Numerical solution of Black-Scholes equation

1 FDM - Before change of variable

Explicit - Put

Implicit - Call

CN - Barrier put

2 Change of variable - Before and after

Change of variable - Before and after

Computation time without and with LU

3 FDM - After change of variable

Explicit - Put

`BS_Explicit`

Implicit - Call

`BS_Implicit`

`BS_Implicit_LU`

`BS_American_Implicit`

`BS_American_Implicit_LU`

CN - Barrier put

`BS_CN`

`BS_CN_LU`

Explicit - Put

Compute put using explicit FDM where $S = 50$, $K = 50$, $r = 0.1$, $T = 1$, $\sigma = 0.4$.

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} = rv$$

Terminal (or initial) condition at T	$v(S_T, T) = (K - S_T)^+$
Boundary condition at S_{min}	$v(S_{min}, t) = (K - S_{min}) * e^{-r(T-t)}$
Boundary condition at S_{max}	$v(S_{max}, t) = 0$

$\boxed{\text{Explicit}}$

```
tspan = linspace(0,T,N_T);
sspan = linspace(S_min,S_max,N_S);
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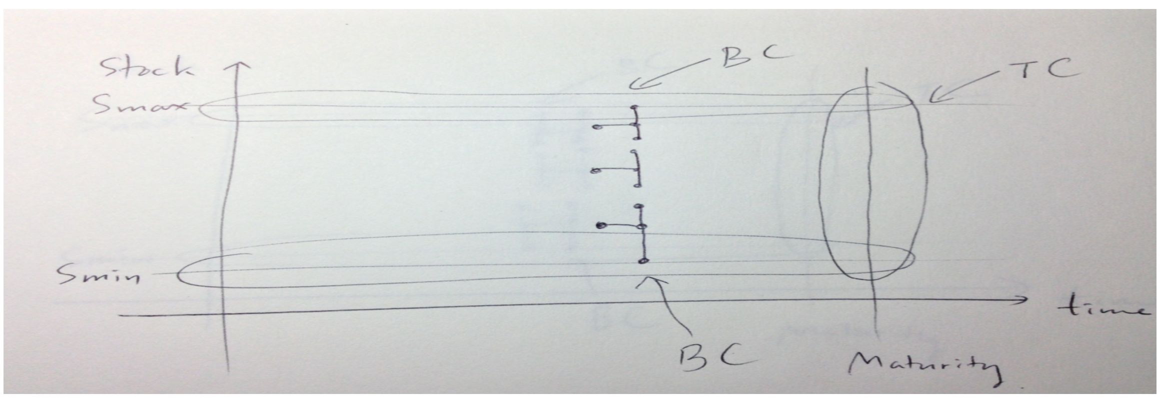
$$\delta t = \frac{T}{N_T - 1}, \qquad \delta s = \frac{S_{max} - S_{min}}{N_S - 1}$$



$$\frac{\partial v}{\partial t} \rightarrow \frac{v_i^{t+1} - v_i^t}{\delta t} + O(\delta t)$$

$$\frac{\partial^2 v}{\partial s^2} \rightarrow \frac{v_{i+1}^{t+1} - 2v_i^{t+1} + v_{i-1}^{t+1}}{(\delta s)^2} + O((\delta s)^2)$$

$$\frac{\partial v}{\partial s} \rightarrow \frac{v_{i+1}^{t+1} - v_{i-1}^{t+1}}{2\delta s} + O((\delta s)^2)$$

$$\begin{array}{lccccccccccccc}
\text{PDE} & \frac{\partial v}{\partial t} & + & \frac{1}{2}\sigma^2 & s^2 & \frac{\partial^2 v}{\partial s^2} & + & r & s & \frac{\partial v}{\partial s} & = & r & v \\
 & \uparrow & & & \uparrow & \uparrow & & & \uparrow & \uparrow & & & \uparrow \\
\text{FDM} & \frac{v_i^{t+1} - v_i^t}{\delta t} & + & \frac{1}{2}\sigma^2 & s_i^2 & \frac{v_{i+1}^{t+1} - 2v_i^{t+1} + v_{i-1}^{t+1}}{(\delta s)^2} & + & r & s_i & \frac{v_{i+1}^{t+1} - v_{i-1}^{t+1}}{2\delta s} & = & r & v_i^{t+1}
\end{array}$$

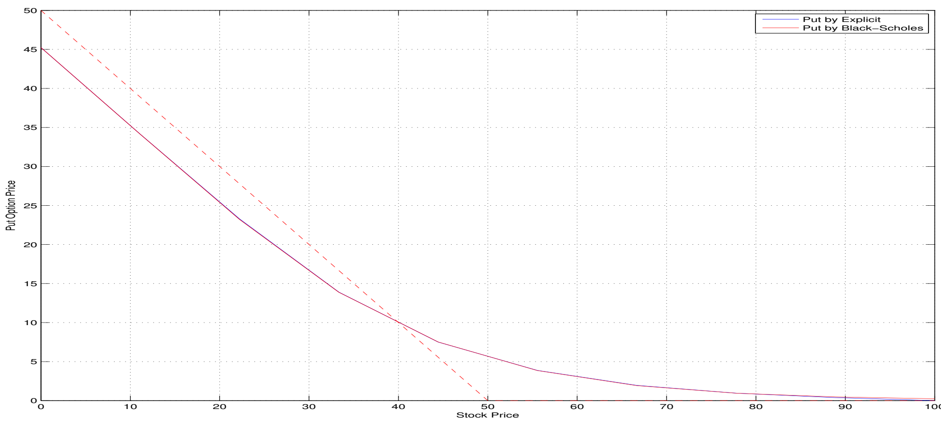
$$v_i^t = a_i v_{i-1}^{t+1} + b_i v_i^{t+1} + c_i v_{i+1}^{t+1}$$

where

$$\begin{aligned} a_i &= \frac{1}{2}\sigma^2 \delta t \left(\frac{S_{min}}{\delta s} + (i-1) \right)^2 - \frac{1}{2} r \delta t \left(\frac{S_{min}}{\delta s} + (i-1) \right) \\ b_i &= 1 - r\delta t - \sigma^2 \delta t \left(\frac{S_{min}}{\delta s} + (i-1) \right)^2 \\ c_i &= \frac{1}{2}\sigma^2 \delta t \left(\frac{S_{min}}{\delta s} + (i-1) \right)^2 + \frac{1}{2} r \delta t \left(\frac{S_{min}}{\delta s} + (i-1) \right) \end{aligned}$$

$$\begin{aligned} &\text{BC at } S_{min} \qquad v_1^t = (K - S_{min}) * e^{-r*\delta t*(N_T - t)} \\ &\text{BC at } S_{max} \qquad v_{N_S}^t = 0 \end{aligned}$$

$$\underbrace{\begin{bmatrix} v_2^t \\ v_3^t \\ \vdots \\ v_{N_S-2}^t \\ v_{N_S-1}^t \end{bmatrix}}_{\text{Unknown}} = \begin{bmatrix} a_2 & b_2 & c_2 & & & & \\ & a_3 & b_4 & c_5 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_{N_S-2} & b_{N_S-2} & c_{N_S-2} & \\ & & & & a_{N_S-1} & b_{N_S-1} & c_{N_S-1} \end{bmatrix} \underbrace{\begin{bmatrix} v_1^{t+1} \\ v_2^{t+1} \\ v_3^{t+1} \\ \vdots \\ v_{N_S-2}^{t+1} \\ v_{N_S-1}^{t+1} \\ v_{N_S}^{t+1} \end{bmatrix}}_{\text{Known}}$$




```
clear all; close all; clc;

S = 50; K = 50; T = 1; r = 0.1; v = 0.4;

Smax = 100; Smin = 0;
NS = 10; NT = 100;
dS = (Smax-Smin)/(NS-1); dT = (T-0)/(NT-1);

P = zeros(NS,NT);
Stock = linspace(Smin,Smax,NS)';
P(1,:)  = (K-Smin)*exp(-r*dT*(NT-(1:NT)));
P(NS,:) = 0;
P(:,NT) = max(K-Stock,0);

a = 0.5*v^2*dT*((Smin/dS)+(1:NS-2)).^2-0.5*r*dT*((Smin/dS)+(1:NS-2));
b = 1-r*dT-v^2*dT*((Smin/dS)+(1:NS-2)).^2;
c = 0.5*v^2*dT*((Smin/dS)+(1:NS-2)).^2+0.5*r*dT*((Smin/dS)+(1:NS-2));
i = [(1:NS-2) (1:NS-2) (1:NS-2)];
j = [(1:NS-2) (2:NS-1) (3:NS)];
s = [a b c];
A = sparse(i,j,s);

for t=NT-1:-1:1
    P(2:end-1,t) = A*P(:,t+1);
end

x = linspace(Smin,Smax,NS)'; y = interp1(Stock, P(:,1), x);
x1 = sort([x; K]); y1 = Put(x1,K,r,v,T); y2 = max(K-x1,0);
plot(x,y,x1,y1,'-r',x1,y2,'--r'); grid on;
legend('Put by Explicit','Put by Black-Scholes')
xlabel('Stock price'); ylabel('Put price')
```

Implicit - Call

Compute call using implicit FDM where $S = 50$, $K = 50$, $r = 0.1$, $T = 1$, $\sigma = 0.4$.

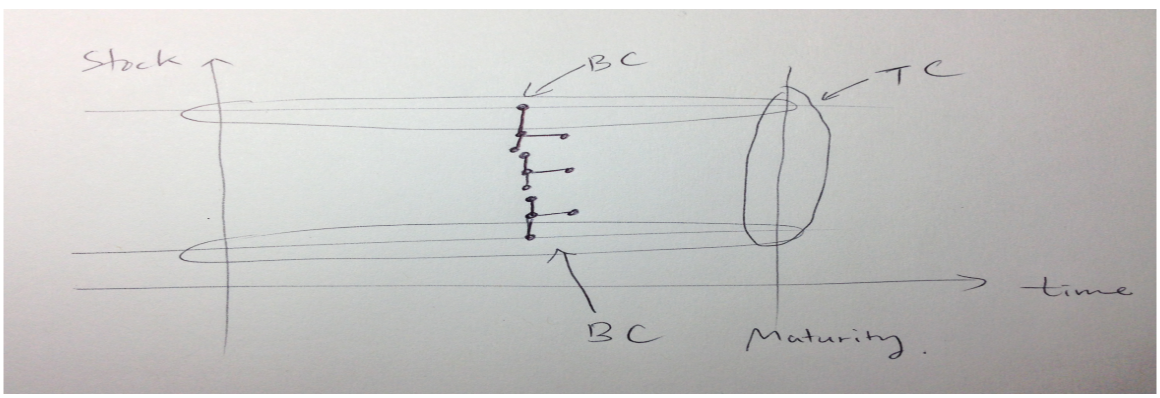
$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} = rv$$

Terminal (or initial) condition at T	$v(S_T, T) = (S_T - K)^+$
Boundary condition at S_{min}	$v(S_{min}, t) = 0$
Boundary condition at S_{max}	$v(S_{max}, t) = (S_{max} - K) * e^{-r(T-t)}$

Implicit

```
tspan = linspace(0,T,N_T);
sspan = linspace(S_min,S_max,N_S);
```

$$\delta t = \frac{T}{N_T - 1}, \qquad \delta s = \frac{S_{max} - S_{min}}{N_S - 1}$$



$$\frac{\partial v}{\partial t} \rightarrow \frac{v_i^{t+1} - v_i^t}{\delta t} + O(\delta t)$$

$$\frac{\partial^2 v}{\partial s^2} \rightarrow \frac{v_{i+1}^t - 2v_i^t + v_{i-1}^t}{(\delta s)^2} + O((\delta s)^2)$$

$$\frac{\partial v}{\partial s} \rightarrow \frac{v_{i+1}^t - v_{i-1}^t}{2\delta s} + O((\delta s)^2)$$

$$\begin{array}{lccccccccccc}
\text{PDE} & \frac{\partial v}{\partial t} & + & \frac{1}{2}\sigma^2 & s^2 & \frac{\partial^2 v}{\partial s^2} & + & r & s & \frac{\partial v}{\partial s} & = r & v \\
 & \uparrow & & & \uparrow & \uparrow & & & \uparrow & \uparrow & & \uparrow \\
\text{FDM} & \frac{v_i^{t+1} - v_i^t}{\delta t} & + & \frac{1}{2}\sigma^2 & s_i^2 & \frac{v_{i+1}^t - 2v_i^t + v_{i-1}^t}{(\delta s)^2} & + & r & s_i & \frac{v_{i+1}^t - v_{i-1}^t}{2\delta s} & = r & v_i^t
\end{array}$$

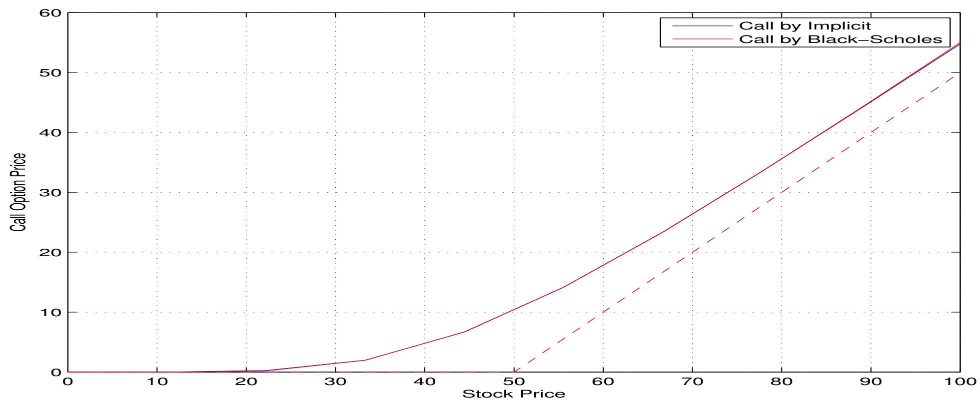
$$v_i^{t+1} = a_i v_{i-1}^t + b_i v_i^t + c_i v_{i+1}^t$$

where

$$\begin{aligned}
a_i &= -\frac{1}{2}\sigma^2\delta t\left(\frac{S_{min}}{\delta s}+(i-1)\right)^2 + \frac{1}{2}r\delta t\left(\frac{S_{min}}{\delta s}+(i-1)\right)\\
b_i &= 1 + r\delta t + \sigma^2\delta t\left(\frac{S_{min}}{\delta s}+(i-1)\right)^2\\
c_i &= -\frac{1}{2}\sigma^2\delta t\left(\frac{S_{min}}{\delta s}+(i-1)\right)^2 - \frac{1}{2}r\delta t\left(\frac{S_{min}}{\delta s}+(i-1)\right)
\end{aligned}$$

$$\begin{aligned}
&\text{BC at } S_{min} \qquad v_1^t = 0\\
&\text{BC at } S_{max} \qquad v_{N_S}^t = (S_{max} - K) * e^{-r*\delta t*(N_T - t)}
\end{aligned}$$

$$\underbrace{\begin{bmatrix} v_2^{t+1} \\ v_3^{t+1} \\ \vdots \\ v_{N_S-2}^{t+1} \\ v_{N_S-1}^{t+1} \end{bmatrix} - \begin{bmatrix} a_2 v_1^t \\ 0 \\ \vdots \\ 0 \\ c_{N_S-1} v_{N_S}^t \end{bmatrix}}_{\text{Known}} = \begin{bmatrix} b_2 & c_2 & & & \\ a_3 & b_3 & c_3 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N_S-2} & b_{N_S-2} & c_{N_S-2} \\ & & & a_{N_S-1} & b_{N_S-1} \end{bmatrix} \underbrace{\begin{bmatrix} v_2^t \\ v_3^t \\ \vdots \\ v_{N_S-2}^t \\ v_{N_S-1}^t \end{bmatrix}}_{\text{Unknown}}$$




```
clear all; close all; clc;

S = 50; K = 50; T = 1; r = 0.1; v = 0.4;

Smax = 100; Smin = 0;
NS = 10; NT = 100;
dS = (Smax-Smin)/(NS-1); dT = (T-0)/(NT-1);

C = zeros(NS,NT);
Stock = linspace(Smin,Smax,NS)';
C(1,:)  = 0;
C(NS,:) = (Stock(end)-K)*exp(-r*dT*(NT-(1:NT)));
C(:,NT) = max(Stock-K,0);

a = -0.5*v^2*dT*((Smin/dS)+(1:NS-2)).^2+0.5*r*dT*((Smin/dS)+(1:NS-2));
b = 1+r*dT+v^2*dT*((Smin/dS)+(1:NS-2)).^2;
c = -0.5*v^2*dT*((Smin/dS)+(1:NS-2)).^2-0.5*r*dT*((Smin/dS)+(1:NS-2));
i = [(2:NS-2) (1:NS-2) (1:NS-3)];
j = [(1:NS-3) (1:NS-2) (2:NS-2)];
s = [a(2:NS-2) b c(1:NS-3)];
A = sparse(i,j,s);

for t=NT-1:-1:1
    d = zeros(NS-2,1); d(1) = a(1)*C(1,t); d(end) = c(end)*C(end,t);
    C(2:end-1,t) = A\(C(2:end-1,t+1)-d);
end

x = linspace(Smin,Smax,NS)'; y = interp1(Stock, C(:,1), x);
x1 = sort([x; K]); y1 = Call(x1,K,T,r,v); y2 = max(x1-K,0);
plot(x,y,x1,y1,'-r',x1,y2,'--r'); grid on;
legend('Call by Implicit','Call by Black-Scholes')
xlabel('Stock price'); ylabel('Call price')
```

CN - Barrier put

Using CN compute knock-out barrier put with $S = 50$, $K = 50$, $r = 0.1$, $T = 1$, $\sigma = 0.4$, where there is a knock-out barrier at 20. If the stock price hit this knock-out barrier or stay below this barrier before maturity, the option pays nothing (knock-out) and expires.

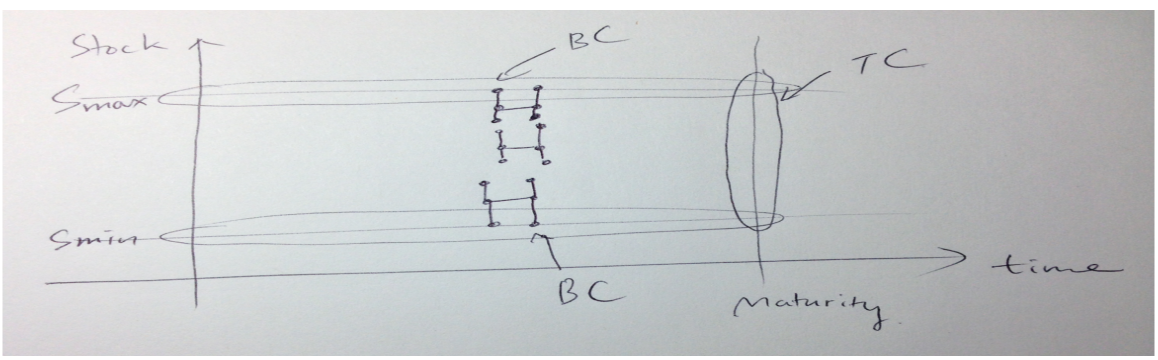
$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} = rv$$

Terminal (or initial) condition at T	$v(S_T, T) = (K - S_T)^+$
Boundary condition at $S_{min} = 20$	$v(S_{min}, t) = 0$
Boundary condition at S_{max}	$v(S_{max}, t) = 0$

$\boxed{\text{CN}}$

```
tspan = linspace(0,T,N_T);
sspan = linspace(S_min,S_max,N_S);
```

$$\delta t = \frac{T}{N_T - 1}, \qquad \delta s = \frac{S_{max} - S_{min}}{N_S - 1}$$



$$\frac{\partial v}{\partial t} \rightarrow \frac{v_i^{t+1} - v_i^t}{\delta t} + O(\delta t)$$

$$\frac{\partial^2 v}{\partial s^2} \rightarrow \frac{1}{2}\left[\frac{v_{i+1}^{t+1} - 2v_i^{t+1} + v_{i-1}^{t+1}}{(\delta s)^2} + \frac{v_{i+1}^t - 2v_i^t + v_{i-1}^t}{(\delta s)^2}\right] + O((\delta s)^2)$$

$$\frac{\partial v}{\partial s} \rightarrow \frac{1}{2}\left[\frac{v_{i+1}^{t+1} - v_{i-1}^{t+1}}{2\delta s} + \frac{v_{i+1}^t - v_{i-1}^t}{2\delta s}\right] + O((\delta s)^2)$$

$$\begin{array}{ccccccc} \frac{\partial v}{\partial t} & & +\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} & & +rs\frac{\partial v}{\partial s} & = & rv \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \frac{v_i^{t+1} - v_i^t}{\delta t} & +\frac{\sigma^2}{4}s_i^2 & \left[\frac{v_{i+1}^{t+1} - 2v_i^{t+1} + v_{i-1}^{t+1}}{(\delta s)^2} + \frac{v_{i+1}^t - 2v_i^t + v_{i-1}^t}{(\delta s)^2}\right] & +\frac{r}{2}s_i & \left[\frac{v_{i+1}^{t+1} - v_{i-1}^{t+1}}{2\delta s} + \frac{v_{i+1}^t - v_{i-1}^t}{2\delta s}\right] & = & rv_i^t \end{array}$$

$$a_i^+ v_{i-1}^{t+1} + b_i^+ v_i^{t+1} + c_i^+ v_{i+1}^{t+1} = a_i v_{i-1}^{t} + b_i v_i^{t} + c_i v_{i+1}^{t}$$

where

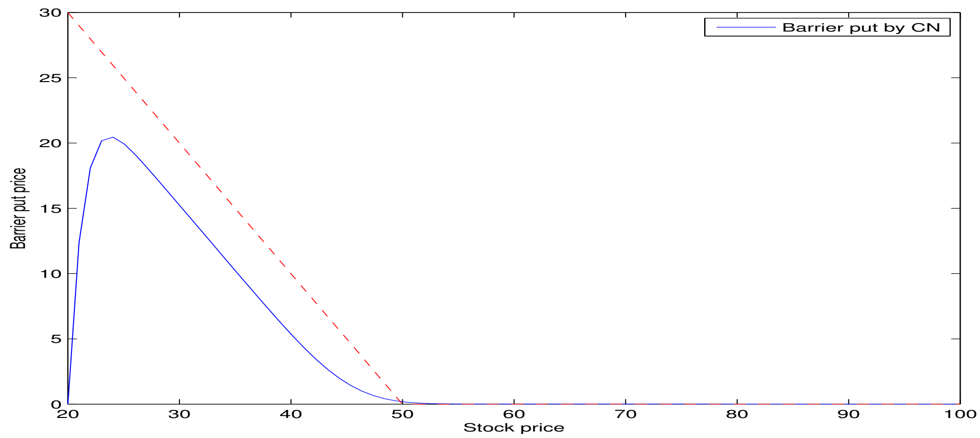
$$\begin{aligned}
a_i &= -\frac{1}{4}\sigma^2 \delta t \left(\frac{S_{min}}{\delta s} + (i-1)\right)^2 + \frac{1}{4} r \delta t \left(\frac{S_{min}}{\delta s} + (i-1)\right) \\
b_i &= 1 + \frac{1}{2} r \delta t + \frac{1}{2}\sigma^2 \delta t \left(\frac{S_{min}}{\delta s} + (i-1)\right)^2 \\
c_i &= -\frac{1}{4}\sigma^2 \delta t \left(\frac{S_{min}}{\delta s} + (i-1)\right)^2 - \frac{1}{4} r \delta t \left(\frac{S_{min}}{\delta s} + (i-1)\right)
\end{aligned}$$

$$a_i^+ = -a_i \qquad b_i^+ = 1 - \frac{1}{2} r \delta t - \frac{1}{2}\sigma^2 \delta t \left(\frac{S_{min}}{\delta s} + (i-1)\right)^2 \qquad c_i^+ = -c_i$$

$$\begin{array}{ll}
\text{BC at } S_{min} & v_1^t = 0 \\
\text{BC at } S_{max} & v_{N_S}^t = 0
\end{array}$$

$$\underbrace{\begin{bmatrix} b_2^+ & c_2^+ & & & \\ a_3^+ & b_4^+ & c_5^+ & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N_S-2}^+ & b_{N_S-2}^+ & c_{N_S-2}^+ \\ & & & a_{N_S-1}^+ & b_{N_S-1}^+ \end{bmatrix} \begin{bmatrix} v_2^{t+1} \\ v_3^{t+1} \\ \vdots \\ v_{N_S-2}^{t+1} \\ v_{N_S-1}^{t+1} \end{bmatrix} + \begin{bmatrix} a_2^+ v_1^{t+1} \\ 0 \\ \vdots \\ 0 \\ c_{N_S-1}^+ v_{N_S}^{t+1} \end{bmatrix} - \begin{bmatrix} a_2 v_1^{t} \\ 0 \\ \vdots \\ 0 \\ c_{N_S-1} v_{N_S}^{t} \end{bmatrix}}_{\text{Known}}$$

$$= \begin{bmatrix} b_2 & c_2 & & & \\ a_3 & b_4 & c_5 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N_S-2} & b_{N_S-2} & c_{N_S-2} \\ & & & a_{N_S-1} & b_{N_S-1} \end{bmatrix} \underbrace{\begin{bmatrix} v_2^{t} \\ v_3^{t} \\ \vdots \\ v_{N_S-2}^{t} \\ v_{N_S-1}^{t} \end{bmatrix}}_{\text{Unknown}}$$




```
clear all; close all; clc;

S=50; K=50; T=1; r=0.1; v=0.4;

Smin=20; Smax=100; NS=80; NT=100; dS=(Smax-Smin)/(NS-1); dT=(T-0)/(NT-1);

P=zeros(NS,NT);
Stock=linspace(Smin,Smax,NS)';
P(1,:)=0; P(NS,:)=0; P(:,NT)=max(K-Stock,0);

a=-0.25*v^2*dT*((Smin/dS)+(1:NS-2)).*2+0.25*r*dT*((Smin/dS)+(1:NS-2));
b=1+0.5*r*dT+0.5*v^2*dT*((Smin/dS)+(1:NS-2)).*2;
c=-0.25*v^2*dT*((Smin/dS)+(1:NS-2)).*2-0.25*r*dT*((Smin/dS)+(1:NS-2));
a_plus=-a; b_plus=1-0.5*r*dT-0.5*v^2*dT*((Smin/dS)+(1:NS-2)).*2; c_plus=-c;
i=[(2:NS-2) (1:NS-2) (1:NS-3)];
j=[(1:NS-3) (1:NS-2) (2:NS-2)];
s=[a(2:NS-2) b c(1:NS-3)];
s_plus=[a_plus(2:NS-2) b_plus c_plus(1:NS-3)];
A=sparse(i,j,s);
A_plus=sparse(i,j,s_plus);

for t=NT-1:-1:1
    d1=zeros(NS-2,1); d1(1)=a_plus(1)*P(1,t+1); d1(end)=c_plus(end)*P(end,t+1);
    d2=zeros(NS-2,1); d2(1)=a(1)*P(1,t); d2(end)=c(end)*P(end,t);
    P(2:end-1,t)=A\(A_plus*P(2:end-1,t+1)+d1-d2);
end

x=linspace(Smin,Smax,NS)'; y=interp1(Stock, P(:,1), x);
x1=sort([x; K]); y2=max(K-x1,0);
plot(x,y,x1,y2,'--r'); legend('Barrier put by CN')
xlabel('Stock price'); ylabel('Barrier put price')
```

Change of variable - Before and after

Time	t	$\rightarrow$	Time to maturity	$\tau = T - t$
Stock price	s	$\rightarrow$	Log stock price	$x = \log s$
LU	Not ready	$\rightarrow$	LU	Ready

	Before	After
BS equation	$V_t + \frac{1}{2}\sigma^2 s^2 V_{ss} + rsV_s = rV$	$-V_\tau + \frac{1}{2}\sigma^2 V_{xx} + \left(r - \frac{1}{2}\sigma^2\right) V_x = rV$
Explicit	$V_t = A_{t+1} * V_{t+1}$	$V_{\tau+1} = A * V_\tau$
Implicit	$B_{t+1} * V_t = V_{t+1}$	$B * V_{\tau+1} = V_\tau$
CN	$B_{t+1} * V_t = A_{t+1} * V_{t+1}$	$B * V_{\tau+1} = A * V_\tau$
LU	LU decomposition not ready	LU decomposition ready

Computation time without and with LU

Time	t	$\rightarrow$	Time to maturity	$\tau = T - t$
Stock price	s	$\rightarrow$	Log stock price	$x = \log s$
LU	Not ready	$\rightarrow$	LU	Ready

```
Case Method         Without LU     With LU
1    Implicite      0.0206981     0.00651466
1    CN             0.00902394     0.00824802
2    Implicite      0.00447663     0.00239277
2    CN             0.00340537     0.00323791
3    Implicite      0.00179192     0.00198815
3    CN             0.00237677     0.00245344
4    Implicite      0.00120422     0.00118971
4    CN             0.00154822     0.00156448
5    Implicite      0.00147729     0.00125095
5    CN             0.00154262     0.00165108
6    Implicite      0.00126445     0.00121238
6    CN             0.00152173     0.00178135
7    Implicite      0.00116916     0.00120145
7    CN             0.00151732     0.00155938
8    Implicite      0.00111789     0.00132641
8    CN             0.0016423     0.00184443
9    Implicite      0.0088877     0.00959593
9    CN             0.0106648     0.0100721
```

```
clear all; close all; clc;

fprintf('Case Method        Without LU     With LU\n')

time_without_LU=zeros(2,9);
time_with_LU=zeros(2,9);

for i=1:9

% Choose an experiment
experiment = i;
switch experiment
    case 1; Smin=1; K=50; T=1; r=0.1; v=0.40;
            Smax=K*exp((r-0.5*v^2)*T+3*v*sqrt(T));
```

```
        ic  = @(x) max(exp(x)-K,0);
        bcl = @(t) 0;
        bcr = @(t) Smax - K*exp(-r*t);
case 2; Smin=1; K=50; T=1; r=0.1; v=0.40;
        Smax=K*exp((r-0.5*v^2)*T+3*v*sqrt(T));
        ic  = @(x) max(K-exp(x),0);
        bcl = @(t) K*exp(-r*t) - Smin;
        bcr = @(t) 0;
case 3; Knock_Out_Barrier_Bottom = 10;
        Smin = Knock_Out_Barrier_Bottom; K=50; T=1; r=0.1; v=0.40;
        Knock_Out_Barrier_Top = 100; Smax = Knock_Out_Barrier_Top;
        ic  = @(x) max(exp(x)-K,0);
        bcl = @(t) 0; % Due to Knock_Out_Barrier_Bottom
        bcr = @(t) 0; % Due to Knock_Out_Barrier_Top
case 4; Knock_Out_Barrier_Bottom = 10;
        Smin = Knock_Out_Barrier_Bottom; K=50; T=1; r=0.1; v=0.40;
        Knock_Out_Barrier_Top = 100; Smax = Knock_Out_Barrier_Top;
        ic  = @(x) max(K-exp(x),0); % This line is modified
        bcl = @(t) 0; % Due to Knock_Out_Barrier_Bottom
        bcr = @(t) 0; % Due to Knock_Out_Barrier_Top
case 5; Smin=1; K=50; T=1; r=0.1; v=0.40;
        Knock_Out_Barrier_Top = 100; Smax = Knock_Out_Barrier_Top;
        ic  = @(x) max(exp(x)-K,0);
        bcl = @(t) 0;
        bcr = @(t) 0; % Due to Knock_Out_Barrier_Top
case 6; Smin=1; K=50; T=1; r=0.1; v=0.40;
        Knock_Out_Barrier_Top = 100; Smax = Knock_Out_Barrier_Top;
        ic  = @(x) max(K-exp(x),0); % This line is modified
        bcl = @(t) K*exp(-r*t) - Smin; % This line is modified
        bcr = @(t) 0; % Due to Knock_Out_Barrier_Top
case 7; Knock_Out_Barrier_Bottom = 10;
        Smin = Knock_Out_Barrier_Bottom; K=50; T=1; r=0.1; v=0.40;
        Smax = K*exp((r-0.5*v^2)*T+3*v*sqrt(T));
        ic  = @(x) max(exp(x)-K,0);
        bcl = @(t) 0; % Due to Knock_Out_Barrier_Bottom
        bcr = @(t) Smax - K*exp(-r*t);
case 8; Knock_Out_Barrier_Bottom = 10;
        Smin = Knock_Out_Barrier_Bottom; K=50; T=1; r=0.1; v=0.40;
        Smax = K*exp((r-0.5*v^2)*T+3*v*sqrt(T));
        ic  = @(x) max(K-exp(x),0); % This line is modified
        bcl = @(t) 0; % Due to Knock_Out_Barrier_Bottom
        bcr = @(t) 0;  % This line is modified
case 9; Smin=1; K=50; T=1; r=0.1; v=0.40;
        Smax=K*exp((r-0.5*v^2)*T+3*v*sqrt(T));
        ic  = @(x) max(exp(x)-K,0);
```

```
            bcl = @(t) 0;
            bcr = @(t) Smax - K*exp(-r*t);
            ee  = @(x) max(exp(x)-K,0);
end

xspan=[log(Smin),log(Smax)]; xspan = real(xspan); Nx=20;
tspan=[0,T]; Nt=50;

% Choose a watch
watch = i;
switch watch
case 1; tic; V1 = BS_Implicit(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(1,i) = toc;
        tic; V2 = BS_CN(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(2,i) = toc;
        tic; V3 = BS_Implicit_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(1,i) = toc;
        tic; V4 = BS_CN_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(2,i) = toc;
case 2; tic; V1 = BS_Implicit(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(1,i) = toc;
        tic; V2 = BS_CN(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(2,i) = toc;
        tic; V3 = BS_Implicit_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(1,i) = toc;
        tic; V4 = BS_CN_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(2,i) = toc;
case 3; tic; V1 = BS_Implicit(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(1,i) = toc;
        tic; V2 = BS_CN(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(2,i) = toc;
        tic; V3 = BS_Implicit_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(1,i) = toc;
        tic; V4 = BS_CN_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(2,i) = toc;
case 4; tic; V1 = BS_Implicit(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(1,i) = toc;
        tic; V2 = BS_CN(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(2,i) = toc;
        tic; V3 = BS_Implicit_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(1,i) = toc;
        tic; V4 = BS_CN_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(2,i) = toc;
case 5; tic; V1 = BS_Implicit(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(1,i) = toc;
```

```
        tic; V2 = BS_CN(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(2,i) = toc;
        tic; V3 = BS_Implicit_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(1,i) = toc;
        tic; V4 = BS_CN_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(2,i) = toc;
case 6; tic; V1 = BS_Implicit(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(1,i) = toc;
        tic; V2 = BS_CN(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(2,i) = toc;
        tic; V3 = BS_Implicit_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(1,i) = toc;
        tic; V4 = BS_CN_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(2,i) = toc;
case 7; tic; V1 = BS_Implicit(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(1,i) = toc;
        tic; V2 = BS_CN(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(2,i) = toc;
        tic; V3 = BS_Implicit_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(1,i) = toc;
        tic; V4 = BS_CN_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(2,i) = toc;
case 8; tic; V1 = BS_Implicit(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(1,i) = toc;
        tic; V2 = BS_CN(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_without_LU(2,i) = toc;
        tic; V3 = BS_Implicit_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(1,i) = toc;
        tic; V4 = BS_CN_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v);
        time_with_LU(2,i) = toc;
case 9; tic; V1 = BS_American_Implicit(xspan,Nx,ic,bcl,bcr,ee,tspan,Nt,r,v);
        time_without_LU(1,i) = toc;
        tic; V2 = BS_American_CN(xspan,Nx,ic,bcl,bcr,ee,tspan,Nt,r,v);
        time_without_LU(2,i) = toc;
        tic; V3 = BS_American_Implicit_LU(xspan,Nx,ic,bcl,bcr,ee,tspan,Nt,r,v);
        time_with_LU(1,i) = toc;
        tic; V4 = BS_American_CN_LU(xspan,Nx,ic,bcl,bcr,ee,tspan,Nt,r,v);
        time_with_LU(2,i) = toc;
end

fprintf('%g    Implicite     %g     %g\n',i,time_without_LU(1,i),time_with_LU(1,i))
fprintf('%g    CN            %g     %g\n',i,time_without_LU(2,i),time_with_LU(2,i))

end
```

Explicit (After) - Put

Compute put using explicit FDM where $S = 50$, $K = 50$, $r = 0.1$, $T = 1$, $\sigma = 0.4$.

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} = rv$$

Terminal (or initial) condition at T	$v(S_T, T) = (K - S_T)^+$
Boundary condition at S_{min}	$v(S_{min}, t) = (K - S_{min}) * e^{-r(T-t)}$
Boundary condition at S_{max}	$v(S_{max}, t) = 0$

Explicit (After)

```
tauspan = linspace(0,T,N_T);
x_min = log(S_min); x_max = log(S_max);
xspan = linspace(x_min,x_max,N_S);
```

$$\delta\tau = \frac{T}{N_T - 1}, \qquad \delta x = \frac{x_{max} - x_{min}}{N_S - 1}$$

$$\underbrace{\begin{bmatrix} v_2^{\tau+1} \\ v_3^{\tau+1} \\ \vdots \\ v_{Nx-2}^{\tau+1} \\ v_{Nx-1}^{\tau+1} \end{bmatrix}}_{\text{Unknown}} = \begin{bmatrix} L' & C' & R' & & & & \\ & L' & C' & R' & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & L' & C' & R' & \\ & & & & L' & C' & R' \end{bmatrix} \underbrace{\begin{bmatrix} v_1^{\tau} \\ v_2^{\tau} \\ v_3^{\tau} \\ \vdots \\ v_{Nx-2}^{\tau} \\ v_{Nx-1}^{\tau} \\ v_{Nx}^{\tau} \end{bmatrix}}_{\text{Known}}$$

where

$$\begin{aligned} L' &= & & \tfrac{1}{2}\sigma^2 \tfrac{\delta\tau}{(\delta x)^2} & & - \left(r - \tfrac{1}{2}\sigma^2\right) \tfrac{\delta\tau}{2(\delta x)} \\ C' &= (1 - r\delta\tau) & & +(-2) * \tfrac{1}{2}\sigma^2 \tfrac{\delta\tau}{(\delta x)^2} & & \\ R' &= & & \tfrac{1}{2}\sigma^2 \tfrac{\delta\tau}{(\delta x)^2} & & + \left(r - \tfrac{1}{2}\sigma^2\right) \tfrac{\delta\tau}{2(\delta x)} \end{aligned}$$

```
function V = BS_Explicit(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v)

% xspan = [log(Smin) log(Smax)]; Log stock price ( x = log S ) interval
% Nx; Number of x nodes

% ic;  initial condition, i.e., payoff as a function form
% bcl; boundary condition at left end as a function form
% bcr; boundary condition at right end as a function form

% tspan = [0 T]; Time to maturity interval
% Nt; Number of t nodes

% r; interest
% v; volatility

x = linspace(xspan(1),xspan(2),Nx);
dx = x(2)-x(1);

t = linspace(tspan(1),tspan(2),Nt);
dt = t(2)-t(1);

rho = dt/dx^2;

L1 =                 0.5*v^2*rho - (r-0.5*v^2)*(dt/(2*dx));
C1 = (1-r*dt) + (-2)*0.5*v^2*rho;
R1 =                 0.5*v^2*rho + (r-0.5*v^2)*(dt/(2*dx));

V       = zeros(Nx,Nt);
V(:,1)  = ic(x');  % initial condition, i.e., payoff
V(1,:)  = bcl(t'); % boundary condition at left end
V(Nx,:) = bcr(t'); % boundary condition at right end

for j=2:Nt
    V(2:Nx-1,j) = L1*V(1:Nx-2,j-1)+C1*V(2:Nx-1,j-1)+R1*V(3:Nx,j-1); % Explicit
end
end
```

Implicit (After) - Call

Compute call using implicit FDM where $S = 50$, $K = 50$, $r = 0.1$, $T = 1$, $\sigma = 0.4$.

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} = rv$$

Terminal (or initial) condition at T	$v(S_T, T) = (S_T - K)^+$
Boundary condition at S_{min}	$v(S_{min}, t) = 0$
Boundary condition at S_{max}	$v(S_{max}, t) = (S_{max} - K) * e^{-r(T-t)}$

Implicit (After)

```
tauspan = linspace(0,T,N_T);
x_min = log(S_min); x_max = log(S_max);
xspan = linspace(x_min,x_max,N_S);
```

$$\delta\tau = \frac{T}{N_T - 1}, \qquad \delta x = \frac{x_{max} - x_{min}}{N_S - 1}$$

$$\underbrace{\begin{bmatrix} v_2^{\tau} \\ v_3^{\tau} \\ \vdots \\ v_{NS-2}^{\tau} \\ v_{NS-1}^{\tau} \end{bmatrix} - \begin{bmatrix} Lv_1^{\tau+1} \\ 0 \\ \vdots \\ 0 \\ Rv_{NS}^{\tau+1} \end{bmatrix}}_{\text{Known}} = \begin{bmatrix} C & R & & & \\ L & C & R & & \\ & \ddots & \ddots & \ddots & \\ & & L & C & R \\ & & & L & C \end{bmatrix} \underbrace{\begin{bmatrix} v_2^{\tau+1} \\ v_3^{\tau+1} \\ \vdots \\ v_{NS-2}^{\tau+1} \\ v_{NS-1}^{\tau+1} \end{bmatrix}}_{\text{Unknown}}$$

where

$$\begin{aligned} L &= & -\tfrac{1}{2}\sigma^2 \tfrac{\delta\tau}{(\delta x)^2} & + \left(r - \tfrac{1}{2}\sigma^2\right) \tfrac{\delta\tau}{2(\delta x)} \\ C &= (1 + r\delta\tau) & +2 * \tfrac{1}{2}\sigma^2 \tfrac{\delta\tau}{(\delta x)^2} & \\ R &= & -\tfrac{1}{2}\sigma^2 \tfrac{\delta\tau}{(\delta x)^2} & - \left(r - \tfrac{1}{2}\sigma^2\right) \tfrac{\delta\tau}{2(\delta x)} \end{aligned}$$

```
function V = BS_Implicit(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v)

% xspan = [log(Smin) log(Smax)]; Log stock price ( x = log S ) interval
% Nx; Number of x nodes

% ic;  initial condition, i.e., payoff as a function form
% bcl; boundary condition at left end as a function form
% bcr; boundary condition at right end as a function form

% tspan = [0 T]; Time to maturity interval
% Nt; Number of t nodes

% r; interest
% v; volatility

x = linspace(xspan(1),xspan(2),Nx); dx = x(2)-x(1);

t = linspace(tspan(1),tspan(2),Nt); dt = t(2)-t(1);

rho = dt/dx^2;

L =          -   0.5*v^2*rho + (r-0.5*v^2)*(dt/(2*dx));
C = (1+r*dt) + 2*0.5*v^2*rho;
R =          -   0.5*v^2*rho - (r-0.5*v^2)*(dt/(2*dx));
e1 = ones(Nx-2,1);
LL = L*e1;
CC = C*e1;
RR = R*e1;

D = [CC RR LL];
d = [ 0  1 -1];
A = spdiags(D,d,Nx-2,Nx-2);

V       = zeros(Nx,Nt);
V(:,1)  = ic(x');  % initial condition, i.e., payoff
V(1,:)  = bcl(t'); % boundary condition at left end
V(Nx,:) = bcr(t'); % boundary condition at right end

for j=2:Nt
    e0 = zeros(Nx-2,1);
    f = e0; f(1) = L*V(1,j); f(end) = R*V(end,j);
    V(2:Nx-1,j) = A\(V(2:Nx-1,j-1)-f); % Implicit
end
end
```

```
function V = BS_Implicit_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v)

% xspan = [log(Smin) log(Smax)]; Log stock price ( x = log S ) interval
% Nx; Number of x nodes

% ic;  initial condition, i.e., payoff as a function form
% bcl; boundary condition at left end as a function form
% bcr; boundary condition at right end as a function form

% tspan = [0 T]; Time to maturity interval
% Nt; Number of t nodes

% r; interest
% v; volatility

x = linspace(xspan(1),xspan(2),Nx); dx = x(2)-x(1);

t = linspace(tspan(1),tspan(2),Nt); dt = t(2)-t(1);

rho = dt/dx^2;

L =          -   0.5*v^2*rho + (r-0.5*v^2)*(dt/(2*dx));
C = (1+r*dt) + 2*0.5*v^2*rho;
R =          -   0.5*v^2*rho - (r-0.5*v^2)*(dt/(2*dx));
e1 = ones(Nx-2,1);
LL = L*e1;
CC = C*e1;
RR = R*e1;

D = [CC RR LL];
d = [ 0  1 -1];
A = spdiags(D,d,Nx-2,Nx-2);
[LLL UUU] = lu(A);

V       = zeros(Nx,Nt);
V(:,1)  = ic(x');  % initial condition, i.e., payoff
V(1,:)  = bcl(t'); % boundary condition at left end
V(Nx,:) = bcr(t'); % boundary condition at right end

for j=2:Nt
    e0 = zeros(Nx-2,1);
    f = e0; f(1) = L*V(1,j); f(end) = R*V(end,j);
    V(2:Nx-1,j) = UUU\(LLL\(V(2:Nx-1,j-1)-f)); % Implicit LU
end
end
```

```
function V = BS_American_Implicit(xspan,Nx,ic,bcl,bcr,ee,tspan,Nt,r,v)

% xspan = [log(Smin) log(Smax)]; Log stock price ( x = log S ) interval
% Nx; Number of x nodes

% ic;  initial condition, i.e., payoff as a function form
% bcl; boundary condition at left end as a function form
% bcr; boundary condition at right end as a function form
% ee;  early exercise payoff as a function form

% tspan = [0 T]; Time to maturity interval
% Nt; Number of t nodes

% r; interest
% v; volatility

x = linspace(xspan(1),xspan(2),Nx); dx = x(2)-x(1);
t = linspace(tspan(1),tspan(2),Nt); dt = t(2)-t(1);
rho = dt/dx^2;

L =          -   0.5*v^2*rho + (r-0.5*v^2)*(dt/(2*dx));
C = (1+r*dt) + 2*0.5*v^2*rho;
R =          -   0.5*v^2*rho - (r-0.5*v^2)*(dt/(2*dx));
e1 = ones(Nx-2,1);
LL = L*e1;
CC = C*e1;
RR = R*e1;

D = [CC RR LL];
d = [ 0  1 -1];
A = spdiags(D,d,Nx-2,Nx-2);

V       = zeros(Nx,Nt);
V(:,1)  = ic(x');  % initial condition, i.e., payoff
V(1,:)  = bcl(t); % boundary condition at left end
V(Nx,:) = bcr(t); % boundary condition at right end

for j=2:Nt
    e0 = zeros(Nx-2,1);
    f = e0; f(1) = L*V(1,j); f(end) = R*V(end,j);
    V(2:Nx-1,j) = A\(V(2:Nx-1,j-1)-f); % Implicit
    V(:,j) = max(V(:,j),ee(x'));
end
end
```

```
function V = BS_American_Implicit_LU(xspan,Nx,ic,bcl,bcr,ee,tspan,Nt,r,v)

% xspan = [log(Smin) log(Smax)]; Log stock price ( x = log S ) interval
% Nx; Number of x nodes

% ic;  initial condition, i.e., payoff as a function form
% bcl; boundary condition at left end as a function form
% bcr; boundary condition at right end as a function form
% ee;  early exercise payoff as a function form

% tspan = [0 T]; Time to maturity interval
% Nt; Number of t nodes

% r; interest
% v; volatility

x = linspace(xspan(1),xspan(2),Nx); dx = x(2)-x(1);
t = linspace(tspan(1),tspan(2),Nt); dt = t(2)-t(1);
rho = dt/dx^2;

L =          -   0.5*v^2*rho + (r-0.5*v^2)*(dt/(2*dx));
C = (1+r*dt) + 2*0.5*v^2*rho;
R =          -   0.5*v^2*rho - (r-0.5*v^2)*(dt/(2*dx));
e1 = ones(Nx-2,1);
LL = L*e1;
CC = C*e1;
RR = R*e1;

D = [CC RR LL];
d = [ 0  1 -1];
A = spdiags(D,d,Nx-2,Nx-2);
[LLL UUU] = lu(A);

V       = zeros(Nx,Nt);
V(:,1)  = ic(x');  % initial condition, i.e., payoff
V(1,:)  = bcl(t'); % boundary condition at left end
V(Nx,:) = bcr(t'); % boundary condition at right end

for j=2:Nt
    e0 = zeros(Nx-2,1);
    f = e0; f(1) = L*V(1,j); f(end) = R*V(end,j);
    V(2:Nx-1,j) = UUU\(LLL\(V(2:Nx-1,j-1)-f)); % Implicit
    V(:,j) = max(V(:,j),ee(x'));
end
end
```

CN (After) - Barrier put

Using CN compute knock-out barrier put with $S = 50$, $K = 50$, $r = 0.1$, $T = 1$, $\sigma = 0.4$, where there is a knock-out barrier at 20. If the stock price hit this knock-out barrier or stay below this barrier before maturity, the option pays nothing (knock-out) and expires.

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} = rv$$

Terminal (or initial) condition at T	$v(S_T, T) = (K - S_T)^+$
Boundary condition at $S_{min} = 20$	$v(S_{min}, t) = 0$
Boundary condition at S_{max}	$v(S_{max}, t) = 0$

CN (After)

```
tauspan = linspace(0,T,N_T);
x_min = log(S_min); x_max = log(S_max);
xspan = linspace(x_min,x_max,N_S);
```

$$\delta\tau = \frac{T}{N_T - 1}, \qquad \delta x = \frac{x_{max} - x_{min}}{N_S - 1}$$

$$\begin{bmatrix} C & R & & & \\ L & C & R & & \\ & \ddots & \ddots & \ddots & \\ & & L & C & R \\ & & & L & C \end{bmatrix} \underbrace{\begin{bmatrix} v_2^{\tau+1} \\ v_3^{\tau+1} \\ \vdots \\ v_{Nx-2}^{\tau+1} \\ v_{Nx-1}^{\tau+1} \end{bmatrix}}_{\text{Unknown}}$$

$$= \underbrace{\begin{bmatrix} C' & R' & & & \\ L' & C' & R' & & \\ & \ddots & \ddots & \ddots & \\ & & L' & C' & R' \\ & & & L' & C' \end{bmatrix} \begin{bmatrix} v_2^{\tau} \\ v_3^{\tau} \\ \vdots \\ v_{Nx-2}^{\tau} \\ v_{Nx-1}^{\tau} \end{bmatrix} + \begin{bmatrix} L'v_1^{\tau} \\ 0 \\ \vdots \\ 0 \\ R'v_{Nx}^{\tau} \end{bmatrix} - \begin{bmatrix} Lv_1^{\tau+1} \\ 0 \\ \vdots \\ 0 \\ Rv_{Nx}^{\tau+1} \end{bmatrix}}_{\text{Known}}$$

where

$$\begin{aligned} L &= -\tfrac{1}{4}\left(\sigma^2 \tfrac{\delta\tau}{(\delta x)^2} - \left(r - \tfrac{1}{2}\sigma^2\right)\tfrac{\delta\tau}{\delta x}\right) & L' &= -L \\ C &= \left(1 + \tfrac{1}{2}\sigma^2 \tfrac{\delta\tau}{(\delta x)^2} + \tfrac{\delta\tau}{2} r\right) & C' &= \left(1 - \tfrac{1}{2}\sigma^2 \tfrac{\delta\tau}{(\delta x)^2} - \tfrac{\delta\tau}{2} r\right) \\ R &= -\tfrac{1}{4}\left(\sigma^2 \tfrac{\delta\tau}{(\delta x)^2} + \left(r - \tfrac{1}{2}\sigma^2\right)\tfrac{\delta\tau}{\delta x}\right) & R' &= -R \end{aligned}$$

```
function V = BS_CN(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v)

% xspan = [log(Smin) log(Smax)]; Log stock price ( x = log S ) interval
% Nx; Number of x nodes

% ic;  initial condition, i.e., payoff as a function form
% bcl; boundary condition at left end as a function form
% bcr; boundary condition at right end as a function form

% tspan = [0 T]; Time to maturity interval
% Nt; Number of t nodes

% r; interest
% v; volatility

x = linspace(xspan(1),xspan(2),Nx); dx = x(2)-x(1);
t = linspace(tspan(1),tspan(2),Nt); dt = t(2)-t(1);
rho = dt/dx^2;

L  =               - 0.5*0.5*v^2*rho + 0.5*(r-0.5*v^2)*(dt/(2*dx));
C  = (1+0.5*r*dt) +     0.5*v^2*rho;
R  =               - 0.5*0.5*v^2*rho - 0.5*(r-0.5*v^2)*(dt/(2*dx));
L1 = -L;
C1 = (1-0.5*r*dt) -     0.5*v^2*rho;
R1 = -R;

e1 = ones(Nx-2,1);
LL = L*e1; CC = C*e1; RR = R*e1; LL1 = L1*e1; CC1 = C1*e1; RR1 = R1*e1;

D = [CC RR LL]; d = [ 0  1 -1]; A = spdiags(D,d,Nx-2,Nx-2);
D1 = [CC1 RR1 LL1]; d1 = [  0   1  -1]; B = spdiags(D1,d1,Nx-2,Nx-2);

V       = zeros(Nx,Nt);
V(:,1)  = ic(x');  % initial condition, i.e., payoff
V(1,:)  = bcl(t'); % boundary condition at left end
V(Nx,:) = bcr(t'); % boundary condition at right end

for j=2:Nt
    e0 = zeros(Nx-2,1);
    f = e0; f(1) = L1*V(1,j-1); f(end) = R1*V(end,j-1);
    g = e0; g(1) = L*V(1,j); g(end) = R*V(end,j);
    V(2:Nx-1,j) = A\(B*V(2:Nx-1,j-1)+f-g); % CN
end
end
```

```
function V = BS_CN_LU(xspan,Nx,ic,bcl,bcr,tspan,Nt,r,v)

% xspan = [log(Smin) log(Smax)]; Log stock price ( x = log S ) interval
% Nx; Number of x nodes

% ic;  initial condition, i.e., payoff as a function form
% bcl; boundary condition at left end as a function form
% bcr; boundary condition at right end as a function form

% tspan = [0 T]; Time to maturity interval
% Nt; Number of t nodes

% r; interest
% v; volatility

x = linspace(xspan(1),xspan(2),Nx); dx = x(2)-x(1);
t = linspace(tspan(1),tspan(2),Nt); dt = t(2)-t(1);
rho = dt/dx^2;

L  =               - 0.5*0.5*v^2*rho + 0.5*(r-0.5*v^2)*(dt/(2*dx));
C  = (1+0.5*r*dt) +     0.5*v^2*rho;
R  =               - 0.5*0.5*v^2*rho - 0.5*(r-0.5*v^2)*(dt/(2*dx));
L1 = -L;
C1 = (1-0.5*r*dt) -     0.5*v^2*rho;
R1 = -R;

e1 = ones(Nx-2,1);
LL = L*e1; CC = C*e1; RR = R*e1; LL1 = L1*e1; CC1 = C1*e1; RR1 = R1*e1;

D = [CC RR LL]; d = [ 0  1 -1]; A = spdiags(D,d,Nx-2,Nx-2);
D1 = [CC1 RR1 LL1]; d1 = [  0   1  -1]; B = spdiags(D1,d1,Nx-2,Nx-2);

[LLL UUU] = lu(A);

V       = zeros(Nx,Nt);
V(:,1)  = ic(x');  % initial condition, i.e., payoff
V(1,:)  = bcl(t'); % boundary condition at left end
V(Nx,:) = bcr(t'); % boundary condition at right end
for j=2:Nt
    e0 = zeros(Nx-2,1);
    f = e0; f(1) = L1*V(1,j-1); f(end) = R1*V(end,j-1);
    g = e0; g(1) = L*V(1,j); g(end) = R*V(end,j);
    V(2:Nx-1,j) = UUU\(LLL\(B*V(2:Nx-1,j-1)+f-g)); % CN
end
end
```