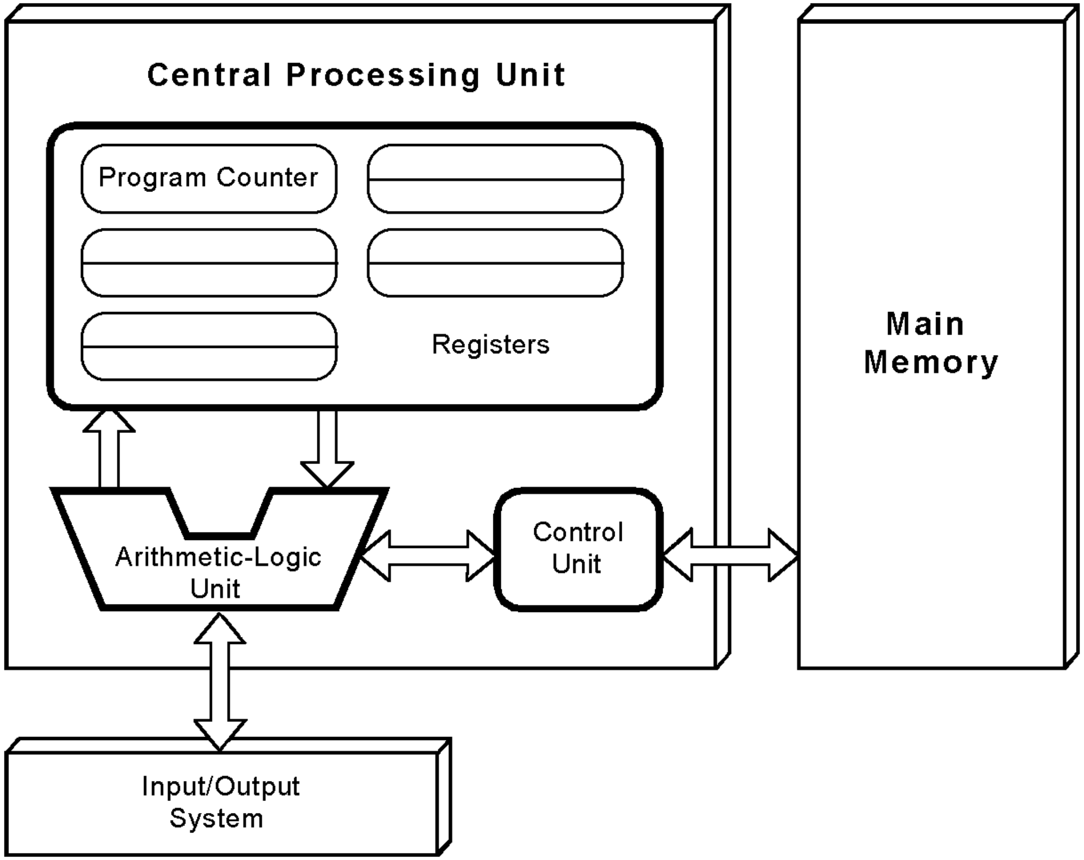
Computer Architecture

ICT 1019Y Week 1 Lecture

Binary Numbers

von Neumann Model



- **How does this run a stored program?**
- **What is the *von Neumann Bottleneck*?**

Converting Between Bases

- The following methods work for converting between *arbitrary* bases
  - We'll focus on converting to/from **binary** because it is the basis for digital computer systems
- Two methods for radix conversion
  - Subtraction method
    - Easy to follow but tedious!
  - Division remainder method
    - Much faster

Subtraction Method: Decimal to Binary

2^0	1
2^1	2
2^2	4
2^3	8
2^4	16
2^5	32
2^6	64
2^7	128
2^8	256
2^9	512
2^{10}	1024
2^{11}	2048

Convert 789_{10} to binary (base 2)

Largest number that fits in 789? (512)	789 – 512 = 277	1xxxxxxxxx
Does 256 fit in 277? (yes)	277 – 256 = 21	11xxxxxxxx
Does 128 fit in 21? (no)	21	110xxxxxxx
Does 64 fit in 21? (no)	21	1100xxxxxx
Does 32 fit in 21? (no)	21	11000xxxxx
Does 16 fit in 21? (yes)	21 – 16 = 5	110001xxxx
Does 8 fit in 5? (no)	5	1100010xxx
Does 4 fit in 5? (yes)	5-4 = 1	11000101xx
Does 2 fit in 1? (no)	1	110001010x
Does 1 fit in 1? (yes)	1-1=**0**	**1100010101**

Division Method: Decimal to Binary

Convert 789_{10} to binary

789 / 2 = 394.5	Remainder of 1
394 / 2 = 197	Remainder of 0
197 / 2 = 98.5	Remainder of 1
98 / 2 = 49	Remainder of 0
49 / 2 = 24.5	Remainder of 1
24 / 2 = 12	Remainder of 0
12 / 2 = 6	Remainder of 0
6 / 2 = 3	Remainder of 0
3 / 2 = 1.5	Remainder of 1
1 / 2 = 0.5 (stop when <1)	Remainder of 1

- Divide by 2 since we're converting to binary (base 2)

Read <u>bottom</u> to <u>top</u>:
$789_{10} = 1100010101_2$

Binary to Decimal

2^0	1
2^1	2
2^2	4
2^3	8
2^4	16
2^5	32
2^6	64
2^7	128
2^8	256
2^9	512
2^{10}	1024
2^{11}	2048

Convert 1011000100_2 to decimal

$= 1x2^9 + 0x2^8 + 1x2^7 + 1x2^6 + 0x2^5 + 0x2^4 + 0x2^3 + 1x2^2 + 0x2^1 + 0x2^0$

$= 512 + 128 + 64 + 4$

= **708**

Binary to Decimal (Faster!)

Convert 1011000100_2 to decimal

1011000100$_2$	0*2 + 1 = 1
1**0**11000100$_2$	1*2 + 0 = 2
10**1**1000100$_2$	2*2 + 1 = 5
101**1**000100$_2$	5*2 + 1 = 11
1011**0**00100$_2$	11*2 + 0 = 22
10110**0**0100$_2$	22*2 + 0 = 44
101100**0**100$_2$	44*2 + 0 = 88
1011000**1**00$_2$	88*2 + 1 = 177
10110001**0**0$_2$	177*2 + 0 = 354
101100010**0**$_2$	354*2 + 0 = **708**

Double your current total and add new digit

Range

- **What is the smallest and largest 8-bit unsigned binary number?**
  - $\mathtt{XXXXXXXX}_2$
  - Smallest = $\mathtt{00000000}_2$ = **0**
  - Largest = $\mathtt{11111111}_2$ = **255**

Converting Between Bases

- What about **fractional values?**
  - Fractional values can be **approximated** in all base systems
  - No guarantee of finding an exact representations under all radices
- Example of an "impossible" fraction:
  - The quantity ½ is exactly representable in the binary and decimal systems, but is not in the ternary (base 3) numbering system

Converting Between Bases

- Fractional values are shown via nonzero digits to the right of the decimal point ("radix point")
  - These represent negative powers of the radix:

$$0.47_{10} = 4 \times 10^{-1} + 7 \times 10^{-2}$$

$$0.11_{2} = 1 \times 2^{-1} + 1 \times 2^{-2}$$
$$= \tfrac{1}{2} + \tfrac{1}{4}$$
$$= 0.5 + 0.25 = 0.75$$

Subtraction Method: Decimal to Binary

Convert 0.8125_{10} to binary

2^{-1}	0.5
2^{-2}	0.25
2^{-3}	0.125
2^{-4}	0.0625
2^{-5}	0.03125
2^{-6}	0.015625

Does 0.5 fit in 0.8125? (yes)	0.8125-0.5 = 0.3125	`.1`
Does 0.25 fit in 0.3125? (yes)	0.3125-0.25 = 0.0625	`.11`
Does 0.125 fit in 0.0625? (no)	0.0625	`.110`
Does 0.0625 fit in 0.0625? (yes)	0.0625-0.0625 = **0**	**`.1101`**

Stop when you reach 0 fractional parts remaining
(**or you have enough binary digits**)

Multiplication Method: Decimal to Binary

Convert 0.8125_{10} to binary

0.8125 * 2 = 1.625	1 (whole number)
0.625 * 2 = 1.25	1
0.25 * 2 = 0.5	0 (no whole number)
0.5 * 2 = 1.0	1

Stop when you reach 0 fractional parts remaining (or you have enough binary digits)

Read <u>top</u> to <u>bottom</u>:
$0.8125_{10} = .1101_2$

Hexadecimal Numbers

- Computers work in binary internally
- Drawback for humans?
  - Hard to read long strings of numbers!
  - Example: $11010100011011_2 = 13595_{10}$
- For compactness and ease of reading, binary values are usually expressed using the **hexadecimal** (base-16) numbering system

Hexadecimal Numbers

A=10
B=11
C=12
D=13
E=14
F=15

- The hexadecimal numbering system uses the numerals 0 through 9 and the letters A through F
  - The decimal number 12 is C_{16}
  - The decimal number 26 is $1A_{16}$
- It is easy to convert between base 16 and base 2, because $16 = 2^4$
- To convert from binary to hexadecimal, group the binary digits into sets of four

Converting Between Bases

- Using groups of 4 bits, the binary number `11010100011011`$_2$ (13595_{10}) in hexadecimal is:

0011	0101	0001	1011
3	5	1	B

Careful!

If the number of bits is not a multiple of 4, pad on the left with zeros.

Thus, <u>safest</u> to <u>start at the right</u> and work towards the left!

Signed Integers

Signed Integer Representation

- To represent signed integers, computer systems use the high-order bit to indicate the sign
  - `0xxxxxxxx` = Positive number
  - `1xxxxxxxx` = Negative number

Value of the number

High order bit /
Most significant bit

- **What have we given up compared to unsigned numbers?**
  - **Range**! With the same number of bits, unsigned integers can express twice as many "positive" values as signed numbers
- Design challenge – How to interpret the *value* field?

Signed Integer Representation

- There are three ways in which signed binary integers may be expressed:
  - Signed magnitude
  - One’s complement
  - Two’s complement
- In an 8-bit word, *signed magnitude* representation places the **absolute value** of the number in the 7 bits to the right of the sign bit.

Signed Integer Representation

- Examples of 8-bit *signed magnitude* representation:
  - +3 = 00000011
  - -3 = 10000011

Sign Bit Magnitude

What if I wanted 16-bit *signed magnitude* representation?

- Computers perform arithmetic operations on signed magnitude numbers in much the same way as humans carry out pencil and paper arithmetic.
  - Ignore the signs of the operands while performing a calculation
  - Apply the appropriate sign after calculation is complete

Signed Integer Representation

- Example: using 8-bit *signed magnitude* binary arithmetic, find 75 + 46
- Convert 75 and 46 to binary
- Arrange as a sum, but separate the (positive) sign bits from the magnitude bits

```
0   1001011
0 + 0101110
  ---------
```

Signed Integer Representation

- Example: using 8-bit *signed magnitude* binary arithmetic, find 75 + 46

- Just as in decimal arithmetic, we find the sum starting with the rightmost bit and work left.

```
0    1001011
0 +  0101110
   ---------
           1
```

Signed Integer Representation

- Example: using 8-bit *signed magnitude* binary arithmetic, find 75 + 46

- In the second bit, we have a carry, so we note it above the third bit.

```
              1
0     1 0 0 1 0 1 1
0 + 0 1 0 1 1 1 0
  ---------------
              0 1
```

Signed Integer Representation

- Example: using 8-bit *signed magnitude* binary arithmetic, find 75 + 46

- The third and fourth bits also give us carries.

```
        1 1 1
0   1 0 0 1 0 1 1
0 + 0 1 0 1 1 1 0
    -------------
          1 0 0 1
```

Signed Integer Representation

- Example: using 8-bit *signed magnitude* binary arithmetic, find 75 + 46

```
          1 1 1
0   1 0 0 1 0 1 1
0 + 0 1 0 1 1 1 0
  ---------------
0   1 1 1 1 0 0 1
```

- Once we have worked our way through all eight bits, we are done.

In this example, I picked two values whose sum would fit into 7 bits (leaving the 8th bit for the sign). If the sum *doesn't* fit into 7 bits, we have a problem.

Signed Integer Representation

- Example: using 8-bit *signed magnitude* binary arithmetic, find 107 + 46.
- The carry from the seventh bit **overflows** and is discarded – no room to store it!
- We get an erroneous result: 107 + 46 = 25.

```
1           1   1 1 1
0       1 1 0 1 0 1 1
0   +   0 1 0 1 1 1 0
        -------------
0       0 0 1 1 0 0 1
```

No magic solution to this overflow problem – you need more bits! (or a smaller number)

Signed Integer Representation

- How do I know what sign to apply to the *signed magnitude* result?
  - Works just like the signs in pencil and paper arithmetic
- **Addition rules**
  - If the **signs are the same**, just add the absolute values together and use the **same sign** for the result
  - If the **signs are different**, use the sign of the **larger number**. Subtract the larger number from the smaller

```
       1 1
1    0101110
1  + 0011001
   ---------
1    1000111
```

- Example: Using *signed magnitude* binary arithmetic, find -46 + -25.
- Because the signs are the same, all we do is add the numbers and supply the negative sign when finished

Signed Integer Representation

- Mixed sign addition (aka **subtraction**) is done the same way
  - Example: Using signed magnitude binary arithmetic, find 46 + -25.
- The sign of the result is the sign of the larger (here: +)
  - Note the “borrows” from the second and third bits.

$$
\begin{array}{cccccccccc}
 & & & 0 & 2 & & & 0 & 2 \\
0 & & 0 & \cancel{1} & 0 & 1 & 1 & \cancel{1} & 0 \\
1 & + & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
0 & & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}
$$

Signed Integer Representation

- Strengths
  - *Signed magnitude* is easy for people to understand
  - **You'll find that, in low-level computer design, "easy for people to understand" doesn't count for very much!**
- Drawbacks
  - Makes computer **hardware** more **complicated** / slower
    - Have to compare the two numbers first to determine the correct sign and whether to add or subtract
  - Has two different representations for zero
    - Positive zero and negative zero
- We can **simplify computer hardware** by using a *complement* system to represent numbers

Signed Integer Representation

- 8-bit *one's complement* representation:
  - + 3 is: 00000011
  - - 3 is: 11111100 (just invert all the bits!)
- In one's complement representation, as with signed magnitude, negative values are indicated by a 1 in the high order bit
- Complement systems are useful because they eliminate the need for subtraction – just complement one and add them together!

Signed Integer Representation

- One's complement is simpler to implement in hardware than signed magnitude
  - Don't need to compare numbers to see which is larger (for mixed signs)
- Still one disadvantage
  - Positive zero and negative zero
- Solution? *Two's complement* representation
  - **Used by all modern systems**

Signed Integer Representation

- To express a value in *two's complement* representation:
  - If the number is **positive**, just convert it to binary and you're **done**
  - If the number is **negative**, find the **one's complement** of the number (i.e. invert bits) and then **add 1**
- Example:
  - In 8-bit binary, 3 is:
    `00000011` *(notice how nothing has changed!)*
  - -3 using one's complement representation is:
    `11111100`
  - Adding 1 gives us -3 in two's complement form:
    `11111101`

Signed Integer Representation

- With two's complement arithmetic, all we do is add the two binary numbers and **discard any carries** from the high order bit

$$
\begin{array}{cc}
 & 1\ 1\ \\
1 \quad & 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0 \\
+ & 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1 \\
\hline
 & 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1
\end{array}
$$

- Example: Using two's complement binary arithmetic, find 48 + -19 = 29

48 in binary is: 00110000
19 in binary is: 00010011,
-19 using one's complement is: 11101100,
-19 using two's complement is: 11101101.

Reminders

For positive numbers, the *signed-magnitude*, *one's complement*, and *two's complement* forms are all **the same**!

In *one's complement* / *two's complement* form, you only need to modify the number if it is **negative**!

Range

- **What is the smallest and largest 8-bit two's complement number?**
  - $XXXXXXXX_2$
  - Smallest (negative) # = 10000000_2 = **-128**
  - Largest (positive) # = 01111111_2 = **127**

Overflow

- **Overflow**: The result of a calculation is too large or small to store in the computer
  - We only have a finite number of bits available for each number
- Can we **prevent** overflow?
  - Not without re-writing your program to use values that can fit within computer memory
- Can we **detect** overflow? Yes!
  - Easy to detect in complement arithmetic
  - A set of rules that you could implement in hardware