UNIT 5 VECTOR ALGEBRA

Unit Structure

5.1 OVERVIEW

In this unit, you learn about a new type of vector multiplication, the vector product and its extension to other products.

5.2 LEARNING OBJECTIVES

When you have successfully completed this unit, you should be able to do the following:

- Work with the various products involving vectors.
- Solve vector equations.
- Differentiate vectors.

5.2 INTRODUCTION

At this stage of your studies, you must have come across vectors. You know how to add and subtract vectors, and how to multiply two vectors (dot product) to obtain a scalar. Make sure you are familiar with these operations before going any further.

In this unit, you learn about a new type of vector multiplication called the vector product. However, before embarking on this subject, it would be appropriate to review briefly the coordinate system we shall be working with, and recall certain concepts.

In the Cartesian system, the reference directions are perpendicular (or *orthogonal*) to each other, and the unit vectors in the Ox, Oy and Oz directions are called $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ respectively (Figure 5.1). The caret (or, hat ^) on top of a vector indicates that it's a unit vector, i.e., its length (or magnitude) is one unit.

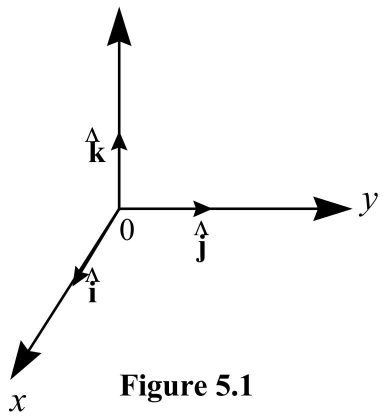


Figure 5.1

It does not matter which unit vector is the "upward vertical", but in this course, the vector $\hat{\mathbf{k}}$ is chosen.

By convention, we always choose a set of **right-handed** reference directions (or, right-handed system of coordinates). This means that if you stand at the origin with your arms stretched out in the direction of the positive x-axis and y-axis, your right arm will be the one along the x-axis, and your left arm will be stretched ahead along the y-axis. Your head will then point in the direction of the positive z-axis.

5.2.1 PROJECTIONS

Let **a** and **b** be two vectors drawn so that they have a common initial point, as shown in Figure 5.2. If we drop a perpendicular from the tip of **a** to the line determined by **b**, we determine a vector called the vector projection of **a** onto **b**, which is labeled **p** in Figure 5.2.

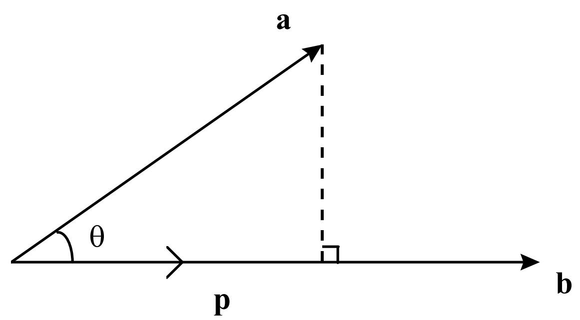


Figure 5.2

The scalar projection of **a** onto **b** (also called the component of **a** along **b**) is the length of the vector projection, and so is denoted by $|\mathbf{p}|$. Let θ be the acute angle between **a** and **b**.

Then,

$$|\mathbf{p}| = |\mathbf{a}|\cos\theta = |\mathbf{a}|\left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|.|\mathbf{b}|}\right) = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{b}|}$$

If θ is obtuse, its cosine is negative and $|\mathbf{p}| = -|\mathbf{a}|\cos\theta$.

In order to find a formula for the vector projection, we note that it is the scalar component of **a** in the direction of **b**.

If θ is acute, then the vector projection has length $|\mathbf{a}|\cos\theta$ and has direction $\mathbf{b}/|\mathbf{b}|$

(i.e. $\hat{\mathbf{b}}$).

If θ is obtuse, the vector projection has length $-|\mathbf{a}|\cos\theta$ and has direction $-\hat{\mathbf{b}}$.

In either case,

$$\mathbf{p} = (|\mathbf{a}|\cos\theta)\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a.b}}{|\mathbf{b}|}\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a.b}}{|\mathbf{b}|^2}\mathbf{b} = \left(\frac{\mathbf{a.b}}{\mathbf{b.b}}\right)\mathbf{b}$$

So we have :

Scalar projection of **a** onto **b** : $\left|\frac{\mathbf{a.b}}{|\mathbf{b}|}\right|$

Vector projection of **a** in the direction of **b** : $\left(\frac{\mathbf{a.b}}{\mathbf{b.b}}\right)\mathbf{b}$

Example 1

Find the scalar and vector projections of $\mathbf{a} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ onto $\mathbf{b} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Solution

Scalar projection : $\left|\frac{(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}).(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})}{|2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}|}\right| = \frac{17}{3}$.

Vector projection : $\left(\frac{(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}).(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})}{(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}).(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})}\right)(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = \frac{17}{3}(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$.

5.2.2 DIRECTION COSINES

Consider a vector, **a**, in three dimensional space. This vector will form angles with the *x*-axis (α), the *y*-axis (β) and the *z*-axis (γ). These angles are called **direction angles** and the cosines of these angles are called **direction cosines**. This is illustrated in fig. 5.3.

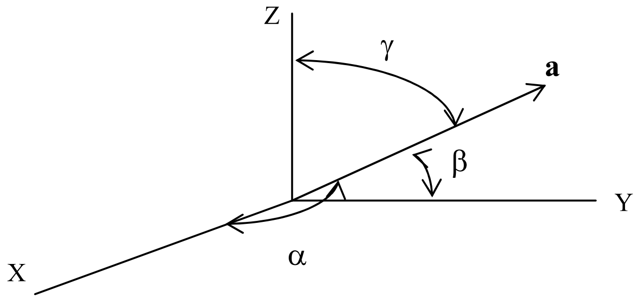


Figure 5.3

The formulae for the direction cosines are :

$$\cos\alpha = \frac{\mathbf{a}\,.\,\mathbf{i}}{|\,\mathbf{a}\,|} = \frac{\mathbf{a}_1}{|\,\mathbf{a}\,|},\ \cos\beta = \frac{\mathbf{a}\,.\,\mathbf{j}}{|\,\mathbf{a}\,|} = \frac{\mathbf{a}_2}{|\,\mathbf{a}\,|}\ \text{ and }\ \cos\gamma = \frac{\mathbf{a}\,.\,\mathbf{k}}{|\,\mathbf{a}\,|} = \frac{\mathbf{a}_3}{|\,\mathbf{a}\,|},$$

where **i**, **j** and **k** are the standard basis vectors. Note that $\mathbf{a}\,.\,\mathbf{i} = a_1$, $\mathbf{a}\,.\,\mathbf{j} = a_2$ and $\mathbf{a}\,.\,\mathbf{k} = a_3$.

Also,

1. The vector $\mathbf{u} = (\cos\alpha, \cos\beta, \cos\gamma)$ is a unit vector.
2. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
3. $\mathbf{a} = |\,\mathbf{a}\,|(\cos\alpha, \cos\beta, \cos\gamma)$

5.3 THE VECTOR PRODUCT

The vector product of two vectors **a** and **b** , written $\mathbf{a} \times \mathbf{b}$ (read **a** cross **b**), or $\mathbf{a} \wedge \mathbf{b}$ (read **a** vec **b**) is defined as

$$\mathbf{a} \wedge \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}} \qquad (1)$$

θ is the smaller angle between the positive directions of the two vectors drawn at a common origin and measured from the direction of **a**; $\hat{\mathbf{n}}$ is a unit vector normal to the plane containing **a** and **b** such that $\mathbf{a}, \mathbf{b}, \hat{\mathbf{n}}$ form a right-handed system. (Figure 5.4)

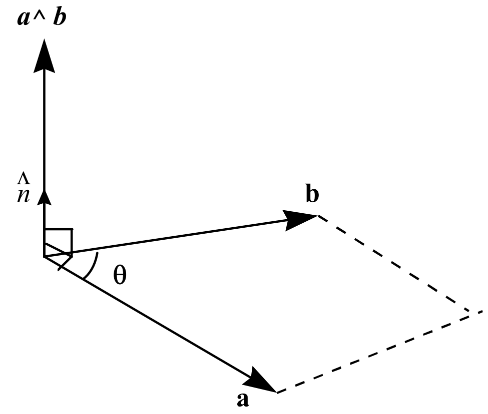


Figure 5.4

The direction of $\hat{\mathbf{n}}$ is obtained by the **right-hand rule**, as follows. Place the palm of your right hand along **a** with your fingers pointing in the direction of **a**, and curl your fingers toward **b.** Then your thumb points in the direction of $\hat{\mathbf{n}}$ (and hence that of $\mathbf{a} \wedge \mathbf{b}$).

According to the definition, we see that the vector product of two non-zero vectors can be a zero vector only when $\theta = 0$, i.e. when the two vectors are parallel. However, if, $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$, then there are 3 distinct possibilities:

$$\mathbf{a} = \mathbf{0} \ , \ \mathbf{b} = \mathbf{0} \ , \ \mathbf{a} \parallel \mathbf{b} \, .$$

In particular, it follows that $\mathbf{a} \wedge \mathbf{a} = \mathbf{0}$.

Again, from the definition ,

$$\mathbf{b} \wedge \mathbf{a} = -(\mathbf{a} \wedge \mathbf{b}) \qquad (2)$$

Hence the vector product, unlike the scalar product, is *not* commutative.

It can be verified from the definition of the vector product that

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c} \ , \qquad (3)$$

$$(\mathbf{a} + \mathbf{b}) \wedge \mathbf{c} = (\mathbf{a} \wedge \mathbf{c}) + (\mathbf{b} \wedge \mathbf{c})$$

$$(\lambda\mathbf{a}) \wedge (\mu\mathbf{b}) = \lambda\mu(\mathbf{a} \wedge \mathbf{b}) \ , \qquad (4)$$

where λ and μ are scalars.

The magnitude of $\mathbf{a} \wedge \mathbf{b}$ is the same as the area of a parallelogram with sides **a** and **b.** Hence the area of the triangle with sides **a** and **b** is $\frac{1}{2}|\mathbf{a} \wedge \mathbf{b}|$.

It is important to note that the vector equation

$$\mathbf{a} \wedge \mathbf{c} = \mathbf{b} \wedge \mathbf{c} \ , \qquad (5)$$

does ***not*** imply $\mathbf{a} = \mathbf{b}$. Consider

$$\mathbf{a} = \mathbf{b} + \lambda\mathbf{c} \ , \qquad (6)$$

where λ is a scalar parameter. Then, on taking cross product with **c** on both sides, we have

$$\begin{aligned} \mathbf{a} \wedge \mathbf{c} &= (\mathbf{b} + \lambda\mathbf{c}) \wedge \mathbf{c} \\ &= \mathbf{b} \wedge \mathbf{c} + \lambda\ \mathbf{c} \wedge \mathbf{c} \\ &= \mathbf{b} \wedge \mathbf{c} \qquad [\ \mathbf{c} \wedge \mathbf{c} = \mathbf{0}\ ] \end{aligned}$$

Consequently (5) is satisfied by any vector with the form (6). In other words, **a** and **b** may differ by any vector parallel to **c**.

Let us now apply the vector product to the orthogonal unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. We then have

$$\hat{\mathbf{i}} \wedge \hat{\mathbf{i}} = \hat{\mathbf{j}} \wedge \hat{\mathbf{j}} = \hat{\mathbf{k}} \wedge \hat{\mathbf{k}} = \mathbf{0} \;\; ,$$

$$\hat{\mathbf{i}} \wedge \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \wedge \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \wedge \hat{\mathbf{i}} = \hat{\mathbf{j}} .$$

These are most easily remembered by noting the cyclic order of the unit vectors.

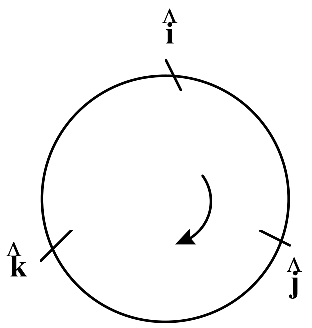


Figure 5.5

From Figure 5.5, we also have (on going anti-clockwise)

$$\hat{\mathbf{i}} \wedge \hat{\mathbf{k}} = -\hat{\mathbf{j}}, \quad \hat{\mathbf{k}} \wedge \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \quad \hat{\mathbf{j}} \wedge \hat{\mathbf{i}} = -\hat{\mathbf{k}} \;\; .$$

<u>**Example 2**</u>

Evaluate $(5\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \wedge (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$.

<u>**Solution**</u>

$$\begin{aligned}
(5\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \wedge (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) &= 5\hat{\mathbf{i}} \wedge \hat{\mathbf{i}} + 5\hat{\mathbf{i}} \wedge \hat{\mathbf{j}} + 10\hat{\mathbf{i}} \wedge \hat{\mathbf{k}} - 6\hat{\mathbf{j}} \wedge \hat{\mathbf{i}} - 6\hat{\mathbf{j}} \wedge \hat{\mathbf{j}} - 12\hat{\mathbf{j}} \wedge \hat{\mathbf{k}} \\
&\quad + 3\hat{\mathbf{k}} \wedge \hat{\mathbf{i}} + 3\hat{\mathbf{k}} \wedge \hat{\mathbf{j}} + 6\hat{\mathbf{k}} \wedge \hat{\mathbf{k}} \\
&= 5(\mathbf{0}) + 5(\hat{\mathbf{k}}) + 10(-\hat{\mathbf{j}}) - 6(-\hat{\mathbf{k}}) - 6(\mathbf{0}) - 12(\hat{\mathbf{i}}) \\
&\quad + 3(\hat{\mathbf{j}}) + 3(-\hat{\mathbf{i}}) + 6(\mathbf{0}) \\
&= -15\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 11\hat{\mathbf{k}} .
\end{aligned}$$

Activity 1

Find the vector product $\mathbf{a} \wedge \mathbf{b}$ when

(i) $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}},\ \mathbf{b} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$,

(ii) $\mathbf{a} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}},\ \mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

A much easier (and quicker) way of calculating the vector product is to write it in determinant form thus

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} , \qquad (7)$$

where $\mathbf{a} = a_1\ \hat{\mathbf{i}} + a_2\ \hat{\mathbf{j}} + a_3\ \hat{\mathbf{k}}$ and $\mathbf{b} = b_1\ \hat{\mathbf{i}} + b_2\ \hat{\mathbf{j}} + b_3\ \hat{\mathbf{k}}$, and evaluate the determinant.

For example, to find $(5\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 7\hat{\mathbf{k}}) \wedge (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ we write

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & -6 & 7 \\ 1 & 1 & 2 \end{vmatrix} .$$

On expanding this determinant, we obtain the result $-19\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 11\hat{\mathbf{k}}$ as before.

Note : $(5\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 7\hat{\mathbf{k}}).(-19\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 11\hat{\mathbf{k}}) = 0$, and $(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}).(-19\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 11\hat{\mathbf{k}}) = 0$.

That the answer is zero is no fluke; it follows from the definition of the vector product. Remember we said the vector product is perpendicular to the other two vectors.

So, if $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$, then $\mathbf{a.c} = 0$ and $\mathbf{b.c} = 0$.

This can therefore be used as a check when evaluating vector products.

An alternative way to evaluate a vector product

We give an alternative method for evaluating the vector product of two three dimensional vectors. You can use this method or the classic method of cofactors presented in the notes related to this tutorial. It should be noted that the method works only on 3x3 determinants.

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

This method says to take the determinant as listed above and then copy the first two columns onto the end as shown below.

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \begin{matrix} \mathbf{i} & \mathbf{j} \\ a_1 & a_2 \\ b_1 & b_2 \end{matrix}$$

We now have three diagonals that move from left to right and three diagonals that move from right to left. We multiply along each diagonal and add those that move from left to right and subtract those that move from right to left.

<u>Example 3</u>

If $\mathbf{a}$ = (2, 1, -1) and $\mathbf{b}$ = (-3, 4, 1) compute $\mathbf{a} \times \mathbf{b}$.

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ -3 & 4 & 1 \end{vmatrix} \begin{matrix} \mathbf{i} & \mathbf{j} \\ 2 & 1 \\ -3 & 4 \end{matrix}$$

= $\mathbf{i}$ (1)(1) + $\mathbf{j}$ (-1)(-3) + $\mathbf{k}$ (2)(4) – $\mathbf{j}$ (2)(1) – $\mathbf{i}$ (-1)(4) – $\mathbf{k}$ (1)(-3) = 5 $\mathbf{i}$ + $\mathbf{j}$ + 11 $\mathbf{k}$

Notice that switching the order of the vectors in the cross product simply changed all the signs in the result. Note as well that this means that the two cross products will point in exactly opposite directions since they only differ by a sign.

<u>Example 4</u>

Find the area of the triangle with vertices at $P(1,2,2), Q(2,-1,1), R(-1,2,3)$.

<u>Solution</u>

Let $\mathbf{PQ} = \mathbf{a}$ and $\mathbf{PR} = \mathbf{b}$. Then area of triangle PQR is

$$\tfrac{1}{2}|\mathbf{a} \wedge \mathbf{b}| = \tfrac{1}{2}\left|(\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) \wedge (-2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})\right|$$
$$= \tfrac{1}{2}\left|-3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 6\hat{\mathbf{k}}\right| = \tfrac{1}{2}\sqrt{46}$$

5.4 ANGLE BETWEEN TWO VECTORS

We can also use the vector product to calculate the angle between two vectors. We have

$$\mathbf{a} \wedge \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \ \hat{\mathbf{n}}$$

Taking modulus throughout yields

$$|\mathbf{a} \wedge \mathbf{b}| = |\mathbf{a}| \ |\mathbf{b}| \ |\sin\theta| \ |\hat{\mathbf{n}}| \ ,$$

$$\therefore \qquad |\sin\theta| = \frac{|\mathbf{a} \wedge \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} \ , \qquad \text{[Note } |\hat{\mathbf{n}}| = 1 \text{ by definition]}$$

Solution

From Activity 1(i), we have $\mathbf{a} \wedge \mathbf{b} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$, hence

$$|\mathbf{a} \wedge \mathbf{b}| = \sqrt{35} \text{ and } \therefore \ |\sin\theta| = \frac{\sqrt{35}}{\sqrt{6}\sqrt{14}} ;$$

whence $\theta = 40.2°$.

Activity 2

Compute the angle between the two vectors in Activity 1(ii), using the vector product.

5.5 TRIPLE PRODUCTS

Products involving three vectors can be formed according to each of the following three types :

(i) $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$

(ii) $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$

(iii) $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$.

5.5.1 TYPE ONE

The first type is merely the product of a scalar (**a.b**) and a vector **c**. The product is therefore a vector along **c** with the magnitude $abc\cos\theta$, where θ is the angle between **a** and **b.**

5.5.2 SCALAR TRIPLE PRODUCT- TYPE TWO

The second type is the scalar product of **a** and ($\mathbf{b} \wedge \mathbf{c}$). Since the product is a scalar, it is called the scalar triple product. Using the determinantal representation of $\mathbf{b} \wedge \mathbf{c}$ as given by (7) we have,

$$\mathbf{a}.(\mathbf{b} \wedge \mathbf{c}) = (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) \;.\; \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) \;.\; [(b_2c_3 - b_3c_2)\hat{\mathbf{i}} + (b_3c_1 - b_1c_3)\hat{\mathbf{j}} + (b_1c_2 - b_2c_1)\hat{\mathbf{k}}]$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{8}$$

Now, the interchange of two rows or two columns of a determinant merely changes the sign of the determinant.
Hence

$$\mathbf{a} \,.\, (\mathbf{b} \wedge \mathbf{c}) = \mathbf{b} \,.\, (\mathbf{c} \wedge \mathbf{a}) = \mathbf{c} \,.\, (\mathbf{a} \wedge \mathbf{b}) \tag{9}$$

and the other products

$$\mathbf{a} \,.\, (\mathbf{c} \wedge \mathbf{b}) = \mathbf{b} \,.\, (\mathbf{a} \wedge \mathbf{c}) = \mathbf{c} \,.\, (\mathbf{b} \wedge \mathbf{a}) \tag{10}$$

have the same magnitude but opposite signs to those given by (9).

In particular, since the dot product of two vectors is commutative, we see from (9) that the dot and the $\wedge$ in a scalar triple product can be swapped.

Thus

$$\begin{aligned} &\mathbf{a} \,.\, (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \,.\, \mathbf{c} = \mathbf{b} \,.\, (\mathbf{c} \wedge \mathbf{a}) = (\mathbf{b} \wedge \mathbf{c}) \,.\, \mathbf{a} \\ &\mathbf{c} \,.\, (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{c} \wedge \mathbf{a}) \,.\, \mathbf{b} = -\, \mathbf{a} \,.\, (\mathbf{c} \wedge \mathbf{b}) \text{, etc.} \end{aligned} \tag{11}$$

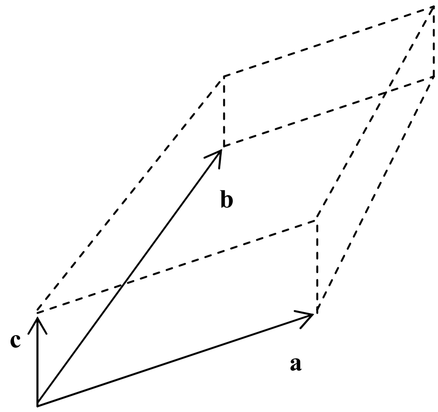
Note that brackets can be omitted since $\mathbf{a} \,.\, \mathbf{b} \wedge \mathbf{c}$ can have no meaning except $\mathbf{a} \,.\, (\mathbf{b} \wedge \mathbf{c})$ and also the cyclic permutation of the vectors.

Activity 3

Given $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{c} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, verify results (11).

5.3.1 GEOMETRIC APPLICATION OF THE CROSS PRODUCT

Suppose we have three vectors **a**, **b** and **c** and we form the three dimensional figure as shown below.



Parallelepiped formed by **a**, **b** and **c**

The area of the front parallelogram of the above shape is given by

$$\text{Area} = |\,\mathbf{a} \wedge \mathbf{b}\,|$$

and the volume of the parallelpiped, the whole three dimensional object, is given by,

$$\text{Volume} = |\mathbf{a} \,.\, (\mathbf{b} \times \mathbf{c})|$$

The absolute value bars are required since the quantity could be negative and volume isn't negative. We can use this volume fact to determine if three vectors lie in the same plane or not. If three vectors lie in the same plane then the volume of the parallelepiped will be zero.

Hence we deduce that ***the condition for*** **a**, **b**, **c** ***to be coplanar*** is

$$\mathbf{a} \,.\, \mathbf{b} \wedge \mathbf{c} = 0 \qquad (12)$$

Example 5

Find the value of α for which the area of the parallelogram formed by the vectors $\mathbf{a} = \hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = \alpha\,\hat{\mathbf{i}} + \alpha\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$, is the least.

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ \alpha & \alpha & 1 \end{vmatrix} = (2+\alpha)\,\hat{\mathbf{i}} - (1+\alpha)\,\hat{\mathbf{j}} - \alpha\,\hat{\mathbf{k}}$$

The area of the parallelogram is given by A = $|\,\mathbf{a} \wedge \mathbf{b}\,| = \sqrt{(2+\alpha)^2 + (1+\alpha)^2 + \alpha^2}$.

Minimizing A with respect to α, we get $\dfrac{dA}{d\alpha} = 6\dfrac{(1+\alpha)}{\sqrt{(2+\alpha)^2 + (1+\alpha)^2 + \alpha^2}} = 0$.

Solving the latter equation, we get $\alpha = 0$.

Activity 4

Show that the vectors **u**, **v**, **w** are coplanar , where

$\mathbf{u} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{v} = -5\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, $\mathbf{w} = -\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$.

Activity 5

Find α if the vectors **p** , **q** , **r** are to be coplanar,

$\mathbf{p} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, $\mathbf{q} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{r} = \alpha\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

5.3.2 THE VECTOR TRIPLE PRODUCT: TYPE THREE

The third type of triple product

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} \text{ and } \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$$

is clearly a vector. Note that now brackets are important as these two products are in general *not* equal.

Consider the vector triple product $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$. Since it is perpendicular to $(\mathbf{a} \wedge \mathbf{b})$, which is itself perpendicular to the plane of **a** and **b**, and is also perpendicular to **c** it follows that $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$ is a vector which is in the plane of **a** and **b**, and perpendicular to **c**. Thus this product must be expressible as a linear combination of **a** and **b** ,

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}.$$

In order to determine the scalars α and β, we first take the scalar product of the equation with **c**,

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} \,.\, \mathbf{c} = (\alpha\mathbf{a} + \beta\mathbf{b}) \,.\mathbf{c}$$

i.e. $$0 = \alpha\mathbf{a.c} + \beta\mathbf{b.c}$$

Let $\beta = \lambda\mathbf{a.c}$, then $\alpha = -\lambda\mathbf{b.c}$, and hence we know that

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \lambda\,[\,(\mathbf{a.c})\mathbf{b} - (\mathbf{b.c})\mathbf{a}]$$

for some scalar λ. Substitution of $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, and so forth, finally yields λ=1. Hence,

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = (\mathbf{a} \,.\, \mathbf{c})\,\mathbf{b} - (\mathbf{b} \,.\, \mathbf{c})\,\mathbf{a} \qquad (13)$$

In a similar way, the vector $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ is seen to be expressible as a linear combination of **b** and **c**, and the identity

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \,.\, \mathbf{c})\,\mathbf{b} - (\mathbf{a} \,.\, \mathbf{b})\,\mathbf{c} \qquad (14)$$

can be deduced from (13).
Consequently we see that $\mathbf{a} \wedge (\,\mathbf{b} \wedge \mathbf{c})$ is neither numerically equal to, nor parallel to $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$.
For example

$$\hat{\mathbf{j}} \wedge (\hat{\mathbf{j}} \wedge \hat{\mathbf{i}}) = \hat{\mathbf{j}} \wedge -\hat{\mathbf{k}} = -\hat{\mathbf{i}}$$

but $$(\hat{\mathbf{j}} \wedge \hat{\mathbf{j}}) \wedge \hat{\mathbf{i}} = \mathbf{0} \qquad \because\ \hat{\mathbf{j}} \wedge \hat{\mathbf{j}} = \mathbf{0}$$

Activity 6

Evaluate $(\mathbf{a} \wedge \mathbf{b})\wedge\mathbf{c}$ and $\mathbf{a} \wedge(\mathbf{b} \wedge \mathbf{c})$ where **a**, **b**, **c** are as in Activity 3.

5.6 RESOLVING A VECTOR INTO TWO COMPONENT VECTORS

From (14), we have, on putting $\mathbf{c} = \mathbf{a}$,

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{a}) = (\mathbf{a} \,.\, \mathbf{a})\mathbf{b} - (\mathbf{a} \,.\, \mathbf{b})\mathbf{a}$$

$$= \mathbf{a}^2\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{a}$$

Hence $$\mathbf{b} = \frac{(\mathbf{a}.\mathbf{b})\mathbf{a}}{\mathbf{a}^2} + \frac{\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{a})}{\mathbf{a}^2} \qquad (15)$$

Thus the vector **b** has been resolved into two component vectors, one parallel to **a**, viz. $(\mathbf{a}.\mathbf{b})\mathbf{a} / \mathbf{a}^2$, and the other perpendicular to **a**.

5.6.1 MULTIPLE PRODUCTS

Vector products involving more than three vectors are readily evaluated in terms of the products considered above.

For example, the product

$$(\mathbf{a} \wedge \mathbf{b}) \,.\, (\mathbf{c} \wedge \mathbf{d})$$

can be considered, as the scalar triple product of **a**, **b** and $\mathbf{c} \wedge \mathbf{d}$. Hence putting $\mathbf{u} = \mathbf{c} \wedge \mathbf{d}$, we have

$$\mathbf{a} \wedge \mathbf{b} \,.\, \mathbf{u} = \mathbf{a} \,.\, \mathbf{b} \wedge \mathbf{u}$$

$$= \mathbf{a} \,.\, [\mathbf{b} \wedge (\mathbf{c} \wedge \mathbf{d})]$$

$$= \mathbf{a} \,.\, [(\mathbf{b} \,.\, \mathbf{d})\mathbf{c} - (\mathbf{b} \,.\, \mathbf{c})\mathbf{d}]$$

Therefore

$$(\mathbf{a} \wedge \mathbf{b}) \,.\, (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{a} \,.\, \mathbf{c})\,(\mathbf{b} \,.\, \mathbf{d}) - (\mathbf{a} \,.\, \mathbf{d})\,(\mathbf{b} \,.\, \mathbf{c}) \qquad (16)$$

This result is known as Lagrange's identity.

Activity 7

Show that $(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d}) = [(\mathbf{a} \wedge \mathbf{b}) \,.\, \mathbf{d}]\, \mathbf{c} - [(\mathbf{a} \wedge \mathbf{b}) \,.\, \mathbf{c}]\, \mathbf{d}$

$$= [(\mathbf{c} \wedge \mathbf{d}) \,.\, \mathbf{a}]\, \mathbf{b} - [(\mathbf{c} \wedge \mathbf{d}) \,.\, \mathbf{b}]\, \mathbf{a}$$

.

At this stage it's worth pointing out that such combinations as
ab , $(\mathbf{a} \wedge \mathbf{b})\mathbf{c}$, and $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ have been left undefined. These quantities known as dyadics are beyond the scope of the present course.

5.7 SOLUTION OF VECTOR EQUATIONS

By making use of the scalar, vector and triple products, we can simplify certain vector equations and hence obtain the unknown(s). Note that it is only possible to divide by a scalar, never by a vector.

Example 6

Solve the vector equation $\mathbf{x} \wedge \mathbf{a} = \mathbf{b} - \mathbf{x}$ for $\mathbf{x}$.

Solution

Scalar multiplication by $\mathbf{a}$ gives

$$\mathbf{a} \,.\, \mathbf{x} \wedge \mathbf{a} = \mathbf{a} \,.\, \mathbf{b} - \mathbf{a} \,.\, \mathbf{x}$$

i.e.
$$\mathbf{a} \,.\, \mathbf{b} - \mathbf{a} \,.\, \mathbf{x} = 0 \qquad \text{(i)}$$

Vector multiplication by $\mathbf{a}$ yields

$$\mathbf{a} \wedge (\mathbf{x} \wedge \mathbf{a}) = \mathbf{a} \wedge \mathbf{b} - \mathbf{a} \wedge \mathbf{x}$$

$$= \mathbf{a} \wedge \mathbf{b} + \mathbf{b} - \mathbf{x} \quad \text{(from given eqn.)}$$

Expanding the l.h.s. we have

$$(\mathbf{a} \,.\, \mathbf{a})\mathbf{x} - (\mathbf{a} \,.\, \mathbf{x})\mathbf{a} = \mathbf{a} \wedge \mathbf{b} + \mathbf{b} - \mathbf{x}$$

or
$$(\mathbf{a} \,.\, \mathbf{a})\mathbf{x} - (\mathbf{a} \,.\, \mathbf{b})\mathbf{a} = \mathbf{a} \wedge \mathbf{b} + \mathbf{b} - \mathbf{x} \quad \text{(on using (i))}$$

The vector $\mathbf{x}$ is now multiplied by scalar factors only; hence

$$\mathbf{x} = \frac{\mathbf{a} \wedge \mathbf{b} + \mathbf{b} + (\mathbf{a}.\mathbf{b})\mathbf{a}}{1 + \mathbf{a}^2}$$

Activity 8

Solve the vector equation $3\mathbf{x} + [\mathbf{x} \wedge (\hat{\mathbf{j}} + \hat{\mathbf{k}})] = 6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

Activity 9

X is an unknown vector which satisfies the equations

$$\mathbf{a} \wedge \mathbf{X} = \mathbf{b}\,, \quad \mathbf{a} \,.\, \mathbf{X} = \lambda\,,$$

where **a** and **b** are known vectors, and λ is a known scalar.

Prove that $\mathbf{X} = (\mathbf{b} \wedge \mathbf{a} + \lambda \mathbf{a}) / \mathbf{a}^2$.

5.8 DIFFERENTIATION OF VECTORS

If the definition of a vector quantity **V** involves a scalar parameter t, the derivative of the vector $\mathbf{V}(t)$ with respect to t is defined as the limit

$$\frac{d\mathbf{V}(t)}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{V}(t+\delta t) - \mathbf{V}(t)}{\delta t}$$

when that limit exists.

From this definition, it follows that the derivative of the product of a scalar $s(t)$ and a vector $\mathbf{V}(t)$ is given by the familiar law

$$\frac{d}{dt}(s\mathbf{V}) = \frac{ds}{dt}\mathbf{V} + s\frac{d\mathbf{V}}{dt}.$$

Hence, if a vector is expressed in terms of its components along the fixed coordinate axes,

$$\mathbf{V} = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}},$$

there follows,

$$\frac{d\mathbf{V}}{dt} = \frac{df}{dt}\hat{\mathbf{i}} + \frac{dg}{dt}\hat{\mathbf{j}} + \frac{dh}{dt}\hat{\mathbf{k}},$$

since $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are constant vectors.

It follows also, from the definition, that the derivative of a product involving two or more vectors is defined as in the corresponding scalar case if the order of the factors is retained.

Thus we obtain the following formulae

$$\frac{d}{dt}(\mathbf{a}.\mathbf{b}) = \frac{d\mathbf{a}}{dt}.\mathbf{b} + \mathbf{a}.\frac{d\mathbf{b}}{dt}$$

$$\frac{d}{dt}(\mathbf{a} \wedge \mathbf{b}) = \frac{d\mathbf{a}}{dt} \wedge \mathbf{b} + \mathbf{a} \wedge \frac{d\mathbf{b}}{dt}$$

$$\frac{d}{dt}(\mathbf{a}.\mathbf{b} \wedge \mathbf{c}) = \frac{d\mathbf{a}}{dt}.\mathbf{b} \wedge \mathbf{c} + \mathbf{a}.\frac{d\mathbf{b}}{dt} \wedge \mathbf{c} + \mathbf{a}.\mathbf{b} \wedge \frac{d\mathbf{c}}{dt}$$

$$\frac{d}{dt}[\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})] = \frac{d\mathbf{a}}{dt} \wedge (\mathbf{b} \wedge \mathbf{c}) + \mathbf{a} \wedge (\frac{d\mathbf{b}}{dt} \wedge \mathbf{c}) + \mathbf{a} \wedge (\mathbf{b} \wedge \frac{d\mathbf{c}}{dt})$$

Once again, note the order of the terms.
<u>The derivative of a vector of constant length, but changing direction</u> , <u>is perpendicular to the vector</u>.
This may be seen by noticing that if **a** has a constant length then

$$\frac{d}{dt}(\mathbf{a}.\mathbf{a}) = \frac{d}{dt}a^2 = 0,$$

and also
$$\frac{d}{dt}(\mathbf{a}.\mathbf{a}) = 2\mathbf{a}.\frac{d\mathbf{a}}{dt}.$$

These results are compatible only if either $\frac{d\mathbf{a}}{dt} = \mathbf{0}$ or $\frac{d\mathbf{a}}{dt}$ is perpendicular to **a**.

[N.B.: $\mathbf{a} \neq \mathbf{0}$ by assumption].

<u>**Example 7**</u>

Let $\mathbf{r} = \alpha \cos\omega t\,\hat{\mathbf{i}} + \beta \sin\omega t\,\hat{\mathbf{j}}$ where α, β, ω are all scalar constants. Then

$$\frac{d\mathbf{r}}{dt} = -\alpha\omega \sin\omega t\,\hat{\mathbf{i}} + \beta\omega \cos\omega t\,\hat{\mathbf{j}}$$

$$\frac{d^2\mathbf{r}}{dt^2} = -\alpha\omega^2 \cos\omega t\,\hat{\mathbf{i}} - \beta\omega^2 \sin\omega t\,\hat{\mathbf{j}}$$

$$= -\omega^2\mathbf{r}$$

or
$$\frac{d^2\mathbf{r}}{dt^2} + \omega^2\mathbf{r} = \mathbf{0} \qquad (\#)$$

This shows that $\mathbf{r} = \alpha\cos\omega t\,\hat{\mathbf{i}} + \beta\sin\omega t\,\hat{\mathbf{j}}$ is a solution of the vector differential equation (#).

Note that if t is time and $\mathbf{r}(t)$ denotes displacement, then velocity is given by

$$\mathbf{v}(t) = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} \qquad [\text{speed} = |\text{velocity}| = |\dot{\mathbf{r}}|\ ],$$

whilst acceleration is given by

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2}.$$

Activity 10

a If $\mathbf{A} = 2\,\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{C} = 3\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k}$, find the projection of $\mathbf{A} - \mathbf{B}$ in the direction of $\mathbf{C}$.

b(i) Determine if the three vectors $\mathbf{a} = (1,4,-7)$, $\mathbf{b}\ (2,-1,4)$ and $\mathbf{c}\ (0,-9,18)$ lie in the same plane or not.

(ii) Find the angle $\alpha = \angle BAC$ of the triangle ABC (Fig. 1) whose vertices are A(1,0,1), B(2,-1,1), C(-2,1,0).

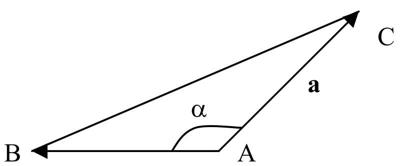


Figure 1

c Consider the vectors given by $\mathbf{v_1} = 2a\,\mathbf{i} - 3\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{v_2} = 3\,a\,\mathbf{i} + a\,\mathbf{j} - 3\,\mathbf{k}$.

(i) Show that for $a = \dfrac{1-\sqrt{13}}{2}$ and $a = \dfrac{1+\sqrt{13}}{2}$, $\mathbf{v_1}$ and $\mathbf{v_2}$ are perpendicular.

(ii) For what value of a lying between $\dfrac{1-\sqrt{13}}{2}$ and $\dfrac{1+\sqrt{13}}{2}$ is $|\mathbf{v_1}\cdot\mathbf{v_2}|$ a maximum?

(iii) Find a such the normal to the plane containing $\mathbf{v_1}$ and $\mathbf{v_2}$ lies only in the X-Y plane shown in Fig. 2.

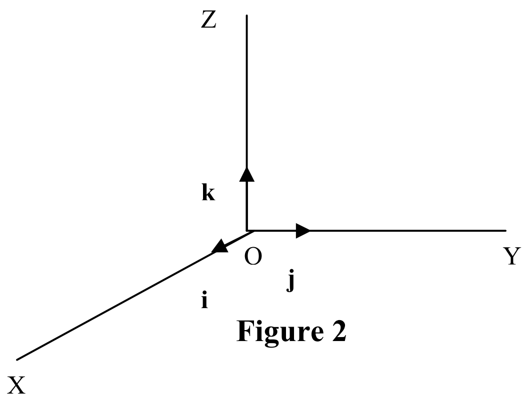


Figure 2

5.9 ANSWERS TO ACTIVITIES

Activity 1

(i) $5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ (ii) $-4\hat{\mathbf{i}} + \hat{\mathbf{j}} + 7\hat{\mathbf{k}}$

Activity 2

$90°$

Activity 3

20

Activity 5

$\alpha = -13/3$

Activity 6

$16\hat{\mathbf{i}} + 17\hat{\mathbf{j}} - 21\hat{\mathbf{k}}$; $40\hat{\mathbf{i}} - 20\hat{\mathbf{j}} - 20\hat{\mathbf{k}}$

Activity 8

$$\mathbf{x} = \frac{1}{33}(45\hat{\mathbf{i}} + 37\hat{\mathbf{j}} - 26\hat{\mathbf{k}})\,]$$

Activity 10

a $\dfrac{-11\,\mathbf{C}}{29}$

b(i) They lie in the same plane, since the volume of the corresponding parallelipiped is zero.;

(ii) $\alpha = 148^\circ\ 31'$.

c (ii) $a = \dfrac{1}{4}$ (iii) a = 0 and $a = \dfrac{-9}{2}$.