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Hangqi Cui - Nada Abdelwahab - Suneh Bhatia

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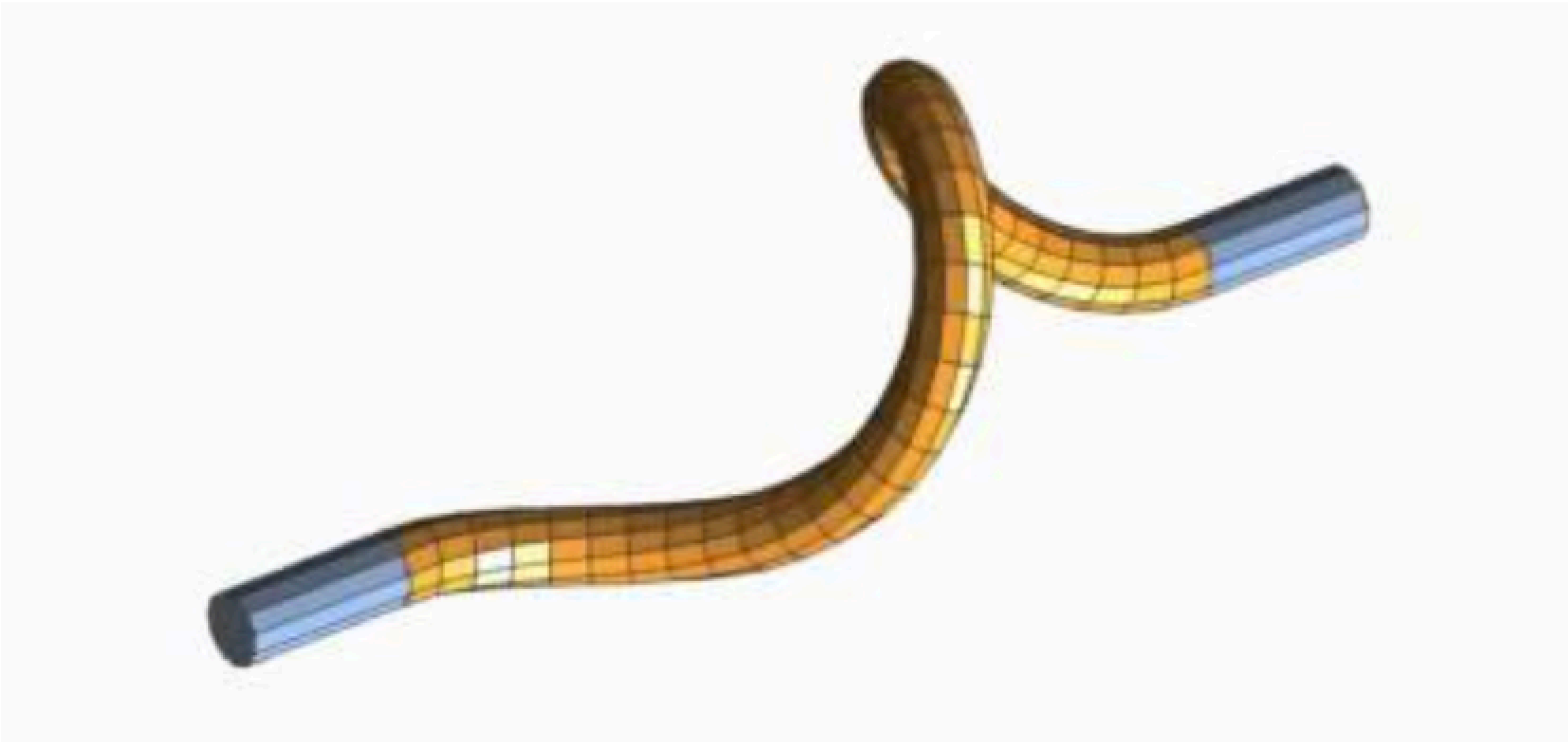

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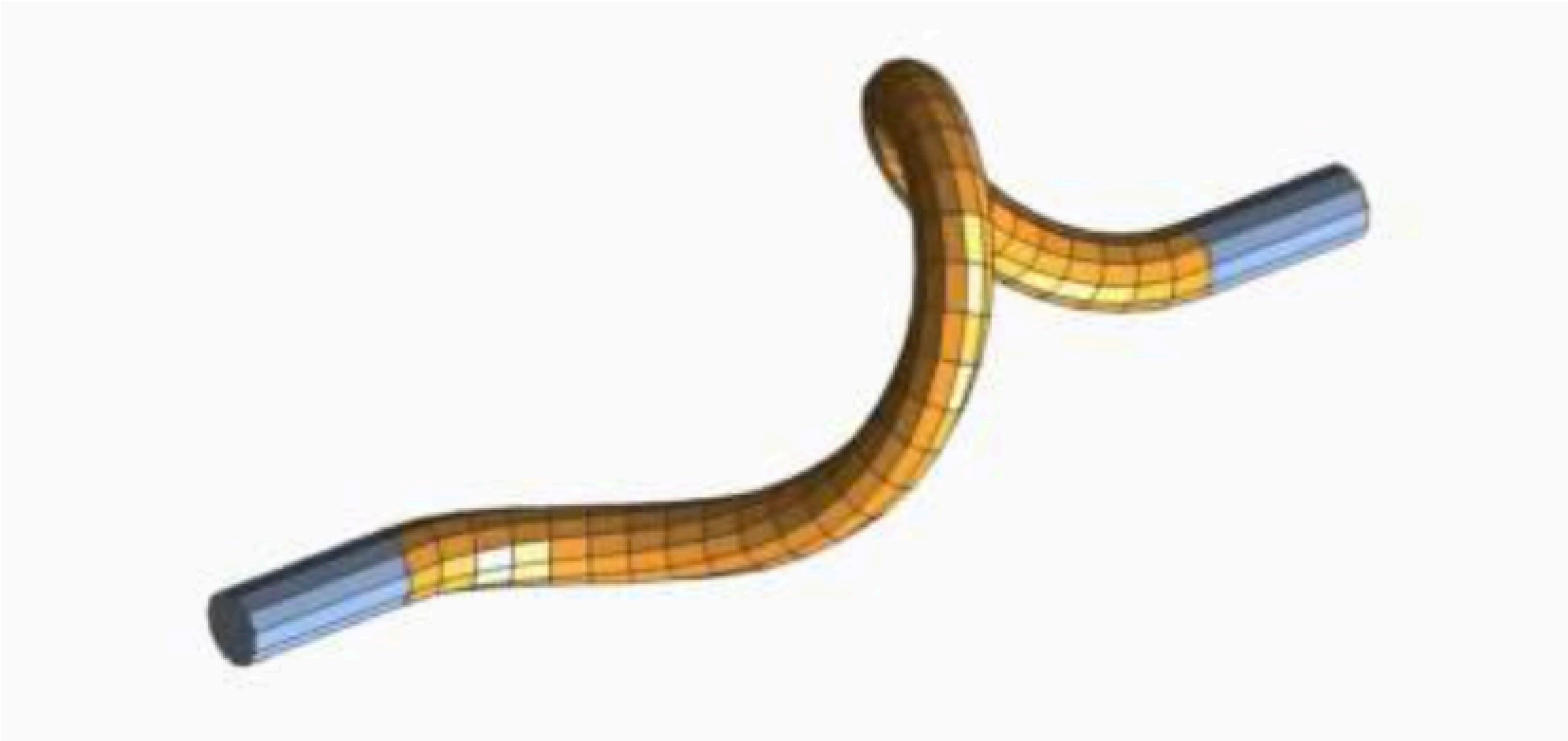

2. Discrete polyline (primals and duals)

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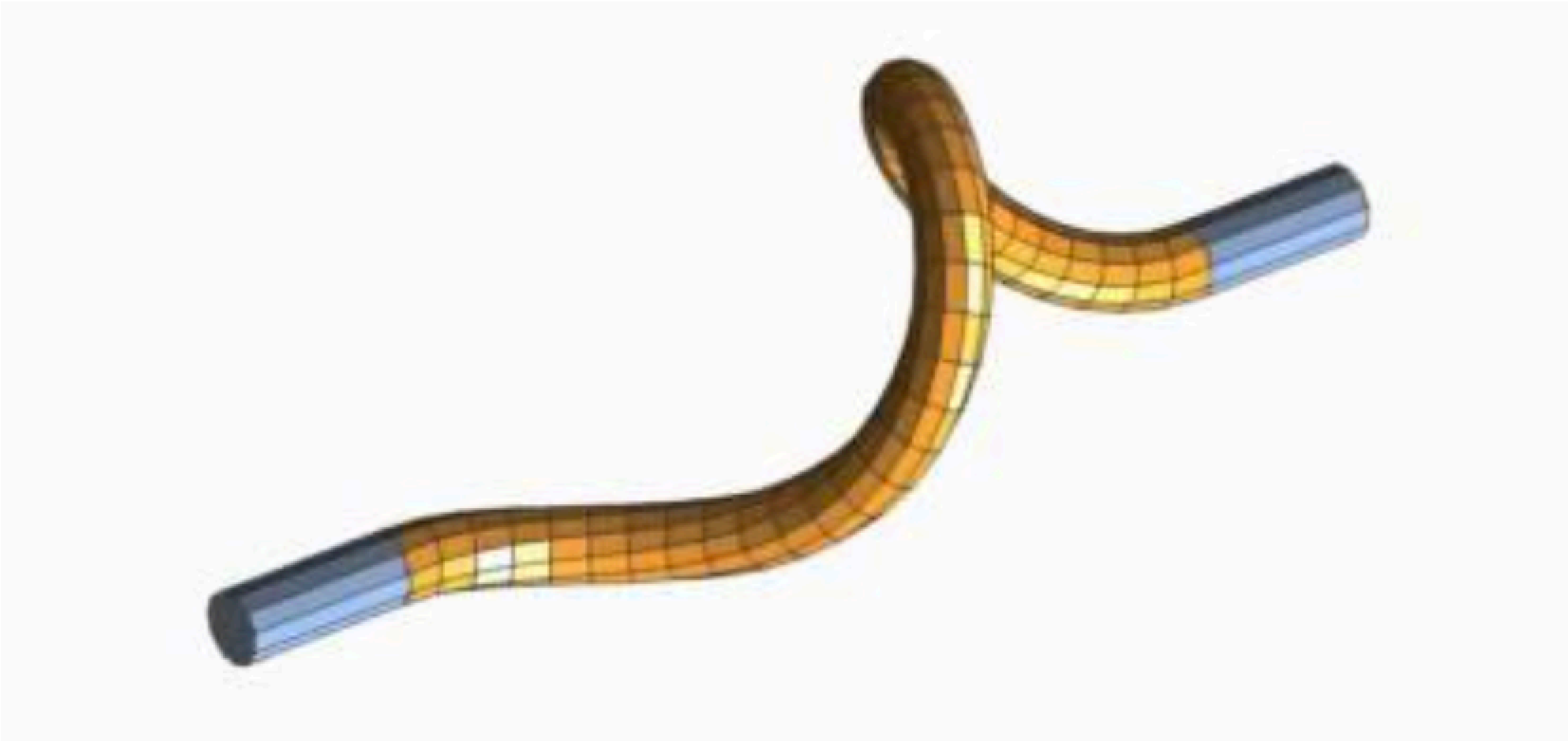

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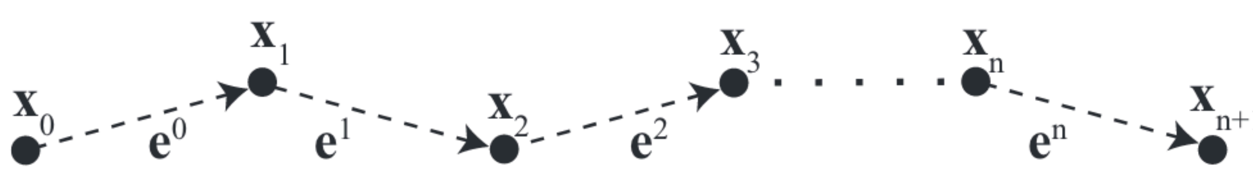

Solve Dynamics!

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$$\text{Energy} \to \text{Force} \to \text{Acceleration} \to \text{Position}$$

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- Isotropic rod
  - Cross section of the rod is of a unit circle

Bishop Frame

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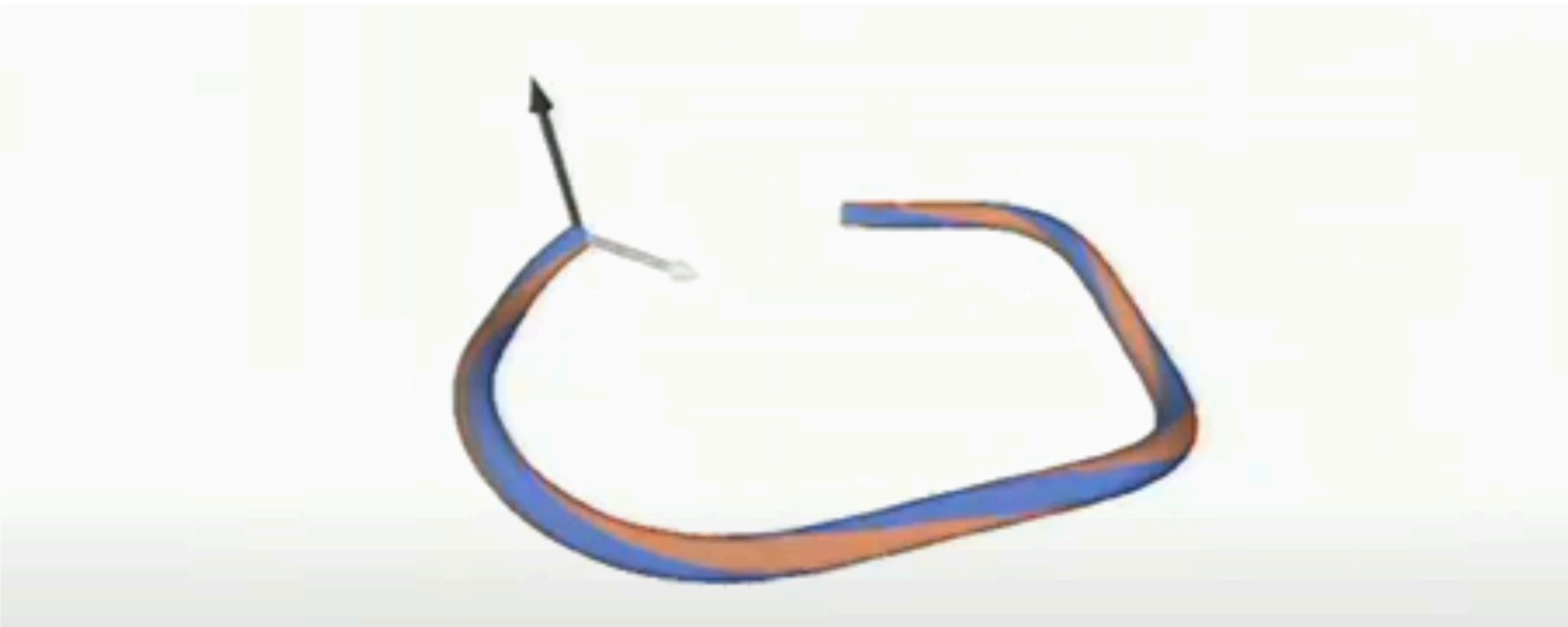
- $B = T \times N$

Bishop Frame



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Parallel Transport

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Material Frame

Simply rotate u and v in twisting angle θ

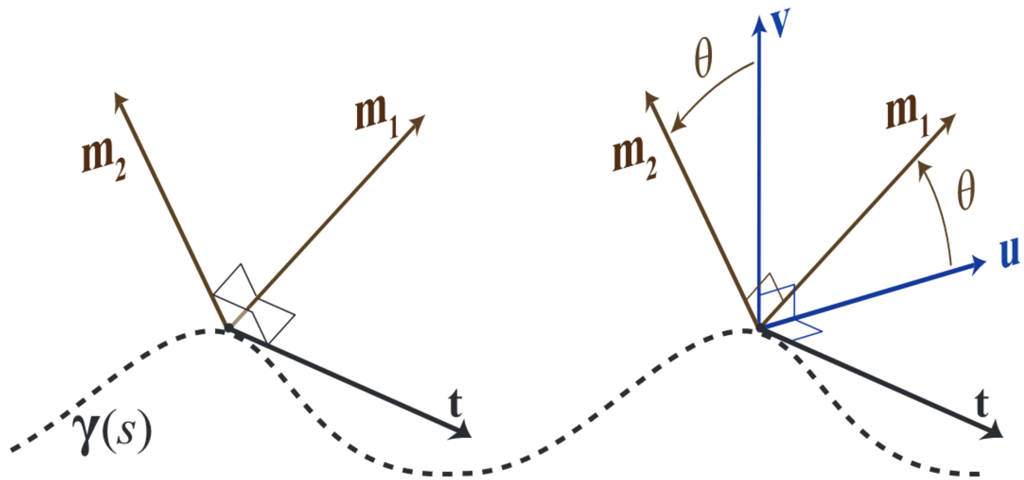

Figure 3: **Adapted framed curve** *(Left)* The configuration of an elastic rod is represented by a curve $\boldsymbol{\gamma}(s)$ and a material frame $\{\mathbf{t}(s), \mathbf{m_1}(s), \mathbf{m_2}(s)\}$. *(Right)* The material frame is encoded by an angle of rotation θ relative to the natural Bishop frame $\{\mathbf{t}(s), \mathbf{u}(s), \mathbf{v}(s)\}$.

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$$\Omega = \kappa b = \frac{2\mathbf{e}^{i-1} \times \mathbf{e}^{i}}{\left|\bar{\mathbf{e}}^{i-1}\right| \left|\bar{\mathbf{e}}^{i}\right| + \mathbf{e}^{i-1} \cdot \mathbf{e}^{i}}$$

Dynamics 101

$$F = dE \qquad E = \int F dx$$

$$F = ma \rightarrow \frac{\partial E}{\partial x} = M\ddot{x}$$

Energies and Forces (continuous case)

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- α, β: Material modulus

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$$E_{bend}(\Gamma) = \frac{1}{2}\sum_{i=1}^{n} \alpha \left(\frac{\kappa \mathbf{b}_i}{\bar{l}_i/2} \right)^2 \frac{\bar{l}_i}{2} = \sum_{i=1}^{n} \frac{\alpha (\kappa \mathbf{b}_i)^2}{\bar{l}_i}$$

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Symplectic Euler

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$$a = \frac{dv}{dt}, \qquad v = \int a dt$$

$$v = \frac{dx}{dt}, \qquad x = \int v dt$$

$$F = ma$$

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  - Remember the curve is inextensible?
- Twist holonomy: Update twisting angles
  - Another assumption: twist is updated simultaneously

Algorithm 1 Discrete elastic rod simulation

Require: $\mathbf{u}^0$ *// Bishop frame vector in frame* $\{\mathbf{t}^0, \mathbf{u}^0, \mathbf{v}^0\}$ *at edge 0*
Require: $\bar{\mathbf{x}}_0 \ldots \bar{\mathbf{x}}_{n+1}$ *// position of centerline in rest state*
Require: $(\mathbf{x}_0, \dot{\mathbf{x}}_0) \ldots (\mathbf{x}_{n+1}, \dot{\mathbf{x}}_{n+1})$ *// initial position/velocity of centerline*
Require: boundary conditions *// free, clamped or body-coupled ends*

precompute $\overline{\boldsymbol{\omega}}_i^j$ using (2)
set quasistatic material frame (§5.1)
while simulating **do**
    apply torque to rigid-body (§8.2)
    integrate rigid-body (external library) *// [Smith 2008]*
    compute forces on centerline (§7.1)
    integrate centerline (§7.2) *// [Hairer et al. 2006]*
    enforce inextensibility and rigid-body coupling (§8)
    collision detection and response *// [Spillmann and Teschner 2008]*
    update Bishop frame (§4.2.2)
    update quasistatic material frame (§5.1)
end while

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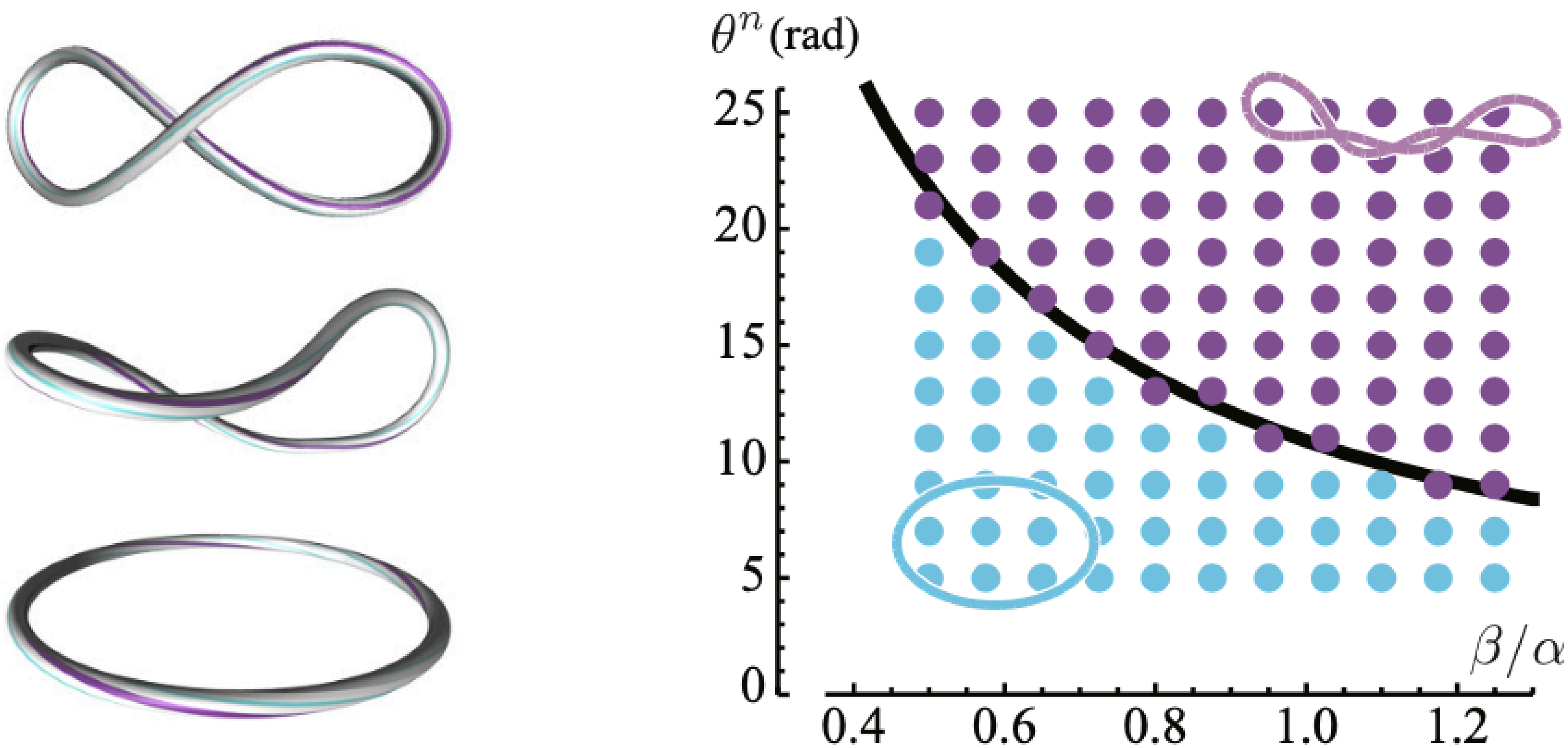
Experiments



Michell's buckling instability of an elastic ring with imposed internal twist θn. Left: above a critical value of θn, the planar, circular shape loses stability and buckles to a non-planar shape. Right: domain of stability of the circular shape with radius R = 1: simulations (dots) compared to theoretical threshold (black curve). Each dot corresponds to a simulation run with particular values of β and θn (α = 1 and n = 50 are fixed), initialized with a slightly perturbed circular shape; dots are colored in light blue when the amplitude of the perturbation decreases in time (stable) and in purple when it increases (unstable).

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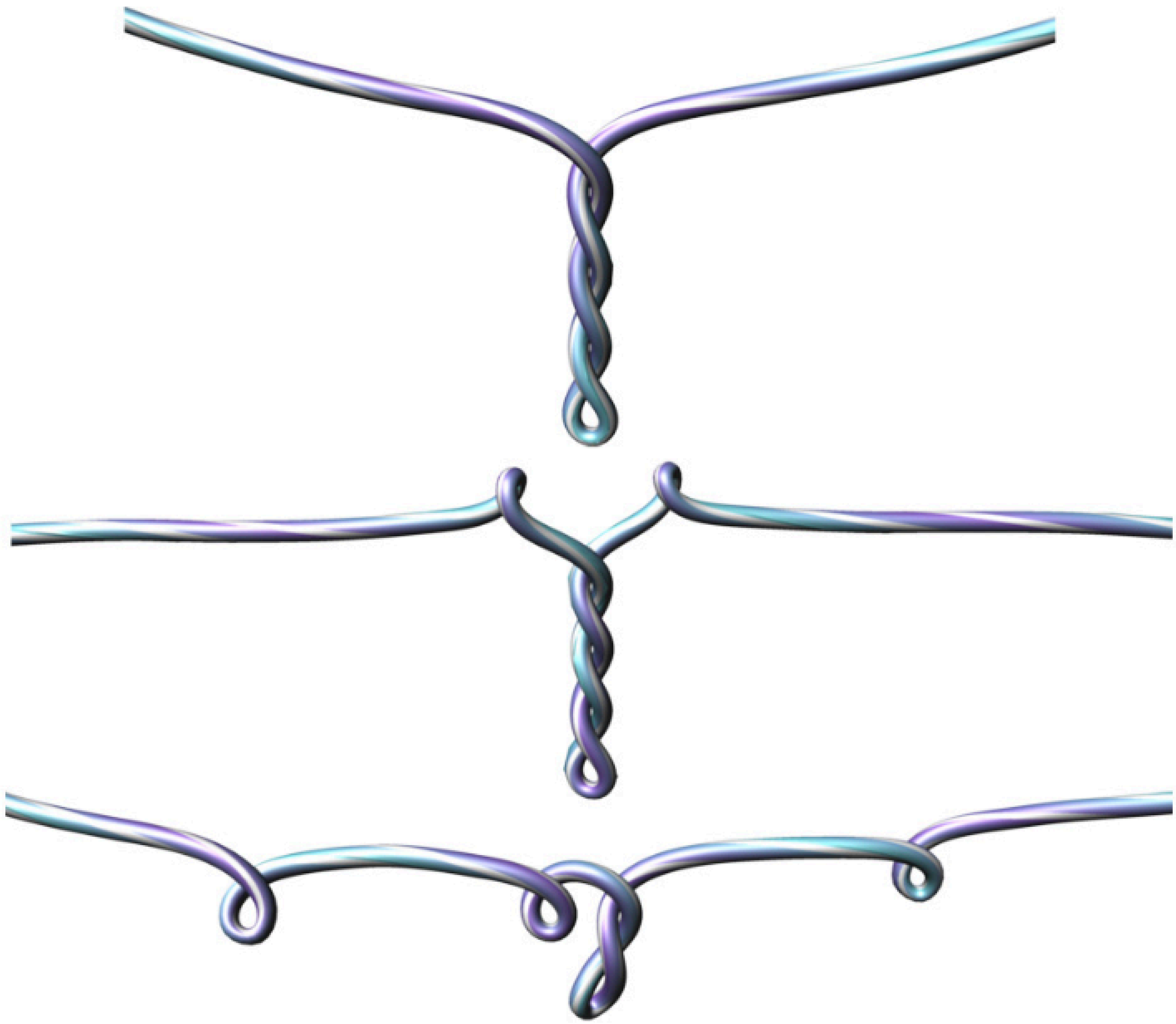

Plectoneme formation: When the ends of a hanging elastic rod are twisted, it takes on structures known as plectonemes. The formation of plectonemes is governed by physical parameters, such as the twist rate, viscosity of the ambient fluid, and gravity.

Applications

Towards Realistic Hair Animation Using Discrete Elastic Rods Mila Grigorova Master Thesis

Applications

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