UNIT 8 SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Unit Structure

8.0 OVERVIEW

In this Unit, we shall study second-order ordinary differential equations. First we consider the solution of homogeneous equations, and then show how to solve inhomogeneous equations.

8.1 LEARNING OBJECTIVES

By the end of this unit, you should be able to do the following:

1. Find the general solution of homogeneous equations.
2. Use D-operators.
3. Obtain the complementary function and particular integral of inhomogeneous second-order linear differential equations.

8.2 INTRODUCTION

We shall study 2^{nd}-order **constant-coefficient** ordinary differential equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c\,y=f(x)\,, \qquad \textbf{(1)}$$

where the coefficients *a*, *b*, *c* are real constants and $a \neq 0$. $f(x)$ is called the **free term** or the **forcing function**.

If $f(x) \equiv 0$, then **(1)** is called a **Homogeneous** differential equation. If $f(x) \neq 0$, then **(1)** is called an **Inhomogeneous** differential equation.

8.3 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

The simplest 2^{nd} -order differential equation is $\frac{d^2y}{dx^2}=0$. To find its general solution we simply integrate twice w.r.t. *x*. Now,

$$\frac{d^2y}{dx^2}=0 \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right)=0\,.$$

Integrating w.r.t. *x*, we have $\frac{dy}{dx}=\int 0\,dx = A\,;$

Integrating again w.r.t. x gives $y=\int A\,dx = Ax+B\,.$

Hence the general solution of $\frac{d^2y}{dx^2}=0$ is $y=Ax+B$, where A and B are arbitrary constants, which can be fixed if we are given two conditions on *x* and *y*.

Example 1

Solve $\dfrac{d^2y}{dx^2} = 0$, given that $y(1) = 2,\ y'(1) = 3$.

The general solution is $y = Ax + B$. The condition $y(1) = 2$ (i.e., $y = 2$ when $x = 1$) yields

$$2 = A \cdot 1 + B$$

The second condition $y'(1) = 3$ (i.e., $y' = 3$ when $x = 1$) gives [since $y'(x) = A$]

$$3 = A.$$

Hence we have $A = 3,\quad B = -1$; therefore the ***particular solution*** is

$$y = 3x - 1.$$

In general, to solve the equation $\dfrac{d^n y}{dx^n} = f(x)$, we just integrate *n* times w.r.t. *x*.

Example 2

Solve $\dfrac{d^2y}{dx^2} = x^2 + \sin x + e^{3x}$.

Integrating once, we have

$$\begin{aligned}\frac{dy}{dx} &= \int (x^2 + \sin x + e^{3x})\, dx \\ &= \frac{x^3}{3} - \cos x + \frac{e^{3x}}{3} + A.\end{aligned}$$

Integrating once more now yields the general solution

$$y = \tfrac{1}{12} x^4 - \sin x + \tfrac{1}{9} e^{3x} + Ax + B\,.$$

[**N.B.** Since we are not given any conditions on *x* and *y*, we can't determine *A* and *B*.]

Activity 1

Solve the following differential equations:

(i) $\dfrac{d^2y}{dx^2} = 0,\quad y(2) = 3,\ y'(2) = -1$ **;**

(ii) $\dfrac{d^2y}{dx^2} = 0,\quad y(0) = 1,\ y'(0) = 2$ **;**

(iii) $\dfrac{d^2y}{dx^2} = 0,\quad y(1) = 2,\ y(2) = 3$ **;**

(iv) $\dfrac{d^2y}{dx^2} = 0,\quad y(-1) = 3,\ y(1) = 2$ **;**

(v) $\dfrac{d^2y}{dx^2} = 5x - 2$ **;**

(vi) $\dfrac{d^2y}{dx^2} = 2\sinh x + 3\cosh x + 4$ **;**

(vii) $\dfrac{d^2y}{dx^2} = 2e^{-x} + 3x - 1,\quad y(0) = 2,\ y'(0) = -3$ **;**

(viii) $\dfrac{d^2y}{dx^2} = 6x - 7,\quad y(1) = 3,\ y(2) = 5$ **;**

We shall now learn how to solve the homogeneous equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c\,y=0\,. \qquad \textbf{(2)}$$

Let us assume that $y = Ae^{mx}$ is a solution of **(2)**. Then

$$\frac{dy}{dx}=mAe^{mx}\,,$$

$$\frac{d^2y}{dx^2}=m^2Ae^{mx}\,.$$

On substituting into **(2)**, we obtain

$$am^2Ae^{mx}+bmAe^{mx}+cAe^{mx}=0\,.$$

Since $Ae^{mx}\neq 0$, this simplifies to

$$am^2+bm+c=0\,.$$

This quadratic equation is called the **Auxiliary Equation (A.E.)** of Eqn **(2)**.

The auxiliary equation can easily be written down. We simply replace *y* by 1, $\frac{dy}{dx}$ by *m*, and $\frac{d^2y}{dx^2}$ by m^2.

Now, the quadratic equation can have either 2 distinct roots, or 2 equal roots, or 2 complex roots. Each case gives a different type of solution to Eqn **(2)**, which is given in the table below.

		General Solution of Eqn **(2)**
Distinct Roots	$m_1, \quad m_2$	$y = A\,e^{m_1x}+B\,e^{m_2x}$
Equal Roots	m_0, m_0	$y=(Ax+B)\,e^{m_0x}$
Complex Roots	$p\pm q\,i$	$y=e^{px}(A\cos qx+B\sin qx)$

Note that the arbitrary constants have been called *A* and *B* here; other symbols could be used.

We shall now consider a few examples to illustrate.

<u>Example 3</u>

Consider the differential equation

$$\frac{d^2y}{dx^2} = y\,, \tag{3}$$

The A.E. is $m^2 - 1 = 0$, which has distinct roots $m = -1, 1$. Hence, from the table, $y = Ae^{-x} + Be^{x}$ is the most general solution of **(3)**.

Example 4

Solve the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0\,.$$

Now the A.E. is $m^2 - 5m + 6 = 0$, whose roots are $m = 2, 3$. Therefore, the general solution is

$$y = Ae^{2x} + Be^{3x}\,.$$

Example 5

Consider the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0\,.$$

The A.E. $m^2 - 6m + 9 = 0$ gives $m = 3, 3$, two repeated roots.

Hence the general solution is

$$y = (A + Bx)e^{3x}\,.$$

Example 6

Solve the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0\,.$$

The A.E. is given by $m^2 - 4m + 13 = 0$, whose roots are $m = 2 \pm 3\,i$. The general solution is

$$y = e^{2x}(A\cos 3x + B\sin 3x)\,.$$

Obtaining Particular Solutions.

First we solve the A.E. and obtain the general solution. Using the 2 conditions given, we then fix the 2 arbitrary constants.

Example 7

Solve the following differential equation

$$3y''-19y'-14y=0, \quad y(0)=1,\ y'(0)=5.$$

The A.E. is $3m^2-19m-14=0$, whose roots are $m=-\frac{2}{3}, 7$. Hence the general solution is $y=Ae^{-\frac{2}{3}x}+Be^{7x}$.

Now $y(0)=1$ yields $A+B=1$.

$y'(x)=-\frac{2}{3}Ae^{-\frac{2}{3}x}+7Be^{7x}$, so that $y'(0)=5$ gives $-\frac{2}{3}A+7B=5$.

Solving simultaneously for *A* and *B*, we see that $A=\frac{6}{23}$, $B=\frac{17}{23}$.

Hence our particular solution is

$$y=\frac{6}{23}e^{-\frac{2}{3}x}+\frac{17}{23}e^{7x}.$$

Example 8

Solve the differential equation

$$y''-6y'+25y=0, \quad y(0)=2,\ y'(0)=5.$$

The A.E. is $m^2-6m+25=0$, whose roots are $m=3\pm 4i$. The general solution is therefore given by

$$y=e^{3x}(A\cos 4x+B\sin 4x).$$

We now determine *A* and *B*.

$y(0)=2 \Rightarrow \quad e^0(A\cos 0+B\sin 0)=2 \quad \Rightarrow A=2$.

So, $y=e^{3x}(2\cos 4x+B\sin 4x)$

Now, $y'(x)=e^{3x}[(4B+6)\cos 4x+(3B-8)\sin 4x]$ on using product rule.

$\therefore \quad y'(0)=5 \Rightarrow e^0[(4B+6)\cos 0+(3B-8)\sin 0]=5$

i.e. $4B+6=5$

$\therefore \quad B=-\frac{1}{4}$

Hence, the particular solution is

$$y=e^{3x}(2\cos 4x-\tfrac{1}{4}\sin 4x).$$

Activity 2

Solve the following differential equations:

(i) $y''-3y'+2y=0$;

(ii) $y''-4y'+4y=0, \quad y(0)=0,\ y'(0)=1$;

(iii) $y''-2y'+2y=0$;

(iv) $y''+y'-6y=0$;

(v) $y''-5y'=0, \quad y(0)=0,\ y'(0)=1$;

(vi) $2y''-3y'+y=0$;

(vii) $4y''-2y'-y=0$;

(viii) $5y''+6y'=0$;

(ix) $y''+4y'+5y=0, \quad y(0)=0,\ y'(0)=1$;

(x) $y''+2ky'+k^2y=0$.

8.4 DIFFERENTIAL OPERATORS

The D-operator

Let D denote the differentiation operator $\frac{d}{dx}$. Then

$$\frac{dy}{dx} = Dy,\ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D(Dy) = D^2 y, \cdots, \frac{d^n y}{dx^n} = D^n y\,.$$

We can now express our differential equations in terms of the D operator. Thus Eqn (1) can be written as

$$(aD^2 + bD + c)\ y = f(x)\,.$$

Similarly, the differential equation

$$5y'' - 3y' + 2y = \sin 3x$$

can be written as

$$(5D^2 - 3D + 2)\ y = \sin 3x\,. \qquad (*)$$

Inverse of the *D* operator

Since D denotes differentiation w.r.t. x, its inverse D^{-1} will represent integration w.r.t. x. In general D^{-n} will mean integrate n times w.r.t. x. The symbol D^{-1} may also be written as $\frac{1}{D}$. In general, it is customary to write D^{-m} as $\frac{1}{D^m}$ when m is a positive integer.

Thus,

$$D^{-2} = \left(\frac{1}{D}\right)^2 = \frac{1}{D}\frac{1}{D}, \text{ i.e., integrate twice.}$$

So,

$$D^{-2}x = \int\left(\int x\,dx\right) = \int \frac{x^2}{2}dx = \frac{x^3}{6};$$

$$D^{-3}x = D^{-1}(D^{-2}x) = \int \frac{x^3}{6}dx = \frac{x^4}{24}.$$

While evaluating $D^{-1} f(x)$, the arbitrary constant of integration may be omitted.

Some Properties of the *D* operator

We shall denote by $L(D)$ a function of the *D* operator, e.g., in the above Eqn (*),

$$L(D) = 5D^2 - 3D + 2\,.$$

Symbolically we can write a differential equation in the form

$$L(D)y = f(x)\,,$$

or,

$$y = \frac{1}{L(D)}\,f(x).$$

Our task is to find ways of evaluating the RHS for different functions $f(x)$.

We first consider some properties of the *D* operator. You can easily verify all these properties.

(i) $L(D)e^{ax} = e^{ax}\,L(a)\,;$

(ii) $L(D)[e^{ax}\,f(x)] = e^{ax}L(D+a)\;f(x)\,;$

(iii) $L(D^2)\cos ax = L(-a^2)\cos ax\,;$

(iv) $L(D^2)\sin ax = L(-a^2)\sin ax\,;$

(v) $L(D^2)\cosh ax = L(a^2)\cosh ax\,;$

(vi) $L(D^2)\sinh ax = L(a^2)\sinh ax\,.$

We note that there are no simple formulae for $L(D)\sin ax$, $L(D)\cos ax$, and the hyperbolics.

Let's see how to use the above properties.

<u>Example 9</u>

(a) Suppose we wish to evaluate $(5D^2 + 2D - 7)e^{3x}$. Of course we could do it from first principles; however, using Property **(i)**, we find that here

$$L(D) = 5D^2 + 2D - 7$$

and $a = 3$.

$\therefore\quad L(a) = L(3) = 5(3^2) + 2(3) - 7 = 44\,.$

Hence, $(5D^2 + 2D - 7)e^{3x} = 44\,e^{3x}\,.$

(b) Prove that

$$\left(D^3 - 2D^2 + D\right)\left(x^2 e^{3x}\right) = e^{3x}\left(12x^2 + 32x + 14\right).$$

Here,

$L(D) = D^3 - 2D^2 + D$, $a = 3$, $f(x) = x^2$. Then, Property **(ii)** gives

$$\begin{aligned}\left(D^3 - 2D^2 + D\right)\left(x^2 e^{3x}\right) &= e^{3x}\left\{(D+3)^3 - 2(D+3)^2 + (D+3)\right\}x^2 \\ &= e^{3x}\left(D^3 + 7D^2 + 16D + 12\right)x^2 \\ &= e^{3x}\left(0 + 7(2) + 16(2x) + 12x^2\right) \\ &= e^{3x}\left(12x^2 + 32x + 14\right).\end{aligned}$$

(c) Evaluate $(3D^2 - 5D + 2)\cos 7x$.

We use Property **(iii)**. We have $L(D) = 3D^2 - 5D + 2$, and $a = 7$.

We therefore replace D^2 by -7^2. [Note: $-7^2 = -49$]

$$\begin{aligned}(3D^2 - 5D + 2)\cos 7x &= [3\,(-49) - 5D + 2]\cos 7x \\ &= (-145 - 5D)\cos 7x \\ &= -145\cos 7x - 5\,D(\cos 7x) \\ &= -145\cos 7x - 5\,\frac{d}{dx}\cos 7x \\ &= -145\cos 7x + 35\sin 7x\end{aligned}$$

8.5 SOLVING THE INHOMOGENEOUS EQUATION

The method of solution of the inhomogeneous equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c\,y= f(x)$$

consists of 3 parts:

1. Find the **Complementary Function** y_C

2. Find the **Particular Integral** y_P

3. The general solution is then given by

$$y = y_C + y_P\,.$$

The complementary function (**C.F.**) is obtained by solving the corresponding homogeneous differential equation, i.e.,

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c\,y=0\,,\ \text{ or }\quad (aD^2+bD+c)\,y=0$$

as we have done above.

The second step, finding the particular integral (**P.I**), is the hardest part. The P.I. is obtained by the following result

$$y_P = \frac{1}{L(D)}\,f(x)\,.$$

The following table summarizes the main results for different forcing functions $f(x)$.

Table of Inverse Operator Techniques

1. $\dfrac{1}{L(D)} ke^{ax}$, k :constant	$\dfrac{k\,e^{ax}}{L(a)}$ $\quad L(a) \neq 0$
2. (i) $\dfrac{1}{L(D^2)} k\cos(ax+b)$ **(ii)** $\dfrac{1}{L(D^2)} k\sin(ax+b)$	$\dfrac{k}{L(-a^2)}\cos(ax+b)$ $L(-a^2) \neq 0$ $\dfrac{k}{L(-a^2)}\sin(ax+b)$
3. $\dfrac{1}{L(D)} P(x)$, where $P(x)$ is a polynomial of degree m	Expand $\dfrac{1}{L(D)}$ in ascending powers of D by the Binomial Theorem as far as the term in D^m. Then operate on $P(x)$.
4. $\dfrac{1}{L(D)} e^{ax}\phi(x)$	$e^{ax}\dfrac{1}{L(D+a)}\phi(x)$ **[Shift Theorem]**
5. $\dfrac{1}{D-m} f(x)$	$e^{mx}\int e^{-mx} f(x)\ dx$
6. $\dfrac{1}{(D-a)^r} e^{ax}$	$\dfrac{x^r}{r!} e^{ax}$, $\quad r = 1, 2, \ldots$
7. $\dfrac{1}{D^2+a^2} k\cos(ax+b)$ $\dfrac{1}{D^2+a^2} k\sin(ax+b)$	$\dfrac{kx}{2a}\sin(ax+b)$ $-\dfrac{kx}{2a}\cos(ax+b)$

We shall now consider a few examples to illustrate.

Example 10

Solve the differential equation

$$2y'' - 5y' - 12y = 9.$$

In terms of the D operator, our equation is

$$(2D^2 - 5D - 12)y = 9.$$

If we compare with the entries in the table, we find that here we need to use the first entry where

$$L(D) = 2D^2 - 5D - 12$$

$$a = 0, \quad k = 9$$

We now find the solution to our equation.

STEP 1. Obtain the Complementary Function y_C.

Now, the A.E. is

$$2m^2 - 5m - 12 = 0.$$

$$\therefore m = -\tfrac{3}{2},\ 4.$$

Hence $\quad y_C = Ae^{-3x/2} + Be^{4x}$.

STEP 2. Find the Particular Integral y_P

From the table,

$$\begin{aligned} y_P &= \frac{1}{2D^2 - 5D - 12}\ 9 \\ &= \frac{1}{0 - 0 - 12}\ 9 \qquad \text{[Putting } D = a = 0\text{]} \\ &= -\frac{3}{4} \end{aligned}$$

STEP 3. Write down the General Solution $y = y_C + y_P$

$$y = Ae^{-3x/2} + Be^{4x} - \tfrac{3}{4}.$$

<u>Example 11</u>

Solve the differential equation

$$y'' + 3y' - 10y = 4e^{-3x}.$$

Now we have

$$L(D) = D^2 - 3D - 10$$

$$k = 4, \quad a = -3$$

A.E. $\qquad m^2 - 3m - 10 = 0 \quad \Rightarrow m = 2, -5$

$\therefore$ C.F. is

$$\therefore \quad y_C = A\,e^{2x} + B\,e^{-5x}.$$

P.I. is

$$
\begin{aligned}
y_P &= \frac{1}{D^2 - 3D - 10}\, 4\, e^{-3x} \\
&= \frac{4\, e^{-3x}}{(-3)^2 - 3(-3) - 10} \quad [\text{Putting } D = a = -3] \\
&= \tfrac{1}{2} e^{-3x}
\end{aligned}
$$

General solution is

$$y = A\, e^{2x} + B\, e^{-5x} + \tfrac{1}{2} e^{-3x} .$$

Example 12

Solve the differential equation

$$\frac{d^2 y}{dx^2} + 6\frac{dy}{dx} + 9y = 2e^{x} - 5e^{-2x} .$$

$$y(0) = 0, \quad y'(0) = 7 .$$

Here we note that the forcing function consists of 2 terms, $2e^{x}$ and $-5e^{-2x}$. To find the P.I., we use entry 1 of the table twice, and add the results, i.e.,

$$y_P = \frac{1}{L(D)} 2e^{x} + \frac{1}{L(D)} (-5e^{-2x}),$$

where $L(D) = D^2 + 6D + 9 = (D+3)^2$.

A.E. $m^2 + 6m + 9 = 0 \quad \Rightarrow m = -3, -3$

C.F. $y_C = (Ax + B)\, e^{-3x}$

P.I.

$$
\begin{aligned}
y_P &= \frac{1}{(D+3)^2}\, 2e^{x} + \frac{1}{(D+3)^2} (-5e^{-2x}) \\
&= \frac{2e^{x}}{(1+3)^2} + \frac{-5e^{-2x}}{(-2+3)^2} \\
&= \tfrac{1}{8} e^{x} - 5e^{-2x}
\end{aligned}
$$

General solution is

$$y = (Ax + B)\, e^{-3x} + \tfrac{1}{8} e^{x} - 5e^{-2x} .$$

We now fix the values of *A* and *B*.

$y(0) = 0 \quad \Rightarrow B + \tfrac{1}{8} - 5 = 0 \quad \Rightarrow B = \tfrac{39}{8}$

Also, we have on differentiating and simplifying

$$y'(x) = \tfrac{1}{8} e^{-3x} \, [-24B + 80\,e^{x} + e^{4x} + A(8 - 24x)]$$

$\therefore \quad y'(0) = 0 \quad \Rightarrow \ \tfrac{1}{8}[-24B + 80 + 1 + 8A] = 0$

$\therefore \quad A = \tfrac{9}{2}$ after putting $B = 39/8$.

Hence, the *Particular Solution* is

$$y = (\tfrac{9}{2}x + \tfrac{39}{8})\,e^{-3x} + \tfrac{1}{8}e^{x} - 5e^{-2x}.$$

<u>Example 13</u>

Solve the differential equation

$$(D^2 + D + 1)y = 5\cos 2x.$$

A.E. $\qquad m^2 + m + 1 = 0 \quad \Rightarrow m = -\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2}\,i$

C.F. $\qquad y_C = e^{-x/2}\left[A\cos\left(\dfrac{\sqrt{3}}{2}x\right) + B\sin\left(\dfrac{\sqrt{3}}{2}x\right)\right]$

We now use entry **2(i)** in the table with $k = 5,\ a = 2,\ b = 0$. Note we replace only D^2 by $-a^2$. The D stays as it is.

$$\begin{aligned} y_P &= \frac{1}{D^2 + D + 1}\,5\cos 2x \\ &= \frac{1}{-2^2 + D + 1}\,5\cos 2x \\ &= \frac{1}{D - 3}\,5\cos 2x \end{aligned}$$

Since the result works only for D^2, the trick is to obtain D^2 in the denominator. This is achieved by multiplying top and bottom by $D + 3$. This then gives

$$
\begin{aligned}
y_P &= \frac{1}{D-3} 5\cos 2x \\
&= \frac{D+3}{D^2-9} 5\cos 2x \\
&= \frac{D+3}{-2^2-9} 5\cos 2x \\
&= -\frac{5}{13}(D+3)\cos 2x \\
&= -\frac{5}{13}(-2\sin 2x + 3\cos 2x)
\end{aligned}
$$

Hence the General solution is

$$y = e^{-\frac{1}{2}x}\left\{A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right\} + \frac{10\sin 2x - 15\cos 2x}{13}.$$

<u>Example 14</u>

Solve the differential equation

$$(D^2 + 5D + 6)y = 4\sin 3x\,,$$

$$y(0) = 0, \quad y'(0) = 0\,.$$

A.E. $\quad m^2 + 5m + 6 = 0 \quad \Rightarrow m = -2,\ -3$

C.F. $\quad y_C = Ae^{-2x} + Be^{-3x}$

Next, we find the Particular Integral. We use entry **2(ii)** in the table with $k = 4,\ a = 3,\ b = 0$.

P.I.

$$
\begin{aligned}
y_P &= \frac{1}{D^2+5D+6}4\sin 3x \\
&= \frac{1}{-3^2+5D+6}4\sin 3x \\
&= \frac{1}{5D-3}4\sin 3x \\
&= \frac{5D+3}{25D^2-9}4\sin 3x \\
&= \frac{5D+3}{25(-3^2)-9}4\sin 3x \\
&= -\frac{2}{117}(5D+3)\sin 3x \\
&= -\frac{2}{39}(5\cos 3x+\sin 3x)
\end{aligned}
$$

Hence, the General Solution is

$$y = Ae^{-2x} + Be^{-3x} - \tfrac{2}{39}(5\cos 3x + \sin 3x)\,.$$

We now determine the arbitrary constants.

$y(0)=0 \quad \Rightarrow \quad A+B-\tfrac{10}{39}=0$ ----- **(i)**

Also,

$y'(x) = -(2A\,e^{-2x} + 3B\,e^{-3x}) + \tfrac{2}{13}(5\sin 3x - \cos 3x)$

$\therefore \quad y'(0)=0 \quad \Rightarrow \quad 2A+3B+\tfrac{2}{13}=0$ -----**(ii)**

Solving **(i)** and **(ii)** simultaneously, we obtain $A=\tfrac{12}{13},\ \ B=-\tfrac{2}{3}$.

Hence, the Particular Solution is given by

$$y = \tfrac{12}{13}e^{-2x} - \tfrac{2}{3}e^{-3x} - \tfrac{2}{39}(5\cos 3x + \sin 3x)\,.$$

Evaluating $\dfrac{1}{L(D)}P(x)$ when $P(x)$ is a polynomial of degree *m*.

In this case we expand the operator $\dfrac{1}{L(D)}$ by the binomial theorem in ascending powers of D as far as the term in D^m.

If $L(D)$ is factorisable, use partial fractions and then expand.
For this purpose, the following binomial expansions are useful:

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + D^4 - \cdots$$
$$(1-D)^{-1} = 1 + D + D^2 + D^3 + D^4 + \cdots$$
$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + 5D^4 - \cdots$$
$$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + 5D^4 + \cdots$$

Note that we need to have our expression in the right form before carrying out the expansion. Thus to expand $(3D+5)^{-1}$ we must first express it as $5^{-1}(1+\frac{3}{5}D)^{-1}$.

Likewise, $(2D-7)^{-1}$ must first be written as $-7^{-1}(1-\frac{2}{7}D)^{-1}$. Also, we must express $(5-4D)^{-2}$ as $5^{-2}(1-\frac{4}{5}D)^{-2}$ and then expand.

Example 15

Evaluate $\dfrac{1}{D-1}x^4$.

Here the degree of the polynomial is 4; we therefore expand $\dfrac{1}{D-1}$ up to D^4.

$$\begin{aligned}\frac{1}{D-1}x^4 &= \frac{1}{-(1-D)}x^4 \\ &= -(1-D)^{-1}\ x^4 \\ &= -(1+D+D^2+D^3+D^4+\cdots)x^4 \\ &= -(x^4+4x^3+12x^2+24x+24).\end{aligned}$$

Note: We just differentiate the previous term as we go along.

Example 16

Solve the differential equation

$$y'' - 3y' + 2y = 7 - 6x - 3x^2 + x^3 .$$

A.E. $m^2 - 3m + 2 = 0 \quad \Rightarrow \quad m = 1, 2$

C.F. $y_C = Ae^x + Be^{2x}$

P.I.

$$y_P = \frac{1}{D^2 - 3D + 2}[7 - 6x - 3x^2 + x^3]$$

Here the degree of the polynomial is 3, therefore expand up to D^3. Also, the operator is factorisable and so we split it into partial fractions before expanding. Thus

$$\frac{1}{D^2 - 3D + 2} = \frac{1}{(D-1)(D-2)} = \frac{1}{D-2} - \frac{1}{D-1} .$$

On using the binomial theorem we obtain

$$(-\tfrac{1}{2} - \tfrac{D}{4} - \tfrac{D^2}{8} - \tfrac{D^3}{16} - \cdots) - (-1 - D - D^2 - D^3 - \cdots)$$

which simplifies to

$$\tfrac{1}{16}(8 + 12D + 14D^2 + 15D^3)$$

Hence, y_P is given by

$$\begin{aligned}
\frac{1}{D^2 - 3D + 2}[7 - 6x - 3x^2 + x^3] &= \tfrac{1}{16}(8 + 12D + 14D^2 + 15D^3)[7 - 6x - 3x^2 + x^3] \\
&= \tfrac{1}{16}[8\,(7 - 6x - 3x^2 + x^3) + 12\,(-6 - 6x + 3x^2) \\
&\quad + 14\,(-6 + 6x) + 15(6)] \\
&= \tfrac{1}{8}[4x^3 + 6x^2 - 18x - 5]
\end{aligned}$$

Note again, we just differentiate previous brackets, while paying attention to the coefficients of the D^i in the first brackets.

General Solution: $y = Ae^x + Be^{2x} + \tfrac{1}{8}(4x^3 + 6x^2 - 18x - 5)$.

Example 17

Solve the differential equation

$$y'' - 3y' + 5y = 2x^3 + 7x - 9\,.$$

A.E. $m^2 - 3m + 5 = 0 \Rightarrow m = \frac{3}{2} \pm \frac{\sqrt{11}}{2}\, i$

C.F. $y_C = e^{3x/2}\,[A\cos\frac{\sqrt{11}}{2}x + B\sin\frac{\sqrt{11}}{2}x]$

P.I.

$$y_P = \frac{1}{D^2 - 3D + 5}(2x^3 + 7x - 9)\,.$$

In this case, the degree of the polynomial is 3, so that we need to expand up to D^3. But now the operator does not factorize, so that we can't use partial fractions. We therefore proceed as follows:

$$\begin{aligned}\frac{1}{D^2 - 3D + 5} = (D^2 - 3D + 5)^{-1} &= \frac{1}{5}\left(1 + \frac{D^2 - 3D}{5}\right)^{-1}\\ &= \frac{1}{5}\left[1 - \left(\frac{D^2 - 3D}{5}\right) + \left(\frac{D^2 - 3D}{5}\right)^2 - \left(\frac{D^2 - 3D}{5}\right)^3 + \cdots\right]\\ &= \frac{1}{5} + \frac{3}{25}D + \frac{4}{125}D^2 - \frac{3}{625}D^3 + \cdots\end{aligned}$$

Hence, the P.I. is

$$\begin{aligned}\frac{1}{D^2 - 3D + 5}(2x^3 + 7x - 9) &= \left(\frac{1}{5} + \frac{3}{25}D + \frac{4}{125}D^2 - \frac{3}{625}D^3\right)(2x^3 + 7x - 9)\\ &= \tfrac{1}{5}(2x^3 + 7x - 9) + \tfrac{3}{25}(6x^2 + 7) + \tfrac{4}{125}(12x) - \tfrac{3}{625}(12)\\ &= \tfrac{1}{625}[250x^3 + 450x^2 + 1115x - 636].\end{aligned}$$

Then, the general solution is given by

$$y = e^{3x/2}\,[A\cos\tfrac{\sqrt{11}}{2}x + B\sin\tfrac{\sqrt{11}}{2}x] + \tfrac{1}{625}[250x^3 + 450x^2 + 1115x - 636]\,.$$

The Shift Theorem

Evaluating $\dfrac{1}{L(D)} e^{ax}\phi(x)$. [Here, $f(x) = e^{ax}\phi(x)$]

We use the formula

$$\boxed{\frac{1}{L(D)} e^{ax}\phi(x) = e^{ax}\frac{1}{L(D+a)}\phi(x)}$$

The D is *shifted* to $D+a$.

Example 18

Evaluate $\dfrac{1}{D^2 - 2D + 1}(e^x x^3)$.

Here, $L(D) = D^2 - 2D + 1, \quad a = 1, \quad \phi(x) = x^3$.

$$\frac{1}{D^2 - 2D + 1} e^x x^3 = \frac{1}{(D-1)^2} e^x x^3 = e^x \frac{1}{[(D+1)-1]^2} x^3$$

$$= e^x \frac{1}{D^2} x^3 = e^x \frac{x^5}{20}. \quad \text{(Integrating twice).}$$

Example 19

Solve the differential equation $y'' - 5y' + 4 = (6x^2 - 7x + 9)e^{3x}$.

A.E. $m^2 - 5m + 4 = 0 \;\Rightarrow\; m = 1, 4$.

C.F. $y_C = Ae^x + Be^{4x}$.

We now need to find the particular integral.

P.I. $y_P = \dfrac{1}{D^2 - 5D + 4}(6x^2 - 7x + 9)e^{3x}$

Here $L(D) = D^2 - 5D + 4, \quad a = 3, \quad \phi(x) = 6x^2 - 7x + 9$.

The **Shift Theorem** gives

$$\frac{1}{D^2-5D+4}(6x^2-7x+9)e^{3x} = e^{3x}\frac{1}{(D+3)^2-5(D+3)+4}(6x^2-7x+9)$$
$$= e^{3x}\frac{1}{D^2+D-2}(6x^2-7x+9)$$
$$= e^{3x}\frac{1}{(D+2)(D-1)}(6x^2-7x+9)$$
$$= e^{3x}\left[\frac{-\frac{1}{3}}{D+2}+\frac{\frac{1}{3}}{D-1}\right](6x^2-7x+9)$$

We now proceed as for Case 3; we expand the operators by the binomial theorem up to the term in D^2 since we have a 2nd –degree polynomial. Thus

$$\frac{-\frac{1}{3}}{D+2}+\frac{\frac{1}{3}}{D-1} = -\frac{1}{3}\left[\tfrac{1}{2}(1+D/2)^{-1}+(1-D)^{-1}\right]$$
$$= -\left[\frac{1}{2}+\frac{D}{4}+\frac{3D^2}{8}+\cdots\right].$$

Now, $-\left[\frac{1}{2}+\frac{D}{4}+\frac{3D^2}{8}\right](6x^2-7x+9) = -3x^2+\tfrac{1}{2}x-\tfrac{29}{4}$.

$$\therefore \quad y_P = e^{3x}(-3x^2+\tfrac{1}{2}x-\tfrac{29}{4}).$$

The general solution is hence given by

$$y = Ae^{x}+Be^{4x}+e^{3x}(-3x^2+\tfrac{1}{2}x-\tfrac{29}{4}).$$

Example 20

Solve the equation $\quad y''+3y'+2y = e^{2x}\sin x, \quad y(0)=1,\ y'(0)=0$.

A.E. $\quad m^2+3m+2=0$, so that $m=-1,-2$.

$$\therefore \quad y_C = Ae^{-x}+Be^{-2x}.$$

Next we find the **P.I.** We have here

$L(D) = D^2+3D+2, \quad a=2, \quad \phi(x)=\sin x.$ Then, by the Shift Theorem

$$y_P = \frac{1}{D^2+3D+2}e^{2x}\sin x = e^{2x}\frac{1}{(D+2)^2+3(D+2)+2}\sin x$$

$$= e^{2x}\frac{1}{D^2+7D+12}\sin x$$

$$= e^{2x}\frac{1}{-1+7D+12}\sin x \text{ (replacing } D^2 \text{ by} -1^2\text{)}$$

$$= e^{2x}\frac{1}{11+7D}\sin x$$

$$= e^{2x}(11-7D)\frac{1}{121-49D^2}\sin x$$

$$= e^{2x}(11-7D)\frac{1}{121+49}\sin x \text{ (replacing } D^2 \text{ by } -1^2\text{)}$$

$$= e^{2x}\frac{1}{170}(11-7D)\sin x$$

$$= \frac{1}{170}e^{2x}(11\sin x - 7\cos x).$$

Therefore, the general solution is

$$y = Ae^{-x} + Be^{-2x} + e^{2x}(11\sin x - 7\cos x)/170.$$

We now fix the arbitrary constants A and B by using the given data $y(0)=1,\ y'(0)=0$.

$$y(0)=1 \quad \Rightarrow \quad A+B-7/170=1. \quad \text{----}\mathbf{(i)}$$

Now, $y'(x) = -Ae^{-x} - 2Be^{-2x} + e^{2x}(29\sin x - 3\cos x)/170$, so that

$$y'(0)=0 \quad \Rightarrow \quad -A-2B-3/170=0. \text{ -----}\mathbf{(ii)}$$

Solving **(i)** and **(ii)** simultaneously for A and B, we obtain $A = 357/170,\ \ B = -18/17$.

Hence, the particular solution is

$$y = \frac{1}{170}\left(357e^{-x} - 180e^{-2x} + e^{2x}\{11\sin x - 7\cos x\}\right).$$

Forcing Function Consisting of Several Functions

When the forcing function $f(x)$ consists of several terms, the P.I. is given by the sum of the P.I.s of each term. Thus,

$$\frac{1}{L(D)}[p(x)+q(x)+r(x)]=\frac{1}{L(D)}[p(x)]+\frac{1}{L(D)}[q(x)]+\frac{1}{L(D)}[r(x)].$$

Example 21

Evaluate $\frac{1}{D^2+3D+2}[e^{2x}\sin x-e^{-3x}+2x^3]$.

Letting $y_P=\frac{1}{D^2+3D+2}[e^{2x}\sin x-e^{-3x}+2x^3]$, we have $y_P=y_1+y_2+y_3$, where

$$y_1=\frac{1}{D^2+3D+2}[e^{2x}\sin x]=\frac{1}{170}e^{2x}(11\sin x-7\cos x),$$

$$y_2=\frac{1}{D^2+3D+2}[-e^{-3x}]=-\frac{1}{2}e^{-3x},$$

$$y_3=\frac{1}{D^2+3D+2}[2x^3]=x^3-\tfrac{9}{2}x^2+\tfrac{21}{2}x-\tfrac{45}{4}.$$

Hence,

$$y_P=\frac{1}{170}e^{2x}(11\sin x-7\cos x)-\frac{1}{2}e^{-3x}+x^3-\frac{9}{2}x^2+\frac{21}{2}x-\frac{45}{4}.$$

SPECIAL CASES

1. Evaluation of $\dfrac{1}{L(D)}ke^{ax}$ when $L(a)=0$

In this case we can proceed in two ways. One is to use the Shift theorem, the other is to use Case 6 in the table.

Example 22

Let's find the P.I. for the equation $y''+3y'+2y=5e^{-2x}$.

$$\begin{aligned} y_P &= \frac{1}{D^2+3D+2}5e^{-2x} \\ &= \frac{1}{(D+2)(D+1)}5e^{-2x} \end{aligned}$$

Clearly, the denominator is zero when we replace D by -2. We therefore proceed as follows.

Replace D by -2 in the factor $(D+1)$ only. We then have

$$y_P = \frac{1}{(D+2)(-1)}5e^{-2x} \qquad \textbf{------ (\#)}$$

Now we use the Shift Theorem with $\phi(x)=5$. So the D is shifted to $(D-2)$. Thus

$$\begin{aligned} y_P &= e^{-2x}\frac{1}{[(D-2)+2](-1)}5 \\ &= -e^{-2x}\frac{1}{D}(5) \qquad \text{[Recall } 1/D \text{ means Integrate]} \\ &= -e^{-2x}(5x) \\ &= -5xe^{-2x}. \end{aligned}$$

Hence the P.I. is $-5xe^{-2x}$.

Alternatively, we can use Result 6 in the table to Eqn. **(#)** above, where $r=1$. This yields again

$$y_P = -5xe^{-2x}.$$

2. Evaluation of $\frac{1}{L(D^2)}k\cos(ax+b)$ and $\frac{1}{L(D^2)}k\sin(ax+b)$ when $L(-a^2)=0$

We use Result 7 in the table.

Example 23

To find the P.I of the equation $y''+4y=3\cos 2x$, we proceed as follows:

$$\begin{aligned} y_P &= \frac{1}{D^2+4}3\cos 2x \\ &= \frac{3x}{2(2)}\sin 2x \qquad \text{[Using Result 7 with } k=3, a=2, b=0\text{]} \\ &= \tfrac{3}{4}x\sin 2x \end{aligned}$$

Activity 3

Solve the following differential equations:

(i) $y''+6y'+5y=2e^{3x}-7$;

(ii) $y''-9y=54e^{3x}, \quad y(0)=-1, y'(0)=18$;

(iii) $y''-y'=2\cosh x$;

(iv) $y''-4y'=8e^{-2x}$;

(v) $y''+y=\sin 2x$;

(vi) $y''-5y'+6y=100\sin 4x$;

(vii) $y''+8y'+25y=48\cos x-16\sin x$;

(viii) $y''+2y'+401y=\sin 20x+40\cos 20x$;

(ix) $y''+y=x^3, y(0)=0, y'(0)=0$;

(x) $y''+2y'=e^{-x}\sin 2x$;

(xi) $y''-y'-2y=44-76x-48x^2$;

(xii) $y''-6y'+9y=54x+18$;

(xiii) $y''+y=4\cos x, \quad y(0)=2, y'(0)=-1$.

8.6 SUMMARY

In this unit, you have studied how to solve homogeneous linear second-order ordinary differential equations and inhomogeneous equations by finding the complementary functions and particular integrals using *D*-operators.

8.7 ANSWERS TO ACTIVITIES

Activity 1

(i) $y = 5 - x$;

(ii) $y = 2x + 1$;

(iii) $y = x + 1$;

(iv) $y = (5 - x)/2$;

(v) $y = \frac{5}{6}x^3 - x^2 + Ax + B$;

(vi) $y = 3\cosh x + 2\sinh x + 2x^2 + Ax + B$;

(vii) $y = 2e^{-x} - x - \frac{1}{2}x^2 + \frac{1}{2}x^3$;

(viii) $y = x^3 + x(11 - 7x)/2$;

Activity 2

(i) $y = Ae^{x} + Be^{2x}$;

(ii) $y = xe^{2x}$;

(iii) $y = e^{x}(A\cos x + B\sin x)$;

(iv) $y = Ae^{2x} + Be^{-3x}$;

(v) $y = \frac{1}{5}(4 + e^{5x})$; **(vi)** $y = Ae^{x} + Be^{x/2}$;

(vii) $y = e^{x/4}(A\cos\sqrt{5}x/4 + B\sin\sqrt{5}x/4)$;

(viii) $y = A + Be^{-6x/5}$;

(ix) $y = e^{-2x}\sin x$;

(x) $y = (A + Bx)e^{-kx}$.

Activity 3

(i) $y = Ae^{-5x} + Be^{-x} + \frac{1}{16}e^{3x} - \frac{7}{5}$;

(ii) $y = (1+9x)e^{3x} - 2e^{-3x}$;

(iii) $y = A + (B + \frac{1}{2}x)e^{x} + (C + \frac{1}{2}x)e^{-x}$;

(iv) $y = Ae^{4x} + B + \frac{2}{3}e^{-2x}$;

(v) $y = A\cos x + B\sin x - \frac{1}{3}\sin 2x$;

(vi) $y = Ae^{2x} + Be^{3x} + 4\cos 4x - 2\sin 4x$;

(vii) $y = e^{-4x}(A\cos 3x + B\sin 3x) + 2\cos x$;

(viii) $y = e^{-x}(A\cos 20x + B\sin 20x) + \sin 20x$;

(ix) $y = x^3 - 6x + 6\sin x$;

(x) $y = A + Be^{-2x} - \frac{1}{5}e^{-x}\sin 2x$;

(xi) $y = Ae^{-x} + Be^{2x} + 24x^2 + 14x - 5$;

(xii) $y = (A + Bx)e^{3x} + 6x + 6$;

(xiii) $y = 2\cos x + (2x-1)\sin x$.