N}}$ is tight with respect to the uniform topology, stronger than the Skorokhod topology, and all limit measures are supported on vaguely continuous trajectories $\pi_\cdot$, that is for each test function $G\in C_c^\infty(\mathbb{R}_\circ^+)$, the map $t\mapsto\langle G,\pi_t\rangle$ is continuous.

Let now $Q$ be any limit measure. We show that $Q$ is supported on weak solutions to the nonlinear PDE (2.11).

*Step 1.* Take any smooth $G$ with compact support in $[0,T]\times\mathbb{R}_\circ^+$, say in $[0,T]\times[a,b]$ such that $0<a<b<\infty$. To obtain the form of the limit equation, recall the martingale $M^{N,G}_t$ and its quadratic variation $\langle M^{N,G}\rangle_t$ introduced in the last section.

Since $G$ is smooth and with compact support, by Lemma 3.1, we have $\mathbb{E}_N\left(M^{N,G}_T\right)^2=\mathbb{E}_N\left(\langle M^{N,G}\rangle_T\right)$ vanishes as $N\to\infty$. Then, by Doob's inequality, for each $\delta>0$,

$$\begin{aligned}&\mathbb{P}_N\big(\sup_{0\le t\le T}\big|\big\langle G_t,\pi^N_t\big\rangle-\big\langle G_0,\pi^N_0\big\rangle-\int_0^t\big(\big\langle\partial_s G_s,\pi^N_s\big\rangle+N^2L\big\langle G_s,\pi^N_s\big\rangle\big)ds\big|>\delta\big)\\ &\le\frac{4}{\delta^2}\mathbb{E}_N\big(\big\langle M^{N,G}\big\rangle_T\big)\ \to\ 0\ \text{ as } N\to\infty.\end{aligned}$$

Recall the computation of $N^2 L \left\langle G_s, \pi_s^N \right\rangle$ in (3.1). Then,

$$\lim_{N\to\infty} \mathbb{P}_N \Bigg( \sup_{0\le t\le T} \Big| \left\langle G_t, \pi_t^N \right\rangle - \left\langle G_0, \pi_0^N \right\rangle - \int_0^t \Bigg( \left\langle \partial_s G_s, \pi_s^N \right\rangle \tag{4.1}$$
$$+ \frac{1}{N} \sum_{k=aN}^{bN} \Big( \Delta_N G_s\Big(\frac{k}{N}\Big) + \frac{\lambda_k - 1}{1/N} \nabla_N G_s\Big(\frac{k}{N}\Big)\Big) \chi_{\{\eta_s(k)>0\}} \Bigg) ds \Big| > \delta \Bigg) = 0.$$

*Step 2.* We would like to replace the nonlinear term $\chi_{\{\eta_s(k)>0\}}$ by a function of the empirical density of particles within a macroscopically small box. To be precise, let $\eta^l(x) = \dfrac{1}{2l+1} \sum_{|y-x|\le l} \eta(y)$, that is the average density of particles in the box centered at $x$ with length $2l+1$.

Recall the coefficient $D_{N,k}^{G,s}$ in (3.2). By the triangle inequality, the $1$ and $2$-block estimates (Lemmas 8.2 and 8.4) give immediately the following replacement lemma.

**Lemma 4.1** (Replacement Lemma)**.** *For each $\delta > 0$ and $0 < a < b < \infty$,*

$$\limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \mathbb{P}_N \Big[ \Big| \frac{1}{N} \sum_{aN\le k\le bN} \int_0^T D_{N,k}^{G,t} \Big( \chi_{\{\eta_t(k)>0\}} - \frac{\eta_t^{\varepsilon N}(k)}{1+\eta_t^{\varepsilon N}(k)} \Big) dt \Big| \ge \delta \Big] = 0.$$

*Step 3.* For each $\varepsilon > 0$, take $\iota_\varepsilon = (2\varepsilon)^{-1} \chi_{[-\varepsilon,\varepsilon]}$. The average density $\eta_t^{\varepsilon N}(k)$ is written as a function of the empirical measure $\pi_t^N$

$$\eta_t^{\varepsilon N}(k) = \frac{2\varepsilon N}{2\varepsilon N + 1} \langle \iota_\varepsilon(\cdot - k/N), \pi_t^N \rangle.$$

Also, as $\lambda_k = e^{-1/N}$ when $\beta = 0$, we have $N(\lambda_k - 1) \sim -1$ (cf. (3.3)).

Then, noting the form of $D_{N,k}^{G,s}$, we get from (4.1) in terms of the induced distribution $Q^N$ that

$$\limsup_{\varepsilon\to 0} \limsup_{N\to\infty} Q^N \Bigg( \Big| \left\langle G_T, \pi_T^N \right\rangle - \left\langle G_0, \pi_0^N \right\rangle - \int_0^T \Bigg( \left\langle \partial_s G_s, \pi_s^N \right\rangle$$
$$+ \frac{1}{N} \sum_{k=aN}^{bN} \Big( \Delta G_s\Big(\frac{k}{N}\Big) - \nabla G_s\Big(\frac{k}{N}\Big)\Big) \frac{\langle \iota_\varepsilon(\cdot - k/N), \pi_s^N \rangle}{\langle \iota_\varepsilon(\cdot - k/N), \pi_s^N \rangle + 1} \Bigg) ds \Big| > \delta \Bigg) = 0.$$

Notice that we replaced $\nabla_N$ and $\Delta_N$ by $\nabla$ and $\Delta$, respectively.

The error in replacing the Riemann sum by an integral is $o(1)$. We get

$$\limsup_{\varepsilon\to 0} \limsup_{N\to\infty} Q^N \Bigg( \Big| \left\langle G_T, \pi_T^N \right\rangle - \left\langle G_0, \pi_0^N \right\rangle - \int_0^T \Bigg( \left\langle \partial_s G_s, \pi_s^N \right\rangle \tag{4.2}$$
$$+ \int_0^\infty \left( \Delta G_s(x) - \nabla G_s(x) \right) \frac{\langle \iota_\varepsilon(\cdot - x), \pi_s^N \rangle}{\langle \iota_\varepsilon(\cdot - x), \pi_s^N \rangle + 1} dx \Bigg) ds \Big| > \delta \Bigg) = 0.$$

Taking $N \to \infty$, along a subsequence, as the set of trajectories in (4.2) is open with respect to the uniform topology, we obtain

$$\limsup_{\varepsilon\to 0} Q \Bigg( \Big| \langle G_T, \pi_T \rangle - \langle G_0, \pi_0 \rangle - \int_0^T \Bigg( \langle \partial_s G_s, \pi_s \rangle$$
$$+ \int_0^\infty \left( \Delta G_s(x) - \nabla G_s(x) \right) \frac{\langle \iota_\varepsilon(\cdot - x), \pi_s \rangle}{\langle \iota_\varepsilon(\cdot - x), \pi_s \rangle + 1} dx \Bigg) ds \Big| > \delta \Bigg) = 0.$$

*Step 4.* We show in Lemma 7.2 that $Q$ is supported on trajectories $\pi_s(dx) = \rho(s,x)dx$ where $\rho \in L^1([0,T]\times\mathbb{R}^+)$. To replace $\langle \iota_\varepsilon(\cdot - x), \pi_s\rangle$ by $\rho(s,x)$, it is enough to show, for all $\delta > 0$, that

$$\limsup_{\varepsilon\to 0} Q\left(\left|\int_0^T\int_0^\infty D_{G,s}\left(\frac{\langle \iota_\varepsilon(\cdot - x),\pi_s\rangle}{\langle \iota_\varepsilon(\cdot - x),\pi_s\rangle + 1} - \frac{\rho(s,x)}{1+\rho(s,x)}\right)dxds\right| > \delta\right) = 0.$$

where $D_{G,s} = \Delta G_s(x) - \nabla G_s(x)$. In fact, considering the Lebesgue points of $\rho$, almost surely with respect to $Q$,

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^\infty D_{G,s}\frac{\langle \iota_\varepsilon(\cdot - x),\pi_s\rangle}{\langle \iota_\varepsilon(\cdot - x),\pi_s\rangle + 1}dxds = \int_0^T\int_0^\infty D_{G,s}\frac{\rho(s,x)}{1+\rho(s,x)}dxds.$$

Now, we have

$$Q\left(\left|\langle G_T, \rho(T,x)\rangle - \langle G_0,\rho(0,x)\rangle - \int_0^T\left(\langle \partial_s G_s, \rho(s,x)\rangle\right.\right.\right.$$
$$\left.\left.\left. + \int_0^\infty \left(\Delta G_s(x) - \nabla G_s(x)\right)\frac{\rho(s,x)}{\rho(s,x)+1}dx\right)ds\right| = 0\right) = 1.$$

*Step 5.* Hence, each $\rho(t,x)$ solves weakly the equation $\partial_t\rho = \partial_x^2\frac{\rho}{\rho+1} + \partial_x\frac{\rho}{\rho+1}$. As we have already remarked that $Q$ is supported on vaguely continuous trajectories (Lemma 7.1), we have that $\rho$ belongs to $\mathscr{C}$.

We claim now that $\rho(t,x)$ satisfies the initial value problem (2.11): Indeed, the initial condition $\rho(0,x) = \rho_0(x)$ holds by (2.9). By Lemma 7.2, we have $\rho(t,x) \le \phi_c(x)$ for all $0 \le t \le T$. The conservation of mass $\int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx$ is proved in Lemma 7.3.

We show in Subsection 10.1 that there is at most one weak solution $\rho$ to (2.11), subject to these constraints. We conclude then that the sequence of $Q^N$ converges weakly to the Dirac measure on $\rho(\cdot,x)dx$. Finally, as $Q^N$ converges to $Q$ with respect to the uniform topology, we have for each $0\le t\le T$ that $\langle G, \pi_t^N\rangle$ weakly converges to the constant $\int G(x)\rho(t,x)dx$, and therefore convergence in probability as stated in Theorem 2.4. □

## 5 Proof outline: Hydrodynamic limits when $\beta > 0$

In this section, we sketch a proof of both Theorems 2.5 and 2.6, following the argument for the $\beta = 0$ case.

*Step 1.* The replacement lemma we need here is simpler than for the case $\beta = 0$, as it relies only on a 1-block estimate. Because of the form of the function $N_\beta\chi_{\{\eta_t(k)>0\}}$, from the 1-block estimate, it is close to $N_\beta\eta_t^l(k)/(1+\eta_t^l(k))$. However, as $N_\beta\eta_t^l(k)$ is of order $O(1)$, and therefore $\eta_t^l(k) = o(1)$, we may replace $N_\beta\eta_t^l(k)/\big(1+\eta_t^l(k)\big)$ by its linearization $N_\beta\eta_t^l(k)$. Then, using smoothness of the test function, $\eta_t^l(k)$ may be replaced by $\eta_t(k)$, so that a 2-blocks estimate is not needed. Moreover, we see as a consequence that a linear PDE arises in the hydrodynamic limit.

Recall the expression $D_{N,k}^{G,t}$ in (3.2).

**Lemma 5.1** (Replacement Lemma)**.** *For each smooth, compactly supported function $G$ on $[0,T]\times\mathbb{R}_\circ^+$, we have*

$$\limsup_{N\to\infty}\mathbb{E}_N\left|\frac{1}{N}\sum_{k=aN}^{bN}\int_0^T D_{N,k}^{G,t}\left(N_\beta\chi_{\{\eta_t(k)>0\}} - N_\beta\eta_t(k)\right)dt\right| = 0.$$

*Proof.* By smoothness of the test function $G$, it suffices to show

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\mathbb{E}_N\left|\frac{1}{N}\sum_{k=aN}^{bN}\int_0^T D_{N,k}^{G,t}\left(N_\beta\chi_{\{\eta_t(k)>0\}}-N_\beta\eta_t^l(k)\right)dt\right|=0,$$

and in turn enough to show that

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}_N\left|\int_0^T D_{N,k}^{G,t}\left(N_\beta\chi_{\{\eta_t(k)>0\}}-N_\beta\eta_t^l(k)\right)dt\right|=0. \tag{5.1}$$

By the 1-block estimate (Lemma 8.2),

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}_N\left|\int_0^T D_{N,k}^{G,t}\left(N_\beta\chi_{\{\eta_t(k)>0\}}-\frac{N_\beta\eta_t^l(k)}{1+\eta_t^l(k)}\right)dt\right|=0.$$

Adding and subtracting $N_\beta\eta_t^l(k)$, noting the uniform bound on $D_{N,k}^{G,t}$ after (3.2), (5.1) will follow if we have

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}_N\int_0^T\left(\frac{N_\beta\left(\eta_t^l(k)\right)^2}{1+\eta_t^l(k)}\right)dt=0.$$

In fact, by attractiveness (2.10), noting that $\mathscr{R}_{c,N}$ is an invariant measure, it will be enough to verify that

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{aN\le k\le bN}E_{\mathscr{R}_{c,N}}\left(\frac{N_\beta\left(\eta^l(k)\right)^2}{1+\eta^l(k)}\right)=0.$$

To this end, for any $l$, $N$, $aN\le k\le bN$, noting that $\mathscr{R}_{c,N}$ is a product measure, we have

$$\begin{aligned} &E_{\mathscr{R}_{c,N}}\left(\frac{N_\beta\left(\eta^l(k)\right)^2}{1+\eta^l(k)}\right)\le E_{\mathscr{R}_{c,N}}\left(N_\beta\left(\eta^l(k)\right)^2\right) \qquad (5.2)\\ &=\frac{N_\beta}{(2l+1)^2}\sum_{|j-k|\le l}E_{\mathscr{R}_{c,N}}\left(\eta(j)\right)^2+\frac{N_\beta}{(2l+1)^2}\sum_{\substack{j\ne m,|j-k|\le l\\|m-k|\le l}}E_{\mathscr{R}_{c,N}}\big(\eta(j)\big)E_{\mathscr{R}_{c,N}}\big(\eta(m)\big).\end{aligned}$$

Recall, under $\mathscr{R}_{c,N}$, that $\{\eta(j)\}$ is a sequence of Geometric variables with parameters $\theta_{j,c}=ce^{-\beta\mathcal{E}_j-j/N}$. We may calculate that (5.2) equals

$$\frac{N_\beta}{(2l+1)^2}\sum_{|j-k|\le l}\left[2\rho_{j,c}^2+\rho_{j,c}\right]+\frac{N_\beta}{(2l+1)^2}\sum_{\substack{j\ne m,|j-k|\le l\\|m-k|\le l}}\rho_{j,c}\rho_{m,c}. \tag{5.3}$$

By the site particle bound (2.8), we have

$$\sup_N\sup_{aN-l\le j\le bN+l}N_\beta\rho_{j,c}<\infty.$$

Also, as $\beta>0$, we have $N_\beta=e^{\beta\mathcal{E}_N}\to\infty$.

Hence, we see that (5.3) is of order $O(N_\beta^{-1}l^{-1}+l^{-1}+N_\beta^{-1})$, which vanishes as $N\to\infty$ and then $l\to\infty$. □

*Step 2.* Now, with the help of this replacement lemma and following Steps 1 and 2 in the proof of Theorem 2.4, we readily have

$$\begin{aligned}\lim_{N\to\infty} Q^N &\Bigg( \big| \left\langle G_T, \pi_T^N \right\rangle - \left\langle G_0, \pi_0^N \right\rangle - \int_0^T \Bigg( \left\langle \partial_s G_s, \pi_s^N \right\rangle \\ &+ \frac{1}{N} \sum_{k=aN}^{bN} \left( \Delta_N G_s \left( \frac{k}{N} \right) + \frac{\lambda_k - 1}{1/N} \nabla_N G_s \left( \frac{k}{N} \right) \right) N_\beta \eta_s(k) \Bigg) ds \big| > \delta \Bigg) = 0.\end{aligned} \tag{5.4}$$

Recall $\alpha(x,\beta) = \lim_{\substack{N\to\infty \\ k/N\to x}} \dfrac{\lambda_k - 1}{1/N}$ equals $-(\beta + x)/x$ when $\mathcal{E}_k \sim \ln k$ and equals $-1$ when $1 \ll \mathcal{E}_k \ll \ln k$ (cf. (3.3)). Then, we may replace $\nabla_N$, $\Delta_N$, and $N(\lambda_k - 1)$ by $\nabla$, $\Delta$, and $a(x,\beta)$ respectively, in (5.4). We obtain

$$\begin{aligned}\lim_{N\to\infty} Q^N \Big[ \big| \left\langle G_T, \pi_T^N \right\rangle - \left\langle G_0, \pi_0^N \right\rangle - \int_0^T \Big( \left\langle \partial_s G_s, \pi_s^N \right\rangle \\ + \ \left\langle \Delta G_s + a(x,\beta) \nabla G_s, \pi_s^N \right\rangle \Big) ds \big| > \delta \Big] = 0.\end{aligned}$$

*Step 3.* Now, the sequence $\{Q^N\}$ is tight with respect to the uniform topology by Lemma 7.1. Let $Q$ be a limit point. Then,

$$Q\Big[ \langle G_T, \pi_T \rangle - \langle G_0, \pi_0 \rangle - \int_0^T \Big( \langle \partial_s G_s, \pi_s \rangle \ + \ \langle \Delta G_s + a(x,\beta) \nabla G_s, \pi_s \rangle \Big) ds = 0 \Big] = 1.$$

Since $Q$ is supported on absolutely continuous trajectories $\pi_t(dx) = \rho(t,x)dx$, where $\rho \in L^1([0,T] \times \mathbb{R}^+)$ by Lemma 7.2, we have that each $\rho(t,x)$ is a weak solution of (2.12) or (2.13), depending on the choice of energy $\mathcal{E}_k$. Using the uniqueness results when $\beta > 0$ shown in Subsection 10.2, we now follow exactly Step 5 of the proof given in $\beta = 0$ case, to obtain the full statements of Theorems 2.5 and 2.6. □

## 6 Proof outline: Hydrodynamic limits for the diagrams

In this section, we prove Corollary 2.7. We will only prove the $\beta = 0$ case where $N_\beta = 1$. The other two cases follow from similar arguments carrying through an additional factor $N_\beta$.

*Step 1.* We will assume the hydrodynamic limit result Theorem 2.4 holds. First, we show that we may extend the limit

$$\lim_{N\to\infty} \mathbb{P}_N \Big[ \Big| \frac{1}{N} \sum_{k=1}^{\infty} g\Big(\frac{k}{N}\Big) \eta_t(k) - \int_0^\infty g(x) \rho(t,x) dx \Big| > \delta \Big] = 0 \tag{6.1}$$

to all $g \in C^\infty(\mathbb{R}_\circ^+)$ supported on $[a,\infty)$ and satisfying $g(x) = g(b)$ for all $x \geq b$ for some $0 < a < b < \infty$. Indeed, fix such a $g$ and take $g_n \in C_c^\infty(\mathbb{R}_\circ^+)$ such that $g_n = g$ on $(0,n)$. Then,

$$\begin{aligned}&\mathbb{P}_N \Big[ \Big| \frac{1}{N} \sum_{k=1}^{\infty} g\Big(\frac{k}{N}\Big) \eta_t(k) - \int_0^\infty g(x) \rho(t,x) dx \Big| > \delta \Big] \\ \leq &\mathbb{P}_N \Big[ \Big| \frac{1}{N} \sum_{k=1}^{\infty} g_n\Big(\frac{k}{N}\Big) \eta_t(k) - \int_0^\infty g_n(x) \rho(t,x) dx \Big| > \frac{\delta}{2} \Big] \\ &+ \mathbb{P}_N \Big[ \Big| \frac{1}{N} \sum_{k=1}^{\infty} \Big( g\Big(\frac{k}{N}\Big) - g_n\Big(\frac{k}{N}\Big) \Big) \eta_t(k) - \int_0^\infty (g(x) - g_n(x)) \rho(t,x) dx \Big| > \frac{\delta}{2} \Big].\end{aligned}$$

Since $g_n$ is compactly supported, by Theorem 2.4, the first term vanishes as $N \to \infty$.

As $\rho \le \phi_c$ and $\phi_c \in L^1(\mathbb{R}^+)$, for $n$ large enough, the second term is bounded from above by

$$\mathbb{P}_N\Big[\Big|\frac{1}{N}\sum_{k=1}^{\infty}\Big(g\Big(\frac{k}{N}\Big)-g_n\Big(\frac{k}{N}\Big)\Big)\eta_t(k)\Big| > \frac{\delta}{4}\Big] \le \mathbb{P}_N\Big[\frac{2\|g\|_\infty}{N}\sum_{k=nN}^{\infty}\eta_t(k) > \frac{\delta}{4}\Big].$$

By attractiveness (2.10) and the Markov inequality, the right-hand side probability is bounded by $(8\|g\|_\infty/\delta)N^{-1}\sum_{k\ge nN}E_{\mathscr{R}_{c,N}}(\eta(k))$. By (2.3), we note $\sum_{k\ge 1}E_{\mathscr{R}_{c,N}}(\eta(k)) = O(N)$. Hence, the above display vanishes as $n \to \infty$ uniformly for $N \ge 1$, and (6.1) is proved.

*Step 2.* Define $\psi(t,x) = \int_x^\infty \rho(t,u)du$. Then, $\psi(t,x)$ belongs to $\mathcal{W}$ and is the unique weak solution of (2.16) as shown in Subsection 10.1. Now, fix any $G \in C_c^\infty(\mathbb{R}_\circ^+)$ and define $g(x) = \int_0^x G(u)du$ for all $x \in \mathbb{R}_\circ^+$. By integration by parts, we have $\int_0^\infty G(x)\psi(t,x)dx = \int_0^\infty g(x)\rho(t,x)dx$.

Recall $\psi_N$ from (2.14). Using summation by parts, we have

$$\begin{aligned}\int_0^\infty G(x)\psi_N(t,x)dx =& \sum_{k=1}^{\infty}\Big[g\Big(\frac{k}{N}\Big)-g\Big(\frac{k-1}{N}\Big)\Big]\psi_N(t,k/N)\\ =& \frac{1}{N}\sum_{k=1}^{\infty}g\Big(\frac{k}{N}\Big)\eta_t(k).\end{aligned}$$

Then, we obtain (2.15) from (6.1) and Corollary 2.7 is proved. □

## 7 Tightness and properties of limit measures

In this section, we obtain tightness of the family of probability measures $\{Q^N\}_{N\in\mathbb{N}}$ on the trajectory space $D([0,T],\mathcal{M})$. Then, we show some properties of the limit measures $Q$.

### 7.1 Tightness

We show that $\{Q^N\}$ is tight with respect to the uniform topology, stronger than the Skorokhod topology on $D([0,T],\mathcal{M})$.

**Lemma 7.1.** $\{Q^N\}_{N\in\mathbb{N}}$ *is relatively compact with respect to the uniform topology. As a consequence, all limit points $Q$ are supported on vaguely continuous trajectories $\pi$, that is for $G \in C_c^\infty(\mathbb{R}_\circ^+)$ we have $t \in [0,T] \mapsto \langle G,\pi_t\rangle$ is continuous.*

*Proof.* Recall the distance $d$ and space of measures $\mathcal{M}$ in Section 2. To show that $\{Q^N\}$ is relatively compact with respect to uniform topology, we show the following items (cf. p. 51 [16]).

1. For each $t \in [0,T]$, $\varepsilon > 0$, there exists a compact set $K_{t,\varepsilon} \subset \mathcal{M}$ such that
$$\sup_N Q^N\left[\pi_\cdot^N : \pi_t^N \notin K_{t,\varepsilon}\right] \le \varepsilon. \tag{7.1}$$
2. For every $\varepsilon > 0$,
$$\lim_{\gamma\to 0}\limsup_{N\to\infty} Q^N\Big[\pi_\cdot^N : \sup_{|t-s|<\gamma} d(\pi_t^N,\pi_s^N) > \varepsilon\Big] = 0. \tag{7.2}$$

We now argue the first condition (7.1). Indeed, since the dynamics is attractive (cf. (2.10)), we have

$$Q^N\Big[\langle 1,\pi_t^N\rangle > A\Big] \le \mathscr{R}_{c,N}\Big[N^{-1}N_\beta\sum_{k\ge 1}\eta(k) > A\Big].$$

Applying Markov's inequality and using the mean particle estimate (2.3), we obtain

$$Q^N\left[\langle 1,\pi_t^N\rangle > A\right] \leq \frac{C}{A}$$

for some constant $C$ independent of $N$ and $A$. Notice that the set $\{\mu \in \mathcal{M} : \langle 1,\mu\rangle \leq A\}$ is compact in $\mathcal{M}$, then the first condition (7.1) is checked by taking $A$ large.

To show the second condition (7.2), it is enough to show a counterpart of the condition for the distributions of $\langle G,\pi_\cdot^N\rangle$ where $G$ is any smooth test function with compact support in $\mathbb{R}_\circ^+$ (cf. p. 54, [16]). In other words, we need to show, for every $\varepsilon > 0$,

$$\lim_{\gamma\to 0}\lim_{N\to\infty} Q^N\Big[\pi_\cdot^N : \sup_{|t-s|<\gamma}\Big|\langle G,\pi_t^N\rangle - \langle G,\pi_s^N\rangle\Big| > \varepsilon\Big] = 0. \tag{7.3}$$

We now show the condition (7.3). Since

$$\left\langle G,\pi_t^N\right\rangle = \left\langle G,\pi_0^N\right\rangle + \int_0^t N^2 L\left\langle G,\pi_s^N\right\rangle ds + M_t^{N,G}$$

we only need to consider the oscillations of $\displaystyle\int_0^t N^2 L\left\langle G,\pi_s^N\right\rangle ds$ and $M_t^{N,G}$ respectively.

Suppose that $G$ has support $[a,b]$ with $0 < a < b < \infty$. Recall the generator computation (3.1). For $N$ large, we have

$$\begin{aligned}
&\sup_{|t-s|<\gamma}\Big|\int_s^t N^2 L\left\langle G,\pi_\tau^N\right\rangle d\tau\Big| \\
=&\sup_{|t-s|<\gamma}\Big|\int_s^t \left\{\frac{N_\beta}{N}\sum_{k=aN}^{bN}\Big(\Delta_N G\,(k/N) + \frac{\lambda_k - 1}{1/N}\nabla_N G\,(k/N)\Big)\chi_{\{\eta_\tau(k)>0\}}\right\} d\tau\Big| \\
\leq& C_G \sup_{|t-s|<\gamma}\int_s^t\left\{\frac{N_\beta}{N}\sum_{k=aN}^{bN}\chi_{\{\eta_\tau(k)>0\}}\right\} d\tau.
\end{aligned}$$

When $\beta = 0$, we have $N_\beta = 1$. Since $\chi_{\{\eta(k)>0\}} \leq 1$, then $\displaystyle\sup_{|t-s|<\gamma}\Big|\int_s^t N^2 L\left\langle G,\pi_\tau^N\right\rangle d\tau\Big| \leq$ $C_G(b-a)\gamma$ vanishes as $\gamma \to 0$.

For other case $\beta > 0$, we bound $\chi_{\{\eta(k)>0\}} \leq \eta(k)$. Then, by conservation of mass,

$$\begin{aligned}
\sup_{|t-s|<\gamma}\left|\int_s^t N^2 L\left\langle G,\pi_\tau^N\right\rangle d\tau\right| &\leq C_G \sup_{|t-s|<\gamma}\int_s^t \left\{\frac{1}{N}\sum_{k=1}^\infty N_\beta\eta_\tau(k)\right\} d\tau \\
&= C_G\gamma\frac{1}{N}\sum_{k=1}^\infty N_\beta\eta_0(k).
\end{aligned}$$

Recall the total expected number of particles is of order $NN_\beta^{-1}$ (cf. (2.6)). By Markov inequality, $Q^N\Big[\sup_{|t-s|<\gamma}\Big|\displaystyle\int_s^t N^2 L\left\langle G,\pi_\tau^N\right\rangle d\tau\Big| > \varepsilon\Big] \leq \dfrac{C_G\gamma}{\varepsilon}\mathbb{E}_N\Big(N^{-1}\sum_{k=1}^\infty N_\beta\eta_0(k)\Big)$, vanishes as $N\uparrow\infty$ and $\gamma\downarrow 0$.

Next, we treat the martingale $M_t^{N,G}$. Trivially, by $\left|M_t^{N,G} - M_s^{N,G}\right| \leq \left|M_t^{N,G}\right| + \left|M_s^{N,G}\right|$, we have

$$\mathbb{P}_N\Big(\sup_{|t-s|<\gamma}\left|M_t^{N,G} - M_s^{N,G}\right| > \varepsilon\Big) \leq 2\mathbb{P}_N\Big(\sup_{0\leq t\leq T}\left|M_t^{N,G}\right| > \varepsilon/2\Big)$$

which, by Chebychev and Doob's inequality, is bounded by

$$\frac{8}{\varepsilon^2}\mathbb{E}_N\Big[\Big(\sup_{0\leq t\leq T}\left|M_t^{N,G}\right|\Big)^2\Big] \leq \frac{32}{\varepsilon^2}\mathbb{E}_N\Big[\big(M_T^{N,G}\big)^2\Big] = \frac{32}{\varepsilon^2}\mathbb{E}_N\langle M^{N,G}\rangle_T.$$

Now, by Lemma 3.1, $\mathbb{E}_N\langle M^{N,G}\rangle_T$ is of order $O(N_\beta N^{-1}) = o(1)$ (cf. (2.2)). Then, we conclude

$$\lim_{\gamma\to 0}\lim_{N\to\infty}\mathbb{P}_N\Big(\sup_{|t-s|<\gamma}\Big|M_t^{N,G}-M_s^{N,G}\Big|>\varepsilon\Big)=0. \qquad \square$$

### 7.2 Properties of limit measures.

By Lemma 7.1, the sequence $\{Q^N\}$ is relatively compact with respect to the uniform topology. Consider any convergent subsequence of $Q^N$ and relabel so that $Q^N \Rightarrow Q$.

We now show some properties of $Q$.

**Lemma 7.2.** *$Q$ is supported on absolutely continuous trajectories whose densities satisfy certain bounds:*

$$Q\left[\pi_\cdot : \pi_t(dx) = \pi(t,x)dx \textit{ with } \pi(t,\cdot)\le \phi_c(\cdot) \textit{ for all } 0\le t\le T\right]=1.$$

*Proof.* Let $C_c^+(\mathbb{R}_\circ^+)$ be the space of nonnegative continuous functions with compact support on $\mathbb{R}_\circ^+$ and we equip it with the topology of uniform convergence on compact sets. Take $\{G_n\}_{n\in\mathbb{N}}$ be a dense sequence of $C_c^+(\mathbb{R}_\circ^+)$. The lemma is equivalent to

$$Q\Big[\langle G_n,\pi_t\rangle \le \int_{\mathbb{R}_\circ^+} G_n(x)\phi_c(x)dx \text{ for all } 0\le t\le T \text{ and } n\in\mathbb{N}\Big]=1.$$

Let $\{t_k\}_{k\in\mathbb{N}}$ be a dense subset of $[0,T]$. Assume for this moment, for any $n,k\in\mathbb{N}$ and $\varepsilon>0$, that

$$Q\Big[\langle G_n,\pi_{t_k}\rangle \le \int_{\mathbb{R}_\circ^+} G_n(x)\phi_c(x)dx+\varepsilon\Big]=1. \tag{7.4}$$

Since $Q$ is supported on vaguely continuous trajectories by Lemma 7.1, we obtain for all $\varepsilon>0$,

$$Q\Big[\langle G_n,\pi_t\rangle \le \int_{\mathbb{R}_\circ^+} G_n(x)\phi_c(x)dx+\varepsilon \text{ for all } 0\le t\le T, n\in\mathbb{N}\Big]=1.$$

Then, we conclude the lemma by taking $\varepsilon\to 0$.

It remains to prove (7.4). Fix $k,n$, and $\varepsilon$ and observe

$$Q^N\Big[\langle G_n,\pi_{t_k}^N\rangle \le \int_{\mathbb{R}_\circ^+} G_n\phi_c dx+\varepsilon\Big] = \mathbb{P}_N\Big[\frac{N_\beta}{N}\sum_{j=1}^\infty G_n(j/N)\eta_{t_k}(j)\le \int_{\mathbb{R}_\circ^+} G_n\phi_c dx+\varepsilon\Big]$$

By attractiveness (cf. Subsection 2.2.1) and the assumption $\nu^N\le \mathscr{R}_{c,N}$, the above display is bounded from below by

$$\mathscr{R}_{c,N}\Big[\frac{N_\beta}{N}\sum_{j=1}^\infty G_n(j/N)\eta(j)\le \int_{\mathbb{R}_\circ^+} G_n\phi_c dx+\varepsilon\Big],$$

which approaches $1$ as $N\to\infty$ by Proposition 2.2. Then, we have

$$\limsup_{N\to\infty} Q^N\left[\langle G_n,\pi_{t_k}^N\rangle \le \int_{\mathbb{R}_\circ^+} G_n\phi_c dx+\varepsilon\right]=1.$$

As compactness of $\{Q^N\}$ was shown in the uniform topology in Lemma 7.1, the distribution of $\langle G_n,\pi_{t_k}^N\rangle$ under $Q^N$ converges weakly to $\langle G_n,\pi_{t_k}\rangle$ under $Q$. Hence, (7.4) follows. $\square$

**Lemma 7.3.** *$Q$ is supported on trajectories with constant total mass:*

$$Q\Big[\pi_\cdot : \langle 1,\pi_t\rangle = \int_0^\infty \rho_0 dx \textit{ for all } 0\le t\le T\Big]=1.$$

*Proof.* Let $\{t_k\}_{k\in\mathbb{N}}$ be a dense subset of $[0,T]$. By compactness in the uniform topology, we have that as $N\to\infty$, the distribution of $\pi_t^N$ under $Q^N$ converges weakly to $\pi_t$ under $Q$. We will show that there exists an increasing sequence of $\{G_n\}_{n\geq 1}\subset C_c(\mathbb{R}_\circ^+)$ such that $\lim_{n\to\infty} G_n(x)=1$ and for all $n,k$,

$$\liminf_{N\to\infty} Q^N\Big[\Big|\langle G_n,\pi_{t_k}^N\rangle-\int_0^\infty \rho_0 dx\Big|>\frac{1}{n}\Big]=0. \tag{7.5}$$

Since $Q^N$ converges to $Q$ with respect to the uniform topolgy (cf. Lemma 7.1), we have $\pi_{t_k}^N$ converges weakly to $\pi_{t_k}$. Then, assuming (7.5), we conclude that

$$Q\Big[\Big|\langle G_n,\pi_{t_k}\rangle-\int_0^\infty \rho_0 dx\Big|>\frac{1}{n}\Big]=0$$

for all $n,k$. Therefore

$$Q\Big[\Big|\langle G_n,\pi_{t_k}\rangle-\int_0^\infty \rho_0 dx\Big|\leq\frac{1}{n},\text{ for all } n,k\Big]=1.$$

Since also $Q$ is supported on vaguely continuous $\pi_\cdot$, we have

$$Q\Big[\Big|\langle G_n,\pi_{t}\rangle-\int_0^\infty \rho_0 dx\Big|\leq\frac{1}{n},\text{ for all } n, 0\leq t\leq T\Big]=1,$$

which clearly implies the lemma.

Now, we focus on the proof of (7.5). For $G\geq 0$

$$\begin{aligned}&Q^N\Big[\Big|\langle G,\pi_{t_k}\rangle-\int_0^\infty \rho_0 dx\Big|>\frac{1}{n}\Big]\\ \leq&Q^N\Big[\langle 1-G,\pi_{t_k}\rangle>\frac{1}{2n}\Big]+Q^N\Big[\Big|\langle 1,\pi_0\rangle-\int_0^\infty\rho_0dx\Big|>\frac{1}{2n}\Big].\end{aligned} \tag{7.6}$$

By (2.7), the variance $\lim_{N\to\infty}\dfrac{N_\beta^2}{N^2}\sum_{k=1}^\infty \mathrm{Var}_{\nu^N}(\eta(k))=0$. Also, by part (1) of Condition 2.3, $\lim_{N\to\infty}\frac{1}{N}\sum_{k\geq 1}\big|N_\beta m_{N,k}-\overline{\rho}_{N,k}\big|=0$. Therefore, by adding and subtracting the mean $m_{N,k}$ inside the absolute value, the second term on the right-hand side of (7.6) vanishes as $N\to\infty$.

We now specify $G_n\in C_c(\mathbb{R}_\circ^+)$ as follows:

$$0\leq G_n\leq 1,\quad G_n=1\text{ on }[a,b]\text{ where }\int_{(0,a)\cup(b,\infty)}\phi_c dx<\frac{1}{3n^2}.$$

Since $\nu^N\leq\mathscr{R}_{c,N}$, by attractiveness (cf. Subsection 2.2.1), for each $t\in[0,T]$, we have

$$Q^N\Big[\langle 1-G_n,\pi_{t_k}\rangle>\frac{1}{2n}\Big]\leq\mathscr{R}_{c,N}\Big[\frac{1}{N}\sum_{\frac{k}{N}\in[0,a)\cup(b,\infty)}N_\beta\eta(k)>\frac{1}{2n}\Big]. \tag{7.7}$$

Recall that $\rho_{k,c}=E_{\mathscr{R}_{c,N}}\eta(k)$ (cf. (2.1)). In Lemma 9.3, it is shown that $\frac{1}{N}\sum_{k\geq 1}\big|N_\beta\rho_{k,c}-\overline{\rho}_{N,k}\big|$, where $\overline{\rho}_{N,k}=N\int_{(k-1)/N}^{k/N}\phi_c(x)dx$, vanishes as $N\to\infty$. Also, $\int_{(0,a)\cup(b,\infty)}\phi_c dx<1/(3n^2)<2/n$, for all $n\geq 1$. Then, by subtracting and adding the mean $N_\beta\rho_{k,c}$, we conclude by Markov inequality and straightforward manipulation that (7.7) vanishes as $N\to\infty$. □

## 8 $1$- and $2$-blocks estimates

In this section, we prove the 1- and 2-block estimate. The statement and proof for the 1-block estimate is written for all three cases of $\beta$ and $\mathcal{E}_k$, while the 2-block estimate assumes $\beta = 0$. In passing, although it is not consequential in this work, we remark that the 2-block estimate may not hold for the other cases; see the beginning of Subsection 8.3 for more comments.

The plan is now to show in the succeeding subsections, a spectral gap bound, and then the $1$ and 2-block estimates.

### 8.1 Spectral gap bound for $1$-block estimate

We obtain now a spectral gap bound to prepare for the 1-block estimate. Define, for $k, l \geq 1$ such that $k - l \geq 1$, the set $\Lambda_{k,l} = \{k - l, k - l + 1, \ldots, k + l\} \subset \mathbb{N}$. Recall that $\theta_k = e^{-\beta\mathcal{E}_k - k/N}$ and $\lambda_k = \dfrac{\theta_{k+1}}{\theta_k}$ (cf. (2.5)). Consider the process restricted to $\Lambda_{k,l}$ generated by $L_{k,l}$ where

$$\begin{aligned} L_{k,l}f(\eta) = \sum_{x,x+1\in\Lambda_{k,l}} &\Big\{\lambda_k \left[f\left(\eta^{x,x+1}\right) - f(\eta)\right] \chi_{\{\eta(x)>0\}} \\ &+ \left[f\left(\eta^{x+1,x}\right) - f(\eta)\right] \chi_{\{\eta(x+1)>0\}}\Big\}. \end{aligned}$$

We will obtain the spectral gap estimate by showing a Poincaré inequality. To state this bound, we need a few more definitions. With respect to product measure $\mu := \mathscr{R}_{c,N}$, let $\mu_{k,l}$ be its restriction to $\Omega_{k,l} = \{0, 1, 2, \ldots\}^{\Lambda_{k,l}}$, that is

$$\mu_{k,l}(\eta) = \prod_{x\in\Lambda_{k,l}} (1 - \theta_{x,c})\theta_{x,c}^{\eta(x)}, \quad \text{where } \theta_{x,c} = ce^{-\beta\mathcal{E}_x - x/N}. \tag{8.1}$$

Let $\mu_{k,l,j}$ be the associated reversible canonical measure on

$$\Omega_{k,l,j} = \{\eta \in \Omega_{k,l} : \sum_{x\in\Lambda_{k,l}} \eta(x) = j\},$$

that is $\mu_{k,l}$ is conditioned so that there are exactly $j$ particles counted in $\Omega_{k,l}$.

The corresponding Dirichlet form is written as

$$E_{\mu_{k,l,j}}\left[f(-L_{k,l}f)\right] = \sum_{x,x+1\in\Lambda_{k,l}} E_{\mu_{k,l,j}}\left[\chi_{\{\eta(x+1)>0\}}\left[f\left(\eta^{x+1,x}\right) - f(\eta)\right]^2\right]. \tag{8.2}$$

The primary method will be to compare with the spectral gap for the standard translation-invariant localized process. Consider the generator $L_l$ on $\Omega_{k,l}$ given by

$$\begin{aligned} L_l f(\eta) =& \frac{1}{2}\sum_{x,x+1\in\Lambda_{k,l}} \Big\{\left[f\left(\eta^{x,x+1}\right) - f(\eta)\right] \chi_{\{\eta(x)>0\}} \\ &+ \left[f\left(\eta^{x+1,x}\right) - f(\eta)\right] \chi_{\{\eta(x+1)>0\}}\Big\}. \end{aligned} \tag{8.3}$$

Let $\nu_\rho$ be the product measure on $\Omega$ with common Geometric marginal on each site $k \in \mathbb{N}$ with mean $\rho$, and let $\nu_l^\rho$ be its restriction to $\Omega_{k,l}$.

Consider $\nu_{l,j}$, the associated canonical measure on $\Omega_{k,l,j}$, with respect to $j$ particles in $\Lambda_{k,l}$, which does not depend on $\rho$. It is well-known that $\nu_l^\rho$ and $\nu_{l,j}$ are both invariant measures with respect to the localized $L_l$ (cf. [1]). The corresponding Dirichlet form is given by

$$E_{\nu_{l,j}}\left[f(-L_l f)\right] = \frac{1}{2}\sum_{x,x+1\in\Lambda_{k,l}} E_{\nu_{l,j}}\left[\chi_{\{\eta(x+1)>0\}}\left[f\left(\eta^{x+1,x}\right) - f(\eta)\right]^2\right]. \tag{8.4}$$

Finally, let $x_1 = \arg\max_{x\in\Lambda_{k,l}} \mathcal{E}_x$ and $x_2 = \arg\min_{x\in\Lambda_{k,l}} \mathcal{E}_x$. Also, for convenience, let $\varepsilon = e^{-1/N}$.

**Lemma 8.1.** *We have the following estimates:*

1. *Uniform bound: For all $\eta \in \Omega_{k,l,j}$, we have*
$$r_{k,l,j,\varepsilon}^{-1} \le \frac{\mu_{k,l,j}(\eta)}{\nu_{l,j}(\eta)} \le r_{k,l,j,\varepsilon} \tag{8.5}$$
*where* $r_{k,l,j,\varepsilon} := \left(\frac{1-ce^{-\beta\mathcal{E}_{x_1}}\varepsilon^{k+l}}{1-ce^{-\beta\mathcal{E}_{x_2}}\varepsilon^{k-l}}\right)^{2l+1} \left(e^{-\beta(\mathcal{E}_{x_2}-\mathcal{E}_{x_1})}\varepsilon^{-2l}\right)^j.$
2. *Poincaré inequality: We have*
$$\mathrm{Var}_{\mu_{k,l,j}}(f) \le C_{k,l,j} E_{\mu_{k,l,j}}\left[f(-L_{k,l}f)\right]$$
*where* $C_{k,l,j} := \frac{C}{2}(2l+1)^2\Big(1+\frac{j}{2l+1}\Big)^2 r_{k,l,j,\varepsilon}^2$ *bounds the inverse of the spectral gap of $-L_{k,l}$ on $\Omega_{k,l,j}$ and $C$ is a universal constant.*
3. *For each $0<a<b<\infty$, $l$ and $j$, we have*
$$\lim_{N\uparrow\infty} \sup_{aN\le k\le bN} r_{k,l,j,\varepsilon} = 1,$$
*and hence* $\sup_{N\ge1}\sup_{aN\le k\le bN} C_{k,l,j} < \infty$.

*Proof.* First, the spectral gap for one dimensional localized symmetric zero range process with rate function $\chi_{\{\cdot>0\}}$ is well known (cf. [20]): For all $j$, with respect to an universal constant $C$,
$$\mathrm{Var}_{\nu_{l,j}}(f) \le C(2l+1)^2\Big(1+\frac{j}{2l+1}\Big)^2 E_{\nu_{l,j}}[f(-L_lf)]. \tag{8.6}$$
Therefore, the inverse of the spectral gap is bounded below by $\Big[C(2l+1)^2\Big(1+\frac{j}{2l+1}\Big)^2\Big]^{-1}$.

To get an estimate with respect to $-L_{k,l}$, we will compare $\mu_{k,l,j}$ with $\nu_{l,j}$. The canonical measure $\nu_{l,j}$ is the measure $\nu_\rho$ conditioned on $j$ particles in $\Lambda_{k,l}$ for any $\rho$. It will be convenient now to choose $\rho$ such that $\frac{\rho}{1+\rho} = \varepsilon$, that is, $\varepsilon$ is the common parameter of the Geometric marginals of $\nu_\rho$.

For $\eta\in\Omega_{k,l,j}$, we have
$$\frac{\mu_{k,l,j}(\eta)}{\nu_{l,j}(\eta)} = \frac{\mu_{k,l}(\eta)}{\nu_l^\rho(\eta)}\frac{\nu_l^\rho(\Omega_{k,l,j})}{\mu_{k,l}(\Omega_{k,l,j})}.$$
Recall that $\theta_{k,c} = ce^{-\beta\mathcal{E}_k - k/N}$. Since $\mu_{k,l}$ (cf. (8.1)) and $\nu_l^\rho$ are product measures,
$$\frac{\mu_{k,l}(\eta)}{\nu_l^\rho(\eta)} = \frac{\prod_{x\in\Lambda_{k,l}}(1-\theta_{x,c})\theta_{x,c}^{\eta(x)}}{\prod_{x\in\Lambda_{k,l}}(1-\varepsilon)\varepsilon^{\eta(x)}}. \tag{8.7}$$
Now, for $\eta\in\Omega_{k,l,j}$, recalling the definitions of $x_1$ and $x_2$ given above, we have
$$\begin{aligned}&\left(1-ce^{-\beta\mathcal{E}_{x_2}}\varepsilon^{k-l}\right)^{2l+1}\left(ce^{-\beta\mathcal{E}_{x_1}}\varepsilon^{k+l}\right)^j \\ &\qquad \le \mu_{k,l}(\eta) \le \left(1-ce^{-\beta\mathcal{E}_{x_1}}\varepsilon^{k+l}\right)^{2l+1}\left(ce^{-\beta\mathcal{E}_{x_2}}\varepsilon^{k-l}\right)^j.\end{aligned}$$
Inputting into (8.7), we obtain
$$\frac{(1-ce^{-\beta\mathcal{E}_{x_2}}\varepsilon^{k-l})^{2l+1}c^je^{-\beta\mathcal{E}_{x_1}j}\varepsilon^{(k+l)j}}{(1-\varepsilon)^{2l+1}\varepsilon^j} \le \frac{\mu_{k,l}(\eta)}{\nu_l^\rho(\eta)} \le \frac{(1-ce^{-\beta\mathcal{E}_{x_1}}\varepsilon^{k+l})^{2l+1}c^je^{-\beta\mathcal{E}_{x_2}j}\varepsilon^{(k-l)j}}{(1-\varepsilon)^{2l+1}\varepsilon^j}.$$

Noting $\mu_{k,l}(\Omega_{k,l,j}) = \sum_{\eta\in\Omega_{k,l,j}} \left[\mu_{k,l}(\eta)/\nu_l^\rho(\eta)\right]\nu_l^\rho(\eta)$, we have

$$\frac{(1-ce^{-\beta\mathcal{E}_{x_2}}\varepsilon^{k-l})^{2l+1}c^j e^{-\beta\mathcal{E}_{x_1}j}\varepsilon^{(k+l)j}}{(1-\varepsilon)^{2l+1}\varepsilon^j} \\ \le \frac{\mu_{k,l}(\Omega_{k,l,j})}{\nu_l^\rho(\Omega_{k,l,j})} \le \frac{(1-ce^{-\beta\mathcal{E}_{x_1}}\varepsilon^{k+l})^{2l+1}c^j e^{-\beta\mathcal{E}_{x_2}j}\varepsilon^{(k-l)j}}{(1-\varepsilon)^{2l+1}\varepsilon^j}.$$

Then, rearranging the formulas establishes (8.5): $r^{-1}_{k,l,j,\varepsilon} \le \left[\mu_{k,l,j}(\eta)/\nu_{l,j}(\eta)\right] \le r_{k,l,j,\varepsilon}$.

Turning now to the Poincaré inequality, from (8.4) and (8.2), using (8.5), we have

$$E_{\nu_{l,j}}\left[f(-L_l f)\right] \le \frac{r_{k,l,j,\varepsilon}}{2} E_{\mu_{k,l,j}}\left[f(-L_{k,l}f)\right]. \tag{8.8}$$

Now, since

$$\begin{aligned}\mathrm{Var}_{\mu_{k,l,j}}(f) = \inf_a E_{\mu_{k,l,j}}\left[(f-a)^2\right] &\le r_{k,l,j,\varepsilon} \inf_a E_{\nu_{l,j}}\left[(f-a)^2\right] \\ &= r_{k,l,j,\varepsilon}\mathrm{Var}_{\nu_{l,j}}(f),\end{aligned}$$

the desired Poincaré inequality follows from (8.6) and (8.8).

For the last item, recall that, for fixed $l$ and $j$,

$$r_{k,l,j,\varepsilon} = \left(\frac{1-ce^{-\beta\mathcal{E}_{x_1}}\varepsilon^{k+l}}{1-ce^{-\beta\mathcal{E}_{x_2}}\varepsilon^{k-l}}\right)^{2l+1}\left(e^{-\beta(\mathcal{E}_{x_2}-\mathcal{E}_{x_1})}\varepsilon^{-2l}\right)^j$$

where $\varepsilon = e^{-1/N}$ and $\mathcal{E}_k = u(\ln k)$ (cf. Condition 2.1). We observe that $\varepsilon \uparrow 1$ as $N \uparrow \infty$. Also, $\mathcal{E}_{x_i} = u(\ln(x_i)) \to \infty$ as $N \uparrow \infty$ given that $aN - l \le x_i \le bN + l$ for $i = 1, 2$. Notice now $\mathcal{E}_{x_2} - \mathcal{E}_{x_1} = u(\ln(x_2)) - u(\ln(x_1)) = u'(y)\ln(x_2/x_1)$ where $y$ is between $\ln(x_2)$ and $\ln(x_1)$ and so $u'(y) \to 0$ or $1$ as $N \uparrow \infty$ (cf. Condition 2.1). Also, as $k - l \le x_1, x_2 \le k + l$, $l$ is fixed, and $aN \le k \le bN$, we have that $\ln(x_2/x_1) \to 0$ as $N \uparrow \infty$. All these comments lead to the claim that $r_{k,l,j,\varepsilon} \to 1$, uniformly over $aN \le k \le bN$, as $N \uparrow \infty$. Moreover, by the form of $C_{k,l,j}$, we see that $C_{k,l,j}$ is uniformly bounded for $aN \le k \le bN$ and $N \ge 1$. □

## 8.2 1-block estimate

Recall $D^{G,s}_{N,k}$ from (3.2). Define

$$V_{k,l}(s,\eta) := D^{G,s}_{N,k}\left(h(\eta(k)) - E_{\nu_{\eta^l(k)}}[h]\right)$$

where $h(x) := \chi_{\{x>0\}}$ and $E_{\nu_\rho}[h] := E_{\nu_\rho}[h(\eta(k))] = \dfrac{\rho}{1+\rho}$.

The 1-block estimate is the following limit.

**Lemma 8.2** (1-block estimate)**.** *We have*

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}_N\left[\left|\int_0^T N_\beta V_{k,l}(s,\eta_s)ds\right|\right] = 0.$$

*Proof.* We separate the argument into 7 steps.

*Step 1.* We first introduce a cutoff of large densities: For any $l$ and $\epsilon > 0$, we may find an $A$ such that for all $t \ge 0$, large $N$, and $aN \le k \le bN$, we have $\mathbb{E}_N(\chi_{\{\eta^l_t(k)>A\}}) < \epsilon N_\beta^{-1}$. Indeed, as $\nu^N \le \mathscr{R}_{c,N}$, by attractiveness (2.10), $\mathbb{E}_N(\chi_{\{\eta^l_t(k)>A\}}) \le E_{\mathscr{R}_{c,N}}(\chi_{\{\eta^l(k)>A\}})$. By Markov's inequality,

$$E_{\mathscr{R}_{c,N}}(\chi_{\{\eta^l(k)>A\}}) \le \frac{1}{A(2l+1)}\sum_{j=k-l}^{k+l} E_{\mathscr{R}_{c,N}}(\eta(k)).$$

Since $N_\beta E_{\mathscr{R}_{c,N}}(\eta(k))$ is uniformly bounded for all $aN \le k \le bN$ and $N \in \mathbb{N}$ by (2.8), it suffices to take $A$ large enough.

Note that $|D^{G,s}_{N,k}| \le C(a,b,G)$ (cf. (3.4)). Then,

$$\begin{aligned}
&\mathbb{E}_N\Big[\Big|\int_0^T V_{k,l}(s,\eta_s)ds\Big|\Big]\\
\le&\mathbb{E}_N\Big[\Big|\int_0^T V_{k,l}(s,\eta_s)\chi_{\{\eta_s^l(k)\le A\}}ds\Big|\Big]+\mathbb{E}_N\Big[\Big|\int_0^T V_{k,l}(s,\eta_s)\chi_{\{\eta_s^l(k)> A\}}ds\Big|\Big]\\
\le&\mathbb{E}_N\Big[\Big|\int_0^T V_{k,l}(s,\eta_s)\chi_{\{\eta_s^l(k)\le A\}}ds\Big|\Big]+C(a,b,G)\mathbb{E}_N\Big[\int_0^T \chi_{\{\eta_s^l(k)> A\}}ds\Big]\\
=&\mathbb{E}_N\Big[\Big|\int_0^T V_{k,l}(s,\eta_s)\chi_{\{\eta_s^l(k)\le A\}}ds\Big|\Big]+C(a,b,G)\int_0^T \mathbb{E}_N\big[\chi_{\{\eta_s^l(k)> A\}}\big]ds\\
\le&\mathbb{E}_N\Big[\Big|\int_0^T V_{k,l}(s,\eta_s)\chi_{\{\eta_s^l(k)\le A\}}ds\Big|\Big]+C(a,b,G)T\epsilon N_\beta^{-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}_N\Big[\Big|\int_0^T N_\beta V_{k,l}(s,\eta_s)ds\Big|\Big]\\
\le&\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}_N\Big[\Big|\int_0^T N_\beta V_{k,l}(s,\eta_s)\chi_{\{\eta_s^l(k)\le A\}}ds\Big|\Big]+C(a,b,G)T\epsilon.
\end{aligned}$$

For convenience, we write

$$\tilde{V}_{k,l,A}(s,\eta) := V_{k,l}(s,\eta)\chi_{\{\eta^l(k)\le A\}}.$$

Then, to prove the lemma, it will be enough to show

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}_N\Big[\Big|\int_0^T N_\beta \tilde{V}_{k,l,A}(s,\eta_s)ds\Big|\Big]=0,$$

and then at the end let $\epsilon \to 0$.

*Step 2.* Define $\Lambda_{k,l}(\eta)$ be the number of particles in $\Lambda_{k,l}$, that is $\Lambda_{k,l}(\eta) := (2l+1)\eta^l(k)$. We would like to replace $\tilde{V}_{k,l,A}(s,\eta)$ by its recentered version:

$$V_{k,l,A}(s,\eta) := D^{G,s}_{N,k}\big(h(\eta(k)) - E_{\mu_{k,l,\Lambda_{k,l}(\eta)}}[h(\eta(k))]\big)\chi_{\{\eta^l(k)\le A\}}.$$

The advantage of working with $V_{k,l,A}$ is that $E_{\mu_{k,l,j}}[V_{k,l,A}] = 0$ for all $k, l, j$. The difference in making such a replacement is less than

$$\mathbb{E}_N\Big[\int_0^T N_\beta \chi_{\{0<\eta_s^l(k)\le A\}}\Big|E_{\mu_{k,l,\Lambda_{k,l}(\eta_s)}}[h(\eta(k))] - E_{\nu_{\eta_s^l(k)}}[h]\Big|ds\Big]. \tag{8.9}$$

In the above, we replaced $\chi_{\{\eta^l(k)\le A\}}$ by $\chi_{\{0<\eta^l(k)\le A\}}$, since $h$ vanishes when $\eta^l(k) = 0$.

By adding and subtracting, (8.9) is bounded by

$$\begin{aligned}
&\mathbb{E}_N\Big[\int_0^T N_\beta \chi_{\{0<\eta_s^l(k)\le A\}}\Big|E_{\mu_{k,l,\Lambda_{k,l}(\eta_s)}}[h(\eta(k))] - E_{\nu_{k,l,\Lambda_{k,l}(\eta_s)}}[h(\eta(k))]\Big|ds\Big]\\
+&\mathbb{E}_N\Big[\int_0^T N_\beta \chi_{\{0<\eta_s^l(k)\le A\}}\Big|E_{\nu_{k,l,\Lambda_{k,l}(\eta_s)}}[h(\eta(k))] - E_{\nu_{\eta_s^l(k)}}[h]\Big|ds\Big] =: A_1 + A_2.
\end{aligned}$$

*Step 3.* Now, by (8.5) and $0 \le h \le 1$, we have

$$\begin{aligned}
&\Big|E_{\mu_{k,l,\Lambda_{k,l}(\eta)}}[h(\eta(k))] - E_{\nu_{k,l,\Lambda_{k,l}(\eta)}}[h(\eta(k))]\Big|\\
&\qquad\le E_{\nu_{k,l,\Lambda_{k,l}(\eta)}}[h(\eta(k))](r_{k,l,j,\varepsilon}-1) \le r_{k,l,j,\varepsilon}-1.
\end{aligned}$$

Then, by $\nu^N \le \mathscr{R}_{c,N}$ and attractiveness (cf. Subsection 2.2.1), and $\chi_{\{\eta^l(k)>0\}} \le \eta^l(k)$, the term $A_1$ is bounded by

$$
\begin{aligned}
(r_{k,l,j,\varepsilon}-1)\mathbb{E}_N\Big[\int_0^T N_\beta \chi_{\{0<\eta^l_s(k)\le A\}} ds\Big] &\le (r_{k,l,j,\varepsilon}-1)\mathbb{E}_{\mathscr{R}_{c,N}}\Big[\int_0^T N_\beta \chi_{\{\eta^l_s(k)>0\}} ds\Big]\\
&\le (r_{k,l,j,\varepsilon}-1)TN_\beta E_{\mathscr{R}_{c,N}}\big[\eta^l(k)\big].
\end{aligned}
$$

By (2.8), $N_\beta E_{\mathscr{R}_{c,N}}\big[\eta^l(k)\big] \le N_\beta \sup_{k-l\le j\le k+l}\rho_{j,c}$ is uniformly bounded for each $l\ge 1$, and $aN\le k\le bN$ for all $N$ large. Hence, for each $l$, $\sup_{aN\le k\le bN} A_1$ vanishes as $N\uparrow\infty$, as $r_{k,l,j,\varepsilon}\to 1$ by item (3) in Lemma 8.1.

On the other hand, by equivalence of ensembles (cf. p.355, [16]), the absolute value in $A_2$ vanishes as $l\to\infty$, uniformly in $k$ as $\nu_{k,l,j}$ and $\nu_{j/2l+1}$ are translation-invariant and do not depend on $k$. Therefore, the term $A_2$ vanishes as well as we take $N\to\infty$, $l\to\infty$ in order.

*Step 4.* Now, the proof of the lemma is reduced to prove

$$
\limsup_{l\to\infty}\limsup_{N\to\infty} N_\beta \sup_{aN\le k\le bN}\mathbb{E}_N\Big[\Big|\int_0^T V_{k,l,A}(s,\eta_s)ds\Big|\Big] = 0.
$$

By the entropy inequality $E_\mu[f]\le H(\mu|\nu)+\log E_\nu[e^f]$ (cf. p.338 [16]) and the assumption $H(\nu^N|\mathscr{R}_{c,N})\le CNN_\beta^{-1}$, we have

$$
\mathbb{E}_N\Big[\Big|\int_0^T V_{k,l,A}(s,\eta_s)ds\Big|\Big] \le \frac{C_0}{\gamma N_\beta} + \frac{1}{\gamma N}\ln \mathbb{E}_{\mathscr{R}_{c,N}}\Big[\exp\Big\{\gamma N\Big|\int_0^T V_{k,l,A}(s,\eta_s)ds\Big|\Big\}\Big].
$$

The absolute value in the right hand side of last inequality can be dropped by using $e^{|x|}\le e^x+e^{-x}$. By Feynman-Kac formula (cf. p.336, [16]),

$$
\frac{1}{\gamma N}\ln \mathbb{E}_{\mathscr{R}_{c,N}}\Big[\exp\Big\{\gamma N\int_0^T V_{k,l,A}(s,\eta_s)ds\Big\}\Big] \le \frac{1}{\gamma N}\int_0^T \lambda_{N,l}(s)ds
$$

where $\lambda_{N,l}(s)$ is the largest eigenvalue of $N^2L+\gamma NV_{k,l,A}(s,\eta)$.

*Step 5.* Fix $s\in[0,T]$; we will omit the argument $s$ to simplify notation. Note the variational formula for $\lambda_{N,l}$:

$$
(\gamma N)^{-1}\lambda_{N,l} = \sup_f\Big\{E_{\mathscr{R}_{c,N}}\left[V_{k,l,A}f\right] - \gamma^{-1}NE_{\mathscr{R}_{c,N}}\Big[\sqrt{f}(-L\sqrt{f})\Big]\Big\},
$$

where the supremum is over all $f$ which are densities with respect to $\mathscr{R}_{c,N}$ (cf. [16], p. 377).

Let $f_{k,l}=E_{\mathscr{R}_{c,N}}\big[f|\Omega_{k,l}\big]$, be the conditional expectation of $f$ given the variables on $\Lambda_{k,l}$. Recall that $\mu_{k,l}$ is the restriction of $\mathscr{R}_{c,N}$ to $\Lambda_{k,l}$, and that $L_{k,l}$ is the localized generator. Since the Dirichlet form $E_{\mathscr{R}_{c,N}}\big[\sqrt{f}(-L\sqrt{f})\big]$ is convex, we have

$$
(\gamma N)^{-1}\lambda_{N,l} \le \sup_{f_{k,l}}\Big\{E_{\mu_{k,l}}\left[V_{k,l,A}f_{k,l}\right] - \gamma^{-1}NE_{\mu_{k,l}}\Big[\sqrt{f_{k,l}}(-L_{k,l}\sqrt{f_{k,l}})\Big]\Big\}.
$$

*Step 6.* We now decompose $f_{k,l}d\mu_{k,l}$ with respect to sets $\Omega_{k,l,j}$ of configurations with total particle number $j$ on $\Lambda_{k,l}$:

$$
E_{\mu_{k,l}}\left[V_{k,l,A}f_{k,l}\right] = \sum_{j\ge 0} c_{k,l,j}(f)\int V_{k,l,A}f_{k,l,j}d\mu_{k,l,j}, \tag{8.10}
$$

where $c_{k,l,j}(f) = \int_{\Omega_{k,l,j}} f_{k,l} d\mu_{k,l}$, and $f_{k,l,j} = c_{k,l,j}(f)^{-1}\mu_{k,l}(\Omega_{k,l,j}) f_{k,l}$; in this expression, $\sum_{j\geq 0} c_{k,l,j} = 1$ and $f_{k,l,j}$ is a density with respect to $\mu_{k,l,j}$.

Straightforwardly, on $\Omega_{k,l,j}$, we have

$$\frac{L_{k,l}\sqrt{f_{k,l}}}{\sqrt{f_{k,l}}} = \frac{L_{k,l}\sqrt{f_{k,l,j}}}{\sqrt{f_{k,l,j}}}.$$

Using (8.10), we write

$$E_{\mu_{k,l}}\left[\sqrt{f_{k,l}}(-L_{k,l}\sqrt{f_{k,l}})\right] = \sum_{j\geq 0} c_{k,l,j}(f) E_{\mu_{k,l,j}}\left[\sqrt{f_{k,l,j}}(-L_{k,l}\sqrt{f_{k,l,j}})\right].$$

Then, we get

$$(\gamma N)^{-1}\lambda_{N,l} \leq \sup_{0\leq j\leq A(2l+1)} \sup_f \left\{ E_{\mu_{k,l,j}}\left[V_{k,l,A} f\right] - \gamma^{-1} N E_{\mu_{k,l,j}}\left[\sqrt{f}(-L_{k,l}\sqrt{f})\right]\right\},$$

where the second supremum is on densities $f$ with respect to $\mu_{k,l,j}$.

*Step 7.* We now use the Rayleigh expansion (cf. [16], pp. 375–376, Appendix 3, Theorem 1.1), where $C_{k,l,j}$ is the uniformly bounded inverse spectral gap estimate of $L_{k,l}$ (cf. Lemma 8.1) and $\|V_{k,l,A}\|_\infty \leq |D^{G,s}_{N,k}| \leq C(a,b,G)$. We have

$$\begin{aligned} &E_{\mu_{k,l,j}}\left[V_{k,l,A} f\right] - \gamma^{-1} N E_{\mu_{k,l,j}}\left[\sqrt{f}(-L_{k,l}\sqrt{f})\right] \\ \leq &\frac{\gamma N^{-1}}{1 - 2C(a,b,G) C_{k,l,j}\,\gamma N^{-1}} E_{\mu_{k,l,j}}\left[V_{k,l,A}(-L_{k,l})^{-1} V_{k,l,A}\right]. \end{aligned} \tag{8.11}$$

The spectral gap estimate of $L_{k,l}$ in Lemma 8.1 also implies that $\|L^{-1}_{k,l}\|_2$, the $L^2(\mu_{k,l,j})$ norm of the operator $L^{-1}_{k,l}$ on mean zero functions, is less than or equal to $C_{k,l,j}$.

Now, by Cauchy-Schwarz and the estimate of $\|L^{-1}_{k,l}\|_2$, we have

$$E_{\mu_{k,l,j}}\left[V_{k,l,A}(-L_{k,l})^{-1} V_{k,l,A}\right] \leq C_{k,l,j} E_{\mu_{k,l,j}}\left[V^2_{k,l,A}\right].$$

Accordingly, retracing our steps, noting (8.11), we have

$$\mathbb{E}_N\left[\Big|\int_0^T N_\beta V_{k,l,A}(\eta_s) ds\Big|\right] \leq \frac{C_0}{\gamma} + \sup_{0\leq j\leq A(2l+1)} \frac{T\gamma N_\beta N^{-1} C_{k,l,j}}{1 - 2C(a,b,G)C_{k,l,j}\,\gamma N^{-1}} E_{\mu_{k,l,j}}\left[V^2_{k,l,A}\right].$$

The last expression vanishes uniformly as $N\to\infty$ for $aN \leq k \leq bN$ and $j \leq A(2l+1)$. The lemma now is proved by letting $\gamma \to \infty$. □

### 8.3 $2$-block estimate

In this subsection, we will restrict to the case $\beta = 0$ where $N_\beta = 0$, since a 2-block estimate is not needed for the other cases. As remarked earlier, the 2-block estimate may not hold when $\beta > 0$. In particular, it is problematic to carry through the factor $N_\beta$ in the estimates of Step 8 in the proof of Lemma 8.4 below; more technically, our bound of the Dirichlet form with respect to the bond connecting the two blocks at a distance $\tau N$ cannot absorb the extra factor $N_\beta$.

Recall the notation $\Lambda_{k,l}$ from the 1-block estimate. For $l \geq 1$ and $l < k < k'$, let $\Lambda_{k,k',l} = \Lambda_{k,l} \cup \Lambda_{k',l}$ for $|k-k'| > 2l$. We introduce the following localized generator $L_{k,k',l}$ governing the coordinates $\Omega_{k,k',l} = \{0,1,2,\dots\}^{\Lambda_{k,k',l}}$. Inside each block, the process moves as before, but we add an extra bond interaction between sites $k+l$ and $k'-l$:

$$\begin{aligned} &L_{k,k',l} f(\eta) \\ = &\sum_{x,x+1\in\Lambda_{k,k',l}} \left\{\lambda_x\left[f\left(\eta^{x,x+1}\right) - f(\eta)\right]\chi_{\{\eta(x)>0\}} + \left[f\left(\eta^{x+1,x}\right) - f(\eta)\right]\chi_{\{\eta(x+1)>0\}}\right\} \\ &+ \frac{\theta_{k'-l}}{\theta_{k+l}}\left[f\Big(\eta^{k+l,k'-l}\Big) - f(\eta)\right]\chi_{\{\eta(k+l)>0\}} + \left[f\Big(\eta^{k'-l,k+l}\Big) - f(\eta)\right]\chi_{\{\eta(k'-l)>0\}}. \end{aligned}$$

Here, as $\beta = 0$, we have $\theta_k = e^{-k/N}$ and $\lambda_k = e^{-1/N}$. As before, the localized measure $\mu_{k,k',l}$ defined by $\mu = \mathscr{R}_{c,N}$ limited to sites in $\Lambda_{k,k',l}$, as well as the canonical measure $\mu_{k,k',l,j}$ on $\Omega_{k,k',l,j} := \{\eta \in \Omega_{k,k',l} : \sum_{x\in\Lambda_{k,k',l}} \eta(x) = j\}$, that is $\mu_{k,k',l}$ is conditioned so that there are exactly $j$ particles counted in $\Omega_{k,k',l}$, are both invariant and reversible for the dynamics.

The corresponding Dirichlet form, with measure $\kappa$ given by $\mu_{k,k',l}$ or $\mu_{k,k',l,j}$, is given by

$$E_\kappa\left[f(-L_{k,k',l}f)\right] = \sum_{x,x+1\in\Lambda_{k,k',l}} E_\kappa\left[\chi_{\{\eta(x+1)>0\}}\left[f\left(\eta^{x+1,x}\right) - f(\eta)\right]^2\right] + E_\kappa\left[\chi_{\{\eta(k'-l)>0\}}\left[f\left(\eta^{k'-l,k+l}\right) - f(\eta)\right]^2\right].$$

Recall also the generator of symmetric zero-range process $L_l$ with respect to $\Lambda_{k,l}$ (cf. (8.3)). Let $L'_l$ be the generator with respect to $\Lambda_{k',l}$. Define, noting $1 \le l < k < k'$, the generator $L_{l,l}$ with respect to $\Lambda_{l,l}$ given by

$$L_{l,l}f(\eta) = L_l + L'_l + \frac{1}{2}\left[f\left(\eta^{k+l,k'-l}\right) - f(\eta)\right]\chi_{\{\eta(k+l)>0\}} + \frac{1}{2}\left[f\left(\eta^{k'-l,k+l}\right) - f(\eta)\right]\chi_{\{\eta(k'-l)>0\}}.$$

When $|k - k'|$ is large, the process governed by $L_{l,l}$ in effect treats the blocks as adjacent, with a connecting bond.

Recall now that $\varepsilon = e^{-1/N}$. Corresponding to the set-up of the gap bound Lemma 8.1, let $\nu^\rho_{l,l}$ be the product of $4l + 2$ Geometric distributions with common parameter $\varepsilon$ and mean $\rho$ such that $\varepsilon = \dfrac{\rho}{1+\rho}$. One may inspect that $\nu^\rho_{l,l}$ is invariant to the dynamics generated by $L_{l,l}$. Let now $\nu_{l,l,j}$ be $\nu^\rho_{l,l}$ conditioned on that the total number of particles in the $4l + 2$ sites is $j$. Note that $\nu_{l,l,j}$ is independent of $\rho$. This canonical measure $\nu_{l,l,j}$ is also invariant to the dynamics. The corresponding Dirichlet form is given by

$$E_{\nu_{l,l,j}}\left[f(-L_{l,l}f)\right] = \frac{1}{2}\sum_{x,x+1\in\Lambda_{k,k',l}} E_{\nu_{l,l,j}}\left[\chi_{\{\eta(x)>0\}}\left[f\left(\eta^{x,x+1}\right) - f(\eta)\right]^2\right] + \frac{1}{2}E_{\nu_{l,l,j}}\left[\chi_{\{\eta(k'-l)>0\}}\left[f\left(\eta^{k'-l,k+l}\right) - f(\eta)\right]^2\right].$$

**Lemma 8.3.** *We have the following estimates:*

1. *Uniform bound: For all $\eta \in \Omega_{k,k',l,j}$, we have*

$$r^{-1}_{k,k',l,j,\varepsilon} \le \frac{\mu_{k,k',l,j}(\eta)}{\nu_{l,l,j}(\eta)} \le r_{k,k',l,j,\varepsilon} \tag{8.12}$$

*where $r_{k,k',l,j,\varepsilon} := \left(\dfrac{1 - c\varepsilon^{k'+l}}{1 - c\varepsilon^{k-l}}\right)^{4l+2} \varepsilon^{-2lj}\varepsilon^{(k-k')j}$.*

2. *Poincaré inequality: For $1 \le l < k < k'$ and fixed $j \ge 0$, we have*

$$\mathrm{Var}_{\mu_{k,k',l,j}}(f) \le C_{k,k',l,j}E_{\mu_{k,k',l,j}}\left[f(-L_{k,k',l}f)\right] \tag{8.13}$$

*where $C_{k,k',l,j} = \dfrac{C}{2}(4l+2)^2\left(1 + \dfrac{j}{4l+2}\right)^2 r^2_{k,k',l,j,\varepsilon}$ for an universal constant $C$.*

3. *For fixed $j$, $l$, and $0 < a < b < \infty$, we have*

$$\limsup_{\tau\downarrow 0}\limsup_{N\uparrow\infty} \sup_{\substack{aN<k<k'\le bN \\ 2l+1\le|k'-k|\le\tau N}} r_{k,k',l,j,\varepsilon} \le 1, \tag{8.14}$$

*and so $\limsup_{\tau\downarrow 0}\limsup_{N\uparrow\infty}\sup_{\substack{aN\le k<k'\le bN \\ 2l+1\le|k'-k|\le\tau N}} C_{k,k',l,j} < \infty$.*

*Proof.* We will compare $\mu_{k,k',l,j}$ with $\nu_{l,l,j}$ and make use of the known Poincaré bound, as in the proof of Lemma 8.1:

$$\mathrm{Var}_{\nu_{l,l,j}}(f) \le C(4l+2)^2\Big(1+\frac{j}{4l+2}\Big)^2 E_{\nu_{l,l,j}}[f(-L_{l,l}f)], \tag{8.15}$$

where $C$ is some universal constant.

For $\eta \in \Omega_{k,k',l,j}$, we have

$$\frac{\mu_{k,k',l,j}(\eta)}{\nu_{l,l,j}(\eta)} = \frac{\mu_{k,k',l}(\eta)}{\nu^{\rho}_{l,l}(\eta)}\frac{\nu^{\rho}_{l,l}(\Omega_{k,k',l,j})}{\mu_{k,k',l}(\Omega_{k,k',l,j})}.$$

Since $\mu_{k,k',l}$ and $\nu^{\rho}_{l,l}$ are product measures, and $\beta = 0$, that is,

$$\mu_{k,k',l}(\eta) = \prod_{x\in\Lambda_{k,k',l}} (1-c\varepsilon^x)c^{\eta(x)}\varepsilon^{x\eta(x)}, \text{ and } \nu^{\rho}_{l,l}(\eta) = \prod_{x\in\Lambda_{k,k',l}} (1-\varepsilon)\varepsilon^{\eta(x)}, \tag{8.16}$$

we have

$$\frac{(1-c\varepsilon^{k-l})^{4l+2}c^j\varepsilon^{(k'+l)j}}{(1-\varepsilon)^{4l+2}\varepsilon^j} \le \frac{\mu_{k,k',l}(\eta)}{\nu^{\rho}_{l,l}(\eta)} \le \frac{(1-c\varepsilon^{k'+l})^{4l+2}c^j\varepsilon^{(k-l)j}}{(1-\varepsilon)^{4l+2}\varepsilon^j}.$$

Consequently,

$$\frac{(1-c\varepsilon^{k-l})^{4l+2}c^j\varepsilon^{(k'+l)j}}{(1-\varepsilon)^{4l+2}\varepsilon^j} \le \frac{\mu_{k,k',l}(\Omega_{k,k',l,j})}{\nu^{\rho}_{l,l}(\Omega_{k,k',l,j})} \le \frac{(1-c\varepsilon^{k'+l})^{4l+2}c^j\varepsilon^{(k-l)j}}{(1-\varepsilon)^{4l+2}\varepsilon^j}.$$

Therefore, $r^{-1}_{k,k',l,j,\varepsilon} \le \dfrac{\mu_{k,k',l,j}(\eta)}{\nu_{l,l,j}(\eta)} \le r_{k,k',l,j,\varepsilon}$ and (8.12) holds.

From (8.12), we have

$$E_{\nu_{l,l,j}}\left[f(-L_{l,l}f)\right] \le \frac{1}{2}r_{k,k',l,j,\varepsilon}E_{\mu_{k,k',l,j}}\left[f(-L_{k,k',l}f)\right]. \tag{8.17}$$

Also, in turn,

$$\begin{aligned}\mathrm{Var}_{\mu_{k,k',l,j}}(f) = \inf_a E_{\mu_{k,k',l,j}}\left[(f-a)^2\right] &\le r_{k,k',l,j,\varepsilon}\inf_a E_{\nu_{l,l,j}}\left[(f-a)^2\right]\\ &= r_{k,k',l,j,\varepsilon}\mathrm{Var}_{\nu_{l,l,j}}(f).\end{aligned}$$

The spectral gap estimate (8.13) now follows from (8.15) and (8.17).

To complete the proof of the lemma, noting that $\varepsilon = e^{-1/N}$, for any fixed $l$, $j$, we see straightforwardly that

$$\limsup_{N\uparrow\infty} \sup_{\substack{aN\le k<k'\le bN\\ 2l+1\le|k'-k|\le\tau N}} r_{k,k',l,j,\varepsilon} \le \sup_{a\le x\le b}\Big((1-ce^{-\tau}e^{-x})/(1-ce^{-x})\Big)^{4l+2}e^{\tau j},$$

which converges to $1$ as $\tau \downarrow 0$. Hence, the limit (8.14) and the desired uniform boundedness of $C_{k,k',l,j}$ both follow. □

We now state and show a $2$-blocks estimate. The scheme is similar to that of the $1$-block estimate. Recall $D^{G,s}_{N,k}$ and its bound for $aN \le k \le bN$ that $|D^{G,s}_{N,k}| \le C(a,b,G)$ (cf. (3.4)).

**Lemma 8.4** (2-block estimate)**.** *We have*

$$\limsup_{l\to\infty}\limsup_{\tau\to 0}\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}_N\Big|\int_0^T D^{G,s}_{N,k}\Big(\frac{\eta^l_s(k)}{1+\eta^l_s(k)} - \frac{\eta^{\tau N}_s(k)}{1+\eta^{\tau N}_s(k)}\Big)ds\Big| = 0.$$

*Proof.* We separate the argument into 9 steps.

*Step 1.* Since $\dfrac{x}{1+x}$ is Lipschitz on $\mathbb{R}^+$ and $D^{G,s}_{N,k}$ is bounded, it is enough to show

$$\limsup_{l\to\infty}\limsup_{\tau\to 0}\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}_N\int_0^T\left|\eta_s^{\tau N}(k)-\eta_s^l(k)\right|ds=0.$$

By the triangle inequality, it will be enough to show that as $N\to\infty$, $\tau\to 0$, and $l\to\infty$,

$$\begin{aligned}\sup_{aN\le k\le bN}\mathbb{E}_N\int_0^T\Big|\eta_s^{\tau N}(k)-\frac{1}{2\tau N+1}\sum_{|x-k|\le\tau N}\eta_s^l(x)\Big|ds\to 0\ \text{ and}\\ \sup_{aN\le k\le bN}\mathbb{E}_N\int_0^T\Big|\frac{1}{2\tau N+1}\sum_{|x-k|\le\tau N}\eta_s^l(x)-\eta_s^l(k)\Big|ds\to 0.\end{aligned}\tag{8.18}$$

*Step 2.* We now show that the first limit in (8.18). Note that

$$\begin{aligned}\Big|\eta^{\tau N}(k)-\frac{1}{2\tau N+1}\sum_{|x-k|\le\tau N}\eta^l(x)\Big|&\le\frac{1}{2\tau N+1}\sum_{\substack{|x-k-\tau N|\le l\\ \text{or }|x-k+\tau N|\le l}}\eta(x)\\&=\frac{2l+1}{2\tau N+1}\left(\eta^l(k-\tau N)+\eta^l(k+\tau N)\right).\end{aligned}$$

Then, the expectation in the first limit in (8.18), given that $\nu^N\le\mathscr{R}_{c,N}$ and that the process is attractive (cf. Subsection 2.2.1), is bounded from above by

$$\frac{2l+1}{2\tau N+1}\int_0^T\mathbb{E}_{\mathscr{R}_{c,N}}\left(\eta_s^l(k-\tau N)+\eta_s^l(k+\tau N)\right)ds$$

For fixed $l$ and $\tau<a$, since $k\ge aN$ and $\beta=0$, we have $E_{\mathscr{R}_{c,N}}[\eta(k)]=\rho_{k,c}=ce^{-k/N}/(1-ce^{-k/N})\le 1$ (cf. (2.1)). Hence, the above display vanishes uniformly in $k$ as $N\to\infty$.

*Step 3.* By a similar argument as in Step 2, we can restrict the $x$ in the summation of the second limit in (8.18) to be $k'$ such that $2l+1\le|k'-k|\le\tau N$. Then, the second limit will follow if we show that

$$\limsup_{l\to\infty}\limsup_{\tau\to 0}\limsup_{N\to\infty}\sup_{\substack{aN\le k<k'\le bN\\ 2l+1\le|k'-k|\le\tau N}}\mathbb{E}_N\int_0^T\left|\eta_s^l(k)-\eta_s^l(k')\right|ds=0.$$

*Step 4.* We will apply a cutoff of large densities first. Let

$$\eta_s^l(k,k')=\eta_s^l(k)+\eta_s^l(k').$$

For any $A$,

$$\begin{aligned}\mathbb{E}_N\int_0^T\left|\eta_s^l(k)-\eta_s^l(k')\right|ds&=\mathbb{E}_N\int_0^T\left|\eta_s^l(k)-\eta_s^l(k')\right|\chi_{\{\eta_s^l(k,k')\le A\}}ds\\&+\mathbb{E}_N\int_0^T\left|\eta_s^l(k)-\eta_s^l(k')\right|\chi_{\{\eta_s^l(k,k')>A\}}ds=I_1+I_2.\end{aligned}$$

As $\nu^N\le\mathscr{R}_{c,N}$ and the process is attractive (cf. Subsection 2.2.1), we may bound the second expectation $I_2$ by

$$\mathbb{E}_N\int_0^T\eta_s^l(k,k')\chi_{\{\eta_s^l(k,k')>A\}}ds\le\frac{T}{A}E_{\mathscr{R}_{c,N}}\left(\eta^l(k,k')\right)^2.\tag{8.19}$$

Recall $\rho_{k,c} = ce^{-k/N}/(1-ce^{-k/N})$ when $\beta = 0$ (cf. (2.1)). Trivially, $\rho_{k,c} \le c/(1-c)$ for all $k$. Note that $\mathscr{R}_{c,N}$ has Geometric marginals, therefore, $E_{\mathscr{R}_{c,N}}[\eta(k)^2] = 2\rho_{k,c}^2 + \rho_{k,c}$ is uniformly bounded. Then, as

$$\left(\eta^l(k,k')\right)^2 \le 2\left((\eta^l(k))^2 + (\eta^l(k'))^2\right) \le 2(2l+1)^{-1} \sum_{x\in\Lambda_{k,l}\cup\Lambda_{k',l}} (\eta(x))^2,$$

we have that (8.19) is of order $O(A^{-1})$ and that the second expectation $I_2$ is negligible.

Hence, it remains to show that

$$\sup_{\substack{aN\le k<k'\le bN\\ 2l+1\le|k'-k|\le\tau N}} \mathbb{E}_N \int_0^T \left|\eta_s^l(k) - \eta_s^l(k')\right| \chi_{\{\eta_s^l(k,k')\le A\}} ds$$

vanishes as we take $N\to\infty$, $\tau\to 0$, and then $l\to\infty$.

*Step 5.* Let

$$V_{k,k',l,A}(\eta) := \left|\eta^l(k) - \eta^l(k')\right| \chi_{\{\eta^l(k,k')\le A\}}.$$

Following the proof of Lemma 8.2, for fixed $l,\tau,N,k,k'$, in order to estimate

$$\mathbb{E}_N \int_0^T V_{k,k',l,A}(\eta_s) ds$$

it suffices to bound

$$(\gamma N)^{-1}\lambda_{N,l} = \sup_f \left\{E_{\mathscr{R}_{c,N}}\left[V_{k,k',l,A} f\right] - \gamma^{-1} N E_{\mathscr{R}_{c,N}}\left[\sqrt{f}(-L\sqrt{f})\right]\right\} \tag{8.20}$$

where the supremum is over all $f$ which are densities with respect to $\mathscr{R}_{c,N}$.

*Step 6.* Recall the generator $L_{k,k',l}$ and its Dirichlet form defined in the beginning of this subsection. Recall also $\mu_{k,k',l}$ is the restriction of $\mathscr{R}_{c,N}$ to $\Lambda_{k,k',l}$. The Dirichlet form with respect to the full generator $L$ under $\mathscr{R}_{c,N}$ is given by

$$E_{\mathscr{R}_{c,N}}\left[f(-Lf)\right] = \sum_{x\ge 1} E_{\mathscr{R}_{c,N}}\left[\chi_{\{\eta(x+1)>0\}}\left(f(\eta^{x+1,x}) - f(\eta(x)\right)^2\right].$$

We now argue the following Dirichlet form inequality:

$$E_{\mathscr{R}_{c,N}}\left[\sqrt{f}(-L_{k,k',l}\sqrt{f})\right] \le (1+\tau N) E_{\mathscr{R}_{c,N}}\left[\sqrt{f}(-L\sqrt{f})\right]. \tag{8.21}$$

First, we observe that

$$\begin{aligned} E_{\mathscr{R}_{c,N}}\left[f(-L_{k,k',l}f)\right] = &\sum_{x,x+1\in\Lambda_{k,k',l}} E_{\mathscr{R}_{c,N}}\left[\chi_{\{\eta(x+1)>0\}}\left[f\left(\eta^{x+1,x}\right) - f(\eta)\right]^2\right] \\ &+ E_{\mathscr{R}_{c,N}}\left[\chi_{\{\eta(k'-l)>0\}}\left[f\left(\eta^{k'-l,k+l}\right) - f(\eta)\right]^2\right]. \end{aligned}$$

Next, by adding and subtracting at most $\tau N$ terms, we have

$$\begin{aligned} &\left[f\left(\eta^{k'-l,k+l}\right) - f(\eta)\right]^2 \\ \le &(k'-k-2l) \sum_{q=0}^{k'-k-2l-1} \left[f\left(\eta^{k'-l,k+l+q}\right) - f\left(\eta^{k'-l,k+l+q+1}\right)\right]^2. \end{aligned}$$

Also, when $\eta(k'-l)>0$, by applying the change of variables $\xi = \eta^{k'-l,k+l+q+1}$ which takes away a particle at $k'-l$ and adds one at $k+l+q+1$, we have (cf. (8.16))

$$\mu(\eta) = \varepsilon^{k'-k-2l-q-1}\mathscr{R}_{c,N}(\xi) \le \mu(\xi).$$

Then, as $\chi_{\{\eta(k'-l)>0\}} = \chi_{\{\xi(k+l+q+1)>0\}}$, we have

$$\begin{aligned}
&E_{\mathscr{R}_{c,N}}\left[\chi_{\{\eta(k'-l)>0\}}\left[f\left(\eta^{k'-l,k+l+q}\right) - f\left(\eta^{k'-l,k+l+q+1}\right)\right]^2\right]\\
=&\sum_{\xi}\mu(\eta)\chi_{\{\xi(k+l+q+1)>0\}}\left[f\left(\xi^{k+l+q+1,k+l+q}\right) - f(\xi)\right]^2\\
\le&E_{\mathscr{R}_{c,N}}\left[\chi_{\{\eta(k+l+q+1)>0\}}\left[f\left(\eta^{k+l+q+1,k+l+q}\right) - f(\eta)\right]^2\right].
\end{aligned}$$

From these observations, (8.21) follows.

*Step 7.* Inputting (8.21) into (8.20), and considering the conditional expectation of $f$ with respect to $\Omega_{k,k',l}$ as in the 1-block estimate proof, for $N$ large, we have

$$(\gamma N)^{-1}\lambda_{N,l} \le \sup_{f_{k.k',l}}\left\{E_{\mu_{k,k',l}}\left[V_{k,k',l,A}f_{k,k',l}\right] - \frac{1}{2\tau\gamma}E_{\mu_{k,k',l}}\left[\sqrt{f_{k,k',l}}\left(-L_{k,k',l}\sqrt{f_{k,k',l}}\right)\right]\right\}$$

where the supremum is over densities with respect to $\mu_{k,k',l}$.

Again, as in the proof of the 1-block estimate, decomposing $f_{k,k',l}d\mu_{k,k'.l}$ along configurations with common total number $j$, we need only to bound

$$\sup_{0\le j\le A(2l+1)}\sup_{f}\left\{E_{\mu_{k,k',l,j}}\left[V_{k,k',l,A}f\right] - \frac{1}{2\tau\gamma}E_{\mu_{k,k',l,j}}\left[\sqrt{f}\left(-L_{k,k',l}\sqrt{f}\right)\right]\right\}$$

where the supremum is over densities with respect to $\mu_{k,k',l,j}$.

*Step 8.* Let

$$\widehat{V}_{k,k',l,A} = V_{k,k',l,A} - E_{\mu_{k,k',l,j}}\left[V_{k,k',l,A}\right].$$

Using the Rayleigh expansion (cf. pp. 375–376, [16]) where the inverse spectral gap $C_{k,k',l,j}$ of $L_{k,k',l}$ is bounded (Lemma 8.3), and $\|\widehat{V}_{k,k',l,A}\|_\infty \le A$, we have

$$\begin{aligned}
&E_{\mu_{k,k',l,j}}\left[\widehat{V}_{k,k',l,A}f\right] - \frac{1}{2\tau\gamma}E_{\mu_{k,k',l,j}}\left[\sqrt{f}\left(-L_{k,k',l}\sqrt{f}\right)\right]\\
\le&\frac{2\tau\gamma}{1-4AC_{k,k',l,j}\,\tau\gamma}E_{\mu_{k,k;,l,j}}\left[\widehat{V}_{k,k',l,A}(-L_{k,k',l})^{-1}\widehat{V}_{k,k',l,A}\right]\\
\le&\frac{2\tau\gamma C_{k,k',l,j}}{1-4AC_{k,k',l,j}\,\tau\gamma}E_{\mu_{k,k',l,j}}\left[\widehat{V}^2_{k,k',l,A}\right] \to 0 \text{ as } \tau\to 0.
\end{aligned}$$

*Step 9.* To finish, we still need to show that $E_{\mu_{k,k',l,j}}\left[V_{k,k',l,A}\right]$ vanishes. In fact, by Lemma 8.3, $E_{\mu_{k,k',l,j}}\left[V_{k,k',l,A}\right] \le r_{k,k',l,j,\varepsilon}E_{\nu_{l,l,j}}\left[V_{k,k',l,A}\right]$ and, for $l$ and $j$ fixed,

$$\limsup_{\tau\downarrow 0}\limsup_{N\uparrow\infty}\sup_{\substack{aN\le k<k'\le bN\\ 2l+1\le|k'-k|\le\tau N}} r_{k,k',l,j,\varepsilon} \le 1.$$

The term $E_{\nu_{l,l,j}}\left[V_{k,k',l,A}\right]$ does not depend on $N$ or $\tau$. By adding and subtracting $j/(2(2l+1))$, we need only bound $E_{\nu_{l,l,j}}\left[\left|\eta^l(k) - j/(2(2l+1))\right|\right]$. By an equivalence of ensemble estimate (cf. p. 355 [16]), $E_{\nu_{l,l,j}}\left[\left|\eta^l(k) - j/(2(2l+1))\right|^2\right] \le C(A)\mathrm{Var}_{\nu_{l,l}^{j/(2(2l+1))}}\left(\eta^l(k)\right)$ (recall $\nu^\rho_{l,l}$ defined before Lemma 8.3). This variance is of order $O(l^{-1})$, since the single site variance $\mathrm{Var}_{\nu_{l,l}^{j/(2(2l+1))}}\left(\eta(k)\right)$ is uniformly bounded for $j/(2(2l+1)) \le A$. Hence, $E_{\nu_{l,l,j}}\left[V_{k,k',l,A}\right]$ is of order $O(l^{-1/2})$, finishing the proof. □

# 9 Properties of the initial measures

In this section, we show key properties of the invariant measures $\mathscr{R}_{c,N}$ in Subsection 9.1, the local equilibria $\mu^N$ in Subsection 9.2, and also of $\nu^N$ in Subsection 9.3.

Recall the three regimes in Subsection 2.1: (1) $\beta = 0$, (2) $\mathcal{E}_k \sim \ln k$ and $0 < \beta < 1$, and (3) $1 \ll \mathcal{E}_k \ll \ln k$ and $\beta > 0$.

### 9.1 Properties of the invariant measures

We first show that $\mathscr{R}_{c,N}$ is indeed an invariant measure. Recall $c_0 = \min_{k\in\mathbb{N}} e^{\beta\mathcal{E}_k}$ (cf. definition before equation (2.1)).

**Lemma 9.1.** *For $0 \le c \le c_0$, we have $\mathscr{R}_{c,N}$ is a reversible, invariant measure.*

*Proof.* When $c = 0$, there are no particles in the system and the statement is trivial. For $0 < c \le c_0$, recall that $\lambda_k = \theta_{k+1}/\theta_k = \theta_{k+1,c}/\theta_{k,c}$, and the definition of the generator $L$ (cf. (2.5)). With respect to functions $f$ and $h$ depending only on a finite number of occupation variables, we need to show that

$$E_{\mathscr{R}_{c,N}}[h(\xi)(Lf)(\xi)] = E_{\mathscr{R}_{c,N}}[(Lh)(\xi)f(\xi)]. \tag{9.1}$$

For any fixed $k \ge 1$, make a change of variable $\eta = \xi^{k,k+1}$ when $\xi(k) > 0$. Then, $\xi = \eta^{k+1,k}$ and $\eta(k+1) > 0$. Using that $\mathscr{R}_{c,N}$ is a product of Geometric marginals with parameters $\{\theta_{k,c}\}$, we have $\chi_{\{\xi(k)>0\}}\dfrac{\mathscr{R}_{c,N}(d\xi)}{\mathscr{R}_{c,N}(d\eta)} = \dfrac{\chi_{\{\eta(k+1)>0\}}}{\lambda_k}$. Therefore,

$$\begin{aligned} E_{\mathscr{R}_{c,N}}\left[\lambda_k f\left(\xi^{k,k+1}\right)h(\xi)\chi_{\{\xi(k)>0\}}\right] &= E_{\mathscr{R}_{c,N}}\left[f(\eta)h(\eta^{k+1,k})\chi_{\{\eta(k+1)>0\}}\right] \\ &= E_{\mathscr{R}_{c,N}}\left[f(\xi)h(\xi^{k+1,k})\chi_{\{\xi(k+1)>0\}}\right], \end{aligned}$$

changing notation from $\eta$ back to $\xi$. With a similar analysis,

$$E_{\mathscr{R}_{c,N}}\left[f\left(\xi^{k+1,k}\right)h(\xi)\chi_{\{\xi(k+1)>0\}}\right] = E_{\mathscr{R}_{c,N}}\left[\lambda_k f(\xi)h(\xi^{k,k+1})\chi_{\{\xi(k)>0\}}\right].$$

Hence,

$$\begin{aligned} &E_{\mathscr{R}_{c,N}}\left[\lambda_k\left(f\left(\xi^{k,k+1}\right) - f(\xi)\right)h(\xi)\chi_{\{\xi(k)>0\}}\right] \\ &\qquad + E_{\mathscr{R}_{c,N}}\left[\left(f\left(\xi^{k+1,k}\right) - f(\xi)\right)h(\xi)\chi_{\{\xi(k+1)>0\}}\right] \\ &= E_{\mathscr{R}_{c,N}}\left[\left(h\left(\xi^{k+1,k}\right) - h\left(\xi\right)\right)f(\xi)\chi_{\{\xi(k+1)>0\}}\right] \\ &\qquad + E_{\mathscr{R}_{c,N}}\left[\lambda_k\left(h\left(\xi^{k,k+1}\right) - h\left(\xi\right)\right)f(\xi)\chi_{\{\xi(k)>0\}}\right], \end{aligned}$$

from which (9.1) follows. □

To prepare to show that $\mathscr{R}_{c,N}$ is a local equilibrium measure, we will need the following. Recall $\theta_k = e^{-\beta\mathcal{E}_k - k/N}$ and $N_\beta = e^{\beta\mathcal{E}_N}$ (cf. (2.2)).

**Lemma 9.2.** *For any fixed $0 \le a < b \le \infty$, we have*

$$\lim_{N\to\infty}\sum_{k=aN}^{bN} N^{-1}N_\beta\theta_k = \lim_{N\to\infty}\frac{1}{N}\sum_{k=aN}^{bN} e^{-\beta(u(\ln k)-u(\ln N))-k/N} = \int_a^b \phi(x)dx$$

*where $\phi = e^{-x}$ in regime (1), $\phi = x^{-\beta}e^{-x}$ in regime (2) and $\phi = e^{-x}$ in regime (3).*

*Proof.* We will show the lemma in regime (2), that is when $\mathcal{E}_k \sim \ln k$ and $0 < \beta < 1$. The other regime (3), when $1 \ll \mathcal{E}_k \ll \ln k$ and $\beta > 0$, can be proved in a similar way. Also regime (1), when $\beta = 0$ is more trivial. We will also suppose $a = 0$, $b = \infty$, as the argument is the same for any other pair $a, b$. Define

$$\Phi_N(x) = \sum_{k=1}^{\infty} e^{-\beta(u(\ln k)-u(\ln N))-k/N}\chi_{(\frac{k-1}{N},\frac{k}{N}]}(x).$$

We need only show that $\lim_{N\to\infty}\int_0^\infty \Phi_N(x)dx = \int_0^\infty \phi(x)dx$ to finish.

By the mean value theorem, $u(\ln k) - u(\ln N) = u'(x_{k,N})\ln\dfrac{k}{N}$, where $x_{k,N}$ is between $\ln k$ and $\ln N$. Fix $\beta_1$ such that $\beta < \beta_1 < 1$. Since $u'(x) \to 1$ as $x \to \infty$, we may find $m_\beta$

such that $0 < u'(x) < \dfrac{\beta_1}{\beta}$, for all $x \geq \ln m_\beta$. Therefore, $\Phi_N(x) \leq x^{-\beta_1} e^{-x}$ for $\frac{m_\beta}{N} < x \leq 1$ and $\Phi_N(x) \leq e^{-x}$ for $x > 1$.

By dominated convergence we obtain $\int_{m_\beta/N}^\infty \Phi_N(x)dx \to \int_0^\infty x^{-\beta} e^{-x} dx$. Also, the remaining term $\int_0^{m_\beta/N} \Phi_N(x)dx \leq \frac{m_\beta N_\beta}{N}$ vanishes as $N_\beta = o(N)$ for $0 < \beta < 1$. This completes the argument. □

**Lemma 9.3.** *For all $c$ such that $0 \leq c < c_0$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^\infty \left| N_\beta \rho_{k,c} - \overline{\rho}_{N,k} \right| = 0 \tag{9.2}$$

*where $\overline{\rho}_{N,k} = N \int_{(k-1)/N}^{k/N} \phi_c(x)dx$. As an immediate consequence, the product invariant measures $\{\mathscr{R}_{c,N}\}_{N\in\mathbb{N}}$, with Geometric marginals, are local equilibrium measures corresponding to $\rho_0 = \phi_c$.*

*Proof.* Recall that $\theta_{k,c} = ce^{-\beta\mathcal{E}_k - k/N}$ and $E_{\mathscr{R}_{c,N}}\eta(k) = \rho_{k,c} = \theta_{k,c}/(1-\theta_{k,c})$ (cf. (2.1)). We now verify the limit (9.2). When $\beta = 0$, we have $N_\beta = 1$ and $\phi_c = \dfrac{ce^{-x}}{1-ce^{-x}}$. Since $\phi_c$ is decreasing, we have $\rho_{k,c} = \dfrac{ce^{-k/N}}{1-ce^{-k/N}} < \overline{\rho}_{N,k}$. Then, the left-hand side of (9.2) equals to

$$\int_0^\infty \frac{ce^{-x}}{1-ce^{-x}} dx - \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^\infty \frac{ce^{-k/N}}{1-ce^{-k/N}},$$

which clearly vanishes as $N \to \infty$ by dominated convergence.

For the remaining two regimes when $\beta > 0$, we will split the summation in (9.2) into two parts: $aN \leq k \leq bN$ and the rest, for an $0 < a < b$ that we will specify. In fact, it will be enough to show, for any $\varepsilon > 0$, that we can find $a > 0$ small enough and $b > 0$ big enough such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{1\leq k\leq aN\\ k>bN}} \left| \frac{N_\beta \theta_{k,c}}{1-\theta_{k,c}} - \overline{\rho}_{N,k} \right| \leq \varepsilon \tag{9.3}$$

and, for all $b > a > 0$, that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=aN}^{bN} \left| \frac{N_\beta \theta_{k,c}}{1-\theta_{k,c}} - \overline{\rho}_{N,k} \right| = 0. \tag{9.4}$$

To verify (9.3),

$$\frac{1}{N} \sum_{\substack{1\leq k\leq aN\\ k>bN}} \left| \frac{N_\beta \theta_{k,c}}{1-\theta_{k,c}} - \overline{\rho}_{N,k} \right| \leq \frac{1}{N} \sum_{\substack{1\leq k\leq aN\\ k\geq bN}} \frac{N_\beta \theta_{k,c}}{1-\theta_{k,c}} + \int_{(0,a)\cup(b,\infty)} \phi_c dx.$$

Recall $c_0 = \min_k e^{\beta\mathcal{E}_k}$. Since $\theta_{k,c} = ce^{-\beta\mathcal{E}_k - k/N} \leq \dfrac{c}{c_0}$, by Lemma 9.2, we have

$$\frac{1}{N} \sum_{\substack{1\leq k\leq aN\\ k>bN}} \frac{N_\beta \theta_{k,c}}{1-\theta_{k,c}} \leq \frac{c_0}{c_0 - c} \frac{1}{N} \sum_{\substack{1\leq k\leq aN\\ k>bN}} N_\beta \theta_{k,c} \to \frac{c_0}{c_0-c} \int_{(0,a)\cup(b,\infty)} \phi_c dx.$$

Then, (9.3) follows as $\phi_c \in L^1(\mathbb{R}^+)$.

It remains to show (9.4). By adding and subtracting, for each $N$ the left side of (9.4) is bounded by

$$\frac{1}{N}\sum_{k=aN}^{bN}\left|\frac{N_\beta\theta_{k,c}}{1-\theta_{k,c}} - N_\beta\theta_{k,c}\right| + \frac{1}{N}\sum_{k=aN}^{bN}\left|N_\beta\theta_{k,c} - c(k/N)^{-\beta u'(\infty)}e^{-k/N}\right|$$
$$+\frac{1}{N}\sum_{k=aN}^{bN}\left|c(k/N)^{-\beta u'(\infty)}e^{-k/N} - \overline{\rho}_{N,k}\right| =: I_1 + I_2 + I_3$$

where $u'(\infty) = \lim_{x\to\infty} u'(x)$ takes value either $0$ or $1$.

The term $I_1$ is trivially bounded by

$$\max_{aN\le k\le bN}\frac{\theta_{k,c}}{1-\theta_{k,c}} N^{-1}\sum_{k=aN}^{bN} N_\beta\theta_{k,c}.$$

Recall that $\theta_{k,c} = ce^{-\beta\mathcal{E}_k - k/N}$ and $\mathcal{E}_k \to \infty$ as $k\to\infty$. Then, $\max_{aN\le k\le bN}\dfrac{\theta_{k,c}}{1-\theta_{k,c}}$ vanishes as $N\to\infty$. Since also $N^{-1}\sum_{k=aN}^{bN} N_\beta\theta_{k,c} \to \int_a^b \phi_c dx < \infty$ (Lemma 9.2) is bounded, the term $I_1$ vanishes.

For term $I_2$, we spell out $N_\beta\theta_{k,c}$ as

$$ce^{-\beta(\mathcal{E}_k - \mathcal{E}_N) - k/N}.$$

By the mean value theorem, we have $\mathcal{E}_k - \mathcal{E}_N = \ln(\frac{k}{N})u'(y_{k,N})$, where $y_{k,N}$ is in between $\ln k$ and $\ln N$. Then, $I_2$ is less than or equal to

$$\max_{aN\le k\le bN}\left\{\left|(k/N)^{\beta(u'(y_{k,N}) - u'(\infty))} - 1\right|\right\}\frac{1}{N}\sum_{k=aN}^{bN} N_\beta\theta_{k,c}.$$

We observed in estimating $I_1$ above that $N^{-1}\sum_{k=aN}^{bN} N_\beta\theta_{k,c}$ is bounded. Hence, $I_2$ vanishes as $N\to\infty$.

We now address the last term $I_3$. Observe, as $\phi_c$ is decreasing, that

$$I_3 \;=\; \int_{a-\frac{1}{N}}^{b}\phi_c(x)dx - \frac{1}{N}\sum_{k=aN}^{bN} c(k/N)^{-\beta u'(\infty)}e^{-k/N},$$

which vanishes as $N\to\infty$ by the dominated convergence theorem. □

We now give a useful mean and variance estimate.

**Lemma 9.4.** *For all $c$ such that $0\le c< c_0$ we have that*

$$E_{\mathscr{R}_{c,N}}\sum_{k=1}^{\infty}\eta(k) = \sum_{k=1}^{\infty}\rho_{k,c} = O\big(NN_\beta^{-1}\big) \text{ and}$$
$$\frac{N_\beta^2}{N^2}\sum_{k=1}^{\infty}\mathrm{Var}_{\mathscr{R}_{c,N}}(\eta(k)) = \frac{N_\beta^2}{N^2}\sum_{k=1}^{\infty}\big[\rho_{k,c}^2 + \rho_{k,c}\big] \to 0. \qquad (9.5)$$

*Proof.* We first consider the means:

$$\frac{N_\beta}{N}E_{\mathscr{R}_{c,N}}\sum_{k=1}^{\infty}\eta(k) = \frac{N_\beta}{N}\sum_{k=1}^{\infty}\rho_{k,c}.$$

By (9.2), $\lim_{N\to\infty}\dfrac{1}{N}\sum_{k=1}^{\infty}|N_\beta\rho_{k,c} - \overline{\rho}_{N,k}| = 0$, where $\overline{\rho}_{N,k} = N\int_{(k-1)/N}^{k/N}\phi_c(x)dx$. As $\dfrac{1}{N}\sum_{k=1}^{\infty}\overline{\rho}_{N,k} = \int_0^\infty \phi_c dx < \infty$, then the estimate on the sum of means in (9.5) follows.

Next, we consider the sum of variances. Since $N_\beta = o(N)$ and $N^{-1}\sum_{k=1}^\infty N_\beta \rho_{k,c} < \infty$ by the first estimate in (9.5), we have that $N_\beta^2 N^{-2}\sum_{k=1}^\infty \rho_{k,c}$ vanishes as $N \to \infty$. For the term $N_\beta^2 N^{-2}\sum_{k=1}^\infty \rho_{k,c}^2$, we use $\sum (N_\beta \rho_{k,c})^2 \le 2(\sum |N_\beta \rho_{k,c} - \overline{\rho}_{N,k}|)^2 + 2\sum \overline{\rho}_{N,k}^2$. Since $N^{-1}\sum |N_\beta \rho_{N,k} - \overline{\rho}_{N,k}| \to 0$, it suffices to show that $\lim_{N\to\infty} \dfrac{1}{N^2}\sum_{k=1}^\infty \overline{\rho}_{N,k}^2 = 0$.

To this end, let $\hat\rho_N = \max_{k\ge 1}\overline{\rho}_{N,k}$. Then, $\dfrac{1}{N^2}\sum_{k=1}^\infty \overline{\rho}_{N,k}^2 \le \dfrac{\hat\rho_N}{N^2}\sum_{k=1}^\infty \overline{\rho}_{N,k}$. Now, $N^{-1}\sum_{k=1}^\infty \overline{\rho}_{N,k} = \int_0^\infty \phi_c dx < \infty$. The desired limit holds since, by absolute continuity of the Lebesgue integral, $N^{-1}\hat\rho_N \to 0$ as $N \to \infty$. □

### 9.2 Properties of local equilibria $\mu^N$

We now observe that the local equilibria $\mu^N$ (cf. Subsection 2.3) satisfy Condition 2.3.

**Proposition 9.5.** *Local equilibrium measures $\mu^N$ satisfy Condition 2.3.*

*Proof.* First, by the definition of $\mu^N$, parts (1) and (2) of Condition 2.3 are met. In Lemma 9.6 below, we show that the relative entropy estimate, part (3), holds. □

It remains to show the relative entropy estimate of $\mu^N$. Note that, for future use, Lemma 9.6 is written for $\mu^N$ as defined in Subsection 2.3 allowing $c = c_0$.

**Lemma 9.6.** *There exists a constant $C$ such that $H(\mu^N|\mathscr{R}_{c,N}) \le CNN_\beta^{-1}$ holds for all $N$.*

*Proof.* Let $\zeta$ and $\chi$ be two Geometric distributions with rate $p$ and $q$ respectively. Assuming $p \le q$ we have

$$H(\zeta|\chi) = \sum_{n\ge 0}(1-p)p^n \ln\left(\frac{1-p}{1-q}\frac{p^n}{q^n}\right) = \ln\left(\frac{1-p}{1-q}\right) + \frac{p}{1-p}\ln\frac{p}{q} \le \ln\frac{1}{1-q}.$$

Suppose now, for $k \ge 1$, that $\zeta = \mu_k^N$ and $p = \theta_{N,k}$ and $\chi = \mathscr{R}_{\beta,c,N,k}$ and $q = c\theta_k$. Note that, by the assumption $\mu^N \le \mathscr{R}_{c,N}$, we have $\theta_{N,k} \le c\theta_k = ce^{-\beta u(\ln k) - k/N}$. Then, as $\mu^N$ and $\mathscr{R}_{c,N}$ is the product over $\{\mu_k^N\}_{k\ge 1}$ and $\{\mathscr{R}_{\beta,c,N,k}\}_{k\ge1}$ respectively, we have

$$H(\mu^N|\mathscr{R}_{c,N}) \le \sum_{k=1}^\infty \ln\frac{1}{1-c_0 e^{-\beta u(\ln k)-k/N}}.$$

When $\beta = 0$, we have

$$H(\mu^N|\mathscr{R}_{c,N}) \le \sum_{k=1}^\infty \ln\frac{1}{1-e^{-k/N}} \le -N\int_0^\infty \ln(1-e^{-x})dx =: CN.$$

For the cases $\beta > 0$, we recall that $c_0 = \min_k e^{\beta \mathcal{E}_k}$. Let $K_0 = \{k_{0,j}\}_{1\le j\le J}$ be the indices where $c_0$ is attained. Since $\mathcal{E}_k$ diverges to $\infty$ as $k\to\infty$, we have that $J$ is finite. The contribution from each $k_{0,j}$ to the relative entropy $H(\mu^N|\mathscr{R}_{c,N})$ is bounded above by

$$\ln\frac{1}{1-e^{-k_{0,j}/N}} = O(\ln N).$$

This order is negligible compared with $NN_\beta^{-1} = Ne^{-\beta\mathcal{E}_N} = Ne^{-\beta u(\ln N)}$ in the two cases when $u'(\ln N) \to 1$ and $0 < \beta < 1$ or when $u'(\ln N)\to 0$ and $\beta > 0$. We will be able to disregard later these $k_{0,j}$'s.

Now, as $u(\ln k) \to \infty$ as $k \to \infty$, find $0 < \alpha < 1$ such that $0 < c_0 e^{-\beta u(\ln k) - k/N} \le \alpha$ for all $N$ and $k \notin K_0$. Using convexity of $-\ln(1-x)$, there exists $c_1 > 0$ such that $-\ln(1-x) \le c_1 x$ on $[0,\alpha]$. Then, we have

$$\sum_{k=1}^\infty \ln\frac{1}{1-c_0 e^{-\beta u(\ln k)-k/N}} \le c_1 c_0 \sum_{k=1}^\infty e^{-\beta u(\ln k) - k/N} + O(\ln N).$$

Multiplying and dividing by the term $NN_\beta^{-1}$, we get

$$H(\mu^N|\mathscr{R}_{c,N}) \le c_1 c_0 N N_\beta^{-1}\left[\sum_{k=1}^\infty \frac{1}{N} e^{-\beta(u(\ln k)-u(\ln N))-k/N} + O(N^{-1}N_\beta \ln N)\right]. \quad (9.6)$$

Now, $N^{-1}N_\beta \ln N$ vanishes as $N\to\infty$ and by Lemma 9.2 the summation in (9.6) approaches a finite limit. The proof is now complete. □

### 9.3 Properties of $\nu^N$ satisfying Condition 2.3

We will establish the items (2.6), (2.7), (2.8), and (2.9). We start with an estimate on the number of particles in the system.

**Lemma 9.7.** *We have that 'the total expected particle bound'* (2.6) *holds.*

*Proof.* Since the total number of particles is conserved we have

$$\frac{N_\beta}{N}\mathbb{E}_N \sum_{k=1}^\infty \eta_t(k) = \frac{N_\beta}{N}\mathbb{E}_N \sum_{k=1}^\infty \eta_0(k) = \frac{1}{N}\sum_{k=1}^\infty N_\beta m_{N,k} = o(1) + \frac{1}{N}\sum_{k=1}^\infty \overline{\rho}_{N,k}$$

by Condition 2.3. However, $N^{-1}\sum_{k=1}^\infty \overline{\rho}_{N,k} = \int_0^\infty \rho_0(x)dx$, which is finite. □

**Lemma 9.8.** *We have that the 'variance bound'* (2.7) *holds.*

*Proof.* By attractiveness (2.10),

$$\mathrm{Var}_{\mathbb{P}_N}(\eta_t(k)) = \mathbb{E}_N[\eta_t^2(k)] - (\mathbb{E}_N\eta_t(k))^2 \le E_{\mathscr{R}_{c,N}}[\eta^2(k)] \le \mathrm{Var}_{\mathscr{R}_{c,N}}(\eta(k)) + \rho_{k,c}^2.$$

Then, by Lemma 9.4, we conclude that $N_\beta^2 N^{-2}\sum_{k=1}^\infty \mathrm{Var}_{\mathbb{P}_N}(\eta_t(k)) \to 0$ as $N\to\infty$. □

**Lemma 9.9.** *We have that the 'site particle bound'* (2.8) *holds.*

*Proof.* First, by attractiveness (2.10), we have that $\mathbb{E}_N\left[\eta_t(k)\right] \le E_{\mathscr{R}_{c,N}}\left[\eta(k)\right] = \rho_{k,c}$ (cf. (2.1)). To bound $N_\beta\rho_{k,c}$, recall that $c_0 = \min_k e^{\beta\mathcal{E}_k}$ and $c < c_0$.

When $\beta = 0$, we have $c_0 = 1$ and $N_\beta = 1$. In this case, we have the desired bound, $N_\beta\rho_{k,c} \le \dfrac{e^{-a}}{1-e^{-a}}$ for all $k \ge aN$.

When $\beta > 0$, using the definition of $c_0$, and that $c < c_0$, we have the denominator $1 - ce^{-\beta\mathcal{E}_k - k/N} \ge 1 - e^{-a}$ as $k \ge aN$. Write $N_\beta e^{-\beta\mathcal{E}_k - k/N} \le e^{-\beta(\mathcal{E}_k - \mathcal{E}_N)-a}$. By the mean value theorem, $\mathcal{E}_k - \mathcal{E}_N = u'(r)\ln(k/N)$ where $r$ is between $aN \le k$ and $N$. By assumption, $u'(r)$ tends to 0 or 1, and $\ln(k/N) \le \ln b$ for $k \le bN$. We conclude then that $N_\beta e^{-\beta\mathcal{E}_k - k/N}$ is uniformly bounded in $N$, and the lemma follows. □

We now address initial convergence.

**Proposition 9.10.** *We have 'initial convergence'* (2.9) *holds.*

*Proof.* By assumption, $\lim_{N\to\infty}\dfrac{1}{N}\sum_{k=1}^\infty |N_\beta m_{N,k} - \overline{\rho}_{N,k}| = 0$. For a test function $G$, since $N^{-1}\sum_{k=1}^\infty G(k/N)\overline{\rho}_{N,k}$ approximates $\int_{\mathbb{R}^+} G(x)\rho_0(x)dx$, it is enough to check that

$$\nu^N\left[\left|N^{-1}\sum_{k=1}^\infty N_\beta(\eta(k) - m_{N,k})\right| > \delta\right].$$

By Chebychev's inequality, we have the upperbound of $\delta^{-2}N_\beta^2 N^{-2}\sum_{k=1}^\infty \mathrm{Var}_{\nu^N}(\eta(k))$, which vanishes by the variance bound in Lemma 9.8. □

## 10 Uniqueness of weak solutions

In this section, we present some uniqueness results for the macroscopic equations in Theorems 2.4, 2.5 and 2.6, governing the particle density $\rho(t,x)$ or the height function $\psi(t,x) := \int_x^\infty \rho(t,u)du$. The methods are based on maximum principles for linear parabolic equations.

We first need a lemma to relate properties of $\psi$ with those of $\rho$. Recall that $\mathscr{C}$ is space of functions $\rho : [0,T] \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $t \in [0,T] \mapsto \rho(t,x)dx \in \mathcal{M}$ is vaguely continuous: Namely, for each $G \in C_c^\infty(\mathbb{R}_\circ^+)$, the map $t \in [0,T] \mapsto \int_0^\infty G(x)\rho(t,x)dx$ is continuous.

Also, recall

$$\mathcal{W} = \left\{\psi \in C\left([0,T] \times \mathbb{R}^+\right) : \text{ for } t \in [0,T],\ \psi(t,\cdot) \text{ is absolutely continuous on } \mathbb{R}_\circ^+\right\}.$$

**Lemma 10.1.** *Let $\rho(t,x) \in \mathscr{C}$. Suppose, for all $t \in [0,T]$, that*

$$\rho(t,\cdot) \le \phi_c(\cdot) \in L^1(\mathbb{R}^+), \quad \int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx < \infty. \tag{10.1}$$

*Let $\psi(t,x) = \int_x^\infty \rho(t,u)du$. Then, $\psi(t,x)$ belongs to the class $\mathcal{W}$ with*

$$\lim_{x\to\infty} \psi(t,x) = 0, \quad 0 \le -\partial_x\psi(t,\cdot) \le \phi_c(\cdot), \quad \psi(t,0) = \psi(0,0). \tag{10.2}$$

*Proof.* The absolute continuity of $\psi(t,\cdot)$ follows from definition of $\psi$ and it is trivial to verify (10.2) from (10.1). To finish, we need only to check that $\psi(t,x)$ is a continuous function on $[0,T] \times \mathbb{R}^+$.

We claim that such continuity will follow if $\psi$ is continuous in $x$ and $t$ separately. Indeed, fix any $(t_0,x_0) \in (0,T) \times \mathbb{R}_\circ^+$ and denote $\psi(t_0,x_0) = a_0$. If $x \mapsto \psi(t,x)$, for each $t$, is continuous at $x_0$, then for any $\epsilon > 0$ there exists $\delta$ such that

$$a_0 - \epsilon \le \psi(t_0, x_0 \pm \delta) \le a_0 + \epsilon.$$

Suppose $t \mapsto \psi(t,x)$, for each $x$, is continuous in $t$, then we may find $\delta'$, such that for all $t$ where $|t - t_0| \le \delta'$, we have

$$\psi(t_0, x_0 \pm \delta) - \epsilon \le \psi(t, x_0 \pm \delta) \le \psi(t_0, x_0 \pm \delta) + \epsilon.$$

Since $x \mapsto \psi(t,x)$, for each $t$, is monotone in $x$, we have, for all $(t,x)$ such that $|t - t_0| \le \delta'$ and $|x - x_0| \le \delta$, that

$$-2\epsilon \le \psi(t,x) - \psi(t_0, x_0 \pm \delta) \le 2\epsilon.$$

Hence, we deduce continuity of $\psi$ at $(t_0,x_0)$. Continuity for boundary points $(t,x)$ on the boundary is verified in the same way.

Now, we focus on showing that $t \mapsto \psi(t,x)$ and $x \mapsto \psi(t,x)$ are both continuous. For any fixed $t \in [0,T]$, $x \mapsto \psi(t,x)$ is continuous on $\mathbb{R}^+$ since $\psi$ is in form $\psi(t,x) = \int_x^\infty \rho(t,u)du$ and $\int_0^\infty \rho(t,u)du < \infty$.

To show continuity in $t$, we first note that $\psi(t,0) = \psi(0,0)$ for all $t \in [0,T]$, and therefore $t \mapsto \psi(t,0)$ is continuous. Fix now any $x_0 > 0$ and $t_0 \in [0,T]$. For any $\epsilon > 0$, using $\rho(t,x) \le \phi_c(x)$ and that $\phi_c \in L^1(\mathbb{R}^+)$, we may find $G$ continuous and with compact support in $\mathbb{R}_\circ^+$ such that for all $t \in [0,T]$,

$$\left|\int_x^\infty \rho(t,u)du - \int_0^\infty G(u)\rho(t,u)du\right| \le \frac{\epsilon}{4}.$$

Then, by the triangle inequality using two applications of the above inequality, we have $|\psi(t, x_0) - \psi(t_0, x_0)|$ is bounded from above by

$$\left|\int_0^\infty G(u)\rho(t,u)du - \int_0^\infty G(u)\rho(t_0,u)du\right| + \frac{\epsilon}{2}.$$

Finally, continuity of $t \mapsto \psi(t, x_0)$ at $t_0$ follows as $\rho \in \mathscr{C}$, namely from the vague continuity of $\rho(t, x)dx$. □

### 10.1 Case: $\beta = 0$

Let $\rho(t, x) \in \mathscr{C}$ with $\rho(0, \cdot) = \rho_0(\cdot)$ be a weak solution of the equation

$$\partial_t \rho = \partial_x^2 \frac{\rho}{\rho+1} + \partial_x \frac{\rho}{\rho+1},$$

that is, for all $G \in C_c^\infty([0,T) \times \mathbb{R}_\circ^+)$,

$$\int_0^\infty G(0,x)\rho_0 dx + \int_0^T \int_0^\infty \left\{\partial_t G\rho + \partial_x^2 G\frac{\rho}{\rho+1} - \partial_x G\frac{\rho}{\rho+1}\right\} dxdt = 0. \tag{10.3}$$

Assume also that $\rho(t, x)$ satisfies (10.1).

**Proposition 10.2.** *We have $\psi(t,x) = \int_x^\infty \rho(t,u)du$ belongs to $\mathcal{W}$ and* (10.2) *holds by Lemma 10.1. In particular, $\psi$ solves weakly the equation*

$$\partial_t \psi = \partial_x \left(\frac{\partial_x \psi}{1 - \partial_x \psi}\right) + \frac{\partial_x \psi}{1 - \partial_x \psi},$$

*that is, for all $G \in C_c^\infty([0,T) \times \mathbb{R}_\circ^+)$*

$$\int_0^\infty G(0,u)\psi_0 dx + \int_0^T \int_0^\infty \left\{\partial_t G\psi - \partial_x G\frac{\partial_x \psi}{1-\partial_x \psi} + G\frac{\partial_x \psi}{1-\partial_x \psi}\right\} dxdt = 0, \tag{10.4}$$

*where $\psi_0(x) = \int_x^\infty \rho_0(u)du$.*

*Moreover, $\psi(t, x)$ is the unique weak solution in the class $\mathcal{W}$ of the initial-boundary value problem* (2.16)*. Consequently, $\rho(t, x)$ is the unique weak solution in $\mathscr{C}$ of the equation* (2.11)*.*

*Proof.* We first show (10.4). Since $\rho(t,x) \le \phi_c(x) \in L^1(\mathbb{R}^+)$ (cf. (10.1)), by straightforward approximations, the test functions admissible for (10.3) may be extended to include all functions of the form $\widehat{G}(t,x) = \int_0^x G(t,u)du$ where $G \in C_c^\infty([0,T) \times \mathbb{R}_\circ^+)$. Then, by integration by parts, (10.4) follows.

We now show $\psi(t, x)$ is the unique weak solution to (2.16) in the space $\mathcal{W}$. Suppose there exist two such weak solutions $\psi_1$, $\psi_2$. Let $\psi = \psi_1 - \psi_2$ and $H(p) = \dfrac{p}{1-p}$. As (10.4) holds for $\psi_1$, $\psi_2$, in the new notation, we have

$$\int_0^T \int_0^\infty \Big\{\partial_t G\psi - \partial_x G\big(H(\partial_x\psi_1) - H(\partial_x\psi_2)\big) + G\big(H(\partial_x\psi_1) - H(\partial_x\psi_2)\big)\Big\}dxdt = 0$$

and

$$\begin{aligned} H(\partial_x\psi_1) - H(\partial_x\psi_2) &= (\partial_x\psi_1 - \partial_x\psi_2)\int_0^1 H'(\tau\partial_x\psi_1 + (1-\tau)\partial_x\psi_2)d\tau \\ &=: (\partial_x\psi_1 - \partial_x\psi_2)\widehat{H}(t,x). \end{aligned}$$

Then, $\psi$ satisfies

$$\int_0^T \int_0^\infty \left\{ \partial_t G \psi - \partial_x G \left( \widehat{H}(t,x) \partial_x \psi \right) + G \left( \widehat{H}(t,x) \partial_x \psi \right) \right\} dxdt = 0,$$

that is, $\psi$ is a weak solution in $\mathcal{W}$ of the linear problem

$$\begin{cases} \partial_t \psi = \partial_x \left( \widehat{H} \partial_x \psi \right) + \widehat{H} \partial_x \psi \\ \psi(0,x) = 0, \quad \psi(t,0) = 0, \\ \lim_{x\to\infty} \psi(t,x) = 0, \quad -\phi_c(\cdot) \le \partial_x \psi(t,\cdot) \le \phi_c(\cdot) \text{ for all } t \in [0,T]. \end{cases} \tag{10.5}$$

To show that $\psi \equiv 0$, and therefore uniqueness of weak solution. it suffices to show, for all $\varepsilon > 0$ and all compact set $D \subset (0,T) \times \mathbb{R}_\circ^+$, that $|\psi| < \varepsilon$ on $D$.

For such a $D$, we may find $0 < a < b < \infty$ where $D \subset Q_{a,b}^T := (0,T) \times (a,b)$. Since $\left|\partial_x \psi(t,\cdot)\right| \le \phi_c(\cdot) \in L^1(\mathbb{R}^+)$ for all $t \in [0,T]$, and $\psi$ vanishes for both $x = 0$ and $x \to \infty$, we can adjust $a$, $b$ so that $|\psi(t,a)| < \varepsilon$ and $|\psi(t,b)| < \varepsilon$ for all $t \in [0,T]$. Then, we have $|\psi| < \varepsilon$ on the parabolic boundary of $Q_{a,b}^T$.

Notice that, on $Q_{a,b}^T$, the PDE in (10.5) is uniformly parabolic and has bounded coefficients: Since $H'(p) = \dfrac{1}{(1-p)^2}$ and $-\phi_c(a) \le \partial_x \psi_1, \partial_x \psi_2 \le 0$ on $Q_{a,b}^T$, we have

$$\frac{1}{(1+\phi_c(a))^2} \le \widehat{H} \le 1 \quad \text{on } Q_{a,b}^T.$$

Then, by a maximum principle (cf. p. 188, [19]), we have $|\psi| < \varepsilon$ on $Q_{a,b}^T$, and therefore on $D$.

Finally, if $\rho(t,x)$ were not unique with respect to (10.3), one could construct two different weak solutions $\psi(t,x)$, which is a contradiction. □

### 10.2 Case $\beta > 0$

Let $\rho(t,x) \in \mathscr{C}$ with $\rho(0,\cdot) = \rho_0(\cdot)$ be a weak solution of

$$\partial_t \rho = \partial_x^2 \rho - \partial_x \Big( \alpha(x,\beta) \rho \Big). \tag{10.6}$$

where $\alpha(x,\beta) = -(\beta + x)/x$ when $\mathcal{E}_k \sim \ln k$ and equals $-1$ when $1 \ll \mathcal{E}_k \ll \ln\ln k$ (cf. (3.3)), and $\rho$ satisfies (10.1).

**Proposition 10.3.** *We have $\psi(t,x) = \int_x^\infty \rho(t,u)du$ belongs to $\mathcal{W}$ and* (10.2) *holds by Lemma 10.1, and solves weakly the equation*

$$\partial_t \psi = \partial_x^2 \psi - \alpha(x,\beta) \partial_x \psi, \tag{10.7}$$

*where $\psi(0,x) = \int_x^\infty \rho_0(u)du$.*

*Then, $\psi(t,x)$ is the unique weak solution in $\mathcal{W}$ of the initial-boundary value problem* (2.17) *when $\mathcal{E}_k \sim \ln k$, and of* (2.18) *when $1 \ll \mathcal{E}_k \ll \ln k$. Consequently, $\rho(t,x)$ is the unique weak solution in $\mathscr{C}$ of the equation* (2.12) *when $\mathcal{E}_k \sim \ln k$ and of* (2.13) *when $1 \ll \mathcal{E}_k \ll \ln k$.*

*Proof.* That $\psi$ solves weakly (10.7) follows, as in the proof of Lemma 10.1, from the assumptions $\rho$ is a weak solution of (10.6) and $\rho \le \phi_c$.

Notice that, in equation (10.7), the coefficient $-\alpha(x,\beta)$ before $\partial_x \psi$ equals $\dfrac{\beta + x}{x}$ when $\mathcal{E}_k \sim \ln k$ and equals 1 when $1 \ll \mathcal{E}_k \ll \ln k$. In both situations, it is bounded on any $[a,b]$ with $0 < a < b < \infty$, even if it blows up at $x = 0$ when $\mathcal{E}_k \sim \ln k$. Then, the same proof of uniqueness given for Lemma 10.1 applies to show uniqueness of weak solutions for the equations (2.17) and (2.18). □

## A Remarks on limits when $c = c_0$

We now make remarks, for the interested reader, on some of the behavior with respect to measures $\mathscr{R}_{c,N}$ at the boundary, when $c = c_0$.

**1.** Lemma 9.4 does not hold for invariant measure $\mathscr{R}_{c_0,N}$. In fact, under $\mathscr{R}_{c_0,N}$, the total number of particles explodes and the associated variance does not vanish in the limit.

**Lemma A.1.** *We have*

$$\frac{N_\beta}{N}\sum_{k=1}^{\infty} E_{\mathscr{R}_{c_0,N}}(\eta(k)) = \frac{N_\beta}{N}\sum_{k=1}^{\infty}\rho_{c_0,k} \to \infty \text{ as } N \to \infty. \tag{A.1}$$

*and*

$$\liminf_{N\to\infty}\frac{N_\beta^2}{N^2}\sum_{k=1}^{\infty}\mathrm{Var}_{\mathscr{R}_{c_0,N}}(\eta(k)) = \liminf_{N\to\infty}\frac{N_\beta^2}{N^2}\sum_{k=1}^{\infty}(\rho_{k,c_0}^2 + \rho_{k,c_0}) > 0. \tag{A.2}$$

*Proof.* To verify these two claims, recall that $\rho_{k,c_0} = \dfrac{c_0 e^{-\beta\mathcal{E}_k - k/N}}{1 - c_0 e^{-\beta\mathcal{E}_k - k/N}}$ and $c_0 = \min_k e^{\beta\mathcal{E}_k}$.

When $\beta = 0$, (A.1) and (A.2) follow from the limits,

$$\frac{N_\beta}{N}\sum_{k=1}^{\infty}\rho_{c_0,k} = \frac{1}{N}\sum_{k=1}^{\infty}\frac{e^{-k/N}}{1-e^{-k/N}} = \sum_{k=1}^{\infty}\frac{1}{N(e^{k/N}-1)} \to \infty,$$

and

$$\frac{N_\beta^2}{N^2}\sum_{k=1}^{\infty}\rho_{k,c_0}^2 = \frac{1}{N^2}\sum_{k=1}^{\infty}\Big(\frac{e^{-k/N}}{1-e^{-k/N}}\Big)^2 \geq \frac{1}{N^2}\Big(\frac{1}{e^{1/N}-1}\Big)^2 \to 1.$$

For the other two cases, when $\beta > 0$, let $k_0$ be an index where $c_0$ is realized, that is $c_0 = e^{\mathcal{E}_{k_0}}$. Now notice, as $N \to \infty$,

$$\frac{1}{N}\rho_{k_0,c_0} = \frac{1}{N}\frac{e^{-k_0/N}}{1-e^{-k_0/N}} \to \frac{1}{k_0}.$$

Then, both (A.1) and (A.2) follow from

$$\frac{N_\beta}{N}\sum_{k=1}^{\infty}\rho_{k,c_0} \geq \frac{N_\beta}{N}\rho_{k_0,c_0}, \quad \frac{N_\beta^2}{N^2}\sum_{k=1}^{\infty}\rho_{k,c_0}^2 \geq \frac{N_\beta^2}{N^2}\rho_{k_0,c_0}^2,$$

and that $N_\beta \to \infty$ as $N \to \infty$. $\square$

**2.** We showed in Proposition 2.2, when $c < c_0$ in the three regimes , that $\phi_c$ corresponds in a sense to the limit shape under the measures $\mathscr{R}_{c,N}$. We now state the same happens when $c = c_0$.

**Lemma A.2.** *We have that the limit* (2.4) *holds when $c = c_0$.*

*Proof.* A main tool in the proof of Proposition 9.10, which applies under measures $\mathscr{R}_{c,N}$ when $c < c_0$, is the variance estimate in Lemma 9.4, which as seen in Lemma A.1 above does not hold. However, since $G$ has compact support, it is enough to make estimates for $k \in [aN, bN]$, where the support of $G$ is contained in $[a, b]$ for $0 < a < b$.

We claim that in all the three regimes,

$$\lim_{N\to\infty}\frac{N_\beta^2}{N^2}\sum_{aN\leq k\leq bN}\mathrm{Var}_{\mathscr{R}_{c_0,N}}(\eta(k)) = 0. \tag{A.3}$$

Indeed, notice that $\mathrm{Var}_{\mathscr{R}_{c_0,N}}(\eta(k)) = \rho_{k,c_0}^2 + \rho_{k,c_0}$ where $\rho_{k,c_0} = \dfrac{c_0 e^{-\beta\mathcal{E}_k - k/N}}{1 - c_0 e^{-\beta\mathcal{E}_k - k/N}}$. Since $N_\beta = o(N)$, the claim (A.3) would follow from the bound $\sup_N \sup_{aN\le k\le bN} N_\beta \rho_{k,c_0} < \infty$. Such a bound holds in fact by the proof of Lemma 9.9.

Hence, under $\mathscr{R}_{c_0,N}$, we conclude $N_\beta N^{-1}\sum G(k/N)(\eta(k) - \rho_{k,c_0}) \to 0$ in probability. To finish, we need only show that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{\infty} G\Big(\frac{k}{N}\Big) N_\beta \rho_{k,c_0} = \int_0^\infty G(x)\phi_{c_0}(x)dx, \tag{A.4}$$

where, we note that the summation of $k$ above is actually on $aN \le k \le bN$. Recall the formula for $\rho_{k,c_0}$ in (2.1).

When $\beta = 0$, we have $N_\beta = 1$ and $c_0 = 1$. Then,

$$N_\beta \rho_{k,c_0} = \frac{e^{-k/N}}{1 - e^{-k/N}} \to \frac{e^{-x}}{1 - e^{-x}} = \phi_{c_0}, \text{ as } N \to \infty, \frac{k}{N} \to x.$$

Then, (A.4) follows from dominated convergence.

However, when $\beta > 0$, note first $N_\beta \theta_{k,c_0} = c_0 e^{-\beta(\mathcal{E}_k - \mathcal{E}_N) - k/N}$ and $\mathcal{E}_k - \mathcal{E}_N = u(\ln k) - u(\ln N)$. By the mean value theorem, $\mathcal{E}_k - \mathcal{E}_N \to \ln(x)\lim_{z\to\infty} u'(z)$ as $N \to \infty$ and $k/N \to x$. Note also that $N_\beta = e^{\beta\mathcal{E}_N} \to \infty$ (cf. (2.2)). Then,

$$N_\beta \rho_{k,c_0} = \frac{N_\beta \theta_{k,c_0}}{1 - \theta_{k,c_0}} \to c_0 e^{-\beta \ln(x)\lim_{z\to\infty} u'(z)} e^{-x} = \phi_{c_0}(x), \text{ as } N \to \infty, \frac{k}{N} \to x.$$

Again, by dominated convergence theorem, (A.4) follows. □

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