Sets, functions and Relations

Algebra of Sets

If a set A has n elements, then it can have 2^n subsets.

Power set of A: It is a set of all the subsets of a set including the empty set.

$$\underbrace{P(A)}_{\text{Power set}} \underbrace{:=}_{\text{defined as}} \{ S \underset{\text{such that}}{:} S \overset{\text{subset}}{\subseteq} A \}$$

Example

1, Given $X = \{a, b\}$ then, $2^2 = 4$,

$$P(X) = \{ \phi, \{a\}, \{b\}, \{a, b\} \}$$

2, Given $Y = \{a, b, c\}$ then,

$$P(Y) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}$$

Properties of Union.

$$A \cup B := \{ x : x \in A \text{ or } x \in B \}$$

Properties

1, $A \subseteq A \cup B$

$B \subseteq A \cup B$

2, $A \cup B = B \cup A$ (Commutative)

3, $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative)

Properties of Intersection.

$A \cap B := \{x : x \in A \text{ and } x \in B\}$

1, $A \cap B \subseteq A$
$A \cap B \subseteq B$

2, $A \cap B = B \cap A$ (Commutative)

3, $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative)

Distributive Law

① $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

② $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof

1, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in (B \cup C)$
$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$
$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$
$\Rightarrow x \in A \cap B$ or $x \in A \cap C$
$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

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(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $x \in (B \cap C)$
$\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$
$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$
$\Rightarrow x \in A \cup B$ and $x \in A \cup C$
$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

Complement

$A' \Rightarrow$ Not in A

$A \cap B' := \{x : x \in A$ and $x \notin B\}$

Note: $A \cap B'$ can also be written as $A - B$

De Morgan's laws:

(1) $(A \cup B)' = A' \cap B'$
(2) $(A \cap B)' = A' \cup B'$

Proof.

(1) $(A \cup B)' = A' \cap B'$

let $x \in (A \cup B)'$
$\Rightarrow x \notin (A \cup B)$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \in A'$ and $x \in B'$
$\Rightarrow$

$\Rightarrow x \in A'$ and B'

$\Rightarrow x \in A' \cap B'$

② $(A \cap B)' = A' \cup B'$

let $x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A'$ or B'

$\Rightarrow x \in A' \cup B'$

Complement laws

1, $A - (B \cup C) = (A - B) \cap (A - C)$

2, $A - (B \cap C) = (A - B) \cup (A - C)$

Proof

$A - (B \cup C) = (A - B) \cap (A - C)$

① Let $x \in A - (B \cup C)$

$\Rightarrow x \in A$ and $x \notin (B \cup C)$

$\Rightarrow x \in A$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in A$ and $x \notin C)$

$\Rightarrow x \in A - B$ and $x \in A - C$

$\Rightarrow x \in (A - B) \cap (A - C)$

(2) $A - (B \cap C) = (A - B) \cup (A - C)$

Let $x \in A - (B \cap C)$

$\Rightarrow x \in A$ and $x \notin (B \cap C)$

$\Rightarrow x \in A$ and $(x \notin B$ or $x \notin C)$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$

$\Rightarrow x \in A - B$ or $x \in A - C$

$\Rightarrow x \in (A - B) \cup (A - C)$

Cartesian Product

Let A and B be non empty sets. Then the cartesian product of A and B is denoted as $A \times B$

$$A \times B := \{ x, y \} ; x \in A \text{ and } y \in B \}$$

The ordered pair (x, y) can also be written as $\langle x, y \rangle$

Example

Given $A = \{2, 3\}$ and $B = \{4, 5, 6\}$. Find $A \times B$.

Solution

$$A \times B = \{ (2,4), (2,5), (2,6), (3,4), (3,5), (3,6) \}$$

Ex 8

If $E=\{5,6\}$, $F=\{1,3\}$ and $G=\{u,r,v\}$, write down the elements of the sets

i) $E \times F$

ii) $F \times F$

iii) $E \times G$

Ex 9

Let $A = \{x \in \mathbb{Z} : 2x-3=7\}$ and,
$B = \{x \in \mathbb{Z} : x^2-10x+24=0\}$

list the elements of

i) A

ii) B

iii) $A \times B$

Relation

A relation R between two sets A and B is a collection of ordered pairs containing one object from each set A and B.

Let $x \in A$ and $y \in B$. If x is related to y by the relation R, then we write xRy

$$R := \{(x,y) : x \in A,\ y \in B \text{ and } xRy\}$$

Example

~~A relation from set x to set~~

Given $A = \{2, 3\}$ and $B = \{4, 5, 6\}$

A relation from set A to set B is defined as

$$a R b \iff a \text{ divides } b \qquad a \in A, b \in B$$

Write down the ordered pairs of the relation.

Solution

$$R = \{(2,4), (2,6), (3,6)\}$$

Example

A relation from set X to set X itself is simply called a relation on set X

Given $X = \{1, 2, 3\}$,

A relation R on set X is defined as

$$x R y \iff x < y \qquad x \in X, y \in X$$

Solution:

$$R = \{(1,2), (1,3), (2,3)\}$$

Equivalence Relation

A relation R on a set X is said to be an equivalence relation if for any a, b, c ∈ X,

i, aRa (reflexive)
ii, $aRb \Rightarrow bRa$ (symmetric)
iii, aRb and $bRc \Rightarrow aRc$ (transitive)

Example

Show that the relation R is an equivalence relation in the set $A = \{1, 2, 3, 4, 5\}$ given by the relation

$$R = \{(a,b) : |a-b| \text{ is even}\}$$

Solution

$R = \{(a,b) : |a-b| \text{ is even}\}$ where $a \in A$, $b \in A$.

Reflexive

$|a-a| = |0| = 0 \rightarrow$ ~~always~~ always even
Thus $|a-a|$ is even

Therefore (a,a) belongs to R.
Hence R is reflexive.

Symmetric

$|a-b| = |b-a|$

We know that $|a-b| = |-(b-a)| = |b-a|$

Therefore if $|a-b|$ is even, $|b-a|$ also is even

Hence, if $(a,b) \in R$, then (b,a) belongs to R.

Transitive

If $|a-b|$ is even then $(a-b)$ is even
Similarly if $|b-c|$ is even, then $(b-c)$ is also even.

Sum of even numbers is also even.

Hence $a-b+b-c$ is even.
$a-c$ is also even.

So, $|a-b|$ and $|b-c|$ are even, then $|a-c|$ is even

Therefore if $(a,b) \in R$ and $(b,c) \in R$, then (a,c) also belongs to R.

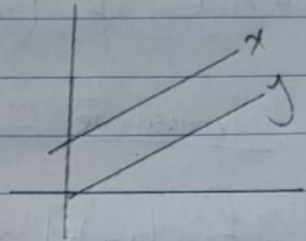
Example

Let X be a set of all straight lives in the Cartesian plane. A relation R is defined as

$$xRy \iff x \text{ is parallel to } y.$$

Show that R is an equivalence relation

Solution



for any $x, y, z \in X$,

i, xRx as x is ~~parel~~ parallel to itself. ∴ Reflective

ii, $xRy \Rightarrow x$ is parallel to y

$\Rightarrow y$ is parallel to x

$\Rightarrow yRx$

∴ Symmetric.

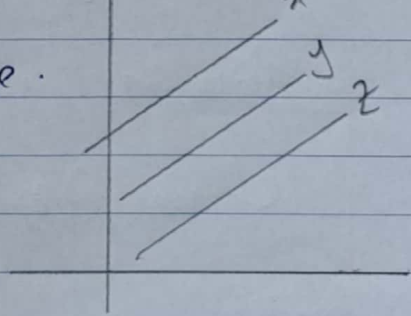
iii, xRy and yRz

$\Rightarrow x$ is parallel to y and y is parallel to z.

$\Rightarrow x$ is parallel to z

$\Rightarrow xRz$

∴ R is transitive.



Ex 9

A relation R is defined as
$$xRy \Leftrightarrow x = y$$
Show that R is an equivalence relation.

Ex 10

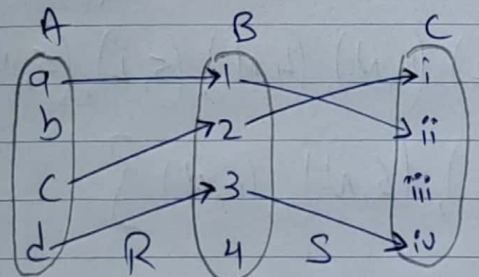
A relation R is defined as
$$xRy \Leftrightarrow |x| = |y|$$
Show that R is an equivalence Relation.

Composition of Relation

Let A, B and C be three sets.
Suppose that R is a relation from A to B and S is a relation from B to C.

The composition of R and S denoted RoS is a relation from A to C if and only if there is $b \in B$ such that aRb and bSc.

e.g



$R = \{ (a,1), (c,2), (d,3) \}$

$S = \{ (1,ii), (2,i), (3,iv) \}$

$RoS = \{ (a,ii), (c,i), (d,iv) \}$

Example

Given $X = \{4, 5, 6\}$

$Y = \{a, b, c\}$

$Z = \{L, M, N\}$

Consider the relation R_1 from X to Y

R_2 from Y to Z

$R_1 = \{(4,a), (4,b), (5,c), (6,a), (6,c)\}$

$R_2 = \{(a,L), (a,N), (b,L), (b,M), (c,L), (c,M), (c,N)\}$

Find the composition of relation $R_1 \circ R_2$

Solution

$$R_1 \circ R_2 = \{(4,L), (4,N), \cancel{(4,L)}, (4,M), (5,L), (5,M),$$
$$(5,N), (6,L), (6,N), \cancel{(6,L)}, (6,M), \cancel{(6,N)}\}$$

Repetition

$$\therefore R_1 \circ R_2 = \{(4,L), (4,N), (4,M), (5,L), (5,M), (5,N),$$
$$(6,L), (6,N), (6,M)\}$$

Ex 11

Let $R = \{(1,2), (3,4), (2,2)\}$

$S = \{(4,2), (2,5), (3,1), (1,3)\}$

Find i, $R \circ S$ ii, $S \circ R$ iii, $R \circ (S \circ R)$ iv, $R \circ R$ v, $S \circ S$