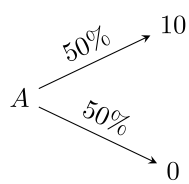
Black-Scholes model - Discrete version

How much is an option contract? - Part 1

Lotto

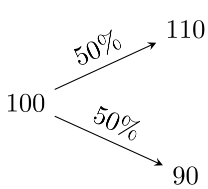
Lotto



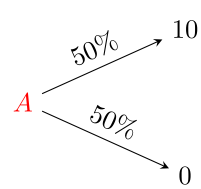
$$A = 10 * 0.5 + 0 * 0.5 = 5$$

Call 100

Stock



Option

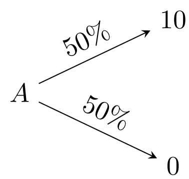


$$A = 10 * 0.5 + 0 * 0.5 = 5 \quad ???$$

How much is an option contract? - Part 2

Lotto

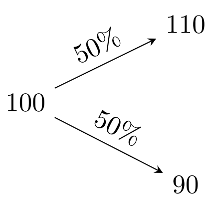
Lotto



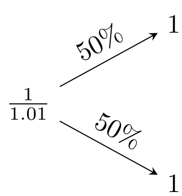
$$A = 10 * 0.5 + 0 * 0.5 = 5$$

Call 100

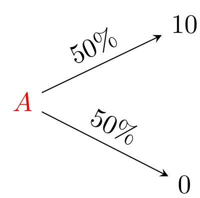
Stock



Bond



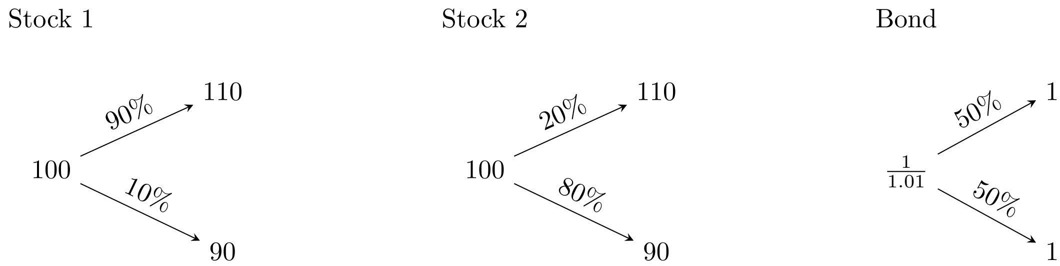
Option



$$A = 10 * 0.5 + 0 * 0.5 = 5 \quad \text{Wrong}$$

$$A = \frac{10 * 0.5 + 0 * 0.5}{1.01} = 4.95 \quad ???$$

How much is an option contract? - Part 3



C_1	Price of call option on stock 1 with strike 100, maturity 1 month
C_2	Price of call option on stock 2 with strike 100, maturity 1 month

$$C_1 > C_2 \quad ???$$

Replicating portfolio

Stock

$$\begin{tikzcd} & SU \\ S \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\ & SD \end{tikzcd}$$

Bond

$$\begin{tikzcd} & 1 \\ \frac{1}{R} \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\ & 1 \end{tikzcd}$$

Option

$$\begin{tikzcd} & A_u \\ A \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\ & A_d \end{tikzcd}$$

Replicating portfolio

$$\begin{tikzcd} & x * SU + y * 1 \\ x * S + y * \frac{1}{R} \arrow[ur] \arrow[dr] & \\ & x * SD + y * 1 \end{tikzcd}$$

Option

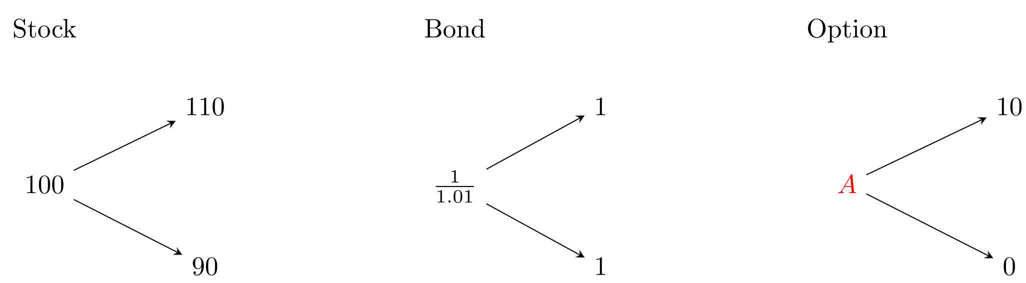
$$\begin{tikzcd} & A_u & \Rightarrow & x * SU + y * 1 = A_u \\ A \arrow[ur] \arrow[dr] & & \Rightarrow & x * S + y * \frac{1}{R} = A \\ & A_d & \Rightarrow & x * SD + y * 1 = A_d \end{tikzcd}$$

$\boxed{\text{Three equations and three unknowns}}$

Stock price up	$x * SU + y * 1 = A_u$
Stock price down	$x * SD + y * 1 = A_d$
Now	$x * S + y * \frac{1}{R} = A$

Example - Replicating portfolio

From the below binomial trees compute the number x of stocks and number y of bonds in the replicating portfolio. Using this replicating portfolio compute the option price A.

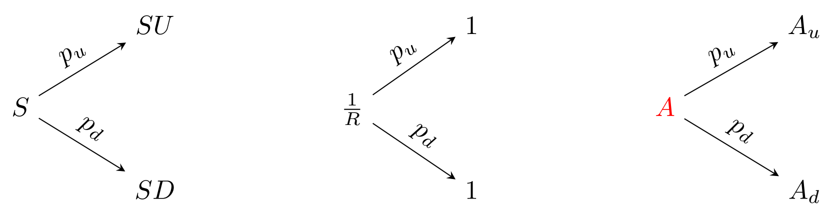


Stock price up	$x * 110 + y * 1 = 10$
Stock price down	$x * 90 + y * 1 = 0$
Now	$x * 100 + y * \frac{1}{1.01} = A$

$$x = 0.5000, \quad y = -45.0000, \quad A = 5.4455$$

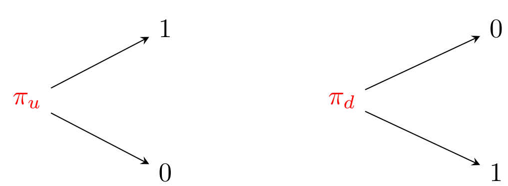
State price

Stock Bond Option



	Replicating portfolio	State price
Number of unknowns	3	3
Number of equations	3	3
Replicating material	Stock and Bond	Up and Down state option
Replicating target	Option	Stock, Bond, Option

Up state option Down state option

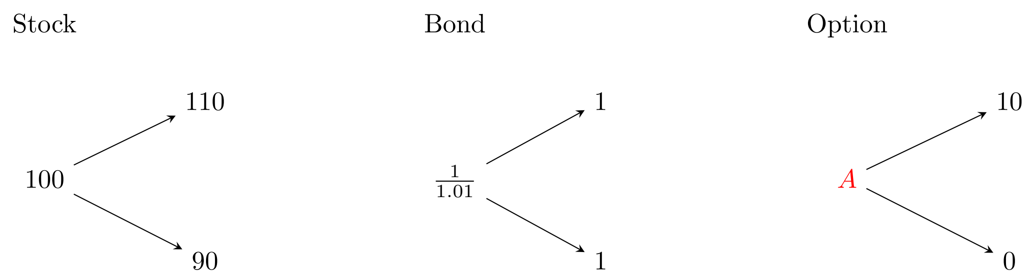


$\boxed{\text{Three equations and three unknowns}}$

$$
\begin{array}{ccccccccc}
\text{Stock} & = & SU & * & \text{Up\_State\_Option} & + & SD & * & \text{Down\_State\_Option} \\
S & = & SU & * & \pi_u & + & SD & * & \pi_d \\
\\
\text{Bond} & = & 1 & * & \text{Up\_State\_Option} & + & 1 & * & \text{Down\_State\_Option} \\
\frac{1}{R} & = & 1 & * & \pi_u & + & 1 & * & \pi_d \\
\\
\text{Option} & = & A_u & * & \text{Up\_State\_Option} & + & A_d & * & \text{Down\_State\_Option} \\
A & = & A_u & * & \pi_u & + & A_u & * & \pi_d
\end{array}
$$

Example - State price

From the below binomial trees compute the up state price π_u and down state price π_d. Using these state prices compute the option price A.



$$
\begin{array}{lccccccccc}
\text{Stock replication} & 100 & = & 110 & * & \pi_u & + & 90 & * & \pi_d \\
\text{Bond replication} & \frac{1}{1.01} & = & 1 & * & \pi_u & + & 1 & * & \pi_d \\
\text{Option replication} & A & = & 10 & * & \pi_u & + & 0 & * & \pi_d
\end{array}
$$

$$\pi_u = 0.5446, \quad \pi_d = 0.4455, \quad A = 5.4455$$

Risk neutral probability

Three equations and three unknowns

Stock replication $S = SU * \pi_u + SD * \pi_d$

Bond replication $\frac{1}{R} = 1 * \pi_u + 1 * \pi_d$

Option replication $A = A_u * \pi_u + A_d * \pi_d$

$$\Updownarrow \quad \pi_u := \frac{q_u}{R}, \quad \pi_d := \frac{q_d}{R}$$

Stock risk neutral valuation $S = \dfrac{SU * q_u + SD * q_d}{R}$

Bond risk neutral valuation $\dfrac{1}{R} = \dfrac{1 * q_u + 1 * q_d}{R}$

Option risk neutral valuation $A = \dfrac{A_u * q_u + A_d * q_d}{R}$

Risk neutral probability

$$\pi_u := \frac{q_u}{R}, \quad \pi_d := \frac{q_d}{R} \quad \Leftrightarrow \quad q_u := R * \pi_u = \frac{R-D}{U-D}, \quad q_d := R * \pi_d = \frac{U-R}{U-D}$$

No free lunch theorem

$$q_u > 0, \; q_d > 0 \quad \Leftrightarrow \quad D < R < U \quad \overset{\text{Theorem}}{\Leftrightarrow} \quad \text{No free lunch}$$

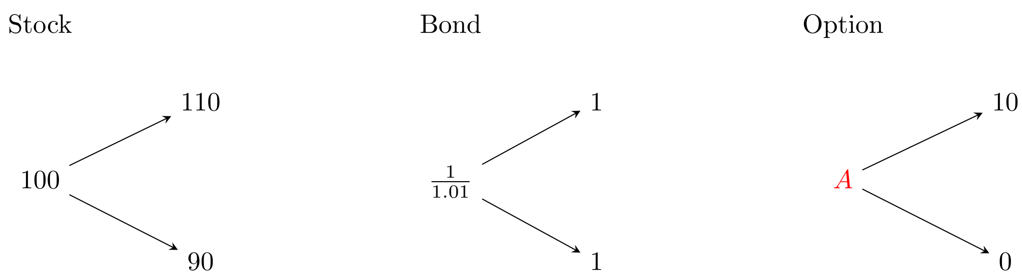
Risk neutral valuation

Risk neutral probability $q_u + q_d = 1, \; q_u > 0, \; q_d > 0$

Risk neutral valuation $A = \dfrac{A_u * q_u + A_d * q_d}{R} = \mathbb{E}^Q\left[\dfrac{A_1}{R}\right]$

Example - Risk neutral probability

From the below binomial trees compute the risk neutral probability q_u of going up and risk neutral probability q_d of going down. Using these risk neutral probabilities compute the option price A.



$$\text{Stock risk neutral valuation} \qquad 100 = \frac{110 * q_u + 90 * q_d}{1.01}$$

$$\text{Bond risk neutral valuation} \qquad \frac{1}{1.01} = \frac{1 * q_u + 1 * q_d}{1.01}$$

$$\text{Option risk neutral valuation} \qquad A = \frac{10 * q_u + 0 * q_d}{1.01}$$

$$q_u = 0.5500, \quad q_d = 0.4500, \quad A = 5.4455$$

Delta hedging

Naked $\xRightarrow{\text{Add } \Delta \text{ stocks}}$ Δ hedged

$$
\begin{tikzcd}
 & -A_u \\
-A \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\
 & -A_d
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & -A_u + \Delta * SU \\
-A + \Delta * S \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \| \ \Delta \text{ hedge} \\
 & -A_d + \Delta * SD
\end{tikzcd}
$$

Two equations and two unknowns

Δ hedged portfolio at maturity $\qquad -A_u + \Delta * SU = -A_d + \Delta * SD$

Δ hedged portfolio now $\qquad \dfrac{-A_u + \Delta * SU}{R} = \dfrac{-A_d + \Delta * SD}{R} = -A + \Delta * S$

Delta

$$\Delta = \frac{A_u - A_d}{SU - SD}$$

Example - Delta hedging

Suppose we have -1 position in option. From the below binomial trees compute the number Δ of stocks that we should hold as a delta hedging. Using this delta hedging portfolio compute the option price A.

Stock

$$\begin{tikzcd} & 110 \\ 100 \arrow[ur] \arrow[dr] & \\ & 90 \end{tikzcd}$$

Bond

$$\begin{tikzcd} & 1 \\ \frac{1}{1.01} \arrow[ur] \arrow[dr] & \\ & 1 \end{tikzcd}$$

Option

$$\begin{tikzcd} & 10 \\ A \arrow[ur] \arrow[dr] & \\ & 0 \end{tikzcd}$$

Naked $\Longrightarrow$ Δ hedged

$$\begin{tikzcd} & -10 \\ -A \arrow[ur] \arrow[dr] & \\ & 0 \end{tikzcd} \qquad \begin{tikzcd} & -10+\Delta*110 \\ -A+\Delta*100 \arrow[ur] \arrow[dr] & \\ & \Delta*90 \end{tikzcd}$$

Δ hedged portfolio at maturity $\quad -10+\Delta*110 = \Delta*90$

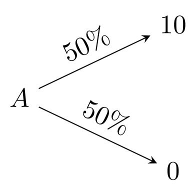
Δ hedged portfolio now $\quad \dfrac{-10+\Delta*110}{1.01} = \dfrac{\Delta*90}{1.01} = -A+\Delta*100$

$$\Delta = \frac{10-0}{110-90} = 0.5, \quad A = 5.4455$$

How much is an option contract? - Part 4

Lotto

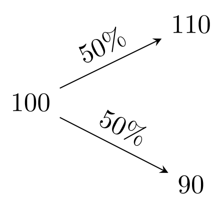
Lotto



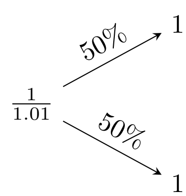
$$A = 10 * 0.5 + 0 * 0.5 = 5$$

Call 100

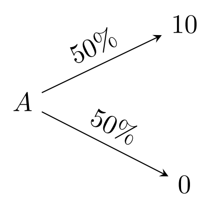
Stock



Bond



Option



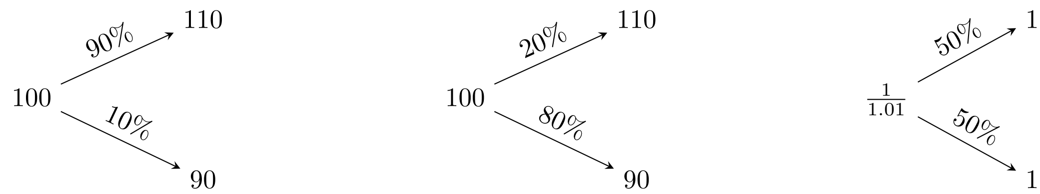
$$A = \frac{10 * \color{red}{0.5} + 0 * \color{red}{0.5}}{1.01} = 4.95 \quad \color{red}{\text{Wrong}}$$

$$q_u = \frac{1.01 - 0.90}{1.10 - 0.90}, \quad q_d = \frac{1.10 - 1.01}{1.10 - 0.90}$$

$$A = \frac{10 * \color{red}{\frac{1.01-0.90}{1.10-0.90}} + 0 * \color{red}{\frac{1.10-1.01}{1.10-0.90}}}{1.01} = 5.45$$

How much is an option contract? - Part 5

Stock 1 Stock 2 Bond



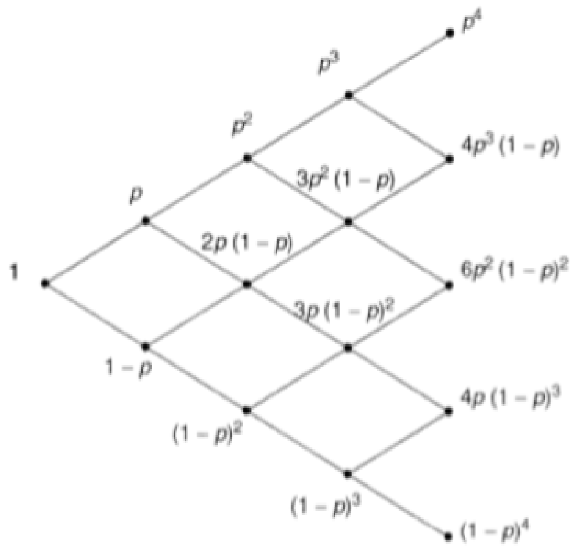
C_1	Price of call option on stock 1 with strike 100, maturity 1 month
C_2	Price of call option on stock 2 with strike 100, maturity 1 month

$C_1 > C_2$ Wrong

$C_1 = C_2$

Multi period binomial model

Multi period binomial model

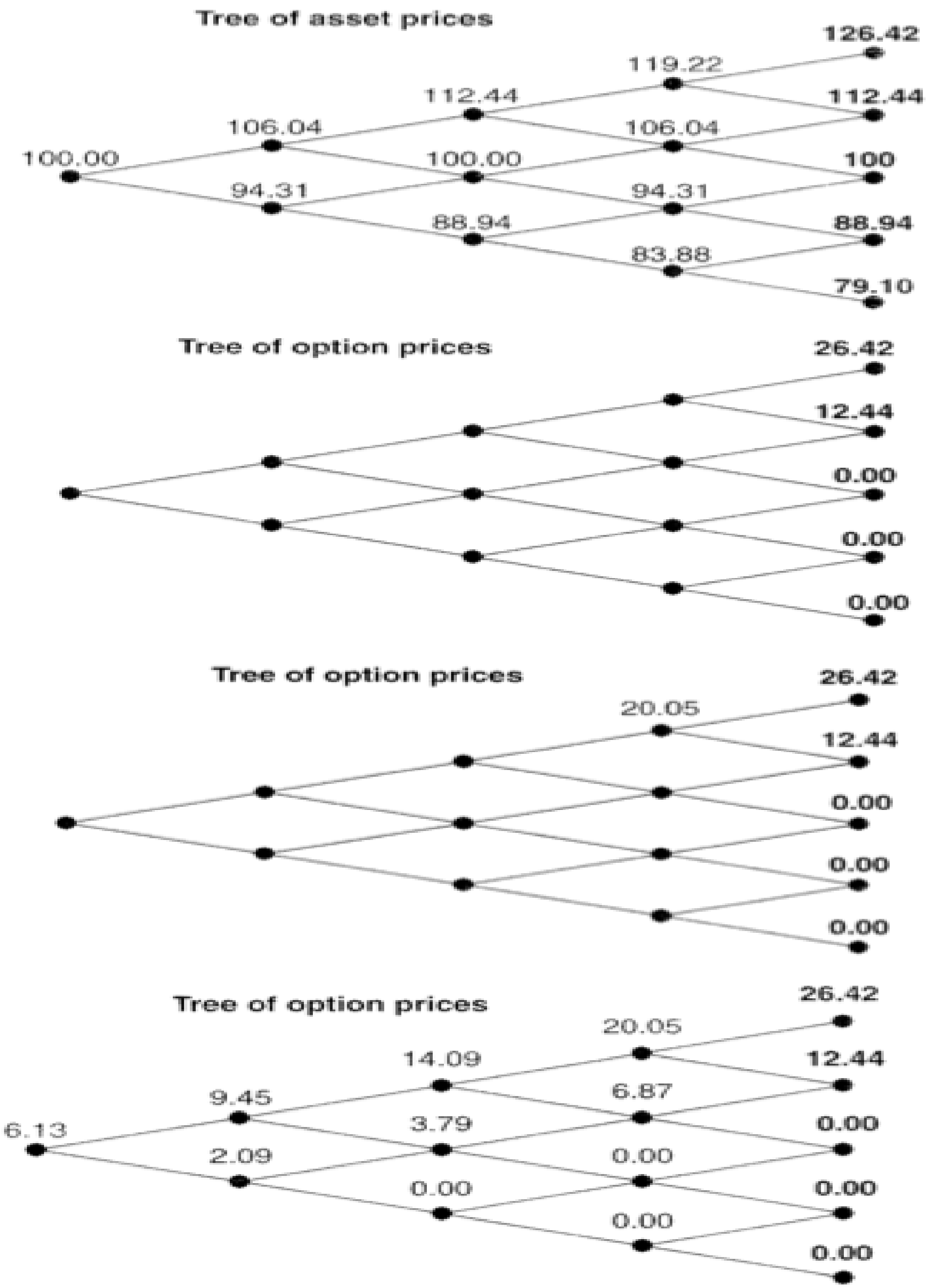


From (2005) Paul Wilmott On Quantitative Finance Volume 1

How to choose parameters

Wilmott	$U = 1 + \sigma\sqrt{dt}$	$D = 1 - \sigma\sqrt{dt}$	$q_u = \frac{1}{2} + \frac{\mu\sqrt{dt}}{2\sigma}$	$q_d = 1 - q_u$
CRR	$U = e^{\sigma\sqrt{dt}}$	$D = U^{-1}$	$q_u = \frac{e^{rdt} - D}{U - D}$	$q_d = 1 - q_u$
JR	$U = e^{(r - \frac{1}{2}\sigma^2)dt + \sigma\sqrt{dt}}$	$D = e^{(r - \frac{1}{2}\sigma^2)dt - \sigma\sqrt{dt}}$	$q_u = \frac{1}{2}$	$q_d = 1 - q_u$

Option valuation - Rolling backward



From (2005) Paul Wilmott On Quantitative Finance Volume 1

Example - Option valuation - Rolling backward - Option

Use JR parametrization and write the code of pricing the put option with $S_0 = 258$, $K = 250$, $T = 1$, $r = 0.03$, $\sigma = 0.20$.

```
clear all; close all; clc;

S=258; K=250; T=1; r=0.03; v=0.20;

M=250;
t=linspace(0,T,M+1);
dt=t(2)-t(1);

% JR parametrization
u=exp((r-0.5*v^2)*dt+v*sqrt(dt)); d=exp((r-0.5*v^2)*dt-v*sqrt(dt));
q=0.5;

% Option payoff at maturity
Stock=S*d.^((0:M)').*u.^((M:-1:0)');
V=max(K-Stock,zeros(size(Stock)));

% Rolling backward
for i=M:-1:1
    V=exp(-r*dt)*(q*V(1:end-1)+(1-q)*V(2:end));
end

Put=V;
fprintf('Put option price  %g\n',Put)
```

Example - Option valuation - Rolling backward - Barrier option

Use JR parametrization and write the code of pricing the barrier put option with $S_0 = 258$, $K = 250$, $T = 1$, $r = 0.03$, $\sigma = 0.20$, where the barrier is constructed at 200. If the stock price hits the barrier or stays below the barrier before maturity, the option expires with no payment.

```
clear all; close all; clc;

S=258; K=250; T=1; r=0.03; v=0.20;
B=200; % Barrier

M=250;
t=linspace(0,T,M+1);
dt=t(2)-t(1);

% JR parametrization
u=exp((r-0.5*v^2)*dt+v*sqrt(dt)); d=exp((r-0.5*v^2)*dt-v*sqrt(dt));
q=0.5;

% Option payoff at maturity
Stock=S*d.^((0:M)').*u.^((M:-1:0)');
V=max(K-Stock,zeros(size(Stock)));

% Rolling backward
for i=M:-1:1
    Stock=S*d.^((0:i-1)').*u.^((i-1:-1:0)');
    V=exp(-r*dt)*(q*V(1:end-1)+(1-q)*V(2:end));
    V(Stock<=B)=0;
end

Put_B=V;
fprintf('Barrier put option price  %g\n',Put_B)
```

Example - Option valuation - Rolling backward - American option

Use JR parametrization and write the code of pricing the American put option with $S_0 = 258$, $K = 250$, $T = 1$, $r = 0.03$, $\sigma = 0.20$.

```
clear all; close all; clc;

S=258; K=250; T=1; r=0.03; v=0.20;

M=250;
t=linspace(0,T,M+1);
dt=t(2)-t(1);

% JR parametrization
u=exp((r-0.5*v^2)*dt+v*sqrt(dt)); d=exp((r-0.5*v^2)*dt-v*sqrt(dt));
q=0.5;

% Option payoff at maturity
Stock=S*d.^((0:M)').*u.^((M:-1:0)');
V=max(K-Stock,zeros(size(Stock)));

% Rolling backward
for i=M:-1:1
    Stock=S*d.^((0:i-1)').*u.^((i-1:-1:0)');
    V=exp(-r*dt)*(q*V(1:end-1)+(1-q)*V(2:end));
    V=max(V,K-Stock);
end

Put_A=V;
fprintf('American put option price  %g\n',Put_A)
```

Option valuation - State price

[**Step 1**] Calculate the state price π_u, π_d or risk neutral probability π_u, π_d.

$$q_u = \frac{R-D}{U-D}, \quad q_d = \frac{U-R}{U-D}, \quad \pi_u = \frac{q_u}{R}, \quad \pi_d = \frac{q_d}{R}$$

[**Step 2**] Calculate the state price of a particular path landing the state i in period n using i number of down arrows and $n-i$ number of up arrows, or the path dependent digital option which pays 1 only when the stock price follows the paths.

$$\pi_u^{n-i}\pi_d^i$$

[**Step 3**] Count the number of paths landing the state i in period n using i number of down arrows and $n-i$ number of up arrows.

$$\binom{n}{i}$$

[**Step 4**] Calculate the state i price or the price of the digital option which pays 1 only when the stock price landing the state i in period n by multiplying the state price of the paths landing the state i in period n (obtained in [Step 2]) and the number of such paths (obtained in [Step 3]).

$$\text{State } i \text{ price} = \binom{n}{i} * \pi_u^{n-i}\pi_d^i$$

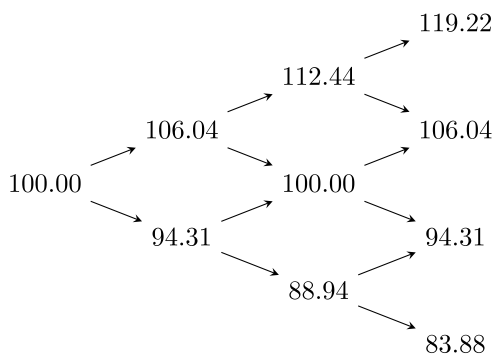
[**Step 5**] Now, let's use these state prices and value an option which pays `Payoff_i` when the underlying stock price lands on the state i for each i. Since this option can be decomposed into the portfolio of `Payoff_i` shares of the state i digital options, and since the state i price is $\binom{n}{i} * \pi_u^{n-i}\pi_d^i$, the option price is given by

$$\text{Option price} = \sum_{i=0}^{n} \binom{n}{i} * \pi_u^{n-i}\pi_d^i * \texttt{Payoff\_i}$$

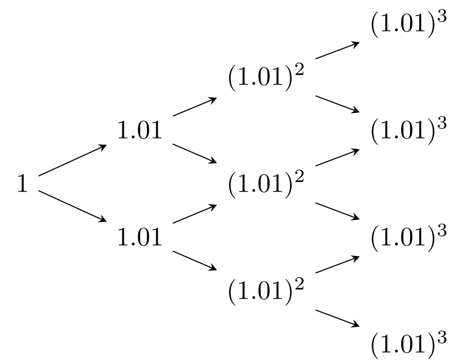
Example - Option valuation - State price

From the below binomial trees compute the up state price π_u and down state price π_d. Using these state prices compute the option price A.

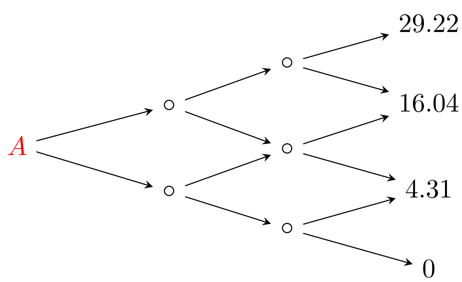
Stock



Bond



Option




```
clear all; close all; clc;

U=106.04/100; D=94.31/100; R=1.01;

q_u=(R-D)/(U-D); q_d=1-q_u;
pi_u=q_u/R; pi_d=q_d/R;

Payoff=[29.22; 16.04; 4.31; 0];
i=(0:3)';
Option_Price=sum(Payoff.*pi_u.^(3-i).*pi_d.^(i).*binomial(3,i))
```

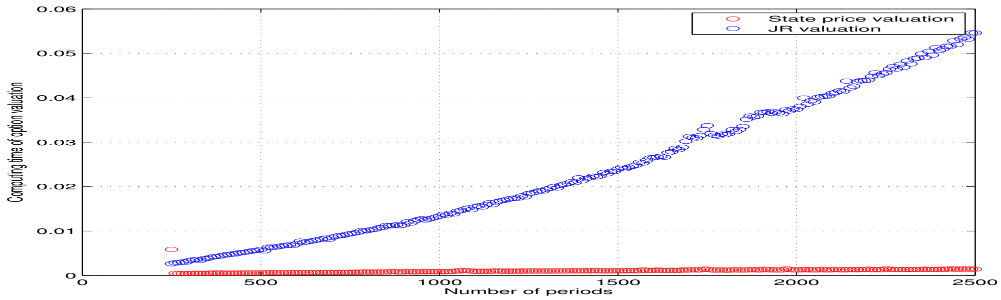


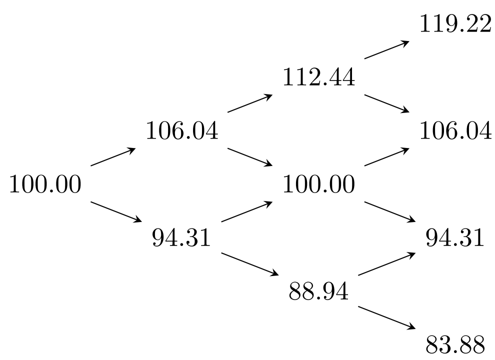
Figure 1: Computing time of state price and JR valuation. This is the power of vectorization.

```
clear all; close all; clc;

S=258; K=250; T=1; r=0.03; mu=0.03; v=0.20;
M=250:10:2500; % M - Number of time grids before maturity

Time_SP=zeros(size(M));
Time_JR=zeros(size(M));
for M=250:10:2500
dt=T/M;
% JR parametrization
q_u=0.5; q_d=1-q_u; U=exp((r-0.5*v^2)*dt+v*sqrt(dt)); D=U*exp(-2*v*sqrt(dt));
% Option value at maturity
V1=max(S*D.^((M:-1:0)').*U.^((0:M)')-K,0); % Call at maturity
V2=max(K-S*D.^((M:-1:0)').*U.^((0:M)'),0); % Put at maturity
% state price valuation time
tic
pi_u=exp(-r*dt)*q_u; pi_d = exp(-r*dt)*q_d;
C_SP=sum(V1.*pi_d.^((M:-1:0)').*pi_u.^((0:M)').*binomial(M,(M:-1:0)'));
P_SP=sum(V2.*pi_d.^((M:-1:0)').*pi_u.^((0:M)').*binomial(M,(M:-1:0)'));
Time_SP((M-240)/10) = toc;
% JR valuation time
tic
for i=M:-1:1
    V1=exp(-r*dt)*(q_u*V1(2:i+1)+q_d*V1(1:i)); % Call backward roll
    V2=exp(-r*dt)*(q_u*V2(2:i+1)+q_d*V2(1:i)); % Put backward roll
end
C_JR=V1; P_JR=V2; Time_JR((M-240)/10)=toc;
end

plot(250:10:2500,Time_SP,'or',250:10:2500,Time_JR,'ob'); grid on
xlabel('Number of periods'); ylabel('Computing time of option valuation')
legend('State price valuation','JR valuation')
```

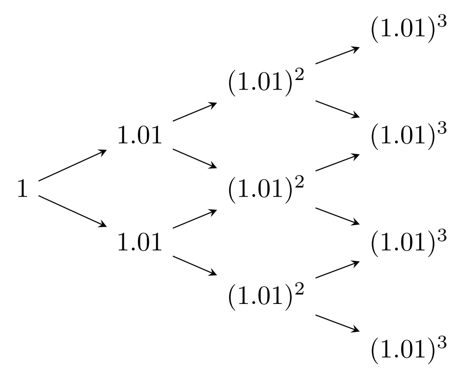
Example - Delta

From the below binomial trees suppose we have -1 position in option at the beginning node. From the below binomial trees compute the number Δ of stocks that we should hold as a delta hedging at the beginning node.

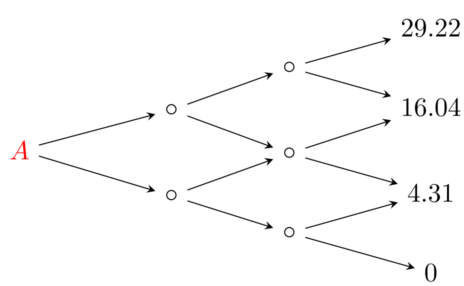
Stock



Bond



Option




```
S=100; U=106.04/100; D=94.31/100; R=1.01;

q_u=(R-D)/(U-D); q_d=1-q_u;
pi_u=q_u/R; pi_d=q_d/R;

Payoff=[29.22; 16.04; 4.31; 0];
i=(0:2)';
Payoff_U=[29.22; 16.04; 4.31];
Payoff_D=[16.04; 4.31; 0];
Option_Price_U=sum(Payoff_U.*pi_u.^(2-i).*pi_d.^(i).*binomial(2,i));
Option_Price_D=sum(Payoff_D.*pi_u.^(2-i).*pi_d.^(i).*binomial(2,i));

Delta=(Option_Price_U-Option_Price_D)/(S*U-S*D)
```