UNIT 6 VECTOR ANALYSIS

Unit Structure

6.0 OVERVIEW

In this Unit, you will study some of the most important tools of Applied Mathematics. These have many applications in the physical sciences and in engineering.

6.1 LEARNING OBJECTIVES

After completing this unit, you should be able to do the following:

- Use the del operator in finding gradient, divergence and curl.
- Work out directional derivatives.
- Find the Laplacian of a scalar and vector field.

6.2 THE VECTOR OPERATOR - del

The symbol ∇ (called del) represents a vector operator of the form

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

in Cartesian coordinates.

When it operates on a differentiable <u>scalar</u> function of the space variables x, y, z, we obtain a quantity called the <u>gradient vector</u> of the function; thus if $\phi = \phi(x, y, z)$ then the gradient vector of ϕ, written grad ϕ, is given by

$$\begin{aligned} \text{grad } \phi &= \nabla\phi \\ &= (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z})\phi \\ &= \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z}\ , \end{aligned}$$

which is a **vector** quantity.

We now show that **grad ϕ is a vector normal to the surface** $\phi(x, y, z) = c$, where c is a constant.

Let $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ be the position vector to any point $P(x, y, z)$ on the surface.

Then $d\mathbf{r} = dx\ \hat{\mathbf{i}} + dy\ \hat{\mathbf{j}} + dz\ \hat{\mathbf{k}}$ lies in the plane tangential to the surface at P. But

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0\ \text{, or}\ \left(\frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}}\right)\mathbf{.}(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}) = 0\ ,$$

i.e. $\nabla\phi . d\mathbf{r} = 0$, so that grad ϕ is perpendicular to $d\mathbf{r}$ and therefore to the surface.

Hence a unit normal $\hat{\mathbf{n}}$ to the surface $\phi(x, y, z) = c$ is given by

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|}\ .$$

We also note that a concise form of the differential of $\phi(x, y, z)$ is

$$\boxed{d\phi = \nabla\phi\ .\ d\mathbf{r}}.$$

Example 1

If $\phi(x,y,z) = x^2y^3 + e^z \sin y - z^4$, then grad ϕ is given by

$$\nabla\phi = (2xy^3)\hat{\mathbf{i}} + (3x^2y^2 + e^z \cos y)\hat{\mathbf{j}} + (e^z \sin y - 4z^3)\hat{\mathbf{k}} \ ,$$

and a unit vector to the surface $x^2y^3 + e^z \sin y - z^4 = const.$ is given by

$$\frac{(2xy^3)\hat{\mathbf{i}} + (3x^2y^2 + e^z \cos y)\hat{\mathbf{j}} + (e^z \sin y - 4z^3)\hat{\mathbf{k}}}{\sqrt{(2xy^3)^2 + (3x^2y^2 + e^z \cos y)^2 + (e^z \sin y - 4z^3)^2}}$$

Example 2

Show that the surfaces $\mathrm{F}(x,\ y,\ z) = 3x^2 + 4y^2 + 8z^2 - 36 = 0$ and $G(x, y, z) = x^2 + 2y^2 - 4z^2 - 6 = 0$ intersect at right angles.

Solution

The family of normals to the surface $\mathrm{F}(x,\ y,\ z)$ are defined by the vector $\nabla\,\mathrm{F} = 6\,x\hat{\mathbf{i}} + 8\,y\,\hat{\mathbf{j}} + 16\,z\,\hat{\mathbf{k}}$ and those to $\mathrm{G}(x,\ y,\ z)$ by $\nabla\,\mathrm{G} = 2\,x\hat{\mathbf{i}} + 4\,y\,\hat{\mathbf{j}} - 8\,z\,\hat{\mathbf{k}}$.

$\nabla\ \mathrm{F}\,.\,\nabla\ \mathrm{G} = 12\ x^2 + 32\ y^2 - 12\ z^2$,
or
$\nabla\,\mathrm{F}\,.\,\nabla\,\mathrm{G}\ = 4\left\{6\left(x^2 + 2\,y^2 - 4\,z^2\right) - \left(3\,x^2 + 4\,y^2 + 8\,z^2\right)\right\} = 0$.

Hence the two surfaces intersect at right angle.

Example 3
Find $\nabla\phi$ if $\phi = 1/r$.

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \ \text{, so } r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla(\frac{1}{r}) = \hat{\mathbf{i}}\frac{\partial}{\partial x}[\frac{1}{\sqrt{x^2 + y^2 + z^2}}] + \hat{\mathbf{j}}\frac{\partial}{\partial y}[\frac{1}{\sqrt{x^2 + y^2 + z^2}}] + \hat{\mathbf{k}}\frac{\partial}{\partial z}[\frac{1}{\sqrt{x^2 + y^2 + z^2}}]$$

$$= -\ \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}} \ = -\frac{\mathbf{r}}{r^3}.$$

In general,

$$\nabla\ \mathrm{f}(r) = \frac{d\mathrm{f}}{dr}\hat{\mathbf{r}} = \frac{1}{r}\frac{d\mathrm{f}}{dr}\mathbf{r}\,.$$

6.2.1 TANGENT PLANE AND NORMAL LINE TO A SURFACE

The equation of the tangent plane to the surface $F(x, y, z) = 0$ at one of its points $P_0(x_0, y_0, z_0)$ is

$$\frac{\partial F}{\partial x}(x - x_0) + \frac{\partial F}{\partial y}(y - y_0) + \frac{\partial F}{\partial z}(z - z_0) = 0$$

and the equations of the normal line at P_0 are

$$\frac{(x - x_0)}{\frac{\partial F}{\partial x}} + \frac{(y - y_0)}{\frac{\partial F}{\partial y}} + \frac{(z - z_0)}{\frac{\partial F}{\partial z}} = 0,$$

the partial derivatives are evaluated at the point P_0 (see Fig.1)

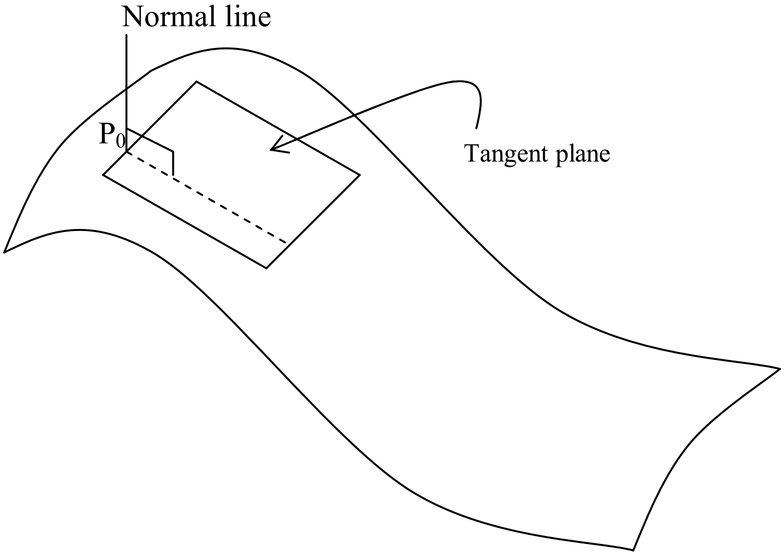

Figure 1 : Tangent Plane and Normal Line

__Example 4__

Find the equation of the tangent plane and the normal line to the surface $z = 3x^2 + 2y^2 - 11$ at (2, 1, 3).

Solution

We put $F(x, y, z) = 3x^2 + 2y^2 - z - 11 = 0$.

At (2, 1, 3), $\frac{\partial F}{\partial x} = 6x = 12, \frac{\partial F}{\partial y} = 4y = 4$ and $\frac{\partial F}{\partial z} = -1$ and the equation of the tangent plane is then $12(x - 2) + 4(y - 1) - (z - 3) = 0$ (which you can simplify). Further, the equation of the normal line is

$$\frac{(x - 2)}{12} = \frac{(y - 1)}{4} = \frac{(z - 3)}{-1}.$$

6.2.2 BASIC PROPERTIES OF THE GRADIENT OPERATOR

Let f and g be differentiable functions. Then

$$\nabla c = \mathbf{0} \quad \text{for any constant } c$$

$$\nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g \quad \text{for constants } \alpha \text{ and } \beta$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(f/g) = \frac{g\nabla f - f\nabla g}{g^2} \qquad g \neq 0$$

$$\nabla(f^n) = nf^{n-1}\nabla f$$

Activity 1

Find the gradient of each of the following functions :

(a) $x^2 + 2yz$ **; (b)** e^{xyz} **; (c)** $x^a y^b z^c$.

Activity 2

Show that the equation of the tangent plane to the surface $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ at the point $P_0(x_0, y_0, z_0)$ is $\frac{x\,x_0}{a^2} - \frac{y\,y_0}{b^2} - \frac{z\,z_0}{c^2} = 1$.

Activity 3

Find a unit normal to each of the following surfaces at the specified points :

(a) $x^2 + y^2 + z^2 = 2(x + y + z)$ at (2,2,0) ;

(b) $z = x^2 + y^2$ at (1,-2,5) .

∇ cannot operate on a vector function (i.e., $\nabla\mathbf{a}$ is left undefined for the moment), but we can define the dot and vector products of ∇ with a vector function of *x*, *y*, *z*. These products are called respectively the *divergence* and *curl* of the vector function. We write

$$\text{div } \mathbf{F} = \nabla.\mathbf{F} = \hat{\mathbf{i}}.\frac{\partial \mathbf{F}}{\partial x} + \hat{\mathbf{j}}.\frac{\partial \mathbf{F}}{\partial y} + \hat{\mathbf{k}}.\frac{\partial \mathbf{F}}{\partial z}.$$

If $\mathbf{F} = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}$, we then have

$$\text{div } \mathbf{F} = \nabla.\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Thus, the divergence of a **vector** function is a *scalar* function.

If the divergence is zero we say that the vector field is <u>*solenoidal.*</u>

Example 5

If $\mathbf{F} = x^2 \cos y\, \hat{\mathbf{i}} - xz \tan y\, \hat{\mathbf{j}} + y^2 e^z\, \hat{\mathbf{k}}$, then

$$\nabla.\mathbf{F} = \frac{\partial}{\partial x}(x^2 \cos y) + \frac{\partial}{\partial y}(-xz \tan y) + \frac{\partial}{\partial z}(y^2 e^z)$$

$$= 2x \cos y - xz \sec^2 y + y^2 e^z \ .$$

If $\mathbf{F} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, then

$$\nabla.\mathbf{F} = \nabla.\mathbf{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3.$$

$$\boxed{\nabla.\mathbf{r} = 3}.$$

Note: This is a very useful result, worth memorizing.

Activity 4

Compute the divergence of the following vector fields :

(a) $y^2\ \hat{\mathbf{i}} + 2xz\,\hat{\mathbf{j}} - xyz\,\hat{\mathbf{k}}$ **, (b)** $z\cosh xy\,\mathbf{i} - \sin yz\,\hat{\mathbf{j}} - x^3z^2\ \hat{\mathbf{k}}$.

Activity 5

Let **F** be the vector field $z\,e^{x+3y}\,\mathbf{i} + 3\,z\,e^{x+3y}\,\mathbf{j} + e^{x+3y}\,\mathbf{k}$. Find Φ with ∇ Φ = **F**.

The **curl** of a **vector** function is written:

$$\text{curl}\,\mathbf{F} = \nabla \wedge \mathbf{F} = \hat{\mathbf{i}} \wedge \frac{\partial \mathbf{F}}{\partial x} + \hat{\mathbf{j}} \wedge \frac{\partial \mathbf{F}}{\partial y} + \hat{\mathbf{k}} \wedge \frac{\partial \mathbf{F}}{\partial z}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note: This is not a true determinant and so rows and /or columns should not be manipulated. Also in the expansion of the determinant, the operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ must precede the components F_1, F_2, F_3 .
When the curl is zero, we say that the vector field is *irrotational*, otherwise it is rotational.

Example 6

If $\mathbf{F} = (5x^2y^2)\ \hat{\mathbf{i}} + (x^2 - 2yz^2)\ \hat{\mathbf{j}} - 2y^2z\,\hat{\mathbf{k}}$, then

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 5x^2y^2 & x^2 - 2yz^2 & -2y^2z \end{vmatrix}$$

$$= \hat{\mathbf{i}} \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - 2yz^2 & -2y^2z \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ 5x^2y^2 & -2y^2z \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ 5x^2y^2 & x^2 - 2yz^2 \end{vmatrix}$$

(on expanding along the **first** row)

$$= \hat{\mathbf{i}}[\frac{\partial}{\partial y}(-2y^2z) - \frac{\partial}{\partial z}(x^2 - 2yz^2)] - \hat{\mathbf{j}}[\frac{\partial}{\partial x}(-2y^2z) - \frac{\partial}{\partial z}(5x^2y^2)]$$

$$+ \hat{\mathbf{k}}[\frac{\partial}{\partial x}(x^2 - 2yz^2) - \frac{\partial}{\partial y}(5x^2y^2)]$$

$$= \hat{\mathbf{i}}(-4yz + 4yz) - \hat{\mathbf{j}}(0-0) + \hat{\mathbf{k}}(2x - 10x^2y)$$

$$= (2x - 10x^2y)\hat{\mathbf{k}} \ .$$

Likewise you can show that

$$\boxed{\nabla \wedge \mathbf{r} = \mathbf{0}}$$

implying that **r** is irrotational.

Example 7

Consider the vector $\mathbf{F} = (3\text{ x} + \alpha\text{ y})\,\mathbf{i} + (\beta\text{ x} + 5\text{ y})\,\mathbf{j}$, where α and β are constants. Determine α and β such that $\nabla \wedge \mathbf{F} = \mathbf{0}$ and for those values of the constants, find Φ such that $\nabla\,\Phi = \mathbf{F}$.

$\nabla \wedge \mathbf{F} = \mathbf{0}$ when $\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ 3x + \alpha y & \beta x + 5y \end{vmatrix} \hat{\mathbf{k}} = (\beta - \alpha)\hat{\mathbf{k}} = \mathbf{0}$ or $\alpha = \beta$.

We must find Φ such that

$$\Phi_x = 3x + \alpha y \ , \tag{1}$$

and

$$\Phi_y = \alpha x + 5y \ . \tag{2}$$

Integrating Eq.(1) gives $\Phi = \dfrac{3x^2}{2} + \alpha\, y\, x + f(y)$ for any function $f(y)$. It follows that

$$\Phi_y = \alpha x + f\,'(y) \tag{3}$$

and comparing Eqs.(2) and (3), we see that we must have $f\,'(y) = 5y$.

It follows that $\Phi = \dfrac{3x^2}{2} + \alpha\, y\, x + 5\,y$.

Activity 6

Find curl **F**, where $\mathbf{F} = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{x^2 + y^2}$.

6.3 FORMULAE INVOLVING ∇

Assuming the partial derivatives of $\mathbf{A}$, $\mathbf{B}$, ϕ , ψ to exist, then

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$$

$$\nabla.(\mathbf{A} + \mathbf{B}) = \nabla.\mathbf{A} + \nabla.\mathbf{B}$$

$$\nabla \wedge (\mathbf{A} + \mathbf{B}) = \nabla \wedge \mathbf{A} + \nabla \wedge \mathbf{B}$$

$$\nabla.(\phi\mathbf{A}) = (\nabla\phi).\mathbf{A} + \phi(\nabla.\mathbf{A})$$

$$\nabla \wedge (\phi\mathbf{A}) = (\nabla\phi) \wedge \mathbf{A} + \phi(\nabla \wedge \mathbf{A})$$

$$\nabla \wedge (\nabla\phi) = \text{curl grad } \phi \equiv \mathbf{0}$$

$$\nabla.(\nabla \wedge \mathbf{A}) = \text{div curl } \mathbf{A} \equiv 0$$

$$\nabla.(\nabla\phi) = \nabla^2\phi \qquad \text{(scalar)Laplacian of } \phi$$

$$\nabla^2\mathbf{A} = \nabla(\nabla.\mathbf{A}) - \nabla \wedge (\nabla \wedge \mathbf{A}) \quad \text{vector Laplacian}$$

These results can all be verified by direct expansion in terms of components along $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

In some cases, however, the proof can be considerably shortened by making use of the fact that the operator ∇ may be substituted for a vector in any vector identity provided that it operates on the same factors in all terms.

6.3.1 THE JACOBI MATRIX

To each differentiable vector field

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}} \ ,$$

there corresponds a **Jacobi matrix**

$$J[\mathbf{F}(x,y,z)] = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix} .$$

The **trace** of this matrix, i.e. the sum of its leading elements, is

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \text{div}\ \mathbf{F}.$$

Its determinant is called the **jacobian** and denoted as $\frac{\partial(F_1, F_2, F_3)}{\partial(x,y,z)}$.

The components of

$$\text{curl}\ \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}$$

are the differences, in the orders indicated, of the elements of the matrix located symmetrically with respect to its leading diagonal.

6.3.2 THE LAPLACIAN ∇^2

In many situations, a physical vector field **F** is the gradient of a scalar ϕ, so that its divergence is related to ϕ by

$$\text{div } \mathbf{F} = \text{div (grad } \phi) = \nabla . \nabla\phi .$$

The combination of operators div grad $= \nabla . \nabla$ is a scalar operator known as the *Laplacian*, written as ∇^2 , and pronounced 'del squared'. It occurs in many partial differential equations. For any function $\phi(x, y, z)$, the (3-dimensional) Laplacian of ϕ is given by

$$\nabla^2\phi = \nabla.(\nabla\phi) = (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial x} + \hat{\mathbf{k}}\frac{\partial}{\partial x}).(\hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial x})$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \ .$$

[It's being assumed that ϕ has continuous second partials.]

In operator form, we have

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}} \ .$$

The partial differential equation $\nabla^2\phi = 0$ is called *Laplace's equation*, and any solution of Laplace's equation is called a *harmonic function* .

Example 8

If $u = 3xy\cos z - e^{-y}\sinh x$, then

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}(3xy\cos z - e^{-y}\sinh x) + \frac{\partial^2}{\partial y^2}(3xy\cos z - e^{-y}\sinh x) +$$

$$\frac{\partial^2}{\partial z^2}(3xy\cos z - e^{-y}\sinh x)$$

$$= -3xy\cos z - 2e^{-y}\sinh x \ .$$

Example 9

Show that $1/r$ is a harmonic function.

Solution

Recall $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. We must show that $\nabla^2(1/r) = 0$.

Method 1

$\nabla^2(1/r) = \nabla^2[(x^2 + y^2 + z^2)^{-1/2}]$

$= \dfrac{\partial^2}{\partial x^2}[(x^2 + y^2 + z^2)^{-1/2}] + \dfrac{\partial^2}{\partial y^2}[(x^2 + y^2 + z^2)^{-1/2}] + \dfrac{\partial^2}{\partial z^2}[(x^2 + y^2 + z^2)^{-1/2}]$

$= [3x^2(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}] + [3y^2(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}] +$

$[3z^2(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}]$

$= 0.$

Method 2

$\nabla^2(1/r) = \nabla.[\nabla(1/r)] = \nabla.[-\tfrac{1}{r^3}\mathbf{r}]$ -Refer to Example 3

$= [\nabla(-\tfrac{1}{r^3})].\mathbf{r} + \{-\tfrac{1}{r^3}\}\ \nabla.\mathbf{r}$

$= [3r^{-5}\mathbf{r}].\mathbf{r} - 3/r^3$

$= 3r^{-5}.r^2 - 3/r^3$

$= 0.$

In general,

$$\boxed{\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)}$$

dashes denoting differentiation w.r.t r.

6.3.3 DIRECTIONAL DERIVATIVES

Suppose $\phi(x, y, z)$ is defined at a point $P(x, y, z)$ on a given space curve C. Let $Q(x+\delta x, y+\delta y, z+\delta z)$ be a neighboring point on C, and let δs denote the length of arc of the curve between those points. Furthermore, let the direction of the arc be that of a unit vector $\hat{\mathbf{s}}$ originating at P (Figure 6.1).

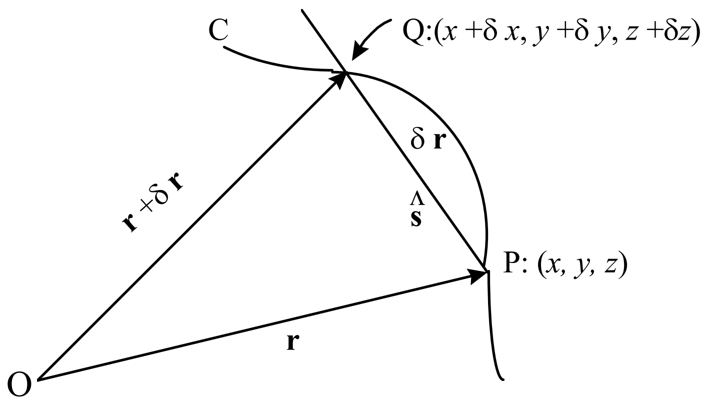

Figure 6.1

Then

$$\lim_{\delta s \to 0} \frac{\delta \phi}{\delta s} = \lim_{\delta s \to 0} \frac{\phi(x+\delta x, y+\delta y, z+\delta z) - \phi(x, y, z)}{\delta s}$$

if it exists, is called the *directional derivative* of ϕ at the point P along the curve C and is given by

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds} \ .$$

In vector form this may be written as,

$$\frac{d\phi}{ds} = \left(\frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{k}}\right) \boldsymbol{.} \left(\frac{dx}{ds}\hat{\mathbf{i}} + \frac{dy}{ds}\hat{\mathbf{j}} + \frac{dz}{ds}\hat{\mathbf{k}}\right) = \nabla\phi \boldsymbol{.} \frac{d\mathbf{r}}{ds}$$

Since δs is by definition just the length of $\delta\mathbf{r}$, it follows that $d\mathbf{r}/ds$ is a unit vector. In particular, if Q approaches P along the arc, then $\delta\mathbf{r} = \delta s\,\hat{\mathbf{s}}$, so that $d\mathbf{r} / ds = \hat{\mathbf{s}}$.
Hence

$$\frac{d\phi}{ds} = \nabla\phi \boldsymbol{.} \hat{\mathbf{s}}$$

which shows that the directional derivative of ϕ in the direction of $\hat{\mathbf{s}}$ is just the scalar projection of $\nabla\phi$ along the direction of $\hat{\mathbf{s}}$. In other words, $\nabla\phi$ has the property that its component along any direction equals the directional derivative of ϕ in that direction.

Note that the directional derivative of ϕ in the direction of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, or $\hat{\mathbf{k}}$ is the corresponding partial derivative of ϕ in that direction.

Since the maximum projection of a vector is the vector itself, it follows that $\nabla\phi$ extends in the direction of the greatest rate of change for its length.

In general, for a function $w = \phi(x, y, z)$, the directional derivative at P(x, y, z) in the direction determined by the angles α, β and γ is given by

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\cos\alpha + \frac{\partial\phi}{\partial y}\cos\beta + \frac{\partial\phi}{\partial z}\cos\gamma .$$

By the direction determined by the angles α, β and γ, it is meant the direction of the vector $(\cos\alpha)\hat{\mathbf{i}} + (\cos\beta)\hat{\mathbf{j}} + (\cos\gamma)\hat{\mathbf{k}}$

__Example 10__

Find the directional derivative of the function $\phi(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

__Solution__

First we find the gradient of ϕ at the point $(2, -1, 1)$:

$$\nabla\phi|_{(2,-1,1)} = \left[\mathbf{y}^2\ \hat{\mathbf{i}} + (2\mathbf{xy} + \mathbf{z}^3)\ \hat{\mathbf{j}} + 3\mathbf{yz}^2\ \hat{\mathbf{k}}\right]|_{(2,-1,1)}$$

$$= \hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

The projection of this vector in the direction of the given vector will be the required directional derivative. The unit vector ($\hat{\mathbf{s}}$) in the given direction is $(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})/3$.
Hence

$$\frac{d\phi}{ds}|_{(2,-1,1)} = (\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}).\ (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})/3\ = -11/3.$$

The negative sign, of course, indicates that ϕ decreases in the given direction.

Example 11

(i) Find the directional derivative of $\varphi = 3xy^3 - 2y^2z + x^2yz$ at $A(1,-3,2)$ in a direction towards $B(3,5,-4)$.

(ii) In what direction from A is the directional derivative a maximum?

(iii) What is the magnitude of the maximum directional derivative ?

Solution

(i) $\nabla\varphi|_{(1,-3,2)} = [(3y^3 + 2xyz)\hat{\mathbf{i}} + (9xy^2 - 4yz + x^2z)\hat{\mathbf{j}} + (x^2y - 2y^2)\hat{\mathbf{k}}]|_{(1,-3,2)}$

$$= -93\hat{\mathbf{i}} + 107\hat{\mathbf{j}} - 21\hat{\mathbf{k}}$$

The required direction is that of **AB**, i.e. $2\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$, so that the unit vector $\hat{\mathbf{s}}$ in this direction is $(2\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) / \sqrt{104}$, i.e. $(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) / \sqrt{26}$.

Hence,

$$\frac{d\varphi}{ds}|_A = (-93\hat{\mathbf{i}} + 107\hat{\mathbf{j}} - 21\hat{\mathbf{k}}).\ (\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) / \sqrt{26} \ = 398 / \sqrt{26}.$$

(ii) The directional derivative is a maximum in the direction $-93\hat{\mathbf{i}} + 107\hat{\mathbf{j}} - 21\hat{\mathbf{k}}$.

(iii) The magnitude of the max. directional derivative is $\left|-93\hat{\mathbf{i}} + 107\hat{\mathbf{j}} - 21\hat{\mathbf{k}}\right| = \sqrt{20539}$.

Activity 7

The electrical potential V at any point (x,y) is given by

$$V = \ln\sqrt{(x-3)^2 + (y-5)^2} \ .$$

(a) Find the rate of change at the point (5,7) in the direction towards the point (3,9).

(b) Show that V changes most rapidly along the set of radial lines through the point (3,5).

6.4 SUMMARY

In this unit, you have studied the gradient operator and its applications. You've also learnt how to find the divergence and curl of a vector field.

6.5 ANSWERS TO ACTIVITIES

Activity 1

(a) $2(x\hat{\mathbf{i}} + z\hat{\mathbf{j}} + y\hat{\mathbf{k}})$

(b) $e^{xyz}(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})$

(c) $x^a y^b z^c (\frac{a}{x}\hat{\mathbf{i}} + \frac{b}{y}\hat{\mathbf{j}} + \frac{c}{z}\hat{\mathbf{k}})$

Activity 3

(a) $(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) / \sqrt{3}$

(b) $(\hat{\mathbf{i}} - 2\hat{\mathbf{j}}) / \sqrt{5}$, or their negative.

Activity 4

(a) $-xy$
(b) $yz\sinh xy - z\cos yz - 2x^3 z$.

Activity 5

$\Phi_x = z\,\mathrm{e}^{x+3y}, \Phi_y = 3z\,\mathrm{e}^{x+3y}, \Phi_z = \mathrm{e}^{x+3y}$. Integrating these equations, give $\Phi = z\,\mathrm{e}^{x+3y} + \mathrm{h}_1(y,z), \Phi = z\,\mathrm{e}^{x+3y} + \mathrm{h}_2(x,y), \Phi = z\,\mathrm{e}^{x+3y} + \mathrm{h}_3(x,z)$ which can all be satisfied if we choose the $h_1 = h_2 = h_3 = 0$. Hence , $\Phi = z\,\mathrm{e}^{x+3y}$.

Activity 6

$$\nabla \wedge \mathbf{F} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ \dfrac{y}{x^2+y^2} & \dfrac{-x}{x^2+y^2} \end{vmatrix} \hat{\mathbf{k}} = (0)\hat{\mathbf{k}} = \mathbf{0}$$

Activity 7

(a) $$\frac{dV}{ds} = \frac{(x-3)}{(x-3)^2+(y-5)^2}\cos(\theta) + \frac{(y-5)}{(x-3)^2+(y-5)^2}\sin(\theta)$$

$\tan(\theta) = \frac{9-7}{3-5} = -1$, in the given direction, θ is in the second quadrant and

therefore, $\cos(\theta) = \frac{-1}{\sqrt{2}}$ and $\sin(\theta) = \frac{1}{\sqrt{2}}$.

Hence, at (5,7) in the indicated direction, $\frac{dV}{ds} = \frac{1}{4}\left(\frac{-1}{\sqrt{2}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right) = 0$

(b) At any point (x_1,y_1) in the direction θ, the directional derivative is given by

$$\frac{(x_1-3)}{(x_1-3)^2+(y_1-5)^2}\cos\theta + \frac{(y_1-5)}{(x_1-3)^2+(y_1-5)^2}\sin\theta.$$

Therefore, V changes most rapidly when

$$\frac{d}{d\theta}\left(\frac{(x_1-3)}{(x_1-3)^2+(y_1-5)^2}\cos\theta + \frac{(y_1-5)}{(x_1-3)^2+(y_1-5)^2}\sin\theta\right) = 0,$$

that is, when $\tan\theta = \frac{(y_1-5)}{(x_1-3)}$, hence the proof.