UNIT 6 HYPERBOLIC FUNCTIONS

Unit Structure

6.0 OVERVIEW

In this Unit, we study a new class of functions related the equations of the hyperbola. The parametric representation of such functions is done by using Hyperbolic functions. These functions are in fact a combination of exponential functions and they have their own set of rules.

6.1 LEARNING OBJECTIVES

By the end of this Unit, you should be able to do the following:

1. Prove the identities related to hyperbolic functions.
2. Find derivatives and integrals of and involving hyperbolic functions.
3. Find derivatives of inverse hyperbolic functions and express integrals in terms of inverse hyperbolic functions.

6.2 HYPERBOLIC FUNCTIONS

The two basic hyperbolic functions $\sinh x$ and $\cosh x$ are defined by

$$\begin{aligned}\sinh x &= \frac{e^x - e^{-x}}{2}, \\ \cosh x &= \frac{e^x + e^{-x}}{2}\end{aligned} \quad , \qquad \textbf{(1)}$$

where x is real. [**Note: (i)** x is not an angle; **(ii)** the *spelling* is important, as $\sin \mathrm{h}x$ means $\sin(\mathrm{h}x)$.]

By analogy with the trig functions, we define

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$

and $\operatorname{cosech} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$.

We notice that $\cosh(-x) = \cosh x$, $\sinh(-x) = -\sinh x$ and $\tanh(-x) = -\tanh x$.

The graphs of the six hyperbolic functions are shown in Figure 1. We immediately see that $\cosh x \geq 1$ and $-1 < \tanh x < +1$, while $\sinh x$ assumes all values. Thus, in solving the equation $(\cosh x + 3)(\cosh x - 5) = 0$, we can't have $\cosh x = -3$ if x is real.

Finally, we note that none of the hyperbolic functions is periodic.

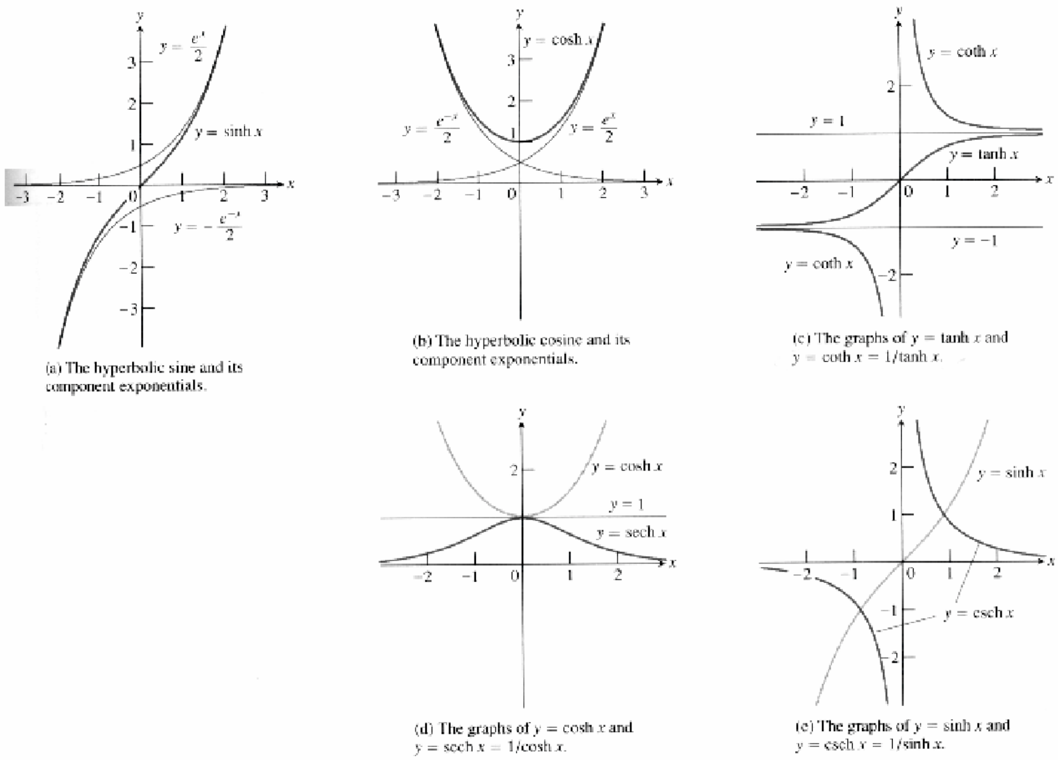

Figure 1: Plots of the hyperbolic functions.

6.3 IDENTITIES

From **(1)**, we obtain

$$\cosh x + \sinh x = e^{x}, \tag{2}$$

and

$$\cosh x - \sinh x = e^{-x}. \tag{3}$$

Mutiplying (2) by (3) gives

$$\cosh^2 x - \sinh^2 x = 1. \tag{4}$$

Dividing both sides of (4) by $\cosh^2 x$ yields

$$1 - \tanh^2 x = \operatorname{sech}^2 x,$$

while division of (4) by $\sinh^2 x$ gives

$$\coth^2 x - 1 = \operatorname{cosech}^2 x.$$

We also have other identities given by

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y\,,$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y\,,$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}\,.$$

Finally, we deduce that

$$\sinh 2x = 2\sinh x \cosh x\,,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x\,,$$

and

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}\,.$$

Also,

$$\sinh 2x = \frac{2\sinh x \cosh x}{\cosh^2 x - \sinh^2 x} = \frac{2\tanh x}{1-\tanh^2 x} \quad \text{[Dividing top \& bottom by } \cosh^2 x\text{]}$$

$$\cosh 2x = \frac{\cosh^2 x + \sinh^2 x}{\cosh^2 x - \sinh^2 x} = \frac{1+\tanh^2 x}{1-\tanh^2 x}\,.$$

Replacing x by $\frac{1}{2}x$ in the last three results and writing t for $\tanh \frac{1}{2}x$, they become

$$\tanh x = \frac{2t}{1+t^2}, \quad \sinh x = \frac{2t}{1-t^2}, \quad \cosh x = \frac{1+t^2}{1-t^2}.$$

These results are particularly useful when evaluating certain integrals.

We shall now consider a few examples relating to the definitions and properties of the hyperbolic functions.

<u>Example 1</u>

Given $\sinh x = 5$, find the values of the other hyperbolic functions.

First we note that if $\sinh x$ is positive, then so are $x, \tanh x$, $\coth x$ and obviously $\operatorname{cosech} x$. Likewise, if it is negative. However, $\cosh x$ and $\operatorname{sech} x$ are always positive for all real x, with $\cosh x \geq 1$, $0 < \operatorname{sech} x \leq 1$.

Here we have

$\operatorname{cosech} x = \dfrac{1}{\sinh x} = \dfrac{1}{5}$.

From the identity $\cosh^2 x - \sinh^2 x = 1$, it follows that $\cosh x = \sqrt{1+\sinh^2 x} = \sqrt{26}$.

Note we take *positive* square roots since $\cosh x \geq 1$**.**

Hence $\operatorname{sech} x = \dfrac{1}{\sqrt{26}}$, $\tanh x = \dfrac{5}{\sqrt{26}}$, $\coth x = \dfrac{\sqrt{26}}{5}$.

<u>Example 2</u>

If $\operatorname{cosech} x = -3$, find the values of **(i)** $\sinh 2x$; **(ii)** $\cosh 2x$; **(iii)** $\tanh 2x$**.**

Since $\operatorname{cosech} x$ is negative so are x, $\sinh x$, $\tanh x$, $\coth x$.

We have immediately $\sinh x = -\dfrac{1}{3}$.

$\therefore \quad \cosh x = \sqrt{1+\sinh^2 x} = +\dfrac{\sqrt{10}}{3}$.

Hence, (i) $\sinh 2x = 2\sinh x \cosh x = -\dfrac{2\sqrt{10}}{9}$;

(ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = \dfrac{11}{9}$;

(iii) $\tanh 2x = \dfrac{\sinh 2x}{\cosh 2x} = -\dfrac{2\sqrt{10}}{11}$.

<u>Example 3</u>

If $\cosh\left(\dfrac{u}{2}\right) = 2$, find $\tanh u$.

Now, $\cosh u = 2\cosh^2\left(\dfrac{u}{2}\right) - 1 = 7$

$\therefore \quad \sinh u = \pm\sqrt{\cosh^2 u - 1} = \pm 4\sqrt{3}$

$$\therefore \quad \tanh u = \pm\frac{4\sqrt{3}}{7}.$$

Example 4

If $\sinh x = \dfrac{3}{4}$ and $\cosh y = \dfrac{5}{4}$, find the values of $\sinh(x \pm y), \cosh(x \pm y), \tanh(x \pm y)$.

$$\sinh x = \frac{3}{4} \Rightarrow \cosh x = +\sqrt{1+\frac{9}{16}} = \frac{5}{4}, \text{ and } \therefore \quad \tanh x = \frac{3}{5};$$

$$\cosh y = \frac{5}{4} \Rightarrow \sinh y = \pm\sqrt{\frac{25}{16}-1} = \pm\frac{3}{4}, \text{ and } \therefore \quad \tanh y = \pm\frac{3}{5}.$$

Hence,

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y = \frac{3}{4}\cdot\frac{5}{4}+\frac{5}{4}\cdot\left(\pm\frac{3}{4}\right) = \frac{15}{8} \text{ or } 0.$$

$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y = \frac{3}{4}\cdot\frac{5}{4}-\frac{5}{4}\cdot\left(\pm\frac{3}{4}\right) = 0 \text{ or } \frac{15}{8}.$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y = \frac{5}{4}\cdot\frac{5}{4}+\frac{3}{4}\cdot\left(\pm\frac{3}{4}\right) = \frac{17}{8} \text{ or } 1.$$

$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y = \frac{5}{4}\cdot\frac{5}{4}-\frac{3}{4}\cdot\left(\pm\frac{3}{4}\right) = 1 \text{ or } \frac{17}{8}.$$

$$\tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{15/8}{17/8} \text{ or } \frac{0}{1} = \frac{15}{17} \text{ or } 0.$$

$$\tanh(x-y) = \frac{\sinh(x-y)}{\cosh(x-y)} = \frac{0}{1} \text{ or } \frac{15/8}{17/8} = 0 \text{ or } \frac{15}{17}.$$

Example 5

If $\sinh x = \dfrac{5}{12}$, find $\sinh\frac{1}{2}x, \cosh\frac{1}{2}x$ and $\tanh\frac{1}{2}x$.

$$\cosh x = +\sqrt{1+\sinh^2 x} = \frac{13}{12}$$

$$\sinh^2 \tfrac{1}{2} x = \tfrac{1}{2}(\cosh x - 1) = \frac{1}{24}$$

$$\therefore \quad \sinh \tfrac{1}{2} x = +\sqrt{\frac{1}{24}} = \frac{1}{2\sqrt{6}}, \quad \because \quad x > 0 .$$

$$\cosh^2 \tfrac{1}{2} x = \tfrac{1}{2}(\cosh x + 1) = \frac{25}{24},$$

$$\therefore \cosh \tfrac{1}{2} x = +\frac{5}{2\sqrt{6}} .$$

Hence $\tanh \tfrac{1}{2} x = \dfrac{1}{2\sqrt{6}} \Big/ \dfrac{5}{2\sqrt{6}} = \dfrac{1}{5}$.

6.3.1 Osborn's Rule

By now, you will have surely noticed the striking similarity between the above identities and those connecting the trigonometric functions. In fact the standard identities are in the same form except that certain signs are changed. Osborn's rule provides a simple way of remembering these changes of signs. The rule is to change the sign of any term containing the product of two sines (or cosecant or tangent or cotangent because these functions all include a sine by implication). See table below.

Trigonometric Identities	**Corresponding Hyperbolic Identities**
$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$1 + \tan^2 x = \sec^2 x$	$1 - \tanh^2 x = \operatorname{sech}^2 x$
$1 + \cot^2 x = \operatorname{cosec}^2 x$	$1 - \coth^2 x = -\operatorname{cosech}^2 x$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x$

$\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$	$\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$
$\sin 3x = 3\sin x - 4\sin^3 x$	$\sinh 3x = 3\sinh x + 4\sinh^3 x$
$\cos 3x = 4\cos^3 x - 3\cos x$	$\cosh 3x = 4\cosh^3 x - 3\cosh x$
$\cos ax \cos bx = \frac{1}{2}[\cos(a+b)x + \cos(a-b)x]$	$\cosh ax \cosh bx = \frac{1}{2}[\cosh(a+b)x + \cosh(a-b)x]$
$\sin ax \cos bx = \frac{1}{2}[\sin(a+b)x + \sin(a-b)x]$	$\sinh ax \cosh bx = \frac{1}{2}[\sinh(a+b)x + \sinh(a-b)x]$
$\sin ax \sin bx = \frac{1}{2}[\cos(a-b)x - \cos(a+b)x]$	$\sinh ax \sinh bx = \frac{1}{2}[\cosh(a+b)x - \cosh(a-b)x]$
$t = \tan\frac{1}{2}x \Rightarrow \tan x = \dfrac{2t}{1-t^2}$, $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$.	$t = \tanh\frac{1}{2}x \Rightarrow \tanh x = \dfrac{2t}{1+t^2}$, $\sinh x = \dfrac{2t}{1-t^2}$, $\cosh x = \dfrac{1+t^2}{1-t^2}$.
$t = \tan x \Rightarrow$ $\sin x = \dfrac{t}{\sqrt{1+t^2}}$, $\cos x = \dfrac{1}{\sqrt{1+t^2}}$	$t = \tanh x \Rightarrow$ $\sinh x = \dfrac{t}{\sqrt{1-t^2}}$, $\cosh x = \dfrac{1}{\sqrt{1-t^2}}$.

NOTE: Osborn's rule holds for *standard* trig identities and must not be applied indiscriminately. Thus, it won't work for identities such as

$$\frac{\sin x}{1+\cos x} + \frac{1-\cos x}{\sin x} = 2\tan\tfrac{1}{2}x.$$

Also, it should not be used for inverses, derivatives and integrals of the hyperbolic functions.

<u>Activity 1</u>

1. Prove that the point $(a\cosh t, b\sinh t)$ lies on one branch of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

[Hence the name hyperbolic functions !]

2. If $\tanh\dfrac{x}{2} = \dfrac{2}{3}$, find $\sinh 2x$ and $\operatorname{sech} 2x$.

3. If $\tanh x = k$, where $|k| < 1$, show that there are two possible values of $\tanh \frac{1}{2} x$, and that their product is $+1$.

4. Show that

$$(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx .$$

5. If $A + B + C = 0$, prove that

(i) $\tanh A + \tanh B + \tanh C + \tanh A \tanh B \tanh C = 0$;

(ii) $1 + \cosh A + \cosh B + \cosh C = 4 \cosh \frac{1}{2} A \cosh \frac{1}{2} B \cosh \frac{1}{2} C$.

6. Prove that if $\theta \neq 0$,

$$\cosh \theta + \cosh 3\theta + \ldots + \cosh(2n-1)\theta = \tfrac{1}{2} \sinh 2n\theta \operatorname{cosech} \theta .$$

8. By considering the Maclaurin series for e^x and e^{-x}, show that

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots ,$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots .$$

6.4 INVERSE HYPERBOLIC FUNCTIONS

In this section we consider the inverse of hyperbolic functions. We note that these functions can also be expressed in terms of logarithmic functions.

If $y = \sinh^{-1} x$, then $\sinh y = x$, and consequently $\cosh y = \sqrt{x^2 + 1}$. Hence, since

$$e^y = \sinh y + \cosh y ,$$

we have

$$e^y = x + \sqrt{x^2 + 1} ,$$

or, alternatively,

$$y = \sinh^{-1} x = \ln[x + \sqrt{x^2 + 1}] \quad \forall x .$$

If $y = \cosh^{-1} x$, then $\cosh y = x$, and consequently $\sinh y = \pm\sqrt{x^2 - 1}$. Hence,

$$e^y = x \pm \sqrt{x^2 - 1},$$

which gives

$$y = \cosh^{-1} x = \ln[x \pm \sqrt{x^2 - 1}], \quad x \geq 1.$$

Now
$$x - \sqrt{x^2 - 1} = \frac{1}{x + \sqrt{x^2 - 1}},$$

so that
$$\ln[x - \sqrt{x^2 - 1}] = -\ln[x + \sqrt{x^2 - 1}].$$

Hence,

$$y = \pm\ln[x + \sqrt{x^2 - 1}].$$

Referring to the graph of $\cosh x$, it will be seen that $\cosh^{-1} x$ is defined only when $x \geq 1$ and that it is two-valued. It is conventional to adopt the positive sign and to write

$$\cosh^{-1} x = \ln[x + \sqrt{x^2 - 1}], \quad x \geq 1.$$

If $y = \tanh^{-1} x$, then $x = \tanh y$, and hence

$$\operatorname{sech} y = \sqrt{1 - x^2},$$

so that

$$\cosh y = \frac{1}{\sqrt{1 - x^2}}, \quad \sinh y = \frac{x}{\sqrt{1 - x^2}}.$$

Again, using

$$e^y = \cosh y + \sinh y,$$

we have

$$e^y = \frac{1}{\sqrt{1 - x^2}} + \frac{x}{\sqrt{1 - x^2}} = \sqrt{\frac{1 + x}{1 - x}}, \quad |x| < 1.$$

Hence,

$$y = \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), \quad |x| < 1.$$

Example 7

$\sinh^{-1} 3 = \ln[3 + \sqrt{3^2 + 1}] = \ln[3 + \sqrt{10}]$;

$\cosh^{-1} 4 = \pm\ln[4 + \sqrt{4^2 - 1}] = \pm\ln[4 + \sqrt{15}]$;

$$\tanh^{-1}(-3/4) = \frac{1}{2}\ln\left(\frac{1-\frac{3}{4}}{1+\frac{3}{4}}\right) = \tfrac{1}{2}\ln\tfrac{7}{4}.$$

Example 8

Simplify $\tanh[2\tanh^{-1} k]$.

$$\tanh[2\tanh^{-1} k] = \frac{2\tanh(\tanh^{-1} k)}{1+\tanh^2(\tanh^{-1} k)} = \frac{2\tanh(\tanh^{-1} k)}{1+[\tanh(\tanh^{-1} k)]^2} = \frac{2k}{1+k^2}.$$

Example 9

Show that $\tanh^{-1}(\sin\theta) = \cosh^{-1}(\sec\theta)$ if $\sin\theta \neq 1$.

$$\tanh^{-1}(\sin\theta) = \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$$

$$= \frac{1}{2}\ln\left(\frac{(1+\sin\theta)^2}{1-\sin^2\theta}\right) \quad \text{[Multiplying top \& bottom by } (1+\sin\theta)\text{]}$$

$$= \ln\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

$$= \ln[\sec\theta + \tan\theta]$$

$$= \ln[\sec\theta + \sqrt{\sec^2\theta - 1}]$$

$$= \cosh^{-1}(\sec\theta).$$

Activity 2

1. Show that

(i) $\operatorname{cosech}^{-1} x = \ln\left(\dfrac{1 \pm \sqrt{1+x^2}}{x}\right), \quad x \neq 0$; [the alternative signs being taken according as $x > 0$ or $x < 0$.]

(ii) $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1$;

(iii) $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right), \quad |x| > 1$.

2. Prove that, for $0 < \theta < \pi$,

$$\sinh^{-1}(\cot\theta) = \ln\cot(\theta/2).$$

3. Given that $\sec\phi + \tan\phi = e^u$, show that

$$u = \sinh^{-1}(\tan\phi).$$

4. Prove that

(i) $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$;

(ii) $2\coth^{-1} e^x = \ln[\operatorname{cosech} x + \coth x]$;

(iii) If $\sinh^{-1} a = 2\cosh^{-1} b$, then $a^2 = 4b^2(b^2 - 1)$.

6.5 SOLUTION OF EQUATIONS

In this section we learn how to solve equations involving hyperbolic functions.

Example 10

Find the real solutions of the following equations:

(i) $\sinh 2x = 4$;

(ii) $2\cosh 3x = 5$;

(iii) $\tanh 4x = -\dfrac{1}{3}$;

(iv) $\operatorname{sech} 3x = \dfrac{1}{2}$;

(v) $\operatorname{cosech} \tfrac{1}{2}x = -5$.

Solution

(i) $\sinh 2x = 4 \Rightarrow 2x = \sinh^{-1} 4 \Rightarrow x = \tfrac{1}{2}\sinh^{-1} 4 = 1.047356$

Alternatively, $x = \tfrac{1}{2}\ln[4+\sqrt{17}]$.

Note: There are no units.

(ii) $2\cosh 3x = 5 \Rightarrow \cosh 3x = 5/2 \Rightarrow x = \tfrac{1}{3}\cosh^{-1}\tfrac{5}{2} = \pm 0.522266$

Alternatively, $x = \pm\tfrac{1}{3}\ln[(5+\sqrt{21})/2]$.

(iii) $\tanh 4x = -\tfrac{1}{3} \Rightarrow x = \tfrac{1}{4}\tanh^{-1}(-1/3) = -0.0866434$

Alternatively, $x = \tfrac{1}{4}\cdot\tfrac{1}{2}\ln\left(\dfrac{2/3}{4/3}\right) = -\tfrac{1}{8}\ln 2$.

(iv) $\operatorname{sech} 3x = 1/2 \Rightarrow \cosh 3x = 2 \Rightarrow x = \tfrac{1}{3}\cosh^{-1} 2 = \pm 0.438986$

Alternatively, $x = \pm\tfrac{1}{3}\ln[2+\sqrt{3}]$.

(v) $\operatorname{cosech} \tfrac{1}{2}x = -5 \Rightarrow \sinh \tfrac{1}{2}x = -1/5 \Rightarrow x = 2\sinh^{-1}(-1/5) = -0.397380$

Alternatively, $x = 2\ln\left(\dfrac{\sqrt{26}-1}{5}\right)$.

Note: For some problems, you may be asked to give your answers as logarithms, in which case you give the log equivalent of the inverse hyperbolic functions; otherwise use a calculator to evaluate the inverses directly and not compute the logs!

Example 11

Solve the following simultaneous equations:

$$2\sinh(x+y)=5$$
$$3\cosh(x-y)=4$$

We have on inverting

$$x+y=\sinh^{-1}\frac{5}{2}$$
$$x-y=\cosh^{-1}\frac{4}{3}$$

Adding, $2x=\sinh^{-1}\frac{5}{2}+\cosh^{-1}\frac{4}{3}\Rightarrow x=\frac{1}{2}(\sinh^{-1}\frac{5}{2}+\cosh^{-1}\frac{4}{3})=1.22123$

Subtracting, $2y=\sinh^{-1}\frac{5}{2}-\cosh^{-1}\frac{4}{3}\Rightarrow y=\frac{1}{2}(\sinh^{-1}\frac{5}{2}-\cosh^{-1}\frac{4}{3})=0.42593$

Example 12

Solve the equation $5\sinh x+3\cosh x=-3$.

Equations of the form $a\sinh x+b\cosh x=c$, where *a, b, c* are constants, are best solved by using the definitions of the hyperbolic functions, i.e., express the equations in terms of exponentials.

When $c=0$, we have immediately $\tanh x=-b/a$, and a real solution exists if $|b/a|<1$.

To come back to our equation, we have on using the definitions:

$$5\left(\frac{e^x-e^{-x}}{2}\right)+3\left(\frac{e^x+e^{-x}}{2}\right)=-3,$$

which on rearranging yields

$$4e^x-e^{-x}+3=0\,.$$

Multiplying throughout by e^x, we obtain a quadratic in e^x,

$$4e^{2x} + 3e^{x} - 1 = 0\,.$$

Letting $X = e^{x}$, we have

$$4X^{2} + 3X - 1 = 0\,,$$

giving $X = -1, \dfrac{1}{4}$.

Now, since $e^{x} > 0$, we have to discard the negative root. Hence, $e^{x} = 1/4$, so that

$$x = -\ln 4\,.$$

Note: We could have factorised directly: $4e^{2x} + 3e^{x} - 1 = (4e^{x} - 1)(e^{x} + 1)$.

Sometimes there are neater or more elegant ways to solve an equation, even though direct substitution might probably work.

<u>Example 13</u>

Solve the equation $7\sinh x \cosh x = -5$.

It's most tempting to substitute for the two hyperbolics; however, careful inspection shows a $\sinh 2x$ hidden in there. Hence our equation can be rewritten as

$$\frac{7}{2}(2\sinh x \cosh x) = -5$$

$$\Rightarrow \frac{7}{2}\sinh 2x = -5\,,$$

$$\Rightarrow \sinh 2x = -10/7\,.$$

$\therefore \quad x = \tfrac{1}{2}\sinh^{-1}(-10/7) = -0.577239$, or in terms of log, $x = \tfrac{1}{2}\ln\left(\dfrac{\sqrt{149} - 10}{7}\right)$.

<u>Example 14</u>

Solve the equation $\cosh 2x - 5\cosh x = 2$.

Using the identity $\cosh 2x = 2\cosh^{2} x - 1$, we obtain the following quadratic in $\cosh x$

$$2\cosh^2 x - 5\cosh x - 3 = 0\,,$$

which factorises as $\quad (2\cosh x + 1)(\cosh x - 3) = 0\,,$

so that either $\cosh x = -1/2$, which is not valid for real *x*, ($\cosh x \geq 1 \quad \forall x \in \Re$);

or $\quad \cosh x = 3$; hence

$$x = \cosh^{-1} 3 = \pm\ln[3 + \sqrt{8}]\,.$$

For some equations the factor formulae often provide a very useful alternative, if not the only means, to solving the equation. The formulae are given in the table above but are reproduced here for ease of reference.

$$\sinh A + \sinh B = 2\sinh\frac{A+B}{2}\cosh\frac{A-B}{2}$$

$$\sinh A - \sinh B = 2\cosh\frac{A+B}{2}\sinh\frac{A-B}{2}$$

$$\cosh A + \cosh B = 2\cosh\frac{A+B}{2}\cosh\frac{A-B}{2}$$

$$\cosh A - \cosh B = 2\sinh\frac{A+B}{2}\sinh\frac{A-B}{2}$$

Example 15

Solve the equation $\quad \cosh(x+1) + \cosh(x-1) = 6$.

The 3rd factor formula above gives

$$2\cosh x \cosh 1 = 6$$

$$\Rightarrow \quad \cosh x = \frac{3}{\cosh 1}\,.$$

Hence, $x = \cosh^{-1}\left(\frac{3}{\cosh 1}\right) = \pm 1.2841$

Example 16

Solve the equation $\sinh 5x - 4\sinh 3x + \sinh x = 0$.

Here, the sum of the sinh's and the fact that $3x = \dfrac{5x + x}{2}$, the semi-sum, suggests we use the factor formulae. Thus

$$\begin{aligned}\sinh 5x - 4\sinh 3x + \sinh x &= (\sinh 5x + \sinh x) - 4\sinh 3x \\ &= 2\sinh 3x \cosh 2x - 4\sinh 3x \text{ , (using the 1}^{\text{st}}\text{ formula)} \\ &= 2\sinh 3x(\cosh 2x - 2).\end{aligned}$$

Hence, our equation is simply

$$2\sinh 3x\,(\cosh 2x - 2) = 0\,.$$

Either $\qquad \sinh 3x = 0 \Rightarrow x = 0$,

Or $\qquad \cosh 2x = 2 \Rightarrow x = \tfrac{1}{2}\cosh^{-1} 2 = \pm\tfrac{1}{2}\ln(2+\sqrt{3})$.

Let's now consider a few simultaneous equations.

Example 17

Solve the equations

$$\begin{aligned}\sinh x \cosh y &= 5 \\ \cosh x \sinh y &= -8\end{aligned}$$

Looks complicated, but the trick is to add and subtract the two equations to obtain

$$\begin{aligned}\sinh(x+y) &= -3 \\ \sinh(x-y) &= 13\end{aligned}$$

Proceeding as in Example 11, we have $x = 0.720563, \quad y = -2.53901$.

Example 18

Solve the equations

$$\cosh x = 3\sinh y\,. \qquad \textbf{(5 (i))}$$

$$2\sinh x = 5 - 6\cosh y \qquad \textbf{(5 (ii))}$$

Here, none of the above techniques will work. We need to eliminate either *x* or *y*, and this is done through the identity $\cosh^2 A - \sinh^2 A = 1$. We therefore square both equations, *bearing in mind that the squaring process often introduces spurious roots*, so that the roots obtained should always be substituted back into the original equations to check if they are valid or not.

Proceeding,

$$\cosh^2 x = 9\sinh^2 y$$
$$4\sinh^2 x = 25 - 60\cosh y + 36\cosh^2 y$$

Clearly it's easier to get rid of *x*. We therefore express $\sinh^2 x$ in terms of $\cosh y$, in the last equation.

Now,

$$\begin{aligned}\sinh^2 x &= \cosh^2 x - 1\\ &= 9\sinh^2 y - 1\\ &= 9\cosh^2 y - 10\end{aligned}$$

The equation then reduces to

$$\cosh y = \frac{13}{12}$$
$$\Rightarrow y = \cosh^{-1}\frac{13}{12} = \pm\ln\frac{3}{2}.$$

The root $-\ln\frac{3}{2}$ is clearly not valid, since it will make $\cosh x$ negative in **(5 (i))**

Hence, we must have $y = +\ln\frac{3}{2}$.

To find *x*, we have from **(5 (ii))**

$$\begin{aligned}x &= \sinh^{-1}[(5 - 6\cosh y)/2]\\ &= \sinh^{-1}[-3/4]\\ &= \ln\tfrac{1}{2}\\ &= -\ln 2.\end{aligned}$$

So, $x = -\ln 2, \quad y = \ln\frac{3}{2}$.

Activity 3

1. Given that

$$\sinh x + 2\cosh x = k ,$$

where k is a positive constant,

(i) find the set of values of k for which a real solution of this equation exists;

(ii) solve the equation when $k = 2$.

2. Find the real solutions of the following equations:

(i) $\begin{array}{r} \tanh(x - y) = \dfrac{1}{4} \\ \operatorname{sech}(x + y) = \dfrac{1}{2} \end{array}$;

(ii) $5\sinh 3x \cosh 3x = -12$;

(iii) $2\cosh 2x = 5\sinh x + 3$;

(iv) $\cosh 5x + 6\sinh 4x = \cosh 3x$;

(v) $\begin{array}{l} \cosh x \cosh y = 5 \\ \sinh x \sinh y = -3 \end{array}$;

(vi) $\begin{array}{l} 2\cosh x + 5\sinh y = 0 \\ 2\sinh x + \cosh y = \dfrac{1}{2} \end{array}$.

(vii) $2\cosh x \cosh 2x = 3\sinh 2x$.

6.6 DIFFERENTIATION OF HYPERBOLIC FUNCTIONS

Now we consider the derivation of the hyperbolic functions.

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = +\sinh x\,,$$

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x\,.$$

$$\begin{aligned}\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) &= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}\\ &= \operatorname{sech}^2 x.\end{aligned}$$

Similarly, it can be shown that

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$$

$$\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x\,.$$

Note the signs for the derivatives of cosh and sech.

All the usual rules of differentiation apply.

<u>Example 19</u>

$$\frac{d}{dx}\tanh 3x = 3\operatorname{sech}^2 3x\,;$$

$$\begin{aligned}\frac{d}{dx}\tan 2x \cosh 3x &= \tan 2x \cdot 3\sinh 3x + \cosh 3x \cdot 2\sec^2 2x\\ &= 3\tan 2x \sinh 3x + 2\cosh 3x \sec^2 2x.\end{aligned}$$

$$\frac{d}{dx}\tan^{-1}(\cosh 5x) = \frac{1}{1+\cosh^2 5x}\cdot 5\sinh 5x\,.$$

$$\frac{d}{dx}\coth^2 3x \;=\; 2\coth 3x \cdot -\operatorname{cosech}^2 3x \cdot 3 \;=\; -6\coth 3x\ \operatorname{cosech}^2 3x\,.$$

Activity 4

1. Differentiate the following functions w.r.t. *x*:

   (i) $\sinh^3(2x+3)$;

   (ii) $x^3 s \tanh 5x$;

   (iii) $\ln(\text{sech } x + \tanh x)$;

   (iv) $\text{cosech}\sqrt{2-5x}$;

   (v) $\ln[\ln \tanh x]$;

   (vi) $(\cosh x)^{\sin x}$.

2. Show that the maximum value of $e^{-2x}\sinh x$ is $\dfrac{1}{3\sqrt{3}}$.

3. Verify that $y = A\cosh \omega x + B\sinh \omega x$ satisfies the differential equation

$$y'' - \omega^2 y = 0.$$

4. Obtain the following series using Maclaurin's expansion:

   (i) $\tanh x = x - \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \cdots$;

   (ii) $\text{sech } x = 1 - \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \cdots$.

To find the **derivatives of the inverse hyperbolic functions** we proceed in the same way as for the inverse trig functions.

Example 20

Find (i) $\dfrac{d}{dx}\sinh^{-1} x$;

(ii) $\dfrac{d}{dx}\tanh^{-1} x$.

(i) Let $y = \sinh^{-1} x$. Then $x = \sinh y$ and $\dfrac{dx}{dy} = \cosh y = \sqrt{1+x^2}$.

$$\therefore \quad \frac{d}{dx}\sinh^{-1} x = \frac{dy}{dx} = 1\bigg/\frac{dx}{dy} = \frac{1}{\sqrt{1+x^2}}.$$

(ii) Let $y = \tanh^{-1} x$. Then $x = \tanh y$ and $\dfrac{dx}{dy} = \operatorname{sech}^2 y = 1 - x^2$

$$\therefore \quad \frac{d}{dx}\tanh^{-1} x = \frac{dy}{dx} = 1\bigg/\frac{dx}{dy} = \frac{1}{1-x^2}, \quad |x| < 1.$$

Likewise, you can show that

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$$

$$\frac{d}{dx}\coth^{-1} x = \frac{1}{1-x^2}, \quad |x| > 1$$

$$\frac{d}{dx}\operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$$

$$\frac{d}{dx}\operatorname{cosech}^{-1} x = \frac{-1}{|x|\sqrt{x^2+1}}, \quad x \neq 0.$$

Note the various restrictions on *x*.

Example 21

$$\frac{d}{dx}\cosh^{-1} e^x = \frac{1}{\sqrt{e^{2x}-1}} \cdot \frac{d}{dx}(e^x) = \frac{e^x}{\sqrt{e^{2x}-1}}.$$

$$\frac{d}{dx}\, 2\tanh^{-1}(\tan \tfrac{1}{2}x) = 2\,\frac{1}{1-\tan^2 \frac{1}{2}x} \cdot \frac{d}{dx}(\tan \tfrac{1}{2}x)$$

$$= 2\,\frac{1}{1-\tan^2 \frac{1}{2}x}\sec^2 \tfrac{1}{2}x \cdot \frac{1}{2} = \frac{\sec^2 \frac{1}{2}x}{1-\tan^2 \frac{1}{2}x} = \sec x.$$

$$\frac{d}{dx}\operatorname{sech}^{-1}(\cos x) = \frac{-1}{\cos x\sqrt{1-\cos^2 x}} \cdot \frac{d}{dx}(\cos x)$$

$$= \frac{\sin x}{\cos x \sin x} = \sec x.$$

$$\frac{d}{dx}\tanh^{-1}(\sin x)=\frac{1}{1-\sin^2 x}\cdot\frac{d}{dx}\left(\sin x\right)=\sec x\,.$$

[By the way, we see that the integral of sec *x* can be expressed in several ways! The same is true for the integral of sech *x*, as we'll see in the next section.]

Activity 5

1. Differentiate the following functions w.r.t. *x*:

   (i) $\cosh^{-1}\sec x$;

   (ii) $\coth^{-1}\sec x$;

   (iii) $\sinh^{-1}(\sin\sqrt{x})$;

   (iv) $\sinh^{-1}2x+\tanh^{-1}\sqrt{x}+\operatorname{sech}^{-1}x^2$.

2. If $\ln y=\sinh^{-1}x$, prove that

$$(1+x^2)\,y''+x\,y'-y=0\,.$$

3. Show that

$$\sinh^{-1}x=x-\frac{x^3}{6}+\frac{3x^5}{40}+\cdots;$$

$$\tanh^{-1}x=x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots.$$

   How about the Maclaurin series of $\cosh^{-1}x$?

6.7 INTEGRATION OF HYPERBOLIC FUNCTIONS

From the derivatives of the hyperbolic functions in the last section, we obtain the following results:

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$\int \operatorname{cosech}^2 x \, dx = -\coth x + C$$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + C \, .$$

We shall now find the integrals of the other four hyperbolic functions.

<u>Example 22</u>

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx = \ln \cosh x + C$$

$$\int \coth x \, dx = \int \frac{\cosh x}{\sinh x} \, dx = \ln \sinh x + C$$

$$\begin{aligned} \int \operatorname{cosech} x \, dx &= \int \frac{1}{\sinh x} \, dx \\ &= \int \frac{1}{2 \sinh \frac{1}{2} x \cosh \frac{1}{2} x} \, dx \qquad \text{(Dividing top \& bottom by } \cosh^2 \tfrac{1}{2} x \text{)} \\ &= \int \frac{\operatorname{sech}^2 \frac{1}{2} x}{2 \tanh \frac{1}{2} x} \, dx \\ &= \ln \tanh \tfrac{1}{2} x + C \end{aligned}$$

$\int \operatorname{sech} x \, dx$ can be found in several ways.

(i)

$$\begin{aligned}\int \operatorname{sech} x \, dx &= \int \frac{2}{e^x + e^{-x}} \, dx \\ &= 2\int \frac{e^x}{e^{2x}+1} \, dx \qquad [\text{Let } u = e^x] \\ &= 2\tan^{-1} e^x + C.\end{aligned}$$

(ii)

$$\begin{aligned}\int \operatorname{sech} x \, dx &= \int \frac{\cosh x}{\cosh^2 x} \, dx \\ &= \int \frac{\cosh x}{1+\sinh^2 x} \, dx \qquad [\text{Let } u = \sinh x] \\ &= \tan^{-1} \sinh x + C.\end{aligned}$$

(iii)

$$\begin{aligned}\int \operatorname{sech} x \, dx &= \int \frac{\operatorname{sech}^2 x}{\operatorname{sech} x} \, dx \\ &= \int \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} \, dx \qquad [\text{Let } u = \tanh x] \\ &= \sin^{-1} \tanh x + C.\end{aligned}$$

(iv)

$$\begin{aligned}\int \operatorname{sech} x \, dx &= \int \frac{dx}{\cosh x} \\ &= \int \frac{dx}{\cosh^2 \frac{1}{2}x + \sinh^2 \frac{1}{2}x} = \int \frac{\operatorname{sech}^2 \frac{1}{2}x}{1+\tanh^2 \frac{1}{2}x} \, dx \quad [\text{Let } u = \tanh \tfrac{1}{2}x] \\ &= 2\tan^{-1} \tanh \tfrac{1}{2}x + C.\end{aligned}$$

All these results are equivalent, the first one being the most popular. Note that when evaluating the definite integral, the answer will be in radians, due to the presence of the inverse trig functions.

Example 23

$$\int_1^2 \operatorname{sech} x \, dx = \left[2\tan^{-1} e^x\right]_1^2 = 2[\tan^{-1} e^2 - \tan^{-1} e^1] = 0.435991$$

NB: Calculator in **RADIAN mode**. Try with the other 3 results to see if you get the same answer.

Most hyperbolic integrals are worked out as their trigonometric counterparts.

Example 24

$$\int \sinh^2 x\, dx = \tfrac{1}{2}\int (\cosh 2x - 1)\, dx = \tfrac{1}{4}\sinh 2x - \tfrac{1}{2}x + C$$

$$\int \tanh^2 3x\, dx = \int (1 - \operatorname{sech}^2 3x)\, dx = x - \tfrac{1}{3}\tanh 3x + C$$

$$\int \operatorname{sech}^4 x\, dx = \int (1 - \tanh^2 x)\operatorname{sech}^2 x\, dx = \tanh x - \tfrac{1}{3}\tanh^3 x + C$$

$$\int e^x \cosh x\, dx = \int e^x\left(\frac{e^x + e^{-x}}{2}\right) dx = \tfrac{1}{2}\int (e^{2x} + 1)\, dx = \tfrac{1}{4}e^{2x} + \tfrac{1}{2}x + C$$

$$\int x\sinh x\, dx = x\cosh x - \int \cosh x\, dx = x\cosh x - \sinh x + C \quad \text{[Integrating by parts]}$$

$$\begin{aligned}\int \sinh^{-1} x\, dx &= \int 1\cdot \sinh^{-1} x\, dx \\ &= x\sinh^{-1} x - \int \frac{x}{\sqrt{x^2+1}}\, dx \qquad \left[\text{Recall } \int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)}\,\right] \\ &= x\sinh^{-1} x - \sqrt{x^2+1} + C.\end{aligned}$$

$$\begin{aligned}\int \tanh^{-1} x\, dx &= \int 1\cdot \tanh^{-1} x\, dx \\ &= x\tanh^{-1} x - \int \frac{x}{1-x^2}\, dx \\ &= x\tanh^{-1} x + \tfrac{1}{2}\ln(1-x^2) + C, \quad |x| < 1.\end{aligned}$$

Activity 6

1. Find the following integrals:

(i) $\int \cosh^2 x\, dx$;

(ii) $\int \cosh^3 x\, dx$;

(iii) $\int \sinh^3 x \cosh^2 x\, dx$;

(iv) $\int \sinh x \ln \cosh^2 x\, dx$;

(v) $\int x^2 \cosh x \, dx$;

(vi) $\int \coth^2 3x \, dx$;

(vii) $\int_2^4 \operatorname{sech} 3x \, dx$;

(viii) $\int \operatorname{sech}^3 x \, dx$;

(ix) $\int_2^3 \cosh^{-1} x \, dx$;

(x) $\int \sinh \ln x \, dx$.

6.8 INTEGRALS IN TERMS OF INVERSE HYPERBOLIC FUNCTIONS

We now consider some standard integrals expressible in terms of the inverse hyperbolic functions given as

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C, \quad x > a > 0$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} + C, \quad |x| < a$$

Example 25

$$\int \frac{dx}{\sqrt{x^2 + 16}} = \sinh^{-1} \frac{x}{4} + C; \qquad \int \frac{dx}{\sqrt{x^2 - 9}} = \cosh^{-1} \frac{x}{3} + C;$$

$$\int \frac{dx}{49 - x^2} = \frac{1}{7} \tanh^{-1} \frac{x}{7} + C; \qquad \int \frac{dx}{6 - x^2} = \frac{1}{\sqrt{6}} \tanh^{-1} \frac{x}{\sqrt{6}} + C.$$

Often a little manipulation is required to cast the integral into the appropriate standard form.

Example 26

$$\int \frac{dx}{\sqrt{3x^2+8}} = \int \frac{dx}{\sqrt{3(x^2+\frac{8}{3})}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{x^2+\frac{8}{3}}} = \frac{1}{\sqrt{3}}\sinh^{-1}\frac{x}{\sqrt{\frac{8}{3}}}+C\,;$$

$$\int \frac{dx}{\sqrt{5x^2-7}} = \int \frac{dx}{\sqrt{5(x^2-\frac{7}{5})}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{x^2-\frac{7}{5}}} = \frac{1}{\sqrt{5}}\cosh^{-1}\frac{x}{\sqrt{\frac{7}{5}}}+C\,;$$

$$\int \frac{dx}{6-7x^2} = \int \frac{dx}{7(\frac{6}{7}-x^2)} = \frac{1}{7}\int \frac{dx}{\frac{6}{7}-x^2}$$

$$= \frac{1}{7}.\frac{1}{\sqrt{\frac{6}{7}}}\tanh^{-1}\frac{x}{\sqrt{\frac{6}{7}}}+C = \frac{1}{\sqrt{42}}\tanh^{-1}\left(\sqrt{\frac{7}{6}}\;x\right)+C.$$

$$\int \frac{dx}{x^2-25} = -\int \frac{dx}{25-x^2} = -\frac{1}{5}\tanh^{-1}\frac{x}{5}+C$$

$$\int \frac{dx}{3x^2-5} = -\int \frac{dx}{5-3x^2} = -\int \frac{dx}{3(\frac{5}{3}-x^2)} = -\frac{1}{3}\int \frac{dx}{\frac{5}{3}-x^2}$$

$$= -\frac{1}{3}.\frac{1}{\sqrt{\frac{5}{3}}}\tanh^{-1}\frac{x}{\sqrt{\frac{5}{3}}}+C = -\frac{1}{\sqrt{15}}\tanh^{-1}\left(\sqrt{\tfrac{3}{5}}\;x\right)+C.$$

Activity 7

Find the following integrals:

(i) $\int \frac{dx}{\sqrt{x^2+3}}$;

(ii) $\int \frac{dx}{\sqrt{x^2-3}}$;

(iii) $\int_3^4 \frac{du}{\sqrt{u^2-5}}$;

(iv) $\int_2^3 \frac{du}{\sqrt{u^2+15}}$;

(v) $\int \frac{dx}{\sqrt{3x^2+4}}$;

(vi) $\int_2^3 \frac{dt}{\sqrt{4t^2-5}}$;

(vii) $\int \frac{dy}{3-y^2}$;

(viii) $\int \frac{dx}{4-5x^2}$;

(ix) $\int \frac{dy}{5y^2-6}$;

(x) $\int_{1/4}^{1/3} \frac{dv}{8-9v^2}$.

Let us now consider integrals with the full quadratic ($b \neq 0$) in the denominator:

Type A $\int \frac{dx}{ax^2+bx+c}$

Type B $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

For **Type A**, if quadratic is factorisable, then use partial fractions. The answer will involve logs. If quadratic is not factorisable, then complete the square. The integral will reduce to either of the following standard forms, with a multiplicative constant:

$$\int \frac{dx}{a^2+X^2}\text{; } \int \frac{dx}{a^2-X^2}\text{ ; } \int \frac{dx}{X^2-a^2},$$

where $X = x+\alpha$. On using the substitution $u = x+\alpha$, the first one is just $\frac{1}{a}\tan^{-1}\frac{X}{a}+C$, while the last two are $\frac{1}{a}\tanh^{-1}\frac{X}{a}+C$ and $-\frac{1}{a}\tanh^{-1}\frac{X}{a}+C$ respectively.

Example 27

$$\int \frac{dx}{5+3x-x^2} = \int \frac{dx}{\frac{29}{4}-(x-\frac{3}{2})^2} = \frac{1}{\sqrt{\frac{29}{4}}}\tanh^{-1}\frac{x-\frac{3}{2}}{\sqrt{\frac{29}{4}}}+C$$

$$= \frac{2}{\sqrt{29}}\tanh^{-1}\frac{2x-3}{\sqrt{29}}+C.$$

$$\int \frac{dx}{6+5x-2x^2} = \int \frac{dx}{2\,[\frac{73}{16}-(x-\frac{5}{4})^2\,]} = \frac{1}{2}\int \frac{dx}{\frac{73}{16}-(x-\frac{5}{4})^2}$$

$$= \frac{1}{2}.\frac{1}{\sqrt{\frac{73}{16}}}\tanh^{-1}\frac{x-\frac{5}{4}}{\sqrt{\frac{73}{16}}}+C$$

$$= \frac{2}{\sqrt{73}}\tanh^{-1}\frac{4x-5}{\sqrt{73}}+C.$$

$$\int \frac{dx}{3x^2+4x-5} = \int \frac{dx}{3\,[(x+\frac{2}{3})^2-\frac{19}{9}]} = -\frac{1}{3}\int \frac{dx}{\frac{19}{9}-(x+\frac{2}{3})^2}$$

$$= -\frac{1}{3}.\frac{1}{\sqrt{\frac{19}{9}}}\tanh^{-1}\frac{x+\frac{2}{3}}{\sqrt{\frac{19}{9}}}+C$$

$$= -\frac{1}{\sqrt{19}}\tanh^{-1}\frac{3x+2}{\sqrt{19}}+C.$$

For **Type B** integrals, always complete the square. The resulting integral will be either of the following standard integrals, with a multiplicative constant:

$$\int \frac{dx}{\sqrt{a^2-X^2}};\quad \int \frac{dx}{\sqrt{X^2+a^2}};\quad \int \frac{dx}{\sqrt{X^2-a^2}}\,.$$

The first is $\sin^{-1}\frac{X}{a}+C$, the 2nd is $\sinh^{-1}\frac{X}{a}+C$, while the last is $\cosh^{-1}\frac{X}{a}+C$.

Example 28

$$\int \frac{dx}{\sqrt{x^2-2x+17}} = \int \frac{dx}{\sqrt{(x-1)^2+16}} = \sinh^{-1}\frac{x-1}{4}+C$$

$$\int \frac{dx}{\sqrt{3x^2+4x-5}} = \int \frac{dx}{\sqrt{3[(x+\frac{2}{3})^2-\frac{19}{9}]}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{(x+\frac{2}{3})^2-\frac{19}{9}}}$$

$$= \frac{1}{\sqrt{3}} \cosh^{-1} \frac{x+\frac{2}{3}}{\sqrt{\frac{19}{9}}} + C$$

$$= \frac{1}{\sqrt{3}} \cosh^{-1} \frac{3x+2}{\sqrt{19}} + C.$$

$$\int \frac{dx}{\sqrt{5x^2-3x+4}} = \int \frac{dx}{\sqrt{5[(x-\frac{3}{10})^2+\frac{71}{100}]}} = \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{(x-\frac{3}{10})^2+\frac{71}{100}}}$$

$$= \frac{1}{\sqrt{5}} \sinh^{-1} \frac{x-\frac{3}{10}}{\sqrt{\frac{71}{100}}} + C$$

$$= \frac{1}{\sqrt{5}} \sinh^{-1} \frac{10x-3}{\sqrt{71}} + C.$$

<u>Activity 8</u>

Find the following integrals:

(i) $\displaystyle\int \frac{dx}{9-x-x^2}$;

(ii) $\displaystyle\int \frac{dx}{2+5x-2x^2}$;

(iii) $\displaystyle\int_{1/3}^{1/2} \frac{dy}{3y+2-3y^2}$;

(iv) $\displaystyle\int \frac{dx}{\sqrt{x^2-x-3}}$;

(v) $\displaystyle\int_1^2 \frac{dx}{\sqrt{1+5x+2x^2}}$;

(vi) $\displaystyle\int_2^3 \frac{dt}{\sqrt{3t^2-4t-2}}$.

Miscellaneous Integrals

Integrals involving $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$ can often be dealt with most easily by the substitutions $x = a\cosh u$ or $x = a\sinh u$ respectively. Note that the substitutions $x = a\sec\theta$ and $x = a\tan\theta$ can also be used.

Example 29

$$\int \sqrt{9+x^2}\ dx$$

Letting $x = 3\sinh u$, so that $dx = 3\cosh u\ du$, we obtain

$$\begin{aligned}\int \sqrt{9+9\sinh^2 u}\ 3\cosh u\ du = 9\int \cosh^2 u\ du &= 9\int \tfrac{1}{2}\ (1+\cosh 2u)\ du \\ &= \tfrac{9}{2}[u+\tfrac{1}{2}\sinh 2u]+C \\ &= \tfrac{9}{2}[u+\sinh u\cosh u]+C \\ &= \frac{9}{2}\left(\sinh^{-1}\frac{x}{3}+\frac{x}{3}\sqrt{1+\frac{x^2}{9}}\right)+C \\ &= \frac{9}{2}\sinh^{-1}\frac{x}{3}+\frac{1}{2}x\sqrt{9+x^2}+C.\end{aligned}$$

Example 30

$$\int \frac{dx}{x^2\sqrt{x^2-a^2}}$$

Putting $x = a\cosh u,\quad dx = a\sinh u\ du,\quad x^2-a^2 = a^2\sinh^2 u$, we obtain

$$\int \frac{a\sinh u\ du}{a^3\cosh^2 u\sinh u} = \frac{1}{a^2}\int \operatorname{sech}^2 u\ du = \frac{1}{a^2}\tanh u + C = \frac{\sqrt{x^2-a^2}}{a^2 x}+C\,.$$

Example 31

$$\int \sqrt{2x^2-3x+1}\ dx$$

Call this I.

We first complete the square. This gives $I = \sqrt{2}\int \sqrt{(x-\tfrac{3}{4})^2 - \tfrac{1}{16}}\, dx$.

Now, letting $x - \tfrac{3}{4} = \tfrac{1}{4}\cosh\theta, \quad dx = \tfrac{1}{4}\sinh\theta\, d\theta$, we have

$$\begin{aligned} I &= \sqrt{2}\int \tfrac{1}{4}\sinh\theta . \tfrac{1}{4}\sinh\theta\, d\theta \\ &= \frac{\sqrt{2}}{16}\int \sinh^2\theta\, d\theta \\ &= \frac{\sqrt{2}}{16}\left[\frac{1}{4}\sinh 2\theta - \frac{\theta}{2}\right] + C. \end{aligned}$$

In terms of *x*:

$$x - \tfrac{3}{4} = \tfrac{1}{4}\cosh\theta \Rightarrow \cosh\theta = 4x - 3 \Rightarrow \theta = \cosh^{-1}(4x-3)$$

$$\sinh 2\theta = 2\sinh\theta\cosh\theta = 2\sqrt{(4x-3)^2 - 1}\,(4x-3)$$

Hence,

$$\begin{aligned} I &= \frac{\sqrt{2}}{16}\left[\frac{1}{4}\cdot 2\sqrt{(4x-3)^2 - 1}\,(4x-3) - \frac{\cosh^{-1}(4x-3)}{2}\right] + C \\ &= \frac{1}{8}(4x-3)\sqrt{2x^2 - 3x + 1} - \frac{\sqrt{2}}{32}\cosh^{-1}(4x-3) + C. \end{aligned}$$

6.9 SUMMARY

In this unit, we have studied a new class of functions called hyperbolic functions and learned about their differentiation and integration and their respective inverses. We have looked into the solution of equations in terms of these functions. Finally, we have learnt how to work out integrals in terms of inverse hyperbolic functions.

6.11 ANSWERS TO ACTIVITIES

Activity 1

2. $\frac{312}{25}, \quad \frac{25}{313}$

Activity 3

1. (i) $k \geq \sqrt{3}$;

   (ii) $x = 0, -\ln 3$.

2. (i) $x = 0.786185, \quad y = 0.530773$;

   (ii) $x = -0.37874$;

   (iii) $x = 1.152652, -0.174504$;

   (iv) $x = -1.81845, 0.$

   (v) $x = \pm\, 2.04281,\ y = \mp\, 0.725851$;

   (vi) $x = \ln \tfrac{3}{4},\ y = \ln \tfrac{2}{3}$;

   (vii) $x = \ln[\tfrac{1}{2}(1+\sqrt{5}\,)], \quad \ln[1+\sqrt{2}]$.

Activity 4

1. (i) $8\cosh(2x+3)\sinh^3(2x+3)$;

(ii) $5x^3\,\mathrm{sec\,h}^2 5x + 3x^2\tan 5x$;

(iii) $\dfrac{\operatorname{sech} x - \tanh x}{1+\sinh x}$;

(iv) $\dfrac{5\coth\sqrt{2-5x}\,\operatorname{cosech}\sqrt{2-5x}}{2\sqrt{2-5x}}$;

(v) Not defined;

(vi) $(\cosh x)^{\sin x}[\cos x\ln\cosh x + \sin x\tanh x]$.

Activity 5

1. (i) $\sec x$;

(ii) $-\operatorname{cosec} x$;

(iii) $\dfrac{\cos\sqrt{x}}{2\sqrt{x[1+\sin^2\sqrt{x}]}}$;

(iv) $\dfrac{1}{2(1-x)\sqrt{x}} + \dfrac{2}{\sqrt{1+4x^2}} - \dfrac{2}{x\sqrt{1-x^4}}$.

3. $\cosh^{-1} x$ has no Maclaurin expansion for real x. Recall $\cosh^{-1} x$ exists for $x \geq 1$.

Activity 6

(i) $\frac{1}{4}(2x+\sinh 2x)+C$;

(ii) $\frac{1}{12}(9\sinh x+\sinh 3x)+C$;

(iii) $\frac{1}{5}\cosh^5 x-\frac{1}{3}\cosh^3 x+C$;

(iv) $\cosh x(\ln\cosh^2 x-2)+C$;

(v) $(x^2+2)\sinh x-2x\cosh x+C$;

(vi) $x-\frac{1}{3}\coth 3x+C$;

(vii) 0.0016484 ;

(viii) $\tan^{-1}(\tanh\frac{1}{2}x)+\frac{1}{2}\operatorname{sech} x\tanh x+C$;

(ix) 1.55795;

(x) $\frac{1}{4}(x^2-2\ln x)+C$.

Activity 7

(i) $\sinh^{-1}\dfrac{x}{\sqrt{3}}+C$;

(ii) $\cosh^{-1}\dfrac{x}{\sqrt{3}}+C$;

(iii) $\ln\left(\dfrac{4+\sqrt{11}}{5}\right)$;

(iv) 0.216878;

(v) $\dfrac{1}{3}\sinh^{-1}\dfrac{\sqrt{3}}{2}x+C$;

(vi) 0.229037;

(vii) $\frac{1}{\sqrt{3}}\tanh^{-1}\frac{y}{\sqrt{3}}+C$;

(viii) $\frac{1}{2\sqrt{5}}\tanh^{-1}\left(\frac{\sqrt{5}}{2}x\right)+C$;

(ix) $-\frac{1}{\sqrt{30}}\tanh^{-1}\left(\sqrt{\tfrac{5}{6}}\;y\right)+C$;

(x) 0.0115309

<u>Activity 8</u>

(i) $\frac{2}{\sqrt{37}}\tanh^{-1}\frac{2x+1}{\sqrt{37}}+C$;

(ii) $\frac{2}{\sqrt{41}}\tanh^{-1}\frac{4x-5}{\sqrt{41}}+C$;

(iii) 0.0612296;

(iv) $\ln\left(2x-1+2\sqrt{x^2-x-3}\right)+C$;

(v) 0.281945;

(vi) 0.415472.