

## PART 8: ADDITIVE FUNCTIONALS AND CENTRAL LIMIT THEOREMS

The occupation time of a set, for instance, is an additive functional of a process. One would like to know the first order and second order behaviors, namely the LLN and CLT for these objects. This is an old subject, and in stochastic particle systems where the configuration space is difficult, interesting scalings and limits.

We focus here mostly on symmetric simple exclusion processes. After a general introduction, we discuss ergodic and fluctuation behaviors. The Kipnis-Varadhan CLT, which has wide application, not only to particle systems, is stated and proved. Comments on the behaviors in asymmetric simple exclusion are also made.

### 1. BASIC PROBLEM AND ERGODIC THEORY

Consider a Markov process  $\eta(t)$  on a state space  $\Sigma$ . Let  $\mu$  be an invariant measure for the process. Let  $f : \Sigma \rightarrow \mathbb{R}$  be an  $L^2(\mu)$  function. The basic problem is to determine the behavior of

$$A_f(t) = \int_0^t f(\eta_s)ds$$

as  $t \uparrow \infty$ .

In the following, we call  $A_f(t)$  an “additive functional” since  $A_f(t+s) - A_f(s)$  under  $\mu$  has the same distribution as  $A_f(t)$  under  $\mu$ .

If  $\mu$  is extremal, then as we have seen, the process started from initial distribution  $\mu$  is ergodic with respect to time shifts, and hence

$$\lim_{t \uparrow \infty} \frac{1}{t} \int_0^t f(\eta_s)ds = E_\mu[f] \quad \mu - \text{a.s.}$$

This covers the case when  $\mu$  is unique, for instance, in positive-recurrent Markov chains. When  $\mu$  is null-recurrent, different behaviors may emerge (see [12]).

The next order question is to ask about the fluctuations: Find the limits of

$$\frac{1}{\sqrt{t}} \int_0^t (f(\eta_s) - E_\mu[f])ds.$$

One expects a Gaussian limit, with sufficient mixing. In the following, we will assume  $f$  is already centered, that is  $E_\mu[f] = 0$  to simplify expressions.

Consider the following example in finite-state irreducible Markov chains. Consider  $\mu = \langle \mu(x) : x \in \Sigma \rangle$  and  $f = \langle f(x) : x \in \Sigma \rangle$  as elements of  $\mathbb{R}^{|\Sigma|}$ . Note that  $(\mu L)(x) = 0$  for all  $x \in \Sigma$  where  $L$  is the generator of the process. Since  $\mu$  is the unique invariant measure,  $\text{Null}(L^T) = \{c\mu : c \in \mathbb{R}\}$ . The orthogonal complement of this null space is  $\text{Range}(L)$ : For  $\phi \in \text{Range}(L)$ ,  $\phi = Lu$ . Then,  $\sum_x \mu(x)(Lu)(x) = 0$  as  $\mu$  is invariant.

Then, given  $E_\mu[f] = \sum_x f(x)\mu(x) = 0$ , we have that  $f \perp \mu$  and hence  $f$  belongs to the range  $\text{Range}(L)$ , and  $f = -Lu$  for some function  $u$ . That is,  $f$  solves a ‘Poisson equation’ with respect to the generator.

We may now write

$$\frac{t}{\sqrt{t}} \int_0^t f(\eta_s) ds = \frac{1}{\sqrt{t}} M(t) + \frac{1}{\sqrt{t}} (u(\eta(0)) - u(\eta(t))) \quad (1.1)$$

where

$$M(t) = u(\eta(t)) - u(\eta(0)) - \int_0^t L u(\eta_s) ds$$

is a martingale with respect to natural sigma-fields. The quadratic variation, as we have observed in an earlier part, equals

$$\langle M \rangle(t) = \int_0^t (L u^2 - 2u L u)(\eta_s) ds$$

whose mean is

$$E_\mu[\langle M \rangle(t)] = 2t E_\mu[u(Lu)] = 2t D(u)$$

where  $D(u)$  is the Dirichlet form.

The second term in (1.1) is on order  $t^{-1/2}$  since  $u$  is a function on finite state space, and hence bounded. The first term, however, can be treated with a martingale central limit theorem.

**Proposition 1.1.** *Let  $(\mathcal{M}_t, \mathcal{F}_t)$  be a mean-zero martingale with stationary increments, and scaled quadratic variation converging to a constant  $\sigma^2$ ,*

$$E \left| \frac{1}{t} \langle \mathcal{M}_t \rangle - \sigma^2 \right| \rightarrow 0.$$

Then,

$$\frac{1}{\sqrt{t}} \mathcal{M}_t \Rightarrow N(0, \sigma^2).$$

**Exercise 1.2.** This theorem can be found in discrete time in many places (see website). From the discrete time version, prove the proposition above, by considering the discrete time martingale  $M_{[t]}$ .

To complete the argument, we note by ergodicity that

$$\frac{1}{t} \langle M \rangle(t) \rightarrow E_\mu[L u^2 - 2u(Lu)] = 2D(u)$$

in both  $L^2(\mu)$  and  $\mu$ -a.s. The martingale  $M(t)$  also has stationary increments when the process is started from  $\mu$ . Hence, by Proposition 1.1, we conclude that

$$\frac{1}{\sqrt{t}} \int_0^t f(\eta_s) ds \Rightarrow N(0, 2D(u)).$$

## 2. OCCUPATION FUNCTIONALS IN STOCHASTIC PARTICLE SYSTEMS

Consider the simple exclusion system on  $\mathbb{Z}^d$  with semigroup and generator on  $L^2(\nu_\rho)$ . We will consider the “additive functional” question for occupation functions  $f(\eta) = \eta(0) - \rho$ . Recall that  $\nu_\rho$  is the Bernoulli product measure with success probability  $\rho \in (0, 1)$ , and the configuration space  $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ . In the following, we will assume that the jump probability  $p$  is finite-range.

Given  $\nu_\rho$  is extremal, we see that the LLN behavior starting under  $\nu_\rho$  is clear: For  $f \in L^2(\nu_\rho)$ ,

$$\lim_{t \uparrow \infty} \frac{1}{t} \int_0^t f(\eta_s) ds = E_{\nu_\rho}[f]$$

both in  $L^2(\nu_\rho)$  and  $\nu_\rho$ -a.s.

However, the fluctuation behavior of  $A(t)$ , with respect to  $f(\eta) = \eta(0) - \rho$  is different depending on the dimension  $d$ , the asymmetry of the jump probability  $p$ , and the density  $\rho$ . A more or less complete theory is known, except for some interesting outlier cases which connect with ‘KPZ’ class phenomenon.

We first consider the types of variances that might be obtained. What is the variance of  $A_f(t)$ , what are its limits, and how does the limit depend on  $f$ ? For a general local function  $f$ , let us try to get a formula for  $\text{Var}(A_f(t))$ . Write, using stationarity, that

$$\begin{aligned}\text{Var}(A_f(t)) &= E_{\nu_\rho} \left[ \left( \int_0^t f(\eta_s) ds \right)^2 \right] \\ &= 2 \int_0^t (t-s) E_{\nu_\rho}[f(\eta_s)f(\eta_0)] ds.\end{aligned}$$

By reversibility of  $\nu_\rho$ , the semigroup  $P_s$  is self-adjoint, and we may express

$$\begin{aligned}E_{\nu_\rho}[f(\eta_s)f(\eta_0)] &= \langle P_s f, f \rangle_{\nu_\rho} \\ &= \langle P_{s/2} f, P_{s/2} f \rangle_{\nu_\rho} = \|P_{s/2} f\|_{L^2(\nu_\rho)}^2 \geq 0.\end{aligned}$$

Then, when the variance is scaled by  $t$ , its limit exists, possibly infinite, and is in form

$$\begin{aligned}\sigma^2(f) &= \lim_{t \uparrow \infty} \frac{1}{t} \text{Var}(A_f(t)) \\ &= \lim_{t \uparrow \infty} 2 \int_0^t \left[ 1 - \frac{s}{t} \right] E_{\nu_\rho}[f(\eta_s)f(\eta_0)] ds \\ &= 2 \int_0^\infty E_{\nu_\rho}[f(\eta_s)f(\eta_0)] ds.\end{aligned}$$

The last equality follows from monotone convergence. This formula is the familiar ‘sum of correlations’ expression with respect to sums of correlated random variables.

**2.1. Symmetric  $p$ .** In symmetric exclusion, any mean-zero, local function can be written as

$$\begin{aligned}f(\eta) &= \sum_B \hat{f}(B) \prod_{x \in B} \frac{\eta(x) - \rho}{\sqrt{\rho(1-\rho)}} \\ &= \sum_{n \geq 1} \sum_{|B|=n} \hat{f}(B) \prod_{x \in B} \frac{\eta(x) - \rho}{\sqrt{\rho(1-\rho)}}.\end{aligned}$$

Here, the decomposition is over ‘degrees’, that is, functions which are the product of  $n$  centered, normalized occupation variables.

This basis is quite central to symmetric simple exclusion because of the ‘duality’ relation. Namely, for a function

$$f_B(\eta) = \prod_{x \in B} \frac{\eta(x) - \rho}{\sqrt{\rho(1-\rho)}}$$

where  $|B| = n$ , we have

$$(P_t f_B)(\eta) = \sum_{|U|=n} p_t(B, U) \prod_{x \in U} \frac{\eta(x) - \rho}{\sqrt{\rho(1-\rho)}}$$

where  $p_t(B, U)$  is the transition probability of  $n$  particle simple exclusion from a configuration  $B$  to configuration  $U$  in time  $t$ .

One way to prove this relation is to observe that

$$Lf(\eta) = \sum_{|U|=n} q(B, U) f_U(\eta)$$

where  $q(B, U)$  is the transition rate of  $n$  particle symmetric simple exclusion.

**Exercise 2.1.** Provide more details in the proof of the duality relation.

At this point, we can use the duality relation, and independence of coordinates under  $\nu_\rho$ , to evaluate more the term

$$\begin{aligned} E_{\nu_\rho}[(P_s f(\eta_0)f(\eta_0))] &= E_{\nu_\rho}\left[\sum_B \hat{f}(B) \sum_U p_t(B, U) f_U(\eta_0) \cdot f(\eta_0)\right] \\ &= \sum_B \sum_U \hat{f}(B) \hat{f}(U) p_t(B, U) \\ &= \sum_{n \geq 1} \sum_{|B|, |U|=n} \hat{f}(B) \hat{f}(U) p_t(B, U). \end{aligned}$$

Now, when  $f(\eta) = \eta(0) - \rho$  is the centered occupation function,  $\hat{f}(B) = 0$  for all  $B \neq \{0\}$ , and  $\hat{f}(\{0\}) = \sqrt{\rho(1-\rho)}$ . Then, the variance

$$\text{Var}(A_f(t)) = 2\rho(1-\rho) \int_0^t (t-s)p_s(0,0)ds$$

where  $p_t(0,0)$  is the return probability of a simple random walk according to jump probabilities  $p$ !

From local limit theorems, we have the following asymptotics for the variance.

**Proposition 2.2.**

$$\text{Var}(A_f(t)) = \begin{cases} \frac{2\rho(1-\rho)}{3\pi} t^{3/2} + o(t^{3/2}) & \text{in } d = 1 \\ \frac{\rho(1-\rho)}{2\pi} t \log t + o(t \log t) & \text{in } d = 2 \\ 2\rho(1-\rho) \int_0^\infty p_t(0,0)dt & \text{in } d \geq 3. \end{cases}$$

**2.2. Asymmetric  $p$ .** By asymmetric, we mean that  $\sum_x xp(x) \neq 0$ . The case that  $p$  is not symmetric but  $\sum_x xp(x) = 0$ , the ‘mean-zero’ asymmetric situation, is largely unexplored, and not considered here.

As for symmetric exclusion, one can understand the general asymptotics by calculating the term  $E_{\nu_\rho}[f(\eta_s)f(\eta_0)]$ . However, the ‘duality relation’ is no longer valid, since the generator does not preserve the ‘degree’ of a function, that is  $L$  applied to a function of  $n$  coordinates is no longer a linear combination of  $n$  coordinate functions.

Write

$$\begin{aligned} E_{\nu_\rho}[f(\eta_s)f(\eta_0)] &= E_{\nu_\rho}[\eta_s(0)\eta_0(0)] - \rho^2 \\ &= \rho\{E_{\nu_\rho}[\eta_s(0)|\eta_0(0)=1] - E_{\nu_\rho}[\eta_s(0)]\} \\ &= \rho(1-\rho)\{E_{\nu_\rho}[\eta_s(0)|\eta_0(0)=1] - E_{\nu_\rho}[\eta_s(0)|\eta_0(0)=0]\}. \end{aligned}$$

By the basic coupling, we can couple two copies of the exclusion process starting from configurations  $\eta' \geq \eta''$  where  $\eta'(x) = \eta''(x)$  for  $x \neq 0$  and  $\eta'(0) = 1$ ,  $\eta''(0) = 0$ .

Recall that  $(\eta'_t, \eta''_t)$  has generator

$$\begin{aligned}\bar{L}(\eta', \eta'') = & \sum_{x,y} p(y-x)1(\eta'(x) = \eta''(x) = 1)[f(\eta'^{x,y}, \eta''^{x,y}) - f] \\ & + \sum_{x,y} p(y-x)1(\eta'(x) = 1, \eta''(x) = 0)[f(\eta'^{x,y}, \eta'') - f].\end{aligned}$$

The  $\eta'_t$  process always majorizes  $\eta''_t$ , and there is exactly one discrepancy, which we label  $R_t$ .

The dynamics of  $R_t$  is as follows: Infinitessimally, it displaces by  $z$  with rate  $p(z)(1 - \eta(R_t + z)) + p(-z)\eta(R_t + z)$ . The part  $p(z)(1 - \eta(R_t + z))$  corresponds to the discrepancy, or ‘second-class’ particle as it is sometimes known, moving by its own intention, and the part  $p(-z)\eta(R_t + z)$  refers to when a particle at  $R_t + z$  likes to move to location  $R_t$  in which case by the basic coupling,  $R_t$  accedes and takes the place  $R_t + z$ .

Then, we have that

$$E_{\nu_\rho}[f(\eta_s)f(\eta_0)] = \rho(1 - \rho)\bar{P}(R_t = 0) \geq 0.$$

Hence, by monotone convergence the limit

$$\sigma_f^2 = 2\rho(1 - \rho) \int_0^\infty \bar{P}(R_t = 0) dt$$

exists.

In mean value, substituting  $\rho$  for  $\eta(R_t + z)$ , we see that the mean infinitessimal drift is

$$\sum z[p(z)(1 - \rho) + p(-z)\rho] = (1 - 2\rho) \sum zp(z).$$

This leads to the conjecture that

$$\sigma_f^2 < \infty \Leftrightarrow \rho \neq 1/2$$

which has been proved.

**Proposition 2.3.** *We have*

$$\text{Var}(A_f(t)) = \sigma_f^2 t + o(t)$$

when  $\rho \neq 1/2$  or when  $d \geq 3$ .

However, when  $\rho = 1/2$ , we have  $\text{Var}(A_f(t)) \geq C_1 t^{3/2}$ . In  $d = 2$ , we have  $\text{Var}(A_f(t)) \geq C_2 t \log \log t$ .

We remark the orders expected when  $\rho \neq 1/2$  in  $d = 1, 2$  are  $t^{4/3}$  and  $t(\log t)^{2/3}$  respectively. These orders connect with certain KPZ class phenomena, and are open.

### 3. CENTRAL LIMIT THEOREMS

Our goal now is to prove asymptotic normality of  $t^{-1/2}A_f(t)$  when  $\sigma_f^2 < \infty$ . Except the two open cases in Proposition 2.3, a central limit theorem also holds when  $A_f(t)$  is normalized by the square root of its variance, although we do not discuss these results here.

We focus on the symmetric case where the Kipnis-Varadhan CLT applies. The Kipnis-Varadhan theorem is a general CLT for additive functionals of reversible Markov processes on general state space. That the result holds under minimal conditions makes it powerful.

**Theorem 3.1.** Consider a Markov process  $\eta_t$  begun with ergodic, reversible invariant measure  $\mu$ . Let  $f : \Omega \rightarrow \mathbb{R}$  be such that  $\sigma_f^2 < \infty$ . Then,

$$\frac{1}{\sqrt{t}} \int_0^t f(\eta_s) ds \Rightarrow N(0, \sigma_f^2).$$

An invariance principle can also be proved under the assumption of the theorem, but we do not discuss this extension here.

The main tool to prove Theorem 3.1 is to approximate  $t^{-1/2} A_f(t)$  by a martingale, and then to use Proposition 1.1. The idea of the proof is to try to write  $f = -Lu$ . However, this is not possible in general. If it were possible, one could use the results in [2]. A resolvent type equation always holds however, that can be worked with. The following, which can be proved by this approach, is sufficient for the martingale approximation.

**Proposition 3.2.** Under the assumptions of Theorem 3.1, there is a martingale  $M(t)$  such that

$$\frac{1}{\sqrt{t}} A_f(t) = \frac{1}{\sqrt{t}} M(t) + \frac{1}{\sqrt{t}} \zeta_t$$

where

$$\lim_{t \uparrow \infty} \frac{1}{t} E_\mu[\zeta^2(t)] = 0.$$

and, in  $L^2(\mu)$ ,

$$\lim_{t \uparrow \infty} \frac{1}{t} \langle M \rangle(t) \Rightarrow \sigma_f^2.$$

**3.1.  $H_1$  and  $H_{-1}$  norms.** Before proving Proposition 3.2, some definitions will be useful. Define, for the core of local functions  $\phi$  and  $\lambda \geq 0$ , the semi-norm  $\|\phi\|_{1,\lambda}$  by

$$\|\phi\|_{1,\lambda}^2 = \langle \phi, (\lambda - L)\phi \rangle_\mu = D(\phi) + \lambda \|\phi\|_{L^2(\mu)}^2.$$

Note that, since  $-L$  is a nonnegative self-adjoint operator,

$$\langle \phi, (\lambda - L)\psi \rangle_\mu = \langle (\lambda - L)^{1/2}\phi, (\lambda - L)^{1/2}\psi \rangle_\mu \leq \|\phi\|_{1,\lambda} \|\psi\|_{1,\lambda}.$$

After modding out by functions with  $\|\phi\|_{1,\lambda} = 0$ , define the space  $\mathcal{H}_{1,\lambda}$  as the completion with respect to  $\|\cdot\|_{1,\lambda}$ .

Now, we define for a local function  $\phi$  that

$$\|\phi\|_{-1,\lambda} = \sup \left\{ \frac{\langle \phi, \psi \rangle_\mu}{\|\psi\|_{1,\lambda}} : \psi \text{ local} \right\}.$$

Again, after modding out by functions  $\|\phi\|_{-1,\lambda} = 0$ , define  $\mathcal{H}_{-1,\lambda}$  as the completion with respect to  $\|\cdot\|_{-1,\lambda}$ . When  $\lambda > 0$ ,  $(\lambda - L)^{-1}$  is a bounded operator, and

$$\|\phi\|_{-1,\lambda}^2 = \langle f, (\lambda - L)^{-1}f \rangle_\mu.$$

Both spaces  $\mathcal{H}_{1,\lambda}$ ,  $\mathcal{H}_{-1,\lambda}$  are Hilbert spaces where the innerproduct is given by polarization:

$$\langle \langle \phi, \psi \rangle \rangle = \frac{1}{4} \{ \|\phi + \psi\| - \|\phi - \psi\| \}.$$

These two norms are dual to each other: For local functions,

$$\langle \phi, \psi \rangle_\mu \leq \|\phi\|_{-1,\lambda} \|\psi\|_{1,\lambda}.$$

When  $\lambda = 0$ ,  $\|\cdot\|_1 := \|\cdot\|_{1,0}$  and  $\|\cdot\|_{-1} := \|\cdot\|_{-1,0}$  are called the  $\mathcal{H}_1$  and  $\mathcal{H}_{-1}$  norms respectively.

We note there is another formula for  $\|\phi\|_{-1,\lambda}$  which may be useful later:

$$\|\phi\|_{-1,\lambda}^2 = \sup \{2\langle \phi, \psi \rangle_\mu - \|\psi\|_{1,\lambda}^2 : \psi \text{ local}\}.$$

Sometimes the  $\mathcal{H}_{-1}$  norm is referred to as the ‘variance’ norm. Since  $\|\phi\|_{-1,\lambda}$  is increasing as  $\lambda \downarrow 0$ , and one can show that  $\lim_{\lambda \downarrow 0} \|\phi\|_{-1,\lambda} = \|\phi\|_{-1}$ , we have

$$\begin{aligned} 2\|\phi\|_{-1}^2 &= 2 \lim_{\lambda \downarrow 0} \|\phi\|_{-1,\lambda}^2 \\ &= 2 \lim_{\lambda \downarrow 0} \langle \phi, (\lambda - L)^{-1} \phi \rangle_\mu \\ &= 2 \lim_{\lambda \downarrow 0} \int_0^\infty e^{-\lambda t} E_\mu[\phi(P_t \phi)] dt \\ &= 2 \int_0^\infty E_\mu[\phi(\eta_t) \phi(\eta_0)] dt = \sigma_f^2. \end{aligned}$$

The last evaluation of the limit follows as  $E_\mu[\phi(P_t \phi)] = E_\mu[\phi(\eta_t) \phi(\eta_0)]$  is nonnegative since  $P_t$  is self-adjoint.

In particular, the condition  $\sigma_f^2 < \infty \Leftrightarrow \|f\|_{-1} < \infty$ .

### 3.2. Step 1: Proof of Proposition 3.2.

Write the resolvent equation

$$f = \lambda u_\lambda - L u_\lambda \quad (3.1)$$

where

$$u_\lambda(\eta) = \int_0^\infty e^{-\lambda t} P_t f(\eta) dt.$$

Let us multiply the resolvent equation by  $u_\lambda$  and take expectation:

$$\langle f, u_\lambda \rangle_\mu = \lambda \|u_\lambda\|_{L^2(\mu)} + D(u_\lambda).$$

Then,

$$\|f\|_{-1} \|u_\lambda\|_1 \geq \lambda \|u_\lambda\|_{L^2(\mu)} + \|u_\lambda\|_1^2$$

which gives immediately that

$$\|u_\lambda\|_1 \leq \|f\|_{-1} \text{ and } \lambda \|u_\lambda\|_{L^2(\mu)}^2 \leq \|f\|_{-1}^2$$

uniformly in  $\lambda > 0$ .

Also, since  $|\langle L\phi, \psi \rangle_\mu| \leq \|\phi\|_1 \|\psi\|_1$ , we have that  $L : \mathcal{H}_1 \rightarrow \mathcal{H}_{-1}$  is a bounded operator, with bound 1. Hence, for all  $\lambda > 0$ ,

$$\|Lu_\lambda\|_{-1} \leq \|u_\lambda\|_1 \leq \|f\|_{-1}.$$

Now, by the uniform boundedness principle, one can find a subsequence which converges weakly to an element  $w \in \mathcal{H}_1$ :

$$u_{\lambda_n} \rightarrow w.$$

For  $\phi$  local, which is dense in  $H_1$ , we have

$$\lambda \langle u_\lambda, \phi \rangle_\mu \leq \sqrt{\lambda} \|f\|_{-1} \|\phi\|_{L^2}$$

and since  $\|\lambda u_\lambda\|_{-1} \leq 2\|f\|_{-1}$ , we conclude that  $\lambda u_\lambda \rightarrow 0$  weakly in  $\mathcal{H}_{-1}$ . Therefore,

$$-Lu_{\lambda_n} \rightarrow f \text{ weakly in } \mathcal{H}_{-1}.$$

**3.3. Step2: Conclusion of proof of Proposition 3.2.** At this point, we would like to claim that

$$\langle f, u_\lambda \rangle \rightarrow \langle f, w \rangle, \quad \langle u_\lambda, -Lu_\lambda \rangle = \|u_\lambda\|_1^2 \rightarrow \|w\|_1^2, \quad \lambda \|u_\lambda\|_{L^2}^2 \rightarrow 0$$

so that  $\langle f, w \rangle = \|w\|_1^2$  and  $f$  satisfies a ‘weak’ Poisson equation. However, the weak convergence shown is not strong enough to conclude this relation immediately.

However, we will show that the convergences can be strengthened to show

**Proposition 3.3.** (i)  $\lim_{\lambda \downarrow 0} \lambda \|u_\lambda\|_{L^2(\mu)}^2 = 0$ .  
(ii) There is a  $w \in \mathcal{H}_1$  such that  $u_\lambda \rightarrow w$  strongly in  $\mathcal{H}_1$ . Also  $\|w\|_1^2 = \langle w, f \rangle_\mu$ .

Given, (i) and (ii) above, we finish the proof of the Proposition 3.2. Write

$$\begin{aligned} \frac{1}{\sqrt{t}} A_f(t) &= M_\lambda(t) + \frac{\lambda}{\sqrt{t}} \int_0^t u_\lambda(\eta_s) ds + u_\lambda(\eta_0) - u_\lambda(\eta_s) \\ &= M_\lambda(t) + \zeta_\lambda(t) \end{aligned}$$

where

$$M_\lambda(t) = u_\lambda(\eta_t) - u_\lambda(\eta_0) - \int_0^t Lu_\lambda(\eta_s) ds$$

is a martingale with quadratic variation

$$\langle M_\lambda \rangle(t) = \int_0^t [Lu_\lambda^2 - 2u_\lambda Lu_\lambda](\eta_s) ds$$

if  $u_\lambda$  is in the domain of  $L$ . Otherwise, one can show that still the expectation of  $\langle M_t \rangle = t \|u_\lambda\|_1^2$  and that it is increasing, and nonnegative.

Now, from this sort of calculation, and invariance of  $\mu$ , we have

$$E_\mu[M_\lambda(t) - M_\theta(t)]^2 = 2t \|u_\lambda - u_\theta\|_1^2 \rightarrow 0$$

as  $\lambda, \theta \downarrow 0$  from (ii). Hence,  $M_\lambda(t) \rightarrow M(t)$  in  $L^2$ . Moreover, by ergodicity and stationary increments,  $M(t)$  is a martingale such that

$$\frac{1}{t} \langle M \rangle(t) \rightarrow E_\mu[\langle M \rangle(1)] \text{ in } L^1.$$

We also evaluate

$$E_\mu[\langle M \rangle(1)] = \lim_{\lambda \downarrow 0} E_\mu[\langle M_\lambda \rangle(1)] = 2 \lim_{\lambda \downarrow 0} \|u_\lambda\|_1^2 = \sigma_f^2.$$

Now, the error  $\zeta_\lambda(t)$  is handled as follows: By the equation  $A_f(t) = M_\lambda(t) + \zeta_\lambda(t)$ , we see that  $\zeta_\lambda(t) \rightarrow \zeta(t)$  as  $\lambda \downarrow 0$  in  $L^2$ . Hence,

$$\zeta(t) = M_\lambda(t) - M(t) + \zeta_\lambda(t).$$

Now, by Schwarz inequality, using that  $\mu$  is invariant, we have by (i) that

$$\begin{aligned} t^{-1} \|\zeta_\lambda(t)\|_{L^2(\mu)}^2 &\leq 3[t^{-1} t^2 \lambda^2 \|u_\lambda\|_{L^2}^2 + 2t^{-1} \|u_\lambda\|_{L^2}^2] \\ &= \frac{C}{t} \|u_{t^{-1}}\|_{L^2}^2 \rightarrow 0 \end{aligned}$$

taking  $\lambda = t^{-1}$ , as  $t \uparrow \infty$ .

On the other hand,

$$\begin{aligned} t^{-1} \|M_{t^{-1}}(t) - M(t)\|_{L^2}^2 &= t^{-1} \lim_{\theta \downarrow 0} \|M_{t^{-1}}(t) - M_\theta(t)\|_{L^2}^2 \\ &= t^{-1} \cdot t \|u_{t^{-1}} - w\|_1^2 \rightarrow 0 \end{aligned}$$

as  $t \uparrow \infty$ .

This finishes the proof of Proposition 3.2.

**3.4. Step 3: Proof of Theorem 3.1.** We now have a representation

$$\frac{1}{\sqrt{t}} \int_0^t f(\eta_s) ds = \frac{1}{\sqrt{t}} M(t) + \frac{1}{\sqrt{t}} \zeta(t).$$

The error vanishes in  $L^2$ .

Hence, we need to identify the variance. By (ii),

$$2\|w\|_1^2 = 2\langle w, f \rangle_\mu = \lim_{\lambda \downarrow 0} 2\langle u_\lambda, f \rangle_\mu = 2\|f\|_{-1}^2 = \sigma_f^2.$$

The martingale central limit theorem, Proposition 1.1, now finishes the proof.

**3.5. Step 4: Proof of Proposition 3.3.** We now recall Mazur's theorem (see [6][Lemma 4.38]). A proof is given at the end.

**Lemma 3.4.** *In a Hilbert space, if  $x_n \rightarrow x$  weakly, there is a convex combination of the  $\{x_1, x_2, \dots, x_n\}$  which converges strongly to  $x$ , that is  $\|x_n - x\| \rightarrow 0$ .*

Now, let  $v_n$  be a convex combination of  $u_{\lambda_n}$  such that  $v_n \rightarrow w$  strongly in  $\mathcal{H}_1$ . Then, as  $-Lu_{\lambda_n} \rightarrow f$  weakly in  $\mathcal{H}_1$  and  $L$  is linear,  $-Lv_n \rightarrow f$  weakly in  $\mathcal{H}_1$ . In fact, since  $\|L(v_n - v_m)\|_{-1} \leq \|v_n - v_m\|$ , as  $L$  is self-adjoint, we see that  $\{-Lv_n\}$  is a Cauchy in  $H_{-1}$ , and hence converges strongly to  $f$ .

Then,

$$\|w\|_1^2 = \lim \|v_n\|_1^2 = \lim \langle v_n, -Lv_n \rangle = \langle w, f \rangle.$$

Now, returning to the subsequence  $u_{\lambda_n}$ , by lower semicontinuity and weak convergence, and (3.1), we have

$$\begin{aligned} \|w\|_1^2 &\leq \liminf \|u_{\lambda_n}\|_1^2 \\ &\leq \liminf \lambda_n \|u_{\lambda_n}\|_{L^2}^2 + \|u_{\lambda_n}\|_1^2 \\ &= \langle f, w \rangle = \|w\|_1^2. \end{aligned}$$

The same calculation can be repeated with 'lim inf' replaced by 'lim sup'.

Therefore, we conclude  $\|u_{\lambda_n}\|_1^2 \rightarrow \|w\|_1^2$ , which means  $u_{\lambda_n} \rightarrow w$  strongly in  $\mathcal{H}_1$ , and also that  $\lambda_n \|u_{\lambda_n}\|_{L^2}^2 \rightarrow 0$ .

By the same arguments, on any subsequence of  $u_\lambda$ , a further subsequence  $u_{\lambda'_n}$  can be found so that  $\lambda'_n \|u_{\lambda'_n}\|_{L^2}^2 \rightarrow 0$ . Hence,  $\lim_{\lambda \downarrow 0} \lambda \|u_\lambda\|_1^2 = 0$ , showing part (i).

To show part (ii), we need to show that the limit  $w$  is obtained on any subsequential limit of  $u_\lambda$  strongly in  $\mathcal{H}_1$ . To this end, as before, suppose  $u_{\lambda'_n}$  is a subsequence converging strongly to  $w'$  in  $\mathcal{H}_1$ . We conclude as before that  $-Lu_{\lambda'_n} \rightarrow f$  strongly in  $\mathcal{H}_{-1}$ . We now show that  $\|w - w'\|_1^2 = 0$  which will finish the claim.

Write, as  $u_{\lambda_n} - u_{\lambda'_n} = \lambda_n u_{\lambda_n} - \lambda'_n u_{\lambda'_n}$ , that

$$\begin{aligned} \|w - w'\|_1^2 &= \lim_n \|u_{\lambda_n} - u_{\lambda'_n}\|_1^2 \\ &= \langle u_{\lambda_n} - u_{\lambda'_n}, -Lu_{\lambda_n} + Lu_{\lambda'_n} \rangle \\ &= \langle u_{\lambda_n} - u_{\lambda'_n}, -\lambda_n u_{\lambda_n} + \lambda'_n u_{\lambda'_n} \rangle \end{aligned}$$

since  $-Lu_{\lambda_n} + Lu_{\lambda'_n} = -\lambda_n u_{\lambda_n} + \lambda'_n u_{\lambda'_n}$  noting (3.1). Now,  $\lambda'_n u_{\lambda'_n}, \lambda_n u_{\lambda_n} \rightarrow 0$  weakly in  $H_{-1}$  (cf. just before subsection 3.3). Hence, we may approximate  $u_{\lambda_n}$  by  $w$  and  $u_{\lambda'_n}$  by  $w'$  to see that the right-hand side vanishes. Hence,  $w = w'$  in  $H_1$ .

This finishes the proof of Proposition 3.3.  $\square$

**Proof of Mazur' Theorem.** Let us assume the limit is  $w = 0$  without loss of generality. From the weak convergence, one can find iteratively a subsequence  $n_k$  such that

$$|\langle\langle u_{n_k}, u_{n_j} \rangle\rangle| \leq k^{-1}, \quad \text{for } 1 \leq j \leq k.$$

Then,

$$\left\| \frac{1}{k} \sum_{j=1}^k u_{n_j} \right\|^2 \leq \frac{1}{k^2} \sum_{j=1}^n \|u_{n_j}\|^2 + \frac{2}{k^2} \sum_{1 \leq i < j \leq n} \frac{1}{j}$$

which vanishes as  $k \uparrow \infty$ . Hence, we can take  $v_n = k^{-1} \sum_{j=1}^n u_{n_j}$  for  $n_k \leq n < n_{k+1}$ .  $\square$

#### 4. NOTES

The Kipnis-Varadhan theorem is a basic theorem in modern probability theory, and aside from the original paper [4], treatments can be found in [5], [6].

There have been some generalizations of the Kipnis-Varadhan theorem to some nonreversible situations [10], [13]. In the nonreversible case, it may be that  $\sigma_f^2 < \infty$  but  $\|f\|_{-1} = \infty$  where the  $\mathcal{H}_{-1}$  norm is with respect to the symmetrized operator  $-S = -(L + L^*)/2$ . An open problem, which would have many applications, is to show in the general nonreversible case that if  $\|f\|_{-1} < \infty$  then a CLT holds.

The asymptotic variances in Proposition 2.2 and associated CLT's were first proved in [3]. For more general additive functionals, when  $f$  is not necessarily the occupation function of a site, CLT's and diffusive variance criteria are proved in [11] for reversible mass conservative systems. When the process is asymmetric, Proposition 2.3 is proved in a combination of papers [7], [1], [9], and associated CLT's are shown in [8].

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