UNIT 2 SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

Unit Structure

2.0 OVERVIEW

This Unit considers different methods for solving a system of linear equations. Cramer's rule for solving three equations in three unknowns provides a method of obtaining the solution immediately. However, we need to evaluate four determinants of order three and so it is not a practical method if the system of equations is a large one. Gauss Elimination method is based upon reducing the augmented matrix corresponding to a given system of equations to row echelon form. The solution is easily obtained by back substitution. The LU- factorization method expresses the coefficient matrix in terms of a lower and an upper triangular matrix. After discussing a method for finding the inverse of a matrix using row operations only, we eventually introduce the concept of rank of a matrix, including the main results associated with it.

2.1 LEARNING OBJECTIVES

By the end of this Unit, you should be able to do the following:

1. Solve matrix equations using

   (i) Cramer's Rule,

   (ii) Gauss Elimination Method,

   (iii) LU-Factorisation Method.

2. Invert a matrix using row operations only.

3. Solve matrix equations by inversion.

4. Determine the rank of a matrix.

2.2 SYSTEMS OF LINEAR EQUATIONS

A system of linear equations can be written in matrix notation:

$$\begin{array}{l} 4x_1 - x_2 + 2x_3 = 15 \\ -x_1 + 2x_2 + 3x_3 = 5 \\ 5x_1 - 7x_2 + 9x_3 = 8 \end{array} \qquad \begin{pmatrix} 4 & -1 & 2 \\ -1 & 2 & 3 \\ 5 & -7 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \\ 8 \end{pmatrix} \qquad A\mathbf{x} = \mathbf{b}. \qquad (1)$$

A defines the matrix of coefficients and is called the coefficient matrix, **x** is a column matrix of three unknowns x_1, x_2, x_3 and **b** is a column matrix of 3 constants.

A set of values of x_1, x_2, x_3, which satisfy all the three equations above is called the solution of the system of equations. If the column vector **b** is appended to the right of the coefficient matrix A in the form $[A\ |\mathbf{b}]$ or simply $[A\ \ \mathbf{b}]$ then the matrix is called the <u>augmented matrix</u>. Here

$$[A\ \ \mathbf{b}] = \begin{pmatrix} 4 & -1 & 2 & 15 \\ -1 & 2 & 3 & 5 \\ 5 & -7 & 9 & 8 \end{pmatrix}.$$

A system of m equations in n unknowns, $A\mathbf{x} = \mathbf{b}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

2.3 CRAMER'S RULE

It is very important to note that Cramer's rule can be used for solving a system of equations only if the coefficient matrix A is invertible; even then for large matrices the method is rather slow. Suppose we need to solve the following system of equations by applying Cramer's rule.

$$\begin{pmatrix} 4 & -1 & 2 \\ -1 & 2 & 3 \\ 5 & -7 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \\ 8 \end{pmatrix}.$$

Here $\quad A = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 2 & 3 \\ 5 & -7 & 9 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 15 \\ 5 \\ 8 \end{pmatrix}.$

First we find D, the determinant of the coefficient matrix A.

$$D = \begin{vmatrix} 4 & -1 & 2 \\ -1 & 2 & 3 \\ 5 & -7 & 9 \end{vmatrix} = 126.$$

Let A_j be the matrix obtained from A by replacing the j^{th} column of A with the vector $\mathbf{b}$.

We then compute D_j, the corresponding determinants of the matrices A_j. Hence

$$D_1 = \begin{vmatrix} 15 & -1 & 2 \\ 5 & 2 & 3 \\ 8 & -7 & 9 \end{vmatrix} = 504, \qquad D_2 = \begin{vmatrix} 4 & 15 & 2 \\ -1 & 5 & 3 \\ 5 & 8 & 9 \end{vmatrix} = 378, \qquad D_3 = \begin{vmatrix} 4 & -1 & 15 \\ -1 & 2 & 5 \\ 5 & -7 & 8 \end{vmatrix} = 126.$$

Then the unique solution vector $\mathbf{x} = (x_1, x_2, x_3)^T$ is given by $x_j = \dfrac{D_j}{D}$, so that

$$x_1 = \frac{504}{126} = 4, \qquad x_2 = \frac{378}{126} = 3, \qquad x_3 = \frac{126}{126} = 1.$$

<u>Activity 1</u>

Solve the following systems of equations with the help of Cramer's rule.

(i) $$\begin{aligned} -x_1 + 3x_2 - 2x_3 &= 7 \\ 3x_1 \qquad + 3x_3 &= -3 \\ 2x_1 + x_2 + 2x_3 &= -1 \end{aligned}$$

(ii) $$\begin{aligned} 2x_1 + 3x_2 - 4x_3 &= -3 \\ 3x_1 - 2x_2 + 5x_3 &= 24 \\ x_1 + 4x_2 - 3x_3 &= -6 \end{aligned}$$

2.4 GAUSS ELIMINATION METHOD

Before introducing the Gauss Elimination method, one should understand the meaning of elementary row operations and the row-echelon form of a matrix.

Definition

There are three types of elementary row operations, namely:

Ro l multiply a row by a non-zero constant.

Ro 2 interchange two rows.

Ro 3 add a multiple of one row to another.

Row Echelon Form

To reduce the matrix A (or augmented matrix $[A \;\; \mathbf{b}]$) to what is called row echelon form, the system of equations becomes easy to deal with. Below are examples of matrices which are in row echelon form:

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & -5 \\ 0 & 7 & 0 & 2 \\ 0 & 0 & 8 & -3 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & -5 \\ 0 & 7 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{etc.}$$

Reduction of a matrix to row echelon form

Step1.

Find the leftmost column of the matrix that is not all zeros.

Step 2.

Interchange rows to get a non-zero entry at the top of this column. It is called a leading (or pivot) entry.

Step 3.

Add suitable multiples of the top row to rows below to make all entries below the leading entry become zero.

Step 4.

Cover up the top row and apply steps 1 to 3 again.

Notes:

1. We can reduce a matrix to its row echelon form by applying **only** elementary row operations to it.

2. Some texts insist that leading entries must be equal to 1 to be in echelon form. We shall not insist on this.

3. For a matrix to be in row echelon form, any rows which are full of zeros, must be grouped together at the bottom of the matrix.

4. Variables corresponding to leading entries are called **leading variables** whereas variables corresponding to non-leading entries are called **non-leading variables** or **free variables**.

Gauss Elimination method is a reduction of the augmented matrix $[A\ \ \mathbf{b}]$ to row echelon form. We will now illustrate the method by an example.

Example 1

Solve the system of equations

$$\begin{aligned}
4x_1 - x_2 + 2x_3 &= 15 \\
-x_1 + 2x_2 + 3x_3 &= 5 \\
5x_1 - 7x_2 + 9x_3 &= 8
\end{aligned}$$

using the Gauss-Elimination Method.

Solution

First we reduce the augmented matrix to its row echelon form

$$[A\ \ \mathbf{b}] = \left[\begin{array}{ccc|c} 4 & -1 & 2 & 15 \\ -1 & 2 & 3 & 5 \\ 5 & -7 & 9 & 8 \end{array}\right]\begin{array}{l} \ldots\ldots\ldots \mathrm{R}_1 \\ \ldots\ldots\ldots \mathrm{R}_2 \\ \ldots\ldots\ldots \mathrm{R}_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & -1 & 2 & 15 \\ 0 & \frac{7}{4} & \frac{7}{2} & \frac{35}{4} \\ 0 & \frac{-23}{4} & \frac{26}{4} & \frac{-43}{4} \end{array}\right]\begin{array}{l} \ldots\ldots\ldots \mathrm{R}_1 \\ \ldots\ldots\ldots \mathrm{R}_2 - \left(\frac{-1}{4}\right)\mathrm{R}_1 = \mathrm{R}_2^1 \\ \ldots\ldots\ldots \mathrm{R}_3 - \frac{5}{4}\mathrm{R}_1 = \mathrm{R}_3^1 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & -1 & 2 & 15 \\ 0 & \frac{7}{4} & \frac{7}{2} & \frac{35}{4} \\ 0 & 0 & \frac{72}{4} & \frac{72}{4} \end{array}\right]\begin{array}{l} \ldots\ldots\ldots \mathrm{R}_1 \\ \ldots\ldots\ldots \mathrm{R}_2^1 \\ \ldots\ldots\ldots \mathrm{R}_3^1 + \frac{23}{7}\mathrm{R}_2^1 = \mathrm{R}_3^2 \end{array}$$

The last augmented matrix is in row echelon form and can be read as:

$$\begin{aligned} 4x_1 - x_2 + 2x_3 &= 15 \\ \frac{7}{4}x_2 + \frac{7}{2}x_3 &= \frac{35}{4} \\ \frac{72}{4}x_3 &= \frac{72}{4} \end{aligned}$$

This is an upper triangular system and we can solve for x_3, x_2 and x_1 by back substitution.

We get

$$x_3 = 1\,, \qquad x_2 = 3\,, \qquad x_1 = 4.$$

Notes:

1. We can understand the Gauss Elimination method as being an application of a series of elementary row operations that reduces the augmented system $[A\ \ \mathbf{b}]$ to an upper triangular system $[U\ \ \mathbf{c}]$, that is ,

$$[A \mid \mathbf{b}] \xrightarrow[\text{Elimination method}]{\text{Gauss}} [U \mid \mathbf{c}].$$

2. Matrices U and A are called row-equivalent, as U is obtained from A only by applying row operations. Since row-equivalent linear systems of equations have the same sets of solutions, it follows that matrix system U has the same solution as matrix system A.

3. Unlike with the calculation of determinants, we are not allowed to perform column operations during Gaussian Elimination.

2.5 LU-FACTORIZATION METHOD

In this method, the coefficient matrix A of the system of equations defined by $A\mathbf{x} = \mathbf{b},$ is factorised into the product of a lower triangular matrix L and an upper triangular matrix U. We write A as

$$A = LU = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}. \tag{2}$$

Using the matrix multiplication rule to multiply the matrices L and U and comparing the elements of the resulting matrix with those of A we get

$$\ell_{i1}u_{1j} + \ell_{i2}u_{2j} + \ell_{i3}u_{3j} = a_{ij} \quad i, j = 1, 2, 3. \tag{3}$$

The 9-equations (3) have 12 unknowns. We therefore choose 3 unknowns, either u_{ii} = 1 or $\ell_{ii} = 1, i = 1, 2, 3.$ Equations (3) become

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

or

$$\begin{aligned} &\ell_{11} = a_{11}\,,\ \ \ell_{11}u_{12} = a_{12}\,,\ \ \ell_{11}u_{13} = a_{13} \\ &\ell_{21} = a_{21}\,,\ \ \ell_{21}u_{12} + \ell_{22} = a_{22}\,,\ \ \ell_{21}u_{13} + \ell_{22}u_{23} = a_{23} \\ &\ell_{31} = a_{31}\,,\ \ \ell_{31}u_{12} + \ell_{32} = a_{32}\,,\ \ \ell_{31}u_{13} + \ell_{32}u_{23} + \ell_{33} = a_{33}. \end{aligned} \tag{5(a)}$$

We note that the first column of the matrix L is identical with the first column of the matrix A, i.e.,

$$\ell_{11} = a_{11}\ , \quad \ell_{21} = a_{21}\ , \quad \ell_{31} = a_{31}. \tag{5(b)}$$

Next we note that the elements of the first row of U are given by

$$u_{12} = \frac{a_{12}}{\ell_{11}}\ , \qquad u_{13} = \frac{a_{13}}{\ell_{11}}. \tag{5(c)}$$

The first column of L and the first row of U have been determined.

We can now proceed to determine the second column of L and the second row of U.

We get

$$\ell_{22} = a_{22} - \ell_{21}\, u_{12}\ , \quad \ell_{32} = a_{32} - \ell_{31}\, u_{12}, \tag{5(d)}$$

and

$$u_{23} = \frac{1}{\ell_{22}}\left(a_{23} - \ell_{21}\, u_{13}\right). \tag{5(e)}$$

Next we find the third column of L. We have

$$\ell_{33} = a_{33} - \ell_{31} u_{13} - u_{32} u_{23}. \tag{5(f)}$$

Having determined the matrices of L and U, the system of equations $A\mathbf{x} = \mathbf{b}$ become

$$LU\mathbf{x} = \mathbf{b}. \tag{6}$$

We write (6) as the following two systems of equations

$$U\mathbf{x} = \mathbf{z}\,, \tag{7(a)}$$

$$L\mathbf{z} = \mathbf{b}\,, \tag{7(b)}$$

where

$$\mathbf{z} = \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix}^T .$$

Equation 7(b) may be written as

$$\begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

The solution values z_1, z_2 and z_3 are given by

$$\begin{aligned} z_1 &= b_1 / \ell_{11} \\ z_2 &= \left(b_2 - \ell_{21}\, z_1\right) / \ell_{22} \\ z_3 &= \left(b_3 - \ell_{31} z_1 - \ell_{32} z_2\right) / \ell_{33} \end{aligned} \tag{8}$$

Having determined z_1, z_2 , z_3 we solve the system of equations 7(a)

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.$$

The solution values x_3, x_2, x_1 are given by

$x_3 = z_{3,}$

$x_2 = z_2 - u_{23}\ x_{3,}$

$x_1 = z_1 - u_{12}\ x_2 - u_{13}\ x_{3.}$ (9)

Thus, the solution values $x_1,\ x_2,\ x_3$ of the system of equation $A\mathbf{x} = \mathbf{b}$ have been determined.

Example 2 Solve the system of equations

$$\begin{bmatrix} 4 & -1 & 2 \\ -1 & 2 & 3 \\ 5 & -7 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 8 \end{bmatrix}$$

by the LU-factorization method.

Solution Using 6(b) and 6(c) we write $A = LU$ as

$$\begin{bmatrix} 4 & -1 & 2 \\ -1 & 2 & 3 \\ 5 & -7 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ -1 & \ell_{22} & 0 \\ 5 & \ell_{32} & \ell_{33} \end{bmatrix}\begin{bmatrix} 1 & -1/4 & 1/2 \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Firstly, we determine the second column of L :

$$\frac{1}{4}+\ell_{22}=2 \quad , \quad \ell_{22}=\frac{7}{4}$$

$$-\frac{5}{4}+\ell_{32}=-7 \ , \quad \ell_{32}=-\frac{23}{4}$$

Secondly, we determine the only unknown in the second row of U, namely u_{23}

$$-\frac{1}{2}+\ell_{22}\,u_{23}=3 \quad , \quad u_{23}=\left(3+\frac{1}{2}\right)\frac{4}{7}=2$$

Finally, we determine the remaining unknown in the third column of L, namely l_{33}:

$$\frac{5}{2}+\ell_{32}u_{23}+\ell_{33}=9$$

$$\ell_{33}=9-\frac{5}{2}-\left(-\frac{23}{4}\right)(2)=9-\frac{5}{2}+\frac{23}{2}=18$$

The matrices L and U become

$$L=\begin{bmatrix} 4 & 0 & 0 \\ -1 & 7/4 & 0 \\ 5 & -23/4 & 18 \end{bmatrix}, U=\begin{bmatrix} 1 & -1/4 & 1/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

From 7(b), $L\mathbf{z}=\mathbf{b}$ is written as

$$\begin{bmatrix} 4 & 0 & 0 \\ -1 & 7/4 & 0 \\ 5 & -23/4 & 18 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} 15 \\ 5 \\ 8 \end{bmatrix},$$

which by using forward substitution, gives

$$z_1 = 15/4\,,\ z_2 = 5\,,\ z_3 = 1\,.$$

Now from 7(a), $U\mathbf{x} = \mathbf{z}$ is written as

$$\begin{bmatrix} 1 & -1/4 & 1/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15/4 \\ 5 \\ 1 \end{bmatrix},$$

which by using backward substitution gives

$$x_1 = 4, x_2 = 3, x_3 = 1.$$

2.6 MATRIX INVERSION BY ROW OPERATIONS

This method is based on the idea that only elementary row operations can be used to reduce an *n* x *n* matrix *A* to the *n* x *n* identity matrix *I*. The next example considers the case when *A* is a 3 x 3 matrix.

Example 3

Find A^{-1} given that $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{pmatrix}$.

Solution:

We consider the augmented matrix [*A*: I] and perform a sequence of elementary row operations so that *A* is carried into I and I is carried into A^{-1}.

$$[A:I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 6 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \ldots\ldots\ldots R_1 \\ \ldots\ldots\ldots R_2 \\ \ldots\ldots\ldots R_3 \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \ldots\ldots\ldots R_2 - R_1 \\ \ldots\ldots\ldots R_3 + R_1 \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \begin{array}{l} \\ \\ \ldots\ldots\ldots R_3 - 2R_2 \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -11 & 8 & -4 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \begin{array}{l} \ldots\ldots\ldots R_1 - 4R_3 \\ \ldots\ldots\ldots R_2 - 2R_3 \\ \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \begin{array}{l} \ldots\ldots\ldots R_1 - 2R_2 \\ \\ \end{array}$$

$$= [I : A^{-1}]$$

Thus,

$$A^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ -7 & 5 & -2 \\ 3 & -2 & 1 \end{bmatrix}.$$

This method can also be used to solve a system of equations of the form:

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 2 \end{bmatrix}.$$

Then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -6 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 0 \\ -7 & 5 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -6 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -27 \\ 11 \end{bmatrix}$$

Hence, $x_1 = 9$, $x_2 = -27$, $x_3 = 11$.

<u>Activity 2</u>

Solve the following systems of equations

(i)	$x_1 + x_2 + x_3 = 1$	(ii)	$-x_2 + x_3 = 1$
	$-x_1 - x_2 + x_3 = 1$		$2x_1 + 2x_2 + 2x_3 = 2$
	$-x_1 + x_2 - x_3 = 2$		$2x_1 + 4x_2 - x_3 = 1$

using (a) Gauss-Elimination Method

(b) LU-Factorisation Method

(c) Matrix Inversion by row operations only.

2.7 RANK OF A MATRIX

Before introducing the rank of a matrix, let us define the terms linear dependence and linear independence of the rows or columns of a rectangular matrix.

Definition

Linear dependence of row and column vectors of a matrix

Let $A = \left[a_{ij}\right]_{m \times n}$ where a_{ij}'s are real or complex numbers. Let $\mathbf{a_1}$, $\mathbf{a_2}$,...,$\mathbf{a_m}$ denote the m row vectors of A. Hence

$$\mathbf{a_1} = \left(a_{11} \quad a_{12} \quad \ldots \quad a_{1n}\right), \mathbf{a_2} = \left(a_{21} \quad a_{22} \quad \ldots \quad a_{2n}\right), \ldots, \mathbf{a_m} = \left(a_{m1} \quad a_{m2} \quad \ldots \quad a_{mn}\right).$$

The row vectors $\mathbf{a_1}$, $\mathbf{a_2}$,...,$\mathbf{a_m}$ are said to be linearly dependent if there exist scalars α_1, α_2,…,α_m , **not all** equal to zero, such that

$$\alpha_1 \mathbf{a_1} + \alpha_2 \mathbf{a_2} + \ldots + \alpha_m \mathbf{a_m} = \mathbf{0} = \left(0 \quad 0 \quad \cdots \quad 0\right). \tag{10}$$

Otherwise, the m row vectors are linearly independent. In this case, **all** $\alpha_i = 0$ in (10).
The linear dependence and independence of the column vectors are defined in a similar manner.

Let $\mathbf{a^1}$, $\mathbf{a^2}$, …,$\mathbf{a^n}$ denote the n column vectors of A. Hence

$$\mathbf{a^1} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \mathbf{a^2} = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{2m} \end{pmatrix}, \ldots, \mathbf{a^n} = \begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{pmatrix}.$$

The column vectors $\mathbf{a^1}$, $\mathbf{a^2}$,…,$\mathbf{a^n}$ are said to be linearly dependent if there exist scalars β_1, β_2, …,β_n, **not all** equal to zero, such that

$$\beta_1 \mathbf{a^1} + \beta_2 \mathbf{a^2} + \ldots + \beta_n \mathbf{a^n} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \qquad (11)$$

Otherwise, the n column vectors are linearly independent. In this case **all** $\beta_i = 0$ in (11).

For instance, if $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$, one can easily verify that the column vectors

$$a^1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad a^2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \text{and } a^3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

are linearly independent. In fact, by setting

$$\beta_1 \mathbf{a^1} + \beta_2 \mathbf{a^2} + \beta_3 \mathbf{a^3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

this will give rise to three equations in β_1, β_2, β_3. Upon solving simultaneously these three equations, the unique solution $\beta_1 = \beta_2 = \beta_3 = 0$ is obtained.

Definition

The rank of a matrix *A* equals the **maximum** number of linearly independent row (or column) vectors of *A*.

Note: A zero matrix is said to have rank zero.

We shall often make use of the next result to compute the rank of a matrix.

Result 1

The rank of a matrix remains unchanged under elementary row (or column) operations such as

(i) Interchange of two rows (or columns).
(ii) Multiplication of a row (or column) by a nonzero constant.
(iii) Addition of a constant multiple of one row (or column) to another row (or column).

From this result, one may deduce that row equivalent matrices have the same rank. Furthermore, the rank of a matrix A can be determined directly after reducing it to echelon form A^*, using elementary row operations. It is exactly equal to the number of **non-zero** rows of A^*.

Result 2

The rank of a matrix A is the same as that of its transpose. In other words, A and A^T have the same rank.

Result 3

For any m x n matrix, rank $A \leq$ smallest of (m,n).

Example 4

Determine the ranks of the following matrices:

(i) $A = \begin{bmatrix} -1 & 2 & 5 \\ 4 & 0 & 3 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ (iii) $C = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 3 & 6 \\ 4 & 6 & 12 \end{bmatrix}$

Solution

(i) The rank of A is 2 since $A^* = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 8 & 23 \end{bmatrix}$ and A^* has two non-zero rows.

(ii) The rank of B is 2 since $B^* = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ and B^* has two non-zero rows.

(iii) The rank of C is 1 since $C^* = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and C^* has one non-zero row.

Note: The rank of an n x n matrix A is n if $|A| \neq 0$. In this case A^* has n non-zero rows.

Otherwise, i.e, if A^* has at least one zero row, $|A^*| = |A| = 0$.

Example 5

By reducing the following matrices to echelon form, determine their ranks.

(i) $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}$ (ii) $B=\begin{bmatrix}5 & 3 & 14 & 4\\ 0 & 1 & 2 & 1\\ 1 & -1 & 2 & 0\end{bmatrix}$ (iii) $C=\begin{bmatrix}1 & 1+i & -i\\ 0 & i & 1+2i\\ 1 & 1+2i & 1+i\end{bmatrix}$

Solution

(i) $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}\begin{matrix}\dots\dots\dots\mathrm{R}_1\\ \dots\dots\dots\mathrm{R}_2\\ \dots\dots\dots\mathrm{R}_3\end{matrix}$

$$\to\begin{bmatrix}1 & 2 & 3\\ 0 & -4 & -8\\ 0 & -3 & -3\end{bmatrix}\begin{matrix}\dots\dots\dots\mathrm{R}_1\\ \dots\dots\dots\mathrm{R}_2-3\mathrm{R}_1=\mathrm{R}_2^1\\ \dots\dots\dots\mathrm{R}_3-2\mathrm{R}_1=\mathrm{R}_3^1\end{matrix}$$

$$\to\begin{bmatrix}1 & 2 & 3\\ 0 & -4 & -8\\ 0 & 0 & 3\end{bmatrix}\begin{matrix}\dots\dots\dots\mathrm{R}_1\\ \dots\dots\dots\mathrm{R}_2^1\\ \dots\dots\dots\mathrm{R}_3^1-\frac{3}{4}\mathrm{R}_2^1=R_3^2\end{matrix}$$

So, the rank of A is 3 since A^* has three non-zero rows.

(ii) $B=\begin{bmatrix}5 & 3 & 14 & 4\\ 0 & 1 & 2 & 1\\ 1 & -1 & 2 & 0\end{bmatrix}\begin{matrix}\dots\dots\dots\mathrm{R}_1\\ \dots\dots\dots\mathrm{R}_2\\ \dots\dots\dots\mathrm{R}_3\end{matrix}$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix} \begin{matrix} \ldots\ldots\ldots\ldots R_1 \leftrightarrow R_3 \\ \ldots\ldots\ldots\ldots R_2 \\ \ldots\ldots\ldots\ldots R_3 \leftrightarrow R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix} \begin{matrix} \ldots\ldots\ldots\ldots R_1 \\ \ldots\ldots\ldots\ldots R_2 \\ \ldots\ldots\ldots\ldots R_3 - 5R_1 = R_3^1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix} \begin{matrix} \ldots\ldots\ldots R_1 \\ \ldots\ldots\ldots R_2 \\ \ldots\ldots\ldots R_3^2 = R_3^1 - 8R_2 \end{matrix}$$

The rank of B is 3 since B^* has three non-zero rows.

(iii) $$C = \begin{bmatrix} 1 & 1+i & -i \\ 0 & i & 1+2i \\ 1 & 1+2i & 1+i \end{bmatrix} \begin{matrix} \ldots\ldots\ldots\ldots R_1 \\ \ldots\ldots\ldots\ldots R_2 \\ \ldots\ldots\ldots\ldots R_3 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1+i & -i \\ 0 & i & 1+2i \\ 0 & i & 1+2i \end{bmatrix} \begin{matrix} \ldots\ldots\ldots\ldots R_1 \\ \ldots\ldots\ldots\ldots R_2 \\ \ldots\ldots\ldots\ldots R_3 - R_1 = R_3^1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1+i & -i \\ 0 & i & 1+2i \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \ldots\ldots\ldots\ldots R_1 \\ \ldots\ldots\ldots\ldots R_2 \\ \ldots\ldots\ldots\ldots R_3^1 - R_2 = R_3^2 \end{matrix}$$

The rank of C is 2 since C^* has two non-zero rows.

Activity 3

Determine the rank of the following matrices:

(i) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & -4 \\ -1 & 0 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 7 \\ 1 & -1 & 5 \\ 1 & 1 & 2 \end{bmatrix}$

2.8 SUMMARY

In this unit you have learnt how a unique solution to linear systems of equations can be obtained using Cramer's rule, the Gauss-Elimination Method and the LU-Factorization Method. You have studied how to obtain the inverse of a square non-singular matrix by using row operations only. Finally the rank of a matrix has been introduced which can easily be determined after reducing the matrix to its row echelon form.

2.9 SUPPLEMENTARY EXERCISES

1. Solve the following systems of equations

$$\begin{aligned} 2x_1 - x_2 + 2x_3 &= 2 \\ x_1 + 10x_2 - 3x_3 &= 5 \\ -x_1 + x_2 + x_3 &= -3 \end{aligned}$$

using

(a) Cramer's rule
(b) Gauss-Elimination Method
(c) LU-Factorisation Method
(d) Matrix Inversion by row operations only.

2. Determine the rank of each of the following matrices:

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 4 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 2 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ -1 & 1 & -1 & 2 \\ -1 & 3 & 3 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 3 & 1 & -1 & 0 \\ 0 & 2 & 1 & 4 \end{bmatrix}$

2.10 ANSWERS TO ACTIVITIES & SUPPLEMENTARY EXERCISES

Activity 1

(i) $x_1 = 2$, $x_2 = 1$, $x_3 = -3$

(ii) $x_1 = 4$, $x_2 = -1$, $x_3 = 2$

Activity 2

(i) $x_1 = -3/2$, $x_2 = 3/2$, $x_3 = 1$

(ii) $x_1 = 4$, $x_2 = -2$, $x_3 = -1$

Activity 3

(i) 2

(ii) 3

Supplementary Exercises

1. $x_1 = 2\ ,\ x_2 = 0\ ,\ x_3 = -1$.

2. (i) 3

   (ii) 3

   (iii) 2

   (iv) 3