高数A期中试题（20241107）

1. 解答下列各题（本题计8+8=16分）.

(1) 若$\lim\limits_{x\to 1}\dfrac{x^2+ax+b}{\sin(x^2-1)}=\dfrac{4}{9}$, 求$a$, b的值.

(2) 设$f(x)$在开区间(c,d)上连续. 证明: 对任意的$x_1,\cdots,x_n\in(c,d)$,
存在$\xi\in(c,d)$使得$f(\xi)=\dfrac{1}{n}\sum\limits_{k=1}^{n}f(x_k)$, 其中$n$是正整数.

2. 解答下列各题（本题计8+8=16分）.

(1) 设$f(x)=\sqrt{x^2+1}\ \arctan x-\ln\left(x+\sqrt{x^2+1}\right)$, 求$\mathrm{d}f(x)$.

(2) 求函数$y=\dfrac{1}{4\sqrt{2}}\ln\dfrac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}-\dfrac{1}{2\sqrt{2}}\arctan\dfrac{\sqrt{2}x}{x^2-1}$的一阶导数.

3. 解答下列各题（本题计5+10+3=18分）.

(1) 设$u(x),v(x)$任意阶可导, 对正整数n, 写出$u(x)v(x)$的n阶导数的莱布尼兹公式.

(2) 设$f(x)=(x^2-3x+2)^{100}\cos\dfrac{\pi x^2}{4}$, 计算$f^{(n)}(1)$, 其中$n=1,2,\cdots,100$.

(3) 计算$f^{(101)}(2)$.

4. 计算下列各题（本题计5+3+5+3=16分）.

(1) $A=\displaystyle\int_0^{2\pi}|\sin x-\cos x|\,\mathrm{d}x$;　　(2) $B=\displaystyle\int_0^{2\pi}\sqrt{1+\sin 2x}\,\mathrm{d}x$;

(3) $I=\displaystyle\int\sqrt{e^x-1}\,\mathrm{d}x$;　　(4) $J=\displaystyle\int\frac{xe^x}{\sqrt{e^x-1}}\,\mathrm{d}x$.

5. 解答下列各题（本题计5+2+5+2=14分）.

(1) 设$f(x)$在$x=a$点可导, 证明$\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a-h)}{h}=2f'(a)$.

(2) 举例说明: 即使$f(x)$于$x=a$点连续且$\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a-h)}{h}$存在,
也不能保证$f'(a)$存在.

(3) 设$f(x)$在$x=a$点可导, 证明$\lim\limits_{h\to 0}\dfrac{f(a+kh)-f(a+h)}{h}=(k-1)f'(a)$,
其中$k\neq 0,1$为常数.

(4) 举例说明: 即使$\lim\limits_{h\to 0}\dfrac{f(a+kh)-f(a+h)}{h}$ $(k\neq 0,1)$存在, 也不能保证$f'(a)$存在.

6. 证明下列各题（本题计10+6+4=20分）.

(1) 设序列$\{x_n\}_{n=1}^{+\infty}$有极限$\lim\limits_{n\to+\infty}x_n=a$. 请**用序列极限的定义**证明: $\lim\limits_{n\to+\infty}\dfrac{1}{n}\sum\limits_{k=1}^{n}x_k=a$.

(2) 设$x_n>0$, $\lim\limits_{n\to+\infty}x_n=a>0$. 证明: $\lim\limits_{n\to+\infty}\sqrt[n]{x_1x_2\cdots x_n}=a$.（不得使用Stolz公式.）

(3) 若$\lim\limits_{n\to+\infty}\dfrac{1}{n}\sum\limits_{k=1}^{n}x_k=a$（有限数）且$\lim\limits_{n\to+\infty}n(x_n-x_{n-1})=0$, 证明$\lim\limits_{n\to+\infty}x_n=a$.

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参考答案

1. 解答下列各题（本题计8+8=16分）.

(1) 若$\lim\limits_{x\to 1}\dfrac{x^2+ax+b}{\sin(x^2-1)}=\dfrac{4}{9}$, 求$a$, b的值.
(2) 设$f(x)$在开区间(c,d)上连续. 证明: 对任意的$x_1,\cdots,x_n\in(c,d)$, 存在$\xi\in(c,d)$使得$f(\xi)=\dfrac{1}{n}\sum\limits_{k=1}^{n}f(x_k)$, 其中$n$是正整数.

解. (1) 当$x\neq\pm 1$时, $\dfrac{x^2+ax+b}{\sin(x^2-1)}=\dfrac{x^2+ax+b}{x^2-1}\cdot\dfrac{x^2-1}{\sin(x^2-1)}$.
因为 $\lim\limits_{x\to 1}\dfrac{x^2-1}{\sin(x^2-1)}=1$,
所以 $\lim\limits_{x\to 1}\dfrac{x^2+ax+b}{x^2-1}=\lim\limits_{x\to 1}\left[\dfrac{x^2+ax+b}{\sin(x^2-1)}\cdot\dfrac{\sin(x^2-1)}{x^2-1}\right]=\dfrac{4}{9}$,
即 $\lim\limits_{x\to 1}\dfrac{x^2+ax+b}{(x-1)(x+1)}=\dfrac{4}{9}$.
所以必须有$\lim\limits_{x\to 1}(x^2+ax+b)=0\ \Rightarrow\ 1+a+b=0$,
于是就有$x^2+ax+b=x^2+ax-1-a=(x-1)(x+1+a)$.
再由 $\lim\limits_{x\to 1}\dfrac{(x-1)(x+1+a)}{(x-1)(x+1)}=\lim\limits_{x\to 1}\dfrac{(x+1+a)}{(x+1)}=\dfrac{(1+1+a)}{(1+1)}=\dfrac{4}{9}$,
得到$a=-10/9,\ b=1/9$.

(2) 对任意的正整数n, 记$A=\min\{x_1,\cdots,x_n\},\ B=\max\{x_1,\cdots,x_n\}$,
$m=\min\{f(x_1),\cdots,f(x_n)\}, M=\max\{f(x_1),\cdots,f(x_n)\}$.
因为$f(x)\in C(c,d)$, 所以, $f(x)\in C[A,B]$. 记$P=\min\limits_{x\in[A,B]}f(x), Q=\max\limits_{x\in[A,B]}f(x)$.
则$P\leqslant m\leqslant M\leqslant Q$, 且$\dfrac{1}{n}\sum\limits_{k=1}^{n}f(x_k)\in[m,M]\subset[P,Q]$,
据连续函数中介值性质, $\exists\xi\in[A,B]\subset(c,d)$ s.t. $f(\xi)=\dfrac{1}{n}\sum\limits_{k=1}^{n}f(x_k),\ \forall n\in\mathbb{Z}^+$. $\square$

2. 解答下列各题（本题计8+8=16分）.

(1) 设$f(x)=\sqrt{x^2+1}\ \arctan x-\ln\left(x+\sqrt{x^2+1}\right)$, 求$\mathrm{d}f(x)$.

(2) 求函数$y=\dfrac{1}{4\sqrt{2}}\ln\dfrac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}-\dfrac{1}{2\sqrt{2}}\arctan\dfrac{\sqrt{2}x}{x^2-1}$的一阶导数.

解. (1) $f'(x)=\dfrac{x}{\sqrt{x^2+1}}\ \arctan x+\sqrt{x^2+1}\ \dfrac{1}{x^2+1}-\dfrac{1}{x+\sqrt{x^2+1}}\ \left(1+\dfrac{x}{\sqrt{x^2+1}}\right)$

$=\dfrac{x}{\sqrt{x^2+1}}\ \arctan x+\dfrac{1}{\sqrt{x^2+1}}-\dfrac{1}{\sqrt{x^2+1}}=\dfrac{x}{\sqrt{x^2+1}}\ \arctan x.$

所以，$\mathrm{d}f(x)=\dfrac{x}{\sqrt{x^2+1}}\ \arctan x\ \mathrm{d}x.$

(2) $y=\dfrac{1}{4\sqrt{2}}\left(\ln(x^2+\sqrt{2}x+1)-\ln(x^2-\sqrt{2}x+1)\right)-\dfrac{1}{2\sqrt{2}}\arctan\dfrac{\sqrt{2}x}{x^2-1}.$

$$y'=\frac{1}{4\sqrt{2}}\left(\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)-\frac{1}{2\sqrt{2}}\ \frac{\left(\frac{\sqrt{2}x}{x^2-1}\right)'}{1+\frac{2x^2}{(x^2-1)^2}}$$

$$=\frac{1}{4}\left(\frac{\sqrt{2}x+1}{x^2+\sqrt{2}x+1}-\frac{\sqrt{2}x-1}{x^2-\sqrt{2}x+1}\right)-\frac{1}{2\sqrt{2}}\ \frac{\sqrt{2}(x^2-1)-\sqrt{2}x\cdot 2x}{(x^2-1)^2}\quad\frac{(x^2-1)^2}{1+x^4}$$

$$=\frac{1}{4}\ \frac{(\sqrt{2}x^3-2x^2+\sqrt{2}x+x^2-\sqrt{2}x+1)-(\sqrt{2}x^3+2x^2+\sqrt{2}x-x^2-\sqrt{2}x-1)}{(x^2+1)^2-2x^2}-\frac{1}{2}\ \frac{-x^2-1}{1+x^4}$$

$$=\frac{1}{2}\ \frac{-x^2+1}{x^4+1}+\frac{1}{2}\ \frac{x^2+1}{1+x^4}=\frac{1}{1+x^4}.$$

□

3. 解答下列各题（本题计5+10+3=18分）.

(1) 设$u(x), v(x)$任意阶可导, 对正整数n, 写出$u(x)v(x)$的n阶导数的莱布尼兹公式.

(2) 设$f(x) = (x^2 - 3x + 2)^{100} \cos \dfrac{\pi x^2}{4}$, 计算$f^{(n)}(1)$, 其中$n = 1, 2, \cdots, 100$.

(3) 计算$f^{(101)}(2)$.

解. (1) $(uv)^{(n)} = \displaystyle\sum_{j=0}^{n} C_n^j u^{(j)}(x) v^{(n-j)}(x),\ n = 1, 2, \cdots.$

(2) $f(x) = (x-1)^{100}(x-2)^{100} \cos \dfrac{\pi x^2}{4}.$

记$u(x) = (x-1)^{100},\ v(x) = (x-2)^{100} \cos \dfrac{\pi x^2}{4},$

则$f^{(n)}(x) = \displaystyle\sum_{j=0}^{n} C_n^j u^{(j)}(x) v^{(n-j)}(x),\ n = 1, 2, \cdots.$

由于$u^{(j)}(1) = 0,\ j \neq 100;\quad u^{(100)}(1) = 100!.$

所以对$n = 1, 2, \cdots, 99,\quad f^{(n)}(1) = \displaystyle\sum_{j=0}^{n} C_n^j u^{(j)}(1) v^{(n-j)}(1) = 0;$

$$f^{(100)}(1) = \sum_{j=0}^{100} C_n^j u^{(j)}(1) v^{(n-j)}(1) = C_{100}^{100} u^{(100)}(1) v^{(0)}(1) = 100! v(1) = 100! \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} 100!.$$

(3) 记$u(x) = (x-2)^{100},\ v(x) = (x-1)^{100} \cos \dfrac{\pi x^2}{4},$

则$u^{(j)}(2) = 0,\ j \neq 100;\quad u^{(100)}(2) = 100!.$

$$f^{(101)}(2) = \sum_{j=0}^{101} C_n^j u^{(j)}(2) v^{(n-j)}(2) = C_{101}^{100} u^{(100)}(2) v'(2) = 101 \times 100! v'(2).$$

$v'(x) = 100(x-1)^{99} \cos \dfrac{\pi x^2}{4} - (x-1)^{100} \dfrac{\pi x}{2} \sin \dfrac{\pi x^2}{4},\quad v'(2) = -100.$

所以, $f^{(101)}(2) = -10100 \times 100!.$ $\square$

4. 计算下列各题（本题计5+3+5+3=16分）.

(1) $A=\int_0^{2\pi}|\sin x-\cos x|\,\mathrm{d}x$; (2) $B=\int_0^{2\pi}\sqrt{1+\sin 2x}\,\mathrm{d}x$;

(3) $I=\int\sqrt{e^x-1}\,\mathrm{d}x$; (4) $J=\int\frac{xe^x}{\sqrt{e^x-1}}\,\mathrm{d}x$.

解. (1) $A \overset{\pi\text{周期}}{=\!=\!=} 2\int_0^{\pi}|\sin x-\cos x|\,\mathrm{d}x$

$$=2\int_0^{\pi}\left|\sqrt{2}\sin(x-\frac{\pi}{4})\right|\,\mathrm{d}x=2\sqrt{2}\int_0^{\frac{\pi}{4}}-\sin(x-\frac{\pi}{4})\,\mathrm{d}x+2\sqrt{2}\int_{\frac{\pi}{4}}^{\pi}\sin(x-\frac{\pi}{4})\,\mathrm{d}x$$

$$=2\sqrt{2}\cos(x-\frac{\pi}{4})\Big|_0^{\frac{\pi}{4}}-2\sqrt{2}\cos(x-\frac{\pi}{4})\Big|_{\frac{\pi}{4}}^{\pi}=2\sqrt{2}(1-\frac{\sqrt{2}}{2})-2\sqrt{2}(-\frac{\sqrt{2}}{2}-1)=4\sqrt{2}.$$

(2) $B=\int_0^{2\pi}|\sin x+\cos x|\,\mathrm{d}x \overset{2\pi\text{周期}}{=\!=\!=\!=} \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}}|\sin x+\cos x|\,\mathrm{d}x$

$$\overset{x=\frac{\pi}{2}+t}{=\!=\!=\!=}\int_0^{2\pi}|\cos t-\sin t|\,\mathrm{d}t=A=4\sqrt{2}.$$

(3) $I=\int\sqrt{e^x-1}\,\mathrm{d}x \xlongequal[x=\ln(1+t^2)]{\sqrt{e^x-1}=t}\int t\,\frac{2t\,\mathrm{d}t}{1+t^2}=2\int\frac{t^2+1-1}{1+t^2}\,\mathrm{d}t$

$$=2t-2\arctan t+c=2\sqrt{e^x-1}-2\arctan\sqrt{e^x-1}+c.$$

(4) $J=\int\frac{xe^x}{\sqrt{e^x-1}}\,\mathrm{d}x=2\int x\,\mathrm{d}\sqrt{e^x-1}=2x\sqrt{e^x-1}-2\int\sqrt{e^x-1}\,\mathrm{d}x$

$$=2x\sqrt{e^x-1}-2I=2x\sqrt{e^x-1}-4\sqrt{e^x-1}+4\arctan\sqrt{e^x-1}+c.$$ □

5. 解答下列各题（本题计5+2+5+2=14分）.

(1) 设$f(x)$在$x=a$点可导，证明$\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a-h)}{h}=2f'(a)$.

(2) 举例说明：即使$f(x)$于$x=a$点连续且$\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a-h)}{h}$存在，也不能保证$f'(a)$存在.

(3) 设$f(x)$在$x=a$点可导，证明$\lim\limits_{h\to 0}\dfrac{f(a+kh)-f(a+h)}{h}=(k-1)f'(a)$，其中$k\neq 0,1$为常数.

(4) 举例说明：即使$\lim\limits_{h\to 0}\dfrac{f(a+kh)-f(a+h)}{h}$ $(k\neq 0,1)$存在，也不能保证$f'(a)$存在.

解. (1) $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a-h)}{h}$

$=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}+\lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-h}=f'(a)+f'(a)=2f'(a)$.

(2) 令$f(x)=|x|$, $x\in\mathbb{R}$.

则$\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a-h)}{h}=0$, 但是$f(x)$于$x=0$点不可导.

(3) $\lim\limits_{h\to 0}\dfrac{f(a+kh)-f(a+h)}{h}$

$=\lim\limits_{h\to 0}\dfrac{f(a+kh)-f(a)}{h}-\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}=kf'(a)-f'(a)=(k-1)f'(a)$.

(4) 令$f(x)=\begin{cases}x+1, & x>0,\\ x, & x\leqslant 0.\end{cases}$

则$\lim\limits_{h\to 0+}\dfrac{f(0+kh)-f(0+h)}{h}=(k-1)$, $\lim\limits_{h\to 0-}\dfrac{f(0+kh)-f(0+h)}{h}=k-1$.

所以，$\lim\limits_{h\to 0}\dfrac{f(0+kh)-f(0+h)}{h}=(k-1)$, 但是$f'(0)$不存在（甚至不连续）.

由于题设条件只有函数在一点可导,故在证明(1)(3)时,若使用诺必达法则就是原则性错误.

6. 证明下列各题（本题计10+6+4=20分）.

(1) 设序列$\{x_n\}_{n=1}^{+\infty}$有极限$\lim\limits_{n\to+\infty} x_n = a$. **请用序列极限的定义**证明: $\lim\limits_{n\to+\infty} \frac{1}{n}\sum\limits_{k=1}^{n} x_k = a$.

(2) 设$x_n > 0$, $\lim\limits_{n\to+\infty} x_n = a > 0$, 证明: $\lim\limits_{n\to+\infty} \sqrt[n]{x_1x_2\cdots x_n} = a$.（不得使用Stolz公式.）

(3) 若$\lim\limits_{n\to+\infty} \frac{1}{n}\sum\limits_{k=1}^{n} x_k = a$（有限数）且$\lim\limits_{n\to+\infty} n(x_n - x_{n-1}) = 0$, 证明$\lim\limits_{n\to+\infty} x_n = a$.

证明. .

(1) 由于$\lim\limits_{n\to+\infty} x_n = a$, 对$\forall \epsilon > 0$, $\exists N_1 \in \mathbb{Z}^+$, 当$n > N_1$时, 就有$|x_n - a| < \frac{\epsilon}{2}$.
当$n > N_1$时,

$$\left|\frac{x_1 + x_2 + \cdots + x_n}{n} - a\right| = \left|\frac{(x_1 - a) + (x_2 - a) + \cdots + (x_n - a)}{n}\right|$$
$$= \left|\frac{(x_1 - a) + (x_2 - a) + \cdots + (x_{N_1} - a) + (x_{N_1+1} - a) + \cdots + (x_n - a)}{n}\right|$$
$$\leqslant \left|\frac{(x_1 - a) + (x_2 - a) + \cdots + (x_{N_1} - a)}{n}\right| + \frac{n - N_1}{n} \cdot \frac{\epsilon}{2}$$
$$\leqslant \left|\frac{(x_1 - a) + (x_2 - a) + \cdots + (x_{N_1} - a)}{n}\right| + \frac{\epsilon}{2}.$$

再取$N_2 \in \mathbb{N}^*$充分大, 使得$n > N_2$时,

$$\left|\frac{(x_1 - a) + (x_2 - a) + \cdots + (x_{N_1} - a)}{n}\right| < \frac{\epsilon}{2}.$$

最后取$N = \max(N_1, N_2)$, 则当$n > N$时, 就有$\left|\frac{x_1 + x_2 + \cdots + x_n}{n} - a\right| < \epsilon$.

这就证明了$\lim\limits_{n\to+\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = a$.

(2) 记$y_n = \sqrt[n]{x_1x_2\cdots x_n}$, 则$\ln y_n = \frac{1}{n}\sum\limits_{k=1}^{n} \ln x_k$.

因$0 < a < +\infty$时, $\lim\limits_{x\to a} \ln x = \ln a$, 所以$\lim\limits_{n\to\infty} x_n = a \Rightarrow \lim\limits_{n\to\infty} \ln x_n = \ln a$.

据前一题结论, $\lim\limits_{n\to\infty} \ln y_n = \ln a$, 即$\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} e^{\ln y_n} = e^{\ln a} = a$.

(3) 记$b_n = n(x_n - x_{n-1})$, 则题设$\lim\limits_{n\to\infty} b_n = 0$, 从而据*(1)*, $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{n} b_k = 0$.

但是$\sum\limits_{k=1}^{n} b_k = \sum\limits_{k=1}^{n} k(x_k - x_{k-1}) = \sum\limits_{k=1}^{n} [kx_k - (k-1)x_{k-1} - x_{k-1}] = nx_n - \sum\limits_{k=1}^{n} x_{k-1}$

所以$\frac{1}{n}\sum\limits_{k=1}^{n} b_k = x_n - \frac{1}{n}\sum\limits_{k=1}^{n} x_{k-1} = x_n - \frac{1}{n}\sum\limits_{k=1}^{n-1} x_k - \frac{x_0}{n} = x_n - \frac{n-1}{n} \times \frac{1}{n-1}\sum\limits_{k=1}^{n-1} x_k - \frac{x_0}{n}$.

所以, $x_n = \frac{1}{n}\sum\limits_{k=1}^{n} b_k + \frac{n-1}{n} \times \frac{1}{n-1}\sum\limits_{k=1}^{n-1} x_k + \frac{x_0}{n}$.

已知$\lim\limits_{n\to\infty} \frac{1}{n-1}\sum\limits_{k=1}^{n-1} x_k = a$, 又$\lim\limits_{n\to\infty} \frac{x_0}{n} = 0$, 所以$\lim\limits_{n\to\infty} x_n = a$. □

如果使用$\frac{1}{n}\sum\limits_{k=1}^{n-1} x_k = \frac{1}{n}\sum\limits_{k=1}^{n} x_k - \frac{x_n}{n}$, 则因为$\lim\limits_{n\to\infty} \frac{x_n}{n}$无法判断而导致困难.