UNIT 4 COMPLEX NUMBERS

Unit Structure

4.0 OVERVIEW

In this Unit, you will be studying some of the more advanced features of Complex Numbers.

Note: Before going through the Unit, make sure you know how to add, subtract, multiply and divide complex numbers, and also how to represent them in an Argand diagram and express them in different forms.

4.1 LEARNING OBJECTIVES

By the end of this Unit, you should be able to do the following:

1. Use De Moivre's Theorem.
2. Find the nth roots of unity.
3. Solve algebraic equations involving complex numbers.

4.2 REVIEW

Let us briefly review the main points on complex numbers.

1. A complex number is a number of the form $a+bi$, where a and b, called the real and imaginary parts respectively, are *real* numbers, and $i=\sqrt{-1}$. We use the abbreviations Re and Im to denote respectively the real and imaginary parts of a complex number. Thus

$$\text{Re}(-2-3i)=-2, \quad \text{Im}(-2-3i)=-3; \qquad \text{Re}\,(8i)=0.$$

NOTE: If z_1 and z_2 are complex quantities, then

$$\text{Re}(z_1 \pm z_2)=\text{Re}(z_1)\pm\text{Re}(z_2)$$

$$\text{Im}(z_1 \pm z_2)=\text{Im}(z_1)\pm\text{Im}(z_2)$$

but in general,

$$\text{Re}(z_1 z_2)\neq\text{Re}(z_1)\,\text{Re}(z_2)$$

$$\text{Re}(z_1/z_2)\neq\text{Re}(z_1)/\text{Re}(z_2)$$

$$\text{Im}(z_1 z_2)\neq\text{Im}(z_1)\;\text{Im}(z_2)$$

$$\text{Im}(z_1/z_2)\neq\text{Im}(z_1)\;/\,\text{Im}(z_2)\,.$$

2. Two complex numbers $z_1=x_1+i\,y_1$, $z_2=x_2+i\,y_2$ are said to be equal if and only if $x_1=x_2$ **and** $y_1=y_2$. Thus, if $-5+\alpha\, i=\beta+7i$, then $\alpha=7, \beta=-5$.

3. Complex numbers can be represented geometrically by means of an Argand diagram.

4. The *complex conjugate* of a complex number $z=x+iy$ is $\bar{z}=x-iy$. The complex conjugate of $\bar{z}$ is *z*. [Some authors use z^* instead of $\bar{z}$.] Geometrically, the complex conjugate of *z* is obtained by reflection of *z* in the real axis. The following properties of the complex conjugate are easily verified:

(i) $\overline{z_1 \pm z_2}=\overline{z_1}\pm\overline{z_2}$

(ii) $\overline{-z}=-\bar{z}$

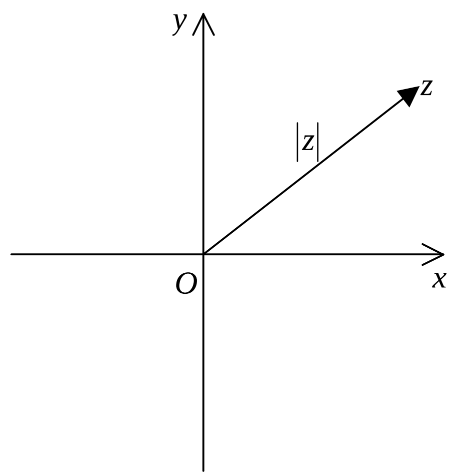
(iii) $\overline{z_1 z_2}=\overline{z_1}\;\overline{z_2}$

(iv) $\overline{(z_1/z_2)}=\overline{z_1}\,/\,\overline{z_2}$

(v) $\overline{(1/z)}=1/\,\overline{z}\,.$

5. For a polynomial equation with *real* coefficients, complex roots occur in conjugate pairs, i.e., one is the conjugate of the other. Thus, if one root of the equation is $-4+7i$, then there will be another root (its conjugate) given by $-4-7i$.

6. **Modulus**. If $z = x + iy$, the *modulus* of z is the non-negative real number $|z|$ defined by $|z| = \sqrt{x^2 + y^2}$. Geometrically, the modulus of z is the distance from z to 0.



More generally, $|z_1 - z_2|$ is the distance between z_1 and z_2 in the complex plane, since

$$|z_1 - z_2| = |(x_1 + iy_1) - (x_2 + iy_2)| = |(x_1 - x_2) + i\,(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

An important result is the identity

$$z\,\bar{z} = |z|^2 .$$

Also, note that

$$|x + y| \le |x| + |y|$$

$$|x - y| \ge \big|\, |x| - |y| \,\big|$$

$$|z - a|^2 = |z|^2 - 2\,\mathrm{Re}(a\,\bar{z}) + |a|^2 .$$

The following properties of the modulus are easily verified using the above identity.

(i) $|z_1 z_2| = |z_1|\,|z_2|$. In general, $|z_1 z_2 \ldots z_r|^n = |z_1|^n \cdot |z_2|^n \ldots |z_r|^n \quad \forall n$.

(ii) $|z^{-1}| = |z|^{-1}$

(iii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$. In general, $\left|\dfrac{z_1}{z_2}\right|^n = \dfrac{|z_1|^n}{|z_2|^n} \quad \forall n$.

For example, to prove **(i)**:

$$|z_1 z_2|^2 = (z_1 z_2)\,\overline{z_1 z_2} = (z_1 z_2)\,\overline{z_1}\;\overline{z_2} = (z_1\bar{z}_1)(z_2\bar{z}_2) = |z_1|^2 |z_2|^2 = (|z_1|\,|z_2|)^2.$$

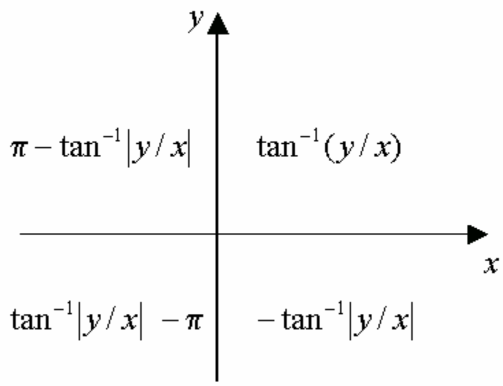
Hence $|z_1 z_2| = |z_1|\,|z_2|$.

<u>Example 1</u> Find $|z|$ when $z = \dfrac{(2-3i)^4}{(1+8i)^2(-5+2i)^3}$.

$$|z| = \frac{|2-3i|^4}{|1+8i|^2\;|-5+2i|^3} = \frac{[\sqrt{2^2+(-3)^2}\,]^4}{[\sqrt{1^2+8^2}\,]^2\;[\sqrt{(-5)^2+2^2}\,]^3} = \frac{169}{65\,(\sqrt{29})^3} = 0.0166485$$

7. **Argument**. If $z = x + iy$, then $\arg z = \tan^{-1}(y/x)$. If $\arg z = \theta$, then the convention used in this manual is that θ is in radians and $-\pi < \theta \le \pi$, i.e. θ must lie between -3.142 and $+3.142$. If the argument of a complex quantity is outside this range, we add or subtract multiples of 2π until it lies in $(-\pi, \pi]$.

Note: **arg(+ve real number) = 0**; **arg (-ve real number)** $= \pi$.



<u>Example 2</u> We find the argument of the following complex numbers, each one lying in a different quadrant. We make use of the above diagram.

(i) $\arg(2+3i) = \tan^{-1}(3/2) = 0.982794$ **(radian)**

(ii) $\arg(-5+4i) = \pi - \tan^{-1}(4/5) = 2.46685$

(iii) $\arg(-3-7i) = \tan^{-1}(7/3) - \pi = -1.97569$

(iv) $\arg(8-5i) = -\tan^{-1}(5/8) = -0.558599$

The properties of arg are like those of log. The results must lie in $(-\pi, \pi]$

$$\arg(z_1 z_2)^n = n[\arg z_1 + \arg z_2]$$

$$\arg(z_1 / z_2)^n = n[\arg z_1 - \arg z_2].$$

Note $\arg(z_1 \pm z_2) \neq \arg z_1 \pm \arg z_2$

<u>Example 3a</u>

$$\begin{aligned}\arg[(2+3i)(8-5i)]^{10} &= 10[\arg(2+3i) + \arg(8-5i)] \\ &= 10[0.982794 + (-0.558599)] \\ &= 4.24195\end{aligned}$$

But this is outside the interval $(-\pi, \pi]$. So, we subtract 2π and see that

$$\arg[(2+3i)(8-5i)]^{10} = -2.0412 \quad \text{to 4 d.p}$$

which does lie in $(-\pi, \pi]$.

Example 3b

$$\begin{aligned}\arg\left(\frac{2+3i}{-5+4i}\right)^{10} &= 10\,[\arg(2+3i)-\arg(-5+4i)] \\ &= 10\,[0.982794-2.46685] \\ &= -14.84056\end{aligned}$$

which is outside $(-\pi,\pi]$. So, we add 2π and get -8.55737. Still outside! So we add another 2π to finally obtain -2.27419.

Hence,

$$\arg\left(\frac{2+3i}{-5+4i}\right)^{10} = -2.2742 \quad \text{to 4 d.p.}$$

8. Complex numbers can be expressed either in the Cartesian form $x+iy$, or in polar form $r(\cos\theta + i\sin\theta)$, or in exponential form $r\,e^{i\theta}$, where $r=\sqrt{x^2+y^2}$ is the modulus of the complex number, and $\theta=\tan^{-1}(y/x)$ is the argument. **Note:** $-\pi<\theta\le\pi$. Work in **Radians.**

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\bar{z}_1 = r_1(\cos\theta_1 - i\sin\theta_1);$$

$$\frac{1}{z_1} = \frac{1}{r_1}(\cos\theta_1 - i\sin\theta_1);$$

$$z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)];$$

$$z_1/z_2 = (r_1/r_2)[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)].$$

Example 4

The complex numbers *w* and *z* are given by

$$w = -2 + 5i, \quad z = -3 - 2i\,.$$

Find:

(i) the polar forms of *w* and *z*;

(ii) $\left|z + w^2\right|$;

(iii) $\left|\dfrac{z^3}{w^5}\right|$;

(iv) $\arg(z^5 / w^4)$.

Answer

(i) $|w| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$; $\arg w = \pi - \tan^{-1}(5/2) = 1.9513$

$$\therefore\ w = \sqrt{29}\,[\cos 1.9513 + i\,\sin 1.9513].$$

$|z| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$; $\arg z = \tan^{-1}(2/3) - \pi = -2.55359$

$$\begin{aligned} \therefore\ z &= \sqrt{13}\,[\cos(-2.55359) + i\ \sin(-2.55359)] \\ &= \sqrt{13}\,[\cos(2.55359) - i\ \sin(2.55359)] \end{aligned} \qquad \because \begin{cases} \cos(-\alpha) = +\cos\alpha \\ \sin(-\alpha) = -\sin\alpha \end{cases}$$

(ii) $z + w^2 = (-2 + 5i) + (-3 - 2i)^2 = (-2 + 5i) + (5 + 12i) = 3 + 17i$

$$\therefore \left|z + w^2\right| = \sqrt{3^2 + 17^2} = \sqrt{298}\,.$$

Note: $\left|z + w^2\right| \neq |z| + \left|w^2\right|$.

(iii) $\left|\dfrac{z^3}{w^5}\right| = \dfrac{|z|^3}{|w|^5} = \dfrac{(\sqrt{13})^3}{(\sqrt{29})^5} = \sqrt{\dfrac{13^3}{29^5}} = 0.0103495$

(iv) $\arg(z^5 / w^4) = 5\arg z - 4\arg w = 5\,(-2.55359) - 4\,(1.9513) = -20.57315$

Now, this value is not in the range $(-\pi, \pi]$. So we keep adding multiples of 2π until the answer is in $(-\pi, \pi]$. Doing this we finally obtain

$$\arg(z^5 / w^4) = -1.7236 \quad \text{to 4 d.p.}$$

9. Finally, we note that the field of complex numbers cannot be ordered; i.e., the relations "greater than" and "less than" cannot be consistently defined for complex numbers. Thus, such statements as

$$352 + 987\,i > 1 + i \text{, and } -79 - 23i < 0$$

are completely meaningless.

In this Unit, a general complex number z will be written in the form

$$z = x + i\,y = r\,(\cos\theta + i\sin\theta) = r\,e^{i\theta}.$$

From **Euler's formula,** $e^{i\theta} = \cos\theta + i\sin\theta$, we have the following results which should be learnt as they are in constant use:

$$\boxed{\begin{aligned} e^{0} &= e^{2\pi i} = e^{4\pi i} = \cdots = e^{2n\pi i} = 1, \\ e^{\pm i\pi} &= -1, \\ e^{\pm i\pi/2} &= \pm i \end{aligned}}$$

The function $e^{i\theta}$ is evidently periodic in θ with period 2π.

We shall now make use of Euler's formula and the above boxed results to work out a few complex quantities. We need the following result:

$$a = e^{\ln a}, \quad a > 0.$$

Example 5

Evaluate 3^i.

We have from above $\quad 3 = e^{\ln 3}$.

$$\therefore \quad 3^i = (e^{\ln 3})^i = e^{i(\ln 3)}$$

$$= \cos(\ln 3) + i\sin(\ln 3) \text{ [Euler, with } \theta = \ln 3\text{, in } \textit{radians}\text{]}$$

$$= 0.45483 + 0.89058\,i.$$

Example 6

Evaluate $(-3)^i$.

Now, we can't write $-3 = e^{\ln(-3)}$. Instead we write

$$-3 = -1 \times 3 = e^{i\pi} \times e^{\ln 3}$$
$$= e^{i\pi + \ln 3}.$$

$$\therefore \quad (-3)^i = (e^{i\pi+\ln 3})^i = e^{-\pi + i\ln 3} = e^{-\pi}.e^{i\ln 3}$$
$$= e^{-\pi}[\cos(\ln 3) + i\sin(\ln 3)]$$
$$= 0.01966 + 0.03849\,i.$$ [Calculator in *Radian* mode!]

Generally, we have for any real $x > 0$,

$$x^i = \cos(\ln x) + i\sin(\ln x), \qquad (\dagger)$$
$$(-x)^i = e^{-\pi}\,[\cos(\ln x) + i\sin(\ln x)].$$

Example 7

Evaluate 5^{2+3i}.

We have $5^{2+3i} = 5^2 \times 5^{3i} = 25 \times (5^3)^i = 25 \times 125^i$

Now $125^i = \cos(\ln 125) + i\sin(\ln 125)$, using (†).

Hence $5^{2+3i} = 25\,[\cos(\ln 125) + i\sin(\ln 125)]$

$$= 2.89163 - 24.83221\,i.$$ [Calculator in *Radian* mode!]

In general, we have for any real $x > 0$,

$$\begin{aligned} x^{a+bi} &= x^a\,[\cos(\ln x^b) + i\sin(\ln x^b)] \\ &= x^a\,[\cos(b\ln x) + i\sin(b\ln x)] \end{aligned}.$$

Complex Powers

Let us now find the complex power of a complex number.

Example 8

Evaluate $(2+3i)^i$.

We first express $(2+3i)$ in the exponential form $re^{i\theta}$.

Thus $2+3i=\sqrt{13}\,e^{i\alpha}$, where $\alpha=\tan^{-1}3/2=0.982794$ [in *Radians*]

$$\begin{aligned}\therefore\quad (2+3i)^i &= (\sqrt{13}\,e^{i\alpha})^i = (\sqrt{13})^i\ e^{i^2\alpha}\\ &= (\sqrt{13})^i\ e^{-\alpha}\\ &= e^{-\alpha}\,[\cos(\ln\sqrt{13})+i\sin(\ln\sqrt{13})], \qquad \text{[using (†)]}\\ &= 0.10642+0.358815\,i\,.\end{aligned}$$

Example 9

Evaluate $(-3+4i)^{2-3i}$.

First $-3+4i=5e^{i\alpha}$, where the argument $\alpha=2.2143$ [in *Radians*]

So $(-3+4i)^{2-3i}=(5e^{i\alpha})^{2-3i}=5^{2-3i}\ e^{2\alpha i+3\alpha}$

Now $5^{2-3i}=5^2\times 5^{-3i}=25\times(5^{-3})^i=25[\cos(\ln 5^{-3})+i\sin(\ln 5^{-3})]$, [using (†)]

And $e^{2\alpha i+3\alpha}=e^{3\alpha}.\,e^{i(2\alpha)}=e^{3\alpha}[\cos 2\alpha+i\sin 2\alpha]$

Putting every thing together, we have

$$\begin{aligned}(-3+4i)^{2-3i} &= 25\,[\cos(\ln 5^{-3})+i\sin(\ln 5^{-3})].e^{3\alpha}(\cos 2\alpha+i\sin 2\alpha)\\ &= (2.89163+24.8322\,i)\times(-214.847-736.619\,i)\\ &= 17670.6-7465.16\,i\ .\end{aligned}$$

In each of the above examples, the power was not real. In the next section we shall learn how to find real powers of complex numbers by using a very important theorem in complex numbers, called De Moivre's theorem.

4.3 De MOIVRE's THEOREM

This is the most important theorem in this Unit. It states that for any rational value of *n*, *one value* of $(\cos\theta + i\sin\theta)^n$ is given by

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

[**NOTE:** The reason for saying "one value" is that there are more than one value for expressions like $(\cos\theta + i\sin\theta)^{\frac{2}{5}}$, i.e., when *n* is not an integer.]

The proof is by Induction and is omitted.

<u>Examples</u>:

(i) $(\cos\theta + i\sin\theta)^9 = \cos 9\theta + i\sin 9\theta$.

(ii) $(\cos\theta + i\sin\theta)^{-3} = \cos(-3\theta) + i\sin(-3\theta) = \cos 3\theta - i\sin 3\theta$.

(iii) $(\cos\theta - i\sin\theta)^7 = \cos 7\theta - i\sin 7\theta \quad [\because \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)]$.

(iv) $\dfrac{(\cos\varphi - i\sin\varphi)^5}{(\cos\varphi + i\sin\varphi)^3} = \dfrac{(\cos\varphi + i\sin\varphi)^{-5}}{(\cos\varphi + i\sin\varphi)^3} = (\cos\varphi + i\sin\varphi)^{-8} = (\cos 8\varphi - i\sin 8\varphi)$.

(v) $\dfrac{(\cos p\alpha + i\sin p\alpha)^m}{(\cos q\beta + i\sin q\beta)^n} = \dfrac{(\cos mp\alpha + i\sin mp\alpha)}{(\cos nq\beta + i\sin nq\beta)} = \cos(mp\alpha - nq\beta) + i\sin(mp\alpha - nq\beta)$.

We shall now relate De Moivre's theorem to Euler's formula. We have

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

So far, the number *n* has been taken as real.

If *n* is the complex number $a + bi$, we *define* $(\cos\theta + i\sin\theta)^{a+bi}$ to be

$$\begin{aligned}(\cos\theta + i\sin\theta)^{a+bi} &= (e^{i\theta})^{a+bi} \\ &= e^{i(a+bi)\theta} \\ &= e^{-b\theta}e^{ia\theta} \\ &= e^{-b\theta}(\cos a\theta + i\sin a\theta).\end{aligned}$$

The real part is $e^{-b\theta}\cos a\theta$ and the imaginary part is $e^{-b\theta}\sin a\theta$. The above result can also be used to raise a complex number to the power of a complex number, as in

Examples 8 and 9 above. Various interesting identities may also be produced. For example, when $\theta = \pi/2,\ a = 0,\ b = 1$, we have $\cos\theta + i\sin\theta = i$, and $a + bi = i$, so

$$i^i = e^{-\pi/2},$$

a real quantity, exhibiting a remarkable relationship between the three fundamental numbers e, i, π.

4.3.1 Applications of De Moivre's Theorem

If $z = x + iy = r(\cos\theta + i\sin\theta)$, then

$$z^n = (x+iy)^n = r^n(\cos n\theta + i\sin n\theta). \qquad \textbf{(\#)}$$

We can therefore use this result to find powers of complex numbers.

Example 10

Express $(1 - i\sqrt{3})^{10}$ in the form $a + bi$.

We'll use the result given by **(#)**. We must first express the number in polar form before using De Moivre's theorem.

Now, $\quad 1 - i\sqrt{3} = 2[\cos(\pi/3) - i\sin(\pi/3)]$

Hence,

$$\begin{aligned}(1 - i\sqrt{3})^{10} &= 2^{10}[\cos(\pi/3) - i\sin(\pi/3)]^{10}\\ &= 2^{10}[\cos(10\pi/3) - i\sin(10\pi/3)]\\ &= -512 + 512\sqrt{3}\,i.\end{aligned}$$

Example 11

Evaluate $\dfrac{(-1-3i)^8}{(4-5i)^6}$.

Here we need the result $z_1/z_2 = (r_1/r_2)[\cos(\theta_1 - \theta_2) + i\ \sin(\theta_1 - \theta_2)]$, where $z_1 = (-1-3i)^8, \quad z_2 = (4-5i)^6$.

In polar form
$$-1-3i=\sqrt{10}\,[\cos(-1.89255)+i\sin(-1.89255)],$$
$$4-5i=\sqrt{41}\,[\cos(-0.896055)+i\sin(-0.896055)]$$

[**Note**: All angles are in radians.]

$$\therefore\quad \frac{(-1-3i)^8}{(4-5i)^6}=\frac{(\sqrt{10})^8[\cos(8\times-1.89255)+i\sin(8\times-1.89255)]}{(\sqrt{41})^6[\cos(6\times-0.896055)+i\sin(6\times-0.896055)]}$$

$$=\frac{10^4}{41^3}[\cos(8\times-1.89255-6\times-0.896055)+i\sin(8\times-1.89255-6\times-0.896055)]$$

$$=-0.1368+0.0483\,i\ \text{(to 4 d.p.)}.$$

ACTIVITY 1

1. Express the following complex numbers in polar form:

(i) $z_1=-1+\sqrt{3}\,i$; **(ii)** $z_2=3i$; **(iii)** $z_3=-7i$; **(iv)** $z_4=-24+7i$.

2. Given that the real and imaginary parts of the complex number $z=x+iy$ satisfy the equation $(2-i)x-(1+3i)y=7$,

(a) Find *x* and *y*.

(b) Express *z* in polar form, and hence find z^{10}.

3. Evaluate

(i) $(-i)^i$; **(ii)** 5^i; **(iii)** $(-10)^i$; **(iv)** 7^{2-3i}; **(v)** $(2+3i)^{3+4i}$;

(vi) $(1+\sqrt{3}\,i)^{10}+(1-\sqrt{3}\,i)^{10}$; **(vii)** $\dfrac{(3-2i)^8}{(7+6i)^5}$.

4. Show that $z=1+i$ is a root of the equation $z^4+3z^2-6z+10=0$. Find the other roots of this equation.

5. Prove that
$$\frac{1+\cos\alpha+i\sin\alpha}{1-\cos\alpha+i\sin\alpha}=\cot\frac{\alpha}{2}\,.e^{i[\alpha-\pi/2]}.$$

4.3.2 More Applications of De Moivre's Theorem

1. Solution of Algebraic Equations

The use of De Moivre's theorem in finding the complex roots of numbers and the equivalent problem of the solution of equations in z is best shown by the following examples.

Example 12

The *n*th roots of unity (i.e. $1^{1/n}$)

To find the n complex roots of the equation

$$z^n - 1 = 0, \qquad (‡)$$

where n is a positive integer, we use the general complex form of unity

$$1 = e^{2k\pi i} = \cos 2k\pi + i \sin 2k\pi,$$

where k is any integer (including zero). De Moivre's theorem now gives

$$z = 1^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},$$

which, on letting $k = 0, 1, 2, \cdots, n-1$, yields the n distinct roots $z_1, z_2, \cdots, z_n$.

These are the roots of (‡), or, in other words, the nth roots of unity. For example, if $n = 6$, the six sixth roots of $z^6 = 1$ are the six values of

$$z = \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6},$$

obtained by putting $k = 0, 1, 2, 3, 4, 5$. Thus we have

$$k = 0, \quad z_1 = \cos 0 + i\sin 0 = 1,$$

$$k = 1, \quad z_2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$k = 2, \quad z_3 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$k = 3, \quad z_4 = \cos\pi + i\sin\pi = -1$$

$$k = 4, \quad z_5 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$k = 5, \quad z_6 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

No new roots are obtained by giving k any other values. For example, the root corresponding to $k = 6$ just reproduces the root corresponding to $k = 0$; similarly $k = 7$ reproduces $k = 1$, $k = -1$ reproduces $k = 5$, and so on.

It is important to notice that in taking roots of complex numbers, the general form

$$z = r\,[\cos(\theta + 2k\pi) + i\,\sin(\theta + 2k\pi)],\ k \text{ any integer},$$

which allows for the many-valuedness of the argument, must always be used.

Using the above ideas, we may find the nth roots of other real numbers as well.

Example 13a

(a) Find the 3 cube roots of 5. Equivalently, solve the equation $z^3 = 5$.

We need to work out the 3 values of $5^{1/3}$.

Now, since

$$1 = e^{2k\pi i} = \cos 2k\pi + i \sin 2k\pi,$$

we have

$$5 = 5e^{2k\pi i} = 5\,[\cos 2k\pi + i \sin 2k\pi]$$

so that

$$\begin{aligned} 5^{1/3} &= 5^{1/3}[\cos 2k\pi + i \sin 2k\pi]^{1/3}, \quad k = 0, 1, 2 \\ &= 5^{1/3}[\cos(2k\pi/3) + i \sin(2k\pi/3)] \end{aligned}$$

Putting $k = 0, 1, 2$, we obtain $5^{1/3} = 1.70998, -0.854988 \pm 1.48088i$.

Example 13b

Solve the equation $z^4 + 7 = 0$. Equivalently, find the four 4th roots of -7.

Clearly, $z = (-7)^{1/4}$, so all we need to find are the 4 values of $(-7)^{1/4}$.

First, we write -7 as $-7 = 7 \times -1$, and express -1 in polar form. Now, $-1 = e^{i\pi} = \cos\pi + i \sin\pi$. Adding the multiple of 2π, we have

$$-1 = \cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)$$

Hence

$$-7 = 7\,[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]$$

$$\therefore \quad \begin{aligned}(-7)^{1/4} &= 7^{1/4}\,[\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/4}, \quad k = 0, 1, 2, 3\\ &= 7^{1/4}\,[\cos(2k+1)\pi/4 + i\sin(2k+1)\pi/4]\end{aligned}$$

i.e., $\quad z = \pm 1.15016 \pm 1.15016\, i$.

We shall now solve a few equations involving complex numbers.

Example 14

Evaluate $\sqrt{5-12i}$. [Alternatively, solve the equation $z^2 = 5-12i$]

Now, $\quad 5-12i = 13\,[\cos(\alpha + 2k\pi) + i\sin(\alpha + 2k\pi)], \quad \alpha = -\tan^{-1}(12/5)$

$$\therefore \quad \begin{aligned}\sqrt{5-12i} &= \sqrt{13}\,[\cos(\alpha + 2k\pi) + i\sin(\alpha + 2k\pi)]^{1/2}, \quad k = 0, 1\\ &= \sqrt{13}\,[\cos(\alpha/2 + k\pi) + i\sin(\alpha/2 + k\pi)]\end{aligned}$$

Putting $k = 0, 1$, we have $\sqrt{5-12i} = \pm\,(3-2i)$.

Example 15

Solve the equation

$$z^4 - 1 = \sqrt{3}\, i\,.$$

We have

$$z^4 = 1 + \sqrt{3}\, i = 2\left\{\cos\left(\frac{\pi}{3} + 2k\pi\right) + i\sin\left(\frac{\pi}{3} + 2k\pi\right)\right\}.$$

Hence, using De Moivre's theorem,

$$z = 2^{1/4}\left\{\cos\left(\frac{\pi}{12} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{12} + \frac{k\pi}{2}\right)\right\}, \quad k = 0, 1, 2, 3.$$

The four roots are therefore

$$k=0,\quad z_1=2^{1/4}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)=1.14869+0.30779\,i$$

$$k=1,\quad z_2=2^{1/4}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)=-0.30779+1.14869\,i$$

$$k=2,\quad z_3=2^{1/4}\left(\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12}\right)=-1.14869-0.30779\,i$$

$$k=3,\quad z_4=2^{1/4}\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)=0.30779-1.14869\,i$$

All other *k*-values (integers) will reproduce these four roots.

Example 16

Evaluate $(-7+4i)^{3/5}$. [Or, Solve the equation $z^5=(-7+4i)^3$]

In polar form, $-7+4i=\sqrt{65}\,[\cos\alpha+i\sin\alpha],\quad \alpha=\pi-\tan^{-1}\tfrac{4}{7}$.

$$\begin{aligned}(-7+4i)^{3/5}&=(\sqrt{65})^{3/5}\,[\cos(\alpha+2k\pi)+i\sin(\alpha+2k\pi)]^{3/5},\quad k=0,1,2,3,4\\ &=65^{3/10}\,[\cos 3(\alpha+2k\pi)/5+i\sin 3(\alpha+2k\pi)/5]\end{aligned}$$

Putting $k=0,1,2,3,4$, we find that the roots are respectively

$-0.00935+3.49842\,i,\ \ 2.06388-2.82479\,i,\ \ -3.33009+1.07218\,i,$

$3.32431+1.08996\,i,\ \ -2.04876-2.83578\,i$

ACTIVITY 2

1. Find the three cube roots of 1 and *i*. [Equivalently, solve the equations: $z^3=1$ and $z^3=i$.]

2. Solve the following equations:

(i) $(1-x)^5=x^5$**; (ii)** $(z+1)^7+(z-1)^7=0$**; (iii)** $z^4+1=\sqrt{3}\,i$**;**

(iv) $z^6-z^3+1=0$**.** [Hint: Let $Z=z^3$.]

4.4 SUMMARY

In this unit you have studied how to manipulate complex numbers. You have been introduced to the important and powerful De Moivre's theorem. You have learnt how to use the theorem to solve algebraic equations and also how to work out the powers of real and complex numbers.

4.5 ANSWERS TO ACTIVITIES

ACTIVITY 1

1 (i) $2\,[\cos(2\pi/3) + i\sin(2\pi/3)]$;

(ii) $3\,[\cos(\pi/2) + i\sin(\pi/2)]$

(iii) $7\,[\cos(\pi/2) - i\sin(\pi/2)]$

(iv) $25\,[\cos(2.858) + i\sin(2.858)]$.

2. (a) $x = 3, \quad y = -1$**;**

(b) $z = \sqrt{10}\,[\cos(0.32175) - i\sin(0.32175)]$**;** $z^{10} = -99712 + 7584\,i$**.**

3. **(i)** $e^{\pi/2}$ $(= 4.81048)$**;** **(ii)** $-0.038632 + 0.999254\,i$ **;**

(iii) $-0.0288756 + 0.0321503\,i$ **; (iv)** $44.2183 + 21.1126\,i$ **;**

(v) $-0.204553 + 0.896623\,i$ **; (vi)** -1024 **; (vii)** $-0.164271 - 0.396056\,i$ **.**

4. $1 - i, \quad -1 \pm 2i$**.**

ACTIVITY 2

1. $1^{1/3} = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$**;** $i^{1/3} = \pm\frac{\sqrt{3}}{2} + \frac{1}{2} i, -i$**.**

2. **(i)** $\dfrac{1}{2} - \dfrac{1}{2} i \tan\dfrac{k\pi}{5}, k = 0, 1, \cdots, 4$ **;**

(ii) $z = \dfrac{\alpha + 1}{\alpha - 1}$, where $\alpha = e^{(2n+1)\pi i/7}$, $n = 0, 1, \cdots, 6$**;**

(iii) $z = 2^{1/4} e^{i\theta}$, where $\theta = \pi/6 + n\pi/2, \quad n = 0, 1, 2, 3$;

(iv) $z = e^{im\pi/9}, \quad m = 1, 5, 7, 11, 13, 17$.