DEPARTMENT OF ECONOMICS, NATIONAL TAIWAN UNIVERSITY
DEPARTMENT OF ECONOMICS, MICHIGAN UNIVERSITY

Solution Manual to Mas-Colell

Chapter 6. Choice Under Uncertainty

Sung-Lin Hsieh

slhsieh@umich.edu

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Choice Under Uncertainty

6.B. EXPECTED UTILITY THEORY

EXERCISE 6.B.1.

Suppose that $L \succ L'$. Then, for all $\alpha \in (0,1)$ and $L'' \in \mathscr{L}$, we have

$$\begin{gathered} L \succ L' \\ \Leftrightarrow \text{not } L' \succsim L \\ \Leftrightarrow \text{not } \alpha L' + (1-\alpha)L'' \succsim \alpha L + (1-\alpha)L'' \\ \Leftrightarrow \alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''. \end{gathered}$$

Similarly, suppose that $L \sim L'$. Then, for all $\alpha \in (0,1)$ and $L'' \in \mathscr{L}$, we have

$$\begin{gathered} L \sim L' \\ \Leftrightarrow L' \succsim L \text{ and } L \succsim L' \\ \Leftrightarrow \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L'' \text{ and } \alpha L' + (1-\alpha)L'' \succsim \alpha L + (1-\alpha)L'' \\ \Leftrightarrow \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''. \end{gathered}$$

Finally, suppose that $L \succ L'$ and $L'' \succ L'''$. Then we have

$$\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'' \succ \alpha L + (1-\alpha)L'''.$$

By transitivity, we have $\alpha L + (1-\alpha)L'' \succ \alpha L + (1-\alpha)L'''$.

Exercise 6.B.2.

Supposed $\succsim$ is represented by a utility function $U(\cdot)$ with expected utility form. Then for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0,1)$ we have

$$\begin{aligned}
&L \succsim L' \\
\Leftrightarrow &U(L) \geq U(L') \\
\Leftrightarrow &\alpha U(L) + (1-\alpha)U(L'') \geq \alpha U(L') + (1-\alpha)U(L'') \\
\Leftrightarrow &U(\alpha L + (1-\alpha)L'') \geq U(\alpha L' + (1-\alpha)L'') \\
\Leftrightarrow &\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.
\end{aligned}$$

It means $\succsim$ satisfies independence axiom.

Exercise 6.B.3.

Since the set of outcomes C is finite, there exist best and worst outcomes in C. Let $\overline{L}$ and $\underline{L}$ denote the degenerate lottery that yields the best and the worst outcome with probability one respectively.

Suppose that $C = \{c_1, \ldots, c_N\}$. It is sufficient to show that $\overline{L} \succsim \sum_{n=1}^{n} a_n c_n \succsim \underline{L}$ for all $a_n \geq 0$ with $\sum_{n=1}^{n} a_n = 1$. Since $\overline{L} \succsim c_n$ for all $n = 1, \ldots, N$, then $\overline{L} \succsim a_1 c_1 + (1-c_1)\overline{L} \succsim a_1 c_1 + a_2 c_2 + (1 - a_1 - a_2)\overline{L}$, and so on. Similarly, it is easy to get $\sum_{n=1}^{n} a_n c_n \succsim \underline{L}$ for all $a_n \geq 0$ with $\sum_{n=1}^{n} a_n = 1$.

Exercise 6.B.4.

Exercise correction:

- [...]the sure outcome C and the lottery of $\underline{A}$ with probability q and D with probability $1-q$.
- *Criterion 2*: [...] an unnecessary evaluation in 15% [...]

(a) $U(A) = 1, U(B) = p, U(C) = q, U(D) = 0$.

(b) p_A, p_B, p_C, p_D denotes the probability of the four outcome.

Under criterion 1, $p_A = 0.99 \times 0.9 = 0.981$, $p_B = 0.99 \times 0.1 = 0.099$, $p_C = 0.01 \times 0.9 = 0.009$, and $p_D = 0.01 \times 0.1 = 0.001$. Hence, $(p_A, p_B, p_C, p_D) = (0.981, 0.099, 0.009, 0.001)$ and the expected utility is $0.099p + 0.009q + 0.981$.

Under criterion 2, $(p_A, p_B, p_C, p_D) = (0.8415, 0.1485, 0.0095, 0.0005)$. The expected utility is $0.1485p + 0.0095q + 0.8415$. Thus, the criterion 1 is preferred if and only if $99p + q < 99$. It is quite easy to satisfy.

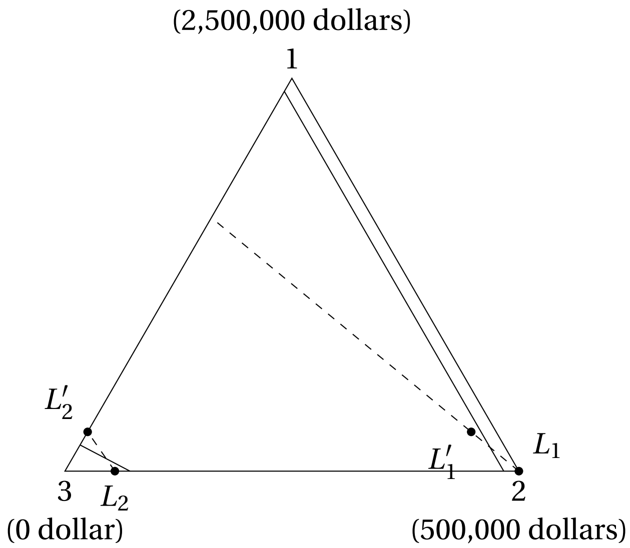
Exercise 6.B.5.

(a) It is shown in Exercise 6.B.1.

(b) Like figure 6.B.5(b), it would cause a contradiction if the indifference curve is not a straight lines. However, in figure 6.B.5, the weak axiom is not violated but the indifference curve is not parallel.

(c) Original independence axiom requires indifference curve to be straight line and parallel, but the betweenness axiom does not require it to be parallel. Hence, the betweenness axiom is weaker than the independence axiom.

(d) If the indifference curve is as following, then it satisfies the betweenness axiom and yield the choice of the Allais paradox.



Exercise 6.B.6.

Given $L = (p_1, \dots, p_N)$ and $L' = (p_1', \dots, p_N')$, assume that a and a' are solutions of the maximize problems $\max_{a \in A} \sum_n p_n u_n(a)$ and $\max_{a \in A} \sum_n p_n' u_n(a)$ respectively. Thus, we have

$$
\begin{aligned}
& U(\alpha L + (1-\alpha) L') \\
=& \max_{a \in A} \sum_n (\alpha p_n + (1-\alpha) p_n') u_n(a) \\
=& \sum_n (\alpha p_n + (1-\alpha) p_n') u_n(a^*) \\
=& \alpha \sum_n p_n u_n(a^*) + (1-\alpha) \sum_n p_n' u_n(a^*) \\
\leq& \alpha \sum_n p_n u_n(a) + (1-\alpha) \sum_n p_n' u_n(a') \\
=& \alpha U(L) + (1-\alpha) U(L')
\end{aligned}
$$

where a^* is the solution of the maximize problem of $\max_{a\in A}\sum_n(\alpha p_n + (1-\alpha)p'_n)u_n(a)$. We prove the concavity.

EXERCISE 6.B.7.

By transitivity, $L > L'$ if and only if $x_L > x_{L'}$. By monotonicity, $x_L > x_{L'}$ if and only if $x_L > x_{L'}$.

6.C. MONEY LOTTERIES AND RISK AVERSION

EXERCISE 6.C.1.

We begin with analyzing the second-order condition of the maximize problem, that is

$$q^2(1-\pi)u''(w-\alpha q) + \pi(1-q)^2 u''(w - D + (\alpha(1-q))) < 0.$$

It means that the first-order condition is decreasing. Then we put $\alpha = D$ into first-order condition with condition $q > \pi$, we would get

$$(\pi - q)u'(w - Dq) < 0$$

. Since we have proven that the first-order condition is decreasing, it means that the optimal α^* which leads the first-order condition equals 0 would be smaller than D. In other words, the individual would not insure completely.

EXERCISE 6.C.2.

(a) We can evaluate his utility from a distribution $F(\cdot)$ is

$$\int u(x)dF(x) = \int \beta x^2 + \gamma x dF(x) = \beta\sigma^2 + \beta\mu^2 + \gamma\mu$$

where $\mu = \int x dF(x)$ is the mean and $\sigma^2 = \int (x-\mu)^2 dF(x)$ is the variance of the distribution.

(b) Supposed that $U(\cdot)$ is compatible with a Bernoulli utility function $u(\cdot)$. That is, $U(F) = \int u(x)dF(x)$ for any distribution function $F(\cdot)$.

For any $x > y$, let $G(\cdot)$ and $H(\cdot)$ be distributions putting probability one at x and y respectively. Hence $u(x) = U(G) = x$ and $u(y) = U(H) = y$. It means $u(\cdot)$ is strictly monotone. However, let $F_0(\cdot)$ be a distribution yielding 0 for sure and $F_\varepsilon(\cdot)$ be a distribution yielding 0 and $\varepsilon > 0$ with equal chance. Thus, $U(F_0) = 0$ and $U(F_\varepsilon) = \frac{\varepsilon}{2} - r\frac{\varepsilon^2}{4}$. It shows that $U(F_\varepsilon)$ would be negative when $\varepsilon > \frac{2}{r}$, a contradiction.

EXERCISE 6.C.3.

First, we claim that (i) implies (iv). Suppose that the decision maker is risk averse. By definition, $\int u(x)dF(x) \le u(\int xdF(x))$ for all distributions $F(\cdot)$. Given some x and ε, let $F(\cdot)$ be a distribution putting probability 0.5 on both $x+\varepsilon$ and $x-\varepsilon$. Hence,

$$\begin{aligned}
&\frac{1}{2}u(x+\varepsilon)+\frac{1}{2}u(x-\varepsilon)\\
=&\int u(x)dF(x)\\
\le& u(\int xdF(x))\\
=&u(x)\\
=&(\frac{1}{2}+\pi(x,\varepsilon,u))u(x+\varepsilon)+(\frac{1}{2}-\pi(x,\varepsilon,u))u(x-\varepsilon).
\end{aligned}$$

In short, we have $\pi(x,\varepsilon,u)(u(x+\varepsilon)-u(x-\varepsilon)) \ge 0$. As long as $u(\cdot)$ is strictly increasing, we will get $\pi(x,\varepsilon,u) \ge 0$ for all x,ε.

Second, we are going to prove that (iv) implies (ii). Suppose that $\pi(x,\varepsilon,u) \ge 0$ for all x,ε. Given some y,z with $y > z$, x denotes $\frac{y+z}{2}$, their middle point, and ε denotes $\frac{y-z}{2}$, half of their distance. By the assumption, we have

$$u(\frac{y+z}{2}) = \frac{1}{2}u(y)+\frac{1}{2}u(z)+\pi(\frac{y+z}{2},\frac{y-z}{2},u)(u(y)-u(z)) \ge \frac{1}{2}u(y)+\frac{1}{2}u(z)$$

for any $y > z$. This statement is called *midpoint concavity.* To get the concavity of $u(\cdot)$, it requires $u(\cdot)$ to be continuous.

In conclusion, we have that (i) is equivalent to (ii) because of the Jensen's inequality. (i) iff (iii) is shown in P.187 (actually, it requires $u(\cdot)$ to be strictly increasing). Since we have proven that (i) implies (iv) as well as (iv) implies (ii) (under continuity), then we have proven that these four statements are equivalent.

EXERCISE 6.C.4.

(a) Given $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{R}^N_+$ and $\alpha' = (\alpha'_1,\dots,\alpha'_N) \in \mathbb{R}^N_+$ with $\alpha \ge \alpha'$. Since $z_i \ge 0$ with probability one for all $i = 1,\dots,N$, it implies $\sum \alpha_i z_i \ge \sum \alpha'_i z_i$ almost surely. Hence, by the monotonicity of $u(\cdot)$, we can infer that $\int u(\sum \alpha_i z_i)dF(z_1,\dots,z_N) \ge \int u(\sum \alpha'_i z_i)dF(z_1,\dots,z_N)$. That is the monotonicity of $U(\cdot)$.

(b) Given $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{R}^N_+$, $\alpha' = (\alpha'_1,\dots,\alpha'_N) \in \mathbb{R}^N_+$ and $t \in (0,1)$. Since $u(\cdot)$ is concave, we have

$$\begin{aligned}
&U(t\alpha+(1-t)\alpha') \\
=&\int u(\sum(t\alpha_i+(1-t)\alpha_i')z_i)dF(z_1,\ldots,z_N) \\
=&\int u(t\sum\alpha_i z_i+(1-t)\sum\alpha_i' z_i)dF(z_1,\ldots,z_N) \\
\leq &t\int u(\sum\alpha_i z_i dF(z_1,\ldots,z_N)+(1-t)\int u(\sum\alpha_i' z_i dF(z_1,\ldots,z_N) \\
=&tU(\alpha)+(1-t)U(\alpha').
\end{aligned}$$

It is the concavity of $U(\cdot)$.

(c) Let $(\alpha^m)_{m\in\mathbb{N}}$ be a sequence of vector in $\mathbb{R}_+^N$ converging to $(\alpha)\in\mathbb{R}_+^N$. There exists a positive number B such that $\alpha^m \leq (B,\ldots,B)$ for every m. Since $u(\cdot)$ is increasing and all z_n are nonnegative almost surely, $\int u(\alpha_1^m z_1+\cdots+\alpha_N^m z_N)dF(z_1,\ldots,z_N)\leq \int u(Bz_1+\cdots+Bz_N)dF(z_1,\ldots,z_N)<\infty$. Thus, $u(\alpha_1^m z_1+\cdots+\alpha_N^m z_N)$ is dominated by a integrable function. By Lebesgue's dominated convergence theorem, we have

$$\lim_{m\to\infty}\int u(\alpha_1^m z_1+\cdots+\alpha_N^m z_N)dF(z_1,\ldots,z_N)=\int \lim_{m\to\infty} u(\alpha_1^m z_1+\cdots+\alpha_N^m z_N)dF(z_1,\ldots,z_N)$$

. Moreover, since $u(\cdot)$ is continuous, $\int \lim_{m\to\infty} u(\alpha_1^m z_1+\cdots+\alpha_N^m z_N)dF(z_1,\ldots,z_N)=\int u(\alpha_1 z_1+\cdots+\alpha_N z_N)dF(z_1,\ldots,z_N)$. In short, we have

$$\lim_{m\to\infty}\int u(\alpha_1^m z_1+\cdots+\alpha_N^m z_N)dF(z_1,\ldots,z_N)=\int u(\alpha_1 z_1+\cdots+\alpha_N z_N)dF(z_1,\ldots,z_N)$$

, and we prove the continuity of $U(\cdot)$.

In fact, the continuity is simply implied by the concavity.

EXERCISE 6.C.5.

(a) If $u(\cdot)$ is concave, it would satisfy $u(\lambda x+(1-\lambda)y)\geq \lambda u(x)+(1-\lambda)u(y)$ for all $x,y\in\mathbb{R}_+^L$ and $\lambda\in(0,1)$. It can be interpreted that the decision maker prefers $\lambda x+(1-\lambda)y$ to a lottery generating x with probability λ and y with probability $1-\lambda$. It coincide the definition of risk averse.

(b) Correction: denote the Bernoulli utility function by $\tilde{u}$, another symbol.
Let $p\gg 0$ be a fixed price vector and w, w' be two different wealth levels and $\lambda\in[0,1]$. Denote the demand function by $x(\cdot)$. Let $x=x(p,w)$ and $x'=x(p,w')$. Then we have $p\cdot(\lambda x+(1-\lambda)x')\leq \lambda w+(1-\lambda)w'$. Thus $u(\lambda x+(1-\lambda)x')\leq \tilde{u}(\lambda w+(1-\lambda)w')$. By assumption, $u(\cdot)$ is concave, then we have

$$u(\lambda x+(1-\lambda)x')\geq \lambda u(x)+(1-\lambda)u(x')=\lambda\tilde{u}(w)+(1-\lambda)\tilde{u}(w')$$

Thus, $\tilde{u}(\lambda w+(1-\lambda)w')\geq \lambda\tilde{u}(w)+(1-\lambda)\tilde{u}(w')$ and it also exhibits risk aversion.

To interpret, since the uncertain wealth leads to uncertain commodities. It means that if the decision maker exhibits risk aversion in the commodities domain, the risk aversion would also exhibit in the wealth domain.

(c) To provide a counterexample, let $L = 2$. Define $u(x) = \sqrt{\max\{x_1, x_2\}}$ and it is not concave. Consider $p = (1,2)$ so that $x(p, w) = (w, 0)$ for all $w \geq 0$. Hence, $\tilde{u} = \sqrt{w}$ for all $w \geq 0$. It is concave.

EXERCISE 6.C.6.

(a) Suppose that $u_2(\cdot)$ is more concave than $u_1(\cdot)$. By Jensen's inequality, for any $F(\cdot)$, we have

$$\begin{aligned}
&u_2(c(F, u_2))\\
=&\int u_2(x)dF(x)\\
=&\int \psi(u_1(x))dF(x)\\
\leq&\psi(\int u_1(x)dF(x))\\
=&\psi(u_1(c(F, u_1)))\\
=&u_2(c(F, u_1)).
\end{aligned}$$

Then $c(F, u_2) \leq c(F, u_1)$ holds if $u_2(x)$ is strictly increasing.

Conversely, suppose that $c(F, u_2) \leq c(F, u_1)$. There always exists $\psi(\cdot)$ mapping the range of $u_1(\cdot)$ to the range of $u_2(\cdot)$ with $\psi : u_1(x) \mapsto u_2(x)$. We are going to prove such $\psi(\cdot)$ is concave. Given $a < b$ and $t \in (0,1)$, let $F(\cdot)$ be a distribution with atom at $u_1(a), u_1(b)$ and height $t, 1-t$ respectively. Hence, since $c(F, u_2) \leq c(F, u_1)$, we have

$$\begin{aligned}
&t\psi(u_1(a)) + (1-t)\psi(u_1(b))\\
=&tu_2(a) + (1-t)u_2(b)\\
=&\int u_2(x)dF(x)\\
=&u_2(c(F, u_2))\\
\leq&u_2(c(F, u_1))\\
=&\psi u_1(c(F, u_1))\\
=&\psi(\int u_1(x)dF(x))\\
=&\psi(tu_1(a) + (1-t)u_1(b)).
\end{aligned}$$

It shows that $\psi(\cdot)$ is concave.

(b) Suppose that $c(F, u_2) \leq c(F, u_1)$. For any lottery $F(\cdot)$, if $\int u_2(x)dF(x) \geq u_2(\bar{x})$, by definition, it implies $u_2(c(F, u_2)) \geq u_2(\bar{x})$. Then $c(F, u_2) \geq \bar{x}$ is implied whenever $u_2(\cdot)$ is strictly increasing. Thus, it implies $c(F,u_1) \geq c(F,u_2) \geq \bar{x}$ and then $\int u_1(x)dF(x) = u_1(c(F,u_1)) \geq u_1(\bar{x})$. Condition (v) holds.

Conversely, supposed that $\int u_2(x)dF(x) \geq u_2(\bar{x})$ implies $\int u_1(x)dF(x) \geq u_1(\bar{x})$. By definition, $\int u_2(x)dF(x) = u_2(c(F,u_2))$. It implies $\int u_1(x)dF(x) \geq u_1(c(F,u_2))$. Thus, we have $u_1(c(F,u_1)) = \int u_1(x)dF(x) \geq u_1(c(F,u_2))$. Then condition (iii) holds if $u_1(\cdot)$ is strictly increasing.

EXERCISE 6.C.7.

(a) Supposed that $c(F,u_2) \leq c(F,u_1)$. Given some x, ε, let $F(\cdot)$ be a lottery winning $x+\varepsilon$ and $x-\varepsilon$ with probability $\frac{1}{2}+\pi(x,\varepsilon,u_2)$ and $\frac{1}{2}-\pi(x,\varepsilon,u_2)$ respectively. Thus, $u_2(c(F,u_2)) = \int u_2(x)dF(x) = u_2(x)$, or $c(F,u_2) = x$. By assumption, we have $u_1(c(F,u_1)) \geq u_1(x)$. Hence,

$$\begin{aligned}
&u_1(c(F,u_1))\\
=&\int u_1(x)dF(x)\\
=&(\frac{1}{2}+\pi(x,\varepsilon,u_2))u_1(x+\varepsilon)+(\frac{1}{2}-\pi(x,\varepsilon,u_2))u_1(x-\varepsilon)
\end{aligned}$$

and

$$\begin{aligned}
&u_1(x)\\
=&(\frac{1}{2}+\pi(x,\varepsilon,u_1))u_1(x+\varepsilon)+(\frac{1}{2}-\pi(x,\varepsilon,u_1))u_1(x-\varepsilon).
\end{aligned}$$

$u_1(c(F,u_1)) \geq u_1(x)$ implies $(\pi(x,\varepsilon,u_2)-\pi(x,\varepsilon,u_1))(u_1(x+\varepsilon)-u_1(x-\varepsilon)) \geq 0$. Then we have $\pi(x,\varepsilon,u_2) \geq \pi(x,\varepsilon,u_1)$ under strictly monotonicity assumption.

(b) Supposed that $\pi(x,\varepsilon,u_2) \geq \pi(x,\varepsilon,u_1)$. Assume that $\pi(x,\cdot,u_1)$ is differentiable for $i = 1,2$. Since $\pi(x,0,u_1) = \pi(x,0,u_1) = 0$, condition (iv) implies that $\frac{\partial\pi(x,0,u_2)}{\partial\varepsilon} \geq \frac{\partial\pi(x,0,u_1)}{\partial\varepsilon}$. By P.190, $r_A(x,u_i) = 4\frac{\partial\pi(x,0,u_i)}{\partial\varepsilon}$. Then we can conclude with $r_A(x,u_2) \geq r_A(x,u_1)$.

EXERCISE 6.C.8.

Since $u(\cdot)$ exhibits decreasing absolute risk aversion, we define $u_1(x) = u(w_1+x)$ and $u_2(x) = u(w_2+x)$, and assume that $w_1 < w_2$. Now we know that $u_1(\cdot)$ is more risk averse than $u_2(\cdot)$. By Example 6.C.2 continued, we can conclude that $\alpha_1^* < \alpha_2^*$. In other words, the decision maker puts more amount of wealth in the risky asset when the wealth is more.

EXERCISE 6.C.9.

(a) The saving decision problem without uncertainty requires the first order condition $u'(w-x_0) = v'(x_0)$. Then we define $f(x) = -u'(w-x)+E[v'(x+y)]$. Note that $f(x^*) = 0$

and $f'(x) \le 0$ because of the concavity. It implies $f(x) \ge 0$ for all $x < x^*$ and $f(x) \le 0$ for all $x > x^*$. Thus, since $f(x_0) = -u'(w - x_0) + E[v'(x_0 + y)] = -v'(x_0) + E[v'(x_0 + y)] > 0$, it implies $x^* > x_0$.

(b) Define $g_i(x) = -v_i'(x)$ for $i = 1,2$. $g_i'(x) = -v_i''(x) > 0$ for $i = 1,2$, so they can be treated as utility functions. Moreover, the coefficient of absolute prudence of v_i is $-\frac{v_i'''(x)}{v_i''(x)} = -\frac{g_i''(x)}{g_i'(x)}$ which equals to the absolute risk aversion of $g_i(x)$. By assumption, we have $r_A(x, g_2) \ge r_A(x, g_1)$. By proposition 6.C.2, we have $\int g_2(x)dF(x) \ge u_2(\bar{x})$ implies $\int g_1(x)dF(x) \ge u_1(\bar{x})$ for any $F(\cdot)$ and $\bar{x}$. Hence, we have $E[g_1(x_0 + y)] < g_1(x_0)$ implies $E[g_2(x_0 + y)] < g_2(x_0)$. In other words, $E[v_1'(x_0 + y)] > v_1'(x_0)$ implies $E[v_2'(x_0 + y)] > v_2'(x_0)$.

In words,

(c) As (b), let $g(x) = -v'(x)$. $v'''(x) > 0$ implies $g''(x) < 0$ and it is strictly risk averse. Thus, by definition, $E[g(x + y)] < g(E[x + y]) = g(x)$. We have $E[v'(x + y)] > v'(x)$ for all x.

(d) The decreasing of the coefficient of absolute risk aversion of $v(\cdot)$ means the negative of first derivative. That is

$$\frac{d}{dx}\frac{-v''(x)}{v'(x)} = \frac{-v'''(x)v'(x) + (v''(x))^2}{(v'(x))^2} < 0.$$

Then $-v'''(x)/v''(x) > -v''(x)/v'(x) > 0$ for all x, and hence $v'''(\cdot) > 0$.

EXERCISE 6.C.10.

Given an utility function $u(\cdot)$, we assume $w_1 > w_2$. By defining $u_1(x) = u(w_1 + x)$ and $u_2(x) = u(w_2 + x)$, we are going to prove that the statements in proposition 6.C.2 and 6.C.3 are parallel.

(i) By definition, $r_A(x, u_2) \ge r_A(x, u_2)$ is equivalent to that $u(\cdot)$ exhibits decreasing absolute risk aversion.

(ii) The statements are trivially equivalent.

(iii) By definition, $u_i(c(F, u_i)) = \int u(w_i + z)dF(z) = u(c_{w_i}) = u_i(c_{w_i} - w_i)$. It implies $c(F, u_i) = c_{w_i} - w_i$. Since $w_1 > w_2$, $c(F, u_2) \le c(F, u_1)$ is equivalent to that $(c_x - x)$ is increasing in x, or $(x - c_x)$ is decreasing in x.

(iv) By the definition of $\pi(w_1, \varepsilon, u)$, we have

$$\begin{aligned}
&u_1(x)\\
=&u(x + w_1)\\
=&(\frac{1}{2} + \pi(x + w_1, \varepsilon, u))u(x + w_1 + \varepsilon) + (\frac{1}{2} - \pi(x + w_1, \varepsilon, u))u(x + w_1 - \varepsilon)\\
=&(\frac{1}{2} + \pi(x + w_1, \varepsilon, u))u_1(\varepsilon) + (\frac{1}{2} - \pi(x + w_1, \varepsilon, u))u_1(-\varepsilon).
\end{aligned}$$

Thus, we have $\pi(x+w_1,\varepsilon,u)=\pi(x,\varepsilon,u_1)$. Similarly, we have $\pi(x+w_2,\varepsilon,u)=\pi(x,\varepsilon,u_2)$. Then we have proven the statements are equivalent.

(v) Note that $\int u_1(x)dF(x)\geq u_1(\bar{x})$ if and only if $\int u(w_1+x)dF(x)\geq u(w_1+\bar{x})$, and likewise for $u_2(\cdot)$. Hence, they are equivalent.

EXERCISE 6.C.11.

Suppose that $w_2>w_1$ are two different wealth level and let $u_i(t)=u(tw_i)$. Then, given the wealth level w_i, we can rewrite the maximization problem

$$\max_{0\leq\alpha\leq w_i}\int u(w_i-\alpha+\alpha z)dF(x)=\max_{0\leq\gamma\leq 1}\int u_i(1-\gamma+\gamma z)dF(x)$$

where γ represents the proportion of wealth invested in the risky asset. The absolute risk aversion of u_i is

$$r_A(t,u_i)=-\frac{u_i''(t)}{u_i'(t)}=-\frac{w_iu''(tw_i)}{u'(tw_i)}=r_R(tw_i,u).$$

Thus, if $rR(x,u)$ s increasing in x, $r_A(t,u_2)$ would be no less than $r_A(t,u_1)$. In other words, $u_2(\cdot)$ would be more risk averse than $u_1(\cdot)$ in the relative sense. In Example 6.C.2, $r_A(t,u_2)\geq r_A(t,u_1)$ implies $\gamma_2^*\leq\gamma_1^*$. That is, the proportion of wealth invested in the risky asset is increasing in the wealth level.

EXERCISE 6.C.12.

Exercise correction: In (c), it should be $\lim_{\rho\to 1}(x^{1-\rho}-1)/(1-\rho)$.

(a) The if part is trivial. To prove the only if part, note that $\frac{u''(\cdot)}{u'(\cdot)}$ is the derivative of $\ln u'(\cdot)$. Hence, supposed that the relative risk aversion equal to $\rho\neq 1$. Then we have

$$\begin{aligned}&(\ln u'(x))'=-x^{-1}\rho\\ \Rightarrow&\ln u'(x)=-\rho\ln x+C_1\\ \Rightarrow&u'(x)=e^{C_1}x^{-\rho}\\ \Rightarrow&u(x)=\frac{e^{C_1}}{1-\rho}x^{1-\rho}+C_2.\end{aligned}$$

In other words, we can say that $u(x)=\beta x^{1-\rho}+\alpha$. Note that we require $u(x)$ to be nondecreasing, so we require $\beta>0$.

(b) In (a), if $\rho=1$, we have $u(x)=e^{C_1}\ln x+C_2$

(c) Since $\lim_{\rho\to 1}x^{1-\rho}-1=\lim_{\rho\to 1}1-\rho=0$ and $\lim_{\rho\to 1}\frac{-x^{1-\rho}\ln x}{-1}$ exists, by L'Hôspital rule,

$$\lim_{\rho\to 1}\frac{x^{1-\rho}-1}{1-\rho}=\lim_{\rho\to 1}\frac{-x^{1-\rho}\ln x}{-1}=\ln x.$$

EXERCISE 6.C.13.

Let $\pi(\cdot)$ be the profit function and $F(\cdot)$ be distribution of the random price. By proposition 5.C.1 (ii), $\pi(\cdot)$ is convex. Then we have $E(\pi(p)) = \int \pi(p)dF(p) \geq \pi(\int p dF(p))\pi(E(p))$ by Jensen's inequality. That is, the firm would prefer the uncertain price.

EXERCISE 6.C.14.

(a) Suppose that $u^*(\cdot)$ is strongly more risk averse than $u(\cdot)$. Then that exist $k > 0$ and a nonincreasing, concave function $v(\cdot)$ such that $u^*(\cdot) = ku(\cdot) + v(\cdot)$. Define $\psi(x) = kx + v(u^{-1}(x))$. Since $\psi(u(x)) = u^*(x)$, by proposition 6.C.2, it is sufficient to show that $\psi(x)$ is concave. In fact, we only have to prove that $v(u^{-1}(x))$ is concave. Since $u(\cdot)$ is increasing and concave, u^{-1} will be convex. Then because $v(\cdot)$ is nonincreasing and convex, we have

$$v(u^{-1}(\alpha x + (1-\alpha)y)) \leq v(\alpha u^{-1}(x) + (1-\alpha)u^{-1}(y)) \leq \alpha v(u^{-1}(x)) + (1-\alpha)v(u^{-1}(y))$$

for all x, y and $\alpha \in (0,1)$.

(b) Suppose that $v(\cdot)$ is not constant. Since $v(\cdot)$ is decreasing and concave, it means that $v(\cdot)$ is not bounded below. Thus, since $u(\cdot)$ is increasing and bounded above, the limit of $u(x+1) - u(x)$ exists and it converges to 0. We can derive the following equation

$$u^*(x+1) - u^*(x) = k(u(x+1) - u(x)) + (v(x+1) - v(x)).$$

With the discussion above, we can infer that $u^*(x+1) - u^*(x)$ would be negative whenever x is sufficiently large, but it contradicts to the assumption that $u^*(\cdot)$ should be increasing.

(c) By (a), we know that, given a utility function $u(\cdot)$, the set of functions which is more strongly risk-aversion is a subset of the set of more Arrow-Pratt risk-aversion.

Moreover, by (b), we give an example to illustrate that there is no more strongly risk-aversion function other than an affine transformation. Meanwhile, the constant absolute risk averse functions, in the Arrow-Pratt sense, are all bounded, and they are not affine transformation of each other.

In conclusion, we can infer that given a utility function $u(\cdot)$, the set of functions which is more strongly risk-aversion is a proper subset of the set of more Arrow-Pratt risk-aversion.

EXERCISE 6.C.15.

(a) If $\min\{a, b\} \geq 1$, the riskless asset would be dominated by the risky asset because risky asset will generate profit greater than the riskless asset does. Thus, the necessary condition would be $\min\{a, b\} < 1$.

(b) If $\pi a + (1-\pi)b \leq 1$, the expected profit from the risky asset would be less than from riskless asset. Besides, the decision maker is a risk averter. The risky asset would be dominated if the condition above hold. Hence, one of the necessary conditions for strictly positive demand of risky asset is $\pi a + (1-\pi)b > 1$.

(c) Denote the demand of risky asset by x. The first-order condition would be

$$\pi(a-1)u'(1-x+xa) + (1-\pi)(b-1)u'(1-x+xb) = 0.$$

(d) Let $f(a,x,\pi) = \pi(a-1)u'(1-x+xa) + (1-\pi)(b-1)u'(1-x+xb)$.Then we have

$$\frac{\partial f(a,x,\pi)}{\partial a} = \pi[u'(1-x+xa) + x(a-1)u''(1-x+xa)] > 0$$
$$\frac{\partial f(a,x,\pi)}{\partial x} = \pi(a-1)^2u''(1-x+xa) + (1-\pi)(b-1)^2u''(1-x+xb) < 0$$

Thus, by the implicit function theorem, we have

$$\frac{dx_1}{da} = -\frac{dx}{da} = \frac{\frac{\partial f(a,x,\pi)}{\partial a}}{\frac{\partial f(a,x,\pi)}{\partial x}} < 0.$$

(e) If the probability of winning small prize become larger, the decision maker should buy more riskless asset. Thus, x_1 should increase. In other words, $dx_1/d\pi$ should be positive.

(f) Similar to (d), we derive

$$\frac{\partial f(a,x,\pi)}{\partial \pi} = (a-1)u'(1-x+xa) - (b-1)u'(1-x+xb) < 0.$$

Thus, we have

$$\frac{dx_1}{d\pi} = -\frac{dx}{d\pi} = \frac{\frac{\partial f(a,x,\pi)}{\partial \pi}}{\frac{\partial f(a,x,\pi)}{\partial x}} > 0.$$

It coincide with the conjecture.

EXERCISE 6.C.16.

(a) It he owns the lottery, then he would be willing to sell it only if the price p_s satisfies

$$u(w+p_s) \geq pu(w+G) + (1-p)u(w+B).$$

Thus, the minimum price equals to $u^{-1}(pu(w+G) + (1-p)u(w+B)) - w$.

(b) If he dose have it, then he would be willing to buy it only if the price p_b satisfies

$$pu(w+G-p_b) + (1-p)u(w+B-p_b) \geq u(w).$$

(c) With the notations in Proposition 6.C.3, we say that $c_w = w + p_s$ and $c_{w-p_b} = w$. By (iii) of the Proposition, it says that $(x-c_x)$ is decreasing in x if $u(\cdot)$ exhibits decreasing absolute risk aversion. $(x-c_x)$ equals to $-p_s$ when x equals to w, and it equals to $-p_b$ when x equals to $w-p_b$. Thus, p_s and p_b are the same only if $u(\cdot)$ exhibits constant absolute risk aversion.

(d) First, p_s satisfies

$$\sqrt{10+p_s} = p\sqrt{20} + (1-p)\sqrt{15}.$$

Then we have

$$p_s = 5[(7-4\sqrt{3})p^2 + (4\sqrt{3}-6)p + 1].$$

On the other hand, p_b should satisfy

$$p\sqrt{20-p_b} + (1-p)\sqrt{15-p_b} = \sqrt{10}.$$

Exercise 6.C.17.

If the individual exhibits constant relative risk aversion, his utility must be either $\beta x^{1-\rho}+\gamma$ or $\beta \ln x + \gamma$. It depends on that the relative risk aversion level is one or not. Assume it is the former. The result can be obtained in the similarly way if the utility function is the later. In period 1, given the initial wealth w_1, the decision maker chooses α_1 to be the solution to the maximization problem

$$\max_{0\le\alpha_1\le 1} \int \beta(((1-\alpha_1)R + \alpha_1 x_2)w_1)^{1-\rho} + \gamma dF(x_2).$$

We assume the solution is α^*. In fact, it is identical to the maximization problem

$$\max_{0\le\alpha\le 1} \int (((1-\alpha)R + \alpha x))^{1-\rho} dF(x). \tag{1}$$

It means the solution is irrelevant to the initial wealth w_1. Hence, in period 0, the decision maker aims to deal with the maximization problem

$$\max_{0\le\alpha_0\le 1} \int\int \beta(((1-\alpha^*)R + \alpha^* x_2)((1-\alpha_0)R + \alpha_0 x_1)w_0)^{1-\rho} + \gamma dF(x_1)dF(x_2)$$

It is identical to the following maximization problem

$$\max_{0\le\alpha_0\le 1} \int \beta(((1-\alpha^*)R + \alpha^* x_2))^{1-\rho} dF(x_2) \int (((1-\alpha_0)R + \alpha_0 x_1))^{1-\rho} dF(x_1)$$

which is identical to (6.1). It means that the optimal solutions of α_0 and α_1 are both α^*.

To show that the result can not be obtained if the preference is constant absolute risk aversion, we provide a counterexample to say that. Let $u(x) = -e^{-ax}$, a utility function exhibiting constant absolute risk aversion, and $R = 1$, the return of riskless asset. In the following discussion, α_t denote the amount, instead of the proportion, of money invested in risky asset. Thus, given the initial wealth w_1, the maximization problem in period 1 can we written as

$$\max_{0\le\alpha_1\le w_1}\int -e^{-((w_1-\alpha_1)+\alpha_1 x_2)}dF(x_2).$$

Similarly, the problem is identical to

$$\max_{0\le\alpha\le w}\int -e^{\alpha(1-x)}dF(x). \tag{2}$$

It means the solution is independent with the initial wealth. Hence, we assume the solution is α^*. Thus, the maximization problem in period 0 would be

$$\max_{0\le\alpha_0\le w_0}\int\int -e^{-(w_0-\alpha_0+\alpha_0 x_1-\alpha^*+\alpha^* x_2}dF(x_2)dF(x_1).$$

which is identical to (6.2). In conclusion, the amount of money invested in the risky asset would be the same in each period under this scenario. It means the proportion of the money invested in risky asset may not be the same. It depends on the initial wealth of each period.

EXERCISE 6.C.18.

(a)

$$r_A(x)=-\frac{u''(x)}{u'(x)}=\frac{1}{2}x^{-1}, r_R(x)=-\frac{xu''(x)}{u'(x)}=\frac{1}{2}$$

Thus, $r_A(5)=\frac{1}{10}$ and $r_R(5)=\frac{1}{2}$.

(b) **Certainty equivalent** $\int u(x)dF(x)=\frac{1}{2}\sqrt{16}+\frac{1}{2}\sqrt{4}=3$. Thus, $c(F,u)=9$.

Probability premium $u(10)=(\frac{1}{2}+\pi(10,6,u))(u(16))+(\frac{1}{2}-\pi(10,6,u))(u(4))$. Thus, $\pi(10,6,u)=\frac{\sqrt{10}-3}{2}$.

(c) **Certainty equivalent** $\int u(x)dF(x)=\frac{1}{2}\sqrt{36}+\frac{1}{2}\sqrt{16}=3$. Thus, $c(F,u)=25$

Probability premium $u(26)=(\frac{1}{2}+\pi(20,10,u))(u(36))+(\frac{1}{2}-\pi(26,10,u))(u(16))$. Thus, $\pi(26,10,u)=\frac{\sqrt{26}-5}{2}$.

EXERCISE 6.C.19.

Given some w, let $\theta=(\theta_1,\dots,\theta_N)$, a column vector, with $\sum_n\theta_n\le w$ be the portfolio of the risky asset. Since every risky asset is normal random variable, the total asset can be considered as a normal random variable with mean $\theta^\intercal\mu+r(w-\sum_n\theta_n)$ and variance $\theta^\intercal V\theta$, say X. Then the expected utility would be $E[-e^{-\alpha X}]$ which is similar to the moment-generating function of X. Specifically,

$$E[-e^{-\alpha X}]=-M_X(-\alpha)=-e^{-\alpha(\theta^\intercal\mu+r(w-\sum_n\theta_n))+\frac{1}{2}\alpha^2\theta^\intercal V\theta}.$$

Thus, the expected utility maximizing problem is equivalent to the maximizing problem of

$$\theta^\top \mu + r(w - \sum_n \theta_n) - \frac{1}{2}\alpha\theta^\top V\theta.$$

By the first order condition, we require

$$\mu - r\mathbf{1} - \frac{1}{2}\alpha(V + V^\top)\theta^* = \mu - r\mathbf{1} - \alpha V\theta^* = \mathbf{0}$$

where $\mathbf{0}$ and $\mathbf{1}$ are $\mathbb{R}^N$ vectors with all 0 and 1 respectively. The first equality holds because the variance-covariance is always symmetric. Thus, we have

$$\theta^* = \frac{1}{\alpha}V^{-1}(\mu - r\mathbf{1}).$$

Exercise 6.C.20.

By definition, $c(L_\epsilon, u) = u^{-1}(\frac{1}{2}u(x+\epsilon) + \frac{1}{2}u(x-\epsilon))$. Thus, we have

$$\frac{\partial c(L_\epsilon, u)}{\partial \epsilon} = \frac{u'(x+\epsilon) - u'(x-\epsilon)}{2u'(u^{-1}(\frac{1}{2}u(x+\epsilon) + \frac{1}{2}u(x-\epsilon)))}$$

and

$$\begin{aligned}\frac{\partial^2 c(L_\epsilon, u)}{\partial \epsilon^2} =& \frac{[\frac{1}{2}u''(x+\epsilon) + \frac{1}{2}u''(x-\epsilon)][u'(u^{-1}(\frac{1}{2}u(x+\epsilon) + \frac{1}{2}u(x-\epsilon)))]}{u'(u^{-1}(\frac{1}{2}u(x+\epsilon) + \frac{1}{2}u(x-\epsilon)))^2} \\ &- \frac{[\frac{1}{2}u'(x+\epsilon) - \frac{1}{2}u'(x-\epsilon)][\frac{u''(u^{-1}(\frac{1}{2}u(x+\epsilon)+\frac{1}{2}u(x-\epsilon)))}{u'(u^{-1}(\frac{1}{2}u(x+\epsilon)+\frac{1}{2}u(x-\epsilon)))}]}{u'(u^{-1}(\frac{1}{2}u(x+\epsilon) + \frac{1}{2}u(x-\epsilon)))^2}\end{aligned}$$

Then we have $\lim_{\epsilon\to 0} \frac{\partial^2 c(L_\epsilon, u)}{\partial \epsilon^2} = \frac{u''(x)}{u'(x)} = -r_A(x)$.

6.D. Comparison of Payoff Distribution in Terms of Return and Risk

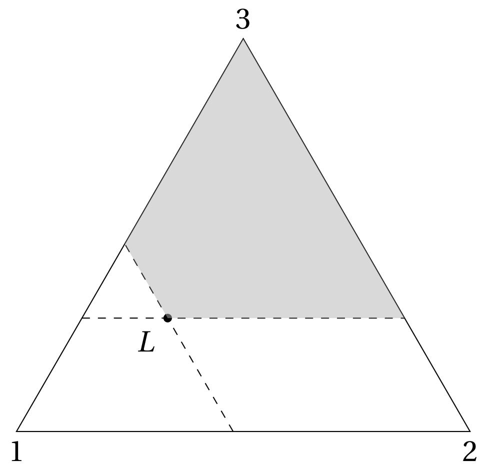
Exercise 6.D.1.

Suppose that $L = \{p_1^L, p_2^L, p_3^L\}$.

(a) Given L, the lotteries we want (p_1, p_2, p_3) should satisfy

$$\begin{aligned} p_1 &\leq p_1^L \\ p_1 + p_2 &\leq p_1^L + p_2^L \end{aligned}$$

Note that the second condition is equivalent to $p_3 \geq p_3^L$.
In the simplex diagram, it would look like



(b) It is as same as above.

EXERCISE 6.D.2.

Since $u(x) = x$ is nondecreasing, by definition, $\int x dF(x) \geq \int x dG(x)$. Let $F(x)$ be a lottery getting 3 dollar for sure, and $G(x)$ be a lottery getting 0 dollar and 4 dollar with equal chance. Hence, the mean of $F(\cdot)$ is 3 and it is 2 for $G(\cdot)$. However, $F(3) = 1 > 0.5 = G(3)$.

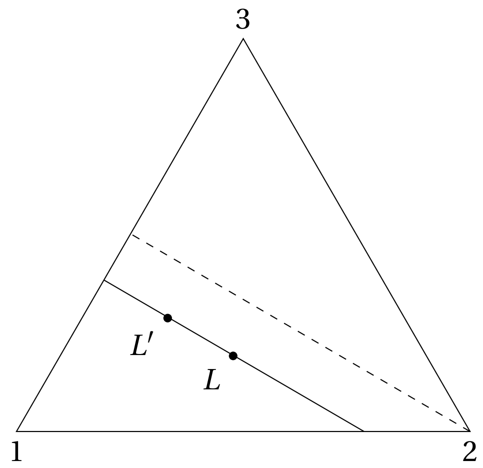
EXERCISE 6.D.3.

An elementary increase in risk from $F(\cdot)$ is also a mean-preserving spread of $F(\cdot)$. In Example 6.D.2, it says that any mean-preserving spread of $F(\cdot)$ is second-order dominated by $F(\cdot)$. Hence, so is any elementary increase in risk from $F(\cdot)$.

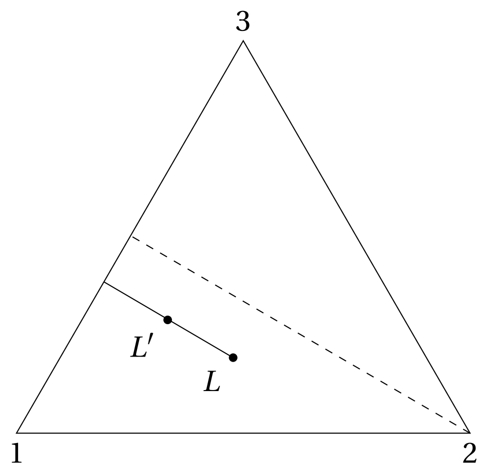
EXERCISE 6.D.4.

Let $L = (p_1, p_2, p_3)$ and $L' = (p'_1, p'_2, p'_3)$.

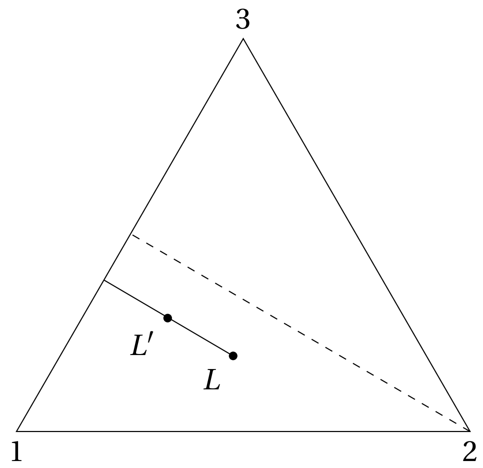
(a) The mean of L and L' are $2 - p_1 + p_3$ and $2 - p'_1 + p'_3$ respectively. If they were the same, then we have $p_1 - p_3 = p'_1 - p'_3$. It means the line of L to L' will be parallel to the height from $2-vertex. In the simplex digram it would look like



(b) For any risk averse preference, it would prefer 2 dollars for sure to a lottery getting 1 dollar and 3 dollar with equal chance. Thus, the indifference curve would be steeper than the height from \$2-vertex. The second-order stochastically dominated lotteries would look like as following



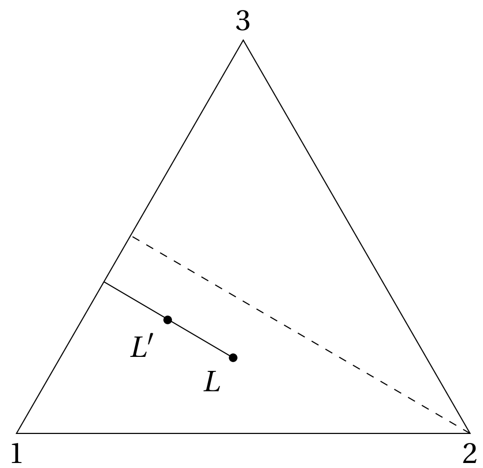
(c) If L' is a mean-preserving spread of L', it means L' keeps the mean of L but more risky. The possible allocation of L' would be like as following



(d) Inequality (6.D.2) holds if and only if

$$p_1' \geq p_1$$
$$p_1' + (p_1' + p_2') \geq p_1 + (p_1 + p_2).$$

Besides, we require they have the same mean, then we also needs $p_1' - p_1 = p_3' - p_3$. This equation is equivalent to $2 * p_1' + p_2' = 2 * p_1 + p_2$, so the second inequality holds. Hence, we only have to restrict $p_1' \geq p_1$. It means that, on the isomean line, L' should be closer to 1\$-dollar node than L. The diagram would look like



In fact, we have three identical diagram form (a) through (c).

6.E. State-dependent Utility

EXERCISE 6.E.1.

Let $x \succsim_R x'$ denote that x is at least as good as x' in the presence of regret and $R(x, x')$ denote the expected regret associated with x relative to x'. Thus, we have

$$\begin{aligned}
R(x,x') &= \frac{1}{3}\sqrt{0}+\frac{1}{3}\sqrt{4}+\frac{1}{3}\sqrt{0} = 1\\
R(x',x) &= \frac{1}{3}\sqrt{0}+\frac{1}{3}\sqrt{0}+\frac{1}{3}\sqrt{3} = \frac{\sqrt{3}}{2}\\
R(x,x'') &= \frac{1}{3}\sqrt{2}+\frac{1}{3}\sqrt{0}+\frac{1}{3}\sqrt{0} = \frac{\sqrt{2}}{2}\\
R(x'',x) &= \frac{1}{3}\sqrt{0}+\frac{1}{3}\sqrt{1}+\frac{1}{3}\sqrt{2} = \frac{\sqrt{2}+1}{2}\\
R(x',x'') &= \frac{1}{3}\sqrt{2}+\frac{1}{3}\sqrt{0}+\frac{1}{3}\sqrt{1} = \frac{\sqrt{2}+1}{2}\\
R(x'',x') &= \frac{1}{3}\sqrt{0}+\frac{1}{3}\sqrt{5}+\frac{1}{3}\sqrt{0} = \frac{\sqrt{5}}{2}.
\end{aligned}$$

We can infer that $x' \succsim_R x$, $x \succsim_R x''$, and $x'' \succsim_R x'$ and $\succsim_R$ violates transitivity.

EXERCISE 6.E.2.

(a) Denote the probability of each state by π_i. Then the expected utility function from the contingent commodity (x_1, x_2) can be written as $U(x_1, x_2) = \pi_1 u(x_1) + \pi_2 u(x_2)$. Since the decision maker is a risk averter, $u(\cdot)$ would be concave. Moreover, $U(\cdot)$ is also concave, and hence the preference would be convex.

(b) By Exercise 6.C.5(a), a concave utility function cab be interpreted as the decision maker exhibiting risk aversion.

(c) Given a sequence $(\alpha^m)_{m\in\mathbb{R}}$ converging to $\alpha \in \mathbb{R}^{\mathbb{N}}{}_+$, then there is a $B \in \mathbb{R}$ such that $\alpha^m \leq (B, \ldots, B)$ for all m. Since all $z_n \geq 0$ almost surely and $u(\cdot)$ is increasing, we can infer that $u(\sum_n \alpha_n^m z_n) \leq u(\sum_n B z_n)$ for all m. We define $u_m(z_1, \ldots, z_N) = u(\sum_n \alpha_n^m z_n)$ and $u_B(z_1, \ldots, z_N) = u(\sum_n B z_n)$. With discussion above, $u_m(\cdot)$ is dominated by $u_B(\cdot)$. That is, $u_m(z_1, \ldots, z_N) \leq u_B(z_1, \ldots, z_N)$ for all possible z_n and all m.

By Lebesgue's Dominated Convergence Theorem and continuity of $u(\cdot)$, we can obtain

$$\begin{aligned}
&\lim_{m\to\infty} U(\alpha^m)\\
=&\lim_{m\to\infty}\int u_m(z_1,\dots,z_N)dF(z_1,\dots,z_N)\\
=&\int \lim_{m\to\infty} u_m(z_1,\dots,z_N)dF(z_1,\dots,z_N)\\
=&\int \lim_{m\to\infty} u(\sum_n \alpha_n^m z_n)dF(z_1,\dots,z_N)\\
=&\int u(\sum_n \alpha_n z_n)dF(z_1,\dots,z_N)\\
=&U(\alpha)
\end{aligned}$$

It proves the continuity of $U(\cdot)$.

EXERCISE 6.E.3.

By definition, $G(\cdot)$ and $G^*(\cdot)$ share the same mean. For any concave $u(\cdot)$, we have $\sum_s \pi_s u(g(s)) \le u(1)$ because $E(g) = 1$. Moreover, we can derive

$$\begin{aligned}
&\sum_s \pi_s u(\alpha g(s) + (1-\alpha))\\
\ge&\alpha\sum_s \pi_s u(g(s)) + (1-\alpha)u(1)\\
\ge&\sum_s \pi_s u(g(s)).
\end{aligned}$$

Then we have $G^*(\cdot)$ second-order stochastically dominates $G(\cdot)$.

The interpretation is that $G^*(\cdot)$ pulls all the possible outcome of $G(\cdot)$ closer to the mean, so it is intuitively preferred by risk averters.

6.F. SUBJECTIVE PROBABILITY THEORY

EXERCISE 6.F.1.

First, we are going to prove that the utility function is unique up to positive affine transformation. Suppose an preference $\succsim$ can be represented by $u(\cdot)$ and $\hat{u}(\cdot)$. In each sate, by proposition 6.B.2, we have that $\pi_s u(\cdot) + \beta_s$ and $\hat{\pi}_s \hat{u}(\cdot) + \hat{\beta}_s$ are the same up to positive affine transformation. So are $u(\cdot)$ and $\hat{u}(\cdot)$.

Second, we are going to show that the Subjective probabilities are uniquely determined. Suppose that $(\pi_1,\dots,\pi_s) \ne (\pi'_1,\dots,\pi'_s)$. Since their sum are both 1, then there are at least 2 states have different probability. Without loss of generality, we assume that $\pi_1 \ne \pi'_1$ and $\pi_2 \ne \pi'_2$. We normalize $u(0) = 0$. Thus, we can find suitable (x_1, x_2) such that $\pi_1 u(x_1) + \pi_2 u(x_2) > 0$ and $\pi'_1 u(x_1) + \pi'_2 u(x_2) > 0$. Thus, we have $(x_1, x_2, \dots, 0)$ is preferred with first kind of Subjective probabilities but $(0,\dots,0)$ is preferred with second kind, a contradiction.

EXERCISE 6.F.2.

(a) Suppose that $P = \{p\}$. Then we have

$$\begin{aligned} & U_W(R) > U_W(H) \\ \Leftrightarrow & 0.49 > p \\ \Leftrightarrow & 0.51 < 1 - p \\ \Leftrightarrow & U_B(R) < U_B(H) \end{aligned}$$

(b) If $P = \{0, 1\}$, then we have $U_W(R) = 0.49 > 0 = U_W(H)$ and $U_B(R) = 0.51 > 0 = U_B(H)$.