Logical Operations and proofs ①

Propositional logic

- It is concerned with statements to which the truth values "True" and "False" can be assigned.

- Some examples of propositions:
1, "Dogs are mammals", it returns truth value "True"
2, "$12+3 = 4-2$", it returns truth value "False"

- The following is not a proposition:

"X is less ~~tha~~ than 3." We cannot say whether the statement is true or false, unless, we give a specific value of X.

Connectives

- In propositional logic generally, we use ~~five~~ six connectives which are:

1, OR ($\vee$)
2, AND ($\wedge$)
3, Negation / NOT ($\neg$ or $\sim$)
4, Implication ($\longrightarrow$) if then
5, If and only iff ($\Longleftrightarrow$)
6, Exclusive OR (XOR) ($\oplus$)

Truth table

- A truth table is a table of rows and columns showing the truth value (either "T" for true or "F" for false) of every possible combination of the given statements (usually represented by p, q, r)

Truth table for the connectives

1, AND ($\wedge$)
p and q ($p \wedge q$) is True if both p, q are True

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2, OR ($\vee$)
~~p and~~ p or q ($p \vee q$) is True if at least one is True.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

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3, Negation ($\neg$)

- Negation is the opposite of the truth value of a statement

p	$\neg p$
T	F
F	T

4, Implies ($\longrightarrow$)

$p \rightarrow q$ is false if the first is ~~te~~ True and second is false.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

5, Equality ($\Leftrightarrow$) or ($\equiv$)

$p \Leftrightarrow q$ is True, if both are False or both are True.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

6, Exclusive OR ($\oplus$)

$p \oplus q$ is True when the inputs are different

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Ex 1

Construct the truth table for $p \to \neg(p \wedge q)$

Ex 2

Construct the truth table for $(p \vee \neg q) \leftrightarrow r$

Tautology

A tautology is a statement that is always True.

example

Prove that the statement $(p \to q) \leftrightarrow (\neg q \to \neg p)$ is a tautology

Solution

p	q	$\neg p$	$\neg q$	$(p \to q)$	$(\neg q \to \neg p)$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Last column contains all true

$\Rightarrow$ Tautology

Contradiction

A contradiction is a statement that is always False

Example

Prove that $p \wedge \neg p$ is a contradiction

Solution

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Last column all false, so it is a contradiction.

Translating between English and Propositional Logic

Eg

Let p = It is raining
q = Mary is sick

Translate the following in propositional logic

i, It is raining and Mary is sick $p \wedge q$

ii, It is not raining $\neg p$

iii, Mary is not sick $\neg q$

iv, It is not the case that both, it is raining and Mary is sick $\neg(p \wedge q)$

v) If it is raining, then ~~the~~ Mary is sick. $p \rightarrow q$

vi) It is raining if and only if Mary is sick. $p \Leftrightarrow q$

Ex 3

Show that the propositions $(p \wedge \neg q) \wedge (\neg p \vee q)$ is a contradiction

Ex 4

for the following proposition indicate whether it is a tautology, a contradiction or neither. Use a truth table to decide upon.

i) $[(A \rightarrow B) \wedge (B \rightarrow \neg A)] \rightarrow A$

ii) $[(\neg B \rightarrow \neg A) \rightarrow (\neg B \rightarrow A) \rightarrow B]$

Ex 5

Show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

Ex 6

Construct a truth table for the statement $[(q \wedge r) \wedge (\neg p \wedge q)] \wedge \neg q$. Hence state whether it forms a tautology or a contradiction.

Ex 7

$p, q,$ and r are propositions. Determine whether $(p \rightarrow q) \rightarrow r$ and $(\neg p \rightarrow r) \wedge (q \rightarrow r)$ are logically equivalent.