Brownian motion

Simulation of Brownian motion

Using a standard normal coin

Number of ticks per year	n
Standard normal coin flip at tick k	X_k
Number of ticks between 0 and t	nt
Cumulative standard normal coin flips up to time t	$\sum_{k=1}^{nt} X_k$
Normalized cumulative standard normal coin flips up to time t	$B_t = \frac{\sum_{k=1}^{nt} X_k}{\sqrt{n}}$

Using a fair coin

Number of ticks per year	n
Fair coin flip at tick k with H as 1 and T as -1	X_k
Number of ticks between 0 and t	nt
Cumulative fair coin flips up to time t	$\sum_{k=1}^{nt} X_k$
Normalized cumulative fair coin flips up to time t	$B_t = \frac{\sum_{k=1}^{nt} X_k}{\sqrt{n}}$

Using an arbitrary coin

Number of ticks per year	n
IID coin flip at tick k	$\frac{X_k-\mu}{\sigma}$
Number of ticks between 0 and t	nt
Cumulative IID coin flips up to time t	$\sum_{k=1}^{nt} \frac{X_k-\mu}{\sigma}$
Normalized cumulative IID coin flips up to time t	$B_t = \frac{\sum_{k=1}^{nt} \frac{X_k-\mu}{\sigma}}{\sqrt{n}}$

where

$$\mathbb{E}X_k = \mu, \quad Var(X_k) = \sigma^2$$

Example - Brownian motion sample path

We flips a fair coin 10 times and we get

$$HHTHTTHHHT$$

Using this construct a Brownian motion sample path up to 1 year.

Time	0/10	1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10	10/10
Coin flip	–	H	H	T	H	T	T	H	H	H	T
Conversion	–	1	1	-1	1	-1	-1	1	1	1	-1
Cum sum	0	1	2	1	2	1	0	1	2	3	2
B_t	0	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{0}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$	$\frac{2}{\sqrt{10}}$

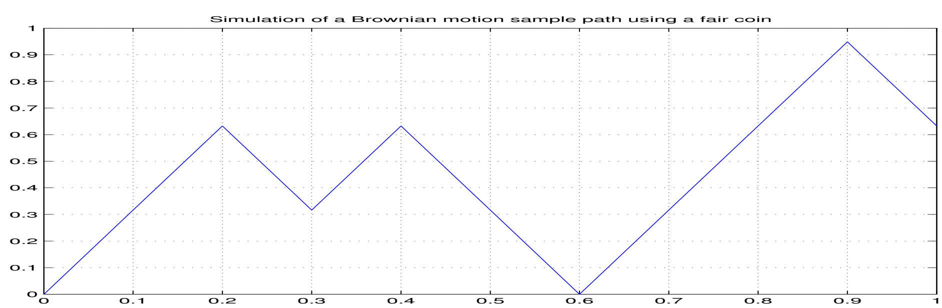

Figure 1: Simulation of a Brownian motion sample path using a fair coin.

```
clear all; close all; clc; rng('default');

M=1; % Number of simulation
n=10; % Number of days per year
T=1; % Number of years in simulation

increment=[1 1 -1 1 -1 -1 1 1 1 -1]';
BM=cumsum(increment)/sqrt(10);
BM=[zeros(1,M); BM];

plot(0:1/n:T,BM); grid on
title('Simulation of a Brownian motion sample path using a fair coin')
```

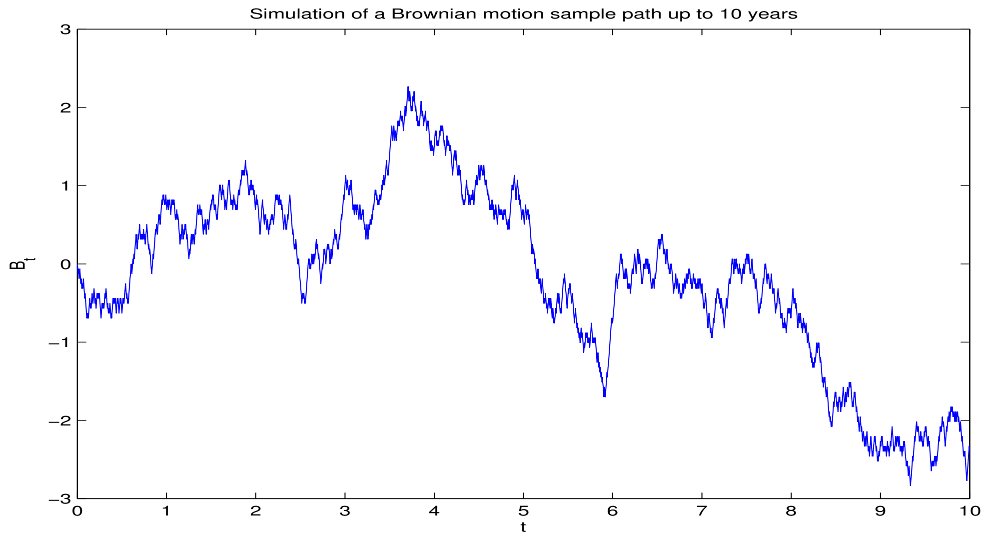

Figure 2: Simulation of a Brownian motion sample path using a fair coin up to 10 years.

```
clear all; close all; clc; rng('default');

M=1; % Number of simulation
n=252; % Number of days per year
T=10; % Number of years in simulation

% Choice of coin
coin=2;
switch coin
    case 1; c=randn(n*T,M); % normal coin
    case 2; c=2*random('bino',ones(n*T,M),0.5*ones(n*T,M))-1; % fair coin
    case 3; c=(random('bino',ones(n*T,M),.4*ones(n*T,M))-0.4)/sqrt(.4*.6); % unfair coin
end

increment=c/sqrt(n);
BM=cumsum(increment); BM=[zeros(1,M); BM];

% Plot of a Brownian motion sample path up to 10 years
plot(0:1/n:T,BM);
xlabel('t'); ylabel('B_t')
title('Simulation of a Brownian motion sample path up to 10 years')
```

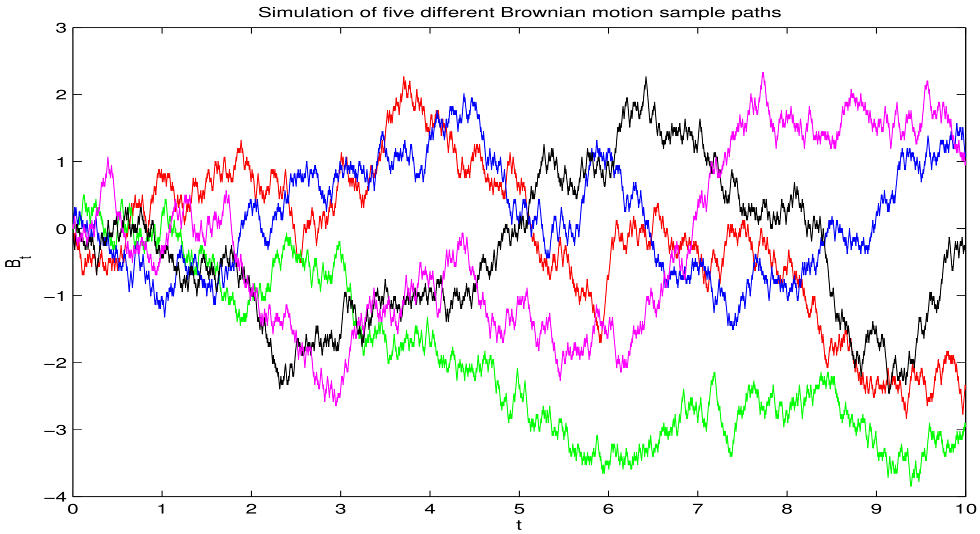

Figure 3: Simulation of five different Brownian motion sample paths.

```
clear all; close all; clc; rng('default');

M=5; % Number of simulation
n=252; % Number of days per year
T=10; % Number of years in simulation

% Choice of coin
coin=2;
switch coin
    case 1; c=randn(n*T,M); % normal coin
    case 2; c=2*random('bino',ones(n*T,M),0.5*ones(n*T,M))-1; % fair coin
    case 3; c=(random('bino',ones(n*T,M),.4*ones(n*T,M))-0.4)/sqrt(.4*.6); % unfair coin
end

increment=c/sqrt(n);
BM=cumsum(increment); BM=[zeros(1,M); BM];

% Plot of five different Brownian motion sample paths
color ='rgbkm';
for i=1:5
    t_temp=0:1/n:T;
    B_temp=BM(:,i)';
plot(t_temp,B_temp,color(i)); hold on
end
xlabel('t'); ylabel('B_t')
title('Simulation of five different Brownian motion sample paths')
```

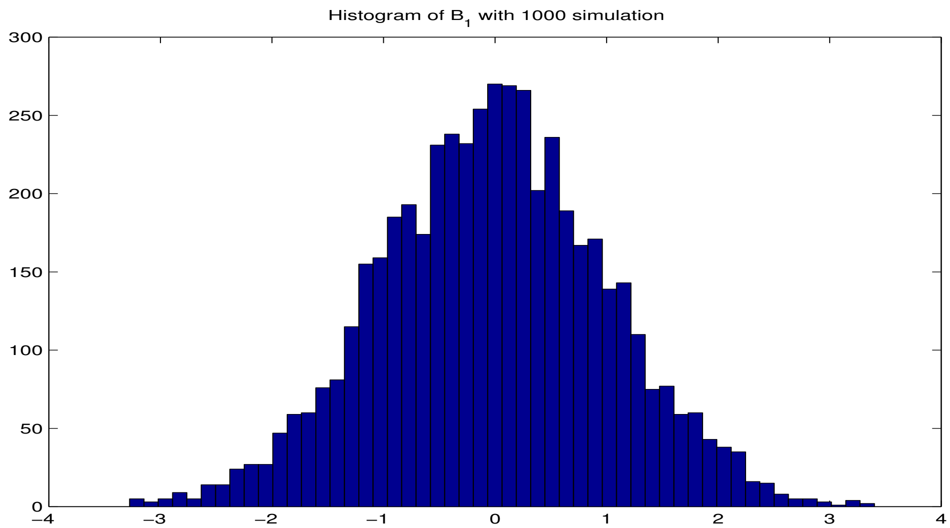

Figure 4: Histogram of B_1 with 5000 simulation.

```
clear all; close all; clc; rng('default');

M=5000; % Number of simulation
n=252; % Number of days per year
T=1; % Number of years in simulation

% Choice of coin
coin=2;
switch coin
    case 1; c=randn(n*T,M); % normal coin
    case 2; c=2*random('bino',ones(n*T,M),0.5*ones(n*T,M))-1; % fair coin
    case 3; c=(random('bino',ones(n*T,M),.4*ones(n*T,M))-0.4)/sqrt(.4*.6); % unfair coin
end

increment=c/sqrt(n);
BM=cumsum(increment); BM=[zeros(1,M); BM];

% Histogram of B_1 with 1000 simulation
Brownian_Motion_at_1=BM(end,:);
hist(Brownian_Motion_at_1,52);
title('Histogram of B_1 with 1000 simulation')
```

Definition of Brownian motion B_t

B_t starts from origin

At time 0 we don't flip any coin. So,

$$B_0 = 0$$

B_t has independent increments

For any $t_0 < t_1 < t_2 < \cdots < t_m$ the coin flips in one time interval $[t_i, t_{i-1}]$ are completely different from the coin flips in other time interval $[t_j, t_{j-1}]$. So,

$$B_{t_i} - B_{s_i} \text{ are all independent}$$

$B_t - B_s$ is normal mean 0 and variance $t - s$

$$\begin{aligned}
& \frac{X_k - \mu}{\sigma} \text{ iid with mean 0, variance 1} \\
\Rightarrow\ & \sum_{k=ns+1}^{nt} \frac{X_k - \mu}{\sigma} \text{ mean 0, variance } n(t-s) \\
\Rightarrow\ & \frac{\sum_{k=ns+1}^{nt} \frac{X_k-\mu}{\sigma}}{\sqrt{n}} \text{ mean 0, variance } t-s \\
\Rightarrow\ & B_t - B_s = \frac{\sum_{k=ns+1}^{nt} \frac{X_k-\mu}{\sigma}}{\sqrt{n}} \quad \sim \quad \mathcal{N}(0, t-s) \text{ by CLT if } n \text{ goes to the infinite}
\end{aligned}$$

Definition

A collection B_t, $t \geq 0$, of random variables is a Brownian motion if

(1) B_t starts from origin

(2) B_t has independent increments

(3) $B_t - B_s$ is normal mean 0 and variance $t - s$