UNIT 3 LINEAR EQUATIONS, EIGENVALUES AND EIGENVECTORS OF A MATRIX

Unit Structure

3.0 OVERVIEW

The beginning of this Unit deals with the consistency and inconsistency of systems of linear equations. The homogeneous and non-homogeneous cases are treated separately. The computation of matrix eigenvalues and eigenvectors is then treated in detail. Finally, similarity transformations and diagonalisation of a matrix are introduced.

3.1 LEARNING OBJECTIVES

By the end of this Unit, you should be able to do the following:

1. Determine when a homogeneous system of linear equations has the trivial solution as the only solution, or has an infinite number of solutions.

2. Determine whether a non-homogeneous of linear equations is consistent or not and if it is consistent whether it has a unique solution or an infinite number of solutions.

3. Compute eigenvalues and eigenvectors of a matrix and use the Cayley-Hamilton theorem.

4. Determine whether a matrix is diagonalisable or not.

3.2 HOMOGENEOUS AND NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

Consider a system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad & \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

where a_{ij} and b_i are scalars for i =1,…, m, j=1,…, n.

In matrix notation, the system of linear equations (1) may be written as

$$A\mathbf{x} = \mathbf{b} \tag{2}$$

where $A = \left[a_{ij}\right]_{m \times n}$, $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ and $\mathbf{b} = [b_1, b_2, \ldots, b_m]^T$.

When the system defined above has at least one solution, it is said to be ***consistent***. Otherwise, the system is said to be ***inconsistent***.

Homogeneous Systems

If $\mathbf{b} = \mathbf{0}$ in (2), the system of linear equations is said to be homogeneous. A homogeneous system of linear equations is always consistent since $A\mathbf{0} = \mathbf{0}$. The solution $\mathbf{x} = \mathbf{0}$, i.e., $x_1 = x_2 = \ldots. = x_n = 0$, is called the *trivial* solution. If $A\mathbf{x} = \mathbf{0}$ has a unique solution, it must be $\mathbf{x} = \mathbf{0}$.

Note: For Gaussian elimination, it is only necessary to reduce the coefficient matrix A (not the augmented system $[\ A \quad \mathbf{0}\ ]$) because a column of zeros is unaltered by row operations.

Result 1

If $m = n$ and A is a non-singular matrix in (2) (i.e., rank of $A = n$), then the system $A\mathbf{x} = \mathbf{0}$ has the trivial solution $\mathbf{x} = \mathbf{0}$, i.e., $x_1 = x_2 = \ldots = x_n = 0$ as the **only** solution.

Result 2

If rank of $A < n$ in (2), then the system $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions.

Example 1

Consider

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 0 \\ -15x_1 + 6x_2 - 5x_3 &= 0 \\ 5x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}$$

Show that the following homogeneous system of linear equations has the trivial solution as the only solution.

Solution

Let $A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$. We use Result 1 to show that rank of A is 3 so that A is then invertible. Now,

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \begin{matrix} \text{..........}R_1 \\ \text{..........}R_2 \\ \text{..........}R_3 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix} \begin{matrix} \text{..........}R_1 \\ \text{..........}R_2 + 5R_1 \\ \text{..........}R_3 - 5/3\,R_1 \end{matrix}$$

$$\rightarrow A^* = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{matrix} \text{.......}R_1 \\ \text{.......}R_2 \\ \text{.......}R_3 + 1/3\,R_2 \end{matrix}$$

Rank of A = 3 since $|A| = |A^*| = 1 \neq 0$.

Example 2

Show that the following homogeneous system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + 3x_2 - 2x_3 + x_4 &= 0 \\ 2x_1 - 3x_3 + 2x_4 &= 0 \\ 3x_1 + 3x_2 + 3x_4 &= 0 \end{aligned}$$

has an infinite number of solutions.

Solution

Clearly, this system has the trivial solution ($x_1 = x_2 = x_3 = x_4 = 0$) as one solution.

Now,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{pmatrix} \begin{array}{l} \ldots\ldots\ldots\ldots\ldots\ldots \mathrm{R}_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \mathrm{R}_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \mathrm{R}_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots \mathrm{R}_4 \end{array}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} \begin{array}{l} \ldots\ldots\ldots\ldots\ldots \mathrm{R}_1 \\ \ldots\ldots\ldots\ldots\ldots \mathrm{R}_2 - \mathrm{R}_1 = \mathrm{R}_2^1 \\ \ldots\ldots\ldots\ldots\ldots \mathrm{R}_3 - 2\mathrm{R}_1 = \mathrm{R}_3^1 \\ \ldots\ldots\ldots\ldots\ldots \mathrm{R}_4 - 3\mathrm{R}_1 = \mathrm{R}_4^1 \end{array}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} \begin{matrix} \ldots\ldots\ldots\ldots\ldots\ldots R_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots R_2^1 \\ \ldots\ldots\ldots\ldots\ldots\ldots R_3^1 + R_2^1 = R_3^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots R_4^1 \end{matrix}$$

$$\rightarrow A^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \ldots\ldots\ldots\ldots\ldots\ldots R_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots R_2^1 \\ \ldots\ldots\ldots\ldots\ldots\ldots R_3^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots R_4^1 - 3/8\, R_3^2 \end{matrix}$$

The rank of A is 3 since A^* has 3 nonzero rows. Since rank of $A = 3 < 4$, using Result 2, we conclude that this system of linear equations has an infinite number of solutions.

In fact, the leading entries 1, 2, -8 correspond to the leading variables x_1, x_2 and x_3 respectively. Since x_4 is a non-leading variable, we introduce the parameter $x_4 = t$ and then express the leading variables x_1, x_2 and x_3 in terms of the parameter t. Thus, after substitution, we obtain $x_1 = -t$, $x_2 = 0$ and $x_3 = 0$ as the general solution to the system of linear equations under consideration. In fact,

$$\mathbf{x} = \begin{pmatrix} -t \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Since t is a parameter and can take any value, the homogeneous system of linear equations has an infinite number of solutions.

Non-Homogeneous Systems

If $\mathbf{b} \neq \mathbf{0}$ in (2), the system of linear equations is said to be non-homogeneous. A non-homogeneous system of linear equations may be **inconsistent**, i.e., it may not possess any solutions at all.

Result 3

If $m = n$ and A is a non-singular matrix in (2) (i.e., rank of $A = n$), then the non-homogeneous system $A\mathbf{x} = \mathbf{b}$ has a **unique** solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Result 4

If A and the augmented matrix $[A \mid \mathbf{b}]$ in (2) has the same rank, then the system of linear equations is consistent. Otherwise, (i.e., rank of $A \neq$ rank of $[A \mid \mathbf{b}]$,) the system of linear equations is inconsistent.

Example 3

Show, without solving, that the system of linear equations

$$\begin{aligned} 5x_1 + 3x_2 + 14x_3 &= 4 \\ x_2 + 2x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

has a unique solution.

Solution

We first find out that the coefficient matrix $A = \begin{pmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{pmatrix}$ has rank 3 since $|A| \neq 0$.

Using Result 3, we conclude that such a system of linear equations has a unique solution

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 4 & -20 & -8 \\ 2 & -4 & -10 \\ -1 & 8 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ 1/3 \end{pmatrix}. \text{(check it)}$$

Example 4

Consider the following system of linear equations.

$$\begin{aligned} 5x_1 + 3x_2 + 14x_3 &= 4 \\ x_2 + 2x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 + 6x_3 &= 2 \end{aligned}$$

Show that the system is inconsistent.

Solution

Let

$$A = \begin{pmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & 6 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 6 & 2 \end{pmatrix}.$$

Rank of A is 3 since $\begin{vmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 12 \neq 0$ (verify it).

Rank of A_1 is 4 since $|A_1| = 4 \neq 0$ (verify it).

Using Result 4, since rank of A is less than rank of A_1, we conclude that such a system is inconsistent.

Example 5

Solve the following system of linear equations.

$$x_1 + x_2 + 3x_3 + x_4 = 2$$
$$x_1 - x_2 + x_3 + x_4 = 4$$
$$x_2 + 2x_3 + 2x_4 = 0$$

Solution

$$[A \quad \mathbf{b}] = \begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ 1 & -1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} = [A^* \quad \mathbf{b}^*].$$

Rank of A = rank of $\begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix}$ = rank of A^* = rank of $\begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} = 3$.

Rank of $[A \quad \mathbf{b}]$ = rank of $\begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ 1 & -1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 & 0 \end{bmatrix}$ = rank of $[A^* \quad \mathbf{b}^*]$ = rank of $\begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} = 3.$

Since Rank of A = Rank of $[A \quad \mathbf{b}]$ = 3, the system of non-homogeneous equations is consistent.

In fact, the leading entries 1, 1, 1 correspond to the leading variables x_1, x_2 and x_3 respectively. Since x_4 is a non-leading variable, we introduce the parameter $x_4 = t$ and then express the leading variables x_1, x_2 and x_3 in terms of the parameter t. Thus, after substitution, we obtain $x_1 = 1+3t$, $x_2 = -2+2t$ and $x_3 = 1-2t$, $x_4 = t$ as the general solution to the system of linear equations under consideration. In fact,

$$\mathbf{x} = \begin{pmatrix} 1+3t \\ -2+2t \\ 1-2t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ -2 \\ 1 \end{pmatrix}.$$

Since t is a parameter and can take any value, the non-homogeneous system of linear equations has an infinite number of solutions.

Activity 1

By considering the ranks of the matrix of coefficients and the augmented matrix, decide in each case whether the given set of equations is consistent or not and, if it is, find all solutions.

(i)
$$\begin{aligned} 3x - 6y &= 5 \\ x - 2y &= 1 \end{aligned}$$

(ii)
$$\begin{aligned} x - y + 2z &= 3 \\ 2x - 3y - z &= -8 \\ 2x + y + z &= 3 \end{aligned}$$

(iii)
$$\begin{aligned} x - 2y - z &= 2 \\ y + z &= 0 \\ x + z &= 2 \\ x + 3y + 4z &= 2 \end{aligned}$$

3.3 EIGENVALUES AND EIGENVECTORS

Consider the matrix equation

$$A\mathbf{x} = \lambda\mathbf{x} \tag{3}$$

where $A = \left[a_{ij}\right]_{3 \times 3}$, $\mathbf{x} = \left[x_1 , x_2 , x_3\right]^T$ and λ is a scalar.

The system of equations (3) can also be expressed in the form

$$\left[A - \lambda I\right]\mathbf{x} = \mathbf{0}, \tag{4a}$$

where I is the 3×3 identity matrix,

or equivalently

$$\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \lambda)x_3 &= 0. \end{aligned} \tag{4b}$$

Definition

We say that λ is an **eigenvalue** of the 3×3 matrix A with corresponding **eigenvector** $\mathbf{x}$ if equations 4(a) or 4(b) hold for some $\mathbf{x} \neq \mathbf{0}$.

Notes:

1. The above definition holds for any $n \times n$ matrix A. In this unit, we shall only deal with 2×2 and 3×3 matrices.

2. The eigenvalue λ is a scalar.

3. λ and $\mathbf{x}$ are in "partnership".

4. It is possible to have a zero eigenvalue, $\lambda = 0$; it is not possible to have a zero eigenvector, $\mathbf{x} = \mathbf{0}$.

5. The eigenvectors associated with a given eigenvalue need not be unique. For instance, the 3×3 identity matrix I has 1 as its only eigenvalue. However all nonzero 3×1 column vectors are eigenvectors of I.

The homogeneous system of equations 4(a) or 4(b) has a non trivial ($\mathbf{x} \neq \mathbf{0}$) solution if and only if the coefficient 3×3 matrix $(A - \lambda I)$ is not invertible or is singular. This occurs if and only if $D(\lambda)$, the determinant of the matrix $(A - \lambda I)$, is equal to zero, i.e.,

$$D(\lambda) \equiv \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0. \qquad (5)$$

This leads to a characterisation of the eigenvalues as solutions to the equation $D(\lambda) = |A - \lambda I| = 0$. The determinant $D(\lambda)$, which is a polynomial in λ, is called the characteristic polynomial of the matrix A. The equation $D(\lambda) = 0$ is called the characteristic equation for A. The roots of this characteristic equation give the eigenvalues of A.

3.4 RELATED RESULTS AND DEFINITIONS

Below is a list of important results. The proofs of these results are beyond the scope of this manual. However you need to know how to apply these results to solve the problems given in the manual. You are thus advised to read these results thoroughly.

Result 1

The eigenvalues of a diagonal matrix A are the diagonal elements of A themselves. For example, the eigenvalues of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ are 1,3 and 4.

Result 2

The eigenvalues of a triangular matrix A (upper or lower triangular) are the diagonal elements of A themselves. For example, the eigenvalues of $\begin{pmatrix} 1 & -5 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 6 & 8 & 4 \end{pmatrix}$ are 1,3 and 4.

Result 3

A matrix is non-singular or invertible if and only if it does not have zero as one of its eigenvalues.

Result 4

If $\mathbf{x}$ is an eigenvector of the matrix A corresponding to the eigenvalue λ, then so is any non-zero scalar multiple of $\mathbf{x}$**.** It follows that to any eigenvalue there is always an infinite number of linearly dependent eigenvectors that correspond to it.

Result 5

For any square matrix A, the eigenvalues of A and A^T are the same. However, the eigenvectors of A^T, in general, are different from the eigenvectors of A.

Result 6

If A is a non-singular matrix, then the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A, the eigenvectors remaining the same.

Result 7

All the eigenvalues of a Hermitian matrix A are real.

Result 8

If $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then

(i) $\mathbf{x}$ is an eigenvector of A^n with eigenvalue λ^n where n is a positive integer.

(ii) $\lambda + \alpha$, where α is a scalar, is an eigenvalue of $A + \alpha I$ with corresponding eigenvector $\mathbf{x}$.

Result 9: The Cayley-Hamilton Theorem.

Every n x n matrix A satisfies its own characteristic equation, i.e., if

$$f(\lambda) = |A - \lambda I| = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_0 = 0,$$

is the characteristic equation of A, then

$$f(A) = A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \ldots + a_0 I = \mathbf{0},$$

where **0** denotes the n x n zero matrix.

Result 10

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of an n x n matrix A and if $f(A)$ is a polynomial, then the eigenvalues of $f(A)$ are $f(\lambda_1)$, $f(\lambda_2)$,…, $f(\lambda_n)$. Also, if $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ, then $\mathbf{x}$ is an eigenvector of f (A) corresponding to the eigenvalue $f(\lambda)$.

***Definition*: (Similar Matrices)**

Let A and B be two n x n matrices. We say that the matrix B is similar to the matrix A if there exists a non-singular n x n matrix P, such that

$$B = P^{-1} AP.$$

Such a transformation of matrix A into matrix B is called a *similarity transformation.*

Properties of similarity transformations:-

1. $P^{-1}(A_1 A_2 \ldots A_n) P = (P^{-1} A_1 P)(P^{-1} A_2 P) \ldots (P^{-1} A_n P)$ where $A_1, A_2, \ldots, A_n$ are matrices of the some order.

2. $P^{-1}(A^n) P = (P^{-1} A P)^n$.

3. Any two similar matrices have the same eigenvalues.

Result 11

If A is an n x n matrix having $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its n distinct eigenvalues, then there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix consisting of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the main diagonal.

Example 6 Determine the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$. Use the Cayley-Hamilton Theorem to find A^{-1}.

Solution. The characteristic equation of A is given by

$$D(\lambda) = \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = \lambda^2 - 25 = 0 .$$

The roots of the characteristic equation are $\lambda_1 = 5$ and $\lambda_2 = -5$.

Thus, 5 and -5 are the eigenvalues of the matrix A.

The system of equations (4a) becomes for $\lambda_1 = 5$,

$$\begin{bmatrix} 3-5 & 4 \\ 4 & -3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{or} -2x_1 + 4x_2 = 0$$
$$4x_1 - 8x_2 = 0.$$

Thus,

$$x_1 = 2\,x_2 \text{ and } x_2 = x_{2,}$$

so that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Hence, by result 4, we can see that any non-zero scalar multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 5. We may thus take $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as a linearly independent eigenvector corresponding to $\lambda = 5$.

For $\lambda = -5$, the system of equations become

$$8\,x_1 + 4\,x_2 = 0$$

$$4\,x_1 + 2\,x_2 = 0.$$

Thus, x_2 = -2 x_1 and $x_1 = x_1$ so that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Hence, by result 4, we can see that any non-zero scalar multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of A with eigenvalue -5. We may thus take $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ as a linearly independent eigenvector corresponding to λ = -5.

By Cayley-Hamilton theorem, $A^2 - 25I = \mathbf{0}$ so that $A^2 = 25I$. Pre-multiplying each side by A^{-1}, we have $A^{-1} = \frac{1}{25}A = \frac{1}{25}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

The following worked example refers to cases where we may have all eigenvalues distinct or some of them repeated. You are strongly advised to study these cases carefully before attempting any related activities or exercises in the tutorial sheets. If you still get difficulties, you should not hesitate to ask for help from your tutor.

<u>Example 6</u>

Let $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$..

For each of the matrices defined above

(i) verify the Cayley-Hamilton Theorem.

(ii) determine their eigenvalues and eigenvectors.

(iii) If possible, define a matrix P such that $P^{-1}AP$ is a diagonal matrix.

Write down the eigenvalues and eigenvectors of the matrices A^2, B^{-1}, $A+4I$ and $(B - 2I)^{-1}$

Solution

Let us consider matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ first of all.

(i) The characteristic equation of A,

$$D(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

Since

$$A^2 = \begin{pmatrix} 0 & 5 & 9 \\ -3 & 4 & 3 \\ -1 & 5 & 10 \end{pmatrix}, \quad A^3 = \begin{pmatrix} -5 & 19 & 32 \\ -7 & 8 & 7 \\ -6 & 19 & 33 \end{pmatrix},$$

we have

$$A^3 - 6A^2 + 11A - 6I = \mathbf{0},$$

which verifies Cayley-Hamilton Theorem.

(ii) Now $D(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$ has roots $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

Thus 1, 2 and 3 are the eigenvalues of the matrix A.

For $\lambda = 1$, the system of equations become

$$\begin{aligned} x_2 + 2x_3 &= 0 \\ -x_1 + x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$$

Thus, $x_2 = -2x_3$, $x_1 = -x_3$ and $x_3 = x_3$ so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = -x_3\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Hence, we may take $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ as a linearly independent eigenvector corresponding to $\lambda = 1$. For $\lambda = 2$ the system of equations become

$$\begin{aligned} -x_1 + x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0. \end{aligned}$$

Thus, $x_1 = x_3$, $x_2 = -x_3$ and $x_3 = x_3$ so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Hence, we may take $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ as a linearly independent eigenvector corresponding to $\lambda = 2$.

For $\lambda = 3$ the system of equations become

$$\begin{aligned} -2x_1 + x_2 + 2x_3 &= 0 \\ -x_1 - x_2 + x_3 &= 0 \\ x_2 &= 0. \end{aligned}$$

Thus, $x_1 = x_3$, $x_2 = 0$ and $x_3 = x_3$ so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Hence, we may take $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ as a linearly independent eigenvector corresponding to $\lambda = 3$.

(iii) It can easily be verified that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly independent.

Define a matrix P as

$$P = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

Since the columns of P are linearly independent, we have $|P| = -2 \neq 0,$ so that P^{-1} exists. Thus,

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

which is a diagonal matrix consisting of the eigenvalues of A on the main diagonal.

Note that if we define

$$P = (\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \text{ then}$$

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Now let us consider matrix $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

(i) The characteristic equation of *B*,

$$D(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

Since

$$B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I, \quad B^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B,$$

we have

$$B^3 - B^2 - B + I = \mathbf{0},$$

which verifies Cayley-Hamilton Theorem.

(ii) Now $D(\lambda) = (\lambda - 1)^2 (\lambda + 1)$ has roots 1,1 and -1.

Thus, 1 and -1 are the eigenvalues of the matrix *B*.

For $\lambda = 1$, the system of equations become

$$\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \\ 0.x_3 &= 0. \end{aligned}$$

Thus, $x_1 = x_2$ and $x_3 = x_3$ so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ do not depend upon one another, we have thus generated two linearly independent eigenvectors corresponding to the repeated eigenvalue $\lambda = 1$. This is not always the case; as we shall see later, sometimes we can only find one linearly independent eigenvector for a repeated eigenvalue.

For $\lambda = -1$ the system of equations become

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \\ 2\,x_3 &= 0. \end{aligned}$$

Thus, $x_1 = -x_2$, $x_3 = 0$ and $x_1 = x_1$ so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \\ 0 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Hence we may take $\mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ as a linearly independent eigenvector corresponding to λ = -1.

(iii) It can easily be verified that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly independent.

Define a matrix P as

$$P = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since the columns of P are linearly independent, we have $|P| = 2 \neq 0,$ so that P^{-1} exists.

Thus,

$$P^{-1}BP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which is a diagonal matrix consisting of the eigenvalues of B on the main diagonal.

Now let us consider matrix $C = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$.

(i) The characteristic equation of C,

$$D(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 1-\lambda & -2 \\ -1 & 0 & -2-\lambda \end{vmatrix} = \lambda^3 - \lambda^2 - 5\lambda - 3 = 0.$$

Since

$$C^2 = \begin{pmatrix} 5 & 3 & -2 \\ 8 & 3 & 4 \\ 0 & -1 & 3 \end{pmatrix}, \quad C^3 = \begin{pmatrix} 18 & 8 & 3 \\ 18 & 11 & -6 \\ -5 & -1 & -4 \end{pmatrix},$$

we have

$$C^3 - C^2 - 5C - 3I = \mathbf{0},$$

which verifies Cayley-Hamilton Theorem.

(ii) Now $D(\lambda) = (\lambda + 1)^2 (\lambda - 3)$ has roots -1,-1 and 3.

Thus -1 and 3 are the eigenvalues of the matrix C.

For $\lambda = -1$, the system of equations become

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 - 2x_3 &= 0 \\ -x_1 - x_3 &= 0. \end{aligned}$$

Thus, $x_1 = -x_3$, $x_2 = 2x_3$ and $x_3 = x_3$ so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ 2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Hence, we may take $\mathbf{x}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ as a linearly independent eigenvector corresponding to the repeated eigenvalue $\lambda = -1$.

For $\lambda = 3$ the system of equations become

$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ 2x_1 - 2x_2 - 2x_3 &= 0 \\ -x_1 - 5x_3 &= 0. \end{aligned}$$

Thus, $x_1 = -5x_3$, $x_2 = -6x_3$ and $x_3 = x_3$ so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5x_3 \\ -6x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix}.$$

Hence, we may take $\mathbf{x}_2 = \begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix}$ as a linearly independent eigenvector corresponding to $\lambda = 3$.

(iii) It can easily be verified that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

It is not possible to define a matrix P such that $P^{-1}CP$ is a diagonal matrix. We say that matrix C is not diagonalisable.

Using Results 6 and 8, we conclude that:

(a) 1, 4 and 9 are the eigenvalues of A^2 with the same eigenvectors as those of A.

(b) B^{-1} has the same eigenvalues and eigenvectors as B $\left(\text{because}\ \frac{1}{1} = 1 \text{ and } \frac{1}{-1} = -1\right)$.

(c) 5, 6 and 7 are the eigenvalues of $A + 4\mathrm{I}$ with the same eigenvectors as those of A.

(d) -1 and -1/3 are the eigenvalues of $(B - 2I)^{-1}$ with the same eigenvectors as those of B.

Note: A matrix may have distinct eigenvalues as well as repeated ones. If all its eigenvalues are distinct, the matrix is diagonalisable since we can define a matrix P consisting of all the corresponding linearly independent eigenvectors. If the matrix has repeated eigenvalues, it may or may not be diagonalisable. It all depends on whether we have enough linearly independent eigenvectors to define our matrix P.

<u>Activity 2</u>

1. Find all eigenvalues and corresponding eigenvectors of the following matrices.

(i) $\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -3 \end{bmatrix}$.

2. Find all eigenvalues and corresponding eigenvectors of the following matrices. Which of them are diagonalisable? If possible, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

(i) $\begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$, **(ii)** $\begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$, (iii) $\begin{bmatrix} 1 & 1 & -2 \\ 4 & 0 & 4 \\ 1 & -1 & 4 \end{bmatrix}$.

3.5 SUMMARY

In this Unit, you have learnt that there are three possible outcomes when solving a system of linear equations, namely:

(i) there is no solution,

(ii) there is a unique solution,

(iii) there are infinitely many solutions.

You have also learnt how to compute eigenvalues and eigenvectors of a matrix and their related properties.

3.6 SUPPLEMENTARY EXERCISES

1. Show that the system of linear equations defined by the matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ -2 & 2 & 0 & 1 \\ 4 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

is consistent if $2b_1 = b_2 + b_3 + b_4$, and solve it completely if $b_1 = b_2 = b_4 = 1, b_3 = 0$.

2. Find the general solution of the system of equations

$$\begin{aligned} x_1 + x_2 - ax_3 &= b \\ 3x_1 - 2x_2 - x_3 &= 1 \\ 4x_1 - 3x_2 - x_3 &= 2 \end{aligned}$$

in each of the three cases (i) a = 1, b = 9; (ii) a = 2, b = -3; (iii) a = 2, b = 0.

3. Find all eigenvalues and corresponding eigenvectors of the following matrices. Which of them are diagonalisable? If possible, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

(i) $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, **(iii)** $\begin{bmatrix} 17 & -10 & -5 \\ 45 & -28 & -15 \\ -30 & 20 & 12 \end{bmatrix}$.

3.7 ANSWERS TO ACTIVITIES & SUPPLEMENTARY QUESTIONS

<u>Activity 1</u>

(i) Ranks 1, 2; inconsistent

(ii) Ranks 3, 3; unique solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 \\ 3/2 \\ 5/2 \end{pmatrix}$

(iii) Ranks 2, 2; infinite solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-t \\ -t \\ t \end{pmatrix}$, t a parameter.

Activity 2

1. (i) 1, $[1,1]^T$.

   (ii) 0, $[3,-2,1]^T$; 2, $[5,2,1]^T$; -3, $[0,1,-2]^T$.

2. (i) 0, $[-2,1]^T$; 1, $[-3,1]^T$. $P = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$

   (ii) -2, $[1,1,0]^T$; 4, $[0,1,1]^T$. Not diagonalisable.

   (iii) 0, $[-1, 3,1]^T$; 2, $[0,2,1]^T$; 3, $[-1, 0, 1]^T$. $P = \begin{bmatrix} -1 & 0 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Supplementary Exercises

1. $x_1 = \frac{1}{3}t, x_2 = -\frac{1}{6}t, x_3 = \frac{1}{6}(6-13t), x_4 = t.$

2. (i) $x_1 = 11,\ x_2 = 10,\ x_3 = 12$

   (ii) $x_1 = t,\ x_2 = t-1,\ x_3 = t+1$

   (iii) No Solution.

3. (i) $3 + 4i$, $[i, 1]^T$; $3 - 4i$, $[-i, 1]^T$. $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

(ii) -1, $[1, 0, 1]^T$; 2, $[1, 3, 1]^T$; 1, $[3, 2, 1]^T$. $P = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

(iii) -3, $[1, 3, -2]^T$; 2, $[1,0,3]^T$, $[0, 1, -2]^T$. $P = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ -2 & 3 & -2 \end{bmatrix}$