UNIT 2 POLAR COORDINATES

Unit Structure

2.0 OVERVIEW

In this Unit, we shall study the Polar Coordinate System, a method of describing the position of a point in a plane.

2.1 LEARNING OBJECTIVES

By the end of this Unit, you should be able to do the following:

1. Explain what the polar coordinate system is.
2. Determine the areas of regions bounded by polar curves.

2.2 INTRODUCTION

Rectangular Cartesian coordinates give the position of a point in a plane relative to two perpendicular lines in the plane. Another method of specifying the position of a point in a plane is by its polar coordinates.

Consider a point P in the xy-plane having Cartesian coordinates (x, y). Join the points O (origin) and P. Let r be the length OP and θ be the angle which OP makes with the positive direction of x-axis. Then (r, θ) are called the polar coordinates of the point P, and we write $P = (r, \theta)$ or $P(r, \theta)$

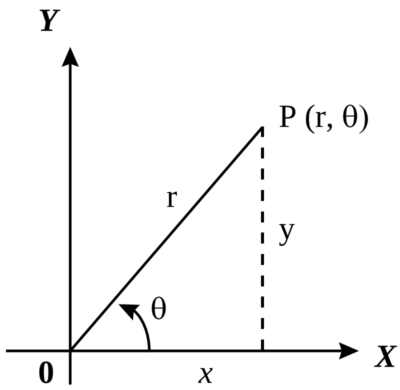


Figure 2.1

In particular, r is called the *radial distance* or directed distance of P from the origin O and θ is called the *polar angle*. Also, O is called the *pole*, the positive x-axis OX is called the *initial line* and $\vec{OP}$ is called the radius vector. As usual, positive angles correspond to anticlockwise rotations.

From Figure 2.1, we find that

$$r^2 = x^2 + y^2 \quad , \qquad \tan\theta = \frac{y}{x} \qquad \textbf{(2.1)}$$

$$x = r\cos\theta \quad , \qquad y = r\sin\theta \qquad \textbf{(2.2)}$$

Relations (2.1) enable us to find the polar coordinates (r,θ) when the Cartesian coordinates (x, y) are known. Conversely, relations (2.2) enable us to find the Cartesian coordinates when the polar coordinates are known. Thus relations (2.1) define the transformation from Cartesian coordinates to polar coordinates and relations (2.2) define the inverse transformation. We note that (2.1) do not determine r,θ uniquely, for $r = \pm\sqrt{x^2 + y^2}$, and θ can take an indefinite number of different values. To obtain a unique correspondence, we take $r = +\sqrt{x^2 + y^2}$ and determine θ as the angle which lies between $-\pi$ and $+\pi$ satisfying the two equations $\cos\theta = x/r$, $\sin\theta = y/r$.

Example 1

Points with polar coordinates $A(2,\frac{\pi}{4})$, $B(-2,\frac{\pi}{2})$, $C(1,\frac{\pi}{2})$ and $D(2,\frac{5\pi}{4})$ are plotted below.

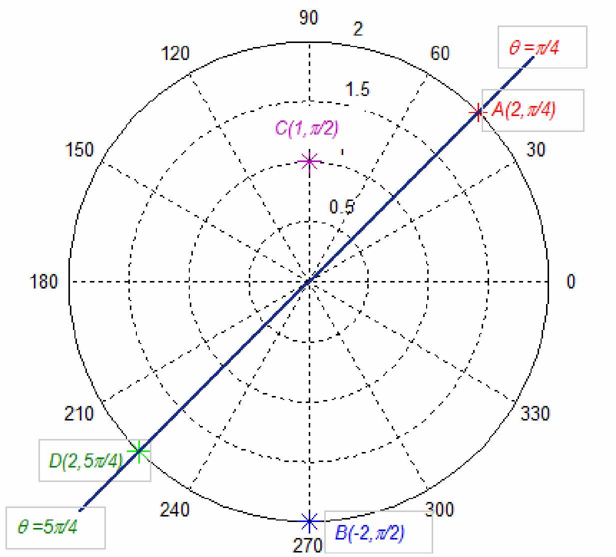


Figure 2.2

We see that the point $B(-2, \frac{\pi}{2})$ is equivalent to $B(2, \frac{3\pi}{2})$. This is due to the following generally result.

Result:

$$P(r, \theta) = P[(-1)^n r, \theta + n\pi)], \quad n \text{ any integer,}$$

and

$$P(-r, \theta) = P(r, \theta + \pi).$$

The result shows that if we have *a negative value* of *r*, then we plot instead the point $(r, \theta + \pi)$, i.e., rotate the point through 180° about the origin. So, $(a, \alpha), (a, 2\pi + \alpha)$ and $(-a, \pi + \alpha)$ all represent the same point.

<u>Example 2</u>

Find Cartesian coordinates of the points with polar coordinates

$$A\left(2, \frac{\pi}{6}\right),\ B\left(2, \frac{\pi}{2}\right) \quad \text{and} \quad C\left(-2\sqrt{2}, \frac{7\pi}{4}\right).$$

<u>Solution</u>

Using $x = r\cos\theta$ and $y = r\sin\theta$, we have,

$$A(x, y) = \left(2\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}\right) = (\sqrt{3}, 1).$$

$$B(x, y) = \left(2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}\right) = (0, 2).$$

$$C(x, y) = \left(-2\sqrt{2}\cos\frac{7\pi}{4}, -2\sqrt{2}\sin\frac{7\pi}{4}\right) = (-2, 2).$$

2.3 POLAR CURVES

A curve whose equation is given in the form $r = f(\theta)$ or $F(r,\theta) = 0$ is called a *polar curve* and can be traced by using the following considerations:

	Observation about $r = f(\theta)$	Properties exhibited by $r = f(\theta)$	Actions to be taken
1	$f(\theta) = f(-\theta)$ E.g. $r = a\cos\theta$	$r = f(\theta)$ is symmetrical about the initial line $(\theta = 0)$	**Step 1:** Plot values of r for $\theta \in [0, \pi]$ **Step 2:** Reflect in the initial line
2	$F(r,\theta) = F(-r,-\theta)$ or $f(\theta) = f(\pi - \theta)$ E.g. $r = a\sin\theta$	$r = f(\theta)$ is symmetrical about the lines $\theta = \pm\pi/2$ (y-axis)	**Step 1:** Plot values of r for $\theta \in [-\pi/2, \pi/2]$ **Step 2:** Reflect in the lines $\theta = \pm\pi/2$
3	$F(r,\theta) = F(-r,\theta)$ E.g. $r^2 = f(\theta)$	$r = f(\theta)$ symmetrical about the pole, i.e., rotation of the curve through 180° about the pole	**Step 1:** Draw the polar curve for $\theta \in [0, \pi]$ **Step 2:** Complete the curve so that we have rotational symmetry of order 2
4	Suppose $\theta = \alpha$ is a solution to $r = f(\theta)$. i.e. $f(\alpha) = 0$	$\theta = \alpha$ is a tangent to the curve $r = f(\theta)$ at the pole **Note:** No solution implies not tangent at pole. E.g. $r = 3 + \cos 2\theta$	**Step 1:** Draw all tangents at the pole **Step 2:** Draw part of the curve where we have the tangents first
5	Suppose $ub = \max(\lvert f(\theta)\rvert)$, ub is the upper bound if it exists	$r = f(\theta)$ lies inside circle $r = ub$	**Step :** Draw a dotted circle as upper bound in order to enclose $r = f(\theta)$
6	Suppose $lb = \min(\lvert f(\theta)\rvert) > 0$, lb is the lower bound	$r = f(\theta)$ lies outside circle $r = lb$	**Step :** Draw a dotted circle as lower bound in order to exclude $r = f(\theta)$
7	If r^2 is negative in a certain interval for θ	$r = f(\theta)$ does **not** lie in that interval	**Step :** Do not draw curve in that interval

Note:

a) By solving the equation for *r* in terms of θ, or for θ in terms of *r*, it is possible to study the nature of variation of *r* with respect to θ and vice versa.

b) If required, a set of corresponding values of *r* and θ may be tabulated. This set gives the specific points through which the curve passes.

Example 3

Sketch the curve $r = a(1+\cos\theta), \quad 0 \le \theta \le 2\pi, a > 0$.

Solution

(v) When θ is changed to $-\theta$, the equation remains unaltered. Therefore, the curve is symmetrical about the initial line $\theta = 0$.

ii) Since $|\cos\theta| \le 1$, we have $r \le 2a$. Hence the entire curve lies within the circle centred at the pole with 2a as the radius.

iii) We have $r = 0$ for $\theta = \pi$. Therefore, the curve passes through the pole at which the tangent is the line $\theta = \pi$.

(v) For $\theta = \pi/2$, we have $r = a$. Therefore, the curve cuts the line $\theta = \pi/2$ at the point with polar coordinates $(a, \pi/2)$.

(vi) As θ increases from 0 to $\pi/2$, r decreases from 2a to a. As θ increases from $\pi/2$ to π, r further decreases from a to 0.

(vii) A set of corresponding values of θ and r is shown below:

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	$2a$	$a\left(1+\frac{1}{\sqrt{2}}\right)$	a	$a\left(1-\frac{1}{\sqrt{2}}\right)$	0

The above observations reveal that the curve is as shown in Fig. 2.3.

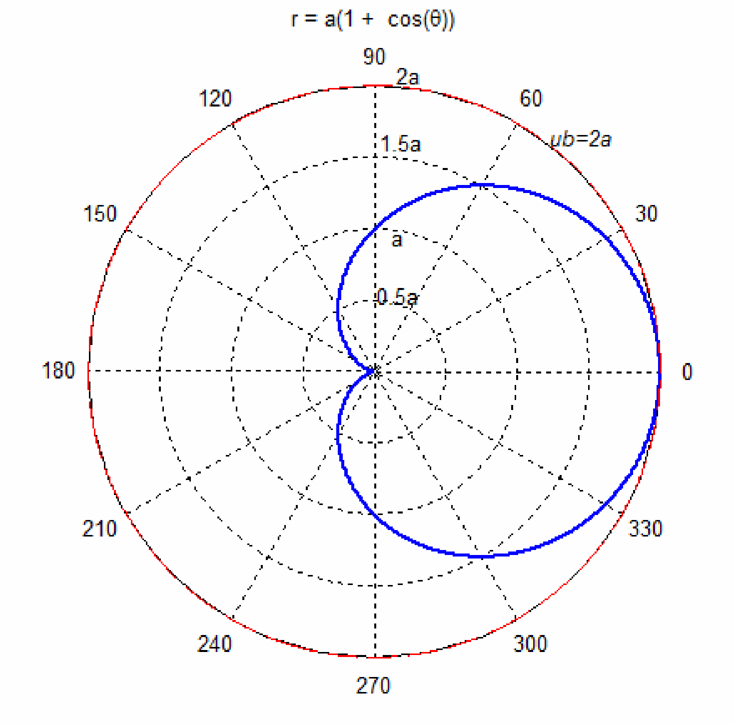


Figure 2.3

This curve is known as the **cardioid,** as it is heart-shaped.

Example 4

Sketch the curve $r^2 = a^2 \cos 2\theta, \quad 0 \le \theta \le 2\pi, a > 0$.

Solution

(i) The equation remains unaltered when θ is changed to $-\theta$. Hence the curve is symmetrical about the initial line, $\theta = 0$.

(ii) The equation remains unaltered when r is changed to $-r$. Hence the curve is symmetrical about the pole.

(iii) $r = 0$ when $\theta = \pi/4$ and $3\pi/4$. Hence the curve passes through the pole, and there are two distinct tangents, $\theta = \pi/4$ and $\theta = 3\pi/4$, at the pole.

(iv) For $\theta = 0$ and $\theta = \pi$, we have $r = a$. Therefore, the curve cuts the lines $\theta = 0$ and $\theta = \pi$ at the points $(a, 0)$ and (a, π) respectively.

(v) Since $r^2 \ge 0$, we must have $\cos 2\theta \ge 0$. Therefore, r is real for $0 \le \theta \le \pi/4$ and $3\pi/4 \le \theta \le \pi$.

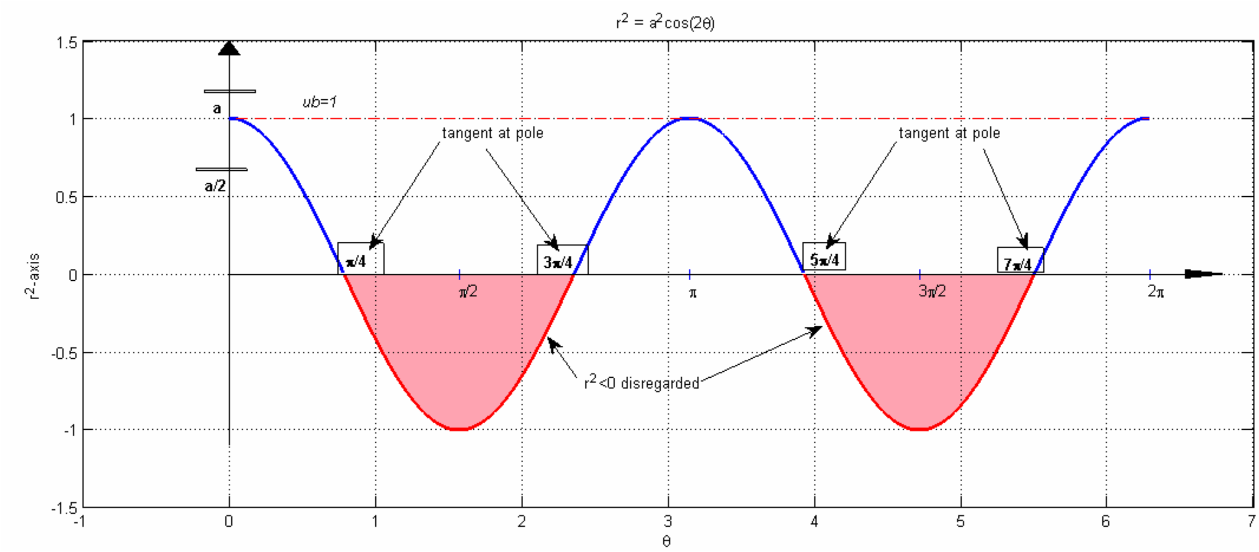


Figure 2.4

Thus, by virtue of the symmetry about the initial line, the entire curve lies between the lines $\theta = -\pi/4$, and $\theta = \pi/4$, and between the lines $\theta = 3\pi/4$ and $\theta = 5\pi/4$. See Figure 2.5.

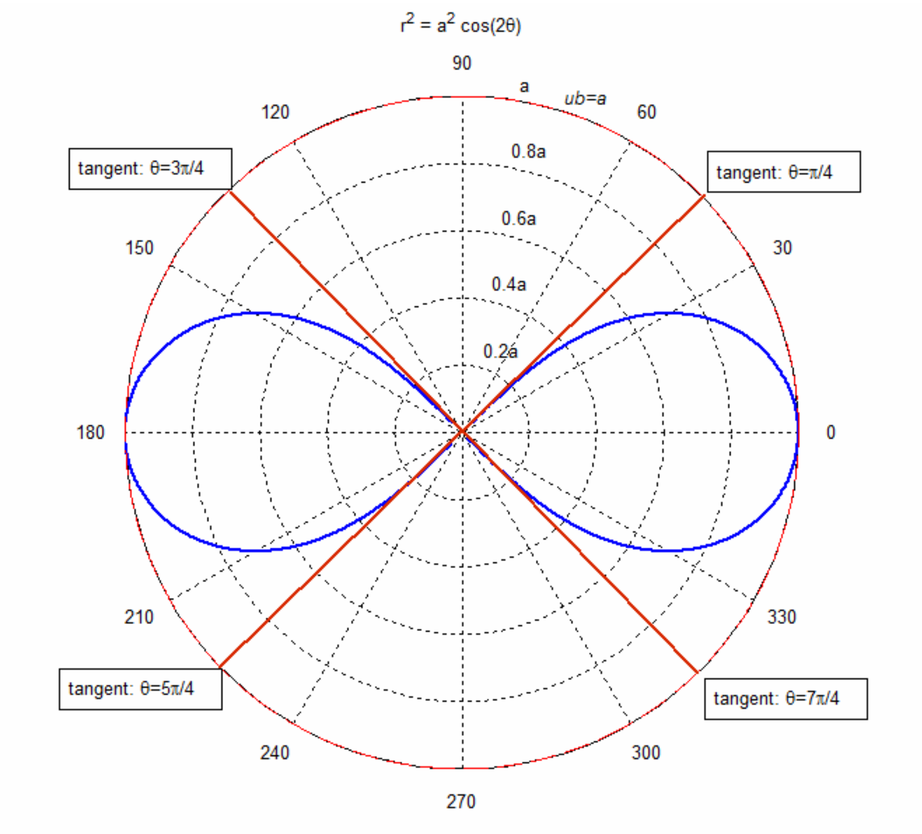


Figure 2.5

Let us consider sketching some polar curves by just observing information like tangents at pole, upper bounds (ub) and lower bounds (lb) (if they exist) using the standard graphs of r v/s θ.

Example 5

Sketch the polar curve (a) $r = \sin\theta, \quad 0 \le \theta \le 2\pi$,

(b) $r = \cos\theta, \quad 0 \le \theta \le 2\pi$.

Solution: **See Figure 2.6**

Example 6

Sketch the polar curve (a) $r = 1 + \sin\theta, \quad 0 \le \theta \le 2\pi$,

(b) $r = 1 + \cos\theta, \quad 0 \le \theta \le 2\pi$.

Solution: **See Figure 2.7**

Example 7

Sketch the polar curve (a) $r = 2 + \sin\theta, \quad 0 \le \theta \le 2\pi$,

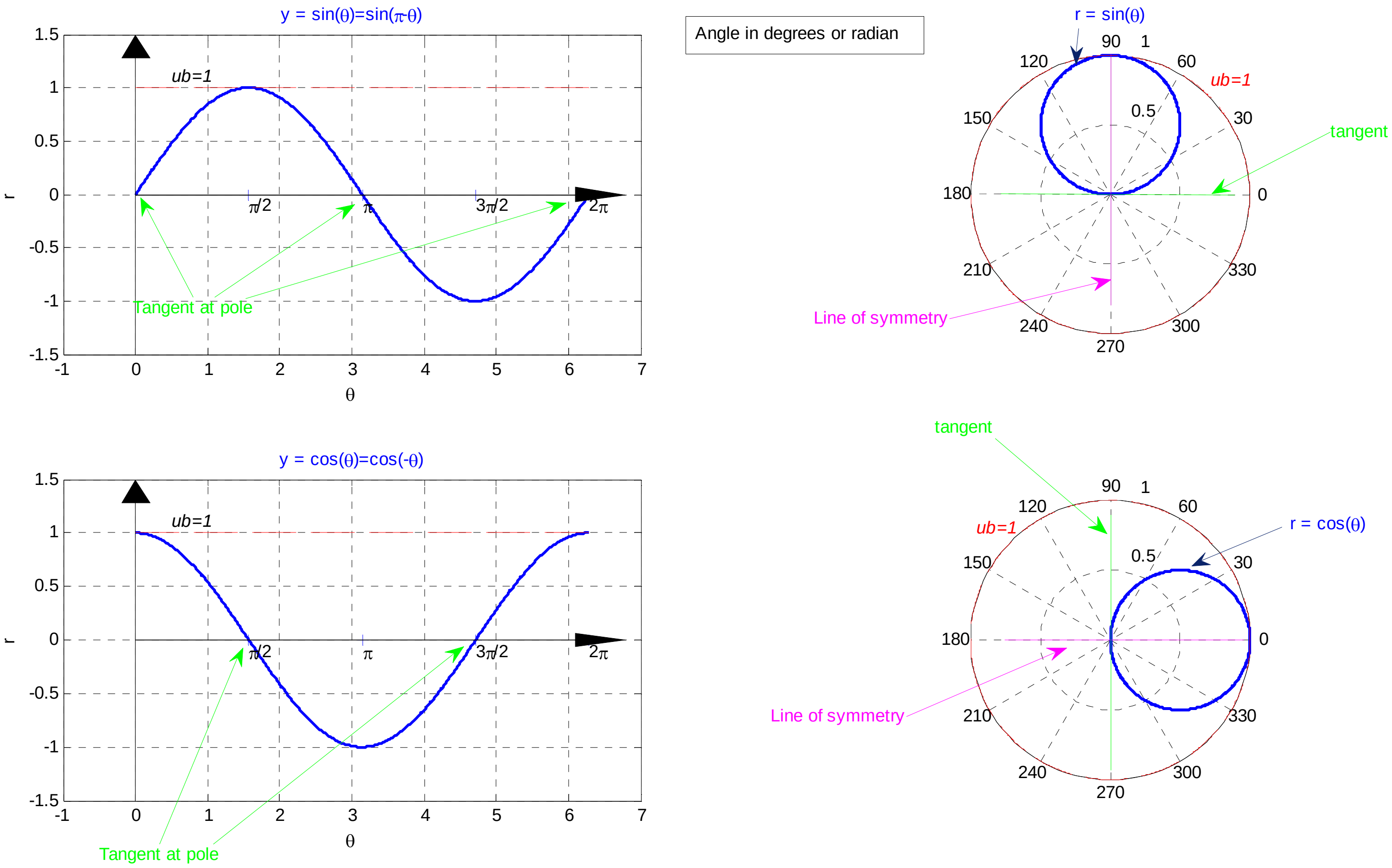
(b) $r = 2 + \cos\theta, \quad 0 \le \theta \le 2\pi$.

Solution: **See Figure 2.8**

Example 8

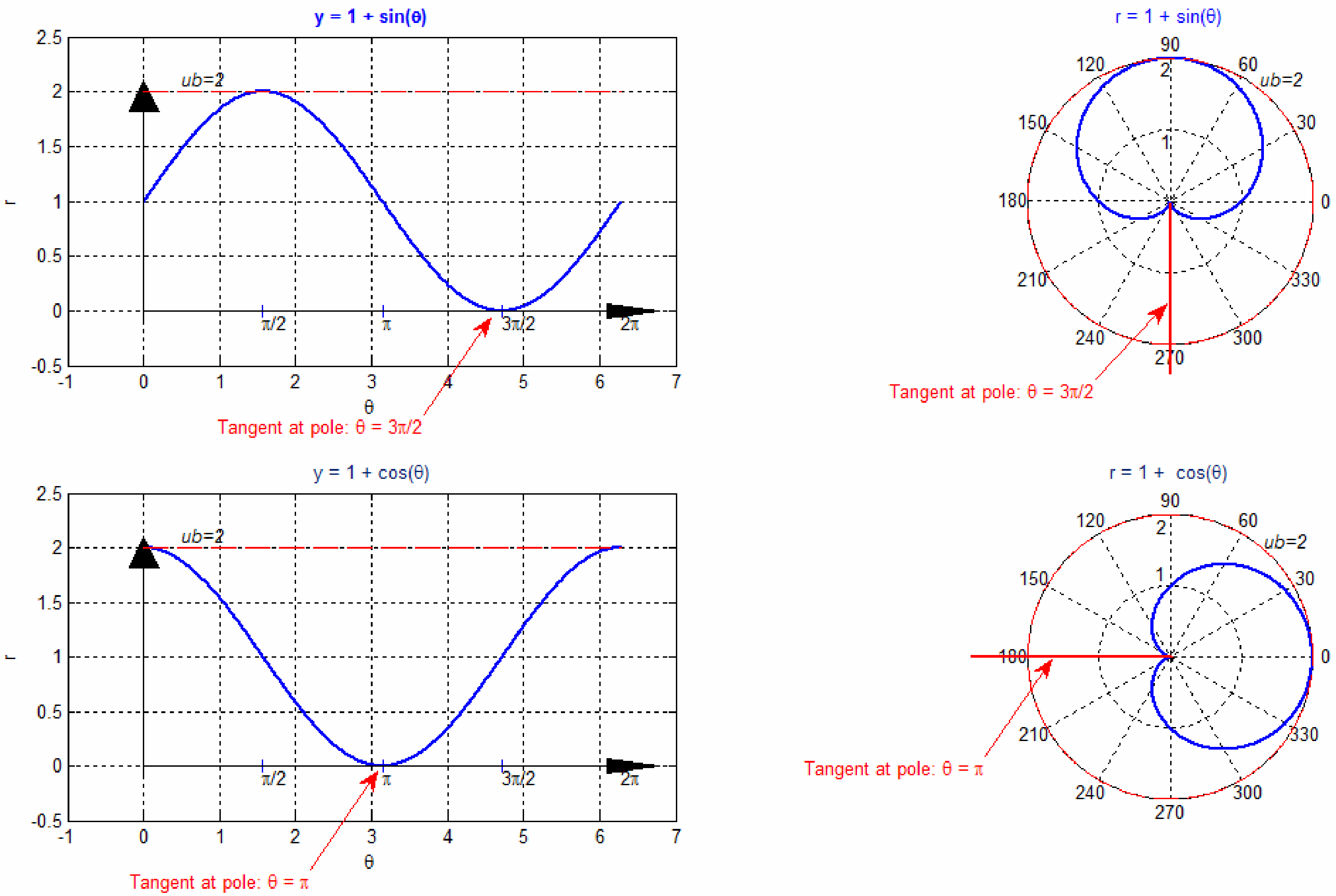
Sketch the polar curve $r = 1 + 2\sin\theta, \quad 0 \le \theta \le 2\pi$.

Solution: **See Figure 2.9**



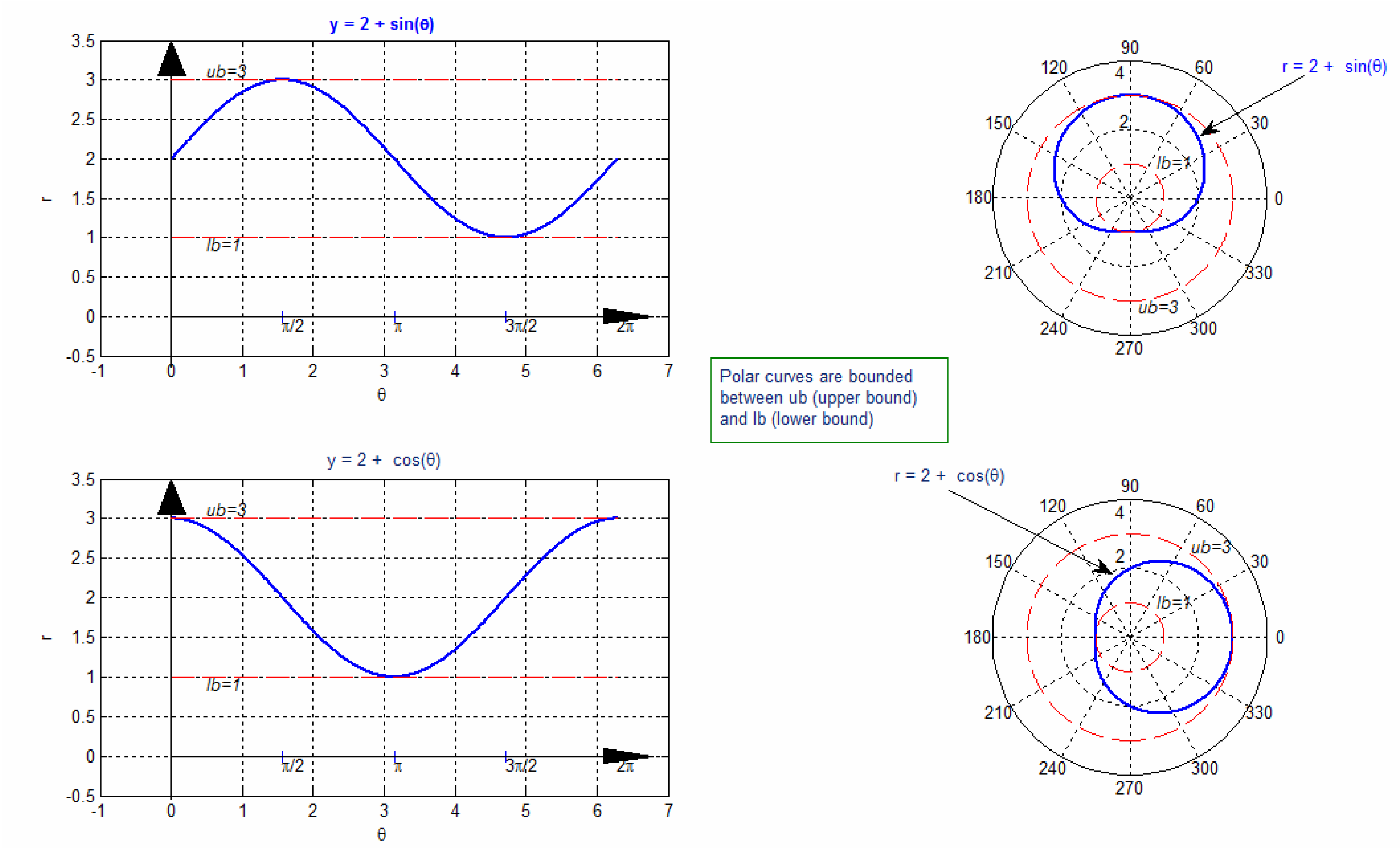
y = sin(θ)=sin(π-θ)
ub=1
Tangent at pole
π/2
π
3π/2
2π
r
θ
Angle in degrees or radian
r = sin(θ)
ub=1
tangent
Line of symmetry
y = cos(θ)=cos(-θ)
ub=1
Tangent at pole
tangent
ub=1
r = cos(θ)
Line of symmetry

Figure 2.6



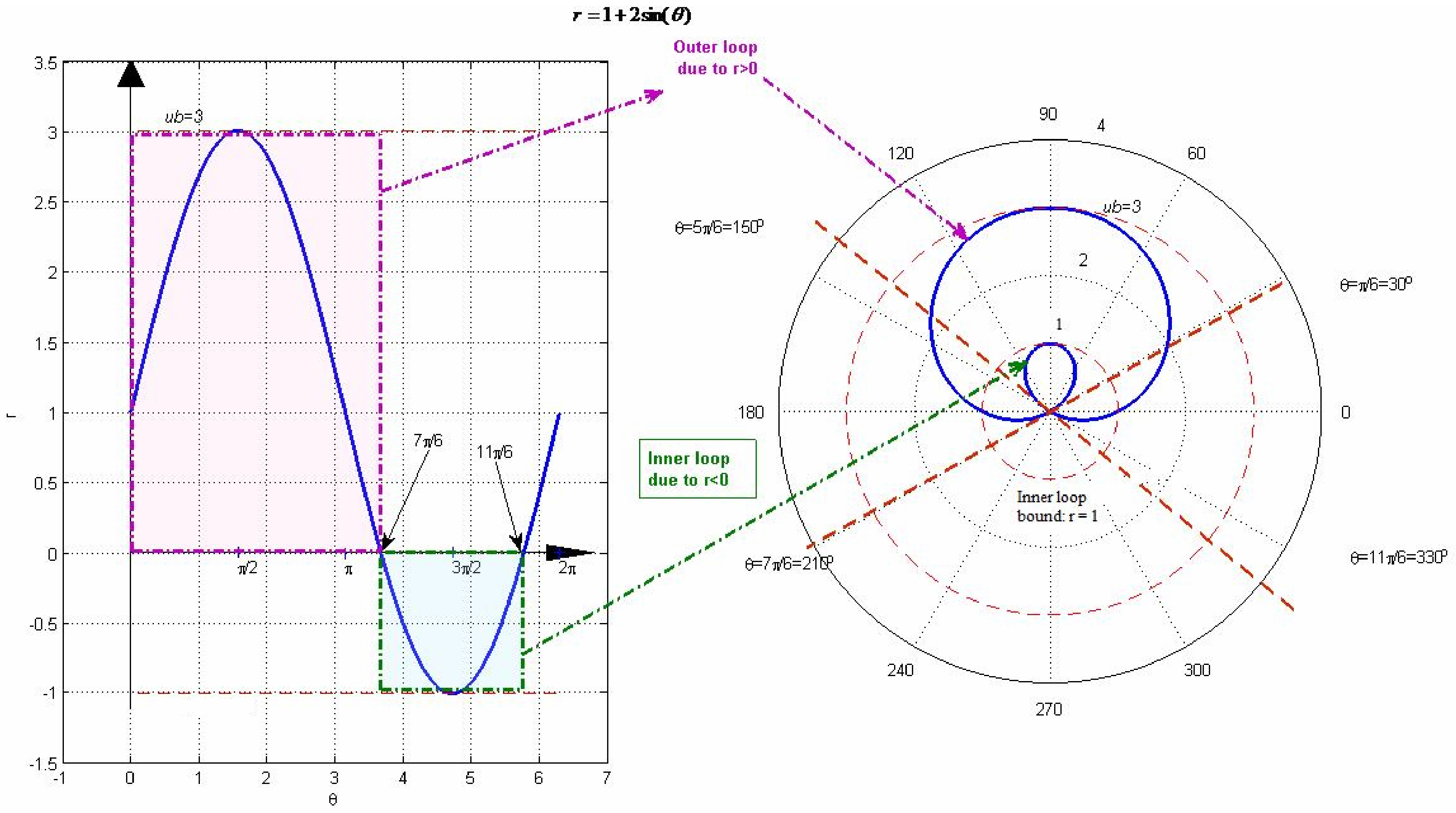
y = 1 + sin(θ)
ub=2
Tangent at pole: θ = 3π/2
r = 1 + sin(θ)
ub=2
Tangent at pole: θ = 3π/2
y = 1 + cos(θ)
ub=2
Tangent at pole: θ = π
r = 1 + cos(θ)
ub=2
Tangent at pole: θ = π

Figure 2.7



y = 2 + sin(θ)
ub=3
lb=1
π/2
π
3π/2
2π
r
θ
r = 2 + sin(θ)
90
60
30
0
330
300
270
240
210
180
150
120
Polar curves are bounded between ub (upper bound) and lb (lower bound)
y = 2 + cos(θ)
r = 2 + cos(θ)

Figure 2.8



$r = 1 + 2\sin(\theta)$
Outer loop due to r>0
Inner loop due to r<0
ub=3
7π/6
11π/6
π/2
π
3π/2
2π
r
θ
θ=5π/6=150°
θ=π/6=30°
θ=7π/6=210°
θ=11π/6=330°
Inner loop bound: r = 1
90
120
60
180
0
240
300
270

Figure 2.9

Note

In the above examples, θ only occurs in a trig function, and therefore it is immaterial whether θ is measured in degrees or radians. However, for curves such as $r = a\theta$, $r = ae^{k\theta}$, it is recommended you work in radians.

Example 9:

Sketch the polar curve $r = 2e^{\frac{\theta}{\pi}}$ for $0 \le \theta \le 2\pi$.

Solution: We find discrete polar points and then interpolate using a smooth curve.

θ	**0**	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$r = 2e^{\frac{\theta}{\pi}}$	**2**	**2.5681**	**3.2974**	**4.2340**	**5.4366**	**6.9807**	**8.9634**	**11.5092**	**14.7781**

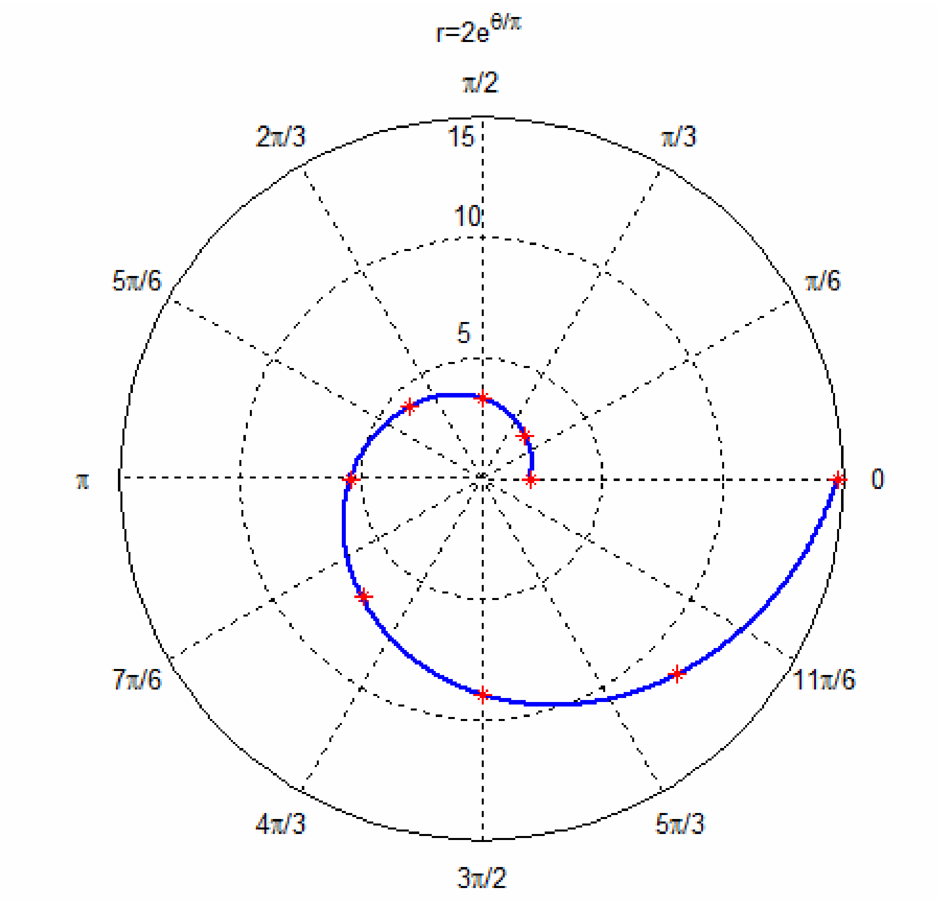


Figure 2.10

Activity 1

1. Express the following equations in polar coordinates:

(i) $x^2 + y^2 = 4x$;

(ii) $x^2 y^2 = (x^2 + y^2)^2$;

(iii) $4x^2 + y^2 = 1$;

(iv) $x^2 + y^2 = 3axy$.

2. Express the following equations in Cartesian coordinates:

(i) $r\cos\theta = a$;

(ii) $r\sin(\theta + \alpha) = a$;

(iii) $r^2 \cos 2\theta = a^2$;

(iv) $r^2 \sin 2\theta = c^2$.

3. Sketch the following curves:

(i) $r = 4\cos\theta$;

(ii) $r = 4\sec\theta$;

(iii) $r = 3\sin\theta$;

(iv) $r = 3\,\mathrm{cosec}\,\theta$;

(v) $r^2 = a^2 \sin 2\theta$;

(vi) $r = 1 + 2\cos\theta$;

(vii) $r^2 \sin 2\theta = c^2$.

2.4 AREA IN POLAR COORDINATES

Consider the region in the plane bounded by the polar curves

$$r = f(\theta)\,,\ \theta = a \quad \text{and} \quad \theta = b\,.$$

We assume that r is a continuous function of θ, $a \le \theta \le b$. To find the area of the region, we divide it into sectors by slicing it along the radial lines $\theta = \theta_i$, $i = 1, \cdots, n$ from the pole to the curve $r = f(\theta)$ as shown in Fig. 2.5.

The area of each sector is approximately $\frac{1}{2} r^2\,\delta\theta$, $\delta\theta$ denoting the sectorial angle and r the radius of the circle of which the sector under consideration is a part. The area of the region is the limit of the sum of the areas of the sectors as n tends to infinity, and is given by the formula

$$\text{Area } = \int_a^b \tfrac{1}{2} r^2\, d\theta = \int_a^b \tfrac{1}{2}\big[f(\theta)\big]^2\, d\theta\,,$$

its derivation being omitted.

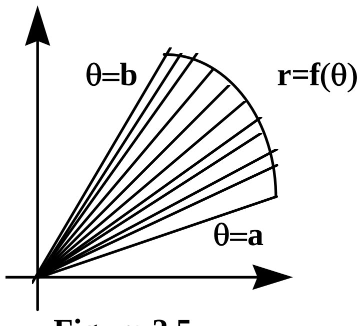


Figure 2.5

The radial lines $\theta = \theta_i$, $i = 1, \cdots, n$ must divide the region into sectors. For any fixed θ between *a* and *b*, the radial lines inclined at angle θ must intersect the region from the pole to the point $(f(\theta), \theta)$ on the boundary curve.

Example 5

Find the area of the circle $r = a$.

The formula gives: Area $= \frac{1}{2}\int_0^{2\pi} a^2 \; d\theta = \pi a^2$.

Example 6

Find the area of the region enclosed by the curve $r = 2\sin\theta$.

Solution

The region and a typical area sector are sketched in Fig. 2.6. The sector runs from the pole to the curve $r = 2\sin\theta$. These sectors will cover the region once as θ ranges from

$\theta = 0$ to $\theta = \pi$.

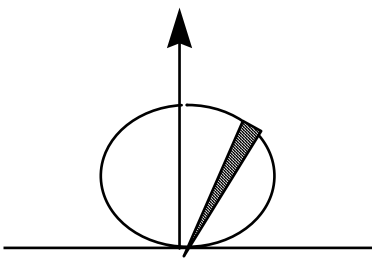


Figure 2.6

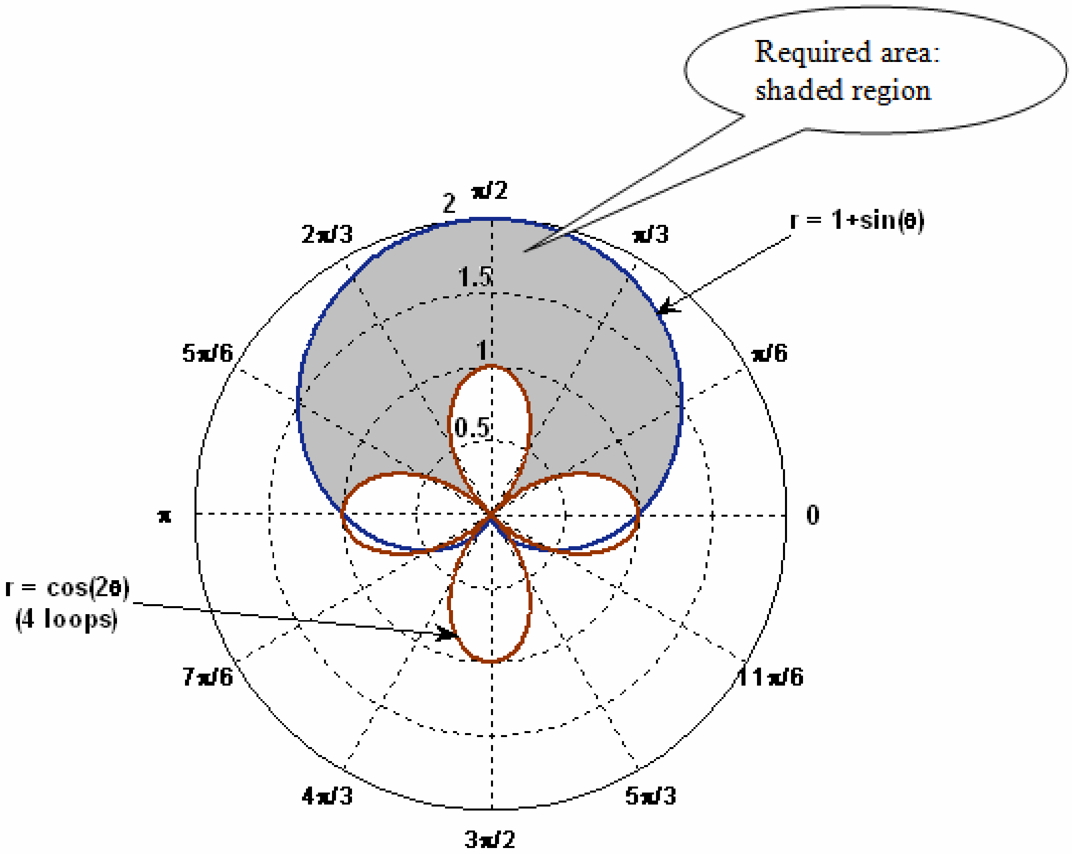
Area $= \frac{1}{2}\int_0^{\pi} r^2 \; d\theta = 2\int_0^{\pi} \sin^2\theta \; d\theta = \pi$.

Example 7

Sketch the polar curves $C_1: \; r = 1 + \sin\theta$ and $C_2: \; r = \cos 2\theta$ for $-\pi \le \theta \le \pi$ on the same diagram.

Show that the area of the region outside C_2 but inside C_1 for $0 \le \theta \le \pi$ is given by $2 + \dfrac{\pi}{2}$.

Solution



Area enclosed by the polar curve

$$= \frac{1}{2}\int_0^{\pi} (1+\sin\theta)^2 - \cos^2 2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \left(1+2\sin\theta+\sin^2\theta\right) - \frac{1}{2}[1+\cos 2\theta] \, d\theta \; = \frac{1}{2}\int_0^{\pi} 1+2\sin\theta - \frac{1}{2}\cos(2\theta) - \frac{1}{2}\cos 4\theta \, d\theta$$

$$= \frac{1}{2}\left[\theta - 2\cos\theta - \frac{1}{4}\sin(2\theta) - \frac{1}{8}\sin 4\theta\right]_0^{\pi} \; = \frac{1}{2}\left[\pi + 2 - (-2)\right] = \frac{1}{2}\pi + 2 \text{ square units}$$

Activity 2

1. Find the area bounded by the curve $r = a\cos 3\theta$.

2. Find the area bounded by the curve $r = 2 + \cos\theta$.

3. Find the area inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

2.5 SUMMARY

In this Unit you have studied a new method of describing the position of a point in a plane, namely, the polar coordinate system. You have learnt how to convert from Cartesian to polars and vice versa, and also how to calculate areas in this new system of coordinates.

2.6 ANSWERS TO ACTIVITIES

Activity 1

1. (i) $r = 4\cos\theta$;

   (ii) $\cos^2\theta\sin^2\theta = 1$;

   (iii) $4r^2\cos^2\theta + \sin^2\theta = 1$;

   (iv) $3a\cos\theta\sin\theta = 1$.

2. (i) $x = a$;

   (ii) $x\sin\alpha + y\cos\alpha = a$;

   (iii) $x^2 - y^2 = a^2$;

   (iv) $2xy = c^2$.

Activity 2

1. $\pi a^2/4$;

2. $9\pi/2$;

3. $2 + \pi/4$.