# SDE

## 1 What is SDE

Stock price move simulation

Vasicek short rate simulation

CIR short rate simulation

## 2 How to solve SDE

SDE for Brownian motion

SDE for drifted Brownian motion

SDE for stock price move

SDE for short rate move

## 3 How to compute the mean and variance of the solution of SDE

Mean and variance of Brownian motion

Mean and variance of drifted Brownian motion

Mean and variance of stock price move

Mean and variance of short rate move

Mean and variance of CIR short rate move

Stock price move simulation

We flips a fair coin 10 times and we get

$$HHTHTTHHHT$$

Using these coin flips construct a Brownian motion sample path $B_t$ up to 1 year. Then, simulate Black-Scholes SDE with initial stock price $S_0 = 100$, mean discrete return $\mu = 0.10$, volatility $\sigma = 0.30$ using the above Brownian motion sample path.

$$\frac{dS}{S} = \mu dt + \sigma dB_t$$

Find the stock price $S_1$ after 1 year.

Divide time interval

Divide time interval of interest into small subintervals with equal length $dt$.

$$t_0 \quad < \quad t_1 \quad < \quad t_2 \quad < \quad \cdots \quad < \quad t_{N-1} \quad < \quad t_N$$

Given info

$$\begin{array}{ccccccccc} S_0 & & S_1 & & S_2 & & \cdots & & S_n \\ t_0 & < & t_1 & < & t_2 & < & \cdots & < & t_n \end{array}$$

Updating rule

$$\begin{array}{ccccccccccc} S_0 & & S_1 & & S_2 & & \cdots & & S_n & & S_{n+1} \\ t_0 & < & t_1 & < & t_2 & < & \cdots & < & t_n & < & t_{n+1} \end{array}$$

$$\begin{array}{ccccccc} \frac{dS}{S} & = & \mu & dt & + & \sigma & dB_t \\ \uparrow & & \uparrow & \uparrow & & \uparrow & \uparrow \\ \frac{S_{n+1}-S_n}{S_n} & & 0.10 & \frac{1}{10} & & 0.30 & \pm\sqrt{\frac{1}{10}} \end{array}$$

| | | | | |
|---|---|---|---|---|
| Time | 0/10 | 1/10 | 2/10 | 3/10 |
| Coin flip | – | $H$ | $H$ | $T$ |
| Conversion | – | 1 | 1 | $-1$ |
| Cum sum | 0 | 1 | 2 | 1 |
| $B_t$ | 0 | $\frac{1}{\sqrt{10}}$ | $\frac{2}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ |
| $dt$ | – | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| $dB_t = B_t - B_{t-dt}$ | – | $\frac{1}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ |
| $\mu * dt + \sigma * dB_t$ | – | $\frac{0.1}{10} + \frac{0.3}{\sqrt{10}}$ | $\frac{0.1}{10} + \frac{0.3}{\sqrt{10}}$ | $\frac{0.1}{10} - \frac{0.3}{\sqrt{10}}$ |
| $S_{t-dt} * (\mu * dt + \sigma * dB_t)$ | – | 10.4868 | 11.5866 | $-10.3602$ |
| $S_t = S_{t-dt} + S_{t-dt} * (\mu * dt + \sigma * dB_t)$ | 100 | 110.4868 | 122.0734 | 111.7132 |

| | | | | |
|---|---|---|---|---|
| Time | 4/10 | 5/10 | 6/10 | 7/10 |
| Coin flip | $H$ | $T$ | $T$ | $H$ |
| Conversion | 1 | $-1$ | $-1$ | 1 |
| Cum sum | 2 | 1 | 0 | 1 |
| $B_t$ | $\frac{2}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ | $\frac{0}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ |
| $dt$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| $dB_t = B_t - B_{t-dt}$ | $\frac{1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ |
| $\mu * dt + \sigma * dB_t$ | $\frac{0.1}{10} + \frac{0.3}{\sqrt{10}}$ | $\frac{0.1}{10} - \frac{0.3}{\sqrt{10}}$ | $\frac{0.1}{10} - \frac{0.3}{\sqrt{10}}$ | $\frac{0.1}{10} + \frac{0.3}{\sqrt{10}}$ |
| $S_{t-dt} * (\mu * dt + \sigma * dB_t)$ | 11.7152 | $-10.4752$ | $-9.5861$ | 10.8399 |
| $S_t = S_{t-dt} + S_{t-dt} * (\mu * dt + \sigma * dB_t)$ | 123.4284 | 112.9532 | 103.3671 | 114.2070 |

| | | | |
|---|---|---|---|
| Time | 8/10 | 9/10 | 10/10 |
| Coin flip | $H$ | $H$ | $T$ |
| Conversion | 1 | 1 | $-1$ |
| Cum sum | 2 | 3 | 2 |
| $B_t$ | $\frac{2}{\sqrt{10}}$ | $\frac{3}{\sqrt{10}}$ | $\frac{2}{\sqrt{10}}$ |
| $dt = dt$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| $dB_t = B_t - B_{t-dt}$ | $\frac{1}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ |
| $\mu * dt + \sigma * dB_t$ | $\frac{0.1}{10} + \frac{0.3}{\sqrt{10}}$ | $\frac{0.1}{10} + \frac{0.3}{\sqrt{10}}$ | $\frac{0.1}{10} - \frac{0.3}{\sqrt{10}}$ |
| $S_{t-dt} * (\mu * dt + \sigma * dB_t)$ | 11.9767 | 13.2327 | $-11.8320$ |
| $S_t = S_{t-dt} + S_{t-dt} * (\mu * dt + \sigma * dB_t)$ | 126.1837 | 139.4164 | 127.5844 |

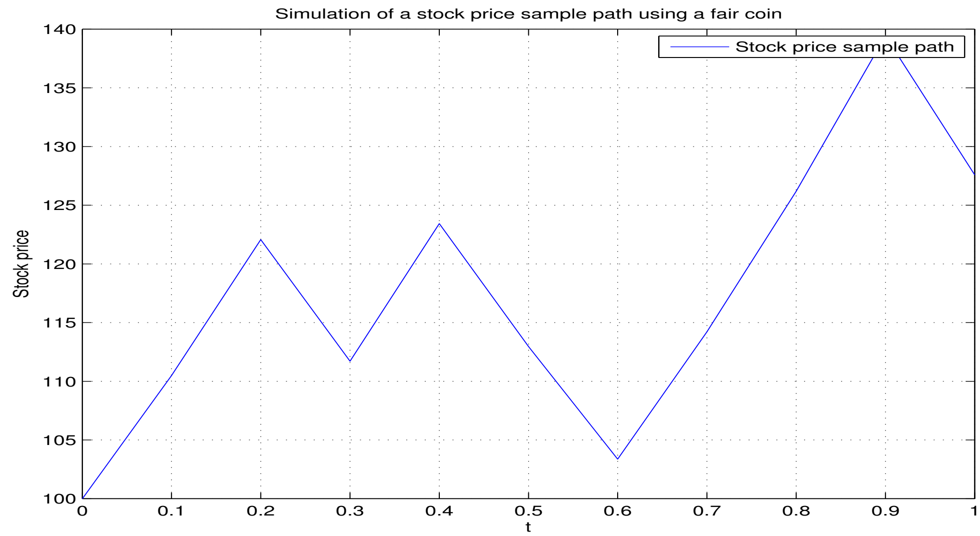


Figure 1: Simulation of a stock price sample path using a fair coin.

```
clear all; close all; clc; rng('default');

mu = 0.10; si = 0.30; S0 = 100;
M = 1; % Number of simulation
n = 10; % Number of days per year
T = 1; % Number of years in simulation

Coin = [1 1 -1 1 -1 -1 1 1 1 -1]';
Brownian_Motion = [zeros(1,M); cumsum(Coin)/sqrt(n)];

db = Brownian_Motion(2:end,:)-Brownian_Motion(1:end-1,:);
dt = (1/n)*ones(size(db));

S = S0*ones(size(Brownian_Motion));
for i=2:ceil(n*T)+1
    S(i,:) = S(i-1,:) + S(i-1).*(mu*dt(i-1,:)+si*db(i-1,:));
end

% Plot of a stock price sample path using a fair coin
plot(0:1/n:T,S); grid on; hold on
legend('Stock price sample path')
xlabel('t'); ylabel('Stock price')
title('Simulation of a stock price sample path using a fair coin')
```

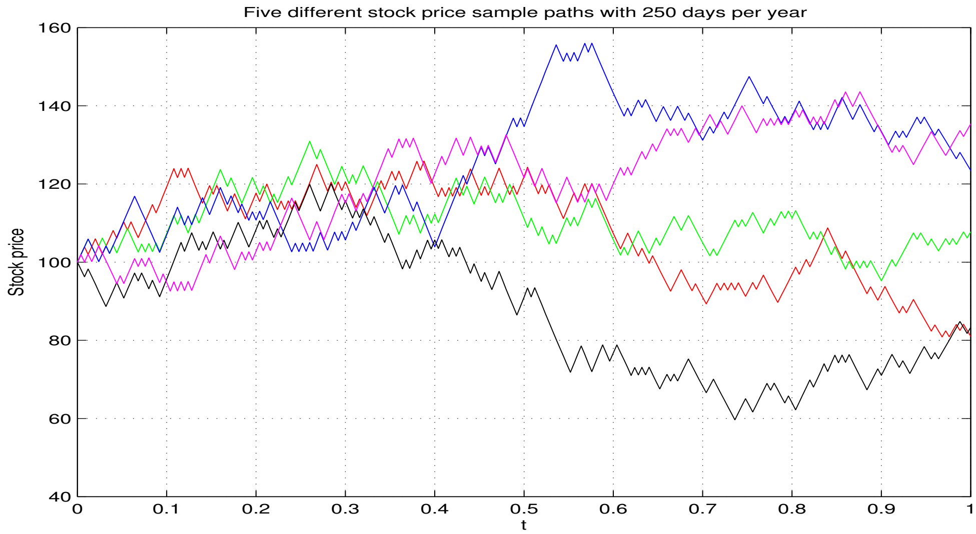


Figure 2: Five different stock price sample paths with 250 days per year.

```
clear all; close all; clc; rng('default');

mu = 0.10; si = 0.30; S0 = 100;
M = 5; % Number of simulation
n = 250; % Number of days per year
T = 1; % Number of years in simulation

x = rand(ceil(n*T),M);
Coin = x; Coin(x<=0.5) = -1; Coin(x>0.5) = 1;
Brownian_Motion = [zeros(1,M); cumsum(Coin)/sqrt(n)];

db = Brownian_Motion(2:end,:)-Brownian_Motion(1:end-1,:);
dt = (1/n)*ones(size(db));

S = S0*ones(size(Brownian_Motion));
for i=2:ceil(n*T)+1
    S(i,:) = S(i-1,:) + S(i-1).*(mu*dt(i-1,:)+si*db(i-1,:));
end

% Plot of five different stock price sample paths up to 1 year
color ='rgbkm';
for i = 1:M
    plot(0:1/n:T,S(1:length(0:1/n:T),i),color(i)); grid on; hold on
end
xlabel('t'); ylabel('Stock price')
title('Five different stock price sample paths with 250 days per year')
```

Vasicek short rate simulation

We flips a fair coin 10 times and we get

$$HHTHTTHHHT$$

Using these coin flips construct a Brownian motion sample path $B_t$ up to 1 year. Then, simulate Vasicek SDE with initial short rate $r_0 = 0.03$, speed of mean reversion $a = 2$, target short rate $b = 0.05$, volatility $\sigma = 0.01$ using the above Brownian motion sample path.

$$dr = a(b - r)dt + \sigma db$$

Find the short rate $r_1$ after 1 year.

$\boxed{\text{Divide time interval}}$

Divide time interval of interest into small subintervals with equal length $dt$.

$$t_0 \;<\; t_1 \;<\; t_2 \;<\; \cdots \;<\; t_{N-1} \;<\; t_N$$

$\boxed{\text{Given info}}$

$$\begin{array}{ccccccccc} r_0 & & r_1 & & r_2 & & \cdots & & r_n \\ t_0 & < & t_1 & < & t_2 & < & \cdots & < & t_n \end{array}$$

$\boxed{\text{Updating rule}}$

$$\begin{array}{ccccccccccc} r_0 & & r_1 & & r_2 & & \cdots & & r_n & & r_{n+1} \\ t_0 & < & t_1 & < & t_2 & < & \cdots & < & t_n & < & t_{n+1} \end{array}$$

$$\begin{array}{cccccccccccc} dr & = & a & ( & b & - & r & ) & dt & + & \sigma & db \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \uparrow \\ r_{n+1} - r_n & & 2 & & 0.05 & & r_n & & \frac{1}{10} & & 0.01 & \pm\sqrt{\frac{1}{10}} \end{array}$$

| | | | | |
|---|---|---|---|---|
| Time | 0/10 | 1/10 | 2/10 | 3/10 |
| Coin flip | – | $H$ | $H$ | $T$ |
| Conversion | – | 1 | 1 | $-1$ |
| Cum sum | 0 | 1 | 2 | 1 |
| $B_t$ | 0 | $\frac{1}{\sqrt{10}}$ | $\frac{2}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ |
| $dt$ | – | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| $dB_t = B_t - B_{t-dt}$ | – | $\frac{1}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ |
| $a*(b-r_{t-dt})*dt+\sigma*dB_t$ | – | 0.0072 | 0.0057 | $-0.0017$ |
| $r_t = r_{t-dt} + a*(b-r_{t-dt})*dt+\sigma*dB_t$ | 0.0300 | 0.0372 | 0.0429 | 0.0412 |

| | | | | |
|---|---|---|---|---|
| Time | 4/10 | 5/10 | 6/10 | 7/10 |
| Coin flip | $H$ | $T$ | $T$ | $H$ |
| Conversion | 1 | $-1$ | $-1$ | 1 |
| Cum sum | 2 | 1 | 0 | 1 |
| $B_t$ | $\frac{2}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ | $\frac{0}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ |
| $dt$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| $dB_t = B_t - B_{t-dt}$ | $\frac{1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ |
| $a*(b-r_{t-dt})*dt+\sigma*dB_t$ | 0.0049 | $-0.0024$ | $-0.0019$ | 0.0048 |
| $r_t = r_{t-dt} + a*(b-r_{t-dt})*dt+\sigma*dB_t$ | 0.0461 | 0.0437 | 0.0418 | 0.0466 |

| | | | |
|---|---|---|---|
| Time | 8/10 | 9/10 | 10/10 |
| Coin flip | $H$ | $H$ | $T$ |
| Conversion | 1 | 1 | $-1$ |
| Cum sum | 2 | 3 | 2 |
| $B_t$ | $\frac{2}{\sqrt{10}}$ | $\frac{3}{\sqrt{10}}$ | $\frac{2}{\sqrt{10}}$ |
| $dt = dt$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| $dB_t = B_t - B_{t-dt}$ | $\frac{1}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ |
| $a*(b-r_{t-dt})*dt+\sigma*dB_t$ | 0.0038 | 0.0031 | $-0.0039$ |
| $r_t = r_{t-dt} + a*(b-r_{t-dt})*dt+\sigma*dB_t$ | 0.0504 | 0.0535 | 0.0496 |

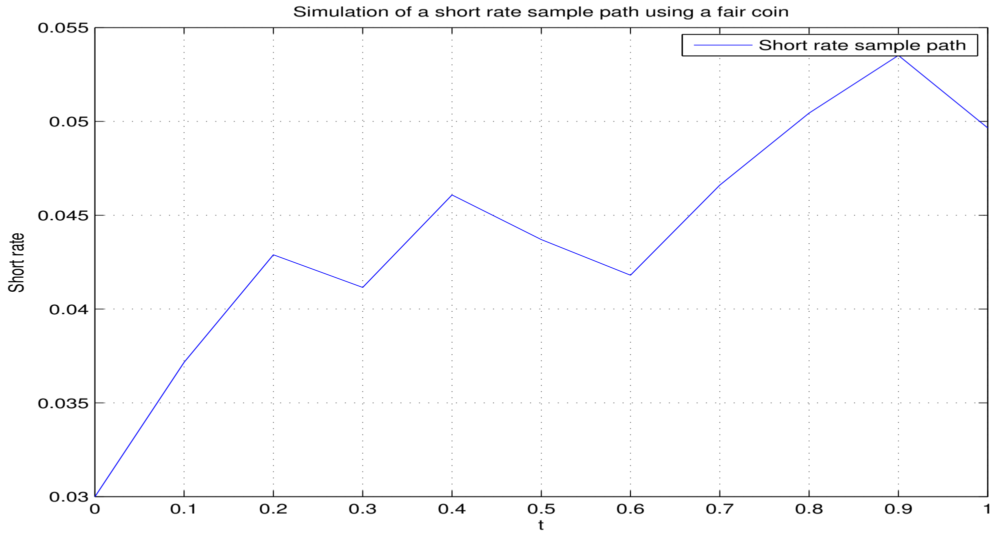


Figure 3: Simulation of a short rate sample path using a fair coin.

```
clear all; close all; clc; rng('default');

a = 2; b = 0.05; si = 0.01; r0 = 0.03;
M = 1; % Number of simulation
n = 10; % Number of days per year
T = 1; % Number of years in simulation

Coin = [1 1 -1 1 -1 -1 1 1 1 -1]';
Brownian_Motion = [zeros(1,M); cumsum(Coin)/sqrt(n)];

db = Brownian_Motion(2:end,:)-Brownian_Motion(1:end-1,:);
dt = (1/n)*ones(size(db));

r = r0*ones(size(Brownian_Motion));
for i=2:n+1
    r(i,:) = r(i-1,:) + a*(b-r(i-1)).*dt(i-1,:) + si.*db(i-1,:);
end

% Plot of a short rate sample path using a fair coin
plot(0:1/n:T,r); grid on; hold on
legend('Short rate sample path')
xlabel('t'); ylabel('Short rate')
title('Simulation of a short rate sample path using a fair coin')
```

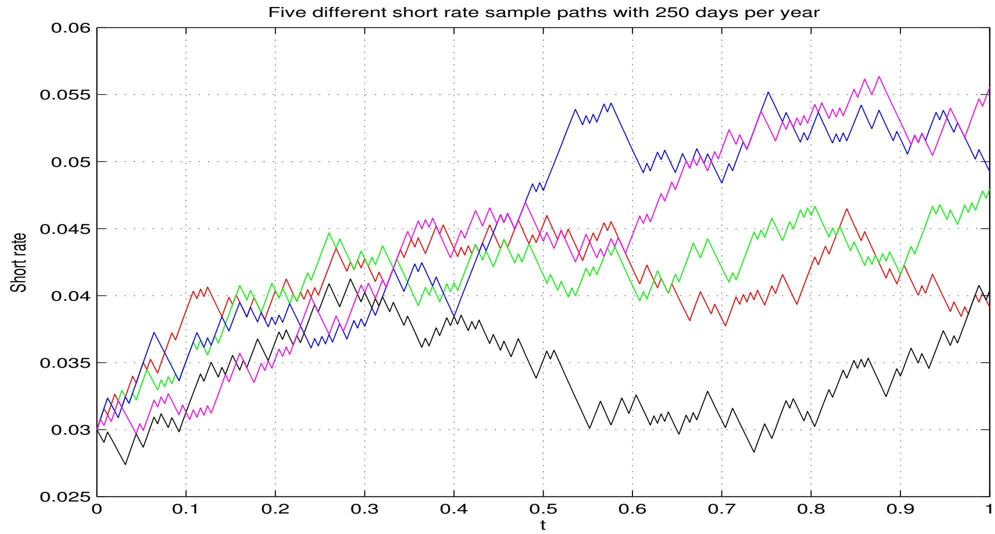


Figure 4: Five different short rate sample paths with 250 days per year.

```
clear all; close all; clc; rng('default');

a = 2; b = 0.05; si = 0.01; r0 = 0.03;
M = 5; % Number of simulation
n = 250; % Number of days per year
T = 1; % Number of years in simulation

x = rand(ceil(n*T),M);
Coin = x; Coin(x<=0.5) = -1; Coin(x>0.5) = 1;
Brownian_Motion = [zeros(1,M); cumsum(Coin)/sqrt(n)];

db = Brownian_Motion(2:end,:)-Brownian_Motion(1:end-1,:);
dt = (1/n)*ones(size(db));

r = r0*ones(size(Brownian_Motion));
for i=2:ceil(n*T)+1
    r(i,:) = r(i-1,:) + a*(b-r(i-1)).*dt(i-1,:) + si.*db(i-1,:);
end

% Plot of five different short rate sample paths up to 1 year
color ='rgbkm';
for i = 1:M
    plot(0:1/n:T,r(1:length(0:1/n:T),i),color(i)); grid on; hold on
end
xlabel('t'); ylabel('Short rate')
title('Five different short rate sample paths with 250 days per year')
```

CIR short rate simulation

We flips a fair coin 10 times and we get

$$HHTHTTHHHT$$

Using these coin flips construct a Brownian motion sample path $B_t$ up to 1 year. Then, simulate CIR SDE with initial short rate $r_0 = 0.03$, speed of mean reversion $a = 2$, target short rate $b = 0.05$, volatility $\sigma = 0.01$ using the above Brownian motion sample path.

$$dr = a(b-r)dt + \sigma\sqrt{r}db$$

Find the short rate $r_1$ after 1 year.

Divide time interval

Divide time interval of interest into small subintervals with equal length $dt$.

$$t_0 \quad < \quad t_1 \quad < \quad t_2 \quad < \quad \cdots \quad < \quad t_{N-1} \quad < \quad t_N$$

Given info

$$\begin{array}{ccccccccc} r_0 & & r_1 & & r_2 & & \cdots & & r_n \\ t_0 & < & t_1 & < & t_2 & < & \cdots & < & t_n \end{array}$$

Updating rule

$$\begin{array}{ccccccccccc} r_0 & & r_1 & & r_2 & & \cdots & & r_n & & r_{n+1} \\ t_0 & < & t_1 & < & t_2 & < & \cdots & < & t_n & < & t_{n+1} \end{array}$$

$$\begin{array}{cccccccccccc} dr & = & a & ( & b & - & r & ) & dt & + & \sigma & \sqrt{r} & db \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \uparrow & \uparrow \\ r_{n+1}-r_n & & 2 & & 0.05 & & r_n & & \frac{1}{10} & & 0.01 & \sqrt{r_n} & \pm\sqrt{\frac{1}{10}} \end{array}$$

| | | | | |
|---|---|---|---|---|
| Time | 0/10 | 1/10 | 2/10 | 3/10 |
| Coin flip | – | $H$ | $H$ | $T$ |
| Conversion | – | 1 | 1 | $-1$ |
| Cum sum | 0 | 1 | 2 | 1 |
| $B_t$ | 0 | $\frac{1}{\sqrt{10}}$ | $\frac{2}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ |
| $dt$ | – | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| $dB_t = B_t - B_{t-dt}$ | – | $\frac{1}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ |
| $\Delta r = a * (b - r_{t-dt}) * dt + \sigma * \sqrt{r_{t-dt}} * dB_t$ | – | 0.0045 | 0.0037 | 0.0018 |
| $r_t = r_{t-dt} + \Delta r$ | 0.0300 | 0.0345 | 0.0382 | 0.0400 |

| | | | | |
|---|---|---|---|---|
| Time | 4/10 | 5/10 | 6/10 | 7/10 |
| Coin flip | $H$ | $T$ | $T$ | $H$ |
| Conversion | 1 | $-1$ | $-1$ | 1 |
| Cum sum | 2 | 1 | 0 | 1 |
| $B_t$ | $\frac{2}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ | $\frac{0}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ |
| $dt$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| $dB_t = B_t - B_{t-dt}$ | $\frac{1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ |
| $\Delta r = a * (b - r_{t-dt}) * dt + \sigma * \sqrt{r_{t-dt}} * dB_t$ | 0.0026 | 0.0008 | 0.0007 | 0.0018 |
| $r_t = r_{t-dt} + \Delta r$ | 0.0426 | 0.0434 | 0.0441 | 0.0459 |

| | | | |
|---|---|---|---|
| Time | 8/10 | 9/10 | 10/10 |
| Coin flip | $H$ | $H$ | $T$ |
| Conversion | 1 | 1 | $-1$ |
| Cum sum | 2 | 3 | 2 |
| $B_t$ | $\frac{2}{\sqrt{10}}$ | $\frac{3}{\sqrt{10}}$ | $\frac{2}{\sqrt{10}}$ |
| $dt = dt$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| $dB_t = B_t - B_{t-dt}$ | $\frac{1}{\sqrt{10}}$ | $\frac{1}{\sqrt{10}}$ | $\frac{-1}{\sqrt{10}}$ |
| $\Delta r = a * (b - r_{t-dt}) * dt + \sigma * \sqrt{r_{t-dt}} * dB_t$ | 0.0015 | 0.0012 | $-0.0004$ |
| $r_t = r_{t-dt} + \Delta r$ | 0.0474 | 0.0486 | 0.0482 |

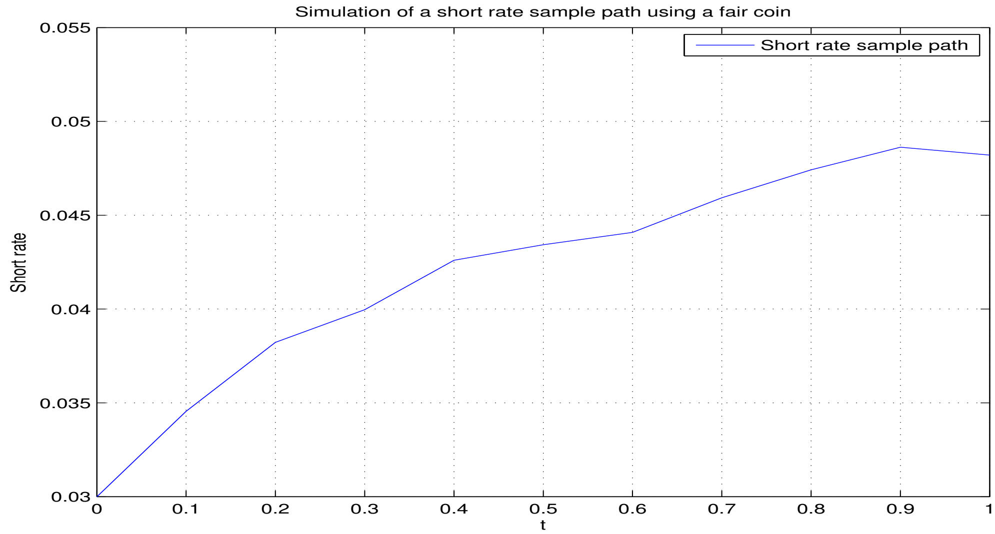


Figure 5: Simulation of a short rate sample path using a fair coin.

```
clear all; close all; clc; rng('default');

a = 2; b = 0.05; si = 0.01; r0 = 0.03;
M = 1; % Number of simulation
n = 10; % Number of days per year
T = 1; % Number of years in simulation

Coin = [1 1 -1 1 -1 -1 1 1 1 -1]';
Brownian_Motion = [zeros(1,M); cumsum(Coin)/sqrt(n)];

db = Brownian_Motion(2:end,:)-Brownian_Motion(1:end-1,:);
dt = (1/n)*ones(size(db));

r = r0*ones(size(Brownian_Motion));
for i=2:n+1
    r(i,:) = r(i-1,:)+a*(b-r(i-1)).*dt(i-1,:)+si.*sqrt(r(i-1)).*db(i-1,:);
end

% Plot of a short rate sample path using a fair coin
plot(0:1/n:T,r); grid on; hold on
legend('Short rate sample path')
xlabel('t'); ylabel('Short rate')
title('Simulation of a short rate sample path using a fair coin')
```

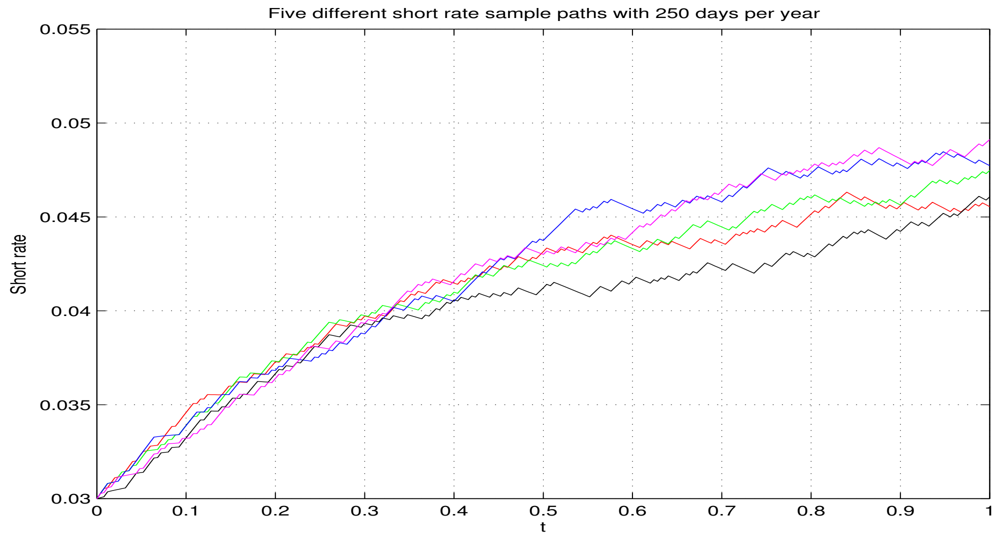


Figure 6: Five different short rate sample paths with 250 days per year.

```
clear all; close all; clc; rng('default');

a = 2; b = 0.05; si = 0.01; r0 = 0.03;
M = 5; % Number of simulation
n = 250; % Number of days per year
T = 1; % Number of years in simulation

x = rand(ceil(n*T),M);
Coin = x; Coin(x<=0.5) = -1; Coin(x>0.5) = 1;
Brownian_Motion = [zeros(1,M); cumsum(Coin)/sqrt(n)];

db = Brownian_Motion(2:end,:)-Brownian_Motion(1:end-1,:);
dt = (1/n)*ones(size(db));

r = r0*ones(size(Brownian_Motion));
for i=2:ceil(n*T)+1
    r(i,:) = r(i-1,:)+a*(b-r(i-1)).*dt(i-1,:)+si.*sqrt(r(i-1)).*db(i-1,:);
end

% Plot of five different short rate sample paths up to 1 year
color ='rgbkm';
for i = 1:M
    plot(0:1/n:T,r(1:length(0:1/n:T),i),color(i)); grid on; hold on
end
xlabel('t'); ylabel('Short rate')
title('Five different short rate sample paths with 250 days per year')
```

**SDE for Brownian motion**

Solve the following SDE:

$$dB_t \quad \text{with} \quad B_0 = 0$$

$$\int_0^T dB_t$$

$$\int_0^T dB_t = \sum dB_t = (B_{dt} - B_0) + (B_{2dt} - B_{dt}) + \cdots + (B_T - B_{T-dt}) = B_T - B_0 = B_T$$

**SDE for drifted Brownian motion**

Solve the following SDE:

$$dX_t = \mu dt + \sigma dB_t \quad \text{with given } X_0$$

$$\int_0^T dX_t = \mu \int_0^T dt + \sigma \int_0^T dB_t$$

$$\begin{aligned}
\int_0^T dX_t &= \sum dX_t = (X_{dt} - X_0) + (X_{2dt} - X_{dt}) + \cdots + (X_T - X_{T-dt}) = X_T - X_0 \\
\int_0^T dt &= \sum dt = (dt - 0) + (2dt - dt) + \cdots + (T - (T - dt)) = T \\
\int_0^T dB_t &= \sum dB_t = B_T
\end{aligned}$$

$$X_T - X_0 = \mu T + \sigma B_T \quad \Rightarrow \quad X_T = X_0 + \mu T + \sigma B_T$$

**SDE for stock price move**

Solve the following SDE:

$$\frac{dS}{S} = \mu dt + \sigma db \quad \text{with given } S_0$$

$$\begin{array}{ll} \text{Change of variable} & f = \log s \\ \text{Ito lemma for } f & df = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma db \end{array}$$

$$df = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma db \quad \Rightarrow \quad \int_0^T df = \left(\mu - \frac{1}{2}\sigma^2\right) \int_0^T dt + \sigma \int_0^T db$$

$$\begin{aligned} \int_0^T df &= \sum df = (f_{dt} - f_0) + \cdots + (f_T - f_{T-dt}) = f_T - f_0 = \log S_T - \log S_0 \\ \int_0^T dt &= T \\ \int_0^T dB_t &= B_T \end{aligned}$$

$$\log S_T - \log S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B_T \quad \Rightarrow \quad S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T}$$

**SDE for short rate move**

Solve the following SDE:

$$dr = a(b-r)dt + \sigma db \quad \text{with given } r_0$$

Integral factor $g = g(t)$

Choose $g$ with $g' - ag = 0$; $g = g(t) = e^{at}$

$$f = f(t,r) = g(t) * r$$

$$\begin{aligned} df &= (abg + (g' - ag)r)\,dt + (\sigma g)\,db \\ &= abe^{at}dt + \sigma e^{at}db \end{aligned}$$

$$abe^{at}dt + \sigma e^{at}db \quad \Rightarrow \quad \int_0^T df = ab\int_0^T e^{at}dt + \sigma\int_0^T e^{at}db$$

$$\begin{aligned} \int_0^T df &= f_T - f_0 = e^{aT}r_T - r_0 \\ \int_0^T e^{at}dt &= \frac{e^{aT}-1}{a} \\ \int_0^T e^{at}dB_t &= \sum e^{at}dB_t = (B_{dt} - B_0) + \cdots + e^{a(T-dt)}(B_T - B_{T-dt}) = \int_0^T e^{at}dB_t \end{aligned}$$

$$e^{aT}r_T - r_0 = b(e^{aT}-1) + \sigma\int_0^T e^{at}dB_t \quad \Rightarrow \quad r_T = r_0e^{-aT} + b(1-e^{-aT}) + \sigma\int_0^T e^{-a(T-t)}dB_t$$

Mean and variance of Brownian motion

Calculate the mean and variance of the solution of the following SDE:

$$dB_t \quad \text{with} \quad B_0 = 0$$

$$\int_0^T dB_t = B_T \quad \Rightarrow \quad \mathbb{E}B_T = 0, \quad Var(B_T) = T$$

**Mean and variance of drifted Brownian motion**

Calculate the mean and variance of the solution of the following SDE:

$$dX_t = \mu dt + \sigma dB_t \quad \text{with given } X_0$$

$$\begin{aligned} X_T - X_0 = \mu T + \sigma B_T \quad &\Rightarrow \quad X_T = X_0 + \mu T + \sigma B_T \\ &\Rightarrow \quad \mathbb{E}X_T = X_0 + \mu T, \quad Var\left(X_T\right) = \sigma^2 Var\left(B_T\right) = \sigma^2 T \end{aligned}$$

Mean and variance of stock price move

Calculate the mean and variance of the solution of the following SDE:

$$\frac{dS}{S} = \mu dt + \sigma db \quad \text{with given } S_0$$

$$\begin{array}{ll} \text{Change of variable} & f = \log s \\ \text{Ito lemma for } f & df = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma db \end{array}$$

$$df = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma db \quad \Rightarrow \quad \int_0^T df = \left(\mu - \frac{1}{2}\sigma^2\right) \int_0^T dt + \sigma \int_0^T db$$

$$\log S_T - \log S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B_T \quad \Rightarrow \quad S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T}$$

$$\begin{aligned} \phi_{\mathcal{N}(\mu,\sigma^2)}(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}}}_{\text{PDF of } N(\mu+\sigma^2 t, \sigma^2)} dx = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \end{aligned}$$

$$\mathbb{E} e^{\sigma B_T} = \phi_{\mathcal{N}(0,\sigma^2 T)}(1) = e^{\frac{1}{2}\sigma^2 T}$$

$$\begin{aligned} &\Rightarrow \quad \mathbb{E} S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T} \mathbb{E} e^{\sigma B_T} = S_0 e^{\mu T} \\ &\Rightarrow \quad \mathbb{E} S_T^2 = S_0^2 e^{2(\mu - \frac{1}{2}\sigma^2)T} \mathbb{E} e^{2\sigma B_T} = S_0^2 e^{2(\mu - \frac{1}{2}\sigma^2)T} e^{\frac{1}{2}(2\sigma)^2 T} = S_0^2 e^{(2\mu + \sigma^2)T} \\ &\Rightarrow \quad Var(S_T) = \mathbb{E} S_T^2 - (\mathbb{E} S_T)^2 = S_0^2 e^{2\mu T} \left(e^{\sigma^2 T} - 1\right) \end{aligned}$$

**Mean and variance of short rate move**

Calculate the mean and variance of the solution of the following SDE:

$$dr = a(b - r)dt + \sigma db \quad \text{with given } r_0$$

Integral factor $g = g(t)$

Choose $g$ with $g' - ag = 0$; $g = g(t) = e^{at}$

$$f = f(t, r) = g(t) * r$$

$$\begin{aligned} df &= (abg + (g' - ag)r)\, dt + (\sigma g)\, db \\ &= abe^{at}dt + \sigma e^{at}db \end{aligned}$$

$$abe^{at}dt + \sigma e^{at}db \quad \Rightarrow \quad \int_0^T df = ab\int_0^T e^{at}dt + \sigma\int_0^T e^{at}db$$

$$e^{aT}r_T - r_0 = b(e^{aT} - 1) + \sigma\int_0^T e^{at}dB_t \;\Rightarrow\; r_T = r_0e^{-aT} + b(1 - e^{-aT}) + \sigma\int_0^T e^{-a(T-t)}dB_t$$

$$\Rightarrow \; \mathbb{E}r_T = r_0e^{-aT} + b(1 - e^{-aT})$$

$$\begin{aligned} Var\left(\int_0^T e^{-a(T-t)}dB_t\right) &\approx Var\left(\sum e^{-a(T-t)}dB_t\right) \\ &= \sum Var\left(e^{-a(T-t)}dB_t\right) \\ &= \sum e^{-2a(T-t)}Var\left(dB_t\right) \\ &= \sum e^{-2a(T-t)}dt \approx \int_0^T e^{-2a(T-t)}dt \end{aligned}$$

$$\Rightarrow \; Var(r_T) = \sigma^2\int_0^T e^{-2a(T-t)}dt = \sigma^2\left[\frac{e^{-2a(T-t)}}{2a}\right]_0^T = \frac{\sigma^2}{2a}\left(1 - e^{-2aT}\right)$$

**Mean and variance of CIR short rate move**

Calculate the mean and variance of the solution of the following SDE:

$$dr = a(b-r)dt + \sigma\sqrt{r}db \quad \text{with given } r_0$$

$$\begin{aligned}
dr = a(b-r)dt + \sigma\sqrt{r}db \quad &\overset{\text{Integrate}}{\Rightarrow} \quad r_t - r_0 = a\int_0^t (b - r_s)ds + \sigma\int_0^t \sqrt{r_s}db \\
&\overset{\text{Take expectation}}{\Rightarrow} \quad \mathbb{E}r_t - r_0 = a\int_0^t (b - \mathbb{E}r_s)ds \\
&\overset{\text{With } g(t) = \mathbb{E}r_t}{\Rightarrow} \quad g(t) - r_0 = a\int_0^t (b - g(s))ds \\
&\overset{\text{Differenciate}}{\Rightarrow} \quad g' = a(b-g) \\
&\overset{\text{Solve ODE}}{\Rightarrow} \quad g = (r_0 - b)e^{-at} + b \\
&\Rightarrow \quad \mathbb{E}r_t = g = (r_0 - b)e^{-at} + b
\end{aligned}$$

With $f = r^2$, by Ito lemma

$$\begin{aligned}
& df = 2r(a(b-r)dt + \sigma\sqrt{r}db) + \sigma^2 r dt \\
\overset{\text{Integrate}}{\Rightarrow} \quad & f(t) - f(0) = (2ab + \sigma^2)\int_0^t r_s ds - 2a\int_0^t r_s^2 ds + 2\sigma\int_0^t r_s^{3/2} db \\
\overset{\text{Take expectation}}{\Rightarrow} \quad & \mathbb{E}f(t) - f(0) = (2ab + \sigma^2)\int_0^t \mathbb{E}r_s ds - 2a\int_0^t \mathbb{E}r_s^2 ds \\
\overset{\text{With } h(t) = \mathbb{E}f(t) = \mathbb{E}r_t^2}{\Rightarrow} \quad & h(t) - h(0) = (2ab + \sigma^2)\int_0^t g(s)ds - 2a\int_0^t h(s)ds \\
\overset{\text{Differenciate}}{\Rightarrow} \quad & h' = (2ab + \sigma^2)g - 2ah \\
\overset{\text{Solve ODE}}{\Rightarrow} \quad & h = (2ab + \sigma^2)\left(\frac{r_0 - b}{a}(e^{-at} - e^{-2at}) + \frac{b}{2a}(1 - e^{-2at})\right) + r_0^2 e^{-2at} \\
\Rightarrow \quad & \mathbb{E}r_t^2 = (2ab + \sigma^2)\left(\frac{r_0 - b}{a}(e^{-at} - e^{-2at}) + \frac{b}{2a}(1 - e^{-2at})\right) + r_0^2 e^{-2at}
\end{aligned}$$