UNIT 1 MATRICES

Unit Structure

1.0 OVERVIEW

Matrices and determinants constitute the structure of matrix algebra that pervades and enriches almost all areas in science and engineering computations. In this Unit, various types of matrices are defined before introducing briefly the topic on the algebra of matrices in general. The determinant of a matrix is then treated in detail, including the main properties associated with it. The co-factor method for finding the inverse of a matrix is also introduced.

1.1 LEARNING OBJECTIVES

By the end of this Unit, you should be able to do the following:

1. Do matrix operations

2. Compute the determinant of a matrix

3. Understand the properties of the determinant of a matrix

4. Determine the inverse of a matrix using the co-factor method.

1.2 LIST OF SPECIAL SYMBOLS

A	a general matrix
a_{ij}	(i, j) entry in matrix A
$\bar{a}_{ij}$	complex conjugate of a_{ij}
$A = [a_{ij}]_{m \times n}$	matrix A of order m by n
A_{ij}	cofactor of a_{ij}
M_{ij}	minor of a_{ij}
D	determinant of a matrix
$\lvert A \rvert$	determinant of A
$Adj(A)$	adjoint matrix A
A^{-1}	inverse of matrix A
$\mathbf{x}$	column vector
x_i	i^{th} component of vector x

$\mathbf{a_i}$	i^{th} row vector of A
$\mathbf{a^j}$	j^{th} column vector of A
R_i	i^{th} row of matrix
C_j	j^{th} column of matrix

1.3 BASIC CONCEPTS AND DEFINITIONS

Matrix

A matrix defines a rectangular array of numbers or symbols. An m x n matrix A has m rows and n columns. a_{ij} = (i, j) entry represents the number in the i^{th} row, j^{th} column.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ or } \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}. \qquad (1)$$

Square Matrix

If the number of rows of a matrix equals the number of columns ($m = n$), then the matrix is called a **square** matrix of order n. The **principal** or, **leading diagonal** of the square matrix defines the entries $a_{11}, a_{12}, \ldots, a_{nn}$. For example, the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

are square matrices of order two and three respectively. The elements a_{11}, a_{22} and b_{11}, b_{22}, b_{33} constitute the principal diagonals of the matrix *A* and *B* respectively.

Diagonal Matrix

A square matrix *A* of the form

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

is called a **diagonal** matrix. Here, $A = \left[a_{ij}\right]_{n \times n}$ with $a_{ij} = 0, \mathrm{i} \neq \mathrm{j}, \mathrm{i}, \mathrm{j} = 1, \ldots, \mathrm{n}$.

Zero Matrix, 0

The zero matrix, **0** is a matrix with all entries zero.

Unit or Identity Matrix

The **identity** or the **unit** matrix is a diagonal matrix in which all the diagonal elements $a_{ii}, \mathrm{i} = 1, \ldots, \mathrm{n}$ are equal to 1. It is denoted by *I* and is written as

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Upper Triangular Matrix

A square matrix U of the form

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix},$$

is called an upper triangular matrix. Note that all the entries below the main diagonal are zero.

Lower Triangular Matrix

A square matrix L of the form

$$L = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

is called a lower triangular matrix. This time, all the entries above the main diagonal are zero.

Transpose of a Matrix

If $A = \left[a_{ij}\right]_{m \times n}$, then the transpose of *A*, denoted by A^T, is obtained by interchanging the rows and columns of the matrix *A*, that is, $A^T = \left[a_{ji}\right]_{n \times m}$. Symbolically,

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

Clearly $\left(A^T\right)^T = A$. For example, if

$$\mathrm{A} = \begin{bmatrix} 1 & 5 & 7 & 6 \\ -2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

then,

$$A^T = \begin{bmatrix} 1 & -2 & 0 \\ 5 & 3 & 0 \\ 7 & 4 & 1 \\ 6 & 5 & 0 \end{bmatrix}.$$

Symmetric Matrix

If $A^T = A,$ the matrix $A = \left[a_{ij}\right]_{n \times n}$ is called symmetric. Examples of symmetric matrices are

$$\begin{pmatrix} -1 & 5 \\ 5 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 0 \\ 4 & 2 & -7 \\ 0 & -7 & -3 \end{pmatrix}, \begin{pmatrix} 2 & 3 & -1 & 9 \\ 3 & 4 & 0 & 2 \\ -1 & 0 & -5 & 8 \\ 9 & 2 & 8 & 1 \end{pmatrix}, \text{etc.}$$

Skew-Symmetric Matrix

A square matrix $A = \left[a_{ij}\right]_{n \times n}$ is called skew-symmetric if $A^T = -A$. Clearly the diagonal elements are zeros because $a_{ii} = -a_{ii}$ if $a_{ii} = 0$ for all i. For example

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix},$$

is a skew-symmetric matrix.

Hermitian Matrix

A square matrix $A = \left[a_{ij}\right]_{n \times n}$ is called a Hermitian matrix if

$$A^T = \overline{A},$$

where $\overline{A} = \left[\overline{a}_{ij}\right]$, and the overbar denotes complex conjugate. Hence

$$a_{ji} = \overline{a}_{ij}, \ \mathrm{i, j} = 1, \ldots, \mathrm{n}$$

for an n x n Hermitian matrix *A*. It follows that the elements along the principal diagonal of a Hermitian matrix are always real. For example, one can easily verify that

$$A = \begin{bmatrix} 2 & -1+i & 0 \\ -1-i & 2 & -i \\ 0 & i & 2 \end{bmatrix},$$

is a Hermitian matrix.

If, in addition, *A* is a real Hermitian matrix, then $A^T = A$, that is, *A* is a symmetric matrix.

<u>Skew-Hermitian Matrix</u>

A square matrix $A = \left[a_{ij}\right]_{n \times n}$ is called a skew-hermitian matrix if $A^T = -\overline{A}$.

Hence, $a_{ji} = -\overline{a}_{ij}, i, j = 1, \ldots, \mathrm{n}.\cdot$ It follows that the elements along the principal diagonal of a skew-hermitian matrix are either zeros or pure imaginaries. For example,

$$A = \begin{bmatrix} 0 & -2-i & 1 \\ 2-i & i & i \\ -1 & i & 2i \end{bmatrix},$$

a Skew-Hermitian matrix.

If, in addition, A is a real Skew-Hermitian, then $A^T = -A$, that is, A is a skew-symmetric matrix.

Note: In general AB ≠ BA, that is, matrix multiplication is **not** commutative.

(1) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix}$. BA is not defined although $AB = \begin{pmatrix} 4 & 4 \\ 5 & 3 \\ 8 & 14 \end{pmatrix}$.

(2) $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix}$. AB is 2×2 whereas BA is 3×3.

(3) $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. AB and BA are of the same size. However

$$\begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix} = AB \neq BA = \begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix}.$$

MATRIX RULES OF ALGEBRA

Addition is commutative

$A + B = B + A$

Associative Laws

$$(A+B)+C = A+(B+C)$$

$$(AB)C = A(BC)$$

$$\lambda(AB) = (\lambda A)B = A(\lambda B)$$

Distributive Laws

$$A(B+C) = AB + AC$$

$$(A+B)C = AC + BC$$

$$\lambda(A+B) = \lambda A + \lambda B$$

Additive Identity

$$A + 0 = A$$

$$A - A = 0$$

$$0A = A0 = 0$$

Multiplicative Identity

$$AI = IA = A$$

1.4 DETERMINANT OF A MATRIX

To each matrix $A = \left[a_{ij}\right]_{n \text{ x } n}$ is associated a unique real number called the **determinant** of *A*. This number contains important information about the matrix and is written as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \qquad (2)$$

Sometimes the capital letter D is used to denote $|A|$.

Note: The determinant of a rectangular *m* x *n* matrix $(m \neq n)$ is not defined.

Definition

For 2 x 2 matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

To extend this to an *n* x *n* matrix the following two definitions are required.

Definition

The *(i, j)* **minor** of *A* is the determinant of the $(n - 1)$ x $(n - 1)$ matrix obtained by removing the i^{th} row and j^{th} column of ***A*****,** denoted M_{ij}.

<u>Example 1</u>

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ 1 & 3 & 1 \end{pmatrix}, \quad M_{13} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix} = 13 \text{ and } M_{21} = \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = 11.$$

Definition

The *(i,j)* **cofactor** of A is

$$A_{ij} = (-1)^{i+j} \, M_{ij}. \qquad (3)$$

From Example 1, $A_{13} = (-1)^{(1+3)} M_{13} = 13$ and $A_{21} = (-1)^{(2+1)} M_{21} = -11$.

It follows that the cofactor A_{ij} is either M_{ij} or - M_{ij} depending on whether the sum i + j is even or odd. A_{ij} can be associated with the **entry** a_{ij} as follows.

1) Cross out the row and the column containing a_{ij}.

2) Evaluate the determinant of the $(n$ - 1) x $(n$ - 1) matrix which remains.

3) Assign a "+" or "-" sign according to the position of a_{ij} on the checkerboard below,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \text{ and in general } \begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Development of a Determinant

If A is an n x n matrix, then the determinant of A can be computed by multiplying the entries of **any** row (or column) by their cofactors and summing the resulting products. If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$|A| = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$ (cofactor expansion by row 1).

$|A| = a_{21}A_{21} + a_{22}A_{22} + \ldots + a_{2n}A_{2n}$ (cofactor expansion by row 2).

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$ (cofactor expansion by row i).

$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}$ (cofactor expansion by column j).

Example 2

Evaluate the determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 5 & 6 & 7 \end{pmatrix}$.

Solution

The cofactor expansion by the first row is given by

$$|A| = 1C_{11} + 2C_{12} + 3C_{13} = 1M_{11} - 2\mathrm{M}_{12} + 3\mathrm{M}_{13} = 1\begin{vmatrix} 4 & 0 \\ 6 & 7 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 5 & 7 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 5 & 6 \end{vmatrix} = -32.$$

Similarly the cofactor expansion by the second row is given by

$$|A| = 0C_{21} + 4C_{22} + 0C_{23} = 4M_{22} = 4\begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} = 4(7-15) = -32.$$

As an exercise, you can check $|A|$ using the cofactor expansion by column one.

Note: It is better to choose to expand the determinant using a row or column which will lead to less or simpler computations. For instance, it is easier and quicker to expand by the row or column that contains zero entries.

Example 3

Solve the equation $\begin{vmatrix} -2-x & 2 & -3 \\ 2 & 1-x & -6 \\ -1 & -2 & -x \end{vmatrix} = 0.$

Solution

The development of the determinant by the first column may be written as

$$\begin{aligned}
&(-2-x)\begin{vmatrix} 1-x & -6 \\ -2 & -x \end{vmatrix} - 2\begin{vmatrix} 2 & -3 \\ -2 & -x \end{vmatrix} - \begin{vmatrix} 2 & -3 \\ 1-x & -6 \end{vmatrix} = 0 \\
&(-2-x)[-x(1-x)-12] - 2[-2x-6] - [-12+3(1-x)] = 0 \\
&\text{or } -x^3 - x^2 + 21x + 45 = 0
\end{aligned}$$

It is easily verified that x = -3, -3 and 5 satisfy the above equation.

Activity 1

1. Evaluate the determinants:

(i) $\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix}$,

(ii) $\begin{vmatrix} 1 & 3 & 3 \\ 2 & 6 & 3 \\ -1 & 1 & 2 \end{vmatrix}$,

(iii) $\begin{vmatrix} 2 & 1 & -1 \\ -1 & 10 & 1 \\ 2 & -3 & 1 \end{vmatrix}$.

2. Solve the equation $\begin{vmatrix} 4x+3 & 2 & 1 \\ x & 3 & 4 \\ 2x-1 & 1 & -1 \end{vmatrix} = 0$.

1.5 COFACTOR METHOD FOR A^{-1}

If $|A| \neq 0,$ we show that A^{-1} exists by constructing it as follows:

First we need to define the adjoint matrix.

Adjoint Matrix

The matrix of cofactors of A defines the matrix whose (i,j) entry is A_{ij}, the (i,j) cofactor of A. The transpose of the matrix of the cofactors is called the **adjoint** matrix and is

$$Adj(A)=\begin{bmatrix} A_{11} & A_{21} & A_{n1} \\ A_{12} & A_{22} & A_{n2} \\ \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & A_{nn} \end{bmatrix},$$

denoted by $Adj(A)$.

For example, if $A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & 1 \\ 3 & 3 & 2 \end{bmatrix}$

then

$$\begin{aligned} A_{11} &= 9, & A_{12} &= -3, & A_{13} &= -9, \\ A_{21} &= -7, & A_{22} &= 5, & A_{23} &= 3, \\ A_{31} &= 8, & A_{32} &= -4, & A_{33} &= 0, \end{aligned}$$

and

$$Adj(A)=\begin{bmatrix} 9 & -7 & 8 \\ -3 & 5 & -4 \\ -9 & 3 & 0 \end{bmatrix}.$$

Definition

$A = \left[a_{ij}\right]_{n \times n}$ is invertible or non-singular if and only if there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. A^{-1} is given by

$$A^{-1} = \frac{1}{|A|} Adj(A). \tag{4}$$

From (4) it follows that A^{-1} exists if and only if $|A| \neq 0$.

If A^{-1} exists, that is A is invertible, then A is called a non-singular matrix. If A^{-1} does not exist then A is called a singular matrix.

Properties of Inverse

- A^{-1} is unique.
- $(A^{-1})^{-1} = A.$
- $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}, \lambda$ a scalar.
- $\left|A^{-1}\right| = \frac{1}{|A|}.$
- If A^{-1}, B^{-1} exist, then $(AB)^{-1} = B^{-1}A^{-1}.$

Note: There is no way to define matrix division. Only invertible matrices have a multiplicative inverse. But be careful! Matrix multiplication is not commutative; if you multiply an equation through by a matrix, you must multiply *both* sides on the right or both sides on the left.

Example 4

If possible, determine the inverse of the following matrices:

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & 1 \\ 3 & 3 & 2 \end{bmatrix}, \mathrm{C} = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}, \mathrm{D} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}.$$

Solution

$$A^{-1} = \frac{1}{(a_{11}a_{22} - a_{21}a_{12})}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

$$B^{-1} = \frac{1}{|B|}Adj(B) = \frac{1}{12}\begin{bmatrix} 9 & -7 & 8 \\ -3 & 5 & -4 \\ -9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 9/12 & -7/12 & 8/12 \\ -3/12 & 5/12 & -4/12 \\ -9/12 & 3/12 & 0 \end{bmatrix}.$$

C^{-1} does not exist because $|C| = 0.$ In fact, C is a singular matrix.

$$D^{-1} = \begin{bmatrix} 1/a_{11} & 0 & 0 \\ 0 & 1/a_{22} & 0 \\ 0 & 0 & 1/a_{33} \end{bmatrix}.$$

Activity 2

For each of the matrices given below, compute the adjoint matrix and hence determine the corresponding inverse matrix.

(i) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$

(ii) $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}.$

1.6 PROPERTIES OF DETERMINANTS

1. If a row (or column) of A consists entirely of zeros, then $|A| = 0$.

2. $|A^T| = |A|$.

3. If $A = I$, then $|A| = 1$.

4. If any two rows (or columns) of A are interchanged (or swapped) producing a matrix B, then $|B| = -|A|$.

5. If two rows (or columns) of A are identical, then $|A| = 0$.

6. If any row (or column) of A is multiplied by a scalar k producing a matrix B, then $|B| = k|A|$. For example,

$$\begin{vmatrix} a_{11} & ka_{12} & a_{13} \\ a_{21} & ka_{22} & a_{23} \\ a_{31} & ka_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

7. If a scalar multiple of one row (or column) is added to some other row (or column) producing a matrix B then $|B| = |A|$. For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + ka_{11} & a_{22} + ka_{12} \end{bmatrix}, \text{ then}$$

$$|B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ ka_{11} & ka_{12} \end{vmatrix} = |A| + k\begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix}$$

$$= |A| + 0 \quad (\text{ using property } \mathbf{5.})$$

$$= |A|.$$

8. If $A_1, A_2, A_3, \ldots$ are all *n* x *n* matrices then $|A_1 A_2| = |A_1||A_2|$ and, in general,

$$|A_1 A_2 A_3 \ldots| = |A_1||A_2||A_3| \ldots$$

9. If *A* is upper triangular, lower triangular, or diagonal, the determinant of *A* is given by the product of the diagonal entries.

10. If the cofactors of one row are multiplied by the entries of a different row, the result is zero.

Example 5

Show that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$.

Solution

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} \text{ (using property } \mathbf{7}.)$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} \text{ (using property } \mathbf{6}.)$$

$$= 0 \text{ (using property } \mathbf{5}.).$$

Activity 3

By using the properties of the determinant, show that

(i) $\begin{vmatrix} x+a & a & a \\ a & x+a & a \\ a & a & x+a \end{vmatrix} = x^2(x+3a)$.

(ii) $\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = (x-y)(x-z)(y-z).$

1.7 SUMMARY

In this Unit, you have studied matrices and their operations, including the determinant of a matrix. You have learnt the different properties of the determinant of a matrix, as well as, applying the co-factor method to obtain the inverse of a square non-singular matrix.

1.8 SUPPLEMENTARY EXERCISES

1. Given that $a \neq 0$, show that for all real x

(i) $\begin{vmatrix} e^{ax} & ae^{ax} \\ xe^{ax} & (1+ax)e^{ax} \end{vmatrix} \neq 0,$

(ii) $\begin{vmatrix} e^{ax}\sin x & e^{ax}\cos x \\ e^{ax}(\cos x + a\sin x) & e^{ax}(a\cos x - \sin x) \end{vmatrix} \neq 0,$

2. Solve the following determinantal equations:

(i) $\begin{vmatrix} x & x & x \\ y & x & x \\ 0 & y & x \end{vmatrix} = 0, x \neq 0.$

(ii) $\begin{vmatrix} x^3 & 5 & 1 \\ 8 & x^2+1 & 1 \\ 1 & x^2+4 & 1 \end{vmatrix} = 0.$

3. By using the properties of the determinant, show that

$$\begin{vmatrix} x_1+y_1 & x_2+y_2 & x_3+y_3 \\ y_1+z_1 & y_2+z_2 & y_3+z_3 \\ z_1+x_1 & z_2+x_2 & z_3+x_3 \end{vmatrix} = 2\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

4. Prove that

$$\frac{d}{dx}\begin{vmatrix} f & g & h \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} f' & g' & h' \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ p' & q' & r' \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ p & q & r \\ u' & v' & w' \end{vmatrix}.$$

1.8 ANSWERS TO ACTIVITIES & SUPPLEMENTARY EXERCISES

Activity 1

1. (i) -2; (ii) 12; (iii) 46.

2. $-\dfrac{26}{15}$.

Activity 2

(i) $A^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$

(ii) $$B^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 1 & -2 \\ -3 & -1 & 5 \\ 0 & 1 & -2 \end{bmatrix}..$$

Supplementary Exercises

2. (i) $x = y$.

   (ii) $x = 1, 2, -2/3$.