Boolean Algebra

Boolean Algebra is a mathematical structure that deals with binary variables and logical operations. It was named after the mathematician George Boole.

Definition

If A is a non-empty set with two binary operations AND $(\cdot)$ and OR $(+)$, two distinct elements 0 and 1 and a unary operation NOT $(')$ the A is called Boolean Algebra if the following basic properties hold for all a, b, c in A.

Identity Laws	$a+0=a$ $a\cdot 1=a$	A1
Commutative laws	$a+b=b+a$ $a\cdot b=b\cdot a$	A2
Associative laws	$(a+b)+c=a+(b+c)$ $(a\cdot b)\cdot c=a\cdot(b\cdot c)$	A3
Distributive laws	$a+(b\cdot c)=(a+b)\cdot(a+c)$ $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$	A4
Complement laws	$a+a'=1$ $a\cdot a'=0$	A5

Laws of Boolean Algebra

1, Idempotent laws:

$a+a=a$ and $a.a=a$ for all $a \in A$

2, Dominance laws:

$a+1=1$ and $a.0=0$ for all $a \in A$

3, Absorption laws:

$a.(a+b)=a$ and $a+a.b=a$ for all $a, b \in A$

4, De Morgan's Laws

$(a+b)'=a'.b'$ and $(a.b)'=a'+b'$ for all $a, b \in A$

5, Double complement or Involution law.

$(a')'=a$ for all $a \in A$

6, Zero and One law

$0'=1$ and $1'=0$

Proofs

1, Idempotent laws:

$a+a=a$ and $a.a=a$ for all $a \in A$

Proof

$$
\begin{aligned}
a &= a+0 && A1 \\
&= a(a.a') && A5 \\
&= (a+a).(a+a') && A4 \\
&= (a+a).1 && A5 \\
&= a+a && A1
\end{aligned}
$$

$$\begin{aligned} a &= a.1 && A1 \\ &= a.(a+a') && A5 \\ &= (a.a)+(a.a') && A4 \\ &= a.a+0 && A5 \\ &= a.a && A1 \end{aligned}$$

2, Dominance laws:

$a+1=1$ and $a.0=0$ for all $a \in A$

Proof

$$\begin{aligned} a+1 &= (a+1).1 && A1 \\ &= (a+1).(a+a') && A5 \\ &= a+1.a' && A4 \\ &= a+a'.1 && A2 \\ &= a+a' && A1 \\ &= 1 && A5 \end{aligned}$$

$$\begin{aligned} a.0 &= a.0+0 && A1 \\ &= a.0+a.a' && A5 \\ &= a.(0+a') && A4 \\ &= a.(a'+0) && A2 \\ &= a+a' && A1 \\ &= 0 && A5 \end{aligned}$$

3, Absorption laws:

$a.(a+b)=a$ and $a+a.b=a$ for all $a,b \in A$.

Proof

$$\begin{aligned} a.(a+b) &= (a+0).(a+b) && A1 \\ &= a+0.b && A4 \\ &= a+b.0 && A2 \\ &= a+0 && \text{by dominance Law} \\ &= a && A1 \end{aligned}$$

$$a + a.b = a.1 + a.b \quad A1$$
$$= a.(1+b) \quad A4$$
$$= a.(b+1) \quad A2$$
$$= a.1 \quad \text{dominance law}$$
$$= a \quad A1$$

4. De Morgan's laws.

$(a+b)' = a'.b' = (a.b)' = a'+b'$ for all $a, b \in A$

Proof

If y is to be the complement of x, by definition we must show that $x+y=1$ and $x.y=0$

$$(a+b) + a'b' = \{(a+b)+a'\}.\{(a+b)+b'\} \quad A4$$
$$= \{b+a+a'\}.\{(a+b)+b'\} \quad A2$$
$$= \{b+(a+a')\}.\{a+(b+b')\} \quad A3$$
$$= (b+1).(a+1) \quad A5$$
$$= 1.1 \quad \text{dominance law}$$
$$= 1 \quad A1$$

$$(a+b).a'b' = a'b'.(a+b) \quad A2$$
$$= a'b'.a + a'b.b \quad A4$$
$$= a.(a'b') + a'.(b'b) \quad A3$$
$$= (a.a').b' + a'.(b'b) \quad A3 \text{ and } A2$$
$$= 0.b' + a'.0 \quad A5$$
$$= b'.0 + a'.0 \quad A2$$
$$= 0 + 0 \quad \text{dominance law}$$
$$= 0 \quad A1$$

So for both we got $a'b'$ is the complement of $(a+b)$. That is $(a+b)' = a'b'$

5, Double complement or Involution law.

$(a')' = a$ for all $a \in A$.

Proof.

$a + a' = 1$ and $a \cdot a' = 0$ A5

$a' + a = 1$ and $a' \cdot a = 0$ A2.

$\therefore$ a is ~~the~~ the complement of a'

That is $(a')' = a$.

6, Zero and One law.

$0' = 1$ and $1' = 0$

Proof

$0' = (aa')'$ A5

$= a' + (a')'$ De Morgan's Law

$= a' + a$ Involution Law

$= a + a'$. A2

$= 1$ A5

Now $(0')' = 1'$

that is $0 = 1'$ or $1' = 0$

Note:

$a' + 1 = 1$

$(b + ab') = b + a$

Rules

1, Complement

$$(A')' = A$$

2, AND

$$A \cdot A = A$$
$$A \cdot 0 = 0$$
$$A \cdot 1 = A$$
$$A \cdot A' = 0$$

3, OR

$$A + A = A$$
$$A + 0 = A$$
$$A + 1 = 1$$
$$A + A' = 1$$

4, Distributive:

$$A + BC = (A+B) \cdot (A+C)$$
$$A \cdot (B+C) = A \cdot B + A \cdot C$$
$$A + A'B = A + B$$
$$A' + AB = A' + B$$

5, De Morgan's Law.

$$(A+B)' = A' \cdot B'$$
$$(A \cdot B)' = A' + B'$$

Example

Use Boolean ~~alg~~ algebra and Truth table to simplify

$$AB + AB'$$

Solution

Let $y = A.B + A.B'$
$= A(B + B')$
$= A.1$ A5
$= A$ A2

Truth table

A	B	¬B	A∧B	A∧¬B	(A∧B)∨(A∧B')
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	F	F	F
F	F	T	F	F	F
1					2

1 and 2 Are same ⇒ Final answer is A.

OR ~~we~~ can use 0 – False
1 – True.

Exercise 15

Use Boolean Algebra and Truth table to simplify
$AB + AB'C + AB'C'$

Logic Gates

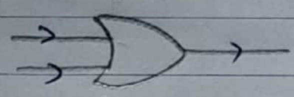
Logic gates are fundamental building blocks of digital circuits. They perform basic logical functions on one or more binary inputs to produce a single binary output.

1, AND Gate

Input  Output

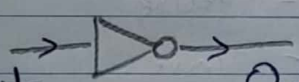
Outputs True (or 1) if all the inputs are True (or 1)

2, OR Gate

Input  Output

Outputs True (or 1) if any one of the inputs is True (or 1)

3, NOT Gate

Input  Output

Returns the complement of the input

eg

Input	Output
T	F
1	0