Heat equation

1 Classification of 2nd order PDE

Classification of 2nd order PDE

2 Explicit FDM

1D heat equation - explicit FDM - Pin down boundary condition

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3 Implicit FDM

1D heat equation - implicit FDM - Pin down boundary condition

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1D heat equation - CN FDM - Pin down boundary condition

Classification of 2nd order PDE

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0$$

$$D = b^2 - 4ac$$

Famous PDE	PDE	Sign of Determinant	Official Name
Heat equation	$u_t = u_{xx}$	$D = b^2 - 4ac = 0$	Parabolic
Wave equation	$u_{tt} = u_{xx}$	$D = b^2 - 4ac > 0$	Hyprbolic
Laplace equation	$u_{xx} + u_{yy} = 0$	$D = b^2 - 4ac < 0$	Elliptic

1D heat equation - explicit FDM - Pin down boundary condition

PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Initial condition

$$u(x,0) = e^{-x^2}$$

Pin down boundary condition

$$u(t,-1) = u(t,1) = 0$$

Explicit FDM

Choose δx and δt so that discretization errors are same order: $(\delta x)^2 = C(\delta t)$. More strongly, for stability we have to choose δx and δt with $\rho := \frac{\delta t}{(\delta x)^2} \leq 0.5$.

$$\frac{\partial u}{\partial t} \to \frac{u_i^{t+1} - u_i^t}{\delta t} + O(\delta t)$$

$$\frac{\partial^2 u}{\partial x^2} \to \frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{(\delta x)^2} + O((\delta x)^2)$$

$$\frac{u_i^{t+1} - u_i^t}{\delta t} = \frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{(\delta x)^2}$$

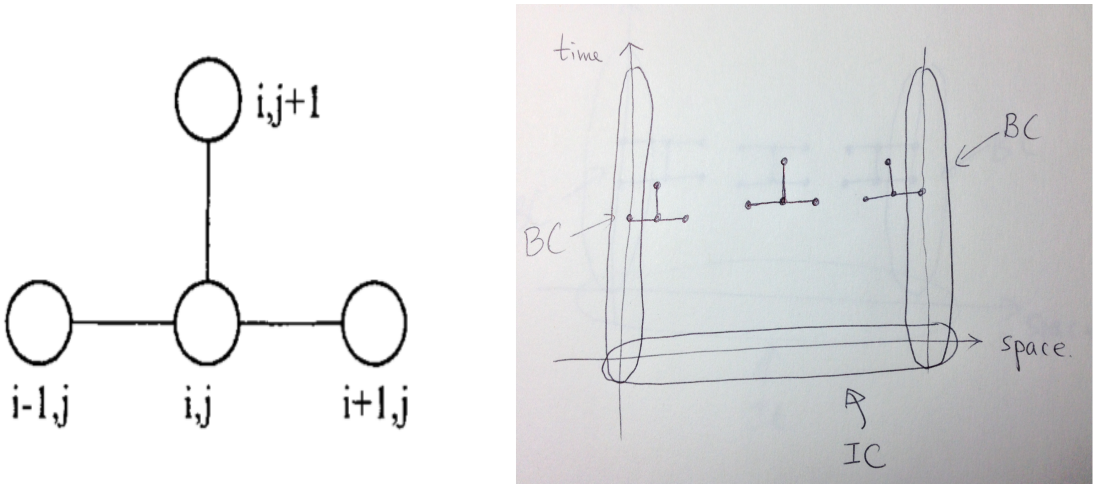


Figure 1: Explicit stenci positioning.

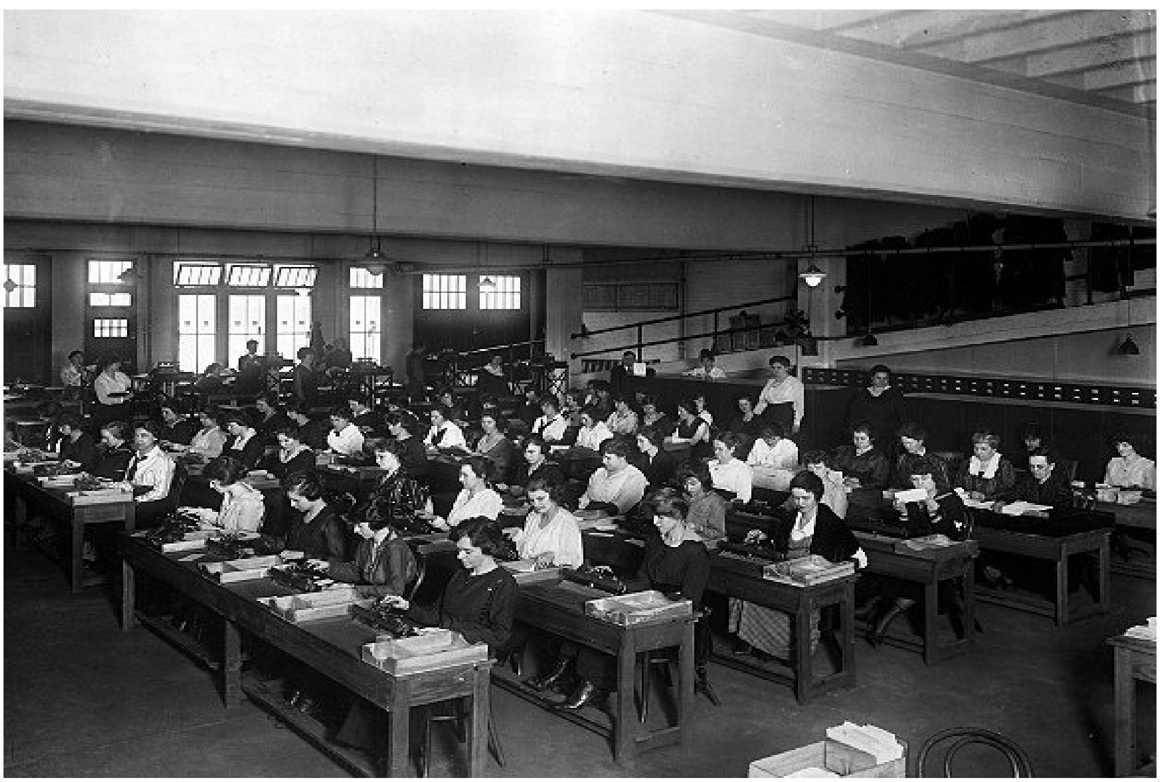


Figure 2: Not Los Alamos, but computers at work in WWII.
Source: http://sigforum.com/eve/forums/a/tpc/f/320601935/m/3130057943/p/2

With $\rho = \delta t/(\delta x)^2$

$$
\underbrace{\begin{bmatrix} u_2^{t+1} \\ u_3^{t+1} \\ \vdots \\ u_{N_x-2}^{t+1} \\ u_{N_x-1}^{t+1} \end{bmatrix}}_{\text{Unknown}} = \begin{bmatrix} \rho & 1-2\rho & \rho & & & & \\ & \rho & 1-2\rho & \rho & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \rho & 1-2\rho & \rho & \\ & & & & \rho & 1-2\rho & \rho \end{bmatrix} \underbrace{\begin{bmatrix} u_1^t \\ u_2^t \\ u_3^t \\ \vdots \\ u_{N_x-2}^t \\ u_{N_x-1}^t \\ u_{N_x}^t \end{bmatrix}}_{\text{Known}}
$$

$$
= \underbrace{\begin{bmatrix} 1-2\rho & \rho & & & \\ \rho & 1-2\rho & \rho & & \\ & \ddots & \ddots & \ddots & \\ & & \rho & 1-2\rho & \rho \\ & & & \rho & 1-2\rho \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} u_2^t \\ u_3^t \\ \vdots \\ u_{N_x-2}^t \\ u_{N_x-1}^t \end{bmatrix}}_{\text{Known}}
$$

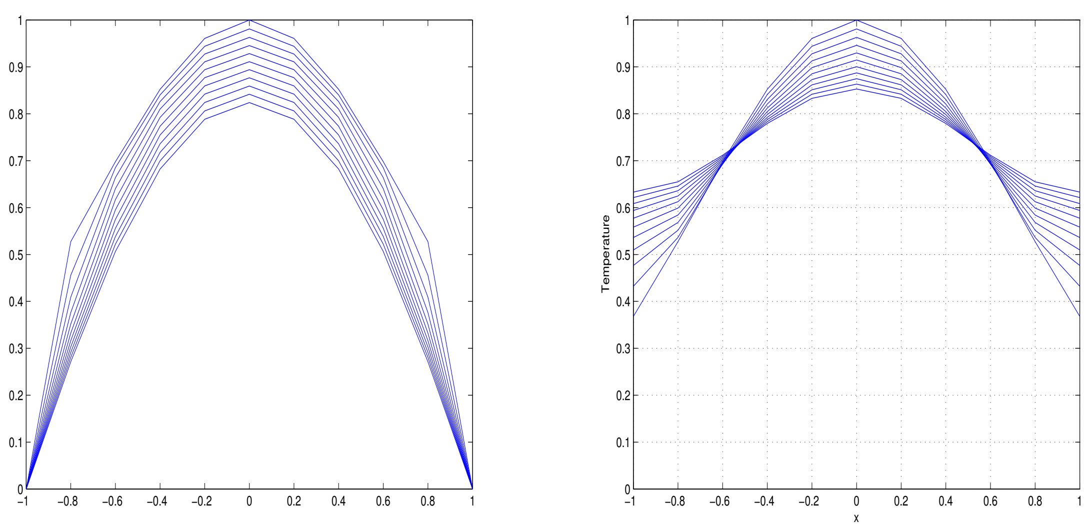


Figure 3: 1D heat equation - explicit FDM - Pin down (left) and No flux boundary condition (right). When we solve a PDE, of course PDE itself important. However, the boundary condition is also equally important. Here, we consider two different boundary conditions; pin down and no flux condition. Even thought they share the common PDE and the common initial condition, due to two different boundary conditions they have two completely different solutions.

```
%% 1D heat equation - explicit FDM - Pin down boundary condition %%%%%%%%%

clear all; close all; clc;

xspan = [-1,1];
Nx = 11;
dx = (xspan(end)-xspan(1))/(Nx-1);
x2 = linspace(xspan(1),xspan(end),Nx);
x = x2(2:end-1);

tspan = [0,0.1];
Nt = 1000+1;
t = linspace(tspan(1),tspan(end),Nt);
dt = t(2)-t(1);

e1 = ones(Nx-2,1);
rho = dt/(dx^2);
D = [(1-2*rho)*e1 rho*e1  rho*e1];
d = [          0     -1       1];
B = spdiags_Lee(D,d,Nx-2,Nx-2);

ic = exp(-x.^2);
u = ic';

plot([-1 x 1],[0 u' 0]); hold on; pause(0.1);

tout = t(1:100:end)';
uout = zeros(length(tout),Nx);
uout(1,:) = [0 u' 0];

for j=2:Nt
    u = B*u;
    if mod(j,100)==1
        plot([-1 x 1],[0 u' 0]); pause(0.1);
        ind = ceil(j/100);
        uout(ind,:) = [0 u' 0];
    end
end

%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

1D heat equation - explicit FDM - No flux boundary condition

PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Initial condition

$$u(x,0) = e^{-x^2}$$

No flux boundary condition

$$u'(t=-1) = u'(t,1) = 0$$

We first put N_x uniform nodes on the interval $[-1,1]$ with mesh dx. To handle the no flux condition we add two ghost nodes at both ends; `x(1) = -dx` and `x(Nx+2) = 1+dx`. With these two ghost nodes we implement the no flus condition as follows.

$$u'(-1,t) = 0 \;\Rightarrow\; \frac{u_3^t - u_1^t}{2dx} = 0 \;\Rightarrow\; u_1^t = u_3^t$$

$$u'(1,t) = 0 \;\Rightarrow\; \frac{u_{Nx+2}^t - u_{Nx}^t}{2dx} = 0 \;\Rightarrow\; u_{Nx+2}^t = u_{Nx}^t$$

At time 0, we use the initial condition and assign values at the N_x uniform nodes on the interval $[-1,1]$, as usual. We also assign values at two ghost nodes not using the initial condition but using the above two equations that are interpretations of no flux condition at both ends.

$$\underbrace{\begin{bmatrix} u_2^{t+1} \\ u_3^{t+1} \\ \vdots \\ u_{N_x-2}^{t+1} \\ u_{N_x+1}^{t+1} \end{bmatrix}}_{\text{Unknown}} = \underbrace{\begin{bmatrix} \rho & 1-2\rho & \rho & & & & \\ & \rho & 1-2\rho & \rho & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \rho & 1-2\rho & \rho & \\ & & & & \rho & 1-2\rho & \rho \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} u_1^t \\ u_2^t \\ u_3^t \\ \vdots \\ u_{N_x}^t \\ u_{N_x+1}^t \\ u_{N_x+2}^t \end{bmatrix}}_{\text{Known}}$$

We get all the values at δt future except two ghost nodes. To assign the values at these two ghost nodes we again use our interpretation of the no flux condition. Of course, when we interpret our computation result, we just remove the ghosts.

```
%% 1D heat equation - explicit FDM - No flux boundary condition %%%%%%%%%%

clear all; close all; clc;

xspan = [-1,1];
Nx = 11;
x = linspace(xspan(1),xspan(end),Nx);
dx = x(2)-x(1);
x = [x(1)-dx x x(end)+dx]; % Add two ghost nodes

tspan = [0,0.1];
Nt = 1000+1;
t = linspace(tspan(1),tspan(end),Nt);
dt = t(2)-t(1);

e1 = ones(Nx+2,1);
rho = dt/(dx^2);
D = [rho*e1 (1-2*rho)*e1 rho*e1];
d = [0 1 2];
B = spdiags_Lee(D,d,Nx,Nx+2);

ic = exp(-x.^2); ic(1) = ic(3); ic(end) = ic(end-2); % No flux condition
u = ic';

plot(x(2:end-1),u(2:end-1)); hold on; pause(0.1); % Plot without two ghosts

tout = t(1:100:end)';
uout = zeros(length(tout),Nx);
uout(1,:) = u(2:end-1)'; % Record without two ghosts

for j=2:Nt

    u_temp = B*u;
    u = [u_temp(2); u_temp; u_temp(end-1)];

    if mod(j,100)==1
        plot(x(2:end-1),u(2:end-1)); pause(0.1); % Plot without two ghosts
        axis([-1 1 0 1]); grid on; xlabel('x'); ylabel('Temperature')
        ind = ceil(j/100);
        uout(ind,:) = u(2:end-1)';
    end

end

%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

1D heat equation - ode45 - Pin down boundary condition

$\boxed{\text{PDE}}$ $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$\boxed{\text{Initial condition}}$ $$u(x,0) = e^{-x^2}$$

$\boxed{\text{Pin down boundary condition}}$ $$u(t,-1) = u(t,1) = 0$$

By the boundary condition $u(t,-1) = u(t,1) = 0$, we have $u_1^t = u_{Nx}^t = 0$ for all time t. So, we consider $[u_2^t, u_3^t, \cdots, u_{N_x-2}^t, u_{N_x-1}^t]$ only. Since $u_1^t = u_{Nx}^t = 0$, we have

$$\frac{d}{dt}\begin{bmatrix} u_2^t \\ u_3^t \\ \vdots \\ u_{N_x-2}^t \\ u_{N_x-1}^t \end{bmatrix} = \frac{1}{(\delta x)^2}\begin{bmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} u_1^t \\ u_2^t \\ u_3^t \\ \vdots \\ u_{N_x-2}^t \\ u_{N_x-1}^t \\ u_{N_x}^t \end{bmatrix}}_{\text{Known}}$$

$$= \frac{1}{(\delta x)^2} \underbrace{\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} u_2^t \\ u_3^t \\ \vdots \\ u_{N_x-2}^t \\ u_{N_x-1}^t \end{bmatrix}$$

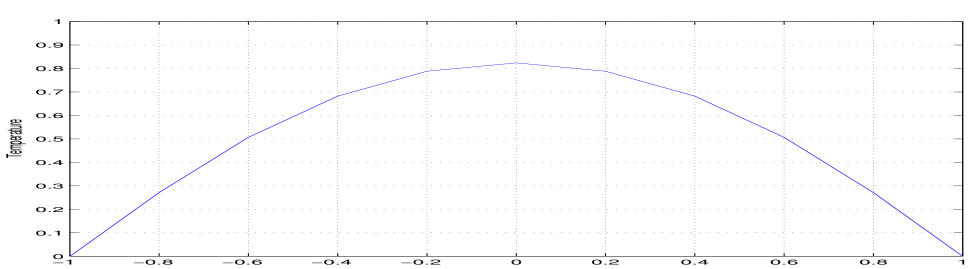


Figure 4: 1D heat equation - ode45 - Pin down boundary condition: Solution at $t = 0.1$.

```
%% 1D heat equation - ode45 - Pin down boundary condition %%%%%%%%%%%%%%%

clear all; close all; clc;

xspan = [-1,1];
Nx = 11;
dx = (xspan(end)-xspan(1))/(Nx-1);
x2 = linspace(xspan(1),xspan(end),Nx);
x = x2(2:end-1);

e1 = ones(Nx-2,1);
D = [-2*e1 e1 e1];
d = [    0 -1  1];
A = spdiags_Lee(D,d,Nx-2,Nx-2);

ode = @(t,u) (1/dx.^2)*A*u;
tspan = 0:0.01:0.1;
ic = exp(-x.^2);

[t u] = ode45(ode,tspan,ic);

e0 = zeros(length(t),1);
x = [-1 x  1];
u = [e0 u e0];

for j=1:length(t)

    plot(x,u(j,:)); grid on; xlabel('x'); ylabel('Temperature')
    axis([-1 1 0 1]);
    pause(0.1)

end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

1D heat equation - ode45 - Periodic boundary condition

PDE	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$
Initial condition	$u(x,0) = e^{-x^2}$
Periodic boundary condition	$u(t,-1) = u(t,1)$

To reflect the boundary condition $u(t,-1) = u(t,1)$, we first put $Nx+1$ uniform grids on $[-1,1]$ and by identifying the first and last grids in reality we work with Nx grids. Since $u_1^t = u_{Nx}^t$, we have

$$\frac{d}{dt}\begin{bmatrix} u_1^t \\ u_2^t \\ \vdots \\ u_{N_x-2}^t \\ u_{N_x}^t \end{bmatrix} = \frac{1}{(\delta x)^2} \underbrace{\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} u_1^{t+1} \\ u_2^{t+1} \\ \vdots \\ u_{N_x-2}^{t+1} \\ u_{N_x}^{t+1} \end{bmatrix}$$

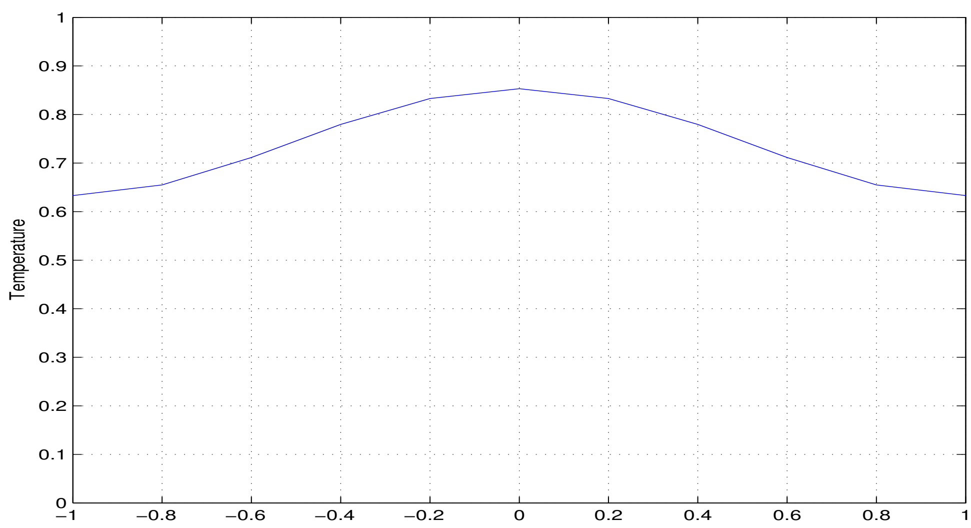


Figure 5: 1D heat equation - ode45 - Periodic boundary condition: Solution at $t = 0.1$.

1D heat equation - fft - Periodic boundary condition

PDE
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Initial condition
$$u(x,0) = e^{-x^2}$$

Periodic boundary condition
$$u(t,-1) = u(t,1)$$

```
%% 1D heat equation - fft - Periodic boundary %%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all; close all; clc;

xspan = [-1,1];
n = 4;
Nx = 2^n;
x2 = linspace(xspan(1),xspan(end),Nx+1);
x = x2(1:end-1);
dx = x(2)-x(1);

om = [0:0.5*Nx-1 -0.5*Nx:-1]/2;
k = 2*pi*om;

ode = @(t,U) -(k.^2)'.*U;
tspan = 0:0.01:0.1;
ic = (fft(exp(-x.^2)))';

[t U] = ode45(ode,tspan,ic);

x = [x  1];
u = zeros(length(t),Nx+1);

for j=1:length(t)
    u_temp = ifft(U(j,:));
    u(j,:) = [u_temp u_temp(1)];
    plot(x,u(j,:)); grid on; xlabel('x'); ylabel('Temperature')
    axis([-1 1 0 1]);
    pause(0.1)
end
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

1D heat equation - implicit FDM - Pin down boundary condition

PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Initial condition

$$u(x,0) = e^{-x^2}$$

Pin down boundary condition

$$u(t,-1) = u(t,1) = 0$$

Implicit FDM

Choose δx and δt so that discretization errors are same order: $(\delta x)^2 = C(\delta t)$. Stability is guaranteed for any $\rho := \frac{\delta t}{(\delta x)^2}$.

$$\frac{\partial u}{\partial t} \rightarrow \frac{u_i^{t+1} - u_i^t}{\delta t} + O(\delta t)$$

$$\frac{\partial^2 u}{\partial x^2} \rightarrow \frac{u_{i+1}^{t+1} - 2u_i^{t+1} + u_{i-1}^{t+1}}{(\delta x)^2} + O((\delta x)^2)$$

$$\frac{u_i^{t+1} - u_i^t}{\delta t} = \frac{u_{i+1}^{t+1} - 2u_i^{t+1} + u_{i-1}^{t+1}}{(\delta x)^2}$$

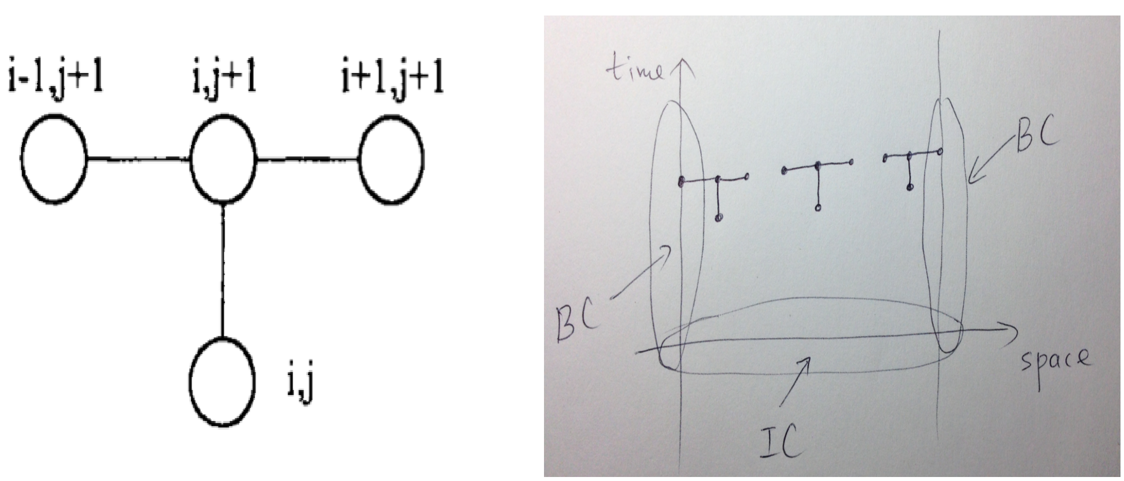


Figure 6: Implicit stenci positioning.

$$\begin{bmatrix} u_2^t \\ u_3^t \\ \vdots \\ u_{Nx-2}^t \\ u_{Nx-1}^t \end{bmatrix} = \begin{bmatrix} -\rho & 1+2\rho & -\rho & & & \\ & -\rho & 1+2\rho & -\rho & & \\ & & \ddots & \ddots & \ddots & \\ & & -\rho & 1+2\rho & -\rho & \\ & & & -\rho & 1+2\rho & -\rho \end{bmatrix} \begin{bmatrix} u_1^{t+1} \\ u_2^{t+1} \\ u_3^{t+1} \\ \vdots \\ u_{Nx-2}^{t+1} \\ u_{Nx-1}^{t+1} \\ u_{Nx}^{t+1} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} u_2^t \\ u_3^t \\ \vdots \\ u_{Nx-2}^t \\ u_{Nx-1}^t \end{bmatrix} - \begin{bmatrix} \rho u_1^{t+1} \\ 0 \\ \vdots \\ 0 \\ \rho u_{Nx}^{t+1} \end{bmatrix}}_{\text{Known}} = \begin{bmatrix} 1+2\rho & -\rho & & & \\ -\rho & 1+2\rho & -\rho & & \\ & \ddots & \ddots & \ddots & \\ & & -\rho & 1+2\rho & -\rho \\ & & & -\rho & 1+2\rho \end{bmatrix} \underbrace{\begin{bmatrix} u_2^{t+1} \\ u_3^{t+1} \\ \vdots \\ u_{Nx-2}^{t+1} \\ u_{Nx-1}^{t+1} \end{bmatrix}}_{\text{Unknown}}$$

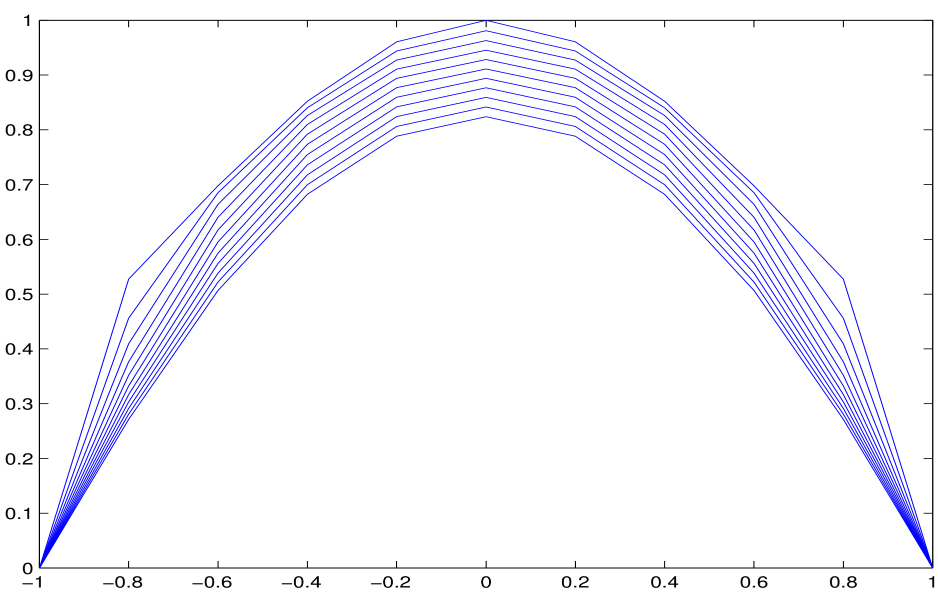


Figure 7: 1D heat equation - implicit FDM - Pin down boundary condition.

```
%% 1D heat equation - implicit FDM - Pin down boundary condition %%%%%%%%%

clear all; close all; clc;

xspan = [-1,1];
Nx = 11;
dx = (xspan(end)-xspan(1))/(Nx-1);
x2 = linspace(xspan(1),xspan(end),Nx);
x = x2(2:end-1);

tspan = [0,0.1];
Nt = 1000+1;
dt = (tspan(end)-tspan(1))/(Nt-1);
t = linspace(tspan(1),tspan(end),Nt);

e1 = ones(Nx-2,1);
rho = dt/(dx^2);
D = [(1+2*rho)*e1 -rho*e1  -rho*e1]; % This line is modified
d = [          0      -1        1]; % This line is modified
A = spdiags_Lee(D,d,Nx-2,Nx-2); % This line is modified

ic = exp(-x.^2);
u = ic';

plot([-1 x 1],[0 u' 0]); hold on; pause(0.1);

tout = t(1:100:end)';
uout = zeros(length(tout),Nx);
uout(1,:) = [0 u' 0];

for j=2:Nt
    u = A\u; % This line is modified
    if mod(j,100)==1
        plot([-1 x 1],[0 u' 0]); pause(0.1);
        ind = ceil(j/100);
        uout(ind,:) = [0 u' 0];
    end
end

%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

1D heat equation - CN FDM - Pin down boundary condition

PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Initial condition

$$u(x,0) = e^{-x^2}$$

Pin down boundary condition

$$u(t,-1) = u(t,1) = 0$$

CN FDM

CN (Crank-Nicolson) discretization of u_{xx} is taking average of Explicit and Implicit discretization of u_{xx}.

$$\frac{\partial u}{\partial t} \rightarrow \frac{u_i^{t+1} - u_i^t}{\delta t} + O(\delta t)$$

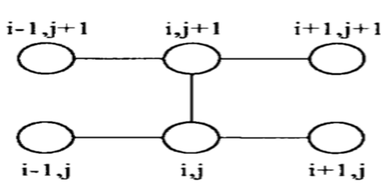
$$\frac{\partial^2 u}{\partial x^2} \rightarrow \frac{1}{2}\left[\frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{(\delta x)^2}\right] + \frac{1}{2}\left[\frac{u_{i+1}^{t+1} - 2u_i^{t+1} + u_{i-1}^{t+1}}{(\delta x)^2}\right] + O((\delta x)^2)$$

$$\frac{u_i^{t+1} - u_i^t}{\delta t} = \frac{1}{2}\left[\frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{(\delta x)^2}\right] + \frac{1}{2}\left[\frac{u_{i+1}^{t+1} - 2u_i^{t+1} + u_{i-1}^{t+1}}{(\delta x)^2}\right]$$

Generalized CN FDM

Generalized CN discretization (θ method) of u_{xx} is taking θ-average of Explicit and Implicit discretization of u_{xx}.

$$\frac{u_i^{t+1} - u_i^t}{\delta t} = \theta\left[\frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{(\delta x)^2}\right] + (1-\theta)\left[\frac{u_{i+1}^{t+1} - 2u_i^{t+1} + u_{i-1}^{t+1}}{(\delta x)^2}\right]$$



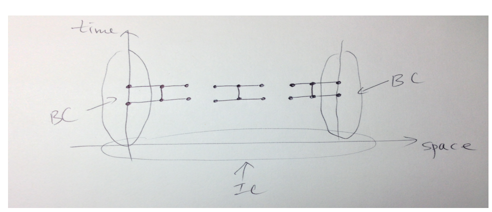


Figure 8: (generalized) Crank-Nicolson stenci positioning.

$$\underbrace{\begin{bmatrix} 1+\rho & -\frac{1}{2}\rho & & & \\ -\frac{1}{2}\rho & 1+\rho & -\frac{1}{2}\rho & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2}\rho & 1+\rho & -\frac{1}{2}\rho \\ & & & -\frac{1}{2}\rho & 1+\rho \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u_2^{t+1} \\ u_3^{t+1} \\ \vdots \\ u_{Nx-2}^{t+1} \\ u_{Nx-1}^{t+1} \end{bmatrix}}_{\text{Unknown}}$$

$$= \underbrace{\begin{bmatrix} \frac{1}{2}\rho u_1^{t+1} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}\rho u_{Nx}^{t+1} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\rho u_1^{t} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}\rho u_{Nx}^{t} \end{bmatrix} + \underbrace{\begin{bmatrix} 1-\rho & \frac{1}{2}\rho & & & \\ \frac{1}{2}\rho & 1-\rho & \frac{1}{2}\rho & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2}\rho & 1-\rho & \frac{1}{2}\rho \\ & & & \frac{1}{2}\rho & 1-\rho \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} u_2^{t} \\ u_3^{t} \\ \vdots \\ u_{Nx-2}^{t} \\ u_{Nx-1}^{t} \end{bmatrix}}_{\text{Known}}$$

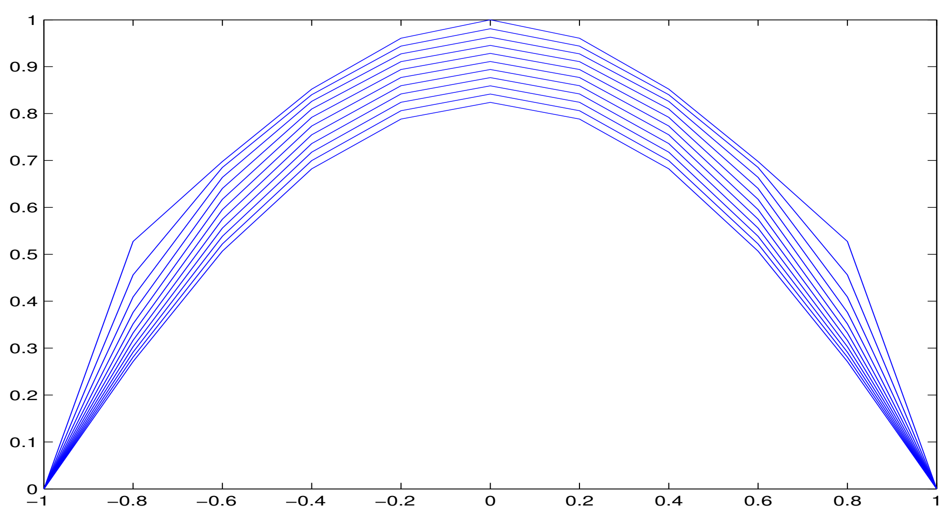


Figure 9: 1D heat equation - CN FDM - Pin down boundary condition.

```
%% 1D heat equation - CN FDM - Pin down boundary condition %%%%%%%%%%%%%%%
clear all; close all; clc;

xspan = [-1,1];
Nx = 11;
dx = (xspan(end)-xspan(1))/(Nx-1);
x2 = linspace(xspan(1),xspan(end),Nx);
x = x2(2:end-1);

tspan = [0,0.1];
Nt = 1000+1;
dt = (tspan(end)-tspan(1))/(Nt-1);
t = linspace(tspan(1),tspan(end),Nt);

e1 = ones(Nx-2,1);
rho = dt/(dx^2);
DA = [(1+rho)*e1 -0.5*rho*e1 -0.5*rho*e1]; % This line is modified
dA = [         0           -1            1]; % This line is modified
A = spdiags_Lee(DA,dA,Nx-2,Nx-2); % This line is modified
DB = [(1-rho)*e1 0.5*rho*e1 0.5*rho*e1]; % This line is modified
dB = [         0         -1           1]; % This line is modified
B = spdiags_Lee(DB,dB,Nx-2,Nx-2); % This line is modified

ic = exp(-x.^2);
u = ic';

plot([-1 x 1],[0 u' 0]); hold on; pause(0.1);

tout = t(1:100:end)';
uout = zeros(length(tout),Nx);
uout(1,:) = [0 u' 0];

for j=2:Nt

    u = A\(B*u); % This line is modified

    if mod(j,100)==1
        plot([-1 x 1],[0 u' 0]); pause(0.1);
        ind = ceil(j/100);
        uout(ind,:) = [0 u' 0];
    end

end
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```