Black-Scholes model - Continuous version

Black-Scholes equation

Multi period binomial model under risk neutral probability measure

Stock

$$
\begin{tikzcd}[row sep=small]
 & Se^{(r-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}} \\
S \arrow[ur] \arrow[dr] & \\
 & Se^{(r-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}}
\end{tikzcd}
$$

Bond

$$
\begin{tikzcd}[row sep=small]
 & e^{rdt} \\
1 \arrow[ur] \arrow[dr] & \\
 & e^{rdt}
\end{tikzcd}
$$

Option

$$
\begin{tikzcd}[row sep=small]
 & A(Se^{(r-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}}, t+dt) \\
A(S,t) \arrow[ur] \arrow[dr] & \\
 & A(Se^{(r-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}}, t+dt)
\end{tikzcd}
$$

$$
\begin{aligned}
\pi_u &\approx \frac{1}{2} + 0 \cdot \sqrt{dt} \\
\pi_d &\approx \frac{1}{2} + 0 \cdot \sqrt{dt}
\end{aligned}
$$

$$A = A(Se^{(r-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}}, t+dt) * \pi_u + A(Se^{(r-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}}, t+dt) * \pi_d$$

We expand up to the dt order. We group terms based on A, A_s, A_{ss}, and A_t: Since $\pi_u + \pi_d = e^{-rdt}$,

$$
\begin{aligned}
A\left[1 - e^{-rdt}\right] &\approx A_s s\left[\pi_u\left(e^{(r-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}} - 1\right) + \pi_d\left(e^{(r-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}} - 1\right)\right] \\
&\quad + \frac{1}{2}A_{ss}s^2\left[\pi_u\left(e^{(r-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}} - 1\right)^2 + \pi_d\left(e^{(r-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}} - 1\right)^2\right] \\
&\quad + A_t\left[e^{-rdt}dt\right]
\end{aligned}
$$

We expand terms up to dt order:

$$
\begin{aligned}
A\,[rdt] &\approx A_s s\left[\pi_u\left(rdt+\sigma\sqrt{dt}\right) + \pi_d\left(rdt-\sigma\sqrt{dt}\right)\right] \\
&\quad + \frac{1}{2}A_{ss}s^2\left[\pi_u\left(rdt+\sigma\sqrt{dt}\right)^2 + \pi_d\left(rdt-\sigma\sqrt{dt}\right)^2\right] + A_t\,[dt] \\
&\approx A_s s\,[rdt] + \frac{1}{2}A_{ss}s^2\left[\sigma^2 dt\right] + A_t\,[dt]
\end{aligned}
$$

Black-Scholes equation

$$A_t + \frac{1}{2}\sigma^2 s^2 A_{ss} + rsA_s = rA$$

From multi period binomial to Black-Scholes - Physical probability measure

Multi period binomial model under physical probability measure

Stock

$$\begin{array}{ccc} & & Se^{(\mu-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}} \\ & \nearrow & \\ S & & \\ & \searrow & \\ & & Se^{(\mu-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}} \end{array}$$

Bond

$$\begin{array}{ccc} & & e^{rdt} \\ & \nearrow & \\ 1 & & \\ & \searrow & \\ & & e^{rdt} \end{array}$$

Option

$$\begin{array}{ccc} & & A(Se^{(\mu-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}}, t+dt) \\ & \nearrow & \\ A(S,t) & & \\ & \searrow & \\ & & A(Se^{(\mu-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}}, t+dt) \end{array}$$

State price

We expand terms up to $\sqrt{dt}$ order:

$$\begin{aligned} \pi_u &\approx \frac{1}{2} - \frac{1}{2}\frac{\mu - r}{\sigma} \cdot \sqrt{dt} \\ \pi_d &\approx \frac{1}{2} + \frac{1}{2}\frac{\mu - r}{\sigma} \cdot \sqrt{dt} \end{aligned}$$

Option price

We expand terms up to dt order:

$$\begin{aligned} A &= A(Se^{(\mu-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}}, t+dt) * \pi_u + A(Se^{(\mu-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}}, t+dt) * \pi_d \\ &\approx \left[A + A_s s\left(e^{(\mu-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}} - 1\right) + \frac{1}{2}A_{ss}s^2\left(e^{(\mu-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}} - 1\right)^2 + A_t dt\right] * \pi_u \\ &\quad + \left[A + A_s s\left(e^{(\mu-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}} - 1\right) + \frac{1}{2}A_{ss}s^2\left(e^{(\mu-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}} - 1\right)^2 + A_t dt\right] * \pi_d \end{aligned}$$

Black-Scholes PDE

We expand terms up to dt order:

$$A_t + \frac{1}{2}\sigma^2 s^2 A_{ss} + rsA_s = rA$$

We group terms based on A, A_s, A_{ss}, and A_t: Since $\pi_u + \pi_d = e^{-rdt}$,

$$\begin{aligned} A\left[1-e^{-rdt}\right] &\approx A_s s\left[\pi_u\left(e^{(\mu-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}}-1\right)+\pi_d\left(e^{(\mu-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}}-1\right)\right] \\ &\quad +\frac{1}{2}A_{ss}s^2\left[\pi_u\left(e^{(\mu-\frac{1}{2}\sigma^2)dt+\sigma\sqrt{dt}}-1\right)^2+\pi_d\left(e^{(\mu-\frac{1}{2}\sigma^2)dt-\sigma\sqrt{dt}}-1\right)^2\right] \\ &\quad +A_t\left[e^{-rdt}dt\right] \end{aligned}$$

We expand terms up to dt order:

$$\begin{aligned} A\left[rdt\right] &\approx A_s s\left[\pi_u\left(\mu dt+\sigma\sqrt{dt}\right)+\pi_d\left(\mu dt-\sigma\sqrt{dt}\right)\right] \\ &\quad +\frac{1}{2}A_{ss}s^2\left[\pi_u\left(\mu dt+\sigma\sqrt{dt}\right)^2+\pi_d\left(\mu dt-\sigma\sqrt{dt}\right)^2\right]+A_t\left[dt\right] \\ &\approx A_s s\left[rdt\right]+\frac{1}{2}A_{ss}s^2\left[\sigma^2 dt\right]+A_t\left[dt\right] \end{aligned}$$

How to solve heat equation

Heat kernel

Heat equation $u_t = \frac{1}{2}\sigma^2 u_{xx}$

Initial condition $u(0,x) = \delta_{x_0}(x)$

Heat kernel $u(t,x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-x_0)^2}{2\sigma^2 t}}$

Superposition

Heat equation $u_t = \frac{1}{2}\sigma^2 u_{xx}$

Initial condition $u(0,x) = f(x)$

Superposition $u(t,x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-x_0)^2}{2\sigma^2 t}} f(x_0)\,dx_0$

How to solve Black-Scholes equation

BS equation

$$\text{BS PDE} \qquad V_t + \frac{1}{2}\sigma^2 s^2 V_{ss} + rsV_s = rV$$

$$\text{Terminal condition} \qquad V(S', T) = g(S')$$

Change of variable

Time t $\rightarrow$ Time to maturity $\tau = T - t$

Stock price s $\rightarrow$ Log stock price $x = \log s$

$\rightarrow$ Appreciated log stock price $y = x + \left(r - \frac{1}{2}\sigma^2\right)\tau$

Option price V $\rightarrow$ Appreciated option price $U = Ve^{r\tau}$

Heat equation

$$\text{Heat equation} \qquad U_\tau = \frac{1}{2}\sigma^2 U_{yy}$$

$$\text{Initial condition} \qquad U(0, y) = g(e^y)$$

Solution

$$V = \underbrace{e^{-rT}}_{\text{Discount}} * \int_0^\infty \underbrace{\frac{1}{S' * \sqrt{2\pi} * \sigma\sqrt{T}} e^{-\frac{1}{2}\left(\frac{\log(S'/S) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2}}_{\text{Risk-neutral prob of jumping from } S \text{ to } S'} \underbrace{g(S')}_{\text{Payoff}} \, dS'$$

Black-Scholes formula

Black-Scholes formula

$$\begin{array}{ll} \text{Call} & SN(d_1) - Ke^{-rT}N(d_2) \\ \text{Put} & -SN(-d_1) + Ke^{-rT}N(-d_2) \end{array}$$

where N is the standard normal CDF, and where d_1 and d_2 are given by

$$\begin{aligned} d_1 &= \frac{\log(S/K) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\log(S/K) + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{aligned}$$

Interpretation of Black-Scholes formula - Part 1

$$\text{Call} \qquad \underbrace{SN(d_1)}_{\text{Long stock}} \underbrace{-Ke^{-rT}N(d_2)}_{\text{Short bond}}$$

$$\text{Put} \qquad \underbrace{-SN(-d_1)}_{\text{Short stock}} \underbrace{+Ke^{-rT}N(-d_2)}_{\text{Long bond}}$$

Interpretation of Black-Scholes formula - Part 2

$$\text{Call} \qquad SN(d_1) - Ke^{-rT} \qquad \underset{\substack{\uparrow \\ \text{Risk-neutral probability that call option is exercised}}}{N(d_2)}$$

$$\text{Put} \qquad -SN(-d_1) + Ke^{-rT} \qquad \underset{\substack{\uparrow \\ \text{Risk-neutral probability that put option is exercised}}}{N(-d_2)}$$

Black-Scholes formula for digital call and put

$$\begin{array}{ll} \text{Digital call} & e^{-rT}N(d_2) \\ \text{Digital put} & e^{-rT}N(-d_2) \end{array}$$

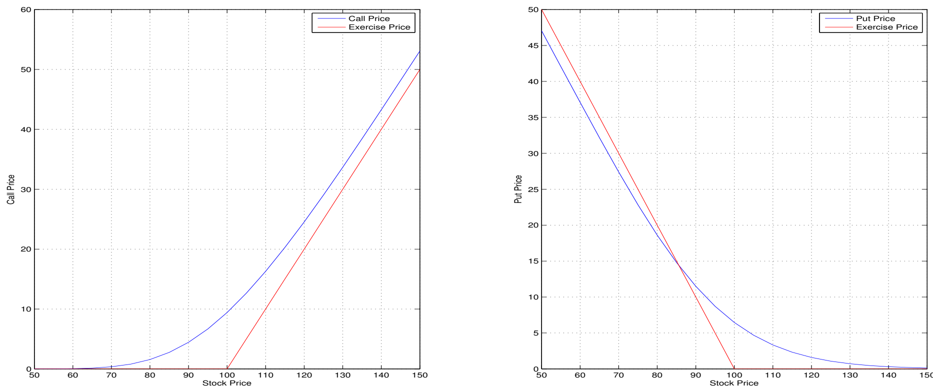


Figure 1: Call (left) and Put (right) option price calculated by Black-Scholes formula; $K = 100$, $T = 1$, $r = 0.03$, $\sigma = 0.2$. Note that the call price is always bigger than the immediate exercise price whereas the put price is sometimes smaller than the immediate exercise price.

```
S = 50:5:150; K = 100; T = 1; r = 0.03; v = 0.2;

subplot(1,2,1)
C = Call(S,K,T,r,v); plot(S,C); grid on; hold on;
S1 = sort([S K]); C1 = max(S1-K,0); plot(S1,C1,'-r');
xlabel('Stock'); ylabel('Call'); legend('Call','Exercise')

subplot(1,2,2)
P = Put(S,K,T,r,v); plot(S,P); grid on; hold on;
S1 = sort([S K]); P1 = max(K-S1,0); plot(S1,P1,'-r');
xlabel('Stock'); ylabel('Put'); legend('Put','Exercise Price')

%% Function needed

function C = Call(S,K,T,r,v,d)
if (nargin<=5), d = 0; end;
d1 = (log(S/K)+(r-d+0.5*v^2)*T)/(v*sqrt(T));
d2 = d1-v*sqrt(T);
C = S*normcdf(d1,0,1) - K*exp(-r*T)*normcdf(d2,0,1);
end

function P = Put(S,K,T,r,v,d)
if (nargin<=5), d = 0; end;
d1 = (log(S/K)+(r-d+0.5*v^2)*T)/(v*sqrt(T));
d2 = d1-v*sqrt(T);
P = - S*normcdf(-d1,0,1) + K*exp(-r*T)*normcdf(-d2,0,1);
end
```

Replication of call

$$\text{Call} \quad \underbrace{SN(d_1)}_{\text{Long stock}} \ \underbrace{-Ke^{-rT}N(d_2)}_{\text{Short bond}}$$

Replication of call

Day 0 $\quad \underbrace{S}_{\text{Stock price}} * \underbrace{N(d_1)}_{\text{Stock position}} + \underbrace{1}_{\text{Bond price}} * \underbrace{-Ke^{-rT} * N(d_2)}_{\text{Bond position}}$

Day i
1. Update d_1 and d_2 using new stock price and time to maturity
2. Hold $N(d_1)$ stocks and put the rest in the bonds

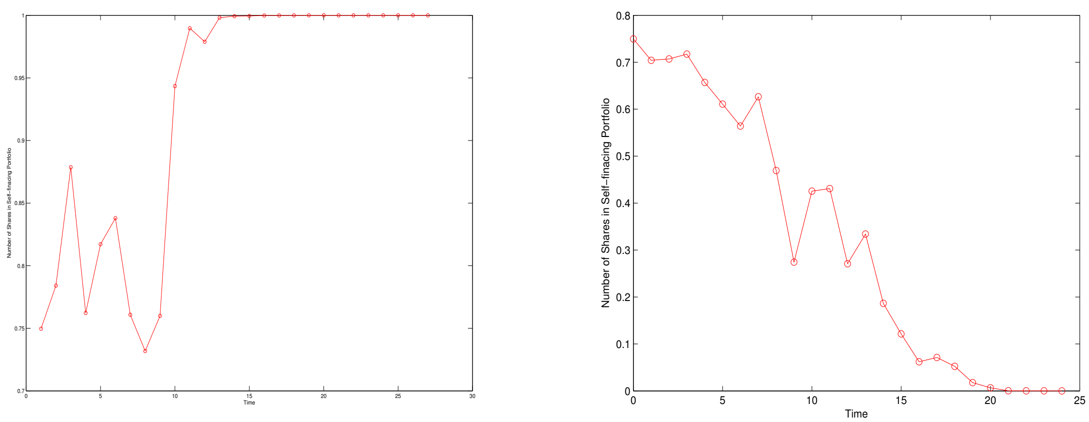


Figure 2: Stock position of self-financing portfolio of replicating CALL45 using default (left) and seed 16 (right).

```
clear all; close all; clc; rng(16); % rng('default');

S=50; K=45; T=0.5; r=0.03; v=0.30;

n=24; % Weekly rebalance
dt=T/n;
tau=T-0*dt; % Time to maturity

% Initialization
d1=zeros(1,n+1);
d2=zeros(1,n+1);
N_Stock=zeros(1,n+1);
```

```
N_Bond=zeros(1,n+1);
STOCK=zeros(1,n+1);
BOND=zeros(1,n+1);
TOTAL=STOCK+BOND;
STOCK_at_END=zeros(1,n+1);
BOND_at_END=zeros(1,n+1);
TOTAL_at_END=STOCK_at_END+BOND_at_END;

% Week 0
d1(1)=(log(S/K)+(r+0.5*v^2)*tau)/(v*sqrt(tau));
d2(1)=d1(1)-v*sqrt(tau);
N_Stock(1)=normcdf(d1(1),0,1);
N_Bond(1)=-K*exp(-r*tau)*normcdf(d2(1),0,1);
STOCK(1)=N_Stock(1)*S;
BOND(1)=N_Bond(1)*1;
TOTAL(1)=STOCK(1)+BOND(1);

for i=1:n
    tau=T-i*dt; % Time to maturity
    S=S*exp((r-0.5*v^2)*dt+v*sqrt(dt)*randn(1,1));
    STOCK_at_END(i)=N_Stock(i)*S;
    BOND_at_END(i)=N_Bond(i)*exp(r*dt);
    TOTAL_at_END(i)=STOCK_at_END(i)+BOND_at_END(i);
    TOTAL(i+1)=TOTAL_at_END(i);
    d1(i+1)=(log(S/K)+(r+0.5*v^2)*tau)/(v*sqrt(tau));
    d2(i+1)=d1(i+1)-v*sqrt(tau);
    N_Stock(i+1)=normcdf(d1(i+1));
    STOCK(i+1)=N_Stock(i+1)*S;
    N_Bond(i+1)=TOTAL(i+1)-STOCK(i+1);
    BOND(i+1)=N_Bond(i+1);
end

S=S*exp((r-0.5*v^2)*dt+v*sqrt(dt)*randn(1,1));
STOCK_at_END(n+1)=N_Stock(n+1)*S;
BOND_at_END(n+1)=N_Bond(n+1)*exp(r*dt);
TOTAL_at_END(n+1)=STOCK_at_END(n+1)+BOND_at_END(n+1);

plot((0:n),N_Stock,'o-r')
xlabel('Time')
ylabel('Number of Shares in Self-finacing Portfolio')

DATA = [(0:n)' TOTAL' STOCK' BOND' TOTAL_at_END' STOCK_at_END' BOND_at_END']';
t = 'WEEK    TOTAL    STOCK      BOND    T_at_END S_at_END  B_at_END\n';
fprintf(t)
fprintf('%3d   %7.2f   %6.2f   %7.2f     %7.2f   %6.2f   %7.2f  \n',DATA)
```

Replication of put

$$\text{Put} \quad \underbrace{-SN(-d_1)}_{\text{Short stock}} \underbrace{+Ke^{-rT}N(-d_2)}_{\text{Long bond}}$$

$\boxed{\text{Replication of put}}$

Day 0 $\quad \underbrace{S}_{\text{Stock price}} * \underbrace{-N(-d_1)}_{\text{Stock position}} + \underbrace{1}_{\text{Bond price}} * \underbrace{Ke^{-rT} * N(-d_2)}_{\text{Bond position}}$

Day i

1. Update d_1 and d_2 using new stock price and time to maturity
2. Hold $-N(-d_1)$ stocks and put the rest in the bonds

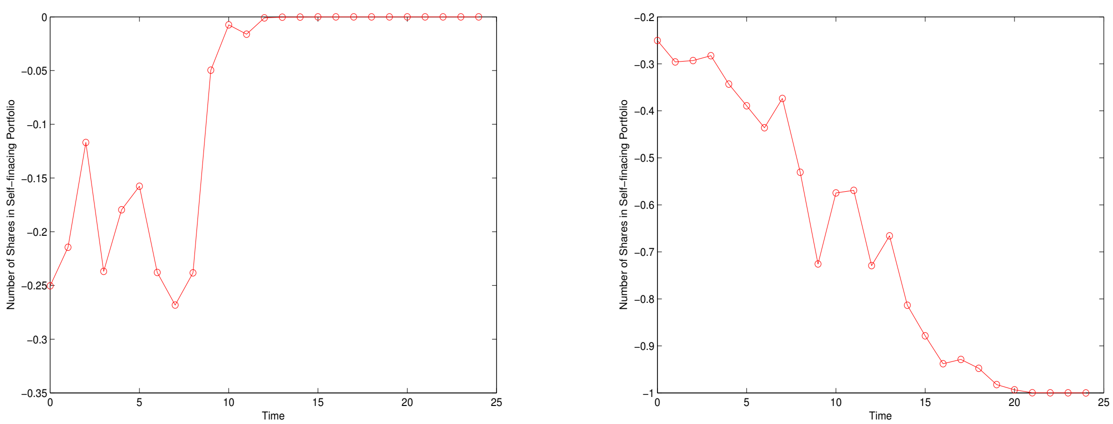


Figure 3: Stock position of self-financing portfolio of replicating PUT45 using default (left) and seed 16 (right).

Replication of protective put

You purchased a stock for \$50 that is now worth \$70
You don't want to sell the stock yet
But you wants to make sure you don't lose the \$20 unrealized gains
Buy a put option for that same stock. This is protective put!

Protective put

$$\text{Protective Put} \quad = \quad \texttt{Stock 1} \quad + \quad \text{Put 1}$$

Protective put at maturity

$$S_T + (K - S_T)^+ = \begin{cases} S_T + 0 = S_T & \text{if } S_T \geq K \\ S_T + (K - S_T) = K & \text{if } S_T \leq K \end{cases}$$

Protective put now

$$\begin{aligned} \texttt{Protective\_Put} &= S + \left[-SN(-d_1) + Ke^{-rT}N(-d_2)\right] \\ &= S(1 - N(-d_1)) + Ke^{-rT}N(-d_2) \\ &= SN(d_1) + Ke^{-rT}N(-d_2) \\ &= \underbrace{S * N(d_1)}_{\text{Long Stock}} + \underbrace{Ke^{-rT}N(-d_2)}_{\text{Long Bond}} \end{aligned}$$

Replication of protective put

Day 0 $\quad \underbrace{S}_{\text{Stock Price}} * \underbrace{N(d_1)}_{\text{Stock Position}} + \underbrace{1}_{\text{Bond Price}} * \underbrace{Ke^{-rT} * N(-d_2)}_{\text{Bond Position}}$

Day i
1. Update d_1 and d_2 using new stock price and time to maturity
2. Hold $N(d_1)$ stocks and put the rest in the bonds

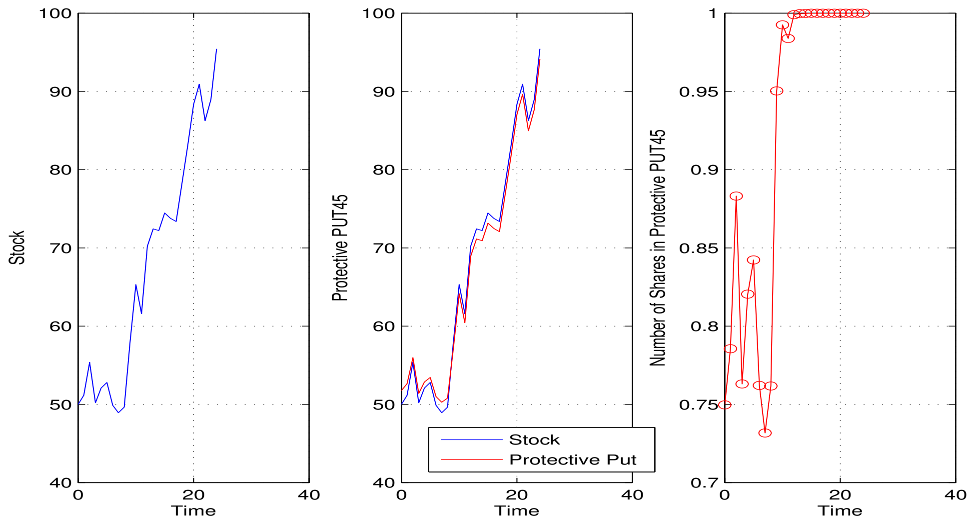


Figure 4: Protective Put45. Stock price move (left), Stock price move and Protective PUT45 move in red (center), and the number of shares in Protective PUT45. When the stock price hits above strike 45, the protective put follows the stock price closely and the protective put becomes the stock at the end.

Replication of covered call

You pay a stock for \$50 and think that it will rise to \$60 within one year
You'd be willing to sell at \$55 within six months, a nice short-term profit
Then, selling a covered call might be an attractive option for you
Sell a call option for that same stock. This is covered call!

Covered call

$$\text{Covered Call} \quad = \quad \texttt{Stock 1} \quad + \quad \text{Call } -1$$

Covered call at maturity

$$S_T - (S_T - K)^+ = \begin{cases} S_T - (S_T - K) = K & \text{if } S_T \geq K \\ S_T + 0 = S_T & \text{if } S_T \leq K \end{cases}$$

Covered call now

$$\begin{aligned} \texttt{Covered\_Call} &= S - \left[SN(d_1) - Ke^{-rT}N(d_2)\right] \\ &= S(1 - N(d_1)) + Ke^{-rT}N(d_2) \\ &= SN(-d_1) + Ke^{-rT}N(d_2) \\ &= \underbrace{S * N(-d_1)}_{\text{Long Stock}} + \underbrace{Ke^{-rT}N(d_2)}_{\text{Long Bond}} \end{aligned}$$

Replication of covered call

Day 0 $\quad \underbrace{S}_{\text{Stock Price}} * \underbrace{N(-d_1)}_{\text{Stock Position}} + \underbrace{1}_{\text{Bond Price}} * \underbrace{Ke^{-rT} * N(d_2)}_{\text{Bond Position}}$

Day i

1. Update d_1 and d_2 using new stock price and time to maturity
2. Hold $N(-d_1)$ stocks and put the rest in the bonds

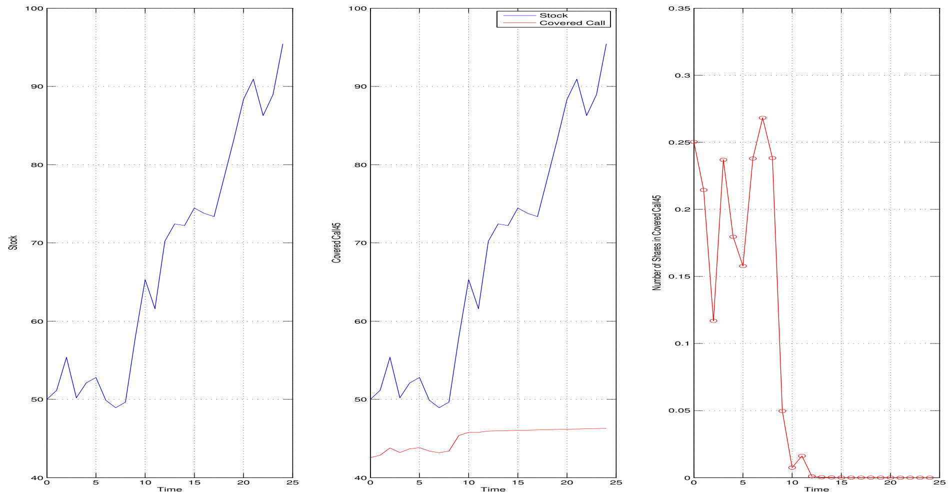


Figure 5: Covered Call45. Stock price move (left), Stock price move and Covered Call45 move in red (center), and the number of shares in Covered Call45. When the stock price goes up, the covered call profit becomes stable and the covered call becomes the bond at the end.

Superposition principle

$\boxed{\text{Option at maturity}}$

$$\sum_{i=1}^{n} \left[\alpha_i (S' - K_i)^+ + \beta_i (K_i - S')^+ \right]$$

$\boxed{\text{Option now}}$

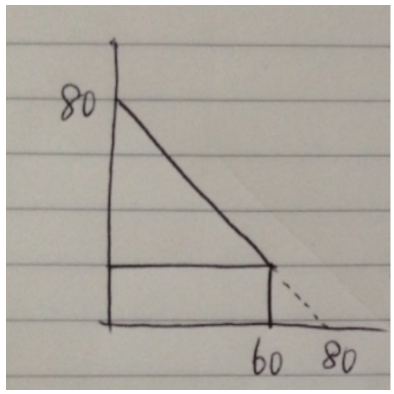
$$\sum_{i=1}^{n} \left[\alpha_i C_i + \beta_i P_i \right]$$

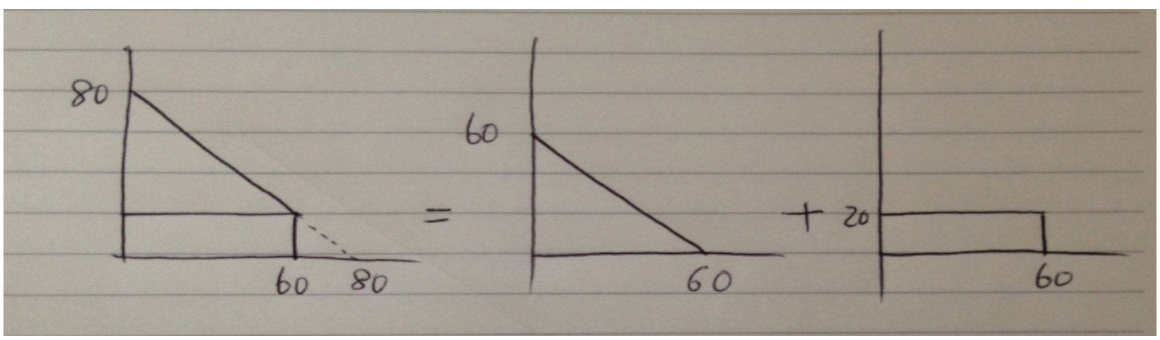
where

C_i	Call price with strike K_i
P_i	Put price with strike K_i

Example - Superposition principle

Compute the price V of the option whose payoff at maturity is given by the below figure, where $S = 100$, $K = 100$, $T = 1$, $r = 0.03$, $\sigma = 0.2$.





$$\text{Option} = \texttt{PUT60} + 20 * \texttt{DIGITAL\_PUT60}$$

```
S = 100; K = 100; T = 1; r = 0.03; v = 0.2;

Option_Value = Put(S,K,T,r,v) + 20 * Digital_Put(S,K,T,r,v)

%% Output

Option_Value =

   15.7754
```

Put-call parity

Put-call parity

Put-call parity	$C - P = S - Ke^{-rT}$
Put-call parity for options on dividend paying stock	$C - P = S - D - Ke^{-rT}$

Interpretation of put-call parity

$$\underbrace{C}_{\text{Call}} - \underbrace{P}_{\text{Put}} = \underbrace{S}_{\text{Stock}} - \underbrace{Ke^{-rT}}_{\text{Bond}}$$

If we know three prices among four, then these three determine the other price.

Proof of put-call parity

Call 1 + Put -1 at maturity	$(S' - K)^+ - (K - S')^+ = S' - K$
Call 1 + Put -1 now	$C - P = S - Ke^{-rT}$

where

S	Stock price now
S'	Stock price at maturity
C	Call price with strike K, maturity T
P	Put price with strike K, maturity T

Proof of put-call parity for options on dividend paying stock

Call 1 + Put -1 at maturity	$(S' - K)^+ - (K - S')^+ = S' - K$
Call 1 + Put -1 now	$C - P = S - D - Ke^{-rT}$

where

D	Present value of all dividends until maturity

Don't exercise American call early, if underlying stock pay no dividend

Lower bounds of American call C_A

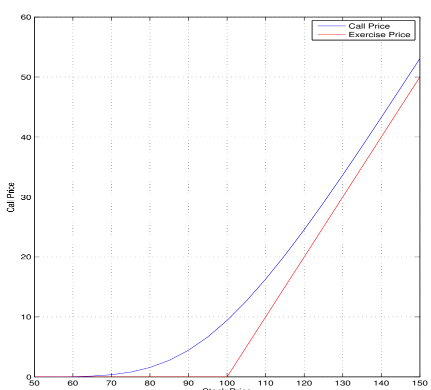
$$\begin{array}{lrcl} \text{From early exercise} & C_A & \geq & (S-K)^+ \\ \text{From put-call parity (Merton)} & C_A & \geq & (S-Ke^{-rT})^+ \end{array}$$

$C_A \geq S - Ke^{-rT} > S - K$ if $S - K > 0$ $\Rightarrow$ Don't exercise American call early

Lower bounds of American put P_A

$$\begin{array}{ll} \text{From early exercise} & P_A \geq -S+K \\ \text{From put-call parity} & P_A \geq P = C - S + Ke^{-rT} \geq -S + Ke^{-rT} \end{array}$$

$$P_A \geq -S + K$$



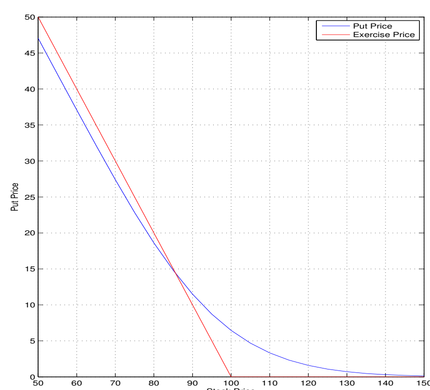


Figure 6: Call (left) and Put (right) option price calculated by Black-Scholes formula; $K = 100$, $T = 1$, $r = 0.03$, $\sigma = 0.2$. Note that the call price is always bigger than the immediate exercise price, that mean, it is better to sell than exercise the call even if the immediate exercise is allowed. However, the put price is sometimes below the immediate exercise price, that mean, it is better to exercise than sell the put if the immediate exercise is allowed.