UNIT-VIII : QUEUING THEORY

INTRODUCTION: "Queuing" is a phenomenon that we come across in our day to day life. For example,

* In a railway station many times we wait in a queue to buy a ticket.

* In a Barber Shop people wait to get Served by the Barber.

* In a T.V machanics shop T.V sets wait in queue for being served by mechanic.

* In a Cinema theatre people wait in queue to get tickets atleast in the intial days of Release of a good movie.

* In a office, letters, drafts written by higher authorities wait for the typist to get typed

* In a car Servicing centre, cars wait in a queue to get serviced.

In all these examples we come across individuals either humanbeings or T.V sets or letters ....etc who wait to get served are called customers and the individuals who serves these customers is called a server.

→ A queuing system can be completely described by

(i) The way in which queue is formed

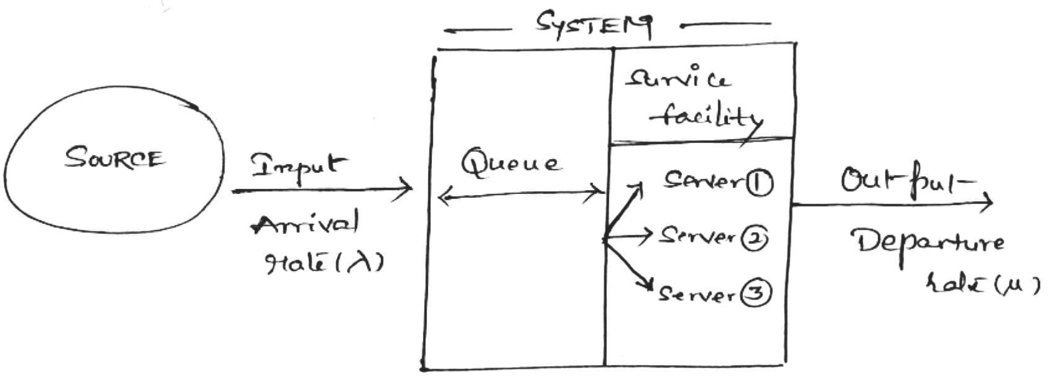
(ii) The way in which the server serves

(iii) The queue discipline

(iv) The queue behaviour.

⟹ The way in which queue is formed is also referred to as the input pattern (or) arrival pattern. Usually customers arrive at a service centre in a random way. The arrival of customers is in general probabilistic or stochastic. Hence whenever we intend to study a queueing system we must know about the probability distribution of the inter arrival time (i.e) the probability distribution of the time gap b/w two successive arrivals. In this chapter we will deal only with the queuing systems in which customers arrive in "poisson" fashion.

⟹ Service pattern: This is also refferred to as service mechanism. It is possible that there may be one server or many servers to serve the customers in queue. The time taken to serve a customer by the server is referred to as service time and this is in general a random variable. The service time distribution that we consider in this chapter is taken as "Negative exponential distribution".



⇒ The queue discipline :- This concerns the way in which customers in the queue are served. The usual type that we come across methodically is "First Come First Served". The queue discipline may be "Last in First out". Sometimes the queue discipline may be random. For example, in the case of Godmen, people will be waiting to have a Darshan in person. They wave at few spotted individuals at random. While many will be waiting, some are picked at random for special treatment. Here the queue discipline may be to handle "service at random".

Sometimes the service may be on priority basis. In a Doctor's clinic while some patients are waiting for service, one among them whose condition is serious may be allowed into the Doctor's chamber.

⇒ Queue behaviour :- The customers behaviour in the queue is nothing but Queue behaviour, which can be classified as follows influences the customer arrivals.

* A customer behaviour is referred to as balking if a customer may not enter the queue in view of its length.

* Sometimes a customer, who is in the queue for a longtime, due to impatience may leave the queue. This customer behaviour is referred to as Reneging.

* In the case of two parallel queues, it is possible that a customers who is in one queue may leave the queue and join another parallel queue. This process is called jockeying.

NOTE: In this chapter, we deal with "FIFO" queue discipline when there is no balking, no reneging and no jockeying.

* Some times, if the server is ready to serve, but there is no customer in the system, the server will be idle. The period during which the server is idle is called the ideal time of the server.

Nomenclature (Symbols and Notations)

n – Number of customers in the system including the one that is served.

m – Number of customers in the queue excluding the one that is served.

$P_n(t)$ – Probability that there are 'n' customers in the system at time 't'.

P_n – Probability that there are 'n' customers in the system at any time.

λ – Number of arrivals of customers per unit time

μ – Number of services of customers per unit time

$$\rho = \frac{\lambda}{\mu} = \text{Traffic Intensity of the System}$$
$$= \text{Utilization factor of the Server.}$$

$L_s = E(n) =$ average number of customers in the system

$L_q = E(m) =$ average number of customers in the queue.

W_s - Average waiting time in the System.

W_q - Average waiting time in the queue.

PURE BIRTH AND DEATH PROCESS:

In Queueing systems; Each arrival is treated as birth, each departure is treated as death. Here a queueing model consisting both arrival and departure is called pure birth and Death process.

Now we are going study about the following Models.

(i) $(M/M/1):(\infty/FIFO)$ Model

(ii) $(M/M/1):(N/FIFO)$ Model.

Here first 'M' denotes that the arrivals, second 'M' denotes that the departures, and '1' denotes there is a single server. '∞' denotes that the arrivals are from an infinite population and there is no limit for the people that are admitted into the System. FIFO - discribes the queue discipline = "First In First out".

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(i) $(M/M/1):(\infty/FIFO)$ MODEL :-

The Measures involving in the $(M/M/1):(\infty/FIFO)$ Model are given below.

1. Probability that service channel is busy (Traffic intensity)

$$\Rightarrow \quad \rho = \frac{\lambda}{\mu}, \quad \text{where } \lambda = \text{No. of arrivals per unit time}$$
$$\mu = \text{No. of services per unit time.}$$

2. Probability that there are no customers in the System

$$\Rightarrow \quad P_0 = 1 - \rho$$

3. Probability that there are 'n' customers in the System

$$\Rightarrow \quad P_n = (1-\rho)\rho^n.$$

4. Average number of customers in the System.

$$\Rightarrow L_s = E(n) = \frac{\rho}{1-\rho} \quad (\text{or}) \quad \frac{\lambda}{\mu - \lambda}.$$

5. Average number of customers in the Queue

$$\Rightarrow L_q = E(m) = \frac{\rho^2}{1-\rho}$$

6. Average waiting time a customer in the System (including Service time)

$$\Rightarrow \quad W_s = \frac{1}{\mu(1-\rho)}$$

7. Average Waiting-time a customer in the queue (Excluding Service time)

$$\Rightarrow \quad W_q = \frac{\rho}{\mu(1-\rho)}$$

8. The probability of the waiting time exceeds 'w_0' in the System is given by,

$$P(W > w_0) = e^{-(\mu-\lambda) w_0}$$

9. The probability of the waiting-time exceeds 'w_0' in the Queue is given by,

$$P(W > w_0) = \rho \cdot e^{-(\mu-\lambda) w_0}$$

where $\rho = \frac{\lambda}{\mu}$.

$\Rightarrow$ $\boxed{\text{NOTE: System = Queue + 1}}$

Problems:-

1. A self service canteen employs one cashier. 8 customers arrive per every 10 minutes on an average. The cashier can serve on average one per minute. Assuming that the arrivals are poisson and the service time distribution is exponential, determine (i) The average number of customers in the system (ii) The average queue length (iii) average time a customer spends in the system. (iv) average waiting-time of each customer.

Sol: Here 8 customers arrive per every 10 minutes

Hence average arrival rate $\lambda = 8/10$ customers/minute

Average service rate $\mu = 1$ customer/minute.

Traffic intensity $\rho = \frac{\lambda}{\mu} = \frac{8/10}{1} = \frac{4}{5}$

(i) Average number of customers in the System

$$\Rightarrow L_s = E(n) = \frac{\rho}{1-\rho} = \frac{4/5}{1-4/5} = \underline{\underline{4}}$$

(ii) Average Queue Length

$$\Rightarrow L_q = E(m) = \frac{\rho^2}{1-\rho} = \frac{(4/5)^2}{1-4/5} = \frac{16}{5} = \underline{\underline{3.2}}$$

(iii) Average time a customer Spends in the System

$$\Rightarrow W_s = \frac{1}{\mu(1-\rho)}$$

$$= \frac{1}{1(1-4/5)} = 5 \text{ minutes.}$$

(iv) Average waiting time of each customer (in queue)

$$\Rightarrow W_q = \frac{\rho}{\mu(1-\rho)} = \frac{4/5}{1(1-4/5)} = 4 \text{ minutes.}$$

<u>NOTE:</u> Since Every <u>measure</u> (formulae) is depending upon the values "λ" and "μ" So first findout the arrival rate and no of services per single unit time.

Pb ②: A T.V repair man finds that the time spent on his jobs has an exponential distribution with mean 30 minutes, He repairs sets in the order in which they arrive. The arrival of the sets is approximately Poisson with an average of 10 per an eight hour day. Find the repairman's idle time each day. How many jobs are ahead of the average set just brought in ?. 4

Sol: The repair man spends 30 minutes per job on an average

Hence the Service rate $\mu = 2$ sets per hour.

The sets arrive at the average rate of 10 per 8 hours

Hence arrival rate $\lambda = \frac{10}{8} = \frac{5}{4}$ sets/hour.

$\therefore$ Traffic Intensity 'ρ' $= \frac{\lambda}{\mu} = \frac{5/4}{2} = \frac{5}{8}$

We know that, the repairman will be idle if there are no sets in the system.

$\therefore$ Probability that there is no set in the system

= Probability of repairman will be idle

$= P_0 = 1 - \rho = 1 - 5/8 = 3/8$

$\therefore$ Expected Idle time for the repairman in an 8 hour day

$= 3/8 \times 8 = 3$ hours.

The number of jobs of the avg. set just brought in

= The avg. number of sets in the system

$= E(n) = L_s = \frac{\rho}{1-\rho} = \frac{5/8}{1-5/8} = 5/3 = 1\frac{2}{3}$ jobs.

Pb 3 : A bank plans to open a single server drive - in banking facility at a certain Centre. It is estimated that 20 customers will arrive each hour on average. If on the avg., it requires 2 minutes to process a customer's transaction, determine

(i) The proportion of time that the system will be idle

(ii) On the average, how long a customer will have to wait before reaching the server.

(iii) The fraction of customers who will have to wait.

Sol: Here, No. of arrivals $= \lambda = 20$ per hour

Each service takes 2 minutes on average

Hence number of services $= \mu = 30$ per hour

$\Longrightarrow$ Traffic Intensity $= \rho = \frac{\lambda}{\mu} = \frac{20}{30} = \frac{2}{3}$.

(i) proportion of time that the system will be idle

$$\Longrightarrow \quad P_0 = 1 - \rho = 1 - 2/3 = 1/3$$

Hence the system will be idle for $(1/3)^{rd}$ of the time.

(ii) Expected waiting time of the customer before reaching the server, $W_q = \frac{\rho}{\mu(1-\rho)}$

$$\Longrightarrow \quad W_q = \frac{2/3}{30(1-2/3)} = \frac{1}{15} \text{ hrs.} = 4 \text{ Min}$$

(iii) the fraction of customers who will have to wait

$$= \text{Length of queue} = L_q = \frac{\rho^2}{1-\rho} = \frac{(2/3)^2}{1-2/3}$$

$$= \frac{4}{3} = 1\tfrac{1}{3} \text{ Customers.}$$

Pb ④: A toll gate is operated on a freeway where cars arrive according to a poisson distribution with mean frequency of 1.2 cars per minute. The time of completing payment follows an Exponential distribution with mean of 20 Seconds. Find (i) The idle-time of the Counter (ii) Average number of cars in the system (iii) Average no. of cars in the queue (iv) Average time a car spends in the system (v) average time a car spends in the queue

(vi) Prob that a car spends more than 30 seconds in the system.

Sol: Here Arrival rate $\lambda = 1.2$ cars/minute

$$\Rightarrow \lambda = 0.02/\text{sec}$$

Service rate is 1 service/20 seconds

$$\therefore \mu = 1/20 \,/\, \text{seconds}$$

$$\Rightarrow \text{Traffic Intensity} = \rho = \frac{\lambda}{\mu} = \frac{0.02}{(1/20)} = 2/5$$

(i) Probability that there are no Customers in the System

$$\Rightarrow P_0 = 1-\rho = 1-2/5 = 3/5$$

Hence $3/5^{th}$ of the time the counter will be idle.

(ii) Average number of cars in the System

$$\Rightarrow L_s = \frac{\rho}{1-\rho} = \frac{2/5}{1-2/5} = 2/3$$

(iii) Average number of cars in the Queue

$$\Rightarrow L_q = \frac{\rho^2}{1-\rho} = \frac{(2/5)^2}{1-2/5} = \frac{4}{15}$$

(iv) Average waiting time in the System

$$\Rightarrow W_s = \frac{1}{\mu(1-\rho)} = \frac{1}{\frac{1}{20}(1-2/5)}$$

$$= \frac{100}{3} \text{ seconds}$$

(v) Average waiting time in the Queue

$$\Rightarrow W_q = \frac{\rho}{\mu(1-\rho)} = \frac{2/5}{1/20(1-2/5)} = \frac{40}{3} \text{ seconds.}$$

(vi) Probability that a car spends more than 30 seconds in the System $\Rightarrow$ $P(W > 30) = e^{-(1/20 - 0.02).30}$

$$= \underline{\underline{0.4065}}$$

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(ii) $(M/M/1):(N/FIFO)$ MODEL :

The measures involving in this model are given as follows.

1. Traffic Intensity $\Rightarrow \rho = \frac{\lambda}{\mu}$

2. Probability of no customers in the system

$$\Rightarrow P_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{N+1}}, & \text{when } \rho \neq 1 \\[2ex] \dfrac{1}{N+1}, & \text{when } \rho = 1 \end{cases}$$

3. Probability of 'n' customers in the system

$$\Rightarrow P_n = P_0 \rho^n$$

$$= \begin{cases} \left(\dfrac{1-\rho}{1-\rho^{N+1}}\right)\rho^n, & \rho \neq 1 \\[2ex] \dfrac{1}{N+1}, & \rho = 1 \end{cases}$$

4. Expected number of customers in the system $[E(n)]$

$$\Rightarrow L_s = \sum_{n=0}^{N} n P_n$$

$$= \sum_{n=0}^{N} n(P_0 \rho^n)$$

$$= P_0 \sum_{n=0}^{N} n \rho^n.$$

5. Expected number of customers in the Queue $[E(m)]$

$$\Rightarrow L_q = L_s - \frac{\lambda}{\mu} \quad (\text{or}) \quad L_q = P_0 \sum_{n=0}^{N} (n-1)\rho^n$$

6. Expected Waiting time of customer in System

$$\Rightarrow W_s = \frac{L_s}{\lambda}$$

7. Expected Waiting time of customer in Queue

$$\Rightarrow W_q = \frac{L_q}{\lambda}.$$

Problems :

Pb ①: A car park contains 5 cars. The arrival of cars in poisson with mean rate of 10 per hour. The length of time each car spends in the car park has negative exponential distribution with mean of 2 hours. How many cars in the car park on avg. and what is the probability of a newly arriving customer finding the car park full and having to park his car elsewhere?

Sol: The capacity of the car park $N = 5$.

cars arrive at the rate of 10 / hour

Here $\lambda = \frac{10}{60}$ per minute $= 1/6$ / Minute

A car stays in the System on average for 2 hours

(i.e) $\mu = 1/2$ cars / hour.

$\Rightarrow \mu = 1/120$ cars / Minute

$\therefore$ Traffic Intensity $\Rightarrow \rho = \frac{\lambda}{\mu} = \frac{1/6}{1/120} = \underline{\underline{20}}$

Here $P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{20-1}{20^6-1}$

Number of cars in the system on average

$$L_s = E(n) = \sum_{n=1}^{5} n\, P_n$$

$$= P_0 \sum_{n=1}^{5} n\, \rho^n$$

$$= P_0 \left[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5 \right]$$

$$= \left(\frac{19}{20^6 - 1} \right) \left[20 + 2(20)^2 + 3(20)^3 + 4(20)^4 + 5(20)^5 \right].$$

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Pb ② : Consider a single server queueing system with poisson input and exponential service time. Suppose the mean arrival rate is 3 calling units per hour with the expected service time as 0.25 hrs and the maximum permissible number of calling units in the system is two. obtain the steady state probability distribution of the number of calling units in the system and then calculate the expected number in the system?

Sol:

λ = Arrival rate = 3 units/hour

Since expected service time is 0.25 hours

$\Rightarrow$ The service rate $\mu = 4$ units/hours

Traffic Intensity $\Rightarrow \rho = \frac{\lambda}{\mu} = 3/4$.

Here $N = 2$

Here $$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-3/4}{1-(3/4)^3}$$

$$= 1/4 \times \frac{64}{37} = \frac{16}{37}$$

The Expected number in the System

$$\Rightarrow E(n) = \sum_{n=0}^{2} n P_n$$

$$= P_0 \sum_{n=0}^{2} n \rho^n$$

$$= \frac{16}{37}\left[\rho + 2\rho^2\right]$$

$$= \frac{16}{37}\left[3/4 + \frac{2 \times 9}{16}\right]$$

$$= 30/37$$

Pb: 3 : At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while other is given signal to leave the station. Trains arrive at the station at an avg. rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming poisson arrivals and exponential service distribution, find the steady state probabilities for the various number of trains in the system. Find also the average waiting time of a new train coming into the yard ?

Sol: Here number of arrivals $= \lambda = 6/\text{hour}$

No. of services $= \mu = 12/\text{hour}$

$$\rho = \frac{\lambda}{\mu} = \frac{6}{12} = 1/2$$

While one train is served only 2 can wait

$$\therefore \quad N = 2 + 1 = 3.$$

Probability that there is no train in the system

$$\Rightarrow P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-1/2}{1-(1/2)^4} = \frac{8}{15}$$

We know that probability for 'n' trains in the system

$$\Rightarrow P_n = P_0 \rho^n.$$

Average number of trains in the system

$$\Rightarrow E(n) = \sum_{n=1}^{3} n P_n$$

$$= P_0 \sum_{n=1}^{3} n \rho^n$$

$$= \frac{8}{15}\left[\rho + 2\rho^2 + 3\rho^3\right]$$

$$= \frac{8}{15}\left[1/2 + 2(1/2)^2 + 3(1/2)^3\right]$$

$$= \underline{\underline{11/15}}$$

∴ On average there are $11/15$ trains in the system.

On avg. the service for a train requires $1/12$ hours = 5 minutes

Since new arrival expects $11/15$ trains in the system, the expected waiting time of a new train entering into system

$$= 5 \times \frac{11}{15} = \underline{\underline{\frac{11}{3} \text{ minutes}}}$$

Pb 4: A one person barber shop has six chairs to accommodate people waiting for hair cut. Assume that customers who arrive when all the six chairs are full leave without entering the shop. customers arrive at the average rate of 3 per hour and spend an average of 15 minutes for service. Find

(a) The probability that a customer can get directly into the barber chair upto arrival.

(b) Expected number of customers waiting for a haircut

(c) Effective arrival rate

(d) The time a customer can expect to spend in the barber shop.

Sol: This is (M/M/1) : (7/FIFO) Model ($\because$ Six chairs for waiting people & one for the person being served)

Here $\lambda = 3/\text{hour}$

$\mu = 4/\text{hour}$

$\rho = \frac{\lambda}{\mu} = 3/4$

(a) $$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-3/4}{1-(3/4)^8} = 0.2778$$

(b) Expected no. of customers waiting for haircut

$$\Rightarrow E(m) = \sum_{n=1}^{7}(n-1)P_n$$

$$= P_0 \sum_{n=1}^{7}(n-1)\rho^n$$

$$= P_0\left[0.\rho + 1.\rho^2 + 2\rho^3 + 3\rho^4 + 4\rho^5 + 5\rho^6 + 6\rho^7\right].$$

$$= 1.36$$

(c) Effective arrival rate

$$= \mu(1 - P_0)$$

$$= 4(1 - 0.2778) = 2.89 / \text{hour}$$

(d) The time a customer expects to spend in the system

$$= \frac{E(m) + 1 - P_0}{\text{Effective arrival rate}}$$

$$= \frac{1.36 + 1 - 0.2778}{2.89}$$

$$= 0.72 \text{ Hours}$$

$$= 43.2 \text{ Minutes}$$