Tutorial 1

Question 1
Calculate the determinant of the five matrices and state which are singular.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 1 & 5 \\ 4 & 2 & 3 \end{bmatrix}, \; \mathbf{AB}^2, \; \mathbf{A} + \mathbf{B}, \; \mathbf{AB} + \mathbf{A}^2.$$

Question 2

(a) Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{bmatrix}$. Verify that $\mathbf{A}^3 = \beta\gamma\mathbf{I_3}$, where $\mathbf{I_3}$ is the identity matrix of order 3. Deduce that $\mathbf{A}$ is non-singular and hence, compute $\mathbf{A}^{-1}$.

(b) Find the value(s) of x for which the following matrix is singular:

$$\begin{bmatrix} x+3 & 2 & 1 \\ x & 3\,x & 4 \\ 2\,x-1 & 2 & 1 \end{bmatrix}$$

.

(c) Given that $\mathbf{M} = \begin{bmatrix} \alpha^2+3 & \alpha & 1 \\ \beta^2+3 & \beta & 1 \\ \gamma^2+3 & \gamma & 1 \end{bmatrix}$, use properties of determinant to show that $|\mathbf{M}| = -(\beta-\alpha)(-\alpha+\gamma)(\gamma-\beta)$.

Question 3
An $n \times n$ matrix $\mathbf{A} = (a_{ij})$ is called *orthogonal* if $\mathbf{AA}^T = \mathbf{I}$.

i Show that the following matrix is orthogonal:

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

ii Show that if $\mathbf{A}$ is orthogonal then $\det(\mathbf{A}) = \pm 1$.

Answers

Question 1

$|\mathbf{A}| = 12$, $|\mathbf{B}| = -3$, $|\mathbf{AB}^2| = 108$ $|\mathbf{A} + \mathbf{B}| = 0$ (singular), $|\mathbf{AB} + \mathbf{A}^2| = 0$ (singular).

Question 2

(a) $\begin{bmatrix} 0 & \beta^{-1} & 0 \\ 0 & 0 & \gamma^{-1} \\ 1 & 0 & 0 \end{bmatrix}$.

(b) $\frac{8}{3}$, 4.

Tutorial 2

Question 1
Let ω be a complex cube root of 1 (this means $\omega^3 = 1$) with $\omega \neq 1$. Prove that $1 + \omega + \omega^2 = 0$. Letting $\mathbf{A}$ be the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

Determine $\mathbf{A}^2$ and $\mathbf{A}^{-1}$.

Question 2
For each $v \in \Re$, define the matrix $\mathbf{A}(v)$ by

$$\mathbf{A}(v) = \begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ \frac{1}{2}v^2 & v & 1 \end{pmatrix}$$

Show that for all v, $w \in \Re$ we have $\mathbf{A}(v + w) = \mathbf{A}(v)\mathbf{A}(w)$. Deduce that each matrix $\mathbf{A}(v)$ is invertible.

Question 3
Determine how the rank of the real matrix

$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & b & -1 \end{pmatrix}$$

depends on the real number b.

Question 4

Find the rank of the following matrices:

(a) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & \alpha^2 + \alpha \\ 1 & 2\alpha & \alpha \end{bmatrix}$, where $1 + \alpha + \alpha^2 = 0$. and $\alpha^3 = 1$.

(b) $\begin{bmatrix} 2 & 1 & 2 & 1 \\ 2 & 2 & 4 & 4 \\ 0 & 1 & 2 & 3 \\ 4 & 5 & 10 & 11 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 3 & 1 & 4 \\ 1 & 5 & 1 & 7 \end{bmatrix}$.

Answers

Question 1

$$\mathbf{A}^2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}, \qquad \mathbf{A}^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & \omega^2/3 & \omega/3 \\ 1/3 & \omega/3 & \omega^2/3 \end{bmatrix}$$

Question 3

(i) $rank = 2\, for\, b\ =\ 3$ and $rank = 3\, for\, b\ \neq\ 3$

Question 4

(a) 2, (b) 2, (c) 3.

Tutorial 3

Question 1
Solve the following system of equations using Gaussian elimination:

(a)
$$x + 2y - 4z = -4$$
$$2x + 5y - 9z = -10$$
$$3x - 2y + 3z = 11$$

(b)
$$x + 2y - 3z = -1$$
$$-3x + y - 2z = -7$$
$$5x + 3y - 4z = 2$$

(c)
$$x + 2y - 3z = 1$$
$$2x + 5y - 8z = 4$$
$$3x + 8y - 13z = 7$$

Question 2
Consider the system
$$2x + 2y + 3z = 0$$
$$x + \alpha y + 3z = 0$$
$$x + 2y + 2z = 0.$$

Find sufficient conditions on α for the system to have

(i) unique solution,

(ii) infinite number of solutions.

Question 3
Find the inverses of the following matrices using elementary row operations:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix}.$$

Question 4

Let $\mathbf{A}$ be each of the following matrices in turn:

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 4 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Find all the eigenvectors of $\mathbf{A}$; determine whether $\mathbf{A}$ is diagonalisable and, if so, find an invertible real matrix X for which $X^{-1}\mathbf{A}X$ is diagonal.

(b) In the case of the first matrix,

(i) find the eigenvalues and eigenvectors of $\mathbf{A}^5$,$\mathbf{A} \ + \ 7\mathbf{I}$ and $(\mathbf{A}$ - $3\mathbf{I})^{-1}$.

(ii) Show that $\mathbf{A}^{-1} \ = \mathbf{A} - \frac{1}{2}\mathbf{A}^2 + \frac{1}{2}\mathbf{I}$ and hence, or otherwise, determine $\mathbf{A}^{-1}$

Answers

Question 1

(a) $x = 2, y = -1, z = 1,$ (b)inconsistent (c)$x = -3 - t, y = 2 + 2t, z = t.$

Question 2

(i)$\alpha \neq 4$ (ii)$\alpha = 4.$

Question 3

$$(a) \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad (b) \begin{bmatrix} 1/10 & -3/5 & 3/10 \\ -1/5 & 1/5 & 2/5 \\ 3/10 & 1/5 & -1/10 \end{bmatrix}, \quad (c) \begin{bmatrix} -1/3 & 0 & 2/3 \\ 1 & -1 & 0 \\ -1/3 & 1 & -1/3 \end{bmatrix}.$$

Question 4

(a)$\{2, (1, 0, 0)^t\}, \{1, (-3, 1, 1)^t\}, \{-1, (1, 3, -3)^t\}$ and $\mathbf{X} = \begin{bmatrix} 1 & 1 & -3 \\ 3 & 0 & 1 \\ -3 & 0 & 1 \end{bmatrix}$

$\{-1, [0, 0, 1]\}, \{2, [1, 1, 1], [1, 1, 1]\}$

$\{1, [1, -1, 0], [0, -1, 1]\}, \{4, [1, 1, 1]\}$ and $\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$

(b)(i)*Each eigenvalue is raised to power* 5 *and the corresponding eigenvectors are unchanged; add* 7 *to each eigenvalue and corresponding eigenvectors are unchanged; subtract* 3 *from each eigenvalue and take reciprocal of each resulting value and corresponding eigenvectors are unchanged.*

(ii) $\mathbf{A}^{-1} = \begin{bmatrix} 1/2 & -1 & -1/2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Tutorial 4

Question 1

Test the following series for convergence:

(i) $\sum_{n=1}^{\infty} \frac{e^n}{3^{n+2}}$,

(ii) $\sum_{n=1}^{\infty} \frac{n-7}{n^2}$,

(iii) $\sum_{n=1}^{\infty} \frac{2}{(2n-1)(2n+1)}$,

(iv) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$,

(v) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}-\sqrt{n^2-1}}$

(vi) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$,

(vii) $\sum_{n=1}^{\infty} ne^{-n}$,

(viii) $\sum_{n=1}^{\infty} \frac{n!}{2n-1!}$

(ix) $\sum_{n=1}^{\infty} \frac{n!}{1.5.9...(4n-3)} x^n$, $x > 0$.

Question 2

Given that $y = (5 - \frac{2}{n})^n$, $n > 0$, by applying L'Hospital's Rule, show that $\lim_{n\to\infty} \ln y = \frac{-2}{5}$. Hence, or otherwise, determine whether the series $\sum_{n=1}^{\infty} \left(\frac{5n-2}{n}\right)^n$ converges or diverges.

Question 3

Use the Taylor series to find a quadratic approximation to each of the following functions at the specified points:

(i) $\sin(3x + 2y)$ at $(\frac{\Pi}{6}, 0)$,

(ii) $\cosh x \cosh y$ at the origin.

Answers

1.(i)convergent; (ii)divergent; (iii)convergent;

(iv)convergent; (v)divergent; (vi)divergent;

(vii)convergent; (viii)convergent; (ix)convergent for $0 < x < 4$.

2. Series converges.

3. (i)$1 - 9/2\ (x - 1/6\,\pi)^2 - 6\,y\,(x - 1/6\,\pi) - 2\,y^2$; (ii) $1 + \frac{x^2}{2} + \frac{y^2}{2}$

Tutorial 5

Question 1
The **triple vector product** $(\mathbf{A}\wedge\mathbf{B})\wedge\mathbf{C})$ and $\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C})$ are usually not equal, although the formulae for evaluating them from components are similar :

$$(\mathbf{A}\wedge\mathbf{B})\wedge\mathbf{C} \;=\; (\mathbf{A}\cdot\mathbf{C})\mathbf{B} \;-\; (\mathbf{B}\cdot\mathbf{C})\mathbf{A}. \tag{1}$$

$$\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C}) \;=\; (\mathbf{A}\cdot\mathbf{C})\mathbf{B} \;-\; (\mathbf{A}\cdot\mathbf{B})\mathbf{C}. \tag{2}$$

Verify each of the formula for the following vectors by evaluating its two sides and comparing the results.

	A	**B**	**C**
a)	$(1,1,-2)^t$	$(-1,0,-1)^t$	$(2,4,-2)^t$
b)	$(1,-1,1)^t$	$(2,1,-2)^t$	$(-1,2,-1)^t$
c)	$(2,1,0)^t$	$(2,-1,1)^t$	$(1,0,2)^t$

Question 2
Show that if **A**, **B**, **C** and **D** are any vectors, then

(a)

$$\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C}) \;+\; \mathbf{B}\wedge(\mathbf{C}\wedge\mathbf{A}) \;+\; \mathbf{C}\wedge(\mathbf{A}\wedge\mathbf{B}) \;=\; \mathbf{0}.$$

(b)

$$\mathbf{A}\wedge\mathbf{B} \;=\; (\mathbf{A}\cdot\mathbf{B}\wedge\mathbf{i})\mathbf{i} \;+\; (\mathbf{A}\cdot\mathbf{B}\wedge\mathbf{j})\mathbf{j} \;+\; (\mathbf{A}\cdot\mathbf{B}\wedge\mathbf{k})\mathbf{k}.$$

Question 3

(a) Given that $\mathbf{A} = \mathbf{2i} + \mathbf{2j} - \mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{k}$ and $\mathbf{C} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, find the scalar and vector projection of $\mathbf{B} \wedge \mathbf{C}$ in the direction of $\mathbf{A}$.

(b) Show that if the lines

$$x = a_1 s + b_1,\ y = a_2 s + b_2,\ z = a_3 s + b_3,\ -\infty < s < \infty,$$

and

$$x = c_1 t + d_1,\ y = c_2 t + d_2,\ z = c_3 t + d_3,\ -\infty < t < \infty,$$

intersect at one point *at least* then

$$\begin{vmatrix} a_1 & c_1 & b_1 - d_1 \\ a_2 & c_2 & b_2 - d_2 \\ a_3 & c_3 & b_3 - d_3 \end{vmatrix} = 0.$$

Question 4

(a) Consider the following vector equations :

$$\begin{aligned} \mathbf{x} \wedge \mathbf{b} &= \mathbf{b} + \alpha\,\mathbf{c}, \\ \mathbf{x} \cdot \mathbf{c} &= \beta, \end{aligned}$$

where $\mathbf{b} = \mathbf{i} + \mathbf{2j} + \mathbf{3k}$, $\mathbf{c} = \mathbf{2i} - \mathbf{j} + \mathbf{k}$, and α and β are scalars. Show that:

(i) $\mathbf{x}$ is a vector through $\frac{1}{3}(5, 5, -5)$ and in the direction of $\mathbf{b}$ and

(ii) $\alpha = \dfrac{-14}{3}$.

(b) Solve the following vector equations for $\mathbf{x}$ and μ:

$$\begin{aligned} \mathbf{x} \wedge \mathbf{u} &= \mathbf{u} + \mu\,\mathbf{v}, \\ \mathbf{x} \cdot \mathbf{v} &= \mathbf{2}, \end{aligned}$$

where $\mathbf{u} = 2\mathbf{i} + -\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, and μ is a scalar.

Answers

Question 1

(a) $(-10,0,-10)^t\,(1) \quad (-12,-4,-8)^t\,(2)$ (b) $(-10,-2,6)^t\,(1) \quad (-9,-2,6)^t\,(2)$
(c)$(-4,-6,2)^t\,(1) \quad (1,-2,-4)^t\,(2)$

Question 3

(a)$\dfrac{-2}{3}\sqrt{3}, \quad \dfrac{2}{3}(-1,-1,1)^t$

Question 4

(b) $\dfrac{-1}{6}(4,-5,-1)^t \quad \mu \; = \; 1$

Tutorial 6

Question 1

(a) If $\mathbf{F} = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$, then show that $\nabla \wedge \mathbf{F} = 0$.

(b) Find $\nabla \wedge \mathbf{S}$ and $\nabla \cdot (\nabla \wedge \mathbf{S})$ for each of the following

(i) $\mathbf{S} = (0, 0, xy)^t$.

(ii) $\mathbf{S} = (-yz, 0, xy)^t$.

Question 2

Let $\omega = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$ and $\mathbf{s} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(i) Find $\mathbf{v} = \omega \wedge \mathbf{s}$

(ii) Show that $\nabla \wedge \mathbf{v} = 2\omega$.

(iii) Show that $div(\mathbf{v}) = 0$

(iv) State with reason, the angle between $\mathbf{v}$ and ω.

Question 3

(a) Find the directional derivative of $z = x^2 - 6y^2$ at $P(7, 2)$ in the direction

(i) $\theta = 45°$,

(ii) $\theta = 135°$.

(b) Find the maximum directional derivative for the same function at P.

Question 4

The temperature T of a heated circular plate at any of its point (x, y) is given by

$$T = \frac{32}{x^2 + y^2 + 1},$$

the origin being the center of the plate. Find the rate of change of T at the point $(2, 3)$, in the direction $\theta = \frac{\pi}{3}$

Question 5

The electrical potential V at any point (x, y) is given by

$$V = \ln\left(\sqrt{(x-1)^2 + (y-2)^2}\right)$$

.

(i) Find the rate of change of V at $(3, 4)$ in the direction towards the point $(4, 5)$.

(ii) Show V changes most rapidly along the set of radial lines through the point $(1, 2)$.

Answers

Question 1

b(i) $(x, -y, 0)^t, \mathbf{0}$, (ii)$(x, -2y, z)^t, \mathbf{0}$.

Question 2

(i) $(\omega_2 z - \omega_3 y, \omega_3 x - \omega_1 z, \omega_1 y - \omega_2 x)^t$, (iv)$\dfrac{\pi}{2}$

Question 3

(i)$-5\sqrt{2}$, (ii)$-19\sqrt{2}$, (b)$2\sqrt{193}$ in the direction $\theta = 300°15'$

Question 4

$-\frac{16}{49} - \frac{24}{49}\sqrt{3}$

Question 5

(i)$\dfrac{\sqrt{2}}{4}$

Tutorial 7

Question 1

(a) Sketch the region of integration of $\displaystyle\int_0^1\int_x^{2x} dydx$.

(b) Exchange the order of integration to express the integral in part (a) in terms of integration in the order $dxdy$ and evaluate it in that case.

(c) Find $\displaystyle\int\int_\Omega \frac{y}{x^2+y^2}dxdy$, where Ω is the shaded region shown in Fig.(1).

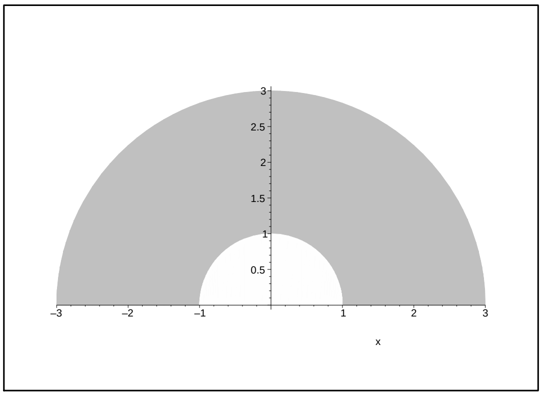

Figure 1: Region Ω

(d) Find $\displaystyle\int\int_\Omega ydydx$, where Ω is the region bounded by the triangle with vertices at $(-2,0)$, $(0,1)$ and $(2,0)$

Question 2

Sketch the corresponding region of integration and evaluate by changing order of the integration :

(a) $\displaystyle\int_0^3\int_{x^2}^{9} dydx$,

(b) $\displaystyle\int_0^1\int_0^{\sqrt{x}} \frac{2xy}{1-y^4}dydx$.

Question 3

Find $\displaystyle\iint_{\Omega}\left(1-\frac{x}{a}-\frac{y}{b}\right)dydx$, where Ω is in the first quadrant bounded by the axes and the line $\dfrac{x}{a}+\dfrac{y}{b}=1$.

Question 4

Consider the following integral

$$I = \int_0^{\sqrt{2}}\int_0^{y}\cos\left(\frac{\pi\left(x^2+y^2\right)}{8}\right)dx\,dy + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-y^2}}\cos\left(\frac{\pi\left(x^2+y^2\right)}{8}\right)dx\,dy$$

(i) Sketch the region of integration defined by the sum of these two integrals.

(ii) By reversing the order of integration, rewrite I as one double integral.

(iii) By using polar coordinates, show that $I = 1$.

Answers

Question 1

(b) $\displaystyle\int_0^1\int_{y/2}^{y} dydx + \int_1^2\int_{y/2}^{1} dydx$, $\dfrac{1}{2}$, (c) 4, (d) $\dfrac{2}{3}$.

Question 2

(a) *18*, (b) $\dfrac{1}{2}$.

Question 3

$\dfrac{ab}{6}$.

Question 4

$I = \int_0^{\sqrt{2}}\int_x^{\sqrt{4-x^2}}\cos\left(\dfrac{\pi\left(x^2+y^2\right)}{8}\right)dy\,dx.$

Tutorial 8

Question 1
Consider the integral

$$I = \int_0^1 \int_0^1 \frac{1}{(1+y^2x)(1+x)} dxdy.$$

The transformation

$$x = u^2 \text{and} y = \frac{v}{u},$$

is applied to I.
Show that the limits in the $u - v$ plane are : $u = 0$, $u = 1$ and $v = 0$, $v = 1$. sketch the region in the $u - v$ plane and hence evaluate the above integral by integrating over an appropriate region in the $u - v$ plane.

Question 2
Consider the integral $I = \int\int_R \frac{(x+y)(x^2+y^2)}{x^4}\, dA$, where R is the region bounded by the lines $y = 0$, $y = x$ and $x + y = \alpha$ with $\alpha > 0$. By applying the transformation $u = x + y$ and $v = \frac{y}{x}$, sketch the corresponding region of integration in the $u - v$. Hence, evaluate I.

Question 3
Show by means of a diagram the area over which the double integral is taken, $I = \int_0^2 \int_{y-2}^{2-y} \frac{x+y}{(x+1)^2} e^{2(x+y)} dxdy$. Apply the transformation of variable $u = x+y$ and $v = \frac{y-1}{x+1}$ to this integral and sketch the region of integration in the $u - v$ plane. Hence, evaluate I.

Question 4
If $f(x, y)$ can be written as $f(x, y) = F(x)G(y)$, then the integral of f over a rectangle R: $a \leq x \leq b$, $c \leq y \leq d$ can be evaluated as a product by the formula

$$\int\int_R f(x, y)\, dxdy = \int_a^b F(x)dx \int_c^d G(y)dy. \qquad (3)$$

Use Eq.(3) to evaluate the following:

(i) $\int_0^1 \int_0^{\frac{\pi}{2}} 6x(\cos(y))^2 dydx.$

(ii) $\int_1^2 \int_0^1 x^3 y^{-3/2} dxdy.$

Question 5

Let $I = \displaystyle\int_0^\infty e^{-(x^2)} dx$ and $I^2 = \displaystyle\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$.

Evaluate the last integral using polar coordinates and solve the resulting equation to show that $\int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$.

Answers

Question 1

$\dfrac{\pi^2}{16}$

Question 2

$\dfrac{4\,\alpha}{3}$

Question 3

$\dfrac{1}{4}\left(e^4 - 1\right)$

Question 4

(b)(i) $\dfrac{3\pi}{4}$,(ii) $\dfrac{2-\sqrt{2}}{4}$

Tutorial 9

Question 1
Evaluate the following triple integrals
(a) $\int_0^1 \int_0^{1-x} \int_0^{2-x} xyz dz dy dx.$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^2 r^2 z \sin\theta dz dr d\theta.$

(c) $\displaystyle\int_0^{\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} \sin 2\phi d\rho d\phi d\theta.$

Question 2
Using triple integrals, find the volume of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

Question 3
Calculate the volume of the region bounded by the following surfaces:

(i) $z = 0$, $x^2 + y^2 = 2$ and $x + y + z = 3$.

(ii) The cylinder $x^2 + y^2 = 16$ and the planes $z = 0$ and $z + y = 4$.

(iii) The sphere $x^2 + y^2 + z^2 = 16$ and the cylinder $x^2 + y^2 = 9$.

Question 4
Find the area of that portion of the surface of the cylinder $x^2 + y^2 = 9$ which lies in the first octant between the planes $z = 0$ and $z = 2x$.

Question 5
Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 4$ inside the upper part of the cone $x^2 + y^2 = z^2$.

Answers

Question 1

(a) $\dfrac{13}{240}$

(b) $\dfrac{2}{3}$

(c) $(2-\sqrt{2})\pi$

Question 2

$\dfrac{1}{6}$

Question 3

(i)$\dfrac{3}{2}\,\pi\dfrac{-4}{3}\,\sqrt{2}$, (ii)$64\pi$, (iii) $\left(\frac{256}{3}\,-32\,\sqrt{3}\right)\pi$

Question 4

18

Question 5

$4\left(2-\sqrt{2}\right)\pi$

Tutorial 10

Question 1
For $\mathbf{F} = x^3y\mathbf{i} + y^2\mathbf{j}$, find $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve $y = x^2$ from $(0,0)$ to $(1,1)$.
Question 2
Evaluate $\displaystyle\int_C (x - y)dx + (y - z)dy$ over the line segment C from $P(1,1,1)$ to $Q(2,4,8)$.
Hint:
C is defined as follows :

$$\mathbf{PQ} = (1,3,7)^t \text{ and any point } (x,y,z)\text{along } C\text{is thus defined}as$$
$$(x = 1 + t,\ y = 1 + 3t,\ z = 1 + 7t,\ \text{where}0\ \le\ t\ \le 1.$$

Question 3

(a) Show that $\mathbf{F} = (3x^2 - 6y^2)\mathbf{i}\ +\ (-12xy + 4y)\mathbf{j}$ is conservative.

(b) Find Φ such $\mathbf{F}\ =\ \nabla\Phi$ (Φ is said to be a potential function for $\mathbf{F}$).

(c) Let C be the curve $x\ =\ 1\ +\ y^3(1 - y)^3$, $0 \le y \le 1$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Question 4
Let $\mathbf{F} = (ax^2y + y^3 + 1)\mathbf{i}\ +\ (2x^3 + by^2x + 2)\mathbf{j}$ be a vector field, where a and b are constants.

(a) Find the values of a and b for which $\mathbf{F}$ is conservative.

(b) Use the values of a and b from (a) to find $f(x,y)$ such that $\mathbf{F}\ =\ \nabla f$.

(c) Using the values of a and b from part (a), compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C such that $x = e^t \cos t$, $y = e^t \sin t$, $0\ \le t\ \le \pi$.

Question 5

(a) Write down the contour integral of $\mathbf{F} = (5x + 3y)\mathbf{i} + (1 + \cos y)\mathbf{j}$, counter-clockwise around the unit circle centered at the origin, in the form $\displaystyle\int_a^b f(t)dt$. (**Do not simplify the integral**.)

(b) Evaluate the line integral using Green's theorem.

Question 6
Consider the region R enclosed by the $x-$axis, $x = 1$ and $y = x^3$.
Travelling in a counterclockwise direction along the boundary C or R, call C_1 the portion of C that goes from $(0,0)$ to $(1,0)$, C_2 the portion of C that goes from $(1,0)$ to $(1,1)$ and C_3 the portion of C that goes from $(1,1)$ to $(0,0)$.

(a) Using Green's theorem, find the total work of $\mathbf{F} = (1 + y^2)\mathbf{i}$ around the boundary C of R, in a counterclockwise direction.

(b) Calculate the work of $\mathbf{F}$ along C_1 and C_2.

(c) Use parts (a) and (b) to find the work along the third side C_3.

Answers

Question 1

1/2

Question 2

-13

Question 3

(b)$f = x^3 - 6y^2x + 2y^2(+constant)$, (c) -4

Question 4

(a) $a = 6$, $b = 3$, (b) $f = 2x^3y + y^3x + x + 2y(+constant)$, (c) $-e^{\pi} - 1$.

Question 5

$\int_0^{2\pi} (5\cos t\ +\ 3\sin t)(-\sin t)dt\ +\ (1\ +\ \cos(\sin t))\cos t dt$, (b) -3π.

Question 6

(a)$-1/7$, (b)along C_1,it is 1, and along C_2,it is 0(c)$-8/7$.

Tutorial 11 - Divergence theorem

Question 1

(a) if $\sum$ is any closed surface enclosing a volume V and $\phi = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, use the Gauss-divergence theorem to prove that $\int\int_{\sum} \phi \cdot \hat{\mathbf{n}} dS = 3V$.

(b) Given that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + (1 - z)\mathbf{k}$, evaluate

$$\int\int \mathbf{r} \cdot \hat{\mathbf{n}} dS$$

over the whole boundary of the region bounded by the paraboloid $z = 13 - x^2 - y^2$ and the plane $z = 9$.

Question 2

Calculate the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (\mathbf{1} + \mathbf{3}z)\mathbf{k}$ out of the portion of the sphere $z^2 = 4 - x^2 - y^2$ in the first octant in the direction away from the origin

Question 3

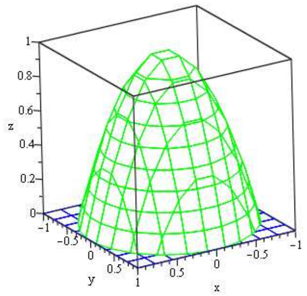

Figure 2: **Plot showing the paraboloid $z = 1 - x^2 - y^2$ and $z = 0$.**

Let S be the **curved part** of the surface formed by the paraboloid $z = 1 - x^2 - y^2$ lying above the xy plane as shown in fig.(2). The basis of the solid formed is the unit disc in the xy plane. Further, let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (\mathbf{1} - \mathbf{2}z)\mathbf{k}$.

(i) Using the divergence theorem, find the flux of $\mathbf{F}$ out of the solid.

(ii) Using direct calculation and taking the upward direction as the one for which the flux is positive, compute the flux of $\mathbf{F}$ through the unit disc in the xy plane ($\hat{\mathbf{n}} = -\hat{\mathbf{k}}$).

(iii) Hence, calculate the flux of $\mathbf{F}$ through S.

Question 4

Use the divergence theorem to compute the flux of $\mathbf{F} = (\mathbf{1} + y^{\mathbf{2}})\mathbf{j}$ out of the curved part of the half-cylinder bounded by $x^2 + y^2 = a^2$, ($y \geq 0$), $z = 0$, $z = b$ and $y = 0$.

Answers

Question 1

8π

Question 2

$\frac{8\pi}{3}$

Question 3

(i) 0, (ii) -2π, (iii) 2π.

Question 4

$2ab + \frac{4a^3b}{3}$.

Tutorial 12 - Stokes' theorem

Question 1
Let $\mathbf{F} = (-6y^2 + 6y)\mathbf{i} + (x^2 - 3z^2)\mathbf{j} - x^2\mathbf{k}$.
Calculate $\nabla \wedge \mathbf{F}$ and use Stokes'theorem to show that the work done by $\mathbf{F}$ along a simple closed curve contained in the plane $x + 2y + z = 1$ is equal to zero.

Question 2
Let $\mathbf{F} = -2xz\mathbf{i} + y^2\mathbf{k}$.

(a) Calculate curl $\mathbf{F}$.

(b) Show that the $\int\int_R \text{curl}\mathbf{F} \cdot \hat{\mathbf{n}}\mathbf{dS} = \mathbf{0}$ for any finite portion R of the unit sphere $x^2 + y^2 + z^2 = 1$.(Take the normal vector $\hat{\mathbf{n}}$ pointing outward).

(c) Show that $\oint_C \mathbf{F} \cdot \mathbf{dr} = \mathbf{0}$ for any simple closed curve C on the unit sphere $x^2 + y^2 + z^2 = 1$.

Question 3
Let $\mathbf{F} = (xz,\ yz + x,\ xy)^t$.

(a) Find $\nabla \wedge \mathbf{F}$.

(b) Let C be the simple closed curve, oriented counterclockwise when viewed from above, $x - y + 2z = 10$. The projection of C on the $xy-$plane is the circle $(x-1)^2 + y^2 = 1$. Use Stokes' theorem to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Question 4
Use Stokes' theorem to find $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $C = C_1 \bigcup C_2$, C_1 is the circumference of the semi circle of radius a, above the $x-$axis and C_2 is the line segment $[-a, a]$ on the $x-$axis and $\mathbf{F} = y^2(a^2 - z^2)\mathbf{i} + ax^2(a - 3z)\mathbf{j} + x^2y^2\mathbf{k}$.

Answers

Question 2

$\nabla \wedge \mathbf{F} = (\mathbf{2y},\ \mathbf{-2x},\ \mathbf{0})^{\mathbf{t}}$. *Question 3*

(a) $\nabla \wedge \mathbf{F} = (\mathbf{x - y},\ \mathbf{x - y},\ \mathbf{1})^{\mathbf{t}}$.

(b) π.

Question 4

$\dfrac{-4a^5}{3}$.

END-OF-TUTORIAL