

Linear Algebra Crash Course

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Contents

1	Vector Spaces and Subspaces	3
2	Linear Independence	3
3	Basis, Representation and Orthonormalization	3
3.1	Basis	3
3.2	Representation	3
3.2.1	Vector	3
3.2.2	Matrix	3
3.3	Orthonormalization	4
3.3.1	Orthonormal	4
3.3.2	Gram-Schmidt process	4
4	Linear Algebraic Equation	4
4.1	Projection	4
4.2	Least Square Method	6
4.3	Generalized Inverses	7
5	Similarity Transformation	7
6	Diagonal Form and Jordan Form	7
6.1	Eigenvalues and Eigenvectors	7
6.2	Eigendecomposition	7
6.3	Multiplicities	7
6.3.1	Algebraic Multiplicities	7
6.3.2	Geometric Multiplicities	7
6.4	Generalized Eigenvectors	7
6.5	Jordan Form	7
7	Functions of a Square Matrix	7
7.1	Polynomials	7
7.2	Cayley-Hamilton Theorem	7
7.3	Exponential	7
8	Sylvester's Equation	7
8.1	Solution with a Vec Operator	7
8.2	Eigenvalues	7

9	Some Useful Formulas	7
9.1	Ranks	7
9.2	Diagonal Expansion	7
9.3	Determinants	7
10	Special Matrices	7
10.1	Symmetric Matrices	7
10.2	Idempotent Matrices	7
11	Quadratic Forms and Positive Definiteness	7
11.1	Quadratic Forms	7
11.2	Positive Definiteness	7
12	Singular Value Decomposition	7
12.1	Singular Value	7
12.2	Singular Value Decomposition	7
13	Norms of Matrices	7
14	Matrix Calculus	7

1. Vector Spaces and Subspaces

2. Linear Independence

If there are only zero α_i for all i which satisfy the equation below, we say a set of vectors $\{x_1, \dots, x_n\}$ is linearly independent.

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0 \quad (1)$$

3. Basis, Representation and Orthonormalization

3.1 Basis

Basis is a set of linearly independent vectors $\{x_1, \dots, x_n\}$ which is used to form an n -dimensional space.
 \iff Basis $\{x_1, \dots, x_n\}$ spans n -dimensional space.

3.2 Representation

In this subsection, we restrict vectors to arrays. Representation of vectors and representation of matrices are introduced differently in this article.

3.2.1 Vector

x is a vector whose basis is I . Q is a new basis on which we want to represent the vector. \bar{x} is a representation for the vector x with respect to the basis Q .¹

$$x = Q\bar{x} \quad (2)$$

We assume the basis Q is formed by stacking linearly independent column vectors q_1, q_2, \dots, q_n .

$$Q := \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix}$$

3.2.2 Matrix

A representation of matrix A is \bar{A} . \bar{A}_i means the i^{th} column of the basis Q .

$$Aq_i = Q\bar{A}_i \quad (3)$$

Why does the representation of a matrix look like this? What we really want to do is actually this: to represent $y = Ax$ in another form $\bar{y} = \bar{A}\bar{x}$. So we should start from the equation $y = Ax$.

$$\begin{aligned} y &= Ax \\ Q\bar{y} &= AQ\bar{x} \end{aligned}$$

Suppose Q is an $n \times n$ matrix. The matrix Q consists of vectors from the basis so the column vectors of the matrix are linearly independent. Then the matrix Q is invertible.

$$\begin{aligned} \bar{y} &= Q^{-1}AQ\bar{x} \\ \bar{y} &= \bar{A}\bar{x} \end{aligned}$$

¹Actually we cannot say Q is a basis. Q is just a matrix made up of vectors of the basis.

We get the equation:

$$\bar{A} = Q^{-1}AQ \quad (4)$$

Next we go forward to get the equation 3.

$$\begin{aligned} AQ &= Q\bar{A} \\ A \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix} &= Q \begin{bmatrix} \bar{A}_1 & \dots & \bar{A}_n \end{bmatrix} \\ \begin{bmatrix} Aq_1 & \dots & Aq_n \end{bmatrix} &= \begin{bmatrix} Q\bar{A}_1 & \dots & Q\bar{A}_n \end{bmatrix} \end{aligned}$$

If we choose i^{th} column then we get the equation 3 $Aq_i = Q\bar{A}_i$.

3.3 Orthonormalization

3.3.1 Orthonormal

Two vectors v_1 and v_2 are **orthogonal** if and only if

$$v_1^T v_2 = 0$$

If all orthogonal vectors in a set have magnitude one then we say them **orthonormal**.

$$\|v_i\| = 1 \quad \text{or} \quad v_i^T v_i = 1$$

3.3.2 Gram-Schmidt process

Suppose we have vectors e_1, e_2, \dots, e_n which we want to orthonormalize. The following steps are called Gram-Schmidt process and are used to make orthonormal vectors q_1, q_2, \dots, q_n .

$$\begin{aligned} v_1 &:= e_1 & q_1 &:= \frac{v_1}{\|v_1\|} \\ v_2 &:= e_2 - q_1^T e_2 & q_2 &:= \frac{v_2}{\|v_2\|} \\ v_3 &:= e_3 - q_1^T e_3 - q_2^T e_3 & q_3 &:= \frac{v_3}{\|v_3\|} \\ & & & \vdots \\ v_n &:= e_n - \sum_{i=1}^{n-1} q_i^T e_n & q_n &:= \frac{v_n}{\|v_n\|} \end{aligned}$$

This process incrementally removes the effect of former vectors from the current vector and normalizes the current vector.

4. Linear Algebraic Equation

A linear algebraic equation is:

$$\mathbf{y}_{n \times 1} = \mathbf{A}_{n \times m} \mathbf{x}_{m \times 1} \quad (5)$$

4.1 Projection

Let $\{z_1, \dots, z_n\}$ be an orthonormal basis for \mathbb{R}^n , $\{z_1, \dots, z_r\}$ be basis for V , and $\{z_{r+1}, \dots, z_n\}$ be basis for V^\perp . Next, we form matrices:

$$Z = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}_{n \times n}, \quad Z_1 = \begin{bmatrix} z_1 & \dots & z_r \end{bmatrix}_{n \times r}, \quad Z_2 = \begin{bmatrix} z_{r+1} & \dots & z_n \end{bmatrix}_{n \times n-r}$$

$C(Z)$ means a space spanned by column vectors of Z . Then the equations below hold.

$$\mathbb{R}^n = C(Z) = \text{span}(\{z_1, \dots, z_n\}), \quad V = C(Z_1) = \text{span}(\{z_1, \dots, z_r\}), \quad V^\perp = C(Z_2) = \text{span}(\{z_{r+1}, \dots, z_n\})$$

Just to be clear, we write relations between them.

$$V \subset \mathbb{R}^n, \quad V^\perp \subset \mathbb{R}^n, \quad V \perp V^\perp$$

Definition of projection: We say a linear mapping $\pi_V : \mathbb{R}^n \rightarrow V$ is a projection onto V when the equation below holds.

$$\forall \mathbf{v} \in V, \quad \forall \mathbf{x} \in \mathbb{R}^n, \quad \langle \mathbf{v}, \pi_V(\mathbf{x}) - \mathbf{x} \rangle = 0$$

Definition of projection matrix: P is a projection matrix when the following equation holds.

$$\pi_V(\mathbf{x}) = P\mathbf{x}$$

$Z_1 Z_1^T$ is a projection matrix onto V . Likewise $Z_2 Z_2^T$ is a projection matrix onto V^\perp . We verify this:
For $\forall \mathbf{x} = Z\alpha \in \mathbb{R}^n$,

$$\begin{aligned} Z_1 Z_1^T \mathbf{x} &= Z_1 Z_1^T Z\alpha \\ &= Z_1 Z_1^T [Z_1 \quad Z_2] \alpha \\ &= Z_1 Z_1^T (Z_1 \alpha_1 + Z_2 \alpha_2) \\ &= Z_1 Z_1^T Z_1 \alpha_1 + Z_1 Z_1^T Z_2 \alpha_2 \\ &= Z_1 \alpha_1 \end{aligned}$$

We used the fact that $Z_1^T Z_1 = I$. Also, notice that $I - Z_1 Z_1^T$ is a projection matrix onto V^\perp and $I - Z_2 Z_2^T$ is a projection matrix onto V .

In the paragraph above, we only consider the case of orthogonal Z s. The next question which arises will be how we form a projection matrix when the matrices don't have orthonormal column vectors. Although a matrix doesn't have orthonormal columns, we can make them be orthonormal. For example, Gram-Schmidt process. So there is always a matrix $A_{r \times r}$ which converts vectors of a matrix $X_{n \times r}$ to be orthonormal.

$$Z_1 = XA$$

Use what we already know:

$$\begin{aligned} Z_1 Z_1^T &= X A A^T X^T \\ &\& \\ I_r &= Z_1^T Z_1 = A^T X^T X A \\ \iff (A^T)^{-1} A^{-1} &= X^T X \\ A A^T &= (X^T X)^{-1} \end{aligned}$$

Finally, we get the equation below.

$$\therefore Z_1 Z_1^T = X (X^T X)^{-1} X^T$$

This is a projection matrix onto V . Likewise, $I - X (X^T X)^{-1} X^T$ is a projection matrix onto V^\perp . We conclude that the projection onto V is:

$$\pi_V(x) = X (X^T X)^{-1} X^T x$$

And its projection matrix is:

$$P_\pi = X (X^T X)^{-1} X^T$$

4.2 Least Square Method

Consider a linear model $y = X\beta + \epsilon$. We want to minimize the square sum of errors $\epsilon^T \epsilon = \epsilon_1^2 + \dots + \epsilon_n^2$.

$$\begin{aligned}\min_{\beta} e^T e &= (y - X\beta)^T (y - X\beta) \\ &= \langle y - X\beta, y - X\beta \rangle \\ &= \langle y - X\beta, y - X\beta \rangle\end{aligned}$$

4.3 Generalized Inverses

5. Similarity Transformation

6. Diagonal Form and Jordan Form

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