UNIT 7 PARTIAL DIFFERENTIATION

Unit Structure

7.0 OVERVIEW

Up to now we have been dealing with the relative change between two related quantities. Thus the area, A, of a circle depends on its radius r and on no other *variable*. Hence, A and r are the two related quantities, and we express this relation symbolically as $A = A(r)$, and say A is a function of r. A is then called the *dependent* variable and r is the *independent* variable. Furthermore, since there is only one independent variable involved, we say A is a function of one variable.

In many cases, more than two quantities are interrelated, e.g., the volume V of a right circular cylinder depends on both its radius r and its height h ($V = \pi r^2 h$). We write this dependence or functional relationship as $V = V(r, h)$. Here, V is the dependent variable and r and h are the independent variables. We now have a function of more than one variable.

In this Unit, we shall study how to find the derivatives of functions of more than one real variable, a procedure known as ***partial differentiation***.

7.1 LEARNING OBJECTIVES

By the end of this Unit, you should be able to do the following:

1. Find all the partial derivatives of a function of several variables.
2. Use the chain rule.
3. Find the differential and total derivative of a function.

7.2 PARTIAL DIFFERENTIATION

Suppose $f(x, y)$ is a real single-valued function of two independent variables *x* and *y*. Then the partial derivative of $f(x, y)$ with respect to *x* is defined as

$$\left(\frac{\partial f}{\partial x}\right)_y = \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x, y) - f(x, y)}{\delta x}\right\}.$$

$\frac{\partial f}{\partial x}$ is read as "curly d f by curly d x" or "partial d f by d x" .

Similarly, the partial derivative of $f(x, y)$ with respect to *y* is defined as

$$\left(\frac{\partial f}{\partial y}\right)_x = \lim_{\delta y \to 0}\left\{\frac{f(x, y+\delta y) - f(x, y)}{\delta y}\right\}.$$

In other words, the partial derivative of $f(x, y)$ w.r.t. *x* may be thought of as the ordinary derivative of $f(x, y)$ w.r.t. *x* obtained by treating *y* as a constant. Similarly, the partial derivative of $f(x, y)$ w.r.t. *y* may be found by treating *x* as a constant and evaluating the ordinary derivative of $f(x, y)$ w.r.t. *y*. The variable which is to be held constant in the differentiation is denoted by a subscript, as shown above.

Alternative notations, however, exist for partial derivatives and one of the more useful and compact of these is to denote $\left(\frac{\partial f}{\partial x}\right)_y$ by f_x and $\left(\frac{\partial f}{\partial y}\right)_x$ by f_y. The subscripts appearing on the f now denote the variables with respect to which $f(x, y)$ is to be differentiated.

The following examples will make things clear.

Example 1

Suppose $f = 3x^2 y - 2x^3 y^3 + x^5 y^4$.

Then, keeping y constant (i.e., pretend it's a number) we find

$$f_x \equiv \left(\frac{\partial f}{\partial x}\right)_y = 6xy - 6x^2 y^3 + 5x^4 y^4.$$

Similarly, keeping x constant,

$$f_y \equiv 3x^2 - 6x^3 y^2 + 4x^5 y^3.$$

Example 2

Suppose now $z = 3\cos(x + y) - \sinh x$.

We obtain $z_x = -3\sin(x + y) - \cosh x$ and $z_y = -3\sin(x + y)$.

Example 3

If $f(x, y) = \tan^{-1}\left(\dfrac{x}{y^2}\right)$, then $f_x \equiv \left(\dfrac{\partial f}{\partial x}\right)_y = \dfrac{1}{1+\left(x/y^2\right)^2} \times \dfrac{\partial}{\partial x}\left(x\, y^{-2}\right)$

$$= \frac{1}{1+\left(x^2/y^4\right)} \times \frac{1}{y^2}$$

$$= \frac{y^2}{y^4\left(1+x^2/y^4\right)}$$

On simplifying, we are out with $f_x = \dfrac{y^2}{x^2 + y^4}$.

$$f_y = \begin{array}{l} \left(\dfrac{\partial f}{\partial x}\right)_y = \dfrac{1}{1+\left(x/y^2\right)^2} \times \dfrac{\partial}{\partial y}\left(x\, y^{-2}\right) \\ = \dfrac{1}{1+x^2/y^4} \times \left(-\dfrac{2x}{y^3}\right) \end{array}.$$

On further simplification, we get $f_y = -\dfrac{2xy}{x^2 + y^4}$.

To obtain the partial derivatives of a function of n independent variables, any $n-1$ of these variables must be held constant and the differentiation carried out w.r.t. the remaining variable. There are therefore n first partial derivatives of such a function.

Example 4

Let's consider a function of 3 variables.

Suppose $F(x, y, z) = e^{-3z}\sin x + x^3y^4z^5$.

Then

$$F_x \equiv \left(\frac{\partial F}{\partial x}\right)_{y,z} = e^{-3z}\cos x + 3x^2y^4z^5;$$

$$F_y \equiv \left(\frac{\partial F}{\partial y}\right)_{x,z} = 4x^3y^3z^5;$$

$$F_z \equiv \left(\frac{\partial F}{\partial z}\right)_{x,y} = -3e^{-3z}\sin x + 5x^3y^4z^4.$$

Activity 1

1. If $z = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$. Find z_x and z_y.

2. Given that $u = \ln\dfrac{x^2 + y^2}{x + y}$. Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

7.2.1 Implicit differentiation

Example 5

Find z_x and z_y given that $x^3z + z\sin x - z^3y = \cos z$..

Here it's easier to differentiate ***implicitly***, rather than make *z* the subject of formula.

Differentiating implicitly w.r.t. *x*, treating *y* as a constant, we have

$3x^2z + x^3 z_x + z_x\sin x + z\cos x - \left(3z^2 z_x y\right) = \left(-\sin z\right)z_x$, so that

$$z_x = \frac{-\left(z\cos x + 3x^2z\right)}{x^3 + \sin x - 3z^2y + \sin z}.$$

Note: Both *x* and *y* are independent variables; so do not differentiate *y* w.r.t. *x*.

Now, differentiating implicitly w.r.t. *y*, treating *x* as a constant, we have $x^3z_y + z_y\sin x - \left(3z^2y\, z_y + z^3\right) = -\left(\sin z\right)z_y$.

Hence,

$$z_y = \frac{z^3}{x^3 + \sin x - 3z^2y + \sin z}.$$

Example 6

Find z_x and z_y if $\cos(xyz) + x^2 y + e^{yz} = 0.$.

Clearly, it's impossible to make *z* subject. We must differentiate *implicitly*.

Treating *y* as constant and differentiating implicitly w.r.t. *x* yields

$$-\sin(xyz)\frac{\partial}{\partial x}(xyz) + 2xy + e^{yz}\ y\, z_x = 0.$$

That is,

$$-\sin(xyz)(yz + xy\, z_x) + 2xy + ye^{yz}\ z_x = 0.$$

On simplifying, we are out with

$$\left(y\,e^{yz} - xy\sin(xyz)\right)\ z_x = \sin(xyz)(yz) - 2xy$$

Hence,

$$z_x = \frac{(yz)\ \sin(xyz) - 2xy}{ye^{yz} - xy\sin(xyz)}.$$

Treating *x* as constant and differentiating implicitly w.r.t. *y*, gives

$$-\sin(xyz)\frac{\partial}{\partial y}(xyz) + x^2 + e^{yz}\frac{\partial}{\partial y}(yz) = 0\,.$$

This simplifies to

$$-\sin(xyz)\left(xz + xy\, z_y\right) + x^2 + e^{yz}\left(z + y\, z_y\right) = 0\,.$$

And hence,

$$z_y = \frac{xz\sin(xyz) - x^2 - ze^{yz}}{ye^{yz}\ \text{-}\ xy\sin(xyz)}.$$

Activity 2

1) For each of the following functions, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(i) $z^2 - e^{z\sin(x)} = x + y^2.$

(ii) $x^4 + y^4 + z^2 + x^2yz = 10.$

7.3 HIGHER-ORDER PARTIAL DERIVATIVES

Provided the first partial derivatives of a function are differentiable we may differentiate them partially to obtain the second partial derivatives. The four second partial derivatives of $f(x, y)$ are therefore

$$f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)_y ,$$

$$f_{yy} \equiv \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)_x ,$$

$$f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_x$$

$$f_{yx} \equiv \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_y .$$

Higher partial derivatives than the second may be obtained in a similar way. In general

$$\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$$

denotes the result of differentiating a function $f(x, y)$ n times w.r.t. y treating x as a constant, and then differentiating this result m times w.r.t. x treating y as a constant.

We consider $f(x, y) = \ln\left(x^2 + y^2\right)$.

$$\left(\frac{\partial f}{\partial x}\right)_y = \frac{2x}{x^2 + y^2}, \quad \left(\frac{\partial f}{\partial y}\right)_x = \frac{2y}{x^2 + y^2} .$$

Hence, differentiating these first derivatives partially, we obtain

$$f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2 + y^2}\right) = \frac{2\left(x^2 + y^2\right) - 4x^2}{(x^2 + y^2)^2} = \frac{2\left(y^2 - x^2\right)}{(x^2 + y^2)^2},$$

$$f_{yy} \equiv \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{2y}{x^2 + y^2}\right) = \frac{2\left(x^2 + y^2\right) - 4y^2}{(x^2 + y^2)^2} = \frac{2\left(x^2 - y^2\right)}{(x^2 + y^2)^2},$$

$$f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{x^2 + y^2}\right) = -\frac{4xy}{(x^2 + y^2)^2},$$

$$f_{yx} \equiv \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2}\right) = -\frac{4xy}{(x^2 + y^2)^2}.$$

The last two results show that the mixed derivatives are equal, i.e., the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are commutative. This is in fact the case for most functions. Verify for yourself by choosing a few functions at random. It can be proved that a sufficient (but not necessary) condition that $f_{xy} = f_{yx}$ at some point (a,b) is that both f_{xy} and f_{yx} are continuous at (a,b).

Lastly, we note from the above example that $f(x,y)$ satisfies the partial differential equation (called **Laplace's equation** in 2 variables) given by $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

Such a function is called a *Harmonic function*.

<u>Example 8</u>

Show that $f(x,y) = e^x \sin y$ is a harmonic function.

We need to show that f satisfies Laplace's equation i.e $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$

$$\frac{\partial^2 f}{\partial x^2} = e^x \sin y.$$

$$\frac{\partial f}{\partial y} = e^x \cos y \qquad\qquad \frac{\partial^2 f}{\partial y^2} = -e^x \sin y.$$

Clearly, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$ Hence, f is harmonic.

<u>Example 9</u>

If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

We note that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$. First, we obtain an expression for $\frac{\partial u}{\partial y}$ or u_y.

$$u_y = x^2 \frac{1}{1+\left(y/x\right)^2}\frac{1}{x} - \left(2y\tan^{-1}\frac{x}{y} + y^2 \frac{1}{1+\left(x/y\right)^2}\frac{\partial}{\partial y}\left(xy^{-1}\right)\right).$$

The latter equation can be written as

$$u_y = \frac{x^2}{1+\left(y/x\right)^2}\frac{1}{x} - \left(2y\tan^{-1}\frac{x}{y} + \frac{y^2}{1+\left(x/y\right)^2} \times \left(-\frac{x}{y^2}\right)\right),$$

$$u_y = \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2}.$$

which simplifies to

$$u_y = x - 2y\tan^{-1}\frac{x}{y}.$$

Now,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x - 2y\tan^{-1}\frac{x}{y}\right) = 1 - \left(\frac{1}{y}\right)\frac{2y}{1+\left(x/y\right)^2} = 1 - \frac{2}{1+\left(x^2/y^2\right)} = 1 - \frac{2y^2}{x^2+y^2}$$

Hence,

$$u_{xy} = \frac{x^2 - y^2}{x^2 + y^2}.$$

Example 10

If $F(x,y,z,t)=e^{-t}\sin xz+y^2z\sin t-x^3z^5$, find $\dfrac{\partial^3 F}{\partial x\partial z\partial t}$.

$$\frac{\partial^3 F}{\partial x\partial z\partial t}=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial t}\right)\right]=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}\left(-e^{-t}\sin xz+y^2z\cos t\right)\right]$$

$$=\frac{\partial}{\partial x}\left[-e^{-t}x\cos xz+y^2\cos t\right]$$

$$=-e^{-t}(x.-z\sin xz+\cos xz)=e^{-t}(xz\sin xz-\cos xz).$$

Example 11

If $\theta=t^n\,e^{-\frac{r^2}{4t}}$ where $\theta=f(r,t)$, find the value of n such that $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)=\dfrac{\partial\theta}{\partial t}$.

$$\frac{\partial\theta}{\partial r}=t^ne^{-\frac{r^2}{4t}}\frac{\partial}{\partial r}\left(-\frac{r^2}{4t}\right)$$

$$=t^n\mathrm{e}^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)=-\frac{1}{2}rt^{n-1}\,e^{-\frac{r^2}{4t}}.$$

Thus we have

$$r^2\frac{\partial\theta}{\partial r}=-\frac{t^nr^3}{2t}e^{\frac{-r^2}{4t}}$$

Therefore,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{3}{2}r^2t^{n-1}\,e^{-\frac{r^2}{4t}}-\frac{1}{2}r^3t^{n-1}\,e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)$$

$$=\quad -\frac{3}{2}r^2t^{n-1}\,e^{-\frac{r^2}{4t}}+\frac{1}{4}r^4t^{n-2}\,e^{-\frac{r^2}{4t}},$$

giving

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)\ =t^{n-2}e^{\frac{-r^2}{4t}}\left(-\frac{3}{2}t+\frac{r^2}{4}\right).$$

(*)

We next obtain $\frac{\partial \theta}{\partial t}$.

$$\begin{aligned}
\frac{\partial \theta}{\partial t} &= nt^{n-1}\, e^{-\frac{r^2}{4t}} + t^n e^{-\frac{r^2}{4t}}\, \frac{\partial}{\partial t}\left(-\frac{r^2}{4t}\right) \\
&= nt^{n-1}\, e^{-\frac{r^2}{4t}} + t^n e^{-\frac{r^2}{4t}} \left(\frac{r^2}{4t^2}\right) \\
&= e^{-\frac{r^2}{4t}}\left(nt^{n-1} + \frac{r^2 t^n}{4t^2}\right) \\
&= t^{n-2} e^{-\frac{r^2}{4t}}\left(nt + \frac{r^2}{4}\right).
\end{aligned}$$

So that,

$$\frac{\partial \theta}{\partial t} = t^{n-2} e^{-\frac{r^2}{4t}}\left(nt + \frac{r^2}{4}\right). \qquad (**)$$

Equating (*) and (**), we get $n = -\frac{3}{2}$.

Activity 3

1. For each of the following functions, find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial y^2}$.

   (i) $z = 3x^5 - 5x^2 y^4 + y^7$

   (ii) $z = \tanh 5x \ \sin 2y$

2. Show that if $\psi = \sin x \ \sin y$, then $\frac{\partial^4 \psi}{\partial x^4} - 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0$.

7.4 FUNCTION OF A FUNCTION: THE CHAIN RULE

We recall that if f is a function of a variable *u*, and *u* is a function of a variable *x*, then

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx},$$

a result known as the ***chain rule***. The result may be immediately extended to the case when *f* is a function of two or more variables. Suppose $f = f(u)$ and $u = u(x, y)$. [This means *f* is a function of *u* and *u* is a function of *x* and *y*.] Then, by the definition of a partial derivative,

$$f_x \equiv \left(\frac{\partial f}{\partial x}\right)_y = \frac{df}{du}\left(\frac{\partial u}{\partial x}\right)_y$$

and

$$f_y \equiv \left(\frac{\partial f}{\partial y}\right)_x = \frac{df}{du}\left(\frac{\partial u}{\partial y}\right)_x .$$

7.5 TOTAL DIFFERENTIALS & TOTAL DERIVATIVES

For a function of a *single* variable, say $y = f(x)$, we define:

(i) dx, called differential of x, by the relation $dx = \delta x$, where δx is the small increment in x. [Note: Some authors use the notation Δx instead of δx.]

(ii) dy, called the differential of y.

Now consider the function $z = f(x, y)$ of the *two* independent variables x and y, and define $dx = \delta x$ and $dy = \delta y$. When x varies while y is held fixed, z is a function of x only and the *partial differential* of z w.r.t. x is defined as

$$d_x z = f_x(x, y)\,dx = \frac{\partial z}{\partial x}\,dx.$$

Likewise, the partial differential of z w.r.t. y is defined as

$$d_y z = f_y(x, y)\,dy = \frac{\partial z}{\partial y}\,dy.$$

The *total differential* dz is defined as the sum of the partial differentials, i.e.,

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

In general, for a function $w = F(x, y, z, \ldots, t)$, the total differential dw is defined as

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz + \cdots + \frac{\partial w}{\partial t}dt.$$

Example 12

Find the total differential of $z = xy^2 + \sin^{-1}(xy)$.

$$\frac{\partial z}{\partial x} = y^2 + \frac{1}{\sqrt{1-(xy)^2}} \times \frac{\partial}{\partial x}(xy)$$

$$= y^2 + \frac{y}{\sqrt{1-(xy)^2}}.$$

The partial derivative of z with respect to y is given by

$$\frac{\partial z}{\partial y} = 2xy + \frac{x}{\sqrt{1-(xy)^2}}.$$

Hence,

$$dz = \left(y^2 + \frac{y}{\sqrt{1-(xy)^2}} \right) dx + \left(2xy + \frac{x}{\sqrt{1-(xy)^2}} \right) dy.$$

Activity 4

Find the total differential of the following functions:

(i) $z = 2y \sin x + x \ln y$.

(ii) $w = e^{xyz} - x^2 \cosh\ 3x + x^2 \cos(2y)$.

7.6 THE TOTAL DERIVATIVE: MORE CHAIN RULE

If $z = f(x, y)$ is a continuous function of the variables x, y with continuous partial derivatives $\partial z / \partial x$ and $\partial z / \partial y$, and if x and y are differentiable functions $x = g(t),\ y = h(t)$ of a variable *t*, then *z* is a function of *t* and dz / dt, called the *total derivative* of z w.r.t. *t*, is given by the *chain rule* as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \qquad (\dagger)$$

Similarly, if $w = f(x, y, z, \ldots)$ is a continuous function of the variables $x, y, z, \ldots$, with continuous partial derivatives, and if $x, y, z, \ldots$ are differentiable functions of a variable *t*, the total derivative of w w.r.t. t is given by

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \cdots \qquad (\ddagger)$$

<u>Example 13</u>

Suppose $z = xy^2 + x^2 y,\ \ x = at^2,\ y = 2at$ where *a* is a constant, find $\dfrac{dz}{dt}$.

$$\frac{\partial z}{\partial x} = y^2 + 2xy; \quad \frac{\partial z}{\partial y} = 2xy + x^2.$$

$$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a\,.$$

Hence the total derivative is given by

$$\frac{dz}{dt} = 2at\left(y^2 + 2xy\right) + 2a\left(2xy + x^2\right).$$

In the above example, the independent variables, and therefore the dependent variables, were all functions of a *single* variable *t*. This explains the use of $\dfrac{d}{dt}$ instead of $\dfrac{\partial}{\partial t}$.

Now, if $z = f(x, y)$ is a continuous function of the variables x, y with continuous partial derivatives $\partial z / \partial x$ and $\partial z / \partial y$, and if x and y are differentiable functions $x = g(r, s),\ y = h(r, s)$ of the variables r and s, then z is a function of r and s with

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}. \qquad (\dagger\dagger)$$

In general, if $w = f(x, y, z, \ldots)$ is a continuous function of the variables $x, y, z, \ldots$, with continuous partial derivatives, and if $x, y, z, \ldots$ are differentiable functions of the variables $r, s, t, \ldots$, then,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} + \cdots$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} + \cdots$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \cdots$$

$$\vdots$$

which is just the chain rule.

Example 14

If $\psi = x^2 \ln y + y^3 \tan x$, where $x = r\cos\theta$ and $y = r\sin\theta$, using the chain rule, find expressions for $\dfrac{\partial\psi}{\partial r}$ and $\dfrac{\partial\psi}{\partial\theta}$ in terms of x, y, θ and r.

$$\frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial\psi}{\partial\theta} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\theta}.$$

We substitute the following derivatives :

$\dfrac{\partial\psi}{\partial x} = 2x\ln y + y^3\sec^2 x$, $\dfrac{\partial\psi}{\partial y} = \dfrac{x^2}{y} + 3y^2\tan x$, $\dfrac{\partial x}{\partial r} = \cos\theta$, $\dfrac{\partial x}{\partial\theta} = -r\sin\theta$, $\dfrac{\partial y}{\partial r} = \sin\theta$ and $\dfrac{\partial y}{\partial\theta} = r\cos\theta$ in the above two equations to obtain

$$\frac{d\psi}{dr} = \left(2x\ln y + y^3\sec^2 x\right)\cos\theta + \left(\frac{x^2}{y} + 3y^2\tan x\right)\sin\theta$$

and

$$\frac{d\psi}{d\theta} = -\left(2x\ln y + y^3\sec^2 x\right)r\sin\theta + \left(\frac{x^2}{y} + 3y^2\tan x\right)r\cos\theta.$$

Example 15

If $V = f\left(xz, \frac{y}{z}\right)$, prove that $z\frac{\partial V}{\partial z} = x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y}$.

We let $V = f(u, v)$ where $u = xz$ and $v = \frac{y}{z}$ and obtain

$$\frac{\partial u}{\partial x} = z, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = x, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = \frac{1}{z}, \frac{\partial v}{\partial z} = -\frac{y}{z^2}.$$

We now write rewrite $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$ in terms of $\frac{\partial V}{\partial u}$ and $\frac{\partial V}{\partial v}$:

$$\begin{aligned}\frac{\partial V}{\partial x} &= \frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x} \\ &= z\frac{\partial V}{\partial u},\end{aligned}$$

$$\begin{aligned}\frac{\partial V}{\partial y} &= \frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y} \\ &= \frac{1}{z}\frac{\partial V}{\partial v}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial V}{\partial z} &= \frac{\partial V}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial z} \\ &= x\frac{\partial V}{\partial u} - \frac{y}{z^2}\frac{\partial V}{\partial v}.\end{aligned}$$

Therefore,

$$\begin{aligned}z\frac{\partial V}{\partial z} &= z\left(x\frac{\partial V}{\partial u} - \frac{y}{z^2}\frac{\partial V}{\partial v}\right) \\ &= x\left(z\frac{\partial V}{\partial u}\right) - y\left(\frac{1}{z}\frac{\partial V}{\partial v}\right) \\ &= x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y}.\end{aligned}$$

Activity 5

1. Given that $\phi = z\sin\left(\frac{y}{x}\right)$, $x = 3\rho^2 + 2\sigma$, $y = 4\rho - 2\sigma^3$, $z = 2\rho^2 - 3\sigma^2$, find $\frac{\partial\phi}{\partial\rho}$ and $\frac{\partial\phi}{\partial\sigma}$.

2. z is a function of x and y. Prove that if $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, then
$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

7.7 SUMMARY

In this Unit, you have learnt of functions of several variables and how to find their partial derivatives. You also studied the total differential and total derivative of a function as well the very important chain rule.

7.8 ANSWERS TO ACTIVITIES

Activity 1

1. $z_x = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$.

$z_y = \frac{x}{x^2 + y^2} - \frac{x}{y\sqrt{y^2 - x^2}}$.

Activity 2

1. i) $z_x = \frac{1 + z\cos x\, e^{z\sin x}}{2z - \sin x\, e^{z\sin x}}$; $z_y = \frac{2y}{2z - e^{z\sin x}\sin x}$.

ii) $z_y = -\frac{2x\left(yz + 2x^2\right)}{2z + x^2 y}$; $z_y = -\frac{\left(x^2 z + 4y^3\right)}{2z + x^2 y}$.

Activity 3

1. i) $z_{xx} = 60x^3 - 10y^4.$

$z_{xy} = -40xy^3.$

$z_{yx} = -40xy^3.$

$z_{yy} = 6y^2\left(7y^3 - 10x^2\right).$

ii) $z_{xx} = -50\,\mathrm{sech}^2\, 5x\ \tanh\ 5x.$

$z_{xy} = 10\,\mathrm{sech}^2\, 5x\ \cos\ 2y.$

$z_{xy} = 10\,\mathrm{sech}^2\, 5x\ \cos\ 2y.$

$z_{yy} = -4\sin 2y\ \tanh 5x.$

Activity 4

i) $dz = \left(2y\cos x + \ln y\right)dx + \left(2\sin x + \dfrac{x}{y}\right)dy.$

ii)

$$dw = \left(yz\,e^{xyz} - 2x\cosh 3x - 3x^2\sinh 3x + 2x\cos 2y\right)dx + \left(xz\,e^{xyz} - 2x^2\sin 2y\right)dy + \left(xy\,e^{xyz}\right)dz.$$

Activity 5

1. $$\frac{\partial\phi}{\partial\rho} = -\frac{6\rho yz}{x^2}\cos\left(\frac{y}{x}\right) + \frac{4z}{x}\cos\left(\frac{y}{x}\right) + 4\rho\sin\left(\frac{y}{x}\right).$$

$$\frac{\partial\phi}{\partial\sigma} = -\frac{2yz}{x^2}\cos\left(\frac{y}{x}\right) - \frac{6\sigma^2 z}{x}\cos\left(\frac{y}{x}\right) - 6\sigma\sin\left(\frac{y}{x}\right).$$