Ito lemma

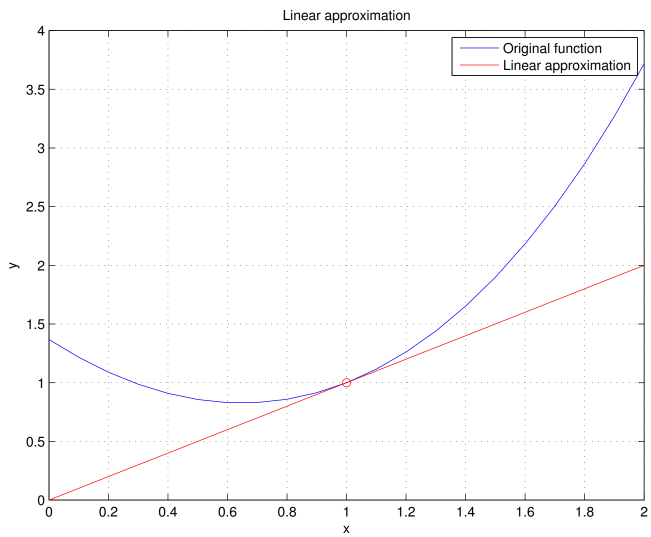
Linear approximation

$$\begin{array}{lrcl} \text{One variable} & df & = & f_x * dx \\ \text{Two variables} & df & = & f_x * dx \quad + \quad f_b * db \end{array}$$

Example - Linear approximation - One variable

Calculate $f(1.1)$ using the linear approximation of f at $x_0 = 1$, where f is given by

$$f(x) = e^{x-1} + (x-1)^2$$




```
clear all; close all; clc;

x=0:0.1:2;
f=exp(x-1)+(x-1).^2; % Original function
g=x; % Linear approximation

plot(x,f,'-',x,g,'-r',1,1,'or'); grid on
legend('Original function','Linear approximation')
xlabel('x'); ylabel('y'); title('Linear approximation')
```

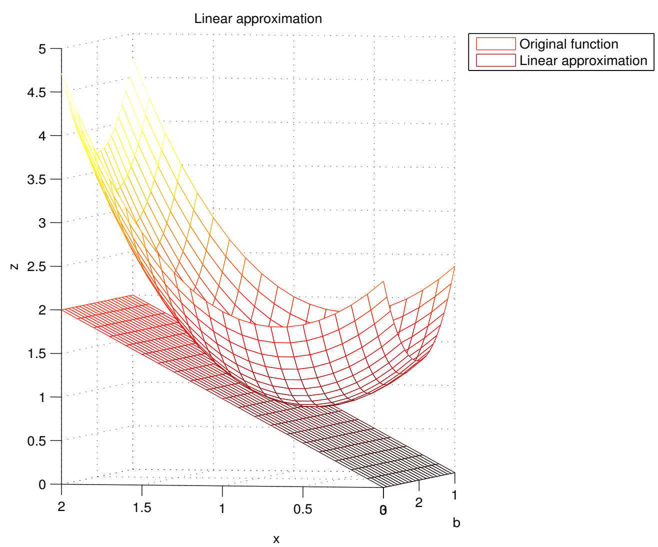
$$\begin{array}{ccccc} df & = & f_x & * & dx \\ \uparrow & & \uparrow & & \uparrow \\ f(x) - f(x_0) & & f_x(x_0) & & x - x_0 \\ & & & & \\ f(1.1) - f(1) & & f_x(1) & & 1.1 - 1 \end{array}$$

$$f(1.1) - 1 = 1 * (1.1 - 1) \quad \Rightarrow \quad f(1.1) = 1.1$$

Example - Linear approximation - Two variable

Calculate $f(1.1, 1.8)$ using the linear approximation of f at $(x_0, b_0) = (1, 2)$, where f is given by

$$f(x) = e^{x-1} + (x-1)^2 + (b-2)^2$$




```
clear all; close all; clc;

x=0:0.1:2; b=1:0.1:3; [X B]=meshgrid(x,b);
F=exp(X-1)+(X-1).^2+(B-2).^2; % Original function
G=X; % Linear approximation

mesh(X,B,F); grid on; hold on
mesh(X,B,G); colormap(hot)
legend('Original function','Linear approximation')
xlabel('x'); ylabel('b'); zlabel('z'); title('Linear approximation')
```

$$\begin{array}{ccccccccc}
df & = & f_x & * & dx & + & f_b & * & db \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
f(x,b) - f(x_0,b_0) & & f_x(x_0,b_0) & & x - x_0 & & f_b(x_0,b_0) & & b - b_0 \\
 & & & & & & & & \\
f(1.1,1.8) - f(1,2) & & f_x(1.1,1.8) & & 1.1 - 1 & & f_b(1.1,1.8) & & 1.8 - 2
\end{array}$$

$$f(1.1, 1.8) - 1 = 1 * (1.1 - 1) + 0 * (1.8 - 2) \quad \Rightarrow \quad f(1.1, 1.8) = 1.1$$

Quadratic approximation

One variable

$$df = f_x * dx \quad + \quad \frac{1}{2} * f_{xx} * (dx)^2$$

Two variable

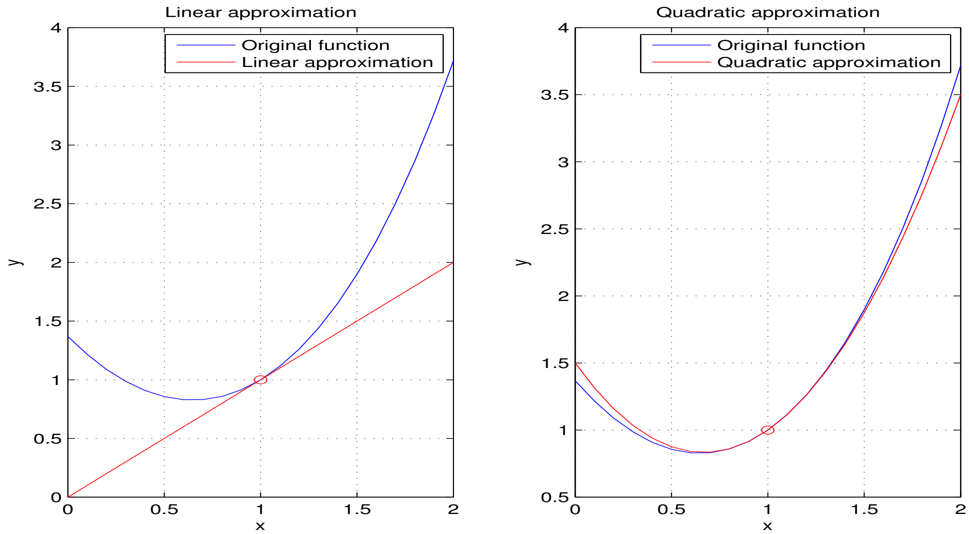
$$df = f_x * dx + f_b * db \quad + \quad \frac{1}{2} * f_{xx} * (dx)^2 + \frac{1}{2} * f_{bb} * (db)^2 + f_{xb} * (dx)(db)$$

Example - Quadratic approximation - One variable

Calculate $f(1.1)$ using the quadratic approximation of f at $x_0 = 1$, where f is given by
$$f(x) = e^{x-1} + (x-1)^2$$

$$\begin{array}{ccccccccccc}
df & = & f_x & * & dx & + & \frac{1}{2} & * & f_{xx} & * & (dx)^2 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
f(x) - f(x_0) & & f_x(x_0) & & x - x_0 & & & & f_{xx}(x_0) & & (x - x_0)^2 \\
 & & & & & & & & & & \\
f(1.1) - f(1) & & f_x(1) & & 1.1 - 1 & & & & f_{xx}(1) & & (1.1 - 1)^2
\end{array}$$

$$f(1.1) - 1 = 1 * (1.1 - 1) + \frac{1}{2} * 3 * (1.1 - 1)^2 \quad \Rightarrow \quad f(1.1) = 1.115$$




```
clear all; close all; clc;

x=0:0.1:2;
f=exp(x-1)+(x-1).^2; % Original function
g=x; % Linear approximation

subplot(121)
plot(x,f,'-',x,g,'-r',1,1,'or'); grid on
legend('Original function','Linear approximation')
xlabel('x'); ylabel('y'); title('Linear approximation')

h=x+1.5*(x-1).^2; % Quadratic approximation

subplot(122)
plot(x,f,'-',x,h,'-r',1,1,'or'); grid on
legend('Original function','Quadratic approximation')
xlabel('x'); ylabel('y'); title('Quadratic approximation')
```

Example - Quadratic approximation - Two variable

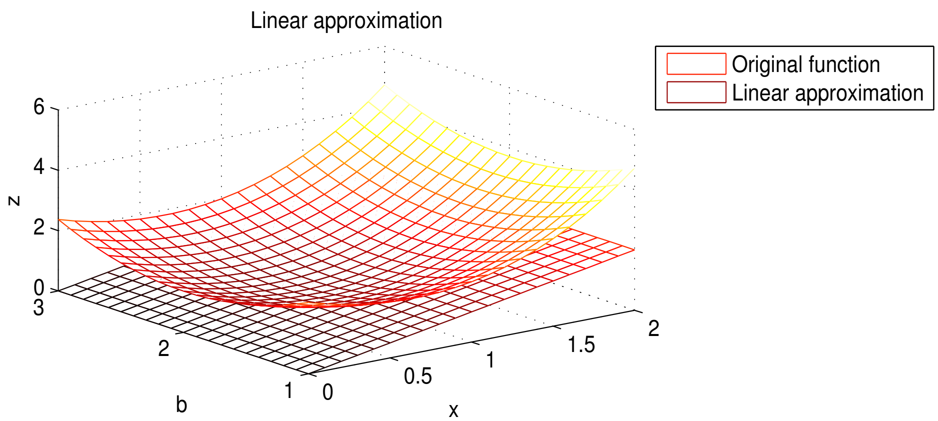
Calculate $f(1.1, 1.8)$ using the quadratic approximation of f at $(x_0, b_0) = (1, 2)$, where f is given by

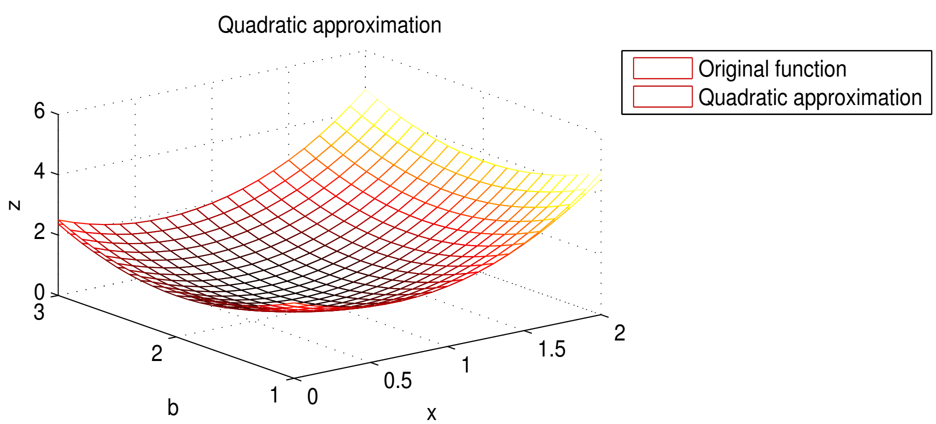
$$f(x) = e^{x-1} + (x-1)^2 + (b-2)^2$$

$$\begin{array}{cccc}
df & = & f_x * dx & +f_b * db \\
\uparrow & & \uparrow & \uparrow \\
f(x,b) - f(x_0,b_0) & & f_x(x_0,b_0) * (x - x_0) & f_b(x_0,b_0) * (b - b_0) \\
 & & & \\
f(1.1,1.8) - f(1,2) & & 1 * (1.1 - 1) & 0 * (1.8 - 2)
\end{array}$$

$$\begin{array}{ccc}
+\frac{1}{2} * f_{xx} * (dx)^2 & +\frac{1}{2} * f_{bb} * (db)^2 & +f_{xb} * (dx)(db) \\
\uparrow & \uparrow & \uparrow \\
\frac{1}{2} * f_{xx}(x_0,b_0) * (x - x_0)^2 & \frac{1}{2} * f_{bb}(x_0,b_0) * (b - b_0)^2 & f_{xb}(x_0,b_0) * (x - x_0) * (b - b_0) \\
 & & \\
\frac{1}{2} * 3 * (1.1 - 1)^2 & \frac{1}{2} * 2 * (1.8 - 2)^2 & 0 * (1.1 - 1) * (1.8 - 2)
\end{array}$$

$$\begin{aligned}
f(1.1, 1.8) &= 1 + 1 * 0.1 + 0 * (-0.2) \\
&\quad + \frac{1}{2} * 3 * (0.1)^2 + \frac{1}{2} * 2 * (-0.2)^2 + 0 * (0.1) * (-0.2) = 1.1550
\end{aligned}$$






```
clear all; close all; clc;

x=0:0.1:2; b=1:0.1:3; [X B]=meshgrid(x,b);
F=exp(X-1)+(X-1).^2+(B-2).^2; % Original function
G=X; % Linear approximation
H=X+1.5*(X-1).^2+(B-2).^2; % Quadratic approximation

subplot(211)
mesh(X,B,F); grid on; hold on;
mesh(X,B,G); colormap(hot)
legend('Original function','Linear approximation')
xlabel('x'); ylabel('b'); zlabel('z'); title('Linear approximation')

subplot(212)
mesh(X,B,F); grid on; hold on
mesh(X,B,H); colormap(hot)
legend('Original function','Quadratic approximation')
xlabel('x'); ylabel('b'); zlabel('z'); title('Quadratic approximation')
```

Ito lemma

Quadratic approximation with box calculus

	dt	db
dt	0	0
db	0	dt

$$
\begin{aligned}
df &= f_t * dt + f_b * db + \frac{1}{2} * f_{tt} * (dt)^2 + \frac{1}{2} * f_{bb} * (db)^2 + f_{tb} * (dt)(db) \\
&= f_t * dt + f_b * db + \frac{1}{2} * f_{tt} * 0 + \frac{1}{2} * f_{bb} * (dt) + f_{tb} * 0
\end{aligned}
$$

Interpretation

Definition	df	$\Rightarrow \int_0^t df = f(t, B_t) - f(0, B_0)$
High school integral	$f_t * dt$	$\Rightarrow \int_0^t f_t * dt = \int_0^t f_t(s, B_s) ds$
Ito integral	$f_b * db$	$\Rightarrow \int_0^t f_b * db = \int_0^t f_b(s, B_s) dB_s$
High school integral	$\frac{1}{2} * f_{bb} * dt$	$\Rightarrow \frac{1}{2} * \int_0^t f_{bb} * dt = \frac{1}{2} \int_0^t f_{bb}(s, B_s) ds$

Ito lemma

$$
f(t, B_t) - f(0, 0) = \underbrace{\int_0^t f_t(s, B_s) ds}_{\text{High school integral}} + \underbrace{\int_0^t f_b(s, B_s) dB_s}_{\text{Ito integral}} + \frac{1}{2} \underbrace{\int_0^t f_{bb}(s, B_s) ds}_{\text{High school integral}}
$$

How to use Ito lemma to calculate Ito integral

[Step 1] Identify integrand g of Ito integral $\int_0^t g(s, B_s)dB_s$

$$\int_0^t g(s, B_s)dB_s$$

[Step 2] Find f with $g = f_b$

$$\int_0^t g(s, B_s)dB_s = \int_0^t f_b(s, B_s)dB_s$$

[Step 3] Apply Ito lemma to f

$$f(t, B_t) - f(0,0) = \underbrace{\int_0^t f_t(s, B_s)ds}_{\text{High school integral}} + \underbrace{\int_0^t f_b(s, B_s)dB_s}_{\text{Ito integral}} + \frac{1}{2}\underbrace{\int_0^t f_{bb}(s, B_s)ds}_{\text{High school integral}}$$

[Step 4] Identify Ito integral $\int_0^t g(s, B_s)dB_s$

$$\underbrace{\int_0^t g(s, B_s)dB_s}_{\text{Ito integral}} = f(t, B_t) - f(0,0) - \underbrace{\int_0^t f_t(s, B_s)ds}_{\text{High school integral}} - \frac{1}{2}\underbrace{\int_0^t f_{bb}(s, B_s)ds}_{\text{High school integral}}$$

Ito lemma is fundamental theorem of calculus in Ito integral computation

Integral computation without fundamental theorem of calculus

$$\int_0^t g(s)ds = \lim_{n\to\infty} \sum_{k=1}^n g\left((k-1)\frac{t}{n}\right)\frac{t}{n}$$

Integral computation with fundamental theorem of calculus

$$\int_0^t g(s)ds = [G]_0^t = G(t) - G(0)$$

where G is an anti-derivative of g, i.e., $G' = g$.

Ito integral computation without Ito lemma

$$\int_0^t g(s,B_s)dB_s = \lim_{n\to\infty} \sum_{k=1}^n g\left((k-1)\frac{t}{n}, B_{(k-1)\frac{t}{n}}\right)\left(B_{k\frac{t}{n}} - B_{(k-1)\frac{t}{n}}\right)$$

Ito integral computation with Ito lemma

$$\underbrace{\int_0^t g(s,B_s)dB_s}_{\text{Ito integral}} = f(t,B_t) - f(0,0) - \underbrace{\int_0^t f_t(s,B_s)ds}_{\text{High school integral}} - \frac{1}{2}\underbrace{\int_0^t f_{bb}(s,B_s)ds}_{\text{High school integral}}$$

where f is an "anti-derivative" of g, i.e., $f_b = g$.

Example - $\int_0^t B_s dB_s$

Calculate the following Ito integral;

$$\int_0^t B_s dB_s$$

[Step 1] Identify integrand g of Ito integral $\int_0^t g(s, B_s) dB_s$

$$\int_0^t B_s dB_s = \int_0^t g(s, B_s) dB_s \quad \Rightarrow \quad g(t, b) = b$$

[Step 2] Find f with $g = f_b$

$$f_b = g = b \quad \Rightarrow \quad f(t, b) = \frac{1}{2} b^2$$

[Step 3] Apply Ito lemma to f

$$f = \frac{1}{2} b^2, \quad f_t = 0, \quad f_b = b, \quad f_{bb} = 1$$

Definition	$df \Rightarrow \int_0^t df = f(t, B_t) - f(0,0) = \frac{1}{2} B_t^2$
High school integral	$f_t * dt \Rightarrow \int_0^t f_t * dt = \int_0^t f_t(s, B_s) ds = 0$
Ito integral	$f_b * db \Rightarrow \int_0^t f_b db = \int_0^t g db = \int_0^t B_s dB_s$
High school integral	$\frac{1}{2} * f_{bb} * dt \Rightarrow \frac{1}{2} * \int_0^t f_{bb} * dt = \frac{1}{2} \int_0^t ds = \frac{1}{2} t$

$$\frac{1}{2} B_t^2 = 0 + \int_0^t B_s dB_s + \frac{1}{2} t$$

[Step 4] Identify Ito integral $\int_0^t g(s, B_s) dB_s$

$$\int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t$$

Example - $\int_0^t s dB_s$

Calculate the following Ito integral;

$$\int_0^t s dB_s$$

[Step 1] Identify integrand g of Ito integral $\int_0^t g(s, B_s) dB_s$

$$\int_0^t s dB_s = \int_0^t g(s, B_s) dB_s \quad \Rightarrow \quad g(t, b) = t$$

[Step 2] Find f with $g = f_b$

$$f_b = g = t \quad \Rightarrow \quad f(t, b) = tb$$

[Step 3] Apply Ito lemma to f

$$f = tb, \quad f_t = b, \quad f_b = t, \quad f_{bb} = 0$$

Definition	$df \Rightarrow \int_0^t df = f(t, B_t) - f(0, 0) = tB_t$
High school integral	$f_t * dt \Rightarrow \int_0^t f_t * dt = \int_0^t f_t(s, B_s) ds = \int_0^t B_s ds$
Ito integral	$f_b * db \Rightarrow \int_0^t f_b db = \int_0^t g db = \int_0^t s dB_s$
High school integral	$\frac{1}{2} * f_{bb} * dt \Rightarrow \frac{1}{2} * \int_0^t f_{bb} * dt = 0$

$$tB_t = \int_0^t B_s ds + \int_0^t s dB_s$$

[Step 4] Identify Ito integral $\int_0^t g(s, B_s) dB_s$

$$\int_0^t s dB_s = tB_t - \int_0^t B_s ds$$

Example - $\int_0^t sB_s dB_s$

Calculate the following Ito integral;

$$\int_0^t sB_s dB_s$$

[Step 1] Identify integrand g of Ito integral $\int_0^t g(s, B_s) dB_s$

$$\int_0^t sB_s dB_s = \int_0^t g(s, B_s) dB_s \quad \Rightarrow \quad g(t, b) = tb$$

[Step 2] Find f with $g = f_b$

$$f_b = g = tb \quad \Rightarrow \quad f(t, b) = \frac{1}{2} tb^2$$

[Step 3] Apply Ito lemma to f

$$f = \frac{1}{2} tb^2, \quad f_t = \frac{1}{2} b^2, \quad f_b = tb, \quad f_{bb} = t$$

Definition	$df \Rightarrow \int_0^t df = f(t, B_t) - f(0, 0) = \frac{1}{2} tB_t^2$
High school integral	$f_t * dt \Rightarrow \int_0^t f_t * dt = \int_0^t f_t(s, B_s) ds = \frac{1}{2} \int_0^t B_s^2 ds$
Ito integral	$f_b * db \Rightarrow \int_0^t f_b db = \int_0^t g db = \int_0^t sB_s dB_s$
High school integral	$\frac{1}{2} * f_{bb} * dt \Rightarrow \frac{1}{2} * \int_0^t f_{bb} * dt = \frac{1}{2} \int_0^t s ds = \frac{1}{4} t^2$

$$\frac{1}{2} tB_t^2 = \frac{1}{2} \int_0^t B_s^2 ds + \int_0^t sB_s dB_s + \frac{1}{4} t^2$$

[Step 4] Identify Ito integral $\int_0^t g(s, B_s) dB_s$

$$\int_0^t sB_s dB_s = \frac{1}{2} tB_t^2 - \frac{1}{2} \int_0^t B_s^2 ds - \frac{1}{4} t^2$$

Example - $\int_0^t B_s^2 dB_s$

Calculate the following Ito integral;

$$\int_0^t B_s^2 dB_s$$

[Step 1] Identify integrand g of Ito integral $\int_0^t g(s, B_s) dB_s$

$$\int_0^t sB_s dB_s = \int_0^t g(s, B_s) dB_s \quad \Rightarrow \quad g(t, b) = b^2$$

[Step 2] Find f with $g = f_b$

$$f_b = g = b^2 \quad \Rightarrow \quad f(t, b) = \frac{1}{3} b^3$$

[Step 3] Apply Ito lemma to f

$$f = \frac{1}{3} b^3, \quad f_t = 0, \quad f_b = b^2, \quad f_{bb} = 2b$$

Definition	$df \Rightarrow \int_0^t df = f(t, B_t) - f(0, 0) = \frac{1}{3} B_t^3$
High school integral	$f_t * dt \Rightarrow \int_0^t f_t * dt = \int_0^t f_t(s, B_s) ds = 0$
Ito integral	$f_b * db \Rightarrow \int_0^t f_b db = \int_0^t g db = \int_0^t B_s^2 dB_s$
High school integral	$\frac{1}{2} * f_{bb} * dt \Rightarrow \frac{1}{2} * \int_0^t f_{bb} * dt = \int_0^t B_s ds$

$$\frac{1}{3} B_t^3 = 0 + \int_0^t B_s^2 dB_s + \int_0^t B_s ds$$

[Step 4] Identify Ito integral $\int_0^t g(s, B_s) dB_s$

$$\int_0^t B_s^2 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s ds$$

How to check SDE solution - Stock price move in BS model - Version 1

$$\text{SDE} \qquad \frac{dS}{S} = rdt + \sigma db \quad \text{with } S_0 \text{ given}$$

$$\text{Solution} \qquad S = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t}$$

$\boxed{\text{Identify } X}$

$$t = t, \quad b = B_t$$

$\boxed{\text{Identify SDE for } X}$

$$dt = dt, \quad db = dB_t$$

$\boxed{\text{Represent } S \text{ in terms of } X}$

$$S = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t} \quad \Rightarrow \quad f(t,x) = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma b}$$

$\boxed{\text{Identify SDE for } S}$

$$\begin{aligned} df &= f_t * dt + f_b * db + \frac{1}{2} * f_{tt} * (dt)^2 + \frac{1}{2} * f_{bb} * (db)^2 + f_{tb} * (dt)(db) \\ &= (r - \frac{1}{2}\sigma^2) f * dt + \sigma f * db + \frac{1}{2} * f_{tt} * 0 + \frac{1}{2} * \sigma^2 f * dt + f_{tb} * 0 \\ &= (r - \frac{1}{2}\sigma^2) f * dt + \sigma f * db + \frac{1}{2} * \sigma^2 f * dt \\ &= f * (rdt + \sigma db) \end{aligned}$$

$$\frac{df}{f} = rdt + \sigma db \quad \Rightarrow \quad \frac{dS}{S} = rdt + \sigma db$$

How to check SDE solution - Stock price move in BS model - Version 2

$$\text{SDE} \qquad \frac{dS}{S} = rdt + \sigma db \quad \text{with } S_0 \text{ given}$$

$$\text{Solution} \qquad S = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t}$$

Identify X

$$X = \left(r - \frac{1}{2}\sigma^2\right) t + \sigma B_t$$

Identify SDE for X

$$dx = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma db$$

Represent S in terms of X

$$S = S_0 e^X \quad \Rightarrow \quad f(t,x) = S_0 e^x$$

Identify SDE for S

$$\begin{aligned}
df &= f_t * dt + f_x * dx + \frac{1}{2} * f_{tt} * (dt)^2 + \frac{1}{2} * f_{xx} * (dx)^2 + f_{tx} * (dt)(dx) \\
&= 0 * dt + f * \left(\left(r - \frac{1}{2}\sigma^2\right) dt + \sigma db\right) + \frac{1}{2} * 0 * 0 + \frac{1}{2} * f * (\sigma^2 dt) + 0 * 0 \\
&= f * \left(\left(r - \frac{1}{2}\sigma^2\right) dt + \sigma db\right) + \frac{1}{2} * f * (\sigma^2 dt) \\
&= f * (rdt + \sigma db)
\end{aligned}$$

$$\frac{df}{f} = rdt + \sigma db \quad \Rightarrow \quad \frac{dS}{S} = rdt + \sigma db$$

How to check SDE solution - Short move move in Vasicek model

$$\begin{aligned}
\text{SDE} \qquad dr &= a(b-r)dt + \sigma db \quad \text{with } r_0 \text{ given} \\
\text{Solution} \qquad r_t &= r_0 e^{-at} + b(1-e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dB_s \\
&= r_0 e^{-at} + b(1-e^{-at}) + \sigma e^{-at} \int_0^t e^{as} dB_s
\end{aligned}$$

Identify X

$$X = \int_0^t e^{as} dB_s$$

Identify SDE for X

$$dx = e^{at} db$$

Represent S in terms of X

$$r = r_0 e^{-at} + b(1-e^{-at}) + \sigma e^{-at} X \quad \Rightarrow \quad f(t,x) = r_0 e^{-at} + b(1-e^{-at}) + \sigma x e^{-at}$$

Identify SDE for S

$$\begin{aligned}
df &= f_t * dt + f_x * dx + \frac{1}{2} * f_{tt} * (dt)^2 + \frac{1}{2} * f_{xx} * (dx)^2 + f_{tx} * (dt)(dx) \\
&= (-af + ab) * dt + \sigma e^{-at} * e^{at} db + \frac{1}{2} * f_{tt} * 0 + \frac{1}{2} * 0 * (dx)^2 + f_{tx} * 0 \\
&= a(b-f) * dt + \sigma db
\end{aligned}$$

$$df = a(b-f) * dt + \sigma db \quad \Rightarrow \quad dr = a(b-r)dt + \sigma db$$