

PART 4: ENTROPY, DIRICHLET FORM, AND AVERAGING PRINCIPLE IN ZERO-RANGE PROCESSES

We now consider another well-studied ‘mass conservative’ interacting particle system, the so called ‘zero-range’ model. Unlike for symmetric simple exclusion, the evolution equation for the empirical measure does not close, and one must invoke an ‘averaging principle’ to approximate a nonlinear term in terms of a function of the empirical measure. We introduce the notion of entropy and Dirichlet form in this context, and at the end give an outline to derive the associated hydrodynamical equation.

A later set of notes will be devoted to proving steps in the outline, the famous so called ‘1-block’ and ‘2-block’ lemmas introduced in [6].

1. RELATIVE ENTROPY

For probability measures μ and ν on a countable space Ω , define the relative entropy of μ with respect to ν as

$$H(\mu|\nu) = \sup_f \{E_\mu[f] - \log E_\nu[e^f]\}$$

where the supremum is over bounded functions $f : \Omega \rightarrow \mathbb{R}$.

Since the expression

$$E_\mu[(f + c)] - \log E_\nu[e^{(f+c)}] = E_\mu[f] - \log E_\nu[e^f] \quad (1.1)$$

for all constants c , the above supremum can be taken over bounded non-negative functions f .

Exercise 1.1. Show that $H(\mu|\nu)$ is non-negative, convex and lower semi-continuous in the argument μ . Hint: For non-negativity, substitute f equal to a constant.

Lemma 1.2 (Entropy inequality). *For all bounded, non-negative functions $f : E \rightarrow \mathbb{R}$ and $\alpha > 0$, we have*

$$E_\mu[f] \leq \frac{1}{\alpha} \{H(\mu|\nu) + \log E_\nu[e^{\alpha f}]\}.$$

Also, when $f = 1(A)$, for $A \subset E$, one has

$$\mu(A) \leq \frac{\log 2 + H(\mu|\nu)}{\log(1 + \frac{1}{\nu(A)})}.$$

Exercise 1.3. Prove this lemma.

Lemma 1.4. *We have $H(\mu|\nu) < \infty \Leftrightarrow \mu \ll \nu$. In the case $H(\mu|\nu) < \infty$, we have*

$$H(\mu|\nu) = \sum_{x \in \Omega} \mu(x) \log \frac{\mu(x)}{\nu(x)}.$$

Proof. If $\mu \ll \nu$, there is an $x_0 \in E$ such that $\mu(x_0) > 0$ but $\nu(x_0) = 0$. In the variational definition of $H(\mu|\nu)$, we can insert function $f(x) = 1(\{x_0\})$ to see $H(\mu|\nu) = \infty$. Hence, $H(\mu|\nu) < \infty \Rightarrow \mu \ll \nu$.

We now show the reverse implication and also prove the equality in the display. Simple computations show that we can approximate $H(\mu|\nu)$ by the supremum in (1.1) taken over functions which vanish except for a finite number of points in Ω .

If $\mu \ll \nu$, we now find the maximum of

$$\langle f(x)|x \in \Omega_k \rangle \in \mathbb{R}^{|\Omega_k|} \mapsto \sum_{x \in \Omega_k} \mu(x)f(x) - \log \left[\sum_{x \in \Omega_k} \nu(x)e^{f(x)} + (1 - \nu(\Omega_k)) \right].$$

Indeed, the functional is concave (check) and takes its maximum where its gradient vanishes. Computing the gradient, we obtain for $x_0 \in \Omega_k$ that

$$\mu(x_0) = \frac{\nu(x_0)e^{f(x_0)}}{\sum_{x \in \Omega_k} \nu(x)e^{f(x)} + (1 - \nu(\Omega_k))}.$$

We can add a constant to the values of f on Ω_k and not change the equation. Hence, we may optimize over f such that

$$\sum_{x \in \Omega_k} \nu(x)e^{f(x)} + (1 - \nu(\Omega_k)) = 1.$$

In this case, we obtain $f(x_0) = \log(\mu(x_0)/\nu(x_0)) + c_k$ where $e^{c_k}\mu(\Omega_k) + (1 - \nu(\Omega_k)) = 1$, which is a well-defined function. As $k \uparrow \infty$, c_k vanishes, and we have the upper bound $H(\mu|\nu) \leq \sum_x \mu(x) \log(\mu(x)/\nu(x))$. Hence, $\mu \ll \nu \Rightarrow H(\mu|\nu) < \infty$.

Given $\mu \ll \nu$, we obtain the lower bound by inserting $f = 1_{x \in \Omega_k} \log \mu(x)/\nu(x)$ where Ω_k is a finite subset of Ω . Optimizing over k , we obtain the inequality $H(\mu|\nu) \geq \sum_{x \in \Omega} \mu(x) \log(\mu(x)/\nu(x))$. \square

2. ENTROPY WITH RESPECT TO MARKOV CHAINS

Let η_t be a continuous time, irreducible Markov chain on a finite space Ω . Let π be the unique invariant measure. Recall P_t and L stand for the semigroup and generator of the process.

Lemma 2.1. *For probability measures μ such that $H(\mu|\pi) < \infty$, we have $\mu P_t \ll \pi$ and*

$$\frac{d}{dt} H(\mu P_t | \pi) = \left\langle \frac{d\mu P_t}{d\pi}, L \log \frac{d\mu P_t}{d\pi} \right\rangle_\pi.$$

Proof. To show that μP_t is absolutely continuous with respect to π , as $|\Omega| < \infty$, we have

$$\max_{x \in \Omega} \mu(x)/\pi(x) = C < \infty.$$

Now, by the invariance $\sum_x \pi(x)P_t(x, y) = \pi(y)$, we have

$$\mu P_t(y) = \sum_{x \in \Omega} \mu(x)P_t(x, y) \leq C \sum_{x \in \Omega} \pi(x)P_t(x, y) = C\pi(y)$$

which shows absolute continuity.

Recall from the backward and forward equations that $(\mu P_t(x))' = \mu P_t L(x) = \mu L P_t(x)$. Then,

$$\begin{aligned} \frac{d}{dt} \sum_x \mu P_t(x) \log \frac{\mu P_t(x)}{\pi(x)} &= \sum_x \mu P_t L(x) \log \frac{\mu P_t(x)}{\pi(x)} \\ &\quad + \sum_x \mu P_t(x) \frac{\pi(x)}{\mu P_t(x)} \mu L P_t(x). \end{aligned}$$

The first term on the right-side equals

$$\begin{aligned} \sum_x \sum_z \mu P_t(z) L(z, x) \log \frac{\mu P_t(x)}{\pi(x)} &= \sum_z \mu P_t(z) L\left(\log \frac{\mu P_t}{\pi}\right)(z) \\ &= \left\langle \frac{d\mu P_t}{d\pi}, L \log \frac{d\mu P_t}{d\pi} \right\rangle_{\pi} \end{aligned}$$

after interchanging the sum on z and x .

The second term on the other hand equals

$$\sum_x \pi(x) \sum_z \mu(z) L P_t(z, x) = \sum_z \mu(z) E_{\pi}[L g_z] = 0$$

where $g_z = P_t(z, \cdot)$ is a real function on Ω for each z , and so $E_{\pi}[L g_z] = 0$ since π is invariant. \square

3. DIRICHLET FORMS

For an irreducible, continuous time Markov process η_t on a finite state space Ω with invariant measure π , define the Dirichlet form on functions $f \in L^2(\pi)$:

$$D(f) = -\langle f, Lf \rangle_{\pi}.$$

We now define the adjoint $L^*(x, y)$ by the relation $\pi(x)L(x, y) = \pi(y)L^*(y, x)$ for $x, y \in \Omega$. Then, a simple computation shows

$$\langle g, Lf \rangle_{\pi} = \langle L^*g, f \rangle_{\pi}.$$

The operators $S = (L + L^*)/2$ and $A = (L - L^*)/2$ may also be defined. If π is reversible, $L = L^* = S$. In particular, S is a reversible operator, $\pi(x)S(x, y) = \pi(y)S(y, x)$, and A is anti-symmetric in that $\pi(x)A(x, y) = -\pi(y)A(y, x)$ for all $x, y \in \Omega$. Recall, also as we have seen before,

$$Lf(x) = \sum_y L(x, y)[f(y) - f(x)].$$

Then, by straightforward calculation, as also $D(f) = \langle L^*f, f \rangle_{\pi}$,

$$\begin{aligned} D(f) &= -\sum_{x,y} \pi(x)f(x)L(x, y)[f(y) - f(x)] \\ &= -\sum_{x,y} \pi(x)f(x)L^*(x, y)[f(y) - f(x)] \\ &= -\frac{1}{2} \sum_{x,y} \pi(x)f(x)S(x, y)[f(y) - f(x)]. \end{aligned}$$

Since S is reversible, by interchanging x and y in the above sum, we obtain the following formula.

Lemma 3.1. *We have*

$$\begin{aligned} D(f) &= \frac{1}{4} \sum_{x,y} \pi(x)S(x, y)[f(y) - f(x)]^2 \\ &= \frac{1}{2} \sum_{x,y} \pi(x)L(x, y)[f(y) - f(x)]^2. \end{aligned} \tag{3.1}$$

We have the following properties of the Dirichlet form. For a non-negative function, let $I(f) = D(\sqrt{f})$.

Lemma 3.2. *We have $f \mapsto I(f)$ is non-negative, convex, and continuous. When $I(f) = 0$, then f is a constant function.*

Proof. Non-negativity follows from the expression (3.1). After squaring out terms, convexity follows by Schwarz inequality applied to the crossterm: Things boil down to showing $\text{Av}\sqrt{f(x)f(y)} \leq \sqrt{\text{Av}f(x)}\sqrt{\text{Av}f(y)}$. Clearly, the Dirichlet form is continuous in its argument as the space is finite.

When $I(f) = 0$, we have $(\sqrt{f(y)} - \sqrt{f(x)})^2 = 0$ for all x, y where $S(x, y) > 0$. Since the chain is irreducible, all values of f must be the same. \square

3.1. Connection between entropy and Dirichlet form in Markov chains.

Recall in Lemma 2.1, that the derivative of $H(\mu P_t | \pi)$ equals

$$\left\langle \frac{d\mu P_t}{d\pi}, L \log \frac{d\mu P_t}{d\pi} \right\rangle_\pi.$$

We now bound this expression in terms of a Dirichlet form.

Lemma 3.3. *We have*

$$\left\langle \frac{d\mu P_t}{d\pi}, L \log \frac{d\mu P_t}{d\pi} \right\rangle_\pi \leq -2D\left(\sqrt{\frac{d\mu P_t}{d\pi}}\right).$$

Proof. The argument follows from an application of the inequality $a(\log b - \log a) \leq 2\sqrt{a}(\sqrt{b} - \sqrt{a})$ for $a, b \geq 0$ which we leave to the reader. \square

4. ZERO-RANGE MODELS

We now discuss another system of interacting random walks, the ‘zero-range’ process, on the d -dimensional torus \mathbb{T}_N^d with N^d locations. Informally, particles interact through their jump times. More specifically, at a vertex with k particles, a particle displaces by y with rate $[g(k)/k]p(y)$ where again p is a finite-range translation-invariant jump probability on \mathbb{T}_N^d and $g : \{0, 1, 2, \dots\} \rightarrow \mathbb{R}_+$ is a non-negative function such that $g(0) = 0$. An alternate description is that each vertex has its own exponential clock which rings at rate $g(k)$ when there are k particles at the vertex. When rung, one of the k particles is selected at random and then it displaces according to p .

Formally, consider $\eta_t = \{\eta_t(x) : x \in \mathbb{T}_N^d\}$ where $\eta_t(x) \in \{0, 1, 2, \dots\}$ denotes the number of particles at x at time $t \geq 0$. The configuration space is $\Omega = \{0, 1, 2, \dots\}^{T_N^d}$. The process is a continuous time Markov chain with generator

$$(Lf)(\eta) = \sum_{x,y} g(\eta(x))p(y)[f(\eta^{x,y}) - f(\eta)]$$

where $\eta^{x,y}$ is the configuration obtained from η by moving a particle from x to y :

$$\eta^{x,y}(z) = \begin{cases} \eta(x) - 1 & \text{when } z = x \\ \eta(y) + 1 & \text{when } z = y \\ \eta(z) & \text{otherwise.} \end{cases}$$

As with the exclusion process, there is an associated family of invariant measures. Consider $\bar{\nu}_\alpha$ to be the product measure with common marginal on $\{0, 1, 2, \dots\}$ given by

$$\kappa_\alpha(k) = \begin{cases} \frac{1}{Z(\alpha)} \frac{\alpha^k}{g(1)\cdots g(k)} & \text{when } k \geq 1 \\ \frac{1}{Z(\alpha)} & \text{when } k = 0. \end{cases}$$

Here, $Z(\alpha)$ is the normalization which converges for $0 \leq \alpha < \limsup g(k)$.

An interesting point is that when $g(k) \equiv k$, this is the model of ‘independent’ random walks considered in the beginning of the course when there is no interaction. In this case, of course κ are Poisson measures.

It will be helpful to index the family in terms of ‘density’. Let $\rho(\alpha)$ be the density, $E_{\bar{\nu}_\alpha}[\eta(0)] = \rho(\alpha)$. One can check that the function $\alpha \mapsto \rho(\alpha)$ is a strictly increasing function which may or may not diverge, depending on the structure of g , when $\alpha \uparrow \limsup g(k)$. In any case, for $0 \leq \alpha < \limsup g(k)$ we can invert $\rho(\alpha)$ and define

$$\nu_\alpha = \bar{\nu}_{\alpha(\rho)}$$

the product measure which places a mean ρ number of particles at each location.

In the following, with respect to a continuous non-negative function $\rho_0 : \mathbb{T}^d \rightarrow \mathbb{R}_+$, define $\nu_{\rho(\cdot)}$ to be the product measure with marginals $\kappa_{\alpha(\rho_0(x/N))}$ over sites $x \in \mathbb{T}_N^d$.

Denote, as before, for fixed $T < \infty$, that \mathbb{P}_κ as the distribution of $\{\eta_t : t \in [0, T]\}$ where η_0 is governed by κ .

Exercise 4.1. Let ρ be a fixed density. Show that $H(\nu_{\rho_0(\cdot)} | \nu_\rho) = O(N^d)$.

5. BASIC COUPLING

We discuss a useful coupling for zero-range processes when g is an increasing function. First, we define the notion of ‘stochastic domination’ on the partially ordered set Ω . We say two probability measures μ_1, μ_2 on a countable set Ω are ordered, $\mu_1 \ll \mu_2$ if $E_{\mu_1}[f] \leq E_{\mu_2}[f]$ for all coordinate-wise increasing functions $f : \Omega \rightarrow \mathbb{R}$.

There is an interesting characterization of ordered measures whose proof we refer to [8][Chapter 2].

Proposition 5.1. *We have $\mu_1 \ll \mu_2$ exactly when there is a bivariate distribution on $\Omega \times \Omega$ such that marginally variables η and ξ are distributed according to μ_1 and μ_2 and $P(\eta \leq \xi) = 1$.*

Lemma 5.2. *The marginal $\kappa_\alpha \ll \kappa_\beta$ exactly when $\alpha \leq \beta$.*

Proof. If $\kappa_\alpha \ll \kappa_\beta$, since $f(\eta) = \eta(0)$ is increasing, $\rho(\alpha) \leq \rho(\beta)$ from which one deduces $\alpha \leq \beta$.

For the converse, since linear combinations of $1([L, \infty))$ are dense in the set of increasing functions, it is enough to show that

$$P_{\kappa_\beta}(\eta \geq L) \geq P_{\kappa_\alpha}(\eta \geq L)$$

for all $L \geq 0$. This is equivalent to showing

$$\sum_{k \geq L} \frac{\beta^k}{g(1) \cdots g(k)} Z(\alpha) \geq \sum_{k \geq L} \frac{\alpha^k}{g(1) \cdots g(k)} Z(\beta)$$

which is equivalent to

$$\sum_{k \geq L} \sum_{\ell \leq L-1} \frac{\beta^k \alpha^\ell}{g(k)! g(\ell)!} \geq \sum_{k \geq L} \sum_{\ell \leq L-1} \frac{\alpha^k \beta^\ell}{g(k)! g(\ell)!}.$$

The last inequality will be shown if term by term the inequality is true. But, $\beta^k \alpha^\ell \geq \alpha^k \beta^\ell$ since $k \geq \ell$. \square

We now show there is a ‘basic coupling’ for the zero-range process when g is increasing, that is there is a coupling between two processes η_t and ξ_t starting from configurations $\mu_1 \ll \mu_2$ respectively such that at all later times $t \geq 0$, η_t and ξ_t marginally are the zero-range processes starting from μ_1 and μ_2 respectively and $\eta_t \leq \xi_t$ a.s.

Let $\bar{\mu}$ be the joint probability on $\Omega \times \Omega$ such that $\eta_0 \leq \xi_0$ a.s. (Proposition 5.1). Consider the Markov process on $\Omega \times \Omega$ with initial distribution $\bar{\mu}$ and generator

$$\begin{aligned} (\bar{L}_N)f(\eta, \xi) &= \sum_{x,y \in \mathbb{T}_N^d} p(y) \min\{g(\eta(x)), g(\xi(x))\} [f(\eta^{x,x+y}, \xi^{x,x+y}) - f] \\ &\quad + \sum_{x,y \in \mathbb{T}_N^d} p(y) (g(\eta(x)) - g(\xi(x)))_+ [f(\eta^{x,x+y}, \xi) - f] \\ &\quad + \sum_{x,y \in \mathbb{T}_N^d} p(y) (g(\xi(x)) - g(\eta(x)))_+ [f(\eta, \xi^{x,x+y}) - f]. \end{aligned}$$

One can see, by inserting a function of coordinate η only or of coordinate ξ only that the marginal processes are as desired.

To check that the set $\eta_t \leq \xi_t$ a.s., recall the joint process is a continuous time Markov chain on finite state space. At time $t = 0$, we have arranged that $\eta_0 \leq \xi_0$. At the next jump time τ , by the specification of the rates, we see that still $\eta_\tau \leq \xi_\tau$. Hence, the joint process remains ordered for all time $t \geq 0$.

Exercise 5.3. There is another way to show that $\eta_t \leq \xi_t$ for all $t \geq 0$ a.s. Compute that $\bar{L}_N 1(\eta \leq \xi) \geq 0$. Then, $\bar{E}_{\bar{\mu}}[1(\eta_t \leq \xi_t)]$ is increasing in t : $\partial \bar{E}_{\bar{\mu}}[1(\eta_t \leq \xi_t)] = \bar{E}_{\bar{\mu}}[\bar{L}_N 1(\eta_t \leq \xi_t)] \geq 0$. Hence,

$$1 \geq \bar{E}_{\bar{\mu}}[1(\eta_t \leq \xi_t)] \geq \bar{E}_{\bar{\mu}}[1(\eta_0 \leq \xi_0)] = 1.$$

This nice proof is similar to that of [7][Theorem II.5.2].

The next exercise will be useful in the later proof of hydrodynamics.

Exercise 5.4. Let $\bar{\rho} = \|\rho_0(\cdot)\|_{L^\infty}$. Show that $\nu_{\rho_0(\cdot)} \ll \nu_{\bar{\rho}}$, and consequently $\mathbb{P}_{\nu_{\rho_0(\cdot)}} \ll \mathbb{P}_{\nu_{\bar{\rho}}}$.

We will need an estimate on $\Psi(\rho)$ if g is a Lipschitz function, $|g(k) - g(l)| \leq C|k - l|$ for all $k, l \geq 0$.

Lemma 5.5. *If g is Lipschitz, then Ψ is also Lipschitz.*

Proof. Let $\delta \geq \beta$. Then, by the basic coupling,

$$\begin{aligned} \Psi(\delta) - \Psi(\beta) &= E_{\nu_\alpha}[g] - E_{\nu_\beta}[g] \\ &= \bar{E}[g(\eta(0)) - g(\xi(0))] \\ &\leq \bar{E}|\eta(0) - \xi(0)| \\ &= \bar{E}[\eta(0) - \xi(0)] = \delta - \beta. \end{aligned}$$

□

We remark one can also bound the derivative $\rho(\alpha)'$ to prove the lemma.

6. WHAT IS THE HYDRODYNAMICAL EQUATION?

Here, we informally compute the generator action to guess the hydrodynamic behavior. We will start the process from initial configurations distributed according to $\nu_{\rho_0(\cdot)}$. Recall π_t^N stands for the empirical measure

$$\pi_t^N = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t(x) \delta_{x/N}.$$

For a smooth $G : \mathbb{T}^d \rightarrow \mathbb{R}$, consider the equation

$$\langle G, \pi_{v(N)t}^N \rangle = \langle G, \pi_0^N \rangle + v(N) \int_0^t L \langle G, \pi_{v(N)s}^N \rangle ds + M_{v(N)t}^G$$

where $M_{v(N)t}^G$ is a martingale.

One may compute the quadratic variation,

$$\begin{aligned} \langle M_{v(N)t}^G \rangle &= v(N) \int_0^t L(\langle G, \pi_{v(N)s}^N \rangle)^2 - 2 \langle G, \pi_{v(N)t}^N \rangle L \langle G, \pi_{v(N)s}^N \rangle ds \\ &= \frac{v(N)}{N^{2d+2}} \sum_{x,y} (\nabla_{x,x+y}^N G)^2 g(\eta(x)) p(y) = O(v(N)N^{-d-2}), \end{aligned}$$

which shows, easily if g is bounded, that the martingale is negligible in the limit.

We now compute that

$$v(N)L \langle G, \pi_{v(N)t}^N \rangle = \frac{v(N)}{N^{d+1}} \sum_{x,y} g(\eta(x)) p(y) [N(G(x+y/N) - G(x/N))].$$

Here, again $v(N) = N^2$ when p is symmetric (or mean-zero), and $v(N) = N$ otherwise.

When p is symmetric, as with respect to symmetric exclusion, one can perform an additional summation-by-parts by combining contributions over bonds $(x, x+y)$ and $(x+y, x)$. The right-side equals

$$\frac{v(N)}{2N^{d+2}} \sum_{x,y} g(\eta(x)) p(y) [N^2(G(x+y/N) - 2G(x/N) + G(x-y/N))].$$

The trouble now is that this is a weighted average of functions $g(\eta(x))$ does not close in terms of the empirical measure. The main point of ‘hydrodynamics’ is to approximate this average by a function of the empirical measure.

What should it be? In the microscopic scale, near point $x \in \mathbb{T}_N^d$ there is lots of local particle movement. One expects the distribution of particles in an $N\epsilon$ neighborhood of x at time N^2t is like $\nu_{\eta^{N\epsilon}(x)}$ where

$$\eta_{N^2t}^\ell(x) = \frac{1}{(2N\epsilon+1)^d} \sum_{|y-x| \leq N\epsilon} \eta_{N^2t}(y)$$

and ϵ is a small parameter.

Then, one might expect that

$$\frac{v(N)}{2N^{d+2}} \sum_{x,y} g(\eta(x)) p(y) \Delta_{x,x+y}^N G \sim \frac{1}{N^d} \sum_{x,y} p(y) \Psi(\eta_{N^2t}^{N\epsilon}(x)) \Delta_{x,x+y}^N G$$

where

$$\Psi(\rho) = E_{\nu_\rho}[g(\eta(0))].$$

Now,

$$\eta_{N^2t}^{N\epsilon} = \langle (2\epsilon)^{-d} 1([x/N - \epsilon, x/N + \epsilon]^d), \pi_{N^2t}^N \rangle$$

which further is approximated if $\pi_{N^2t}^N \sim \pi(t, u)du$ by

$$\frac{1}{(2\epsilon)^d} \int 1_{[-\epsilon, \epsilon]^d}(u - x/N) \pi(t, u) du.$$

Putting this together, one ‘closes’ the equation and obtains

$$\int_{\mathbb{T}^d} G(u) \pi(t, u) du = \int_{\mathbb{T}^d} G(u) \rho_0(u) du + \frac{1}{2} \int_0^t \Delta G(u) \Psi(\pi(t, u)) du$$

which is the weak formulation of

$$\partial_t \rho(t, u) = \Delta_C \Psi(\rho(t, u)) \quad \text{and} \quad \rho(0, u) = \rho_0(u)$$

where Δ_C is as defined before,

$$\Delta_C = \sum_{i \leq j \leq d} C_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$$

and $C_{i,j} = \sum_z z_i z_j p(z)$.

7. ASSUMPTIONS AND STATEMENT OF RESULT

We will make assumptions which will help simplify the proof. See the notes for details on extensions.

We will assume that g is a Lipschitz function, which is bounded and increasing:

- $|g(k+1) - g(k)| \leq a_0$ for $k \geq 0$
- $\sup_{k \geq 0} g(k) \leq a_1$,
- $g(k+1) \geq g(k)$ for $k \geq 0$.

Also, we assume that p is nearest-neighbor and symmetric:

- $p(e) = (2d)^{-1}$ for coordinate unit vectors e .

Theorem 7.1. *We have for all $G \in C^2(\mathbb{T}^d)$ and $\delta > 0$ that*

$$\lim_{N \uparrow \infty} \mathbb{P}_{\nu_{\rho_0}(\cdot)} \left[\left| \langle G, \pi_{N^2t}^N \rangle - \int_{\mathbb{T}^d} G(u) \rho(t, u) du \right| > \delta \right] = 0$$

where ρ is the unique weak solution of the hydrodynamic equation

$$\partial_t \rho = \frac{1}{2} \Delta \rho \quad \text{and} \quad \rho(0, u) = \rho_0(u).$$

7.1. Strategy. We now separate the proof into two rough steps.

Step 1. We will again consider the measures Q^N which govern the trajectories of empirical distributions $\pi_{N^2t}^N$ on $D([0, T]; M_+(\mathbb{T}^d))$. The first task is to show that $\{Q^N\}$ are tight, and all limit points are concentrated on trajectories of measures with densities $\pi(t, u)du$.

Step 2. We show that all limit points are supported on weak solutions to the hydrodynamical equation in $L^2([0, T] \times \mathbb{T}^d)$. It is a result in PDE that such weak solutions are unique. Hence, the measures Q^N converge to a single limit point. One concludes convergence at a fixed time, as for simple exclusion processes.

The development on entropy, Dirichlet forms, and coupling will be useful in proving Step 2 in the next set of notes.

8. NOTES

Much of the development of entropy and its use in various applications, including hydrodynamics, is due to Varadhan. We have followed the scheme in [7][Appendix 1].

On the other hand the basic coupling, and its use in particle systems analysis was much developed by Liggett. Section 5 owes much to [8][Chapter 2].

In terms of hydrodynamics, one may weaken the assumptions on g . In particular, the increasing or boundedness assumptions are not needed when p is mean-zero. There are roughly four different proofs of the hydrodynamic behavior. One way, due to Guo-Papanicolaou-Varadhan [6] is to be outlined in the next set of notes. Another way, H.T. Yau's method of relative entropy, assumes that the weak solution $\rho = \rho(t, u)$ exists and is continuous, and then shows that the relative entropy of $\nu_{\rho_0(\cdot)}$ with respect to $\nu_{\rho(\cdot)}$ vanishes. One may use a Hopf-Lax formula, and the basic coupling, to prove in $d = 1$ the hydrodynamical limit for systems with drift [11], [1], [2]. Also, a method using a logarithmic Sobolev inequality and compensated compactness ideas can be employed in $d = 1$ [4], [5]. The first two methods are discussed in [7]. See also [3] for a nice treatment of the GPV method.

At this point, we comment on why the scaling limit is called a ‘hydrodynamic limit’. The origins go back to the study of N particles with certain positions $\{q^\ell = \langle q_i^\ell : i = 1, 2, 3 \rangle\}_{\ell=1}^N$ and momenta $\{p^\ell = \langle p_i^\ell : i = 1, 2, 3 \rangle\}_{\ell=1}^N$ moving on a torus \mathbb{T}^3 according to Newtonian dynamics:

$$\frac{dq_i^\ell}{dt} = \frac{\partial H}{\partial p_i^\ell}, \quad \text{and} \quad \frac{dp_i^\ell}{dt} = -\frac{\partial H}{\partial q_i^\ell}$$

where the Hamiltonian is

$$H = H_N = \frac{1}{2} \sum_{\ell} \sum_i |p_i^\ell|^2 + \frac{1}{2} \sum_{\ell, k} V(N(q^\ell - q^k)).$$

In the system, mass, the three components of momentum, and energy are conserved. One can form the empirical measures corresponding to these quantities:

$$\begin{aligned} \xi_t^0 &= \frac{1}{N^3} \sum_{\ell} \delta(q^\ell(t)) \\ \xi_t^i &= \frac{1}{N^3} \sum_{\ell} \delta(q^\ell(t)) p_i^\ell(t) \quad \text{for } i = 1, 2, 3 \\ \xi_t^4 &= \frac{1}{N^3} \sum_{\ell} \delta(q^\ell(t)) h^\ell(t) \end{aligned}$$

where energy of the ℓ th particle is

$$h^\ell(t) = \frac{1}{2} \sum_i |p_i^\ell(t)|^2 + \frac{1}{2} \sum_k V(N(q^\ell(t) - q^k(t))).$$

On the torus, one can define a 5 parameter family of Gibbs measures, corresponding to density, velocity, and energy levels, which are invariant for the dynamics. One can take infinite volume limits under some conditions, e.g. Dobrushin-Lanford-Ruelle conditions, and try to define local equilibrium measures. The goal is to show limits $\xi_t^j \rightarrow x_i(u, t) du$ for $j = 0, 2, 3, 4$ where $\langle x_i : i = 0, 1, 2, 3, 4 \rangle$ satisfies

an equation

$$\frac{\partial x}{\partial t} + \nabla_u F(x) = 0.$$

Here F is a 5×3 matrix, and the equation is Euler's equation. Such a limit is open! In computing $d/dt \xi^i$, one has to 'close' the expressions in terms of functions of the empirical measures. Formally, one can do this, as above for zero-range processes, and derive the form of F .

Perhaps, the best result is in [9] where a small amount of noise is added to the dynamics to make it into a reasonable Markov process where local averaging can be done, and a rigorous limit can be proved. See also [12][Part 1], [10] for a comprehensive discussions.

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