Ito integral

1 Ito integral

Construction of Ito integral

Ito integral $\int_0^1 B_s dB_s$

Ito integral $\int_0^1 s dB_s$

Ito integral $\int_0^1 s B_s dB_s$

2 High school integral

Construction of high school integral

High school integral $\int_0^1 B_s ds$

High school integral $\int_0^1 s ds$

High school integral $\int_0^1 s B_s ds$

Construction of Ito integral

Stock price at beginning of day s	B_s
Stock position at beginning of day s	$f(s, B_s)$
P&L of 1 stock at end of day s	$dB_s = B_{s+ds} - B_s$
P&L of stock position at end of day s	$f(s, B_s)dB_s$
Cumulative P&L of stock positions up to end of day t	$\int_0^t f(s, B_s)dB_s$

Ito integral $\int_0^1 B_s dB_s$

We flips a fair coin 10 times and we get

$$HHTHTTHHHT$$

Using these coin flips construct a Brownian motion sample path B_t up to 1 year and then the following Ito integral.

$$\int_0^1 B_s dB_s$$

Time	0/10	1/10	2/10	3/10	4/10	5/10
Coin flip	–	H	H	T	H	T
Conversion	–	1	1	-1	1	-1
Cum sum	0	1	2	1	2	1
B_t (Stock price at time t)	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$
$dB_t = B_t - B_{t-dt}$ (P&L/Share at time t)	–	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{10}}$
$f(t, B_t) = B_{t-dt}$ (Stock position at time t)	–	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$
$f(t, B_t)dB_t$ (P&L at time t)	–	0	$\frac{1}{10}$	$\frac{-2}{10}$	$\frac{1}{10}$	$\frac{-2}{10}$
$\int_0^t f(s, B_s)dB_s$ (Cum P&L up to time t)	–	0	$\frac{1}{10}$	$\frac{-1}{10}$	$\frac{0}{10}$	$\frac{-2}{10}$

Time	6/10	7/10	8/10	9/10	10/10
Coin flip	T	H	H	H	T
Conversion	-1	1	1	1	-1
Cum sum	0	1	2	3	2
B_t (Stock price at time t)	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$
$dB_t = B_t - B_{t-dt}$ (P&L/Share at time t)	$\frac{-1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{10}}$
$f(t, B_t) = B_{t-dt}$ (Stock position at time t)	$\frac{1}{\sqrt{10}}$	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$
$f(t, B_t)dB_t$ (P&L at time t)	$\frac{-1}{10}$	$\frac{0}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{-3}{10}$
$\int_0^t f(s, B_s)dB_s$ (Cum P&L up to time t)	$\frac{-3}{10}$	$\frac{-3}{10}$	$\frac{-2}{10}$	$\frac{0}{10}$	$\frac{-3}{10}$

Therefore,

$$\int_0^1 B_s dB_s = \frac{-3}{10}$$

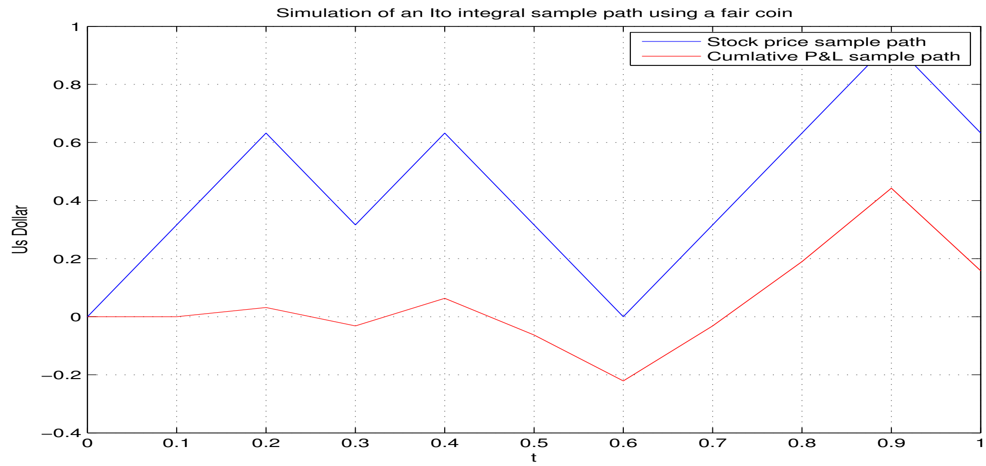

Figure 1: Simulation of an Ito integral sample path $\int_0^1 B_s dB_s$ using a fair coin.

```
clear all; close all; clc; rng('default');

M=1; % Number of simulation
n=10; % Number of days per year
T=1; % Number of years in simulation

% Choice of coin
c=[1 1 -1 1 -1 -1 1 1 1 -1]';

increment=c/sqrt(n);
S=cumsum(increment); S=[zeros(1,M); S];
PL_Per_Share=increment; PL_Per_Share=[zeros(1,M); PL_Per_Share];

% Choice of stock position
position=1;
switch position
    case 1; p=[zeros(1,M); S(1:end-1,:)]; % B_s
    case 2; p=[zeros(2,M); cumsum(ones(ceil(n*T)-1,M))/n]; % s
    case 3; p=[zeros(2,M); cumsum(ones(ceil(n*T)-1,M))/n].*[zeros(1,M); S(1:end-1,:)]; % sB_s
end

PL=p.*PL_Per_Share;
Cumlative_PL=cumsum(PL);

plot(0:1/n:T,S,0:1/n:T,Cumlative_PL,'r'); grid on
xlabel('t'); ylabel('Us Dollar');
legend('Stock price sample path','Cumlative P&L sample path')
title('Simulation of an Ito integral sample path using a fair coin')
```

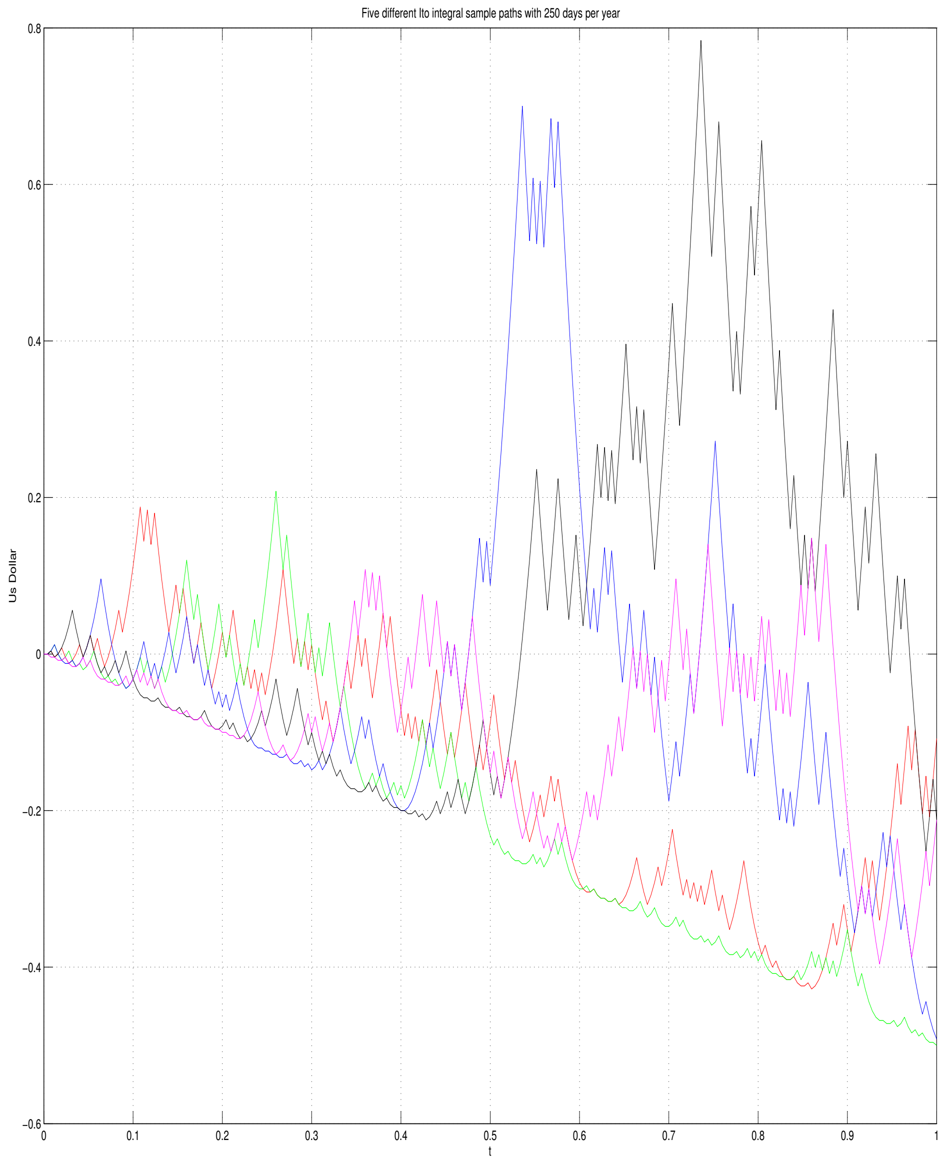

Figure 2: Five different Ito integral sample paths $\int_0^1 B_s dB_s$

```
clear all; close all; clc; rng('default');

M=5; % Number of simulation
n=252; % Number of days per year
T=1; % Number of years in simulation

% Choice of coin
coin=2;
switch coin
    case 1; c=randn(n*T,M); % normal coin
    case 2; c=2*random('bino',ones(n*T,M),0.5*ones(n*T,M))-1; % fair coin
    case 3; c=(random('bino',ones(n*T,M),.4*ones(n*T,M))-0.4)/sqrt(.4*.6); % unfair coin
end

increment=c/sqrt(n);
S=cumsum(increment); S=[zeros(1,M); S];
PL_Per_Share=increment; PL_Per_Share=[zeros(1,M); PL_Per_Share];

% Choice of stock position
position=1;
switch position
    case 1; p=[zeros(1,M); S(1:end-1,:)]; % B_s
    case 2; p=[zeros(2,M); cumsum(ones(ceil(n*T)-1,M))/n]; % s
    case 3; p=[zeros(2,M); cumsum(ones(ceil(n*T)-1,M))/n].*[zeros(1,M); S(1:end-1,:)]; % sB_s
end

PL=p.*PL_Per_Share;
Cumlative_PL=cumsum(PL);

% Plot of five different Ito integral sample paths with 250 days per year
color='rgbkm';
for i=1:M
    plot(0:1/n:T,Cumlative_PL(1:length(0:1/n:T),i),color(i));
    grid on; hold on
end
xlabel('t'); ylabel('Us Dollar');
title('Five different Ito integral sample paths')
```

Ito integral $\int_0^1 s dB_s$

We flips a fair coin 10 times and we get

$$HHTHTTHHHT$$

Using these coin flips construct a Brownian motion sample path B_t up to 1 year and then the following Ito integral.

$$\int_0^1 s dB_s$$

Time	0/10	1/10	2/10	3/10	4/10	5/10
Coin flip	–	H	H	T	H	T
Conversion	–	1	1	−1	1	−1
Cum sum	0	1	2	1	2	1
B_t	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$
$dB_t = B_t - B_{t-dt}$	–	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{10}}$
$f(t, B_t) = t - dt$	–	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$
$f(t, B_t)dB_t$	–	0	$\frac{1}{(10)^{3/2}}$	$\frac{-2}{(10)^{3/2}}$	$\frac{3}{(10)^{3/2}}$	$\frac{-4}{(10)^{3/2}}$
$\int_0^t f(s, B_s)dB_s$	–	0	$\frac{1}{(10)^{3/2}}$	$\frac{-1}{(10)^{3/2}}$	$\frac{2}{(10)^{3/2}}$	$\frac{-2}{(10)^{3/2}}$

Time	6/10	7/10	8/10	9/10	10/10
Coin flip	T	H	H	H	T
Conversion	−1	1	1	1	−1
Cum sum	0	1	2	3	2
B_t	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$
$dB_t = B_t - B_{t-dt}$	$\frac{-1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{10}}$
$f(t, B_t) = t - dt$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$
$f(t, B_t)dB_t$	$\frac{-5}{(10)^{3/2}}$	$\frac{6}{(10)^{3/2}}$	$\frac{7}{(10)^{3/2}}$	$\frac{8}{(10)^{3/2}}$	$\frac{-9}{(10)^{3/2}}$
$\int_0^t f(s, B_s)dB_s$	$\frac{-7}{(10)^{3/2}}$	$\frac{-1}{(10)^{3/2}}$	$\frac{6}{(10)^{3/2}}$	$\frac{14}{(10)^{3/2}}$	$\frac{5}{(10)^{3/2}}$

Therefore,

$$\int_0^1 s dB_s = \frac{5}{(10)^{3/2}}$$

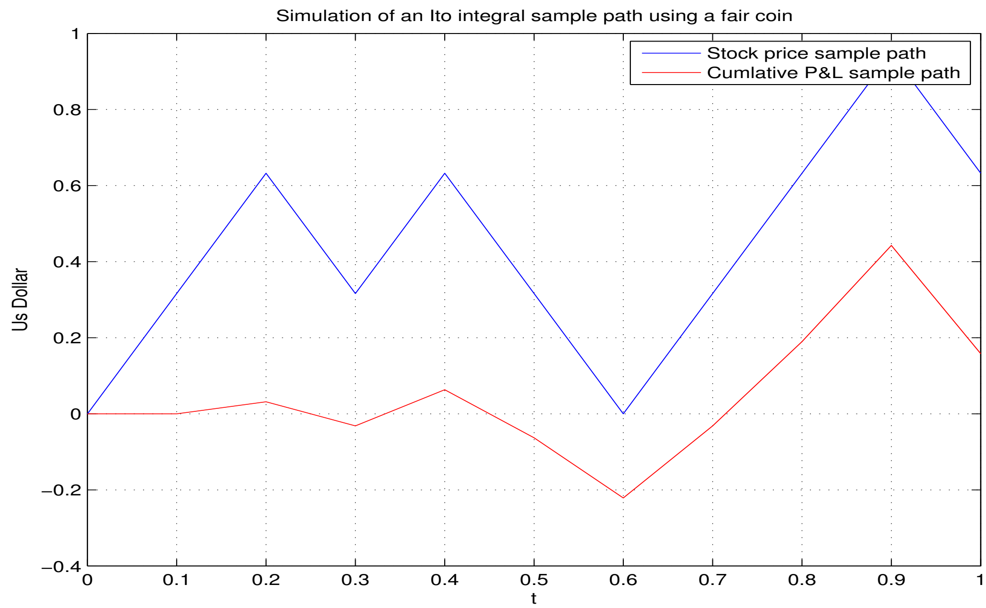

Figure 3: Simulation of an Ito integral sample path $\int_0^1 sdB_s$ using a fair coin.

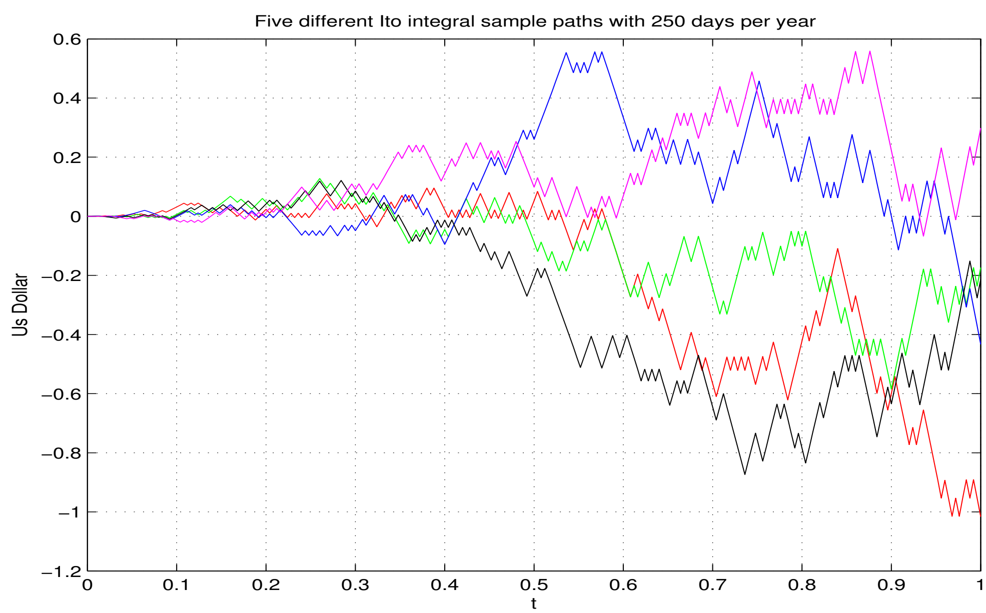

Figure 4: Five different Ito integral sample paths $\int_0^1 sdB_s$

Ito integral $\int_0^1 sB_s dB_s$

We flips a fair coin 10 times and we get

$$HHTHTTHHHT$$

Using these coin flips construct a Brownian motion sample path B_t up to 1 year and then the following Ito integral.

$$\int_0^1 sB_s dB_s$$

Time	0/10	1/10	2/10	3/10	4/10	5/10
Coin flip	–	H	H	T	H	T
Conversion	–	1	1	-1	1	-1
Cum sum	0	1	2	1	2	1
B_t	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$
$dB_t = B_t - B_{t-dt}$	–	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{10}}$
$h(t, B_t) = t - dt$	–	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$
$g(t, B_t) = B_{t-dt}$	–	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$
$f(t, B_t) = h(t, B_t)g(t, B_t)$	–	0	$\frac{1}{(10)^{3/2}}$	$\frac{4}{(10)^{3/2}}$	$\frac{3}{(10)^{3/2}}$	$\frac{8}{(10)^{3/2}}$
$f(t, B_t)dB_t$	–	0	$\frac{1}{10^2}$	$\frac{-4}{10^2}$	$\frac{3}{10^2}$	$\frac{-8}{10^2}$
$\int_0^t f(s, B_s)dB_s$	–	0	$\frac{1}{10^2}$	$\frac{-3}{10^2}$	$\frac{0}{10^2}$	$\frac{-8}{10^2}$

Time	6/10	7/10	8/10	9/10	10/10
Coin flip	T	H	H	H	T
Conversion	-1	1	1	1	-1
Cum sum	0	1	2	3	2
B_t	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$
$dB_t = B_t - B_{t-dt}$	$\frac{-1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{10}}$
$h(t, B_t) = t - dt$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$
$g(t, B_t) = B_{t-dt}$	$\frac{1}{\sqrt{10}}$	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$
$f(t, B_t) = h(t, B_t)g(t, B_t)$	$\frac{5}{(10)^{3/2}}$	$\frac{0}{(10)^{3/2}}$	$\frac{7}{(10)^{3/2}}$	$\frac{16}{(10)^{3/2}}$	$\frac{27}{(10)^{3/2}}$
$f(t, B_t)dB_t$	$\frac{-5}{10^2}$	$\frac{0}{10^2}$	$\frac{7}{10^2}$	$\frac{16}{10^2}$	$\frac{-27}{10^2}$
$\int_0^t f(s, B_s)dB_s$	$\frac{-13}{10^2}$	$\frac{-13}{10^2}$	$\frac{-6}{10^2}$	$\frac{10}{10^2}$	$\frac{-17}{10^2}$

Therefore,

$$\int_0^1 sB_s dB_s = \frac{-17}{10^2}$$

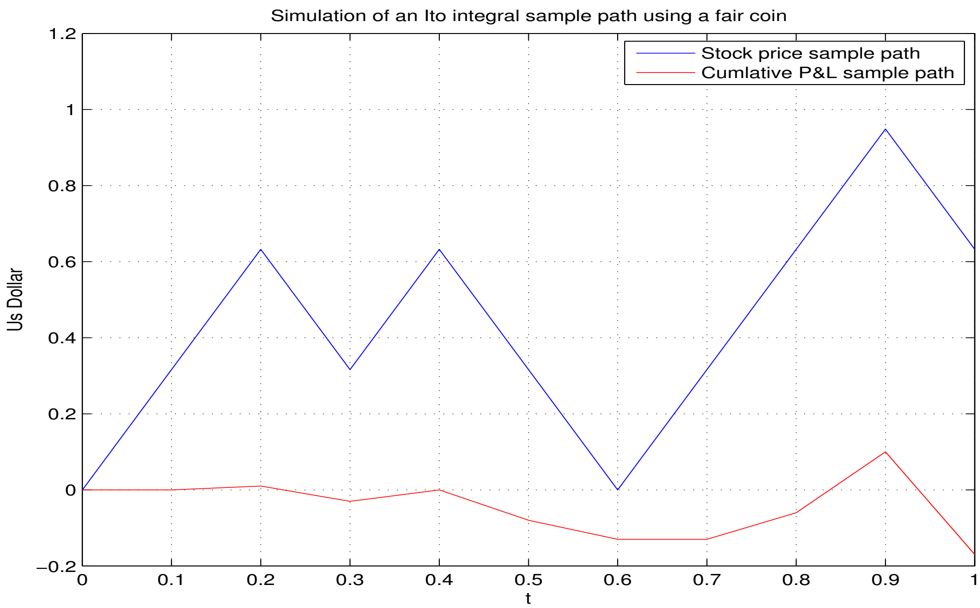

Figure 5: Simulation of an Ito integral sample path $\int_0^1 sB_s dB_s$ using a fair coin.

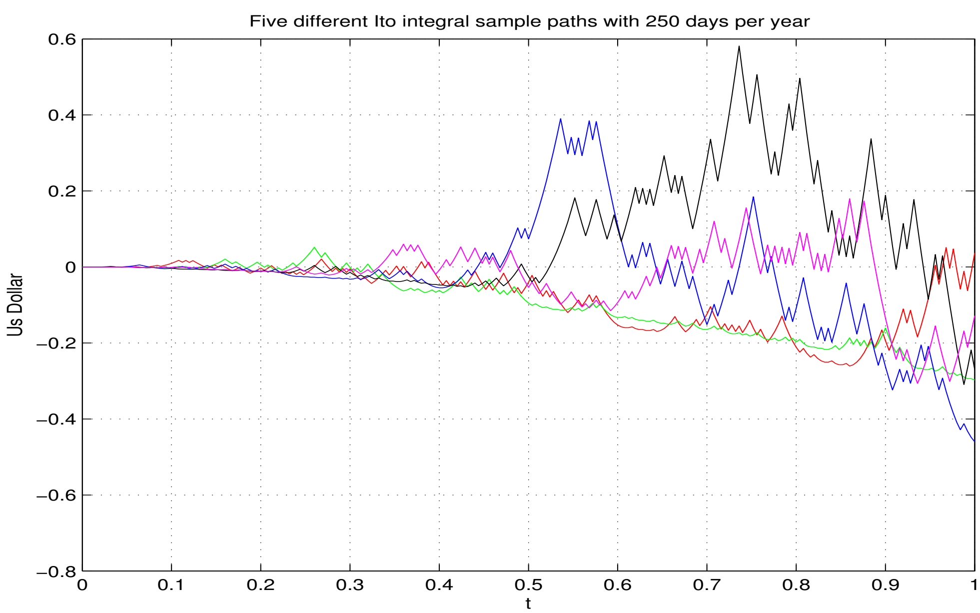

Figure 6: Five different Ito integral sample paths $\int_0^1 sB_s dB_s$

Construction of high school integral

Stock price at beginning of day s	B_s
MMA position at beginning of day s	$f(s, B_s)$
P&L of 1 MMA at end of day s	$ds = (s + ds) - (s)$
P&L of MMA position at end of day s	$f(s, B_s)ds$
Cumulative P&L of MMA positions up to end of day t	$\int_0^t f(s, B_s)ds$

High school integral $\int_0^1 B_s ds$

We flips a fair coin 10 times and we get

$$HHTHTTHHHT$$

Using these coin flips construct a Brownian motion sample path B_t up to 1 year and then the following high school integral.

$$\int_0^1 B_s ds$$

Time	0/10	1/10	2/10	3/10	4/10	5/10
Coin flip	–	H	H	T	H	T
Conversion	–	1	1	-1	1	-1
Cum sum	0	1	2	1	2	1
B_t (Stock price)	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$
$dt = dt$ (P&L/Bond share)	–	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
$f(t, B_t) = B_{t-dt}$ (Bond position)	–	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$
$f(t, B_t)dB_t$ (P&L)	–	0	$\frac{1}{(10)^{3/2}}$	$\frac{2}{(10)^{3/2}}$	$\frac{1}{(10)^{3/2}}$	$\frac{2}{(10)^{3/2}}$
$\int_0^t f(s, B_s)dB_s$ (Cum P&L)	–	0	$\frac{1}{(10)^{3/2}}$	$\frac{3}{(10)^{3/2}}$	$\frac{4}{(10)^{3/2}}$	$\frac{6}{(10)^{3/2}}$

Time	6/10	7/10	8/10	9/10	10/10
Coin flip	T	H	H	H	T
Conversion	-1	1	1	1	-1
Cum sum	0	1	2	3	2
B_t (Stock price)	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$
$dt = dt$ (P&L/Bond share)	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
$f(t, B_t) = B_{t-dt}$ (Bond position)	$\frac{1}{\sqrt{10}}$	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$
$f(t, B_t)dB_t$ (P&L)	$\frac{1}{(10)^{3/2}}$	$\frac{0}{(10)^{3/2}}$	$\frac{1}{(10)^{3/2}}$	$\frac{2}{(10)^{3/2}}$	$\frac{3}{(10)^{3/2}}$
$\int_0^t f(s, B_s)dB_s$ (Cum P&L)	$\frac{7}{(10)^{3/2}}$	$\frac{7}{(10)^{3/2}}$	$\frac{8}{(10)^{3/2}}$	$\frac{10}{(10)^{3/2}}$	$\frac{13}{(10)^{3/2}}$

Therefore,

$$\int_0^1 B_s ds = \frac{13}{(10)^{3/2}}$$

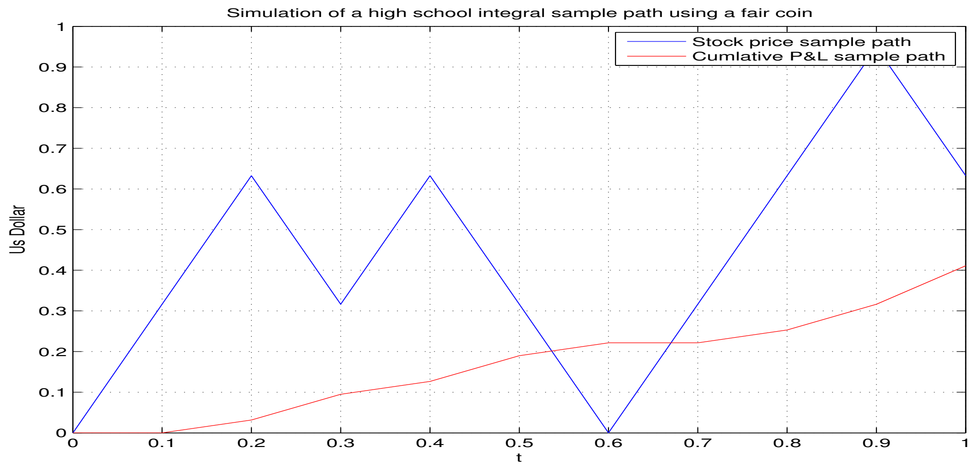

Figure 7: Simulation of a high school integral sample path $\int_0^1 B_s ds$ using a fair coin.

```
clear all; close all; clc; rng('default');

M=1; % Number of simulation
n=10; % Number of days per year
T=1; % Number of years in simulation

% Choice of coin
c=[1 1 -1 1 -1 -1 1 1 1 -1]';

increment=c/sqrt(n);
S=cumsum(increment); S=[zeros(1,M); S];
PL_Per_Bond_Share=[(1/n)*ones(1,M); (1/n)*ones(n*T,M)];

% Choice of bond position
position=1;
switch position
    case 1; p=[zeros(1,M); S(1:end-1,:)]; % B_s
    case 2; p=[zeros(2,M); cumsum(ones(ceil(n*T)-1,M))/n]; % s
    case 3; p=[zeros(2,M); cumsum(ones(ceil(n*T)-1,M))/n].*[zeros(1,M); S(1:end-1,:)]; % sB_s
end

PL=p.*PL_Per_Bond_Share;
Cumlative_PL=cumsum(PL);

plot(0:1/10:1,S,0:1/10:1,Cumlative_PL,'-r'); grid on
xlabel('t'); ylabel('Us Dollar');
legend('Stock price sample path','Cumlative P&L sample path')
title('Simulation of a high school integral sample path using a fair coin')
```

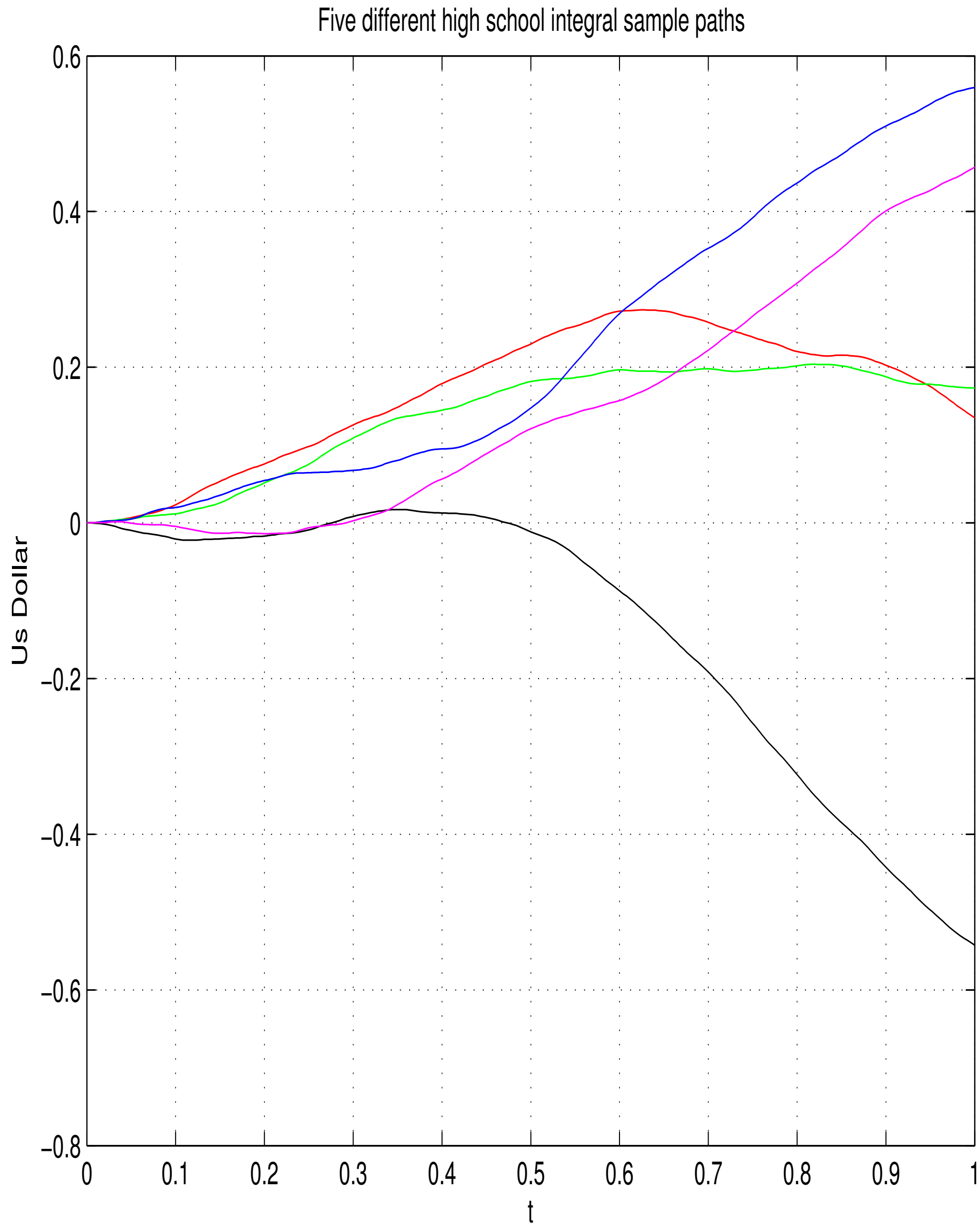

Figure 8: Five different high school integral sample paths $\int_0^1 B_s ds$

```
clear all; close all; clc; rng('default');

M=5; % Number of simulation
n=252; % Number of days per year
T=1; % Number of years in simulation

% Choice of coin
coin=2;
switch coin
    case 1; c=randn(n*T,M); % normal coin
    case 2; c=2*random('bino',ones(n*T,M),0.5*ones(n*T,M))-1; % fair coin
    case 3; c=(random('bino',ones(n*T,M),.4*ones(n*T,M))-0.4)/sqrt(.4*.6); % unfair coin
end

increment=c/sqrt(n);
S=cumsum(increment); S=[zeros(1,M); S];
PL_Per_Bond_Share=[(1/n)*ones(1,M); (1/n)*ones(n*T,M)];

% Choice of bond position
position=1;
switch position
    case 1; p=[zeros(1,M); S(1:end-1,:)]; % B_s
    case 2; p=[zeros(2,M); cumsum(ones(ceil(n*T)-1,M))/n]; % s
    case 3; p=[zeros(2,M); cumsum(ones(ceil(n*T)-1,M))/n].*[zeros(1,M); S(1:end-1,:)]; % sB_s
end

PL=p.*PL_Per_Bond_Share;
Cumlative_PL=cumsum(PL);

% Plot of five high school integral sample paths with 250 days per year
color='rgbkm';
for i=1:M
    plot(0:1/n:T,Cumlative_PL(1:length(0:1/n:T),i),color(i));
    grid on; hold on
end
xlabel('t'); ylabel('Us Dollar');
title('Five different high school integral sample paths')
```

High school integral $\int_0^1 sds$

We flips a fair coin 10 times and we get

$$HHTHTTHHHT$$

Using these coin flips construct a Brownian motion sample path B_t up to 1 year and then the following high school integral.

$$\int_0^1 sds$$

Time	0/10	1/10	2/10	3/10	4/10	5/10
Coin flip	–	H	H	T	H	T
Conversion	–	1	1	-1	1	-1
Cum sum	0	1	2	1	2	1
B_t (Stock price)	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$
$dt = dt$ (P&L/Bond share)	–	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
$f(t, B_t) = t - dt$ (Bond position)	–	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$
$f(t, B_t)dB_t$ (P&L)	–	0	$\frac{1}{(10)^2}$	$\frac{2}{(10)^2}$	$\frac{3}{(10)^2}$	$\frac{4}{(10)^2}$
$\int_0^t f(s, B_s)dB_s$ (Cum P&L)	–	0	$\frac{1}{(10)^2}$	$\frac{3}{(10)^2}$	$\frac{6}{(10)^2}$	$\frac{10}{(10)^2}$

Time	6/10	7/10	8/10	9/10	10/10
Coin flip	T	H	H	H	T
Conversion	-1	1	1	1	-1
Cum sum	0	1	2	3	2
B_t (Stock price)	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$
$dt = dt$ (P&L/Bond share)	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
$f(t, B_t) = t - dt$ (Bond position)	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$
$f(t, B_t)dB_t$ (P&L)	$\frac{5}{(10)^2}$	$\frac{6}{(10)^2}$	$\frac{7}{(10)^2}$	$\frac{8}{(10)^2}$	$\frac{9}{(10)^2}$
$\int_0^t f(s, B_s)dB_s$ (Cum P&L)	$\frac{15}{(10)^2}$	$\frac{21}{(10)^2}$	$\frac{28}{(10)^2}$	$\frac{36}{(10)^2}$	$\frac{45}{(10)^2}$

Therefore,

$$\int_0^1 sds = \frac{45}{(10)^2} = 0.45$$

Remember we take only 10 ticks in a year. If we takes many ticks, this integral will converge to 0.5, which is a well-known integral in high school.

$$\int_0^1 sds = 0.5$$

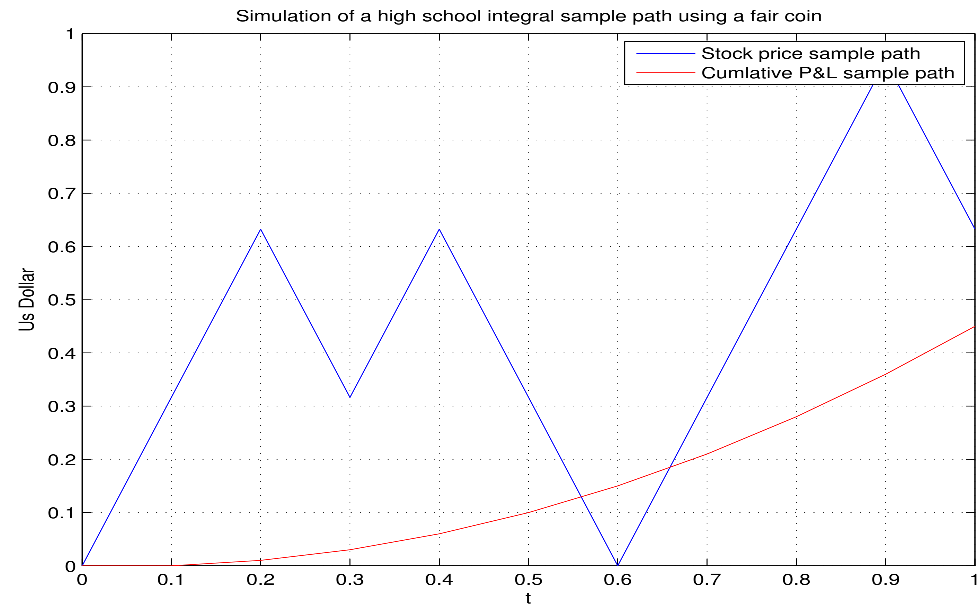

Figure 9: Simulation of a high school integral sample path $\int_0^1 s ds$ using a fair coin.

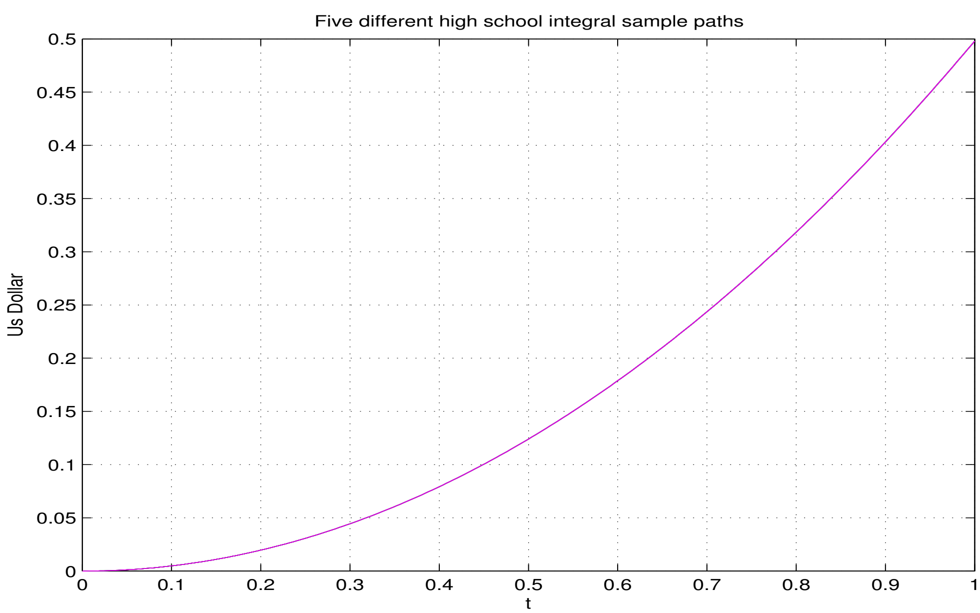

Figure 10: Five different high school integral sample paths $\int_0^1 s ds$

High school integral $\int_0^1 sB_s ds$

We flips a fair coin 10 times and we get

$$HHTHTTHHHT$$

Using these coin flips construct a Brownian motion sample path B_t up to 1 year and then the following high school integral.

$$\int_0^1 sB_s ds$$

Time	0/10	1/10	2/10	3/10	4/10	5/10
Coin flip	–	H	H	T	H	T
Conversion	–	1	1	-1	1	-1
Cum sum	0	1	2	1	2	1
B_t	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$
$dt = dt$	–	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
$h(t, B_t) = t - dt$	–	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$
$g(t, B_t) = B_{t-dt}$	–	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$
$f(t, B_t) = h(t, B_t)g(t, B_t)$	–	0	$\frac{1}{(10)^{3/2}}$	$\frac{4}{(10)^{3/2}}$	$\frac{3}{(10)^{3/2}}$	$\frac{8}{(10)^{3/2}}$
$f(t, B_t)dB_t$	–	0	$\frac{1}{(10)^{5/2}}$	$\frac{4}{(10)^{5/2}}$	$\frac{3}{(10)^{5/2}}$	$\frac{8}{(10)^{5/2}}$
$\int_0^t f(s, B_s)dB_s$	–	0	$\frac{1}{(10)^{5/2}}$	$\frac{5}{(10)^{5/2}}$	$\frac{8}{(10)^{5/2}}$	$\frac{16}{(10)^{5/2}}$

Time	6/10	7/10	8/10	9/10	10/10
Coin flip	T	H	H	H	T
Conversion	-1	1	1	1	-1
Cum sum	0	1	2	3	2
B_t	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$
$dt = dt$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
$h(t, B_t) = t - dt$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$
$g(t, B_t) = B_{t-dt}$	$\frac{1}{\sqrt{10}}$	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$
$f(t, B_t) = h(t, B_t)g(t, B_t)$	$\frac{5}{(10)^{3/2}}$	$\frac{0}{(10)^{3/2}}$	$\frac{7}{(10)^{3/2}}$	$\frac{16}{(10)^{3/2}}$	$\frac{27}{(10)^{3/2}}$
$f(t, B_t)dB_t$	$\frac{5}{(10)^{5/2}}$	$\frac{0}{(10)^{5/2}}$	$\frac{7}{(10)^{5/2}}$	$\frac{16}{(10)^{5/2}}$	$\frac{27}{(10)^{5/2}}$
$\int_0^t f(s, B_s)dB_s$	$\frac{21}{(10)^{5/2}}$	$\frac{21}{(10)^{5/2}}$	$\frac{28}{(10)^{5/2}}$	$\frac{44}{(10)^{5/2}}$	$\frac{71}{(10)^{5/2}}$

Therefore,

$$\int_0^1 sB_s ds = \frac{71}{(10)^{5/2}}$$

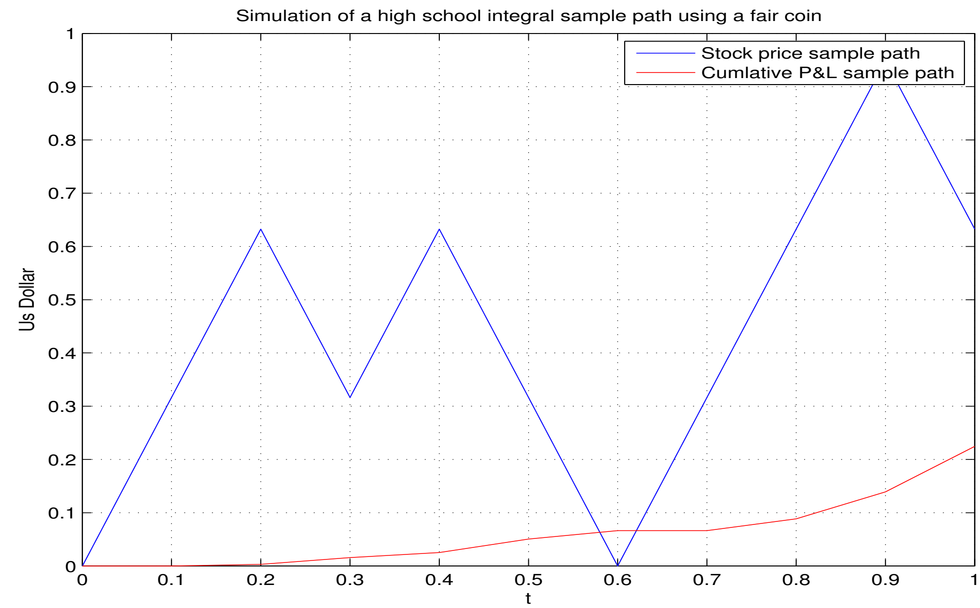

Figure 11: Simulation of a high school integral sample path $\int_0^1 sB_s ds$ using a fair coin.

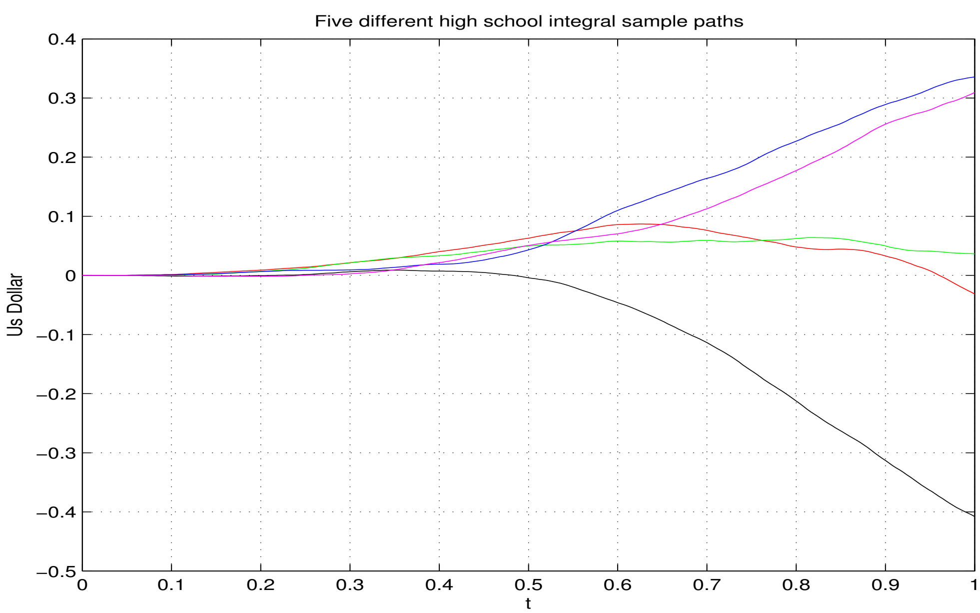

Figure 12: Five different high school integral sample paths $\int_0^1 sB_s ds$