# UNIT-II : RANDOM VARIABLES

**Random Variable (Def):** A real valued function $X$, defined on a sample space "S" of a random experiment (i.e) $X : S \to \mathbb{R}$ is called a "Random Variable".

* Here the set of all possible values of a random variable is called as "Range of X".

**Ex:** In the experiment of tossing of two coins,

the sample space is : $S = \{HH, HT, TH, TT\}$

$\therefore$ We can assign uniform probability "$1/4$" to each element of 'S'.

Define a Random Variable $X : S \to \mathbb{R}$ by

$$X = \text{Number of Heads.}$$

$$\therefore \quad X(HH) = 2, \quad X(HT) = X(TH) = 1, \quad \& \quad X(TT) = 0.$$

Here $X : \{HH, HT, TH, TT\} \longrightarrow \{0, 1, 2\}$

(i.e) Range of $X = \{0, 1, 2\}$.

Also $P(X=0)$ = probability of no heads = $1/4$

$P(X=1)$ = probability of one head

$= P(HT \text{ or } TH)$

$= 1/4 + 1/4 = 1/2$

$P(X=2)$ = probability of getting two heads

$= 1/4$.

* Random Variables are usually classified into two types.

(i) Discrete Random Variable

(ii) Continuous Random Variable.

## Discrete Random Variable (Def):

A discrete random variable is a random variable with finite range.

Ex: $X$ = Sum of the spots on two dice is discrete.

Since $X$ can assume only the values $2,3,4,5,6,7,8,9,10,11 \& 12$.

Here the number of possibilities is 11 (finite).

## Continuous Random Variable (Def):

If the range of the random variable '$X$' is an interval of real numbers then $X$ is a continuous random variable.

Ex: $X$ = Length of time that it takes for a computer program to run is continuous

Let us assume that it is reasonable to expect that the value of $X$ is lessthan '4' minutes then

$\therefore X = (0\ 4)$.

## Probability distribution function:

The probability distribution function $f(x)$ of a random-variable '$X$' is a discription of the set of all possible values of '$X$' along with the probabilities associated with each possible values '$x$' of '$X$'.

* Here we can classify probability distribution function into two types.

(i) Probability Mass function (Discrete probability distribution)

(ii) Probability density function (Continuous probability distribution).

Probability Mass function: If 'X' is a discrete random Variable then it's probability distribution function "f(x)" is said to be probability mass function if it satisfies the following conditions.

(i) $f(x) \geq 0 \quad \forall x$

(ii) $\sum_x f(x) = 1$

(iii) $P(X=x) = f(x)$.

Probability Density function: If 'X' is a continuous random variable then it's Probability distribution function 'f(x)' is said to be probability density function if it satisfies the following conditions.

(i) $f(x) \geq 0 \quad \forall x$

(ii) $\int_{-\infty}^{\infty} f(x) = 1$

(iii) $P(a < X < b) = \int_a^b f(x)\,dx$ (Area under f(x) b/w the ordinates $x=a$ & $x=b$).

**NOTE**: If 'X' is a continuous Random Variable then

$$P(a < X \leq b) = P(a \leq X < b) = P(a \leq X \leq b) = P(a < X < b)$$

(i.e) Inclusion (or) Non-inclusion of end points doesnot change the probability.

Def (Cummulative Distribution function): The cummulative distribution function of a random variable 'X' is defined as,

$F(x) = P(X \leq x)$, where '$x$' is any real number.

* The cummulative distribution function of a Discrete random Variable 'X' is defined by $F(x) = P(X \leq x) = \sum_{x_i \leq x}' f(x_i)$

Where '$f(x)$' is probability mass function of 'X'.

And it follows that $F(-\infty) = 0$ & $F(\infty) = 1$.

* The cummulative distribution function of a Continuous random Variable 'X' is defined by

$$F(x) = \int_{-\infty}^{x} f(x)\,dx,$$ where '$f(x)$' is probability density function of X.

And it follows that $F(-\infty) = 0$ & $F(\infty) = 1$

Also Here $P(a < X < b) = F(b) - F(a)$.

## MATHEMATICAL EXPECTATION :-

If 'X' is a random variable then Expectation of "$X^r$" is defined as,

$$E(X^r) = \begin{cases} \sum_i' x_i^r f(x_i), & \text{if } X \text{ is Discrete} \\ \int_{-\infty}^{\infty} x^r f(x)\,dx, & \text{if } X \text{ is Continuous} \end{cases}$$

**Mean** : If X is a random Variable then 'expectation of X' or 'Mean' is given by,

$$\mu = E(X) = \begin{cases} \sum_i' x_i f(x_i), & \text{if } X \text{ is Discrete} \\ \int_{-\infty}^{\infty} x f(x)\,dx, & \text{if } X \text{ is Continuous} \end{cases}$$

**Variance:** The Variance of 'X' is given by

$$V(X) = \sigma^2 = E[(X-\mu)^2]$$

$$= \begin{cases} \sum_i (x_i - \mu) f(x_i), & \text{if } X \text{ is Discrete} \\ \int_{-\infty}^{\infty} (x-\mu) f(x)\, dx, & \text{if } X \text{ is Continuous} \end{cases}$$

* Here $\mu$ & $\sigma^2$ are the notations of Mean & Variance respectively.

**Standard Deviation:** It is denoted by '$\sigma$' and it is the Positive square root of Variance.

→ **Note:** Variance $\sigma^2 = E(X^2) - [E(X)]^2$

We know that $\sigma^2 = E[(X-\mu)^2]$

$$= E(X^2 + \mu^2 - 2\mu X)$$

$$= E(X^2) + E(\mu^2) - E(2\mu X).$$

$$= E(X^2) + \mu^2 - 2\mu\, E(X)$$

$$= E(X^2) + \mu^2 - 2\mu^2 \quad (\because \mu = E(X))$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - [E(X)]^2$$

**Result:** If X is a Random variable then Prove that

(i) $E(aX+b) = a\,E(X) + b$

(ii) $V(aX+b) = a^2\, V(X)$.

Proof: case(i): If 'X' is Discrete random Variable.

Consider $E(aX+b) = \sum_i' (ax_i+b) f(x_i)$, where $f(x)$ is p.m.f of X.

$$= \sum_i' a x_i f(x_i) + \sum_i' b f(x_i)$$

$$= a \sum_i' x_i f(x_i) + b \sum_i' f(x_i)$$

$$= a.E(x) + b.1 \qquad \left(\because \sum_i' x_i f(x_i) = E(x)\right.$$

$$= a E(x) + b \qquad \left.\sum_i' f(x_i) = 1\right)$$

And, $V(aX+b) = E[(aX+b)^2] - [E(aX+b)]^2$

$$= \sum_i' (ax_i+b)^2 f(x_i) - \left(\sum_i' (ax_i+b) f(x_i)\right)^2$$

$$= \sum_i' a^2 x_i^2 f(x_i) + \sum_i' b^2 f(x_i) + \sum_i' 2ab x_i f(x_i)$$

$$- \left[\sum_i' a x_i f(x_i) + \sum_i' b f(x_i)\right]^2$$

$$= a^2 \sum_i' x_i^2 f(x_i) + b^2 \sum_i' f(x_i) + 2ab \sum_i' x_i f(x_i)$$

$$- a^2\left(\sum_i' x_i f(x_i)\right)^2 - b^2\left(\sum_i' f(x_i)\right)^2$$

$$- 2ab \sum_i' x_i f(x_i) \sum_i' f(x_i)$$

$\left(\because \sum_i' x_i^2 f(x_i) = E(x^2)\right.$

$\sum_i' x_i f(x_i) = E(x)$

$\left.\sum_i' f(x_i) = 1\right)$

$$= a^2 E(x^2) + \cancel{b^2} + \cancel{2ab E(x)} - a^2 (E(x))^2$$

$$- \cancel{b^2} - \cancel{2ab E(x)}(1)$$

$$= a^2 \left\{E(x^2) - [E(x)]^2\right\}$$

$$= a^2 V(x).$$

case (ii): If 'X' is Continuous random Variable.

Consider $E(ax+b) = \int_{-\infty}^{\infty} (ax+b) f(x) dx$

where $f(x)$ is p.d.f of x.

$$= \int_{-\infty}^{\infty} ax\, f(x)\,dx + \int_{-\infty}^{\infty} b\, f(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x\, f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= a\,E(X) + b\cdot 1 \qquad \left(\because \int_{-\infty}^{\infty} x\,f(x)\,dx = E(X)\right.$$

$$\therefore\ E(aX+b) = a\,E(X) + b \qquad \left.\int_{-\infty}^{\infty} f(x)\,dx = 1\right)$$

Also $\quad V(aX+b) = E[(aX+b)^2] - [E(aX+b)]^2$

$$= \int_{-\infty}^{\infty} (ax+b)^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} (ax+b)\, f(x)\,dx\right]^2$$

$$= a^2\int_{-\infty}^{\infty} x^2 f(x)\,dx + b^2\int_{-\infty}^{\infty} f(x)\,dx + 2ab\int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$- a^2\left(\int_{-\infty}^{\infty} x\, f(x)\,dx\right)^2 - b^2\int_{-\infty}^{\infty} f(x)\,dx$$

$$\left(\because \int_{-\infty}^{\infty} x^2 f(x)\,dx = E(X^2)\right. \qquad - 2ab\left(\int_{-\infty}^{\infty} x\,f(x)\,dx\right)\left(\int_{-\infty}^{\infty} f(x)\,dx\right)$$

$$\int_{-\infty}^{\infty} x\,f(x)\,dx = E(X)$$

$$\left.\int_{-\infty}^{\infty} f(x)\,dx = 1\right) \qquad = a^2 E(X^2) + \cancel{b^2} + 2ab\,\cancel{E}(X) - a^2(E(X))^2 - \cancel{b^2}(1) - 2ab\,E(\cancel{X})\,(1)$$

$$= a^2\{E(X^2) - (E(X))^2\}$$

$$= a^2\, V(X).$$

Hence proved.

## Problems:

1. Two dice are thrown, Let 'X' assign to each point $(a,b)$ in 'S' the Maximum of its numbers (i.e) $X[(a,b)] = \text{Max}\{a,b\}$, Find the probability distribution.

**Sol:** When two dice are thrown, the Sample space,

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix} \right\}$$

$\therefore n(S) = 36$

Here $X[(a,b)] = \text{Max}\{a,b\}$ for all $(a,b) \in S$.

$\therefore X(1,1) = 1, \quad X(1,2) = 2 \quad X(1,3) = 3 \quad X(1,4) = 4 \quad X(1,5) = 5 \quad X(1,6) = 6$

$\ldots\ldots\ldots\ldots\ldots\ldots X(6,5) = 6, \quad X(6,6) = 6.$

Hence $X = \{1, 2, 3, 4, 5, 6\}$

Also $P(X=1) = P[(1,1)] = 1/36$

$P(X=2) = P[(1,2)\ (2,1)\ (2,2)] = 3/36 = 1/12$

$P(X=3) = P[(1,3)\ (3,1)\ (2,3)\ (3,2)\ (3,3)] = 5/36.$

$P(X=4) = P[(1,4)\ (4,1)\ (2,4)\ (4,2)\ (3,4)\ (4,3)\ (4,4)]$

$= 7/36$

lly $P(X=5) = 9/36$

$P(X=6) = 11/36.$

Required probability Distribution is :

| $X = x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(X=x)$ | $1/36$ | $3/36$ | $5/36$ | $7/36$ | $9/36$ | $11/36$ |

(2) For the above problem find Mean, Variance and Standard deviation:

Sol: Mean $= \mu = \sum_i' x_i f(x_i)$

$= 1\times 1/36 + 2\times 3/36 + 3\times 5/36 + 4\times 7/36 + 5\times 9/36 + 6\times 11/36$

$= \frac{161}{36} = 4.472.$

Variance $= \sigma^2 = \sum_i' (x_i - \mu)^2 f(x_i).$

$= (1-4.472)^2 \times \frac{1}{36} + (2-4.472)^2 \times \frac{3}{36} + \cdots + (6-4.472)^2 \times \frac{11}{36}$

$= 1.92$

S.D $= \sigma = \sqrt{1.92} = 1.38$

(1) A random variable 'X' has the following probability function.

| X=x : | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| P(X=x) : | 0 | K | 2K | 2K | 3K | $K^2$ | $2K^2$ | $7K^2+K$ |

Then find (i) The value of 'K' (ii) Mean & Variance

(iii) $P(X<6)$ (iv) $P(X\geq 6)$ (v) $P(0<X<5)$

(vi) If $P(X\leq K) > 1/2$, find the Minimum value of K.

Sol: (i) Since $\sum_{x=0}^{7}{}' P(X=x) = 1$

$\Rightarrow \quad 0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1$

$$\Rightarrow \quad 10K^2 + 9K^2 - 1 = 0$$

$$\Rightarrow \quad (10K-1)(K+1) = 0$$

$$\therefore \quad K = 1/10 = \underline{\underline{0.1}} \quad (\text{since } p(X=x) \geq 0,\ K \neq -1)$$

(ii) $$\text{Mean} = \mu = \sum_{x=0}^{7} x\, p(X=x)$$

$$= 0(0) + 1(K) + 2(2K) + 3(2K) + 4(3K) + 5(K^2) + 6(2K^2) + 7(7K^2+K)$$

$$= 66K^2 + 30K$$

$$= 66(0.1)^2 + 30(0.1) \quad (\because K = 0.1)$$

$$= \underline{\underline{3.66}}$$

(iii) $$\text{Variance} = \sum_{x=0}^{7} x^2 p(X=x) - (\mu)^2$$

$$= 1^2(K) + 4(2K) + 9(2K) + 16(3K) + 25(K^2) + 36(2K^2) + 49(7K^2+K) - (3.66)^2.$$

$$= 440K^2 + 124K - (3.66)^2$$

$$= \frac{440}{100} + \frac{124}{10} - (3.66)^2$$

$$= 4.4 + 12.4 - 13.3956$$

$$= 3.4044.$$

(iii) $$P(X<6) = \sum_{x=0}^{5} P(X=x)$$

$$= 0 + K + 2K + 2K + 3K + K^2$$

$$= 8K + K^2 = 0.8 + 0.01 = \underline{\underline{0.81}}$$

(iv) $P(X \geq 6) = 1 - P(X<6)$

$= 1 - 0.81 = \underline{\underline{0.19}}$

(v) $P(0 < X < 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= K + 2K + 2K + 3K$

$= 8K$

$= \underline{\underline{0.8}}$ $(\because K = 0.1)$

(vi) Required minimum value of 'K' is obtained as below

If $K = 1 \Rightarrow P(X \leq 1) = P(X=0) + P(X=1) = K = 0.1 \ngtr 1/2$

If $K = 2 \Rightarrow P(X \leq 2) = P(X \leq 1) + P(X=2) = K + 2K$
$= 3K = 0.3 \ngtr 1/2$

If $K = 3 \Rightarrow P(X \leq 3) = 5K = 0.5 \ngtr 1/2$

If $K = 4 \Rightarrow P(X \leq 4) = 8K = 0.8 > 0.5$ ✓

$\therefore$ Required $\underline{\underline{K = 4}}$.

(3) Find the mean and variance of the uniform probability distribution given by

$$f(x) = \frac{1}{n} \quad \text{for } x = 1, 2, 3, \ldots\ldots n$$

Sol: The Given Probability distribution is:

| $X = x$ | 1 | 2 | . . . . . | $n$ |
|---|---|---|---|---|
| $f(x) = P(X=x)$ | $1/n$ | $1/n$ | . . . . . | $1/n$ |

(i) $\text{Mean} = \sum_{x=0}^{n} x\, f(x) = 1/n + 2/n + \cdots\cdots + \frac{n}{n}$

$$= \frac{1}{n}\left(1+2+3+\cdots+n\right)$$

$$= \frac{1}{n}\cdot\frac{n(n+1)}{2}$$

$$= (n+1)/2$$

(ii) Variance $= \sum x^2 f(x) - (\text{Mean})^2$

$$= \left[1^2\cdot 1/n + 2^2\cdot 1/n + \cdots + n^2\cdot\frac{1}{n}\right] - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}.$$

$$= \underline{\underline{\frac{n^2-1}{12}}}.$$

④ If the probability density of a random variable is given by

$$f(x) = \begin{cases} K(1-x^2), & \text{for } 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

then find the values of

(i) K (ii) $P(0.1 < X < 0.2)$ (iii) $P(X > 0.5)$ (iv) Mean & Variance.

Sol: (i) We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} K(1-x^2)\,dx = 1$$

$$\Rightarrow K\left[x - x^3/3\right]_0^1 = 1$$

$$\Rightarrow K\left\{(1-1/3)-0\right\} = 1$$

$$\Rightarrow \underline{\underline{K = 3/2}}.$$

(ii) $P(0.1 < X < 0.2) = \int_{0.1}^{0.2} f(x)\,dx$

$$= \int_{0.1}^{0.2} K(1-x^2)\,dx$$

$$= 3/2 \left(x - x^3/3\right)_{0.1}^{0.2}$$

$$= 3/2 \left\{\left(0.2 - \frac{0.008}{3}\right) - \left(0.1 - \frac{0.001}{3}\right)\right\}$$

$$= 0.2965.$$

(iii) $P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx = \int_{0.5}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$

$$= \int_{0.5}^{1} K(1-x^2)\,dx + \int_{1}^{\infty} 0\,dx \quad (\text{elsewhere})$$

$$= 3/2 \int_{0.5}^{1} (1-x^2)\,dx \quad (\because K = 3/2)$$

$$= 3/2 \left(x - x^3/3\right)_{0.5}^{1}$$

$$= 3/2 \left\{(1 - 1/3) - \left((0.5) - \frac{(0.5)^3}{3}\right)\right\}$$

$$= 3/2 \left\{2/3 - 0.4583\right\}$$

$$= 0.3125.$$

(iv) Mean $= E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{0}^{1} x\, K(1-x^2)\,dx \quad (\because f(x) = 0 \text{ otherwise in } 0<x<1)$$

$$= 3/2 \int_{0}^{1} (x - x^3)\,dx$$

$$= 3/2 \left(\frac{x^2}{2} - \frac{x^4}{4}\right)_{0}^{1}$$

$$= 3/2 \left\{(1/2 - 1/4) - (0-0)\right\} = \underline{\underline{3/8}}.$$

$$\text{Variance} = V(x) = E(x^2) - [E(x)]^2.$$

$$\text{First} \quad E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \frac{3}{2}\int_0^1 x^2(1-x^2)\,dx \quad (\because K = 3/2)$$

$$= 3/2 \int_0^1 (x^2 - x^4)\,dx$$

$$= 3/2 \left(x^3/3 - x^5/5\right)_0^1$$

$$= 3/2 \left\{ (1/3 - 1/5) - 0 \right\}$$

$$= 3/2\,(2/15) = \underline{3/15} = 1/5$$

$$\therefore \quad V(x) = 1/5 - (3/8)^2$$

$$= 1/5 - 9/64 = 19/320 = 0.0593.$$

$$\therefore \quad \text{Standard Deviation} = \sqrt{V(x)}$$

$$= \sqrt{0.0593} = 0.243$$

(5) If a random Variable has the p.d.f $f(x) = \begin{cases} 2e^{-2x}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases}$

then find the probabilities that will take on a value

(i) b/w 1 and 3 (ii) greater than 0.5.

<u>sol:</u> (i) $P(1 \le X \le 3) = \int_1^3 f(x)\,dx$

$$= \int_1^3 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_1^3$$

$$= 2\left[\frac{e^{-6}}{-2} + \frac{e^{-2}}{2}\right) = -\left(e^{-6} - e^{-2}\right)$$

$$= e^{-2} - e^{-6}.$$

(ii) $P(X \geq 0.5) = \int_{0.5}^{\infty} f(x)\,dx$

$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_{0.5}^{\infty}$$

$$= -\left(e^{-\infty} - e^{-1}\right) = \underline{\underline{e^{-1}}}.$$

⑥ The probability density '$f(x)$' of a continuous R.V is given by $f(x) = Ke^{-|x|}$, $-\infty < x < \infty$ then find

(i) K (ii) Mean & Variance (iii) $P(0 < X < 4)$.

<u>sol:</u> Given $f(x) = Ke^{-|x|}$, $-\infty < x < \infty$

(i) We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ ($\because$ Total probability is one)

$$\Rightarrow \int_{-\infty}^{\infty} Ke^{-|x|}\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} Ke^{-|x|}\,dx + \int_{0}^{\infty} Ke^{-|x|}\,dx = 1$$

( Since,
$|x| = -x$ if $x < 0$
$= x$ if $x > 0$ )

$$\Rightarrow K\left[\int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx\right] = 1$$

$$\Rightarrow K\left\{\left(e^{x}\right)_{-\infty}^{0} + \left(-e^{-x}\right)_{0}^{\infty}\right\} = 1$$

$$\Rightarrow K.\left\{\left(e^{0} - e^{-\infty}\right) + \left(-e^{-\infty} + e^{0}\right)\right\} = 1$$

$$\Rightarrow K(1+1) = 1 \quad (\because e^{-\infty} = 0)$$

$$\Rightarrow K = 1/2$$

$\therefore \quad f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$

(ii) Mean $= \int_{-\infty}^{\infty} x\, f(x)\, dx$

$= \frac{1}{2}\int_{-\infty}^{\infty} x\, e^{-|x|}\, dx = \frac{1}{2}\left\{ \int_{-\infty}^{0} x\, e^{-|x|}\, dx + \int_{0}^{\infty} x\, e^{-|x|}\, dx \right\}$

$= \frac{1}{2}\left\{ \int_{-\infty}^{0} x\, e^{x}\, dx + \int_{0}^{\infty} x\, e^{-x}\, dx \right\}$

$= \frac{1}{2}\left\{ \left(x e^{x} - e^{x}\right)_{-\infty}^{0} + \left(x e^{-x} + e^{x}\right)_{0}^{\infty} \right\}$

$= \frac{1}{2}\{ 0 \} = \underline{\underline{0}}.$

$\therefore \quad$ Mean $= E(X) = 0.$

Variance $= E(X^2) - [E(X)]^2.$

First $\quad E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 \cdot e^{-|x|}\, dx$

$= \frac{1}{2}\cdot 2\int_{0}^{\infty} x^2 e^{-|x|}\, dx \quad$ ($\because$ Integrand is Even)

$= \int_{0}^{\infty} x^2 e^{-x}\, dx$

$= \left[ x^2 \cdot \frac{e^{-x}}{-1} - 2x \frac{e^{-x}}{1} + 2\frac{e^{-x}}{-1} \right]_{0}^{\infty}$

$= [0 - (-2)]$

$= \underline{\underline{2}}.$

$\therefore \quad V(X) = 2 - (0)^2$

$= \underline{\underline{2}}$ ✓

(iii) $P(0 \le X \le 4) = \frac{1}{2}\int_0^4 e^{-|x|}\,dx$

$= \frac{1}{2}\int_0^4 e^{-x}\,dx$ $\quad (\because$ In $0 < x < 4$ $|x| = x)$

$= \frac{1}{2}\left(-e^{-x}\right)_0^4$

$= \frac{1}{2}\left[e^{0} - e^{-4}\right]$

$=$ 0.4908 (approximately).

**Note:** We know that if 'X' is a continuous random variable and $f(x)$ is it's density function then the cummulative distribution function is given by.

$$F(x) = \int_{-\infty}^{x} f(x)\,dx.$$

$$\therefore \boxed{f(x) = \frac{d}{dx}[F(x)]}$$

(7) If $F(x)$ is the distribution function of X given by

$$F(x) = \begin{cases} 0 & , \text{ if } x \le 1 \\ K(x-1)^4 & , \text{ if } 1 < x \le 3 \\ 1 & , \text{ if } x > 3. \end{cases}$$

Determine (i) $f(x)$, (ii) K (iii) Mean of $f(x)$.

**Sol:** (i) We know that $f(x) = \frac{d}{dx}[F(x)]$

$$\therefore f(x) = \begin{cases} 0, & \text{if } x \le 1 \\ 4K(x-1)^3, & \text{if } 1 < x \le 3 \\ 0, & \text{if } x > 3 \end{cases}$$

(ii) We Know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ ($\because$ Total Probability is unity).

$$\Rightarrow \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{1}^{3} f(x)\,dx = 1 \quad (\text{since } f(x) = 0 \text{ in } x \le 1 \text{ and } x > 3)$$

$$\Rightarrow \int_{1}^{3} 4K(x-1)^3\,dx = 1$$

$$\Rightarrow 4K\left(\frac{(x-1)^4}{4}\right)_1^3 = 1$$

$$\Rightarrow K(16-0) = 1 \Rightarrow K = \underline{1/16}.$$

$$\therefore f(x) = \begin{cases} 0, & \text{if } x \le 1 \\ \frac{1}{4}(x-1)^3, & \text{if } 1 < x \le 3 \\ 0, & \text{if } x > 3. \end{cases}$$

(iii) Mean of $f(x) = \int_{-\infty}^{\infty} x\, f(x)\,dx$

$$= \int_{-\infty}^{1} x f(x)\,dx + \int_{1}^{3} x f(x)\,dx + \int_{3}^{\infty} x f(x)\,dx$$

$$= \frac{1}{4}\int_{1}^{3} x(x-1)^3\,dx$$

put $x-1 = t \Rightarrow dx = dt$

$x = (1+t)$

as $x \to 1 \Rightarrow t \to 0$

$x \to 3 \Rightarrow t \to 2$

$$= \frac{1}{4}\int_{0}^{2} (1+t)t^3\,dt$$

$$= \frac{1}{4}\int_{0}^{2} (t^3 + t^4)\,dt$$

$$= 1/4\left[\frac{t^4}{4} + \frac{t^5}{5}\right]_0^2 = \frac{1}{4}\left\{\left(\frac{2^4}{4} + \frac{2^5}{5}\right) - 0\right\}$$

$$= \frac{1}{4}\left(\frac{13}{20}\right) = \underline{\underline{2.6}}.$$

Def (MEDIAN): If '$x$' is 'Median' of a random Variable'

Then $P(X < x) \leq 1/2$ and $P(X > x) \leq 1/2$.

In particular, If $X$ is a Continuous random variable th
a point "$x = M$" is said to be Median of 'X'
if $\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = 1/2$, Here

$f(x)$ is the p.d.f of 'X'.

Def (MODE): A point '$x$' is said to be 'Mode' of r.v 'X'
if $f(x)$ attains its Maximum at '$x$', Here $f(x)$ is
the Probability distribution function of 'X'.

((i.e) $f'(x) = 0$ & $f''(x) < 0$).

Problem: If 'X' is a Continuous random Variable having p.d.f

$$f(x) = \begin{cases} \frac{1}{2}\sin x, & 0 \leq x \leq \pi \\ 0, & \text{Elsewhere} \end{cases}$$

Then find Mean, Median, Mode and $P(0 < X < \pi/2)$.

Sol:

Given is $f(x) = \begin{cases} \frac{1}{2}\sin x, & 0 \leq x \leq \pi \\ 0, & \text{Elsewhere} \end{cases}$

(i) Mean $= \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{2}\int_{0}^{\pi} x \sin x\,dx$

$$= \frac{1}{2}\left[x(-\cos x) + \sin x\right]_0^{\pi}$$

$$= \underline{\underline{\pi/2}}$$

<u>Median</u>: If 'M' is the Median of 'X' then

$$\int_0^M f(x)\,dx = \int_M^{\pi} f(x)\,dx = 1/2$$

Consider, $\int_0^M f(x)\,dx = 1/2$

$$\Rightarrow \frac{1}{2}\int_0^M \sin x\,dx = 1/2$$

$$\Rightarrow \left[-\cos x\right]_0^M = 1$$

$$\Rightarrow 1 - \cos M = 1$$

$$\Rightarrow \cos M = 0 \Rightarrow M = \pi/2$$

$\therefore$ Median $= \underline{\underline{\pi/2}}$

<u>Mode</u>: Here $f(x) = \frac{1}{2}\sin x$

$$\therefore f'(x) = \frac{1}{2}\cos x = 0$$

$$\Rightarrow \cos x = 0 \Rightarrow x = \pi/2$$

Now $f''(x) = -\frac{1}{2}\sin x$

$$\therefore \left[f''(x)\right]_{x=\pi/2} = -1/2 < 0$$

$x = \pi/2$ is the Maximum point of $f(x)$.

$\therefore$ Mode $= \underline{\underline{\pi/2}}$

For this problem Mean = Median = Mode = $\pi/2$.

Also $P(0 < X < \pi/_2) = \int_0^{\pi/2} f(x)\,dx$

$$= \int_0^{\pi/2} \tfrac{1}{2} \sin x \, dx$$

$$= \tfrac{1}{2} \left(-\cos x\right)_0^{\pi/2}$$

$$= \tfrac{1}{2} \left[ 0+1 \right] = \tfrac{1}{2}$$

$\therefore \quad P(0 < X < \pi/_2) = 1/2$

# UNIT-III : DISTRIBUTIONS

In previous unit we studied about 'random Var and corresponding distributions. But in this unit we will study about 'Three' Theoritical distributions namely

(i) Binomial Distribution

(ii) Poisson Distribution

(iii) Normal Distribution.

## (i) BINOMIAL DISTRIBUTION:

This distribution is due to "James Bernoulli" in 1700. The following are the assumptions of "Bernoulli".

(a) An Experiment is repeated for 'n' times (i.e) n trials where 'n' is fixed integer.

(b) Let in every trial the outcomes are classified into two ways namely "SUCCESS' and "FAILURE'. also

Let $p$ = Probability of success

$q$ = Probability of failure, such that

$p+q=1$, in every trial.

(c) Let all the outcomes are independent.

NOTE: The trials satisifying the above three assumptions (a), (b) & (c) are called "Bernoulli trials".

## Derivation of Binomial Distribution:-

Let us consider an experiment is repeated for 'n' time

(i.e) No. of trials $= n$.

Out of 'n' trials, let us assume '$x$' times it is succeded.

and therefore '$n-x$' is the number of failures.

Also Let $p$ = probability of success

$q$ = probability of failure, such that $p+q=1$.

$\therefore$ Probability of '$x$' successess $= p.p.\ldots p$ ($x$ times)

$= p^x$

Then probability of '$n-x$' failures $= q.q.\ldots.q$ ($n-x$ times)

$= q^{n-x}$.

$\therefore$ Probability of '$x$' success and '$n-x$' failures is $= p^x.q^{n-x}$

But this can happends in $\binom{n}{x}$ ways.

Hence Total probability of '$x$' success in $n$ trials

$= \binom{n}{x} p^x.q^{n-x}$.

It is denoted by $b(x;n,p)$.

$$\therefore \boxed{b(x;n,p) = \binom{n}{x} p^x.q^{n-x},} \text{ where } x = 0,1,2,\ldots\ldots$$

NOTE: 1. Binomial distribution is a Discrete distribution.

2. $$\sum_{x=0}^{n} b(x;n,p) = 1.$$

Consider, $(p+q)^n = \binom{n}{0} p^0 q^{n-0} + \binom{n}{1} p^1 q^{n-1} + \cdots \cdots + \binom{n}{n} p^n q^{n-n}$

$$= b(0;n,p) + b(1;n,p) + \cdots \cdots + b(n;n;p)$$

$$= \sum_{x=0}^{n} b(x;n,p)$$

but $p+q=1$

$$\therefore \sum_{x=0}^{n} b(x;n,p) = 1$$

PROBLEMS:

1. Find the probability of getting 5 heads when we toss a coin '9' times.

Sol: Here coin is tossed for '9' times

$$\therefore n = 9$$

Let $p$ = probability of getting Head $= 1/2$

$$\therefore q = 1-p = 1/2.$$

Hence probability of getting 5 heads

$$= b(5;9,1/2)$$

$$= \binom{9}{5} (1/2)^5 (1/2)^{9-5}$$

$$= \binom{9}{5} (1/2)^9.$$

## MEAN & VARIANCES OF BINOMIAL DISTRIBUTION (B.D):

(i) Mean :

The Mean of B.D is given by,

$$\mu = E(X) = \sum_{x=0}^{n} x\, P(X=x)$$

$$= \sum_{x=0}^{n} x\, b(x;n,p)$$

$$= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^{x} \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{(n-x)!\,x!}\, p^{x} \cdot q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n \cdot (n-1)!}{((n-1)-(x-1))!\,(x-1)!} \cdot p^{x-1} \cdot p \cdot q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{[(n-1)-(x-1)]!\,(x-1)!}\, p^{x-1} \cdot q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} \cdot q^{(n-1)-(x-1)}$$

$$= np\,(p+q)^{n-1}$$

$$= np(1) = np.$$

$$\therefore \quad \text{Mean} = E(X) = np$$

(ii) Variance : The Variance of B.D is given by

$$\sigma^2 = V(X) = E(X^2) - [E(X)]^2 \longrightarrow (1)$$

$$\text{First } E(X^2) = \sum_{x=0}^{n} x^2\, P(X=x)$$

$$= \sum_{x=0}^{n} x^2\, b(x;n,p)$$

$$= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} \left[x(x-1) + x\right] \cdot \frac{n!}{(n-x)!\, x!}\, p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \cdot \frac{n!}{(n-x)!\, x!}\, p^x q^{n-x} + \sum_{x=0}^{n} x \frac{n!}{(n-x)!\, x!}\, p^x \cdot q^{n-x}$$

$$= n(n-1) \cdot p^2 \cdot \sum_{x=2}^{n} \frac{(n-2)!}{[(n-2)-(x-2)]!\,(x-2)!} \cdot p^{x-2} \cdot q^{(n-2)-(x-2)}$$

$$+ \sum_{x=0}^{n} x\, b(x; n, p).$$

$$= n(n-1)p^2 \cdot \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} \cdot q^{(n-2)-(x-2)} + E(X)$$

$$= n(n-1)p^2 \cdot (p+q)^{n-2} + np$$

$$\therefore E(X^2) = n(n-1)p^2 + np.$$

But by ① $\Rightarrow$ $V(X) = E(X^2) - [E(X)]^2$

$$= n(n-1)p^2 + np - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$= npq \quad (\because p+q=1)$$

$\therefore$ Variance $= V(X) = npq.$

NOTE: The Standard deviation of Binomial Distribution is given by $\sigma = \sqrt{V(X)} = \sqrt{npq}$.

## Mode of Binomial Distribution:

"Mode is the value of '$x$' at which $P(x)$ has maximum value."

Let '$x$' be the mode of Binomial Distribution

$\therefore$ $P(x) \geq P(x+1)$, $P(x) \geq P(x-1)$

(iff) $P(x) \geq P(x+1)$ and $P(x) = \binom{n}{x} p^x . q^{n-x}$

then $x \geq (n+1)p - 1 \longrightarrow$ ①

(after simplification)

Taking $P(x) \geq P(x-1)$

then $x \leq (n+1)p \longrightarrow$ ②

by ① & ② : $(n+1)p - 1 \leq x \leq (n+1)p$

$\Rightarrow$ Hence, * If $(n+1)p$ is not an integer, then Mode is the integral part of $(n+1)p$. In this case the distribution is called unimodal.

* If $(n+1)p$ is an integer then both $(n+1)p$ and $(n+1)p - 1$ will represent modes. In this case the distribution is bimodal.

$\Rightarrow$ RECURRENCE FORMULA FOR BINOMIAL DISTRIBUTION:

$$P(X = x+1) = \frac{(n-x)}{(x+1)} \frac{p}{q} P(X = x)$$

PROOF: We Know that for B.D

$$P(X=x) = \binom{n}{x} p^x \cdot q^{n-x}, \text{ for } x=0,1,2,\ldots\ldots n.$$

Consider, $P(X=x+1) = \binom{n}{x+1} p^{(x+1)} \cdot q^{n-(x+1)}$

$$= \frac{n!}{(n-x-1)!(x+1)!} \cdot p^x \cdot p \cdot q^{n-x} \cdot q^{-1}$$

$$= \frac{(n-x)\, n!}{(n-x)!\ (x+1)x!} \cdot \frac{p}{q} \cdot p^x \cdot q^{n-x}$$

$$= \frac{(n-x)}{(x+1)} \cdot \frac{p}{q} \cdot \frac{n!}{(n-x)!\,x!}\, p^x \cdot q^{n-x}$$

$$= \frac{(n-x)}{(x+1)} \cdot \frac{p}{q} \cdot \binom{n}{x} p^x \cdot q^{n-x}$$

$$\therefore\ P(X=x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot P(X=x)$$

## Problems:

1. A die is thrown 6 times. If getting an even number is a success, Find the probabilities of (i) Atleast One success, (ii) $\leq 3$ successes (iii) 4 Successes.

**Sol:** In a single throw of die,

$$p = \text{probability of an even number} = \frac{3}{6} = \frac{1}{2}.$$

$$\therefore \quad q = 1 - p = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

Here $n$ = number of trials $= 6$.

(i) $P(x \geq 1) = 1 - P(x = 0)$

$$= 1 - \binom{6}{0}(\tfrac{1}{2})^0(\tfrac{1}{2})^{6-0} = 1 - (\tfrac{1}{2})^6 = \frac{63}{64}$$

(ii) $P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= \binom{6}{0}(1/2)^6 + \binom{6}{1}(1/2)^6 + \binom{6}{2}(1/2)^6 + \binom{6}{3}(1/2)^6$$

$$= \left(\frac{1}{2}\right)^6 \left[\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3}\right]$$

$$= \frac{21}{32}.$$

(iii) $P(X=4) = \binom{6}{4}\left(\frac{1}{2}\right)^6 = \frac{15}{64}.$

2. Ten coins are thrown simultaneously. Find the probability of getting atleast seven heads.

sol: Here $p$ = probability getting a head $= 1/2$

$q = 1 - p = 1 - 1/2 = 1/2$

Also $n = 10$.

Probability of getting atleast 7 heads

$$\Rightarrow P(X \ge 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= \binom{10}{7}(1/2)^7(1/2)^3 + \binom{10}{8}(1/2)^8(1/2)^2 + \binom{10}{9}(1/2)^9(1/2)^1 + \binom{10}{10}(1/2)^{10}.$$

* Since $P(X=x) = \binom{n}{x} p^x \cdot q^{n-x}$

$$= \left(\frac{1}{2}\right)^{10}\left[\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right]$$

$$= \frac{(120 + 45 + 10 + 1)}{2^{10}}$$

$$= \frac{176}{1024} = 0.1719.$$

3. It has been claimed that in 60% of all solar heat installations the utility bill is reduced by atleast one-third. Accordingly, what are the probabilities that the utility bill will be reduced by atleast one third in (i) four of five installations
(ii) Atleast four of five installations.

Sol: Here $p$ = The probability that the solar heat installations the utility bill is reduced by one-third

$= 60\% = 0.6$

$\therefore q = 1 - p = 1 - 0.6 = 0.4.$

(i) Here $n = 5,\ x = 4$

$\therefore$ Required probability $\Rightarrow P(X=4) = \binom{5}{4}(0.6)^4 \cdot (0.4)^{5-4}$

$= 0.2592.$

(ii) Required probability $\Rightarrow P(X \geq 4) = P(X=4) + P(X=5)$

$= \binom{5}{4}(0.6)^4 \cdot (0.4) + \binom{5}{5}(0.6)^5$

$= 0.2592 + 0.07776$

$= \underline{0.337}.$

4. If 3 of 20 tyres are defective and 4 of them are randomly choosen for inspection, what is the probability that only one of the defective tyre will be included.

Sol: Let $p$ = Probability of a defective tyre $= \frac{3}{20}$

$\therefore q = 1 - p = 1 - \frac{3}{20} = \frac{17}{20}.$

Given $n = 4.$

Required probability $\Rightarrow P(X=1) = \binom{4}{1}\left(\frac{3}{20}\right)^1 \cdot \left(\frac{17}{20}\right)^{4-1}$

$= \underline{0.3685}.$

5. Determine the Binomial distribution for which the mean is '4' and Variance '3', and hence find the 'Mode' of B.D?

Sol: Given that Mean of B.D $= 4$ (i.e) $np = 4 \rightarrow ①$

Variance of B.D $= 3$ (i.e) $npq = 3 \rightarrow ②$

Now $① \div ② \Rightarrow \frac{np}{npq} = \frac{4}{3}$

$\Rightarrow \frac{1}{q} = \frac{4}{3} \Rightarrow q = \frac{3}{4}$.

$\therefore p = 1 - q = 1 - 3/4 = 1/4$.

$\therefore$ by $① \Rightarrow np = 4$

$\Longrightarrow n(1/4) = 4$

$\Longrightarrow n = 16$.

$\therefore$ Required Binomial Distribution is: $(p+q)^n$

$\Longrightarrow (1/4 + 3/4)^{16}$

The Mode of Binomial distribution is $= (n+1)p$.

$= (16+1)(1/4)$

$= 4.25$, which is not an integer.

$\therefore$ The Required Mode $=$ integral part of $(4.25)$

$= 4$

6. The mean and variance of B.D are 4 and $\frac{4}{3}$ respectively. Find $P(X \geq 1)$?

Sol: Here $np = 4$ ; $npq = 4/3$

$\therefore \frac{npq}{np} = \frac{4/3}{4} \Rightarrow q = 1/3 \quad \therefore p = 2/3$.

Also $np = 4 \Longrightarrow n(2/3) = 4$

$\Longrightarrow n = 6$

$\therefore P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X=0) = 1 - \binom{6}{0}(2/3)^0 (1/3)^6$

$= 1 - (1/3)^6 = 0.9986$

7. Out of 800 families with 5 children each, how many would you expect to have (a) 3 boys (b) 5 girls (c) 2 or 3 boys ? Assume equal probabilities for boys and girls.

Sol: Let $p$ = Probability of Boy $= 1/2$

$\therefore q = 1-p = 1-1/2 = 1/2$.

Also $n = 5$

We know that for B.D $\Rightarrow P(X=x) = \binom{n}{x} p^x \cdot q^{n-x}$.

(a) $P(3 \text{ boys}) = P(X=3) = \binom{5}{3}(1/2)^3(1/2)^{5-3}$

$= \underline{5/16}$.

(b) $P(5 \text{ girls}) = P(0 \text{ boys})$ ($\because$ Total children $\Rightarrow n=5$).

$= \binom{5}{0}(1/2)^0 (1/2)^{5-0} = \underline{1/32}$.

(c) $P(2 \text{ or } 3 \text{ boys}) = P(2 \text{ boys}) + P(3 \text{ boys})$

$= P(X=2) + P(X=3)$

$= \binom{5}{2}(1/2)^5 + \binom{5}{3}(1/2)^5$

$= \underline{5/8}$

$\therefore$ (a) Out of 800 families, the probability of number of families having 3 boys $= 800 \times P(3 \text{ boys})$

$= 800 \times 5/16 = \underline{250 \text{ families}}$

(b) Out of 800 families the probability of number of families having 5 girls $= 800 \times P(5 \text{ girls})$

$= 800 \times 1/32 = \underline{25 \text{ families}}$

(c) Expected number of families with 2 or 3 boys

$= 800 \times P(2 \text{ or } 3 \text{ boys})$

$= 800 \times 5/8$

$= \underline{500 \text{ families}}$.

8. Find the Maximum '$n$' such that the probability of getting no head in tossing a coin is greater than 0.1.

Sol: Let $p$ = probability of getting Head $= 1/2$

$\therefore q = 1 - p = 1 - 1/2 = 1/2$.

Given that $P(X=0) > 0.1$

(i.e) $\binom{n}{0} p^0 . q^{n-0} > 0.1$

$\Longrightarrow \quad (1/2)^n > 0.1 \quad (\because p = q = 1/2)$

For $n=1$, $1/2 > 0.1$ (True)

For $n=2$, $1/4 > 0.1$ (True)

For $n=3$, $1/8 > 0.1$ (True)

But For $n=4$, $1/16 < 0.1$ (False).

$\therefore$ Required $n = \text{Max}\{1, 2, 3\} = 3$

$\therefore \underline{n = 3}$.

9. Determine the probability of getting a sum of 9 exactly twice in 3 throws with a pair of fair dice.

Sol: In a single throw of a pair of fair dice, a sum of '9' can occur in 4 ways: (3,6), (4,5), (5,4), (6,3) out of $6 \times 6 = 36$ ways.

$\therefore p$ = probability of getting a sum of '9' in one throw

$= 4/36 = 1/9 \quad \therefore q = 1 - p = 8/9$.

Given $n = 3$.

Required probability $\Rightarrow P(X=2) = \binom{3}{2}(1/9)^2 \cdot (8/9)^{3-2}$

$= 3 \times \frac{1}{81} \times \frac{8}{9}$

$= \frac{8}{243}$

$= \underline{0.033}$.

## FITTING OF BINOMIAL DISTRIBUTION:

Problems:

Pb①. Fit a Binomial Distribution for the following data?

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f$ | 2 | 14 | 20 | 34 | 22 | 8 |

Sol: Here $n=5$ and $N = \sum_i f_i$

$$= 2+14+20+34+22+8 = 100.$$

We know that, $\text{Mean} = \dfrac{\sum_i f_i x_i}{\sum_i f_i}$

$$\Rightarrow \text{Mean} = (0+14+40+102+88+40)/100$$

$$= \frac{284}{100} = 2.84$$

But Mean of B.D is $\Rightarrow np = 2.84$

$$\Rightarrow 5p = 2.84 \quad (\because n=5)$$

$$\Rightarrow p = 0.568.$$

$$\therefore q = 1-p = 1-0.568 = 0.432.$$

We know that, $P(X=x) = \binom{n}{x} p^x . q^{n-x}$

Now

$$P(X=0) = \binom{5}{0}(0.568)^0.(0.432)^5 = 0.015$$

$$P(X=1) = \binom{5}{1}(0.568)^1.(0.432)^4 = 0.0989$$

$$P(X=2) = \binom{5}{2}(0.568)^2.(0.432)^3 = 0.260$$

$$P(X=3) = \binom{5}{3}(0.568)^3.(0.432)^2 = 0.341$$

$$P(X=4) = \binom{5}{4}(0.568)^4.(0.432)^1 = 0.224$$

$$P(X=5) = \binom{5}{5}(0.568)^5.(0.432)^0 = 0.059.$$

Expected Frequencies : $E(X=x) = N \times P(X=x)$.

$$\therefore E(X=0) = 100 \times 0.015 = 1.5$$

$$E(X=1) = 100 \times 0.0989 = 9.89$$

$$E(X=2) = 100 \times 0.260 = 26$$

$E(X=3) = 100 \times 0.341 = 34.1$, $E(X=4) = 100 \times 0.224 = 22.4$

$E(X=5) = 100 \times 0.059 = 5.9$

$\therefore$ $E(X=0) = 1.5$; $E(X=1) = 9.89$; $E(X=2) = 26$

$E(X=3) = 34.1$; $E(X=4) = 22.4$; $E(X=5) = 5.9$

Required fitted Expected frequencies are:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f$ | 2 | 14 | 20 | 34 | 22 | 8 |
| $E(X=x)$ | $1.5 \simeq 1$ | $9.89 \simeq 10$ | 26 | $34.1 \simeq 34$ | $22.4 \simeq 22$ | $5.9 \simeq 6$ |

NOTE: We can fit Binomial Distribution by using the "Recurrence relation". (i.e) $P(X=x+1) = \frac{(n-x)}{(x+1)} \cdot \frac{p}{q} P(X=x)$.

Now Let us observe the following Example problem.

Pb: (2) Fit a Binomial Distribution for the following data by using Recurrence relation.

| $X=x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f$ | 30 | 62 | 46 | 10 | 2 |

Sol: Here $n$ = no. of trials $= 4$ and $N = \sum_i f_i = 150$

$\therefore$ Mean $= \frac{\sum_i f_i x_i}{\sum_i f_i} = \frac{0(30)+1(62)+2(46)+3(10)+2(4)}{150}$

$= \frac{192}{150}$

but Mean of Binomial distribution is '$np$'

$\therefore np = \frac{192}{150}$

$\Rightarrow 4p = \frac{192}{150} \Rightarrow p = 0.32$

$\therefore q = 1 - p = 0.68$.

But, $P(X=x) = \binom{n}{x} p^x \cdot q^{n-x}$

$$= \binom{4}{x} (0.32)^x \cdot (0.68)^{4-x}$$

$$P(X=0) = \binom{4}{0} (0.32)^0 (0.68)^4 = 0.2138$$

But by Recurrence relation,

$$P(X=x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot P(X=x)$$

$$\therefore \quad P(X=1) = \frac{4-0}{0+1} \cdot \frac{0.32}{0.68} P(X=0)$$

$$= 4 \times \frac{0.32}{0.68} \times 0.2138 = 0.4024$$

$$P(X=2) = \frac{4-1}{1+1} \times \frac{0.32}{0.68} \times P(X=1)$$

$$= \frac{3}{2} \times \frac{0.32}{0.68} \times 0.4024 = 0.2840$$

$$P(X=3) = \frac{4-2}{2+1} \times \frac{0.32}{0.68} \times 0.2840 = 0.0891$$

$$P(X=4) = \frac{4-3}{3+1} \times \frac{0.32}{0.68} \times 0.0891 = 0.01048$$

$\therefore$ Expected Frequencies:

$E(X=0) = 150 \times P(X=0)$

$= 150 \times 0.2138 \simeq 32$

$E(X=1) = 150 \times P(X=1)$

$= 150 \times 0.4024 \simeq 60$

$E(X=2) = 150 \times P(X=2)$

$= 150 \times 0.2840 \simeq 43$

$E(X=3) = 150 \times P(X=3)$

$= 150 \times 0.0891 \simeq 13$

$E(X=4) = 150 \times P(X=4)$

$= 150 \times 0.01048 \simeq 2$

Required fitted Expected frequencies are:

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f$ | 30 | 62 | 46 | 10 | 2 |
| $E(X=x)$ | 32 | 60 | 43 | 13 | 2 |

POISSON DISTRIBUTION:

Poisson Distribution is due to "Simeon Denis Poisson". and it is the "discrete probability distribution" of discrete random variable which has no upperbound. It is defined for Non-negative values of 'x' as follows.

$$f(x,\lambda) = P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ for } x=0,1,2,\ldots$$

Here $\lambda > 0$ is called the parameter of the distribution.

NOTE: Poisson distribution is suitable for 'rare' events for which the probability of occurence 'p' is very small and the number of trials 'n' is very large.

Examples for Rare events:

1. Number of printing mistakes per page
2. Number of accidents on a highway
3. Number of defectives in a productive Centre . . . etc.

Result: In poisson Distribution $\sum_x' f(x,\lambda) = 1$

(Because, $\sum_x' f(x,\lambda) = \sum_x' \frac{e^{-\lambda}\lambda^x}{x!}$

$$= e^{-\lambda} \sum_x' \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \cdot e^{\lambda} = 1 \text{ )}$$

MEAN of Poisson Distribution:

$$\text{MEAN} = E(X) = \sum_x' x\, P(X=x)$$

$$= \sum_x' x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_x' \frac{e^{-\lambda}\lambda^{x-1}\cdot\lambda}{(x-1)!}$$

$$= \lambda \sum_x' \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \lambda(1) = \underline{\lambda}.$$

## Variance of Poisson Distribution :

$$\text{Variance} = V(X) = E(X^2) - [E(X)]^2 \longrightarrow (1)$$

Now, $$E(X^2) = \sum_x' x^2 \cdot P(X=x)$$

$$= \sum_x' x^2 \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= \sum_x' [x(x-1)+x] \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= \sum_x' \frac{e^{-\lambda} \cdot \lambda^x}{(x-2)!} + \sum_x' x \frac{e^{-\lambda} \cdot \lambda^x}{x!} (\text{Mean})$$

$$= \lambda^2 \cdot \sum_x' \frac{e^{-\lambda} \cdot \lambda^{x-2}}{(x-2)!} + \lambda.$$

$$= \lambda^2 (1) + \lambda$$

$$= \lambda^2 + \lambda.$$

$$\therefore (1) \Rightarrow V(X) = E(X^2) - [E(X)]^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \underline{\underline{\lambda}}$$

**NOTE:** In poisson Distribution Mean and Variance are coinside.

## Recurrence relation for Poisson :-

**St:** $$P(X = x+1) = \frac{\lambda}{x+1} P(X=x).$$

**Proof:** Consider $$P(X = x+1) = \frac{e^{-\lambda} \cdot \lambda^{x+1}}{(x+1)!}$$

$$= \frac{e^{-\lambda} \cdot \lambda^x \cdot \lambda}{(x+1)\, x!}$$

$$= \frac{\lambda}{x+1} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= \frac{\lambda}{x+1} \cdot P(X=x).$$

Pb: Fit a Poisson distribution for the following data.

| | | | | | |
|---|---|---|---|---|---|
| $x$: | 0 | 1 | 2 | 3 | 4 |
| $f(x)$: | 109 | 65 | 22 | 3 | 1 |

Sol: Here $N = \sum_i f_i = 109+65+22+3+1 = 200$

$$\text{Mean} = \frac{\sum_i f_i x_i}{\sum_i f_i} = \frac{0+65+44+9+4}{200}$$

$$= \frac{122}{200} = 0.61$$

Mean of poisson Distribution $\Rightarrow \lambda = 0.61$

But $$P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$\therefore$ Expected frequencies are, $E(X=x) = N \times P(X=x)$

$$= N \times \frac{e^{-\lambda} \lambda^x}{x!}$$

$\therefore$ $$E(X=0) = 200 \times \frac{e^{-0.61} \cdot (0.61)^0}{0!} = 108.67 \simeq 109$$

$$E(X=1) = 200 \times \frac{e^{-0.61} \cdot (0.61)^1}{1!} = 66.29 \simeq 66$$

$$E(X=2) = 200 \times \frac{e^{-0.61} (0.61)^2}{2!} = 20.22 \simeq 20$$

$$E(X=3) = 200 \times \frac{e^{-0.61} (0.61)^3}{3!} = 4.11 \simeq 4$$

$$E(X=4) = 200 \times \frac{e^{-0.61} (0.61)^4}{4!} = 0.63 \simeq 1.$$

## NORMAL DISTRIBUTION:

Normal distribution is one of the most widely used continuous probability distribution, The normal distribution was first discovered by an English mathematician <u>Demoivre</u> in 1733 later it was developed by <u>Laplace</u> in 1774 and <u>Gauss</u> in 1809.

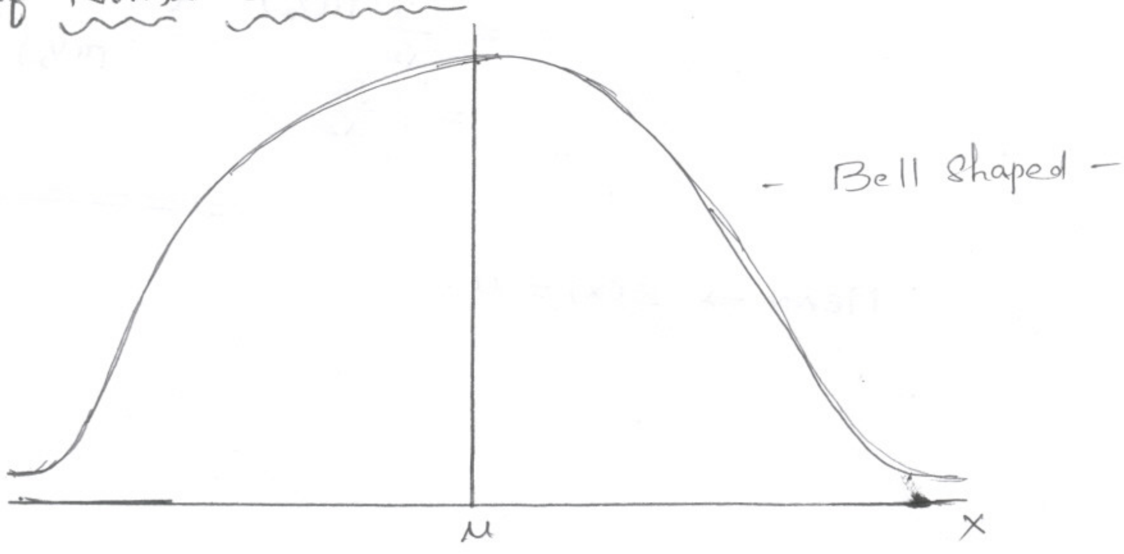
NOTE: <u>Normal distribution</u> is also refferred as the <u>Gaussian distribution</u>.

<u>Definition</u>: A continuous random variable 'X' is said to have a Normal distribution with parameters '$\mu$' and $\sigma^2$, if its density function is given by,

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \text{where } -\infty < x < \infty \text{ and } \sigma > 0$$

NOTE: A continuous random variable 'X' with parameters $\mu$ and $\sigma^2$ follows N.D is denoted by $X \sim N(\mu,\sigma^2)$ (or) $X \sim N(\mu,\sigma)$.

Graph of Normal distribution :-



## MEAN and VARIANCE OF N.D:

(i) MEAN: $E(X) = \int_{-\infty}^{\infty} x\, f(x;\mu,\sigma^2)\,dx$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $z = \frac{x-\mu}{\sigma}$

$\Rightarrow dz = \frac{dx}{\sigma} \Rightarrow dx = \sigma dz$

$x = (\mu + \sigma z)$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{z^2}{2}} (\sigma dz)$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz$$

$$= 0 + \frac{2\mu}{\sqrt{2}\sqrt{\pi}} \int_0^{\infty} e^{-z^2/2} dz \quad \left(\because z e^{-z^2/2} \text{ is odd function, } e^{-z^2/2} \text{ is Even fun}\right)$$

$$= \frac{\sqrt{2}\,\mu}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{\sqrt{2}} = \mu.$$

Since $\int_0^{\infty} e^{-z^2/2} dz = \frac{\sqrt{\pi}}{\sqrt{2}}$

because put $z^2/2 = t \Rightarrow z\,dz = dt$

$\Rightarrow dz = \frac{dt}{z} = \frac{dt}{\sqrt{2t}}$

$$\Rightarrow \int_0^{\infty} e^{-z^2/2} dz = \int_0^{\infty} e^{-t} \cdot \frac{dt}{\sqrt{2}\sqrt{t}} \quad \left(\because \text{As } z \to 0 \Rightarrow t \to 0,\ z \to \infty \Rightarrow t \to \infty\right)$$

$$= \frac{1}{\sqrt{2}} \int_0^{\infty} e^{-t} \cdot t^{1/2-1} dt$$

$$= \frac{1}{\sqrt{2}} \Gamma(1/2) \quad \left(\because \Gamma(n) = \int_0^{\infty} e^{-t} t^{n-1} dt,\ \Gamma(1/2) = \sqrt{\pi}\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}}$$

$\therefore$ MEAN $\Rightarrow E(X) = \mu$.

(ii) <u>VARIANCE</u>: $V(X) = E(X^2) - [E(X)]^2$.

Now $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x;\mu,\sigma^2)\,dx$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} x^2 \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $z = \frac{x-\mu}{\sigma}$

$\Rightarrow dz = \frac{dx}{\sigma}$.

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu+\sigma z)^2 \cdot e^{-\frac{z^2}{2}} (\cancel{\sigma}\,dz)$$

$$= \frac{1}{\sqrt{2\pi}}\left[\mu^2 \cdot \int_{-\infty}^{\infty} e^{-z^2/2} dz + \sigma^2 \int_{-\infty}^{\infty} z^2 \cdot e^{-z^2/2} dz + 2\mu\sigma \int_{-\infty}^{\infty} z e^{-z^2/2} dz\right]$$

$$= \frac{\mu^2}{\sqrt{2\pi}} \cdot 2\int_0^{\infty} e^{-z^2/2} dz + \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-z^2/2} dz + 0 \Big]$$

$$= \frac{2\mu^2}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{2} + \frac{2\sigma^2}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{2}$$

$\left( \because z e^{-z^2/2} \text{ is odd} \quad \left.\begin{matrix} e^{-z^2/2} \\ z^2 e^{-z^2/2} \end{matrix}\right\} \text{ are even.} \right)$

$$E(X^2) = \mu^2 + \sigma^2$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2$$

$$= \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$

$\therefore$ Variance $\Rightarrow$ <u>$V(X) = \sigma^2$</u>.

**MODE:** We know that 'Mode' is the value of '$x$' for which '$f(x)$' is maximum. (i.e) $f'(x) = 0$ & $f''(x) < 0$

Here $f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot \left[-\left(\frac{x-\mu}{\sigma^2}\right)\right] = 0$$

$$\Rightarrow -\frac{1}{\sqrt{2\pi}\cdot\sigma}\left(\frac{x-\mu}{\sigma^2}\right) = 0 \quad \left(\because e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \neq 0\right)$$

$$\Rightarrow \quad x - \mu = 0$$

$$\Rightarrow \quad x = \mu$$

Again $\quad f''(x) = \frac{1}{\sqrt{2\pi}\,\sigma^2}\left[ -\frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^2 . e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} - e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)} . \frac{1}{\sigma} \right]$

when $x$

$$< 0$$

$$\therefore \left[f''(x)\right]_{x=\mu} < 0$$

$\therefore$ $f(x)$ is Maximum at $x = \mu$

$\therefore$ Mode of Normal distribution $= \mu$.

MEDIAN OF N.D :- Suppose '$M$' is the Median of N.D then

$$\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}.$$

Now take, $\displaystyle\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}$

$$\Rightarrow \int_{-\infty}^{M} \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{M} \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2} \rightarrow (A)$$

Here $\displaystyle\int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$ (On simplification)

$$\therefore (A) \Rightarrow \quad \frac{1}{2} + \int_{\mu}^{M} \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\Rightarrow \int_{\mu}^{M} \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0 \Rightarrow M = \mu$$

$\left(\because \displaystyle\int_a^b f(x)\,dx = 0 \iff a = b\right)$

$\therefore$ Median $= \mu$.

Hence for Normal Distribution,

$$\text{MEAN} = \text{MODE} = \text{MEDIAN}. \checkmark$$

Note: Normal Distribution is a Symmetrical distribution.

Mean deviation from the mean for N.D:

Def: If $x$ is the continuous random variable and $f(x)$ be its corresponding distribution then the mean deviation from the mean is defined by $\int_{-\infty}^{\infty} |x-\mu| f(x)\,dx$

Mean Deviation from the mean for N.D

$$\Rightarrow \int_{-\infty}^{\infty} |x-\mu|\, f(x;\mu,\sigma^2)\,dx$$

$$= \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} \cancel{\sigma}|z| \cdot e^{-z^2/2} (\sigma\, dz)$$

put $\frac{x-\mu}{\sigma} = z$ $\Rightarrow dx = \sigma dz$

($\because |z| = z$ in $[0,\infty)$)

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} z \cdot e^{-z^2/2}\, dz$$

put $\frac{z^2}{2} = t$ $\Rightarrow 2z\,dz = 2dt$ $\Rightarrow z\,dz = dt$

$$= \frac{\sqrt{2}\,\sigma}{\sqrt{\pi}} \int_0^{\infty} e^{-t}\, dt$$

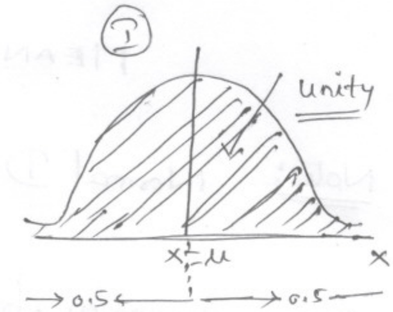
$$= \frac{\sqrt{2}\,\sigma}{\sqrt{\pi}} \left(-e^{-t}\right)_0^{\infty} = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sigma$$

$$\simeq \frac{4}{5}\sigma \checkmark$$

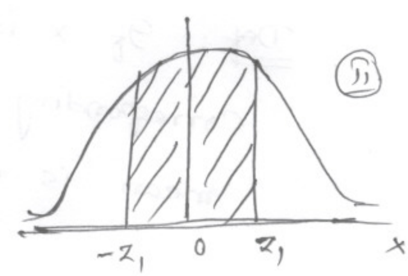
$\therefore$ Mean Deviation from the mean of N.D $= \frac{4}{5}\sigma$ (approx.)

# AREA UNDER NORMAL CURVE:

1. The total area under normal curve is <u>unity</u> (i.e) $\int_{-\infty}^{\infty} f(x;\mu,\sigma^2)\,dx = 1$.




2. Since it is symmetrical distribution Here the area from $Z=0$ to $Z=Z_1$ is same as the area from $Z=-Z_1$ to $Z=0$. This property is called <u>Symmetry property</u>.




(i.e) $$\int_0^{Z_1} \phi(z)\,dz = \int_{-Z_1}^{0} \phi(z)\,dz.$$

By Taking $Z = \frac{x-\mu}{\sigma}$ (Standard Normal variate) then the standard Normal curve is formed. Now the curve corresponding to (I) is the curve (II).

3. Here $$P(\mu < X < x_1) = \int_{\mu}^{x_1} f(x;\mu,\sigma^2)\,dx$$
$$= \int_{\mu}^{x_1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $Z = \frac{x-\mu}{\sigma} \Rightarrow dx = \sigma\, dZ$

If $x = \mu \Rightarrow Z = 0$

$x = x_1 \Rightarrow Z = \frac{x_1-\mu}{\sigma} = Z_1$ (say)

$$\therefore \quad P(\mu < X < x_1) = \int_0^{Z_1} \left[\frac{1}{\sqrt{2\pi}} e^{-Z^2/2}\right] dZ$$
$$= \int_0^{Z_1} \phi(z)\,dz, \quad \text{where } \phi(Z) = \frac{1}{\sqrt{2\pi}} e^{-Z^2/2}$$
$$= A(Z_1) \text{ (say)}$$

$$\therefore \boxed{P(\mu < X < x_1) = P(0 < Z < z_1) = A(z_1).}$$

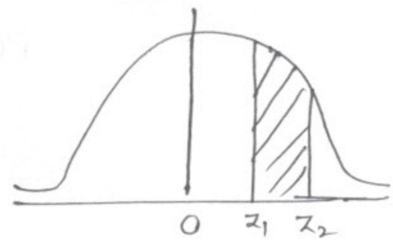
4. The probability that the normal variate X with mean '$\mu$' and S.D $\sigma^2$ lies b/w the two specified values $x_1$ & $x_2$ with $x_1 \le x_2$ can be obtained using the area under the standard Normal curve as follows.

Step ①: Perform the change of scale $Z = \frac{x-\mu}{\sigma}$ and find $z_1$ & $z_2$ corresponding to the values $x_1$ & $x_2$.

Step ②: ⓐ To find $P(x_1 \le X \le x_2) = P(z_1 \le Z \le z_2)$

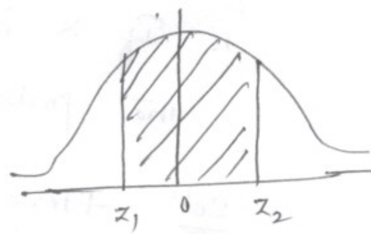
Case ①: If $z_1, z_2$ both are positive (or) Negative then

$$P(x_1 \le X \le x_2) = P(z_1 \le Z \le z_2)$$
$$= A(z_2) - A(z_1)$$
$$= \int_0^{z_2} \phi(z)dz - \int_0^{z_1} \phi(z)dz.$$




Case ②: If $z_1 < 0$ & $z_2 > 0$ then

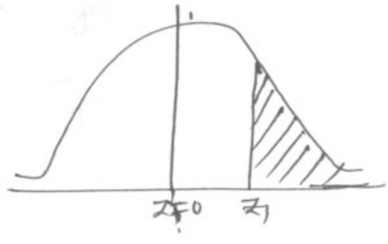
$$P(x_1 \le X \le x_2) = P(z_1 \le Z \le z_2)$$
$$= A(z_1) + A(z_2)$$
$$= \int_0^{z_1} \phi(z)dz + \int_0^{z_2} \phi(z)dz$$




ⓑ To find $P(X > x_1) = P(Z > z_1)$

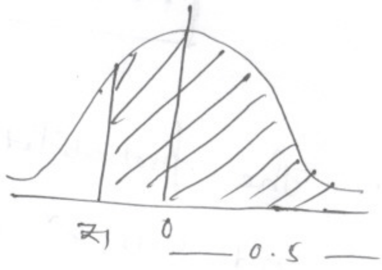
Case ①: If $z_1 > 0$ then

$$P(X > x_1) = P(Z > z_1) = 0.5 - A(z_1)$$
$$= 0.5 - \int_0^{z_1} \phi(z)dz$$



Case ② : If $z_1 < 0$ then

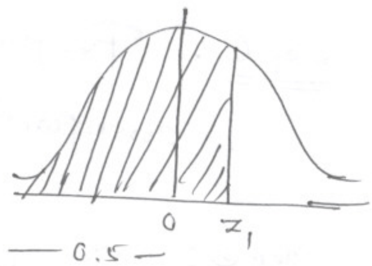
$$P(X > x_1) = P(Z > z_1).$$
$$= 0.5 + A(z_1)$$
$$= 0.5 + \int_0^{z_1} \phi(z)\,dz$$




ⓒ To find $P(X < x_1) = P(Z < z_1)$

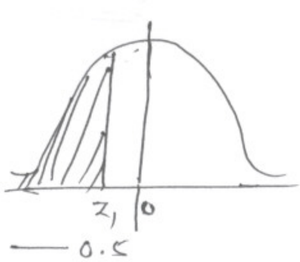
Case ① : If $z_1 > 0$ then

$$P(X < x_1) = P(Z < z_1) = 0.5 + A(z_1)$$
$$= 0.5 + \int_0^{z_1} \phi(z)\,dz.$$




Case ② : If $z_1 < 0$ then

$$P(X < x_1) = P(Z < z_1) = 0.5 - A(z_1)$$
$$= 0.5 - \int_0^{z_1} \phi(z)\,dz$$




Problems :

1. If X is a Normal Variate with mean 30 and S.D 5, find the probabilities that (i) $26 < X < 40$ (ii) $X \geq 45$

Sol: Here $\mu = 30$, $\sigma = 5$

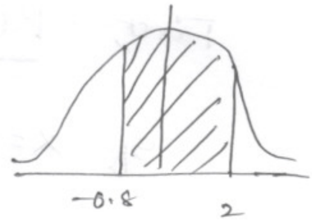
Let $Z = \dfrac{X - \mu}{\sigma} = \dfrac{X - 30}{5}$

(i) $P(26 < X < 40) = ?$

If $X = 26$ then $Z = \dfrac{26 - 30}{5} = -0.8$

If $X = 40$ then $Z = \dfrac{40 - 30}{5} = 2$

$\therefore\ P(26 < X < 40) = P(-0.8 < Z < 2)$




$$= \int_{-0.8}^{0} \phi(z)\,dz + \int_{0}^{2} \phi(z)\,dz$$

$$= \int_{0}^{0.8} \phi(z)\,dz + \int_{0}^{2} \phi(z)\,dz$$

(Symmetry property)

$$= A(0.8) + A(2)$$

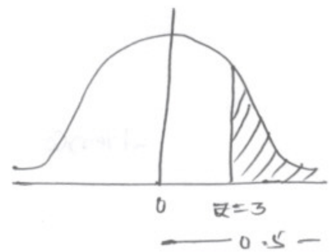
$$= 0.2881 + 0.4772 \quad \text{(By Tables)}$$

$$= \underline{\underline{0.7653}}$$

(ii) $\underline{\underline{P(X \geq 45) = ?}}$

If $X = 45$ then $Z = \dfrac{45-30}{5} = 3$

$\therefore\ P(X \geq 45) = P(Z \geq 3) = 0.5 - \int_{0}^{3} \phi(z)\,dz$




$$= 0.5 - A(3)$$

$$= 0.5 - 0.4987 \quad \text{(By Tables)}$$

$$= \underline{\underline{0.0013}}.$$

Pb ②: Given that the mean heights of students in the class is 158 cms with S.D of 20 cms, Find How many students lie b/w 150 cms to 170 cms if there are 100 students in the class?

Sol:

Here Mean $\Rightarrow \mu = 158, \sigma = 20$

let $z = \frac{x - \mu}{\sigma} = \frac{x - 158}{20}$

First, $P(150 < X < 170) = ?$

If $X = 150 \Rightarrow Z = \frac{150-158}{20} = -0.4$

$X = 170 \Rightarrow Z = \frac{170-158}{20} = 0.6$

$\therefore P(150 < X < 170) = P(-0.4 < Z < 0.6)$


$= \int_{-0.4}^{0} \phi(z)dz + \int_{0}^{0.6} \phi(z)dz$

(By symmetry) $= \int_{0}^{0.4} \phi(z)dz + \int_{0}^{0.6} \phi(z)dz$

$= A(0.4) + A(0.6)$

$= 0.1554 + 0.2257$ (by Tables)

$= \underline{0.3811}$

Hence Number of students who height lie b/w 150 cms to 170 cms

$= 100 \times P(150 < X < 170)$

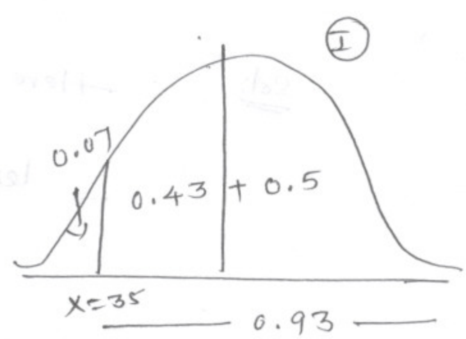
$= 100 \times 0.3811$

$= 38.11 \simeq 38.$

Pb ③: In a distribution exactly Normal 7% of the items are under 35 and 89% are under 63, then find Mean & S.D of the distribution?

Sol: Given that, $P(X < 35) = 7\%$
$= 0.07$

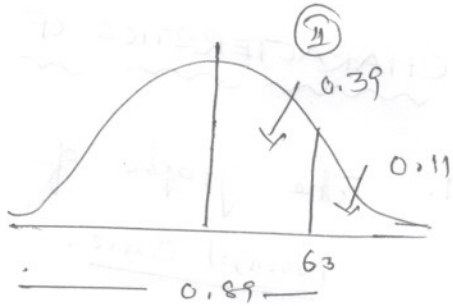
$\Rightarrow P(X > 35) = 0.93$



And $P(X<63) = 89\%$

$= 0.89$

$\Rightarrow P(X>63) = 0.11$




By (I) $\Rightarrow \int_0^{z_1} \phi(z)\,dz = 0.43$

$\Rightarrow z_1 = -1.48$ (by Tables)

$\Rightarrow \dfrac{35-\mu}{\sigma} = -1.48$ (since it is in Negative side)

$\Rightarrow \mu = 35 + 1.48\sigma \longrightarrow$ (a)

By (II) $\Rightarrow \int_0^{z_2} \phi(z)\,dz = 0.39$

$\Rightarrow z_2 = 1.23$

$\Rightarrow \dfrac{63-\mu}{\sigma} = 1.23$

$\Rightarrow \mu = -1.23\sigma + 63 \longrightarrow$ (b)

Solving (a) & (b): $\mu = 50.3$

$\sigma = 10.3$.

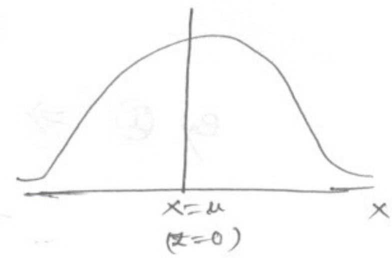
NOTE: The above three models are three different models based on Normal distribution. (Areas).

—— (x) ——

## CHARACTERSTICS OF NORMAL DISTRIBUTION:

1. The graph of N.D $y = f(x)$ in $xy$-plane is known as Normal Curve.

2. The curve is bell shaped and symmetrical about the line $X = \mu$ (or) $Z = 0$.




3. The mean = median = mode for N.D.

4. Linear Combination of independent Normal Variates is also a Normal variate.

   (i.e) If $X_1, X_2$ are two independent normal variates then $\underline{a_1 X_1 + a_2 X_2}$ is also a normal variate ($a_1, a_2$ are constants)

5. Since $f(x)$ being the probability, can never be negative no portion of the curve lies below the X-axis.

6. Normal curve is asymptotic to both positive X-axis and negative X-axis.

7. The N.C extends from $-\infty$ to $\infty$, Thus the total area under normal curve is unity.

8. Area under the N.C is distributed as follows.

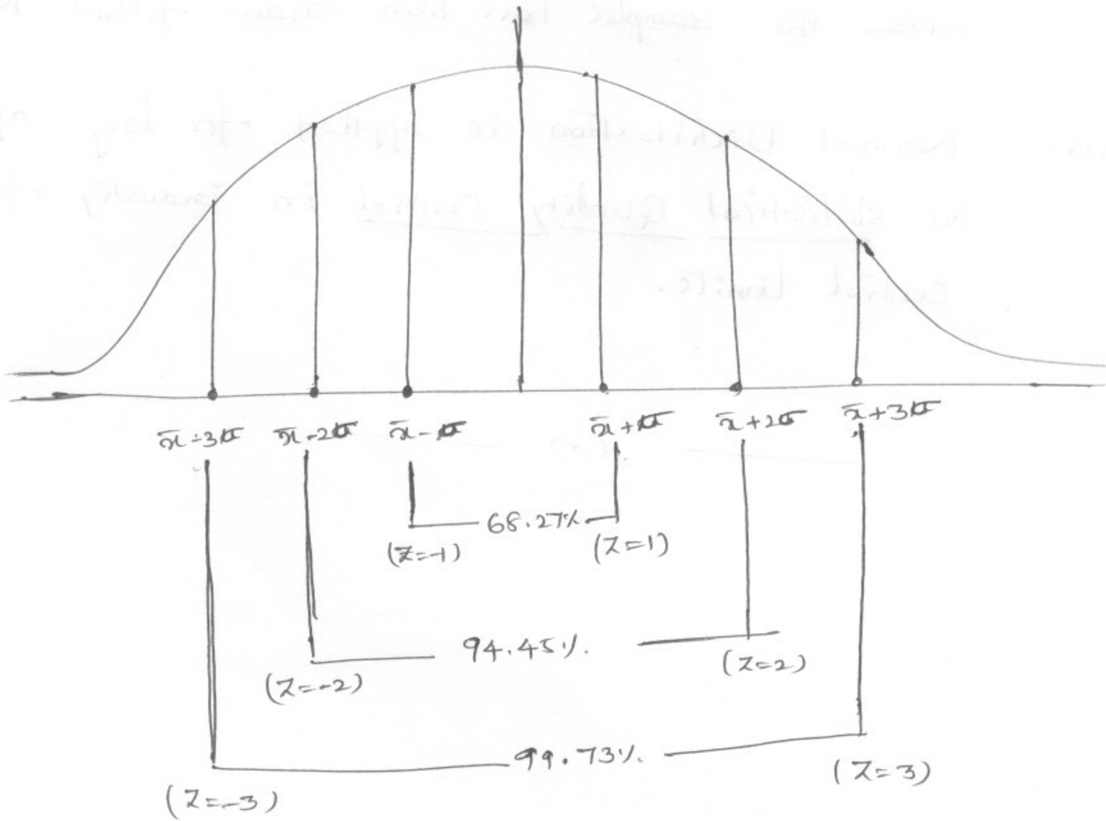
   (i) 68.27% of area is lies b/w $\mu - \sigma$ and $\mu + \sigma$
   (i.e) ($Z = -1$, to $Z = 1$)

   (ii) 94.45% of area is lies b/w $\mu - 2\sigma$ and $\mu + 2\sigma$
   ((i.e) $Z = -2$ to $Z = 2$)

(iii) 99.73% of area lies b/w $\mu-3\sigma$ and $\mu+3\sigma$

((i.e) $Z=-3$ to $Z=3$)

(i.e)



## IMPORTANCE OF NORMAL DISTRIBUTION:-

Normal distribution plays a major role in statistical theory, It has a widest practical area at application than other distribution machanism.

(i) Many of distributions occuring in practice like Binomial, poisson.....etc can be approximated by N.D.

(ii) Many of the distributions of sample statistic for example distribution of sample mean, sample variance......etc tends to normality for large samples. and such they can be studied with the help of normal curve.

(iii) The entire theory of small sample tests based on the fundamental assumption that the parent populations from which the samples have been drawn follows N.D.

(iv) Normal Distribution is applied for large applications in statistical Quality Control in industry for setting control limits.

(x)

## Fitting of Normal Distribution:

We can observe the fitting of N.D by taking the help of following example.

(Pb): Fit a Normal Distribution for the following data

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_i$: | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 | 90 | 95 | 100 |
| $f_i$: | 2 | 3 | 5 | 9 | 10 | 12 | 7 | 2 | 3 | 1 | 0 |

Sol: Here, Mean, $\bar{x} = \frac{\sum_1' f_i x_i}{\sum_1' f_i} = \frac{3845}{54} = 71.2 \simeq 71$

$$S.D \Rightarrow \sigma = \sqrt{\frac{\sum_1' f_i x_i^2}{\sum_1' f_i} - \left(\frac{\sum_1' f_i x_i}{\sum_1' f_i}\right)^2}$$

$$\simeq 10.$$

Also we know that $Z = \frac{X-\mu}{\sigma}$, $N = \sum_1' f_i = 54$

Let $x_1$ – Lower Limit of class interval

$x_2$ – Upper Limit of class interval

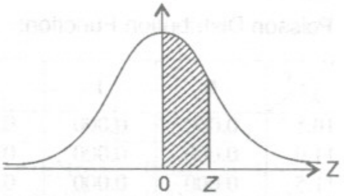
$Z_1, Z_2$ are values of $z$ corresponding to $x_1, x_2$ respectively.

$P(X=x)$ = Area b/w $Z_1$ & $Z_2$.

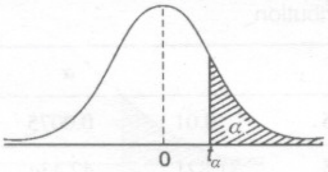
Now observe the following table.

| $F_i$ | (x) Mid | $x_1 - x_2$ | $z_1 - z_2$ | Area b/w $z_1$ & $z_2$ $P(x=a)$ | Expected Frequencies $(E(x=a))$ $N * P(x=a)$ |
|---|---|---|---|---|---|
| 2 | 50 | 47.5 – 52.5 | -2.35 to -1.85 | 0.4906 – 0.4678 = 0.0228 | 54 × 0.0228 = 1.23 ≃ 1 |
| 3 | 55 | 52.5 – 57.5 | -1.85 to -1.35 | 0.4678 – 0.4115 = 0.053 | 54 × 0.053 = 3.04 ≃ 3 |
| 5 | 60 | 57.5 – 62.5 | -1.35 to -0.85 | 0.4115 – 0.3023 = 0.1092 | 54 × 0.1092 = 5.89 ≃ 6 |
| 9 | 65 | 62.5 – 67.5 | -0.85 to -0.35 | 0.3023 – 0.1368 = 0.1655 | 54 × 0.1655 = 8.93 ≃ 9 |
| 10 | 70 | 67.5 – 72.5 | -0.35 to 0.15 | 0.1368 + 0.0596 = 0.1964 | 54 × 0.1964 = 10.6 ≃ 11 |
| 12 | 75 | 72.5 – 77.5 | 0.15 to 0.65 | -0.0590 + 0.2422 = 0.1826 | 54 × 0.1826 = 9.86 ≃ 10 |
| 7 | 80 | 77.5 – 82.5 | 0.65 to 1.15 | -0.2422 + 0.3749 = 0.1327 | 54 × 0.1327 = 7.10 ≃ 7 |
| 2 | 85 | 82.5 – 87.5 | 1.15 to 1.65 | -0.3749 + 0.4505 = 0.0756 | 54 × 0.0756 = 4.08 ≃ 4 |
| 3 | 90 | 87.5 – 92.5 | 1.65 to 2.15 | -0.4505 + 0.4842 = 0.033 | 54 × 0.033 = 1.819 ≃ 2 |
| 1 | 95 | 92.5 – 97.5 | 2.15 to 2.65 | -0.4842 + 0.4960 = 0.0119 | 54 × 0.0119 = 0.67 ≃ 1 |
| 0 | 100 | 97.5 – 102.5 | 2.65 to 3.15 | -0.4960 + 0.4992 = 0.0032 | 54 × 0.0032 = 0.17 ≃ 0 |

3. Areas under the Standard Normal Curve
from 0 to $z$
**(Normal Tables)**




| z | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .0000 | .0040 | .0080 | .0120 | .0160 | .0199 | .0239 | .0279 | .0319 | .0359 |
| 0.1 | .0398 | .0438 | .0478 | .0517 | .0557 | .0596 | .0636 | .0675 | .0714 | .0754 |
| 0.2 | .0793 | .0832 | .0871 | .0910 | .0948 | .0987 | .1026 | .1064 | .1103 | .1141 |
| 0.3 | .1179 | .1217 | .1256 | .1293 | .1331 | .1368 | .1406 | .1443 | .1480 | .1517 |
| 0.4 | .1554 | .1591 | .1628 | .1664 | .1700 | .1736 | .1772 | .1808 | .1844 | .1879 |
| 0.5 | .1916 | .1950 | .1985 | .2019 | .2054 | .2088 | .2123 | .2157 | .2190 | .2224 |
| 0.6 | .2258 | .2291 | .2324 | .2357 | .2389 | .2422 | .2454 | .2486 | .2518 | .2649 |
| 0.7 | .2580 | .2612 | .2642 | .2673 | .2704 | .2734 | .2764 | .2794 | .2823 | .2852 |
| 0.8 | .2881 | .2910 | .2939 | .2967 | .2996 | .3023 | .3051 | .3078 | .3106 | .3133 |
| 0.9 | .3159 | .3186 | .3212 | .3238 | .3264 | .3289 | .3315 | .3340 | .3365 | .3389 |
| 1.0 | .3413 | .3438 | .3461 | .3485 | .3508 | .3531 | .3554 | .3577 | .3599 | .3621 |
| 1.1 | .3643 | .3665 | .3686 | .3708 | .3729 | .3749 | .3770 | .3790 | .3810 | .3830 |
| 1.2 | .3849 | .3869 | .3888 | .3907 | .3925 | .3944 | .3962 | .3980 | .3997 | .4015 |
| 1.3 | .4032 | .4049 | .4066 | .4082 | .4099 | .4115 | .4131 | .4147 | .4162 | .4177 |
| 1.4 | .4192 | .4207 | .4222 | .4236 | .4251 | .4265 | .4279 | .4292 | .4306 | .4319 |
| 1.5 | .4332 | .4345 | .4357 | .4370 | .4382 | .4394 | .4406 | .4418 | .4429 | .4441 |
| 1.6 | .4452 | .4463 | .4474 | .4484 | .4495 | .4505 | .4515 | .4525 | .4535 | .4545 |
| 1.7 | .4654 | .4564 | .4573 | .4582 | .4591 | .4599 | .4608 | .4616 | .4625 | .4633 |
| 1.8 | .4641 | .4649 | .4656 | .4664 | .4671 | .4678 | .4686 | .4693 | .4699 | .4706 |
| 1.9 | .4713 | .4719 | .4726 | .4732 | .4738 | .4744 | .4750 | .4756 | .4761 | .4767 |
| 2.0 | .4772 | .4778 | .4783 | .4788 | .4793 | .4798 | .4803 | .4808 | .4812 | .4817 |
| 2.1 | .4821 | .4826 | .4830 | .4834 | .4838 | .4842 | .4846 | .4850 | .4854 | .4857 |
| 2.2 | .4861 | .4864 | .4868 | .4871 | .4875 | .4878 | .4881 | .4884 | .4887 | .4890 |
| 2.3 | .4893 | .4896 | .4898 | .4901 | .4904 | .4906 | .4909 | .4911 | .4913 | .4916 |
| 2.4 | .4918 | .4920 | .4922 | .4925 | .4927 | .4929 | .4931 | .4932 | .4934 | .4936 |
| 2.5 | .4938 | .4940 | .4941 | .4943 | .4945 | .4946 | .4948 | .4949 | .4951 | .4952 |
| 2.6 | .4953 | .4955 | .4956 | .4957 | .4959 | .4960 | .4961 | .4962 | .4963 | .4964 |
| 2.7 | .4965 | .4966 | .4967 | .4968 | .4969 | .4970 | .4971 | .4972 | .4973 | .4974 |
| 2.8 | .4974 | .4975 | .4976 | .4977 | .4979 | .4978 | .4979 | .4979 | .4980 | .4981 |
| 2.9 | .4981 | .4982 | .4982 | .4983 | .4984 | .4984 | .4985 | .4985 | .4986 | .4986 |
| 3.0 | .4987 | .4987 | .4987 | .4988 | .4988 | .4989 | .4989 | .4989 | .4990 | .4990 |
| 3.1 | .4990 | .4991 | .4991 | .4991 | .4992 | .4992 | .4992 | .4992 | .4993 | .4993 |
| 3.2 | .4993 | .4993 | .4994 | .4994 | .4994 | .4994 | .4994 | .4995 | .4995 | .4995 |
| 3.3 | .4995 | .4995 | .4995 | .4996 | .4996 | .4996 | .4996 | .4996 | .4996 | .4997 |
| 3.4 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4998 |
| 3.5 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4098 | .4998 | .4998 | .4998 |
| 3.6 | .4998 | .4998 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 |
| 3.7 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 |
| 3.8 | .4999 | 4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 |
| 3.9 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 |



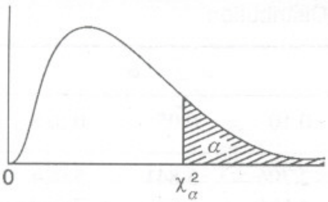

4. $t_\alpha$-Critical Values of the $t$-Distribution

n=1

| | α | | | | | | |
|---|---|---|---|---|---|---|---|
| ν | 0.40 | 0.30 | 0.20 | 0.15 | 0.10 | 0.05 | 0.025 |
| 1 | 0.325 | 0.727 | 1.376 | 1.963 | 3.078 | 6.314 | 12.706 |
| 2 | 0.289 | 0.617 | 1.061 | 1.386 | 1.886 | 2.920 | 4.303 |
| 3 | 0.277 | 0.584 | 0.978 | 1.250 | 1.638 | 2.353 | 3.182 |
| 4 | 0.271 | 0.569 | 0.941 | 1.190 | 1.533 | 2.132 | 2.776 |
| 5 | 0.267 | 0.559 | 0.920 | 1.156 | 1.476 | 2.015 | 2.571 |
| 6 | 0.265 | 0.553 | 0.906 | 1.134 | 1.440 | 1.943 | 2.447 |
| 7 | 0.263 | 0.549 | 0.896 | 1.119 | 1.415 | 1.895 | 2.365 |
| 8 | 0.262 | 0.546 | 0.889 | 1.108 | 1.397 | 1.860 | 2.306 |
| 9 | 0.261 | 0.543 | 0.883 | 1.100 | 1.383 | 1.833 | 2.262 |
| 10 | 0.260 | 0.542 | 0.879 | 1.093 | 1.372 | 1.812 | 2.228 |
| 11 | 0.260 | 0.540 | 0.876 | 1.088 | 1.363 | 1.796 | 2.201 |
| 12 | 0.259 | 0.539 | 0.873 | 1.083 | 1.356 | 1.782 | 2.179 |
| 13 | 0.259 | 0.537 | 0.870 | 1.079 | 1.350 | 1.771 | 2.160 |
| 14 | 0.258 | 0.537 | 0.868 | 1.076 | 1.345 | 1.761 | 2.145 |
| 15 | 0.258 | 0.536 | 0.866 | 1.074 | 1.341 | 1.753 | 2.131 |
| 16 | 0.258 | 0.535 | 0.865 | 1.071 | 1.337 | 1.746 | 2.120 |
| 17 | 0.257 | 0.534 | 0.863 | 1.069 | 1.333 | 1.740 | 2.110 |
| 18 | 0.257 | 0.534 | 0.862 | 1.067 | 1.330 | 1.734 | 2.101 |
| 19 | 0.257 | 0.533 | 0.861 | 1.066 | 1.328 | 1.729 | 2.093 |
| 20 | 0.257 | 0.533 | 0.860 | 1.064 | 1.325 | 1.725 | 2.086 |
| 21 | 0.257 | 0.532 | 0.859 | 1.063 | 1.323 | 1.721 | 2.080 |
| 22 | 0.256 | 0.532 | 0.858 | 1.061 | 1.321 | 1.717 | 2.074 |
| 23 | 0.256 | 0.532 | 0.858 | 1.060 | 1.319 | 1.714 | 2.069 |
| 24 | 0.256 | 0.531 | 0.857 | 1.059 | 1.318 | 1.711 | 2.064 |
| 25 | 0.256 | 0.531 | 0.856 | 1.058 | 1.316 | 1.708 | 2.060 |
| 26 | 0.256 | 0.531 | 0.856 | 1.058 | 1.315 | 1.706 | 2.056 |
| 27 | 0.256 | 0.531 | 0.855 | 1.057 | 1.314 | 1.703 | 2.052 |
| 28 | 0.256 | 0.530 | 0.855 | 1.056 | 1.313 | 1.701 | 2.048 |
| 29 | 0.256 | 0.530 | 0.854 | 1.055 | 1.311 | 1.699 | 2.045 |
| 30 | 0.256 | 0.530 | 0.854 | 1.055 | 1.310 | 1.697 | 2.042 |
| 40 | 0.255 | 0.529 | 0.851 | 1.050 | 1.303 | 1.684 | 2.021 |
| 60 | 0.254 | 0.527 | 0.848 | 1.045 | 1.296 | 1.671 | 2.000 |
| 120 | 0.254 | 0.526 | 0.845 | 1.041 | 1.289 | 1.658 | 1.980 |
| ∞ | 0.253 | 0.524 | 0.842 | 1.036 | 1.282 | 1.645 | 1.960 |

$t_\alpha$-Critical Values of the $t$-Distribution

| | $\alpha$ | | | | | | |
|---|---|---|---|---|---|---|---|
| $\nu$ | 0.02 | 0.015 | 0.01 | 0.0075 | 0.005 | 0.0025 | 0.0005 |
| 1 | 15.895 | 21.205 | 31.821 | 42.434 | 63.657 | 127.322 | 636.590 |
| 2 | 4.849 | 5.643 | 6.965 | 8.073 | 9.925 | 14.089 | 31.598 |
| 3 | 3.482 | 3.896 | 4.541 | 5.047 | 5.841 | 7.453 | 12.924 |
| 4 | 2.999 | 3.298 | 3.747 | 4.088 | 4.604 | 5.598 | 8.610 |
| 5 | 2.757 | 3.003 | 3.365 | 3.634 | 4.032 | 4.773 | 6.869 |
| 6 | 2.612 | 2.829 | 3.143 | 3.372 | 3.707 | 4.317 | 5.959 |
| 7 | 2.517 | 2.715 | 2.998 | 3.203 | 3.499 | 4.029 | 5.408 |
| 8 | 2.449 | 2.634 | 2.896 | 3.085 | 3.355 | 3.833 | 5.041 |
| 9 | 2.398 | 2.574 | 2.821 | 2.998 | 3.250 | 3.690 | 4.781 |
| 10 | 2.359 | 2.527 | 2.764 | 2.932 | 3.169 | 3.581 | 4.587 |
| 11 | 2.328 | 2.491 | 2.718 | 2.879 | 3.106 | 3.497 | 4.437 |
| 12 | 2.303 | 2.461 | 2.681 | 2.836 | 3.055 | 3.428 | 4.318 |
| 13 | 2.282 | 2.436 | 2.650 | 2.801 | 3.012 | 3.372 | 4.221 |
| 14 | 2.264 | 2.415 | 2.624 | 2.771 | 2.977 | 3.326 | 4.140 |
| 15 | 2.249 | 2.397 | 2.602 | 2.746 | 2.947 | 3.286 | 4.073 |
| 16 | 2.235 | 2.382 | 2.583 | 2.724 | 2.921 | 3.252 | 4.015 |
| 17 | 2.224 | 2.368 | 2.567 | 2.706 | 2.898 | 3.222 | 3.965 |
| 18 | 2.214 | 2.356 | 2.552 | 2.689 | 2.878 | 3.197 | 3.922 |
| 19 | 2.205 | 2.346 | 2.539 | 2.674 | 2.861 | 3.174 | 3.883 |
| 20 | 2.197 | 2.336 | 2.528 | 2.661 | 2.845 | 3.153 | 3.849 |
| 21 | 2.189 | 2.328 | 2.518 | 2.649 | 2.831 | 3.135 | 3.819 |
| 22 | 2.183 | 2.320 | 2.508 | 2.639 | 2.819 | 3.119 | 3.792 |
| 23 | 2.177 | 2.313 | 2.500 | 2.629 | 2.807 | 3.104 | 3.768 |
| 24 | 2.172 | 2.307 | 2.492 | 2.620 | 2.797 | 3.091 | 3.745 |
| 25 | 2.167 | 2.301 | 2.485 | 2.612 | 2.787 | 3.078 | 3.725 |
| 26 | 2.162 | 2.296 | 2.479 | 2.605 | 2.779 | 3.067 | 3.707 |
| 27 | 2.158 | 2.291 | 2.473 | 2.598 | 2.771 | 3.057 | 3.690 |
| 28 | 2.154 | 2.286 | 2.467 | 2.592 | 2.763 | 3.047 | 3.674 |
| 29 | 2.150 | 2.282 | 2.462 | 2.586 | 2.756 | 3.038 | 3.659 |
| 30 | 2.147 | 2.278 | 2.457 | 2.581 | 2.750 | 3.030 | 3.646 |
| 40 | 2.125 | 2.250 | 2.423 | 2.542 | 2.704 | 2.971 | 3.551 |
| 60 | 2.099 | 2.223 | 2.390 | 2.504 | 2.660 | 2.915 | 3.460 |
| 120 | 2.076 | 2.196 | 2.358 | 2.468 | 2.617 | 2.860 | 3.373 |
| $\infty$ | 2.054 | 2.170 | 2.326 | 2.432 | 2.576 | 2.807 | 3.291 |

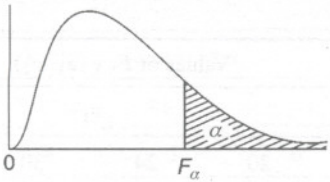



ritical Values of the Chi-squared Distribution

| | | | | α | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 995 | 0.99 | 0.98 | 0.975 | 0.95 | 0.90 | 0.80 | 0.75 | 0.75 | 0.50 |
| 04393 | 0.03157 | 0.03628 | 0.03982 | 0.00393 | 0.0158 | 0.0642 | 0.102 | 0.148 | 0.455 |
| 0100 | 0.0201 | 0.0404 | 0.0506 | 0.103 | 0.211 | 0.446 | 0.575 | 0.713 | 1.386 |
| 0717 | 0.115 | 0.185 | 0.216 | 0.352 | 0.584 | 1.005 | 1.213 | 1.424 | 2.366 |
| 207 | 0.297 | 0.429 | 0.484 | 0.711 | 1.064 | 1.649 | 1.923 | 2.195 | 3.357 |
| 412 | 0.554 | 0.752 | 0.831 | 1.145 | 1.610 | 2.343 | 2.675 | 3.000 | 4.351 |
| 676 | 0.872 | 1.134 | 1.237 | 1.635 | 2.204 | 3.070 | 3.455 | 3.828 | 5.348 |
| 989 | 1.239 | 1.564 | 1.690 | 2.167 | 2.833 | 3.822 | 4.255 | 4.671 | 6.346 |
| 344 | 1.646 | 2.032 | 2.180 | 2.733 | 3.490 | 4.594 | 5.071 | 5.527 | 7.344 |
| 735 | 2.088 | 2.532 | 2.700 | 3.325 | 4.168 | 5.380 | 5.899 | 6.393 | 8.343 |
| 156 | 2.558 | 3.059 | 3.247 | 3.940 | 4.865 | 6.179 | 6.737 | 7.267 | 9.342 |
| 603 | 3.053 | 3.609 | 3.816 | 4.575 | 5.578 | 6.989 | 7.584 | 8.148 | 10.341 |
| 074 | 3.571 | 4.178 | 4.404 | 5.226 | 6.304 | 7.807 | 8.438 | 9.034 | 11.340 |
| 565 | 4.107 | 4.765 | 5.009 | 5.892 | 7.042 | 8.634 | 9.299 | 9.926 | 12.340 |
| 075 | 4.660 | 5.368 | 5.629 | 6.571 | 7.790 | 9.467 | 10.165 | 10.821 | 13.339 |
| 601 | 5.229 | 5.985 | 6.262 | 7.261 | 8.547 | 10.307 | 11.036 | 11.721 | 14.339 |
| 142 | 5.812 | 6.614 | 6.908 | 7.962 | 9.312 | 11.152 | 11.912 | 12.624 | 15.338 |
| 697 | 6.408 | 7.255 | 7.564 | 8.672 | 10.085 | 12.002 | 12.792 | 13.531 | 16.338 |
| 265 | 7.015 | 7.906 | 8.231 | 9.390 | 10.865 | 12.857 | 13.675 | 14.440 | 17.338 |
| 844 | 7.633 | 8.567 | 8.907 | 10.117 | 11.651 | 13.716 | 14.562 | 15.352 | 18.338 |
| 434 | 8.260 | 9.237 | 9.591 | 10.851 | 12.443 | 14.578 | 15.452 | 16.266 | 19.337 |
| 034 | 8.897 | 9.915 | 10.283 | 11.591 | 13.240 | 15.445 | 16.344 | 17.182 | 20.337 |
| 643 | 9.542 | 10.600 | 10.982 | 12.338 | 14.041 | 16.314 | 17.240 | 18.101 | 21.337 |
| 260 | 10.196 | 11.293 | 11.688 | 13.091 | 14.848 | 17.187 | 18.137 | 19.021 | 22.337 |
| 886 | 10.856 | 11.992 | 12.401 | 13.848 | 15.659 | 18.062 | 19.037 | 19.943 | 23.337 |
| 520 | 11.524 | 12.697 | 13.120 | 14.611 | 16.473 | 18.940 | 19.939 | 20.867 | 24.337 |
| 60 | 12.198 | 13.409 | 13.844 | 15.379 | 17.292 | 19.820 | 20.843 | 21.792 | 25.336 |
| 308 | 12.879 | 14.125 | 14.573 | 16.151 | 18.114 | 20.703 | 21.749 | 22.719 | 26.336 |
| 461 | 13.565 | 14.847 | 15.308 | 16.928 | 18.939 | 21.588 | 22.657 | 23.647 | 27.336 |
| 21 | 14.256 | 15.574 | 16.047 | 17.708 | 19.768 | 22.475 | 23.567 | 24.577 | 28.336 |
| 787 | 14.953 | 16.306 | 16.791 | 18.493 | 20.599 | 23.364 | 24.478 | 25.508 | 29.336 |

$\chi^2_\alpha$-Critical Values of the Chi-squared Distribution

| | $\alpha$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\nu$ | 0.30 | 0.25 | 0.20 | 0.10 | 0.05 | 0.025 | 0.02 | 0.01 | 0.005 | 0.001 |
| 1 | 1.074 | 1.323 | 1.642 | 2.706 | 3.841 | 5.024 | 5.412 | 6.635 | 7.879 | 10.827 |
| 2 | 2.408 | 2.773 | 3.219 | 4.605 | 5.991 | 7.378 | 7.824 | 9.210 | 10.597 | 13.815 |
| 3 | 3.665 | 4.108 | 4.642 | 6.251 | 7.815 | 9.348 | 9.837 | 11.345 | 12.838 | 16.268 |
| 4 | 4.878 | 5.385 | 5.989 | 7.779 | 9.488 | 11.143 | 11.668 | 13.277 | 14.860 | 18.465 |
| 5 | 6.064 | 6.626 | 7.289 | 9.236 | 11.070 | 12.832 | 13.388 | 15.086 | 16.750 | 20.517 |
| 6 | 7.231 | 7.841 | 8.558 | 10.645 | 12.592 | 14.449 | 15.033 | 16.812 | 18.548 | 22.457 |
| 7 | 8.383 | 9.037 | 9.803 | 12.017 | 14.067 | 16.013 | 16.622 | 18.475 | 20.278 | 24.322 |
| 8 | 9.524 | 10.219 | 11.030 | 13.362 | 15.507 | 17.535 | 18.168 | 20.090 | 21.955 | 26.125 |
| 9 | 10.656 | 11.389 | 12.242 | 14.684 | 16.919 | 19.023 | 19.679 | 21.666 | 23.589 | 27.877 |
| 10 | 11.781 | 12.549 | 13.442 | 15.987 | 18.307 | 20.483 | 21.161 | 23.209 | 25.188 | 29.588 |
| 11 | 12.899 | 13.701 | 14.631 | 17.275 | 19.675 | 21.920 | 22.618 | 24.725 | 26.757 | 31.264 |
| 12 | 14.011 | 14.845 | 15.812 | 18.549 | 21.026 | 23.337 | 24.054 | 26.217 | 28.300 | 32.909 |
| 13 | 15.119 | 15.984 | 16.985 | 19.812 | 22.362 | 24.736 | 25.472 | 27.688 | 29.819 | 34.528 |
| 14 | 16.222 | 17.117 | 18.151 | 21.064 | 23.685 | 26.119 | 26.873 | 29.141 | 31.319 | 36.123 |
| 15 | 17.322 | 18.245 | 19.311 | 22.307 | 24.996 | 27.488 | 28.259 | 30.578 | 32.801 | 37.697 |
| 16 | 18.418 | 19.369 | 20.465 | 23.542 | 26.296 | 28.845 | 29.633 | 32.000 | 34.267 | 39.252 |
| 17 | 19.511 | 20.489 | 21.615 | 24.769 | 27.587 | 30.191 | 30.995 | 33.409 | 35.718 | 40.790 |
| 18 | 20.601 | 21.605 | 22.760 | 25.989 | 28.869 | 31.526 | 32.346 | 34.805 | 37.156 | 42.312 |
| 19 | 21.689 | 22.718 | 23.900 | 27.204 | 30.144 | 32.852 | 33.687 | 36.191 | 38.582 | 43.820 |
| 20 | 22.775 | 23.828 | 25.038 | 28.412 | 31.410 | 34.170 | 35.020 | 37.566 | 39.997 | 45.315 |
| 21 | 23.858 | 24.935 | 26.171 | 29.615 | 32.671 | 35.479 | 36.343 | 38.932 | 41.401 | 46.797 |
| 22 | 24.939 | 26.039 | 27.301 | 30.813 | 33.924 | 36.781 | 37.659 | 40.289 | 42.796 | 48.268 |
| 23 | 26.018 | 27.141 | 28.429 | 32.007 | 35.172 | 38.076 | 38.968 | 41.638 | 44.181 | 49.728 |
| 24 | 27.096 | 28.241 | 29.553 | 33.196 | 36.415 | 39.364 | 40.270 | 42.980 | 45.558 | 51.179 |
| 25 | 28.172 | 29.339 | 30.675 | 34.382 | 37.652 | 40.646 | 41.566 | 44.314 | 46.928 | 52.620 |
| 26 | 29.246 | 30.434 | 31.795 | 35.563 | 38.885 | 41.923 | 42.856 | 45.642 | 48.290 | 54.052 |
| 27 | 30.319 | 31.528 | 32.912 | 36.741 | 40,113 | 43.194 | 44.140 | 46.963 | 49.645 | 55.476 |
| 28 | 31.391 | 32.620 | 34.027 | 37.916 | 41,337 | 44.461 | 45.419 | 48.278 | 50.993 | 56.893 |
| 29 | 32.461 | 33.711 | 35.139 | 39.087 | 42.557 | 45.722 | 46.693 | 49.588 | 52.336 | 58.302 |
| 30 | 33.530 | 34.800 | 36.250 | 40.256 | 43.773 | 46.979 | 47.962 | 50.892 | 53.672 | 59.703 |




6. Critical Values of the *F*-Distribution

Values of $F_{0.05}(\nu_1, \nu_2)$

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 161.4 | 199.5 | 215.7 | 224.6 | 230.2 | 234.0 | 236.8 | 238.9 | 240.5 |
| 2 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 |
| 3 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 |
| 13 | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 |
| 21 | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 |
| 22 | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 |
| 23 | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 |
| 24 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 |
| 25 | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 |
| 26 | 4.23 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 |
| 27 | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 |
| 28 | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 |
| 29 | 4.18 | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 |
| 60 | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 |
| 120 | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.17 | 2.09 | 2.02 | 1.96 |
| ∞ | 3.84 | 3.00 | 2.60 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 |

Critical Values of the *F*-Distribution

Values of $F_{0.05}(\nu_1, \nu_2)$

| | $\nu_1$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\nu_2$ | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | $\infty$ |
| 1 | 241.9 | 243.9 | 245.9 | 248.0 | 249.1 | 250.1 | 251.1 | 252.2 | 253.3 | 254.3 |
| 2 | 19.40 | 19.41 | 19.43 | 19.45 | 19.45 | 19.46 | 19.47 | 19.48 | 19.49 | 19.50 |
| 3 | 8.79 | 8.74 | 8.70 | 8.66 | 8.64 | 8.62 | 8.59 | 8.57 | 8.55 | 8.53 |
| 4 | 5.96 | 5.91 | 5.86 | 5.80 | 5.77 | 5.75 | 5.72 | 5.69 | 5.66 | 5.63 |
| 5 | 4.74 | 4.68 | 4.62 | 4.56 | 4.53 | 4.50 | 4.46 | 4.43 | 4.40 | 4.36 |
| 6 | 4.06 | 4.00 | 3.94 | 3.87 | 3.84 | 3.81 | 3.77 | 3.74 | 3.70 | 3.67 |
| 7 | 3.64 | 3.57 | 3.51 | 3.44 | 3.41 | 3.38 | 3.34 | 3.30 | 3.27 | 3.23 |
| 8 | 3.35 | 3.28 | 3.22 | 3.15 | 3.12 | 3.08 | 3.04 | 3.01 | 2.97 | 2.93 |
| 9 | 3.14 | 3.07 | 3.01 | 2.94 | 2.90 | 2.86 | 2.83 | 2.79 | 2.75 | 2.71 |
| 10 | 2.98 | 2.91 | 2.85 | 2.77 | 2.74 | 2.70 | 2.66 | 2.62 | 2.58 | 2.54 |
| 11 | 2.85 | 2.79 | 2.72 | 2.65 | 2.61 | 2.57 | 2.53 | 2.49 | 2.45 | 2.40 |
| 12 | 2.75 | 2.69 | 2.62 | 2.54 | 2.51 | 2.47 | 2.43 | 2.38 | 2.34 | 2.30 |
| 13 | 2.67 | 2.60 | 2.53 | 2.46 | 2.42 | 2.38 | 2.34 | 2.30 | 2.25 | 2.21 |
| 14 | 2.60 | 2.53 | 2.46 | 2.39 | 2.35 | 2.31 | 2.27 | 2.22 | 2.18 | 2.13 |
| 15 | 2.54 | 2.48 | 2.40 | 2.33 | 2.29 | 2.25 | 2.20 | 2.16 | 2.11 | 2.07 |
| 16 | 2.49 | 2.42 | 2.35 | 2.28 | 2.24 | 2.19 | 2.15 | 2.11 | 2.06 | 2.01 |
| 17 | 2.45 | 2.38 | 2.31 | 2.23 | 2.19 | 2.15 | 2.10 | 2.06 | 2.01 | 1.96 |
| 18 | 2.41 | 2.34 | 2.27 | 2.19 | 2.15 | 2.11 | 2.06 | 2.02 | 1.97 | 1.92 |
| 19 | 2.38 | 2.31 | 2.23 | 2.16 | 2.11 | 2.07 | 2.03 | 1.98 | 1.93 | 1.88 |
| 20 | 2.35 | 2.28 | 2.20 | 2.12 | 2.08 | 2.04 | 1.99 | 1.95 | 1.90 | 1.84 |
| 21 | 2.32 | 2.25 | 2.18 | 2.10 | 2.05 | 2.01 | 1.96 | 1.92 | 1.87 | 1.81 |
| 22 | 2.30 | 2.23 | 2.15 | 2.07 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.78 |
| 23 | 2.27 | 2.20 | 2.13 | 2.05 | 2.01 | 1.96 | 1.91 | 1.86 | 1.81 | 1.76 |
| 24 | 2.25 | 2.18 | 2.11 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.79 | 1.73 |
| 25 | 2.24 | 2.16 | 2.09 | 2.01 | 1.96 | 1.92 | 1.87 | 1.82 | 1.77 | 1.71 |
| 26 | 2.22 | 2.15 | 2.07 | 1.99 | 1.95 | 1.90 | 1.85 | 1.80 | 1.75 | 1.69 |
| 27 | 2.20 | 2.13 | 2.06 | 1.97 | 1.93 | 1.88 | 1.84 | 1.79 | 1.73 | 1.67 |
| 28 | 2.19 | 2.12 | 2.04 | 1.96 | 1.91 | 1.87 | 1.82 | 1.77 | 1.71 | 1.65 |
| 29 | 2.18 | 2.10 | 2.03 | 1.94 | 1.90 | 1.85 | 1.81 | 1.75 | 1.70 | 1.64 |
| 30 | 2.16 | 2.09 | 2.01 | 1.93 | 1.89 | 1.84 | 1.79 | 1.75 | 1.68 | 1.62 |
| 40 | 2.08 | 2.00 | 1.92 | 1.84 | 1.79 | 1.74 | 1.69 | 1.64 | 1.58 | 1.51 |
| 60 | 1.99 | 1.92 | 1.84 | 1.75 | 1.70 | 1.65 | 1.59 | 1.53 | 1.47 | 1.39 |
| 120 | 1.91 | 1.83 | 1.75 | 1.66 | 1.61 | 1.55 | 1.50 | 1.43 | 1.35 | 1.25 |
| $\infty$ | 1.83 | 1.75 | 1.67 | 1.57 | 1.52 | 1.46 | 1.39 | 1.32 | 1.22 | 1.00 |

ritical Values of the *F*-Distribution

Values of $F_{0.01}(\nu_1, \nu_2)$

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 4052 | 4999.5 | 5403 | 5625 | 5764 | 5859 | 5928 | 5981 | 6022 |
| 2 | 98.50 | 99.00 | 99.17 | 99.25 | 99.30 | 99.33 | 99.36 | 99.37 | 99.39 |
| 3 | 34.12 | 30.82 | 29.46 | 28.71 | 28.24 | 27.91 | 27.67 | 27.49 | 27.35 |
| 4 | 21.20 | 18.00 | 16.69 | 15.98 | 15.52 | 15.21 | 14.98 | 14.80 | 14.66 |
| 5 | 16.26 | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.46 | 10.29 | 10.16 |
| 6 | 13.75 | 10.92 | 9.78 | 9.15 | 8.75 | 8.47 | 8.26 | 8.10 | 7.98 |
| 7 | 12.25 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.99 | 6.84 | 6.72 |
| 8 | 11.26 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.18 | 6.03 | 5.91 |
| 9 | 10.56 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.61 | 5.47 | 5.35 |
| 10 | 10.04 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.20 | 5.06 | 4.94 |
| 11 | 9.65 | 7.21 | 6.22 | 5.67 | 5.32 | 5.07 | 4.89 | 4.74 | 4.63 |
| 12 | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.64 | 4.50 | 4.39 |
| 13 | 9.07 | 6.70 | 5.74 | 5.21 | 4.86 | 4.62 | 4.44 | 4.30 | 4.19 |
| 14 | 8.86 | 6.51 | 5.56 | 5.04 | 4.69 | 4.46 | 4.28 | 4.14 | 4.03 |
| 15 | 8.68 | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.14 | 4.00 | 3.89 |
| 16 | 8.53 | 6.23 | 5.29 | 4.77 | 4.44 | 4.20 | 4.03 | 3.89 | 3.78 |
| 17 | 8.40 | 6.11 | 5.18 | 4.67 | 4.34 | 4.10 | 3.93 | 3.79 | 3.68 |
| 18 | 8.29 | 6.01 | 5.09 | 4.58 | 4.25 | 4.01 | 3.84 | 3.71 | 3.60 |
| 19 | 8.18 | 5.93 | 5.01 | 4.50 | 4.17 | 3.94 | 3.77 | 3.63 | 3.52 |
| 20 | 8.10 | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.70 | 3.56 | 3.46 |
| 21 | 8.02 | 5.78 | 4.87 | 4.37 | 4.04 | 3.81 | 3.64 | 3.51 | 3.40 |
| 22 | 7.95 | 5.72 | 4.82 | 4.31 | 3.99 | 3.76 | 3.59 | 3.45 | 3.35 |
| 23 | 7.88 | 5.66 | 4.76 | 4.26 | 3.94 | 3.71 | 3.54 | 3.41 | 3.30 |
| 24 | 7.82 | 5.61 | 4.72 | 4.22 | 3.90 | 3.67 | 3.50 | 3.36 | 3.26 |
| 25 | 7.77 | 5.57 | 4.68 | 4.18 | 3.85 | 3.63 | 3.46 | 3.32 | 3.22 |
| 26 | 7.72 | 5.53 | 4.64 | 4.14 | 3.82 | 3.59 | 3.42 | 3.29 | 3.18 |
| 27 | 7.68 | 5.49 | 4.60 | 4.11 | 3.78 | 3.56 | 3.39 | 3.26 | 3.15 |
| 28 | 7.64 | 5.45 | 4.57 | 4.07 | 3.75 | 3.53 | 3.36 | 3.23 | 3.12 |
| 29 | 7.60 | 5.42 | 4.54 | 4.04 | 3.73 | 3.50 | 3.33 | 3.20 | 3.09 |
| 30 | 7.56 | 5.39 | 4.51 | 4.02 | 3.70 | 3.47 | 3.30 | 3.17 | 3.07 |
| 40 | 7.31 | 5.18 | 4.31 | 3.83 | 3.51 | 3.29 | 3.12 | 2.99 | 2.89 |
| 60 | 7.08 | 4.98 | 4.13 | 3.65 | 3.34 | 3.12 | 2.95 | 2.82 | 2.72 |
| 120 | 6.85 | 4.79 | 3.95 | 3.48 | 3.17 | 2.96 | 3.79 | 2.66 | 2.56 |
| $\infty$ | 6.63 | 4.61 | 3.78 | 3.32 | 3.02 | 2.80 | 2.64 | 2.51 | 2.41 |

Critical Values of the *F*-Distribution

Values of $F_{0.01}(\nu_1, \nu_2)$

| $\nu_2$ \ $\nu_1$ | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6056 | 6106 | 6157 | 6209 | 6235 | 6261 | 6287 | 6313 | 6339 | 6366 |
| 2 | 99.40 | 99.42 | 99.43 | 99.45 | 99.46 | 99.47 | 99.47 | 99.48 | 99.49 | 99.50 |
| 3 | 27.23 | 27.05 | 26.87 | 26.69 | 26.60 | 26.50 | 26.41 | 26.32 | 26.22 | 26.13 |
| 4 | 14.55 | 14.37 | 14.20 | 14.02 | 13.93 | 13.84 | 13.75 | 13.65 | 13.56 | 13.46 |
| 5 | 10.05 | 9.89 | 9.72 | 9.55 | 9.47 | 9.38 | 9.29 | 9.20 | 9.11 | 9.02 |
| 6 | 7.87 | 7.72 | 7.56 | 7.40 | 7.31 | 7.23 | 7.14 | 7.06 | 6.97 | 6.88 |
| 7 | 6.62 | 6.47 | 6.31 | 6.16 | 6.07 | 5.99 | 5.91 | 5.82 | 5.74 | 5.65 |
| 8 | 5.81 | 5.67 | 5.52 | 5.36 | 5.28 | 5.20 | 5.12 | 5.03 | 4.95 | 4.86 |
| 9 | 5.26 | 5.11 | 4.96 | 4.81 | 4.73 | 4.65 | 4.57 | 4.48 | 4.40 | 4.31 |
| 10 | 4.85 | 4.71 | 4.56 | 4.41 | 4.33 | 4.25 | 4.17 | 4.08 | 4.00 | 3.91 |
| 11 | 4.54 | 4.40 | 4.25 | 4.10 | 4.02 | 3.94 | 3.86 | 3.78 | 3.69 | 3.60 |
| 12 | 4.30 | 4.16 | 4.01 | 3.86 | 3.78 | 3.70 | 3.62 | 3.54 | 3.45 | 3.36 |
| 13 | 4.10 | 3.96 | 3.82 | 3.66 | 3.59 | 3.51 | 3.43 | 3.34 | 3.25 | 3.17 |
| 14 | 3.94 | 3.80 | 3.66 | 3.51 | 3.43 | 3.35 | 3.27 | 3.18 | 3.09 | 3.00 |
| 15 | 3.80 | 3.67 | 3.52 | 3.37 | 3.29 | 3.21 | 3.13 | 3.05 | 2.96 | 2.87 |
| 16 | 3.69 | 3.55 | 3.41 | 3.26 | 3.18 | 3.10 | 3.02 | 2.93 | 2.84 | 2.75 |
| 17 | 3.59 | 3.46 | 3.31 | 3.16 | 3.08 | 3.00 | 2.92 | 2.83 | 2.75 | 2.65 |
| 18 | 3.51 | 3.37 | 3.23 | 3.08 | 3.00 | 2.92 | 2.84 | 2.75 | 2.66 | 2.57 |
| 19 | 3.43 | 3.30 | 3.15 | 3.00 | 2.92 | 2.84 | 2.76 | 2.67 | 2.58 | 2.49 |
| 20 | 3.37 | 3.23 | 3.09 | 2.94 | 2.86 | 2.78 | 2.69 | 2.61 | 2.52 | 2.42 |
| 21 | 3.31 | 3.17 | 3.03 | 2.88 | 2.80 | 2.72 | 2.64 | 2.55 | 2.46 | 2.36 |
| 22 | 3.26 | 3.12 | 2.98 | 2.83 | 2.75 | 2.67 | 2.58 | 2.50 | 2.40 | 2.31 |
| 23 | 3.21 | 3.07 | 2.93 | 2.78 | 2.70 | 2.62 | 2.54 | 2.45 | 2.35 | 2.26 |
| 24 | 3.17 | 3.03 | 2.89 | 2.74 | 2.66 | 2.58 | 2.49 | 2.40 | 2.31 | 2.21 |
| 25 | 3.13 | 2.99 | 2.85 | 2.70 | 2.62 | 2.54 | 2.45 | 2.36 | 2.27 | 2.17 |
| 26 | 3.09 | 2.96 | 2.81 | 2.66 | 2.58 | 2.50 | 2.42 | 2.33 | 2.23 | 2.13 |
| 27 | 3.06 | 2.93 | 2.78 | 2.63 | 2.55 | 2.47 | 2.38 | 2.29 | 2.20 | 2.10 |
| 28 | 3.03 | 2.90 | 2.75 | 2.60 | 2.52 | 2.44 | 2.35 | 2.26 | 2.17 | 2.06 |
| 29 | 3.00 | 2.87 | 2.73 | 2.57 | 2.49 | 2.41 | 2.33 | 2.23 | 2.14 | 2.03 |
| 30 | 2.98 | 2.84 | 2.70 | 2.55 | 2.47 | 2.39 | 2.30 | 2.21 | 2.11 | 2.01 |
| 40 | 2.80 | 2.66 | 2.52 | 2.37 | 2.29 | 2.20 | 2.11 | 2.02 | 1.92 | 1.80 |
| 60 | 2.63 | 2.50 | 2.35 | 2.20 | 2.12 | 2.03 | 1.94 | 1.84 | 1.73 | 1.60 |
| 120 | 2.47 | 2.34 | 2.19 | 2.03 | 1.95 | 1.86 | 1.76 | 1.66 | 1.53 | 1.38 |
| ∞ | 2.32 | 2.18 | 2.04 | 1.88 | 1.79 | 1.70 | 1.59 | 1.47 | 1.32 | 1.00 |