UNIT 7 SKETCHING CURVES AND SURFACES IN TWO AND THREE DIMENSIONAL SPACE

Unit Structure

7.0 OVERVIEW

In this Unit, you will study three different types of three dimensional space data representation coordinate systems. These are the rectangular, cylindrical, and spherical coordinate systems.

7.1 LEARNING OBJECTIVES

When you have successfully completed this Unit, you should be able to do the following:

- Sketch two dimensional (2D) and three dimensional (3D) curves and surfaces.
- Describe regions in 3D using the Cartesian, Cylindrical and Spherical coordinate systems.
- Apply transformations using the curvilinear coordinate system in general.
- Obtain the Jacobian of a transformation.

7.2 THREE DIMENSIONAL (CARTESIAN) RECTANGULAR COORDINATE SYSTEM

The 3D rectangular coordinate system is one of the most common coordinate system used and it is an extension of the 2D rectangular coordinate system by introducing the z-axis. A point (x, y, z) is plotted such that it is x units parallel to the x-axis, y units parallel to the y-axis and z units parallel to the z-axis. Usually, we draw axes at 120^0, and use the same scale on each axis for 3D diagrams. The z-axis is drawn with positive direction upward; the positive x-axis is drawn to the left and downward while the positive y-axis is drawn to the right and downward (see Figure 7.1). In fact, the orientation of axes obeys the right-hand rectangular coordinate system.

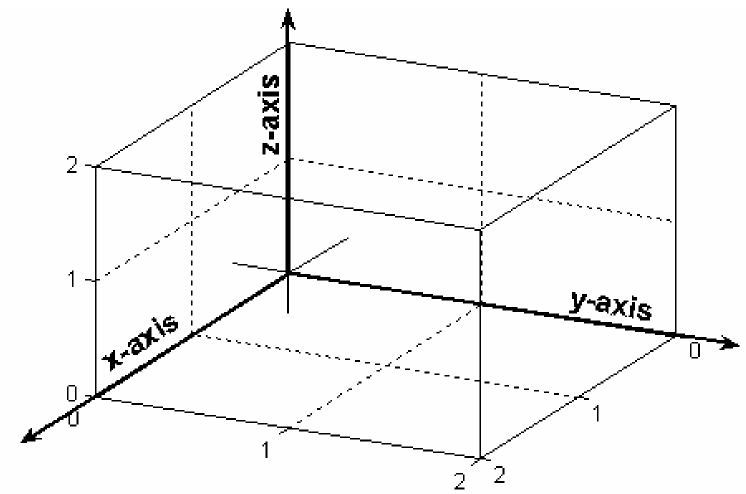


Figure 7.1

Example 1

Plot the points A (1, 1, 0), B (0, 2, 1), C (1, 0, 1) and D (2, 2, 2).

Solution

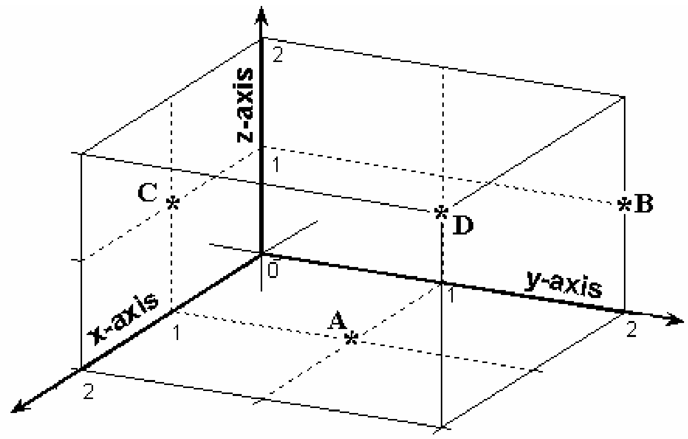


Figure 7.2

Note: Three dimensional space is divided into eight equal portions called <u>octants</u> by the coordinate planes. The first octant is that portion that has all three coordinates positive.

7.3 SKETCHING CURVES

In order to understand concepts found in topics like multiple integral (unit 8), you should be versatile in sketching and understanding the geometry of some important types of three dimensional curves and surfaces. Furthermore, you should be able to describe region of interests in 3D space.

USEFUL PLANES

Let $\vec{\mathbf{r}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the position vector of any point R on a plane and suppose $\vec{\mathbf{n}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a non-zero vector perpendicular (or normal vector) to the plane. Then the equation of a plane is given by $\vec{\mathbf{r}}.\vec{\mathbf{n}} = d$ or $ax + by + cz = d$ where d is a real number.

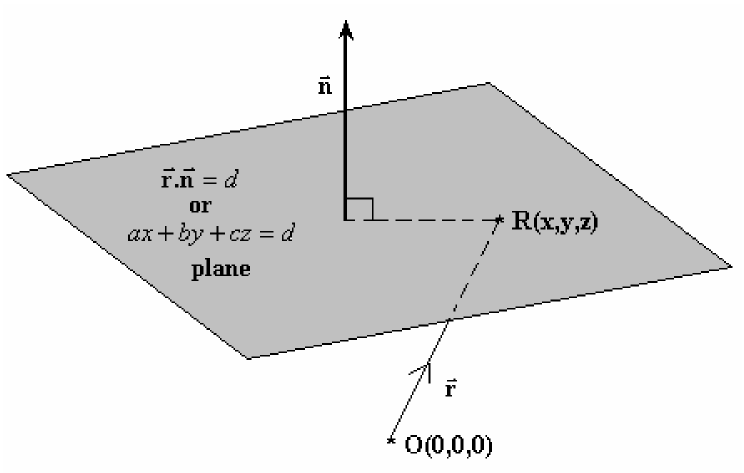
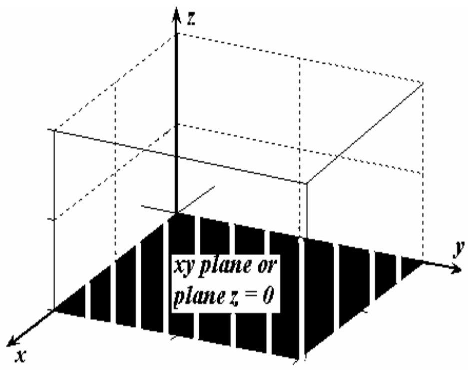


Figure 7.3

<u>Particular Planes to consider</u>:

Type I:

***xy* plane or** $z = 0$	***xz* plane or** $y = 0$	***yz* plane or** $x = 0$
contains O(0,0,0)	contains O(0,0,0)	contains O(0,0,0)
$\vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ **or *z*-axis**	$\vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ **or *y*-axis**	$\vec{\mathbf{n}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ **or *x*-axis**



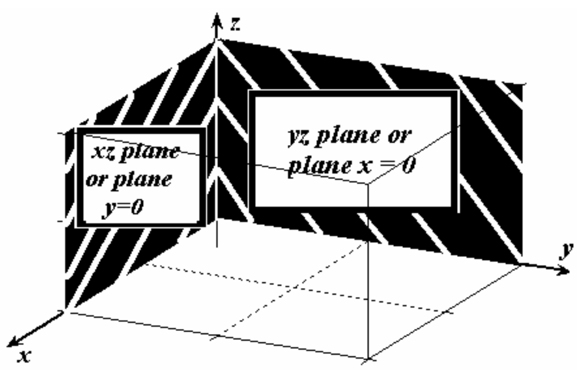


Figure 7.4 (i) **Figure 7.4 (ii)**

Type II:

$z = a$	$y = a$	$x = a$
contains O(0,0,a) and parallel to ***xy* plane**	contains O(0,a,0) and parallel to ***xz* plane**	contains O(a,0,0) and parallel to ***yz* plane**
$\vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ **or *z*-axis**	$\vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ **or *y*-axis**	$\vec{\mathbf{n}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ **or *x*-axis**

Example 2

a) Sketch $y + z = 2$ on two-dimensional coordinate system.

b) Sketch the curve $y + z = 2$, $x = 3$ in three dimensional space.

Solution

(a)

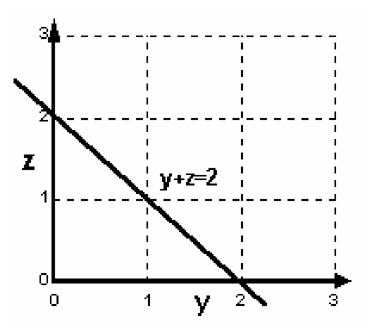


Figure 7.5

(b) The curve (or here line) consists of all points with coordinates (*x,y,z*) that satisfy both equations. In fact, the curve is found in the plane $x = 3$. Hence,

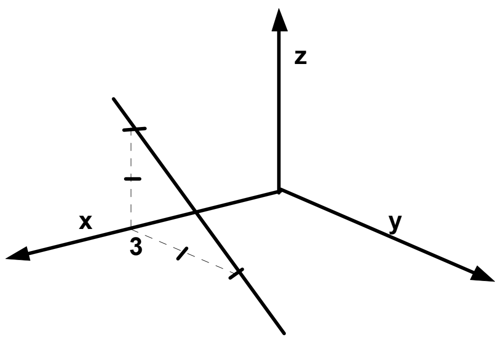


Figure 7.6

Example 3

Sketch the curve $z = 2$, $y = x^2 - z^2$

Solution

The equation z = 2 implies that the curve $y = x^2 - z^2$ is in the plane $z = 2$ and hence we target to sketch $y = x^2 - z^2$ in the plane $z = 2$. By putting $z = 2$, the equation $y = x^2 - z^2$ is simplified to $y = x^2 - 4$ which is a parabola as shown in figure 7.7 (i).

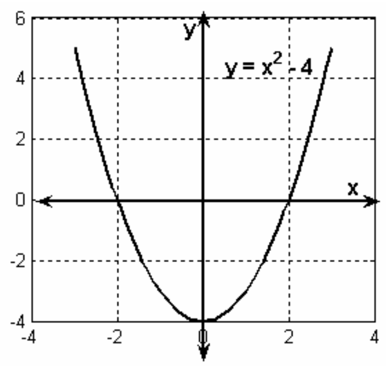


Figure 7.7 (i) xy trace

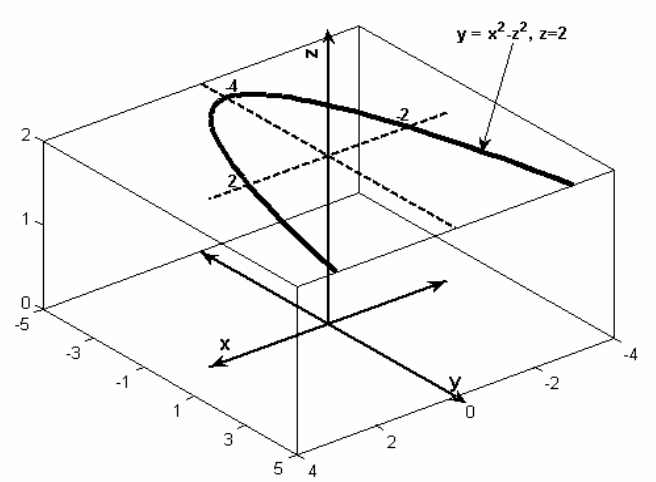


Figure 7.7 (ii)

Activity 1

Sketch the curves with equations:

a) $x = -2$, $\quad 3z - 2y = 6$

b) $y = 2$, $\quad x^2 + z^2 = 4$

7.4 SKETCHING SURFACES

We now consider sketching surfaces. Generally, it is difficult to sketch a three-dimensional surface on a two-dimensional surface such as a piece of paper. However, as with all sketches, the idea is to illustrate a general character.

The intersection of a surface with a plane is called a **trace** of the surface in the plane. The intersection of a surface with a coordinate plane $(xy-, yz-, xz-)$ is called **a coordinate trace.**

To sketch a surface, proceed as follows:

a) Obtain the coordinates of the points of intersection (called *intercepts*) of a surface with each of the coordinate axes.
   For example, to obtain the x-intercept, put the other two variables y and z zero.

b) Obtain the trace of the surface in any two of the coordinates plane. The traces in the coordinate plane are the boundary edges of the surface.
   For example, to obtain the trace in the xy-plane, set $z = 0$ in the equation of surface. Similarly set $x = 0$ and $y = 0$ for yz and xz planes, respectively.

PLANES

The general equation of a plane in three-dimensional space is

$$ax + by + cz + d = 0$$

where $(a,b,c) \neq (0,0,0)$ as illustrated before in figure 7.3.

The coordinate traces are straight lines.

Example 4

Sketch the plane $3x + 3y + 2z = 6$.

Solution

The x-intercept is (2,0,0) which is found by setting $y = 0$ and $z = 0$ in $3x+3y+2z=6$. Similarly, we find that the y-intercept and z-intercept are (0,2,0) and (0,0,3) respectively. The coordinates traces are as follows:

Put $z = 0$, $y = 2-x$ is the xy-coordinate trace as shown in figure 7.8 (i).

Put y = 0, $z = 3-\frac{3}{2}x$ is the xz-coordinate trace as shown in figure 7.8 (ii).

Put x = 0, $z = 3-\frac{3}{2}y$ is the yz-coordinate trace as shown in figure 7.8 (iii).

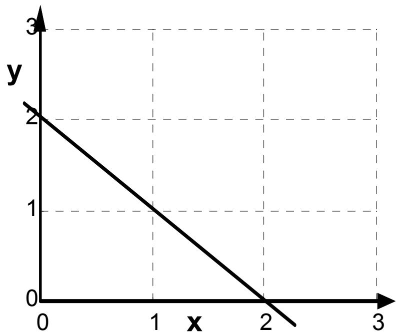


Figure 7.8(i) xy trace

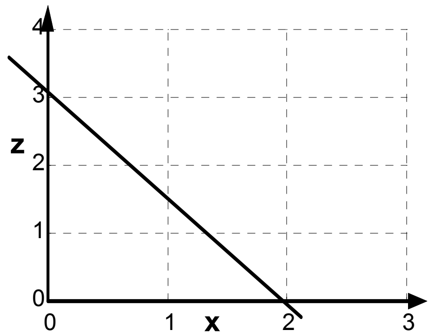


Figure 7.8 (ii) xz trace

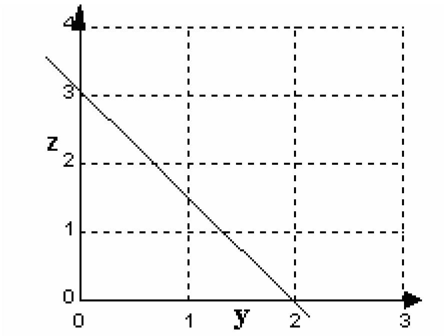


Figure 7.8(iii) yz trace

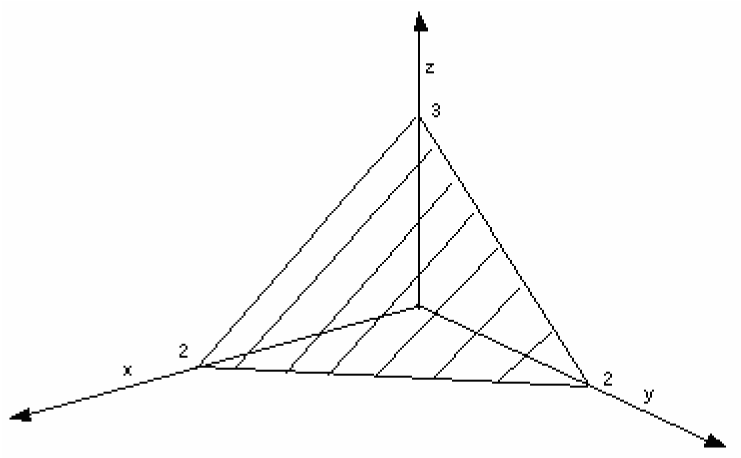


Figure 7.8(iv)

Figure 7.8 (iv) shows a sketch of the plane $3x+3y+2z=6$.

Example 5

Sketch the graph of

a) $-2x+3y=3$

b) $z=2y+1$

Solution

(a)

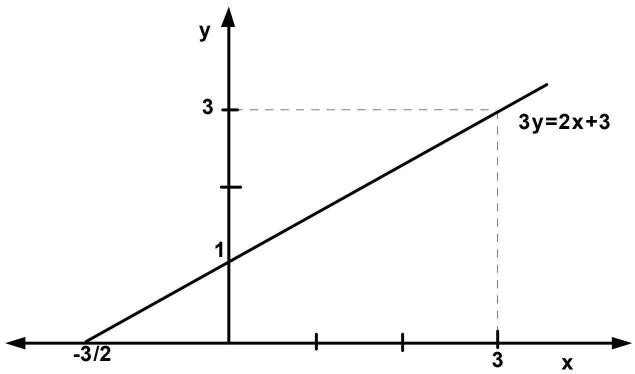


Figure 7.9 (i) xy trace

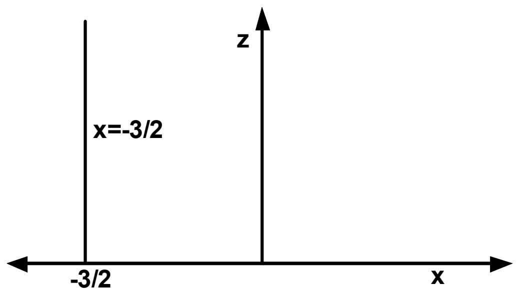


Figure 7.9 (ii) xz trace

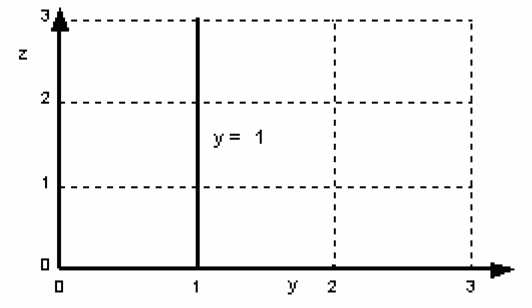


Figure 7.9 (iii) yz trace

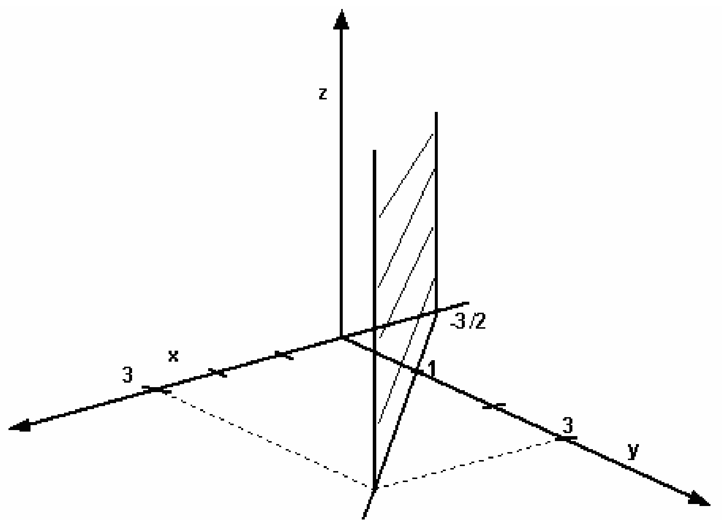


Figure 7.9 (iv)

(b) The y-intercept and z-intercept are both (0,0,0)

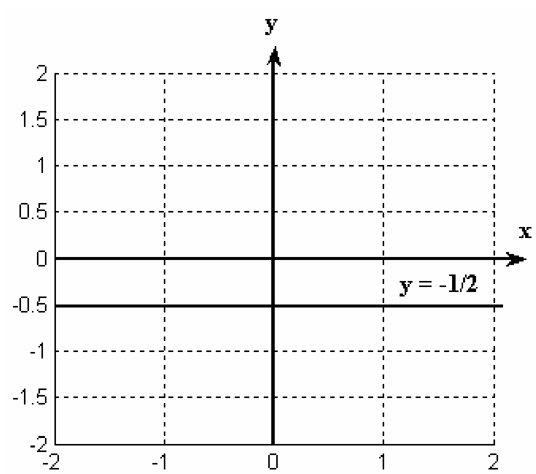


Figure 7.10 (i) xy trace

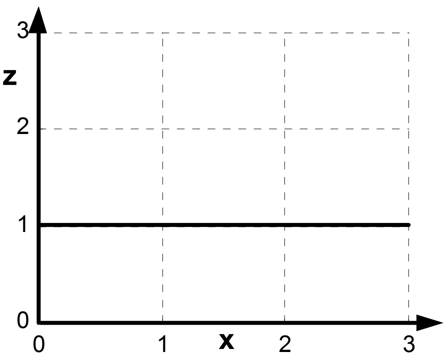


Figure 7.10 (ii) xz trace

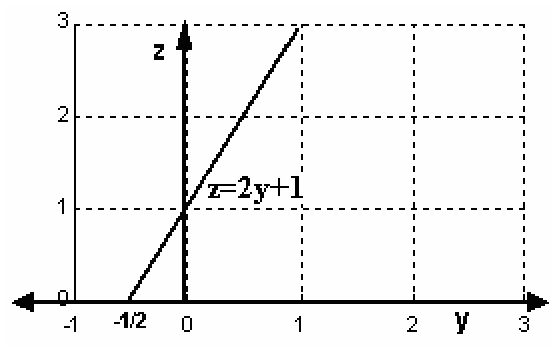


Figure 7.10 (iii) yz trace

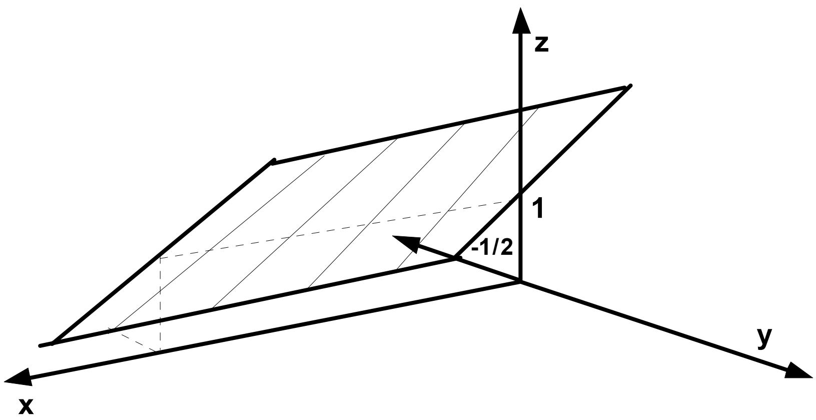


Figure 7.10 (iv)

Figure 7.10 (iv) shows the graph of $z = 2y + 1$.

GENERALISED CYLINDER

A **generalised cylinder** is a surface in three dimensional space that is generated by a collection of parallel lines or curves.

The graph in three dimensional space of an equation that has one (or more) of the three variables (x,y,z) **missing** will be a generalised cylinder, with generating lines parallel to the axis of the missing variable (s).

For example the *plane* $x + y = 2$ (has z missing) is a generalised cylinder with generating lines parallel to z-axis. The *plane* $z = 2y+1$ is also a generalised cylinder with generating lines parallel to x-axis.

Example 6

Sketch the graphs of

a) $x^2 + y^2 = 81$

b) $z = 3x^2$.

Solution

(a) The surface $x^2 + y^2 = 81$ is a generalised cylinder (since z is missing) with generating lines parallel to z-axis. Therefore, it is *enough* to obtain the x and y intercepts and the xy coordinate trace.

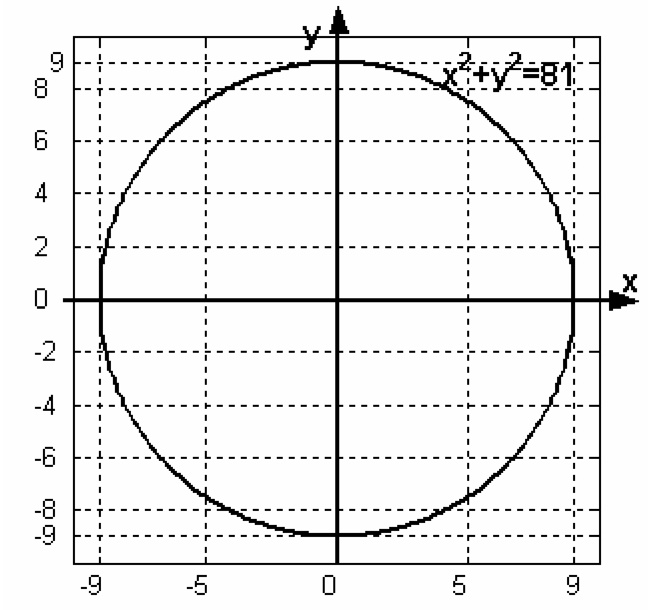


Figure 7.11 (i) xy trace

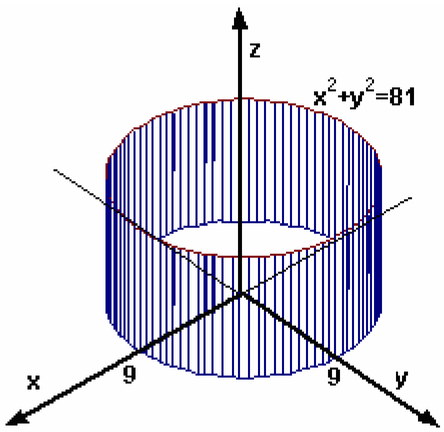


Figure 7.11 (ii) Graph of $x^2 + y^2 = 81$.

(b) The surface $z = 3x^2$ is a generalised cylinder (since y is missing) with generating lines parallel to y-axis. Hence it is enough to obtain the xz trace.

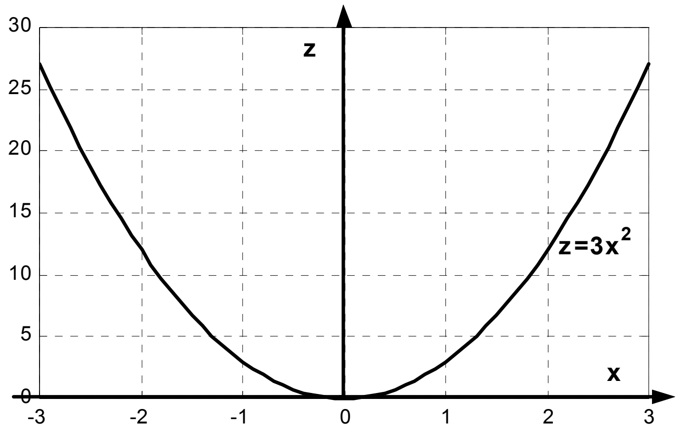


Figure 7.11 (iii) xz trace

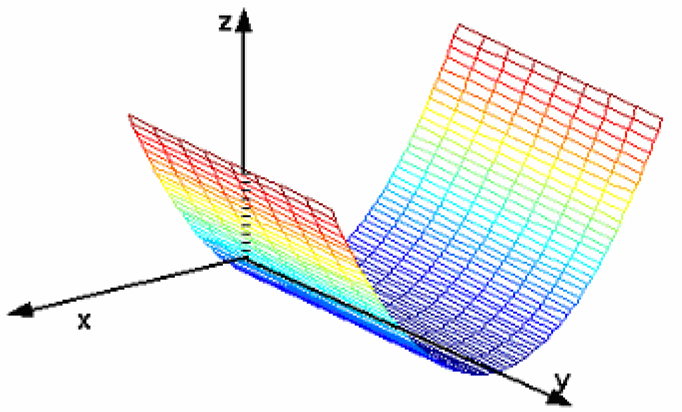


Figure 7.11 (iii) Graph of $z = 3x^2$

QUADRIC SURFACES

Quadric surfaces are surfaces that have equations of the type

$$ax^2 + by^2 + cz^2 + dx + ey + fz + g = 0$$

where $(a,b,c) \neq (0,0,0)$.

Most of these surfaces can be sketched by drawing only one coordinate trace and a typical trace in a perpendicular plane. You should look for traces that are circles or ellipses as they are relatively easy to draw and they give a good indication of the character of the surface.

Typical examples of quadric surfaces are paraboloids, cone, ellipsoid, hyperboloid of two sheets, hyperboloid of one sheet and hyperbolic paraboloid.

Example 7

Sketch the graph of the paraboloid $z = 2x^2 + 2y^2$

Solution

Let us sketch the *yz*-trace, i.e $z = 2y^2$.

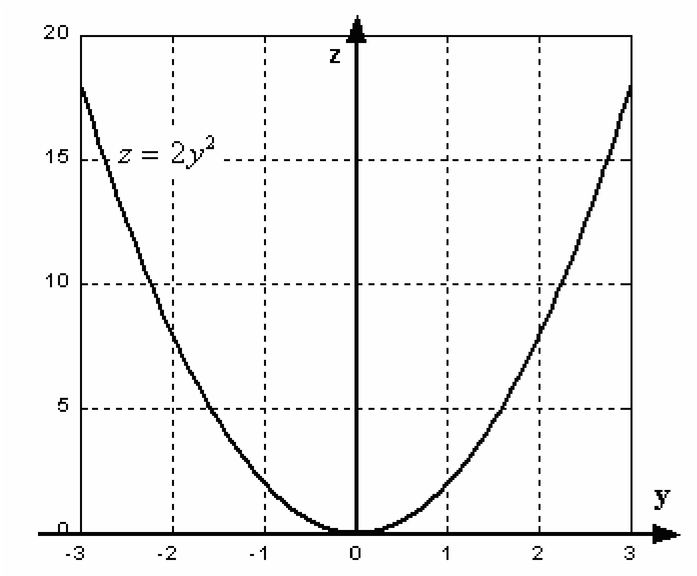


Figure 7.12 (i) xy trace

Now, we also need to sketch the trace of the surface in a plane perpendicular to the *yz* plane. Such a plane is $z = a$ (e.g. *xy* plane). If we let $z = a$, then the equation becomes $x^2 + y^2 = \frac{a}{2}$. These represent circles that are parallel to *xy* plane with radius $\sqrt{\frac{a}{2}}$.

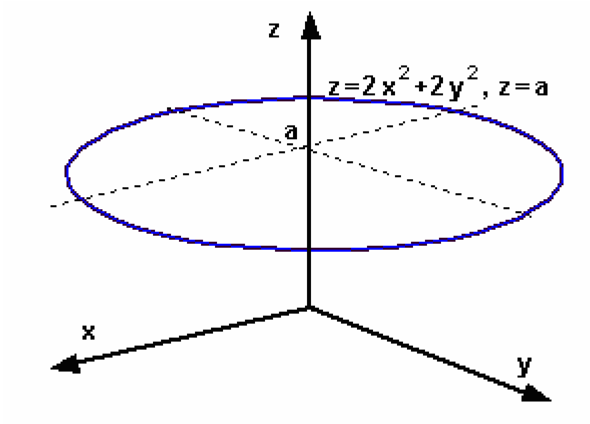


Figure 7.12 (ii)

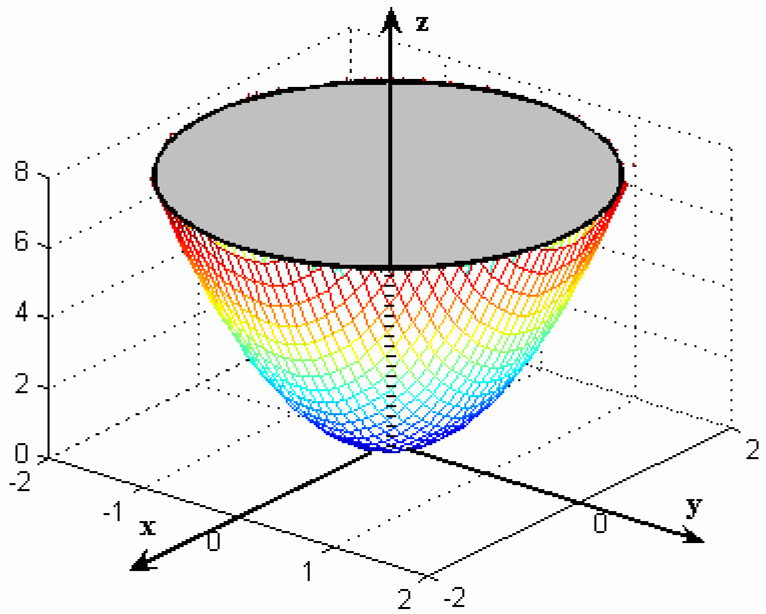


Figure 7.12 (iii) Paraboloid $z = 2x^2 + 2y^2$

Example 8

Sketch the cone $9x^2 - y^2 + 9z^2 = 0$

Solution

Let us sketch the *yz* trace (i.e $-y^2 + 9z^2 = 0$ or $y = \pm 3z$)

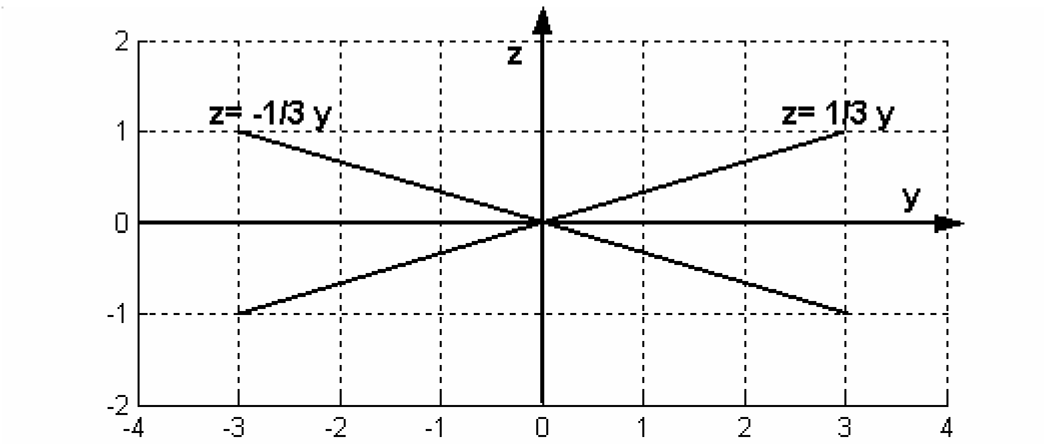


Figure 7.13 (i) yz trace

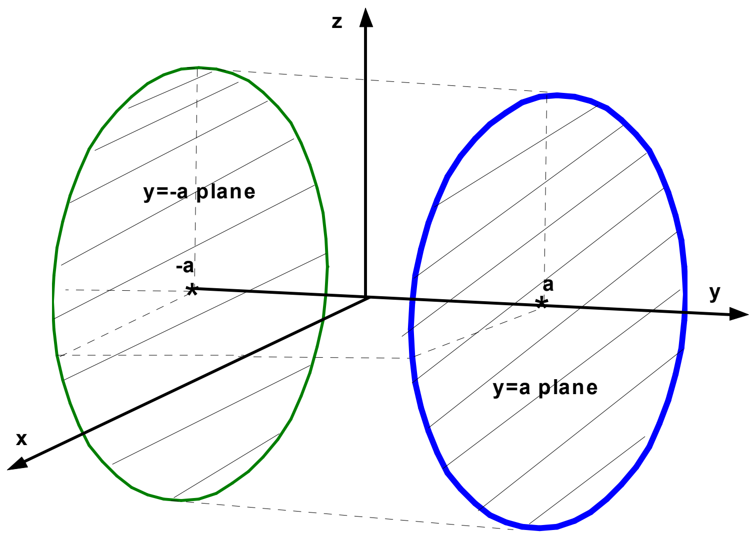


Figure 7.13 (ii) trace in $y = \pm a$

Plane perpendicular to the *yz* trace are $z = \pm a$ or $y = \pm a$. We would prefer to sketch traces in $y = \pm a$ as $9x^2 + 9z^2 = a^2$ represent circles parallel to *xz* plane.

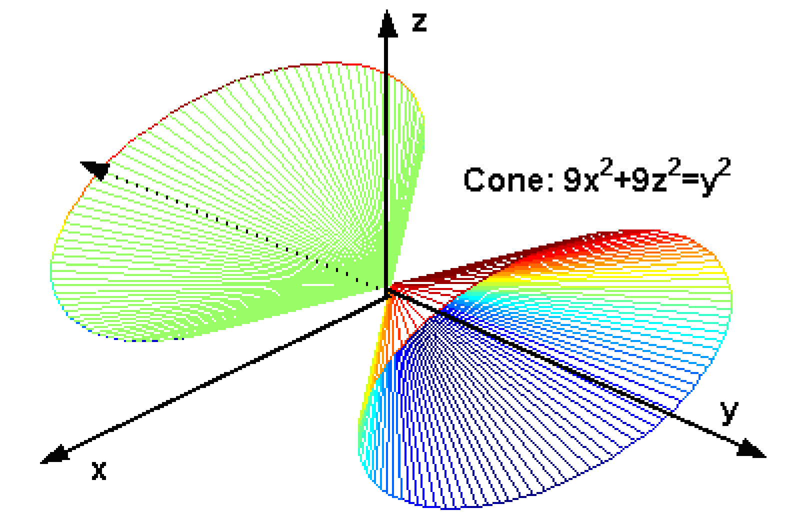


Figure 7.13 (iii) Cone $9x^2 - y^2 + 9z^2 = 0$

Activity 2

Sketch the graphs of the equations:

a) $y = x^2 + z^2$

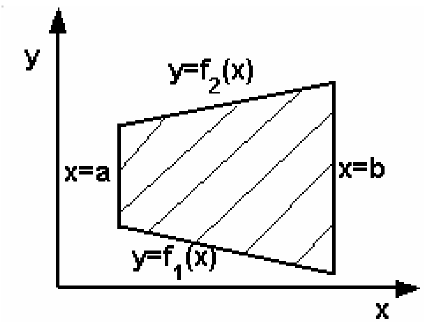
b) $4x^2 + 4y^2 + z^2 = 4$

7.5 REGIONS IN THE PLANE

In the next Unit, you will study integrals over certain types of regions in the plane and in three dimensional space. It is important that the regions be described systematically as discussed below:

7.5.1 REGIONS IN THE PLANE - *CARTESIAN COORDINATES*

Consider the region in the plane bounded by $y = f_1(x)$, $y = f_2(x)$, $x = a$ and $x = b$ where f_1 and f_2 are continuous with $f_1 \le f_2$ for $a \le x \le b$.



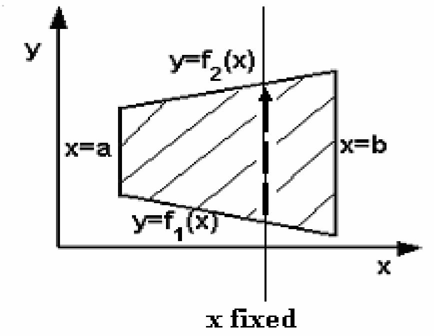


Figure 7.14 (i) **Figure 7.14 (ii)**

Such a region will be called *vertically simple* or *simple in direction of y-axis*, if each of $f_1(x)$ and $f_2(x)$ is given by a single formula that holds for x between a and b.

- How to ***describe*** such a region?

We describe the region as For x fixed, y varies from $y = f_1(x)$ to $y = f_2(x)$, x varies from $x = a$ to $x = b$.

Note that if **an arrow is shot vertically** (vertical lines have x fixed), it enters the region at $y = f_1(x)$ and leaves at $y = f_2(x)$ (see Figure 7.14 (ii)).

Example 9

Sketch and describe the region in the plane bounded by $y = 4- x^2$ and $y = -5$.

Solution

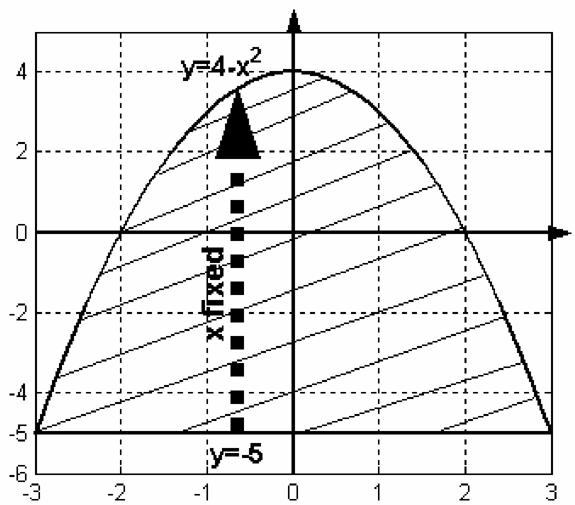


Figure 7.15

Region is described as

For x fixed, y varies from $y = -5$ to $y = 4-x^2$,

x varies from $x = -3$ to $x = 3$.

Example 10

Sketch and describe the region bounded by $x = \frac{1}{4}$ $y = 1-x$, $y = \ln(x)$.

Solution

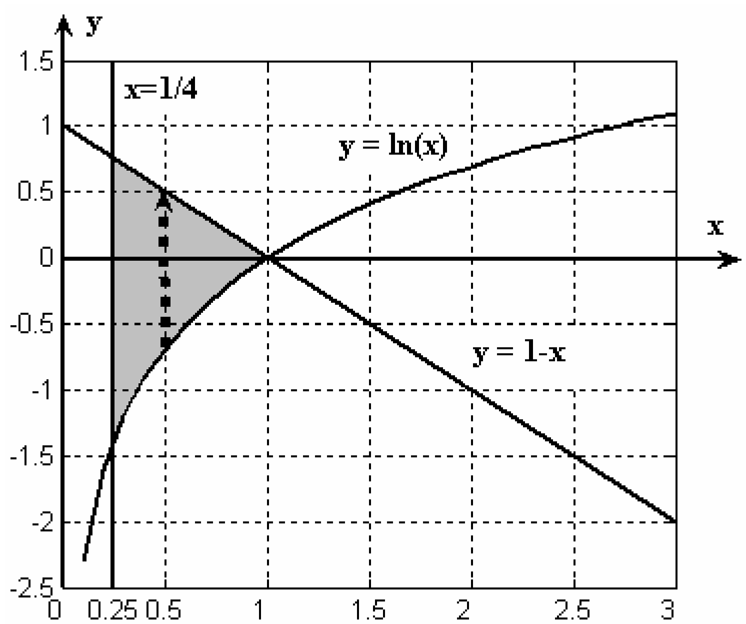


Figure 7.16

Note: $y = 1 - x$ and $y = \ln(x)$ intersect at $1 - x = \ln(x)$ *or* $x = 1$ (by inspection)

Region is described as

For x fixed, y varies from $y = \ln(x)$ to $y = 1 - x$,

x varies from $x = \frac{1}{4}$ to $x = 1$.

A region bounded by $x = g_1(y)$, $x = g_2(y)$, $y = a$ and $y = b$ where g_1 and g_2 are continuous with $g_1(y) \leq g_2(y)$ for $a \leq y \leq b$. Such a region is called *horizontally simple*, or *simple in the direction of the x-axis*, if each of $g_1(y)$ and $g_2(y)$ is given by a simple formula as y varies from a and b.

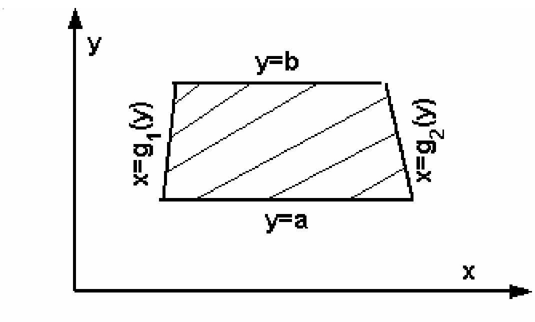


Figure 7.17 (i)

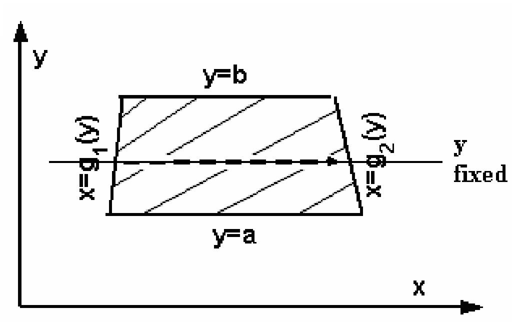


Figure 7.17 (ii)

The region is described as:

For y fixed, x varies from $x = g_1(y)$ to $x = g_2(y)$, y varies from $y = a$ to $y = b$.

Note that if an **arrow is shot horizontally**, it enters the region at $x = g_1(y)$ and leaves it at $x = g_2(y)$ (see figure 7.17 (ii)).

Example 11

Sketch and describe the region in the plane bounded by $x = -4y^2 + 8y - 3$ and $x = -2y^2 + 4y - 2$

Solution

In order to sketch the graphs, we first apply completion of square to both equations. We thus obtain $x = -4y^2 + 8y - 3 = 1 - 4(y-1)^2$ and $x = -2y^2 + 4y - 2 = -2(y-1)^2$.

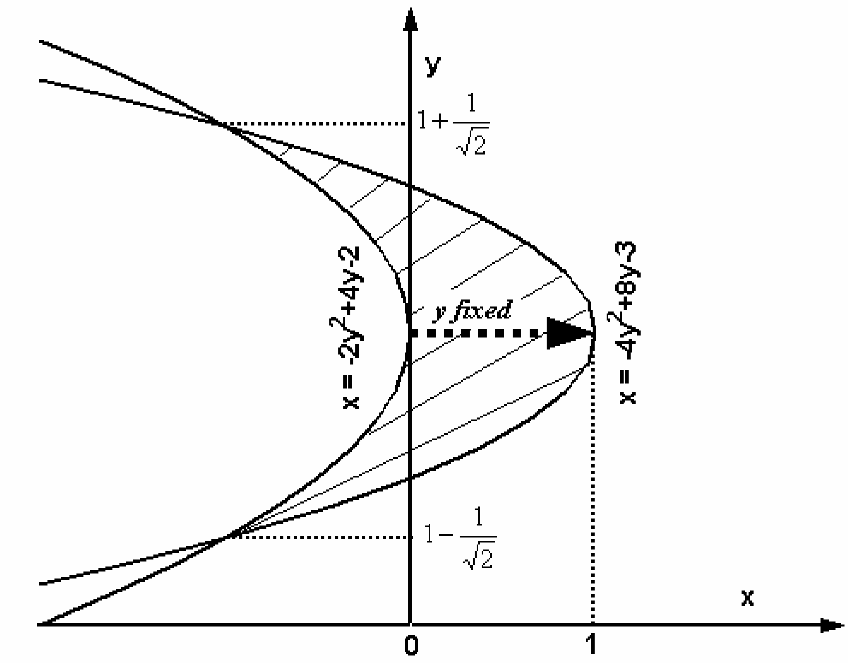


Figure 7.18

$x = -4y^2 + 8y - 3$ and $x = -2y^2 + 4y - 2$ intersect at $-4y^2 + 8y - 3 = -2y^2 + 4y - 2$

or $1 - 4(y-1)^2 = -2(y-1)^2$ or $y = 1 \pm \frac{1}{\sqrt{2}}$

The region is described as:

For y fixed, x varies from $x = -2y^2 + 4y - 2$ to $x = -4y^2 + 8y - 3$,

y varies from $y = 1 - \frac{1}{\sqrt{2}}$ to $y = 1 + \frac{1}{\sqrt{2}}$.

Example 12

Sketch and describe the region in the plane bounded by the equation $x^2 + y^2 = a^2$ in the first quadrant.

Solution

$x^2 + y^2 = a^2$ represents the circle with center (0,0) and radius a.

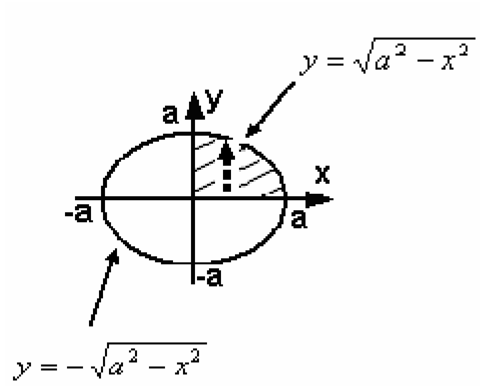


Figure 7.19

The region is described as :

For x fixed, y varies from $y = 0$ to $y = \sqrt{a^2 - x^2}$,

(Note $x^2 + y^2 = a^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$. Since in the first quadrant

$y \geq 0$, we take $y = +\sqrt{a^2 - x^2}$),

x varies from $x = 0$ to $x = a$.

Example 13

Sketch and describe the region in the plane bounded by $y^2 + x^2 - 2x = 3$ in the first quadrant.

Solution

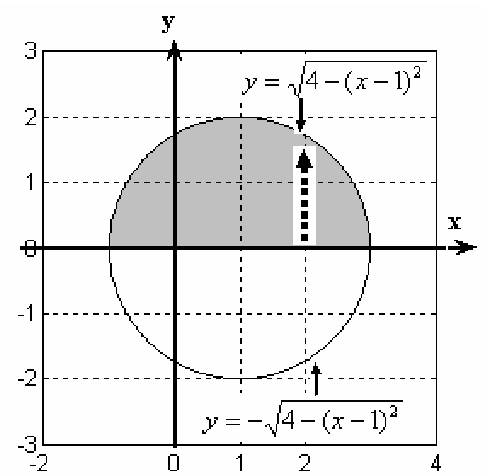


Figure 7.20

$y^2 + x^2 - 2x = 3 \Rightarrow y^2 + (x-1)^2 = 4$ which represents a circle centre (1,0) and radius 2 units and the required region is the upper semicircle.

The region is described as :

For x fixed, y varies from $y = 0$ to $y = \sqrt{4-(x-1)^2}$

x varies from $x = 0$ to $x = 2$

Activity 3

Sketch and describe the regions in the *plane* bounded by the equations:

a) $y = x^2 + 1,\ y = x,\ x = 0,\ x = 1$

b) $x + 2y = 2,\ y = 0,\ x = 0$

7.5.2 REGIONS IN THE PLANE - *POLAR COORDINATES*

Some regions (e.g. circles) can be conveniently described in terms of polar coordinates.

Example 14

Sketch and describe in terms of polar coordinates the region bounded by $x^2 + y^2 = a^2$ in first quadrant.

Solution

The region is shown in Figure 7.19

Before describing, we first transform the given equations of the boundaries in terms of polar coordinates (Recall that $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$)

$x^2 + y^2 = a^2 \Rightarrow r^2 = a^2$ or $r = a$

The x-axis corresponds to line $\theta = 0$

The y-axis corresponds to line $\theta = \pi/2$

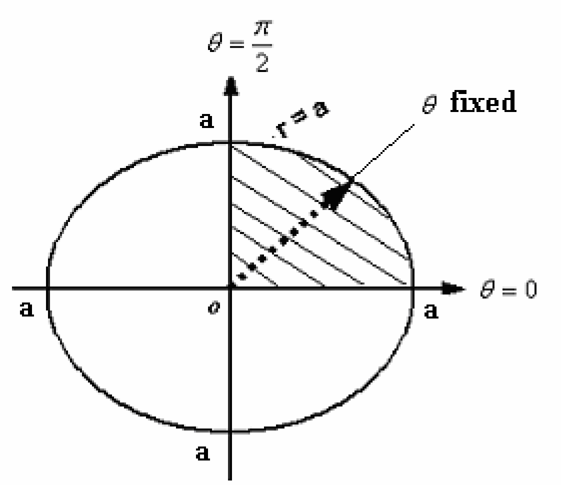


Figure 7.21

The region is described as :

For θ fixed, r varies from $r = 0$ to $r = a$

θ varies from $\theta = 0$ to $\theta = \pi/2$

Example 15

Sketch and describe in terms of polar coordinates the region bounded by the equation $y^2 + x^2 - 2x = 3$ in the first quadrant. The region is sketched in Figure 7.20.

$x^2 + y^2 - 2x = 3 \Rightarrow r^2 - 2\cos\theta = 3$ or $r = 3 + 2\cos\theta$

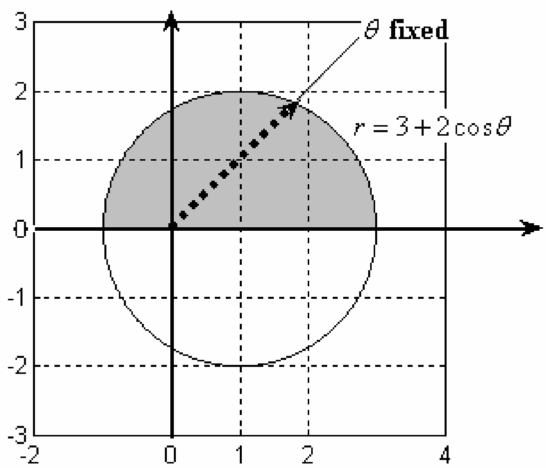


Figure 7.22

The region is described as:

For θ fixed, r varies from $r = 0$ to $r = 3 + 2\cos\theta$,

θ varies from $\theta = 0$ to $\theta = \pi/2$

7.6 REGIONS IN THREE DIMENSIONAL SPACE

7.6.1 REGIONS IN THREE DIMENSIONAL SPACE - *CARTESIAN COORDINATES*

The same ideas that were used to describe regions in the plane can be used to describe regions in three dimensional space.

Often, we will deal with regions in three-dimensional space that can be described as :

For x and y fixed, z varies from $z = h_1(x, y)$ to $z = h_2(x, y)$.

For x fixed, y varies from $y = g_1(x)$ to $y = g_2(x)$,

x varies from $x = a$ to $x = b$

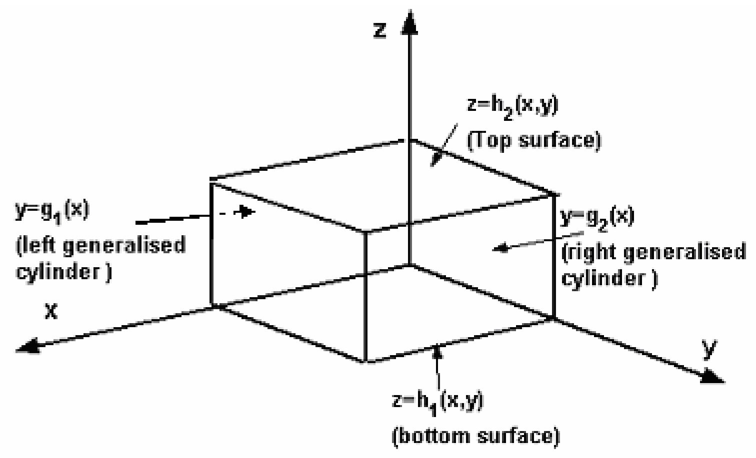


Figure 7.23

Steps to sketch and describe a region in three dimensional space.

a) Sketch the coordinate traces of each surface.

b) Find the points of intersection of traces in each coordinate plane.

c) Sketch the region; draw all the boundary curves.

d) Identify the equations of top and bottom surfaces. The equations of the top and bottom surfaces must involve the variable z. We must be given exactly 2 equations that involve the variable z.

e) Identify the side *cylinders*. The equations of the side cylinders must involve y and cannot involve z. *If* the top and bottom surfaces intersect, equations of side cylinders are obtained by eliminating z from equations of top and bottom surface.

f) Identify front and back planes.

Example 16

Sketch and describe the region bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes in the first octant.

Solution

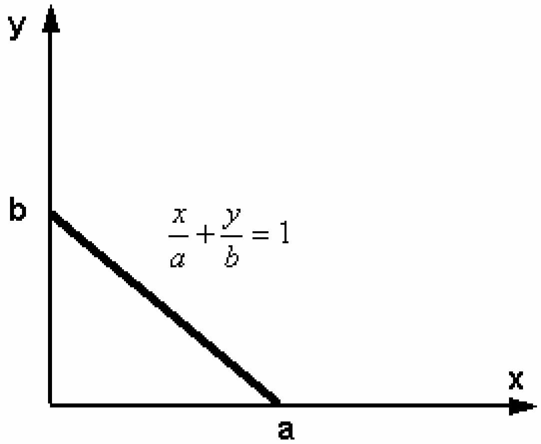


Figure 7.24 (i) xy trace

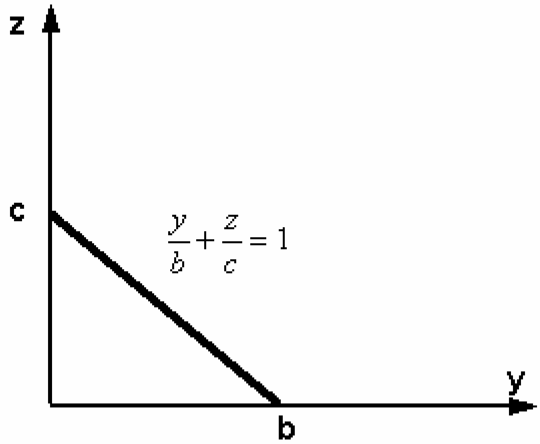


Figure 7.24 (ii) yz trace

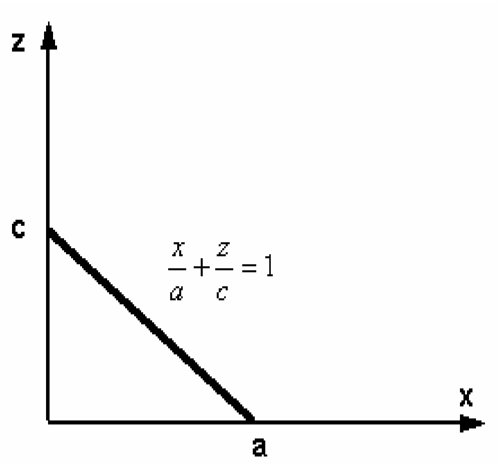


Figure 7.24 (iii) xz trace

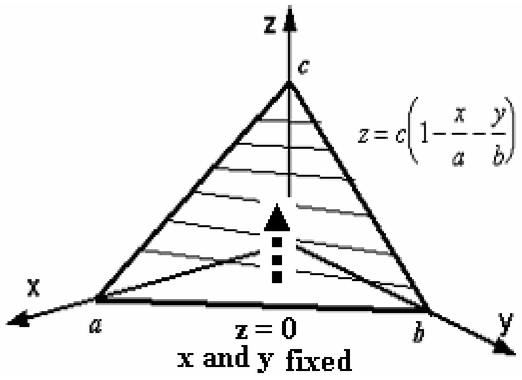


Figure 7.24 (iv)

The top and bottom surfaces are $z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$ and $z = 0$ respectively.

The region is described as:

For x and y fixed, z varies from $z = 0$ to $z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$ (see figure 7.24 (iv))

Using the xy trace,

For x fixed, y varies from $y = 0$ to $y = b\left(1-\frac{x}{a}\right)$ (see figure 7.24 (v))

x varies from $x = 0$ to $x = a$.

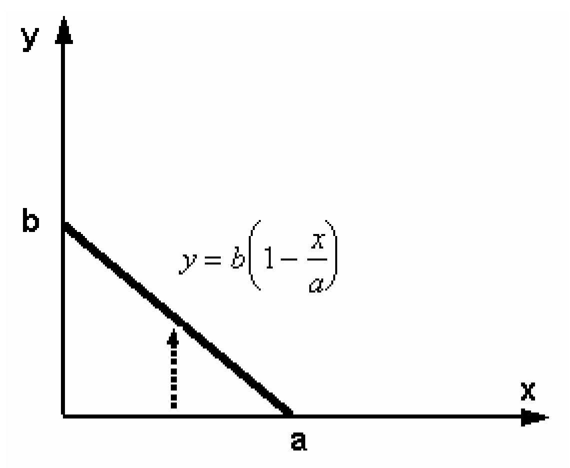


Figure 7.24 (v)

Example 17

Sketch and describe the region bounded by the cylinder $x^2 + y^2 = \frac{1}{4}$ and between the planes $z = 0$ to $\frac{x}{2} + \frac{y}{2} + z = 1$.

Solution

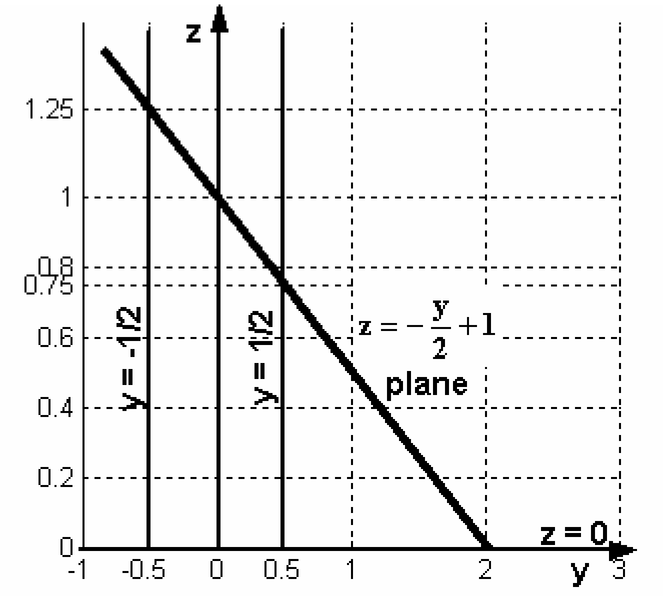


Figure 7.25 (i) yz trace

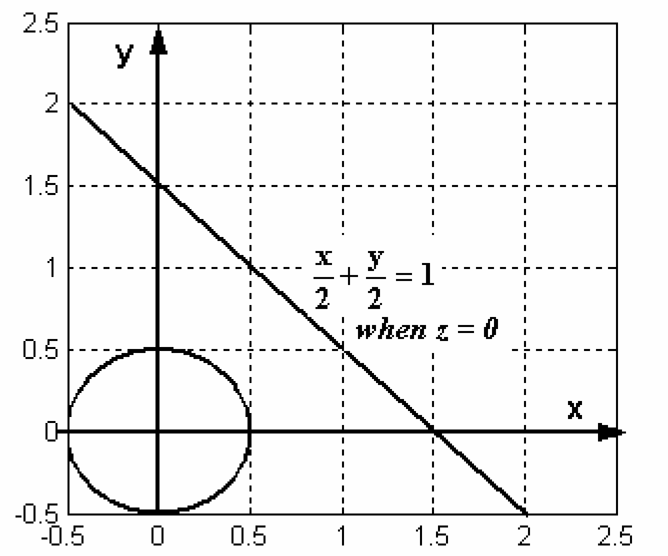


Figure 7.25 (ii) xy trace

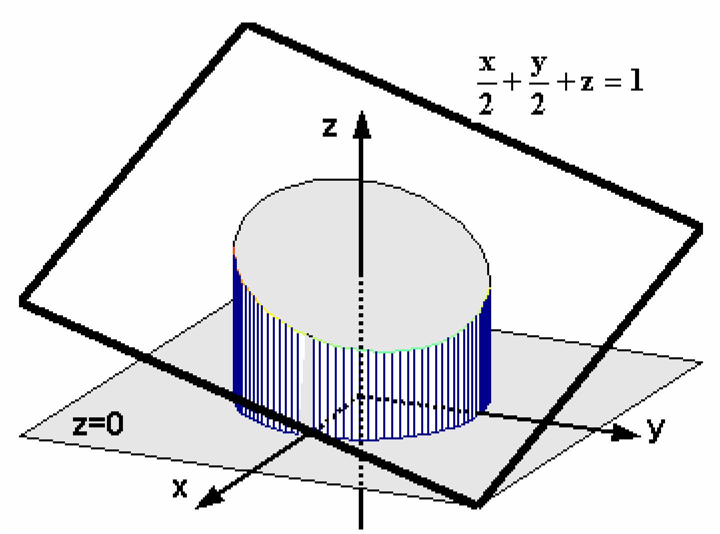


Figure 7.25 (iii) Complete region of interest

The region is described as:

For x and y fixed, z varies from $z = 0$ to $z = 1 - \frac{x}{2} - \frac{y}{2}$.

Using the xy trace,

For x fixed, y varies from $y = -\sqrt{\frac{1}{4} - x^2}$ to $y = \sqrt{\frac{1}{4} - x^2}$

x varies from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$

Note: If the region of interest was in the 1st octant, then the sketch would have been as shown in figure 7.25 (iv)

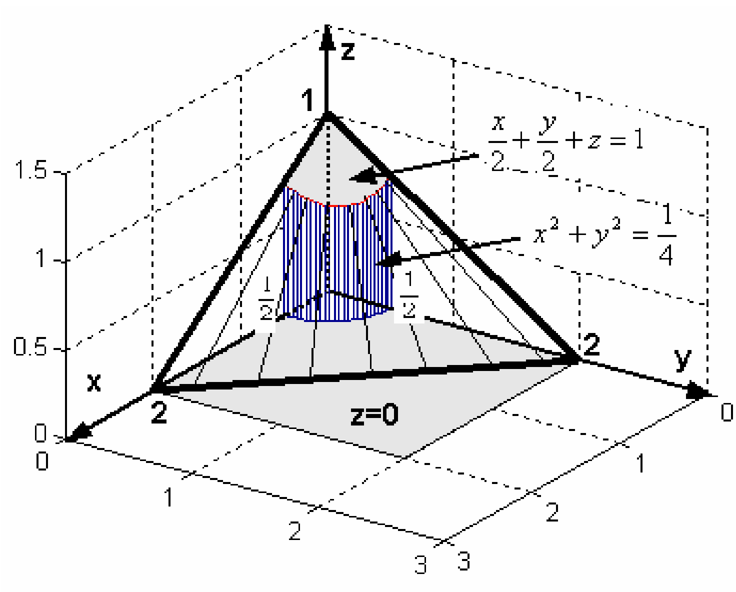


Figure 7.25 (iv) Region of interest in 1st octant

Example 18

Sketch and describe the region in the first octant inside both the cone $9x^2 + 9y^2 = z^2$ and the sphere $x^2 + y^2 + z^2 = 9$.

Solution

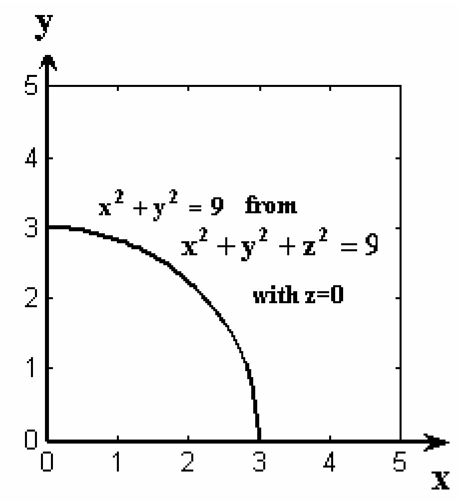


Figure 7.26 (i) xy trace

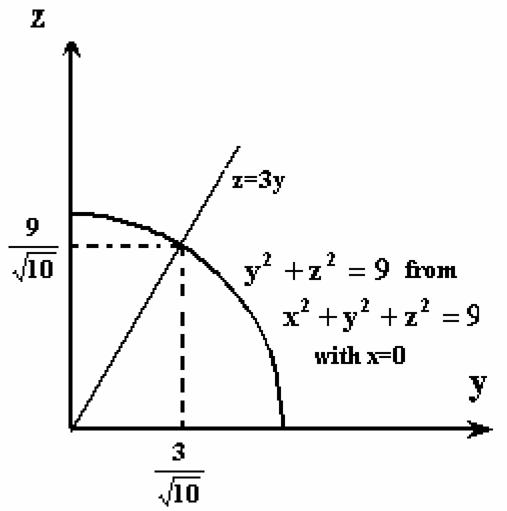


Figure 7.26 (ii) yz trace

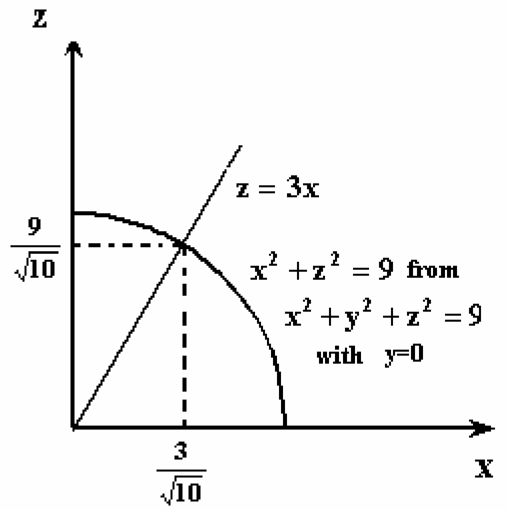


Figure 7.26 (iii) xz trace

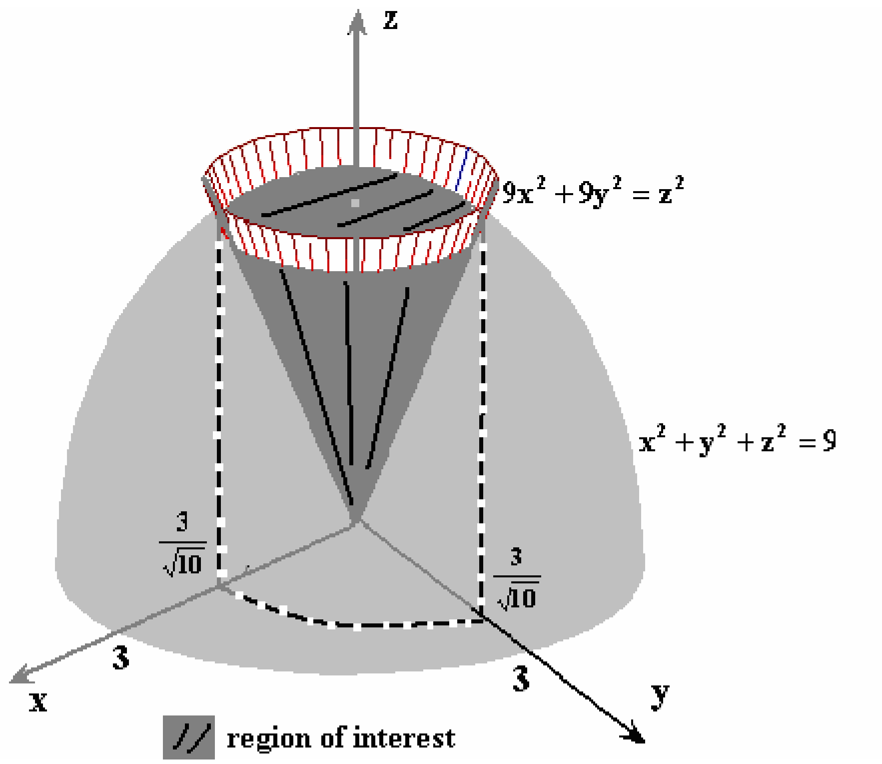


Figure 7.26 (iv)

Important:

a) There are two equations involving z and cone $z^2 = 9x^2 + 9y^2$ represents the bottom surface while the sphere $x^2 + y^2 + z^2 = 9$ represents the top surface.

b) The top and bottom surfaces are intersecting to form a right side edge. The equation of the side cylinder containing the side edge is obtained by eliminating z from the equations of the two surfaces:

$$9x^2 + 9y^2 = 9 - x^2 - y^2$$

or

$$10x^2 + 10y^2 = 9 \quad \text{or} \quad x^2 + y^2 = 9/10$$

This represents a circle with center (0,0) and radius $3/\sqrt{10}$ in the xy plane.

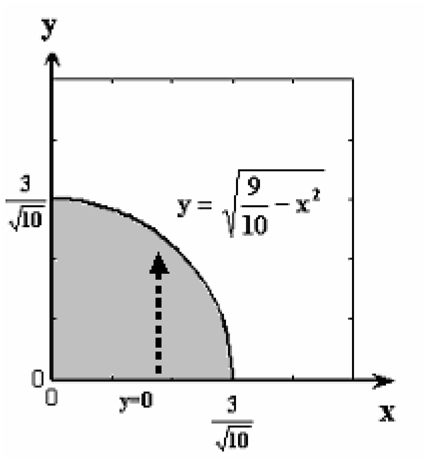


Figure 7.26 (v)

The region is described as:

For x and y fixed, z varies from $z = \sqrt{9x^2 + 9y^2}$ to $z = \sqrt{9 - x^2 - y^2}$ (see figure 7.26 (v))

For x fixed, y varies from $y = 0$ to $y = \sqrt{\dfrac{9}{10} - x^2}$

x varies from $x = 0$ to $x = \dfrac{3}{\sqrt{10}}$.

Activity 4

Sketch and describe (using Cartesian coordinates) the regions in the three-dimensional space bounded by the given equation:

a) below $x + y + z = 3$, above $z = x$, in the first octant

b) Inside $x^2 + y^2 + 1$, outside $z^2 = 4x^2 + 4y^2$.

7.6.2 REGIONS IN THREE DIMENSIONAL SPACE - *CYLINDRICAL COORDINATES*

The **cylindrical coordinates system** is analogous to the **polar coordinates system**. We consider the cylindrical coordinates given by

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$
$$x^2 + y^2 = r^2$$

The numbers (r, θ, z) are called the Cylindrical Coordinates of the point (x,y,z) and are represented below :

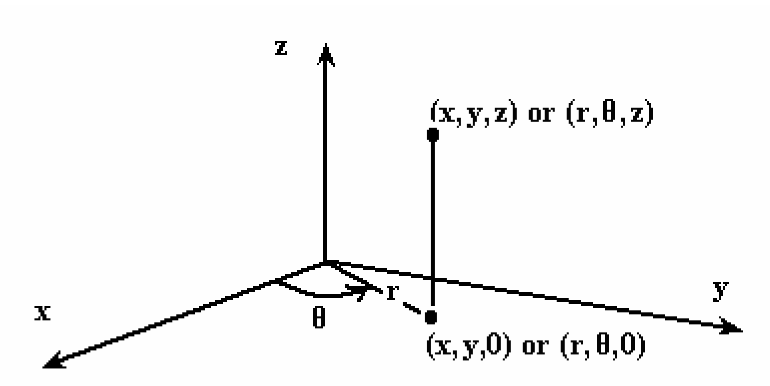


Figure 7.27: Cylindrical coordinate system

The Cylindrical coordinates of a region are usually taken with values of (r, θ, z) such that

$$r \geq 0$$

$$0 \leq \theta \leq 2\pi$$

$$z \text{ is arbitrary}$$

The regions to be described in terms of cylindrical coordinates are usually written as:

For r and θ fixed, z varies from $z = h_1(r, \theta)$ to $z = h_2(r, \theta)$ For θ fixed, r varies from $r = g_1(\theta)$ to $r = g_2(\theta)$ θ varies from $\theta = \theta_1$ to $\theta = \theta_2$

where $z = h_1(r, \theta)$ and $z = h_2(r, \theta)$ are equations of the bottom and top surfaces respectively (see figure 7.23).

The vertical generalized cylinders $r = g_1(\theta)$, $r = g_2(\theta)$, $\theta = \theta_1$ and $\theta = \theta_2$ form the vertical side boundaries of the region and give the projection of the region onto the xy plane.

Example 19

Express the equation of (i) the cylinder $x^2 + y^2 = 25$ and (ii) the sphere $x^2 + y^2 + z^2 = 9$ in terms of cylindrical coordinates.

Solution

(i) $x^2 + y^2 = 25$

$\Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = 25$

$\Rightarrow r^2 = 25$ or $r = 5$

Therefore the equation of the cylinder is $r = 5$.

(ii) $x^2 + y^2 + z^2 = 9$

$\Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 + z^2 = 9$

$\Rightarrow r^2 + z^2 = 9$

Therefore the equation of the sphere is $r^2 + z^2 = 9$.

The three basic surfaces associated with cylindrical coordinates are:

(i) $r = a$ represents a right circular cylinder

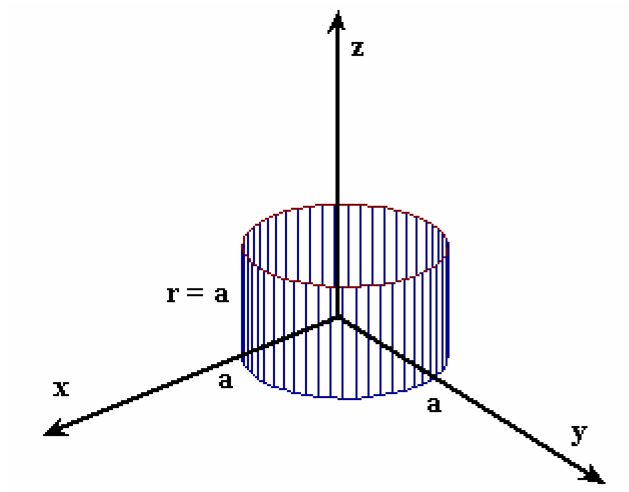


Figure 7.28

(ii) $\theta = \theta_0$ represents a half plane edge along the z-axis.

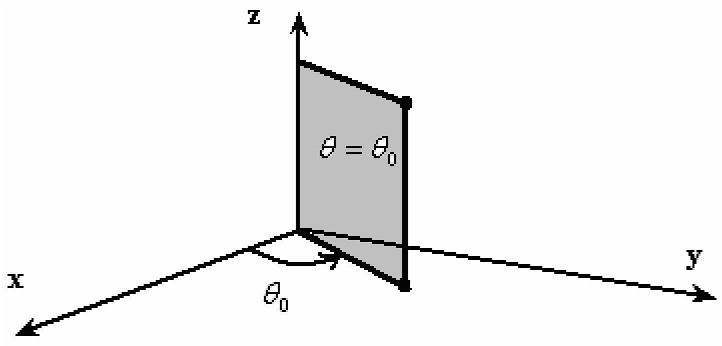


Figure 7.29

(iii) $z = z_0$ represents a horizontal plane. (See figure 7.4(i) where $z_0 = 0$)

<u>Example 20</u>

Sketch and use the cylindrical coordinates to describe the region in the first octant that is inside the cylinder $x^2 + y^2 = 1$ and below the plane $x + y + z = 4$.

<u>Solution</u>

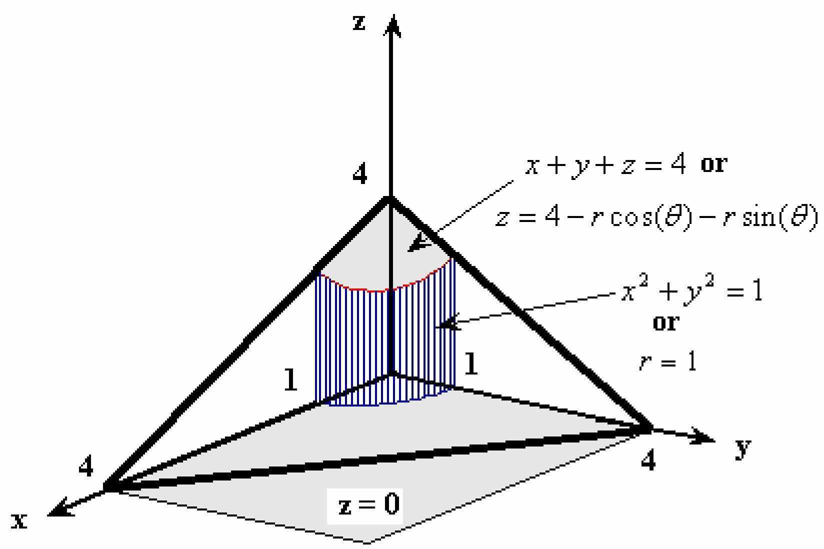


Figure 7.30

First, we express the given equations in terms of cylindrical coordinates:

$x^2 + y^2 = 1 \Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = 1$ or $r = 1$

$x + y + z = 4 \Rightarrow r\cos\theta + r\sin\theta + z = 4$

or $z = 4 - r\cos\theta - r\sin\theta$

Note: The bottom surface is the plane $z = 0$ and the top surface is plane

$z = 4 - r\cos\theta - r\sin\theta$.

$r = 1$ is the side cylinder.

The region is described as:

For θ and r fixed, z varies from $z = 0$ to $z = 4 - r\cos\theta - r\sin\theta$ (see figure 7.30)

For θ fixed, r varies from $r = 0$ to $r = 1$

θ varies from $\theta = 0$ to $\theta = \pi/2$

Example 21

Sketch and use cylindrical coordinates to describe the region inside the cone $z = 3\sqrt{x^2 + y^2}$ and below the plane $z = 3$

Solution

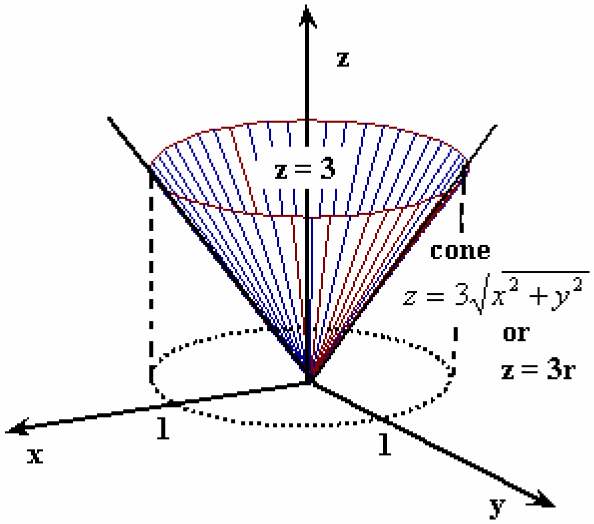


Figure 7.31

Clearly, the bottom is that of the cone

$$z = 3\sqrt{x^2 + y^2} = 3\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = 3r$$

The top surface is the plane $z = 3$

The top and bottom surfaces intersect to form a side edge contained in the cylinder given by

$z = 3 = 3r$ or $r = 1$

Hence the region is described as:

For r and θ fixed, z varies from $z = 3r$ to $z = 3$ (see figure 7.31)

For θ fixed, r varies from $r = 0$ to $r = 1$

θ varies from $\theta = 0$ to $\theta = 2\pi$

Example 22

Sketch and use cylindrical coordinates to describe the region that is inside the paraboloid $z = x^2 + y^2$ and below $z = 4$.

Solution

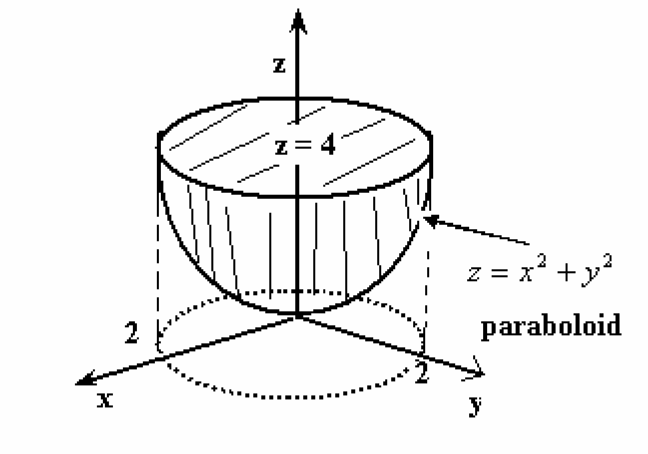


Figure 7.32

The bottom surface is that of the paraboloid $z = x^2 + y^2$ which has cylindrical equation $z = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

The top surface is the plane $z = 4$

The top and bottom surfaces intersect to form a side edge contained in the cylinder given by $z = r^2 = 4$ or $r = 2$.

Thus the region is described as:

For r and θ fixed, z varies from $z = r^2$ to $z = 4$.

For θ fixed, r varies from $r = 0$ to $r = 2$.

θ varies from $\theta = 0$ to $\theta = 2\pi$

Activity 5

Sketch and use cylindrical coordinates to describe the regions :

(a) inside $x^2 + y^2 = 4$, below $z = 2x + 2y$, in the first octant

(b) $z = 4 - x^2 - y^2, z = 0$

7.6.3 REGIONS IN THREE DIMENSIONAL SPACE - *SPHERICAL COORDINATES*

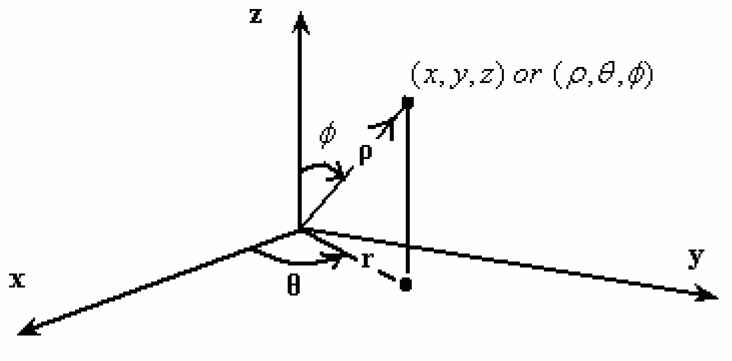


Figure 7.33

Let (x,y,z) be a point whose corresponding spherical coordinates are given by the parameters (ρ,θ,ϕ).

What are (i) ρ **(ii)** θ **(iii)** ϕ**?**

(i) The spherical coordinates ρ is the distance of the point *(x,y,z)* from the origin or $\rho=\sqrt{x^2+y^2+z^2}$. We denote this by ρ to distinguish it from polar coordinate *r* in the *xy*-plane.

(ii) θ is the same angle as with polar coordinates.

(iii) ϕ is the angle between the positive *z*-axis and the ray from the origin to the point (*x,y,z*).

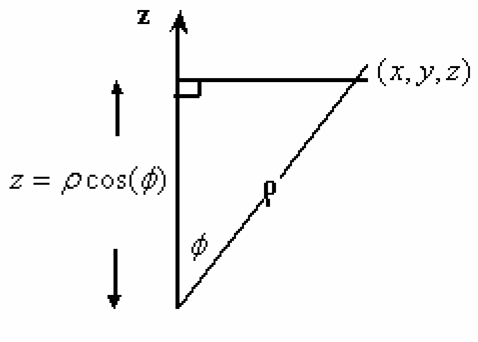


Figure 7.34

Figure 7.34 shows a simple derivation of $z=\rho\cos(\phi)$ by considering the projection of ρ onto the *z*-axis with ϕ included between the *z*-axis and the ρ line.

On the other hand, knowing that $\rho^2 = x^2 + y^2 + z^2$, we have

$$x^2 + y^2 = \rho^2 - z^2 = \rho^2 - \rho^2 cos^2\phi = \rho^2 sin^2\phi,$$

so that polar r $(= x^2 + y^2)$ is given by:

$$r = \sqrt{x^2 + y^2} = \rho \sin \phi.$$

Using $x = r\cos\theta$ and $y = r\sin\theta$, we obtain the following relationship between (x,y,z) and (ρ, θ, ϕ):

$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}}$$

The ρ coordinate is taken to be non-negative and $0 \leq \phi \leq \pi$.

The basic surfaces associated with spherical coordinates are :

(i) $\rho = a$ represents a sphere with center (0,0,0) and radius a

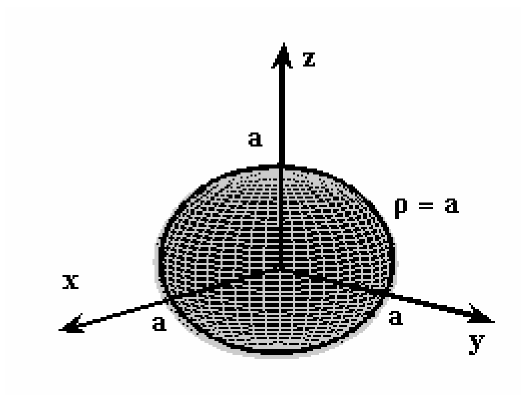


Figure 7.35

(ii) $\phi = \alpha$ represents a cone and note that $\alpha \neq \pi/2$ since $\phi = \pi/2$ is the xy plane

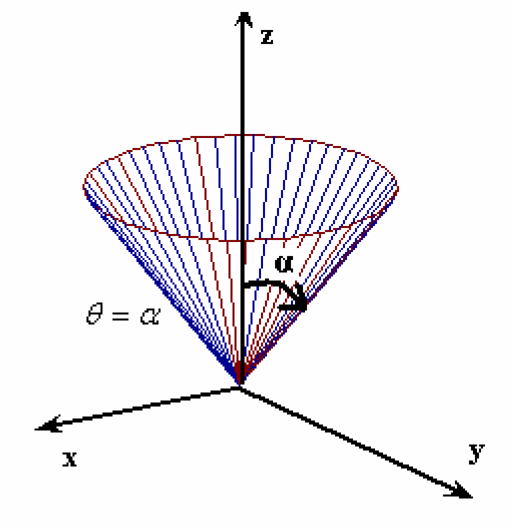


Figure 7.36

$\theta = \theta_0$ represents a half plane with edge along the z-axis as shown in Figure 7.33.

Example 23

Sketch and use spherical coordinates to describe the region that is inside both the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 9$.

Solution

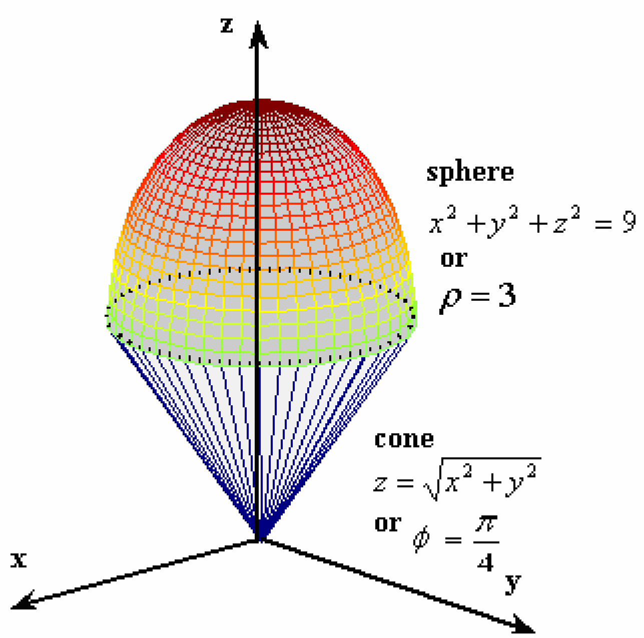


Figure 7.37

The spherical equation corresponding to $x^2 + y^2 + z^2 = 9$ is $\rho = 3$ (since $x^2 + y^2 + z^2 = \rho^2$)
Note that the figure 7.37 shows only half of the sphere, that is, a hemisphere.

The spherical equation corresponding to $z = \sqrt{x^2 + y^2}$ is obtained as follows:

$$z^2 = x^2 + y^2$$

$$\rho^2\cos^2\phi = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$$

$$\Rightarrow \quad \cos^2\phi = \sin^2\phi$$

$$\Rightarrow \quad \tan^2\phi = 1$$

or $\phi = \pi/4$ $\quad$ ($\tan\phi > 0$ here as $z > 0$)

The region is described as:

For θ and ϕ fixed, ρ varies from $\rho = 0$ to $\rho = 3$

For θ fixed, ϕ varies from $\phi = 0$ to $\phi = \pi/4$

θ varies from $\theta = 0$ to $\theta = 2\pi$.

Example 24

Sketch and use spherical coordinates to describe the region that is inside the sphere $x^2 + y^2 + z^2 = 4$ and outside $x^2 + y^2 = 1$

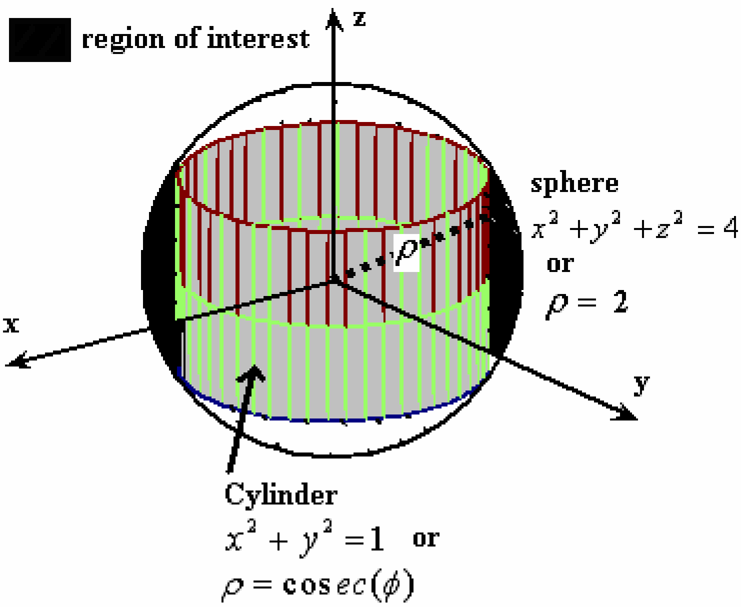


Figure 7.38

Note: An arrow shot from the origin enters the region at a point on the cylinder and leaves the region at a point on the sphere.

The spherical equations of the cylinder $x^2 + y^2 = 1$ is obtained as follows :

$(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = 1$

or $\rho^2 \sin^2\phi = 1$

or $\rho = \operatorname{cosec}\phi$

The spherical equation of the sphere is $\rho = 2$

Points of intersection of the sphere $\rho = 2$ and the cylinder $\rho = \operatorname{cosec}\phi$ satisfy

$\operatorname{cosec}\phi = 2$ which implies $\phi = \pi/6$ or $\phi = 5\pi/6$

Thus the region can be described as:

For θ and ϕ fixed, ρ varies from $\rho = \operatorname{cosec}\phi$ to $\rho = 2$

For θ fixed, ϕ varies form $\phi = \pi/6$ to $\phi = 5\pi/6$

θ varies from $\theta = 0$ to $\theta = 2\pi$

Activity 6

Sketch and use spherical coordinates to describe the regions:

(a) inside $x^2 + y^2 + z^2 = 25$, above $z = 3$

(b) $z = \sqrt{9 - x^2 - y^2}$, $z = 0$

7.7 TRANSFORMATIONS AND MAPPINGS

The main idea behind using polar, cylindrical, or spherical coordinates is that region of interest can be easily described and they can simplify work. We now extend the concept of transformation of coordinates in this section.

Let variables x and y connected with some other variables u and v by means of the relations

$$\left.\begin{aligned} u &= f_1(x,y) \\ v &= f_2(x,y) \end{aligned}\right\} \quad (1)$$

Note : These equations associate a point (x,y) in the xy-coordinate plane with the point $(u(x,y),v(x,y))$ in the uv-coordinate plane.

Where the functions (1) are continuous and possess continuous first partial derivatives with respect to x and y in some region of the xy-plane. Moreover, assume that the equations (1) can be solved for x and y in terms of u and v to yield

$$\left.\begin{aligned} x &= \phi_1(u,v) \\ y &= \phi_2(u,v) \end{aligned}\right\} \quad (2)$$

If u and v are assigned some fixed values, say u_o and v_o, the equations (*1*)

$u_o = f_1(x,y)$

$v_o = f_2(x,y)$

determine two curves which will intersect in a point (x_o,y_o) such that, from (2)

$x_o = \phi_1(u_o,v_o)$

$y_o = \phi_2(u_o,v_o)$

See Figure 7.39.

Thus the pair of numbers (u_o,v_o) determine the point (x_o,y_o) in the xy-plane.

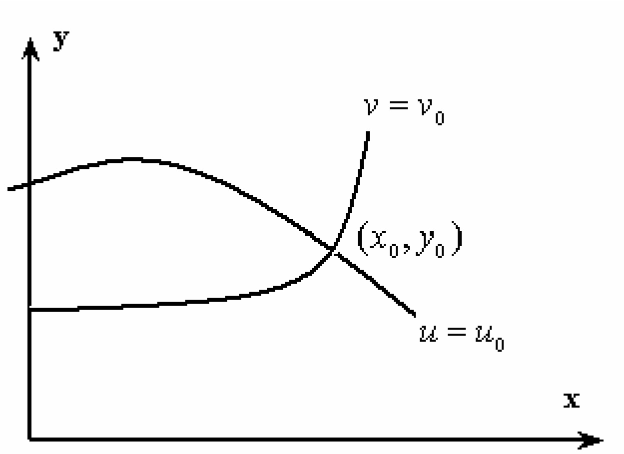


Figure 7.39

If u and v are assigned a sequence of constant values

$(u_1,v_1), (u_2,v_2), (u_3,v_3), \ldots\ldots\ldots\ldots, (u_n,v_n), \ldots\ldots$

there will be determined a network of curves that will intersect in the points

$(x_1,y_1), (x_2,y_2), (x_3,y_3), \ldots\ldots\ldots\ldots, (x_n,y_n), \ldots\ldots$

See Figure 7.40.

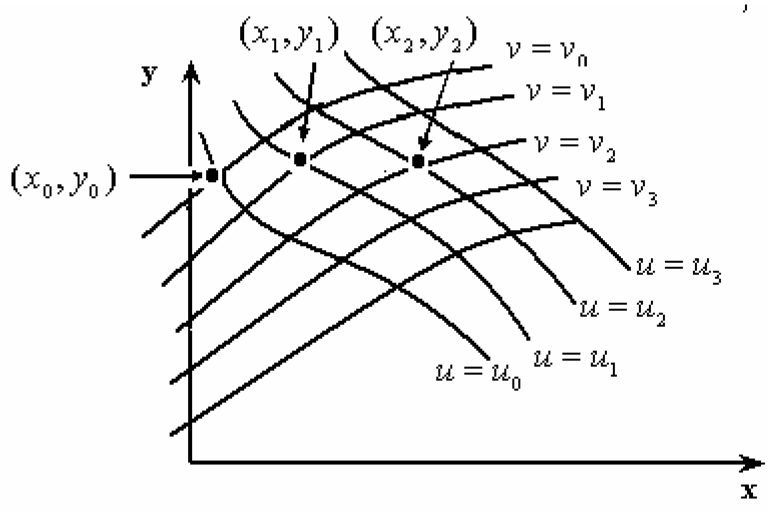


Figure 7.40

Corresponding to any point whose rectangular coordinates are (x,y), there will be a pair of curves $u = u_i$. and $v = v_i$ where u_i and v_i are constants. which pass through this point. The totality of numbers (u,v) defines a ***curvilinear coordinate system***, and the curves themselves are called ***coordinate lines***.

Example 25

Consider

$$\left.\begin{aligned} u &= \sqrt{x^2 + y^2} \\ v &= \tan^{-1}\frac{y}{x} \end{aligned}\right\} \quad (1)$$

The family of curves $u = const.$ is a family of circles. The family $v = const.$ is a family of straight lines passing through the origin. The coordinate system, in this case, is the ordinary polar coordinate system (See Figure7.41).

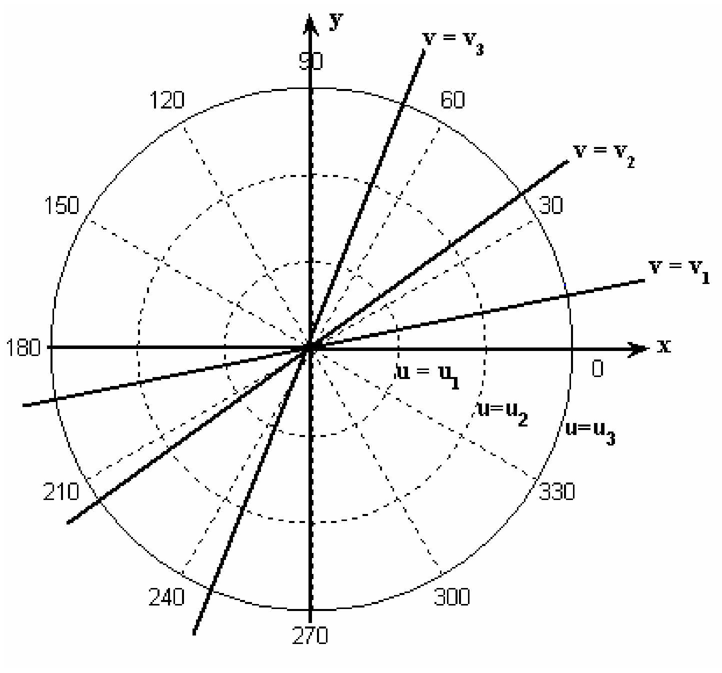


Figure 7.41

The **inverse** of the transformation (1) is

$$\left.\begin{aligned} x &= u\cos(v) \\ y &= u\sin(v) \end{aligned}\right\} \quad (2)$$

and the **Jacobian** of the transformation (2) is

$$\boxed{J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$$

Thus $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v = u$

The Jacobian of the transformation (1) is $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{\left(x^2+y^2\right)^{\frac{3}{2}}} + \frac{y^2}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{u}$$

Note that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\partial(x,y)/\partial(u,v)}$ for this particular case.

<u>Activity 7</u>

1. Discuss the curvilinear coordinate system defined by the relations

   x = u + v, y = u -v

   and describe the region in the *uv*-plane corresponding to the square

   x = 1, x =2, y = 1, y =2.

<u>Hint</u> : Utilize the following pictures :

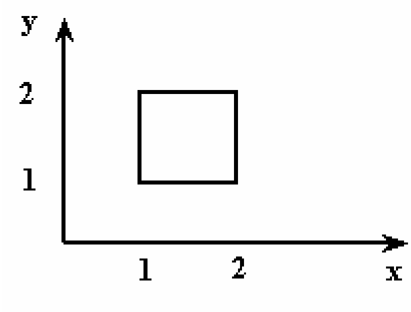


Figure 7.42 (i)

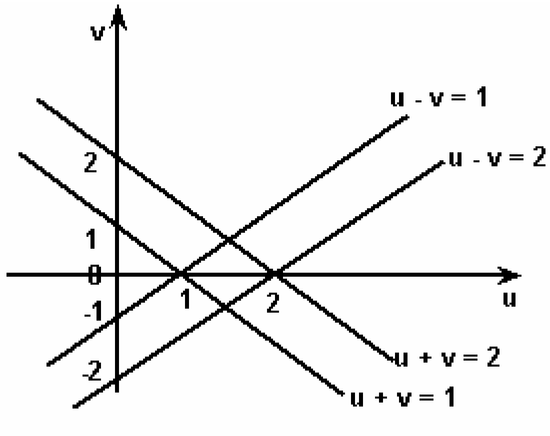


Figure 7.42 (ii)

<u>Hint</u> : Calculate the Jacobian J, and comment on the result.

<u>Theorem</u>

Given the transformation

$$\left.\begin{aligned} x &= f\,(u,v) \\ y &= g\,(u,v) \end{aligned}\right\}\ (1)$$

and the inverse transformation

$$\left.\begin{aligned} u &= F\,(x,y\,) \\ v &= G\,(x,y) \end{aligned}\right\}\ (2)$$

then the Jacobian j of (2) is the reciprocal of the Jacobian J of (1)

Let us consider the element of area dA in the curvilinear coordinate system (u,v) (see Figure 7.46) bounded by the quadrilateral p_1, p_2, p_3, p_4, the boundary of which if formed by the curves:

$u = f_1(x,y)$ $\qquad\qquad$ $u + du = f_1(x,y)$

$v = f_2(x,y)$ $\qquad\qquad$ $v + dv = f_2(x,y)$

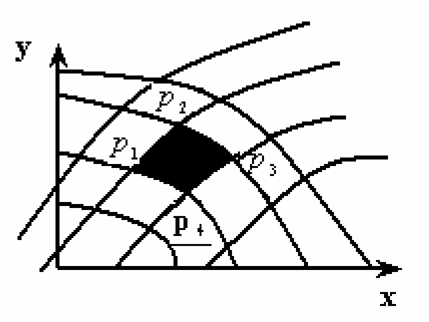


Figure 7.43

The positive quantities du and dv are assumed to be finite but may be chosen as small as desired.

The rectangular coordinates of the point $P_1(x_1, y_1)$ may be calculated from

$x_1 = \phi_1 (u,v)$

$y_1 = \phi_2 (u,v)$

Then it can be shown that $\boxed{dA = \pm J\, du\, dv}$ where $J = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$

Activity 8

1. Consider the mapping $x = au$, $y = bv$, where $a > 0, b > 0$
   a) Find out what region in the uv-plane corresponds to the region in the xy-plane bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

b) Calculate the Jacobian and state the geometrical significance of the result.

2. A mapping of the xy-plane into the uv-plane is defined by

$$u = \frac{x}{1-x-y} \qquad v = \frac{y}{1-x-y}$$

a) Find $\dfrac{\partial(u,v)}{\partial(x,y)}$ b) Find $\dfrac{\partial(x,y)}{\partial(u,v)}$

c) What are the u-curves and v curves in the xy-plane?

d) Find the region R in the xy-plane corresponding to the square in the uv-plane bounded by the lines

$U = -\frac{1}{2}$, $U = -1$, $V = -1$, $V = -\frac{3}{2}$

Recall that for $U = u(x,y)$, $v = v(x,y)$

$$dx \quad dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du \quad dv$$

For polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, one has

$$dx \quad dy = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr \quad d\theta = r \quad dr \quad d\theta \qquad (*)$$

In three dimensions, it may be shown that for

$U = u(x,y,z)$, $V = v(x,y,z)$, $\omega = (x,y,z)$

$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,\omega)}\right| du\,dv\,d\omega, \text{ where } \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} & \frac{dx}{d\omega} \\ \frac{dy}{du} & \frac{dy}{dv} & \frac{dy}{d\omega} \\ \frac{dz}{du} & \frac{dz}{dv} & \frac{dz}{d\omega} \end{vmatrix}$$

3. For $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$

show that

$$\boxed{dx\,dy\,dz = r^2 \sin\phi\,dr\,d\phi\,d\theta} \qquad (**)$$

(This is a transformation to spherical polar coordinates.)

We will be making a lot of use of (*) and (**) later when studying double and triple integration.

4. For $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Show that

$$\boxed{dx\,dy\,dz = r\,dr\,d\theta\,dz}$$

(This is a transformation to cylindrical coordinates and will also be very useful.)

7.8 SUPPLEMENTARY EXERCISES

1. Sketch the curves with equations

   (a) $z = 2,\ x^2 + y^2 = 2z$; (b) $y = 2,\ z = y^2 - x^2$; (c) $z = 1,\ z = y^2 - x^2$

2. Sketch the surfaces with equations

   (a) $2x + y + 3z = 6$; (b) $3x + 2z = 6$; (c) $z = x + 2y$

3. Sketch and describe the regions in three dimensional space using Cartesian coordinates

   (a) inside $x^2 + y^2 = 4$ and $y^2 + z^2 = 4$, in the first octant.

   (b) $y = 3,\ z = 2, z = 2x$, in the first octant.

   (c) $x= 2$, above $y = 2z$, below $y + 2z = 4$, in the first octant.

4. Sketch and use cylindrical coordinates to describe the regions:

   (a) inside $x^2 + y^2 = 9$, below $z = \sqrt{x^2 + y^2}$, above $z = 0$.

   (b) inside $x^2 + y^2 + z^2 = 16$, $x + y = 4$ ($x + y \leq 4$), in the first octant.

   (c) $x^2 + y^2 = 2x,\quad z = 4,\quad z = 0.$

5. Sketch and use spherical coordinates to describe the regions:

   (a) inside $x^2 + y^2 + z^2 = 1$, in the first octant

   (b) inside $x^2 + y^2 = 16$, below $z = \sqrt{x^2 + y^2}$, above $z = 0$

   (c) inside $z = 2\sqrt{x^2 + y^2}$, below $x + z = 2$

6. Sketch and describe the regions in three dimensional space using Cartesian coordinates

   a) inside $x^2 + y^2 + z^2 = 16$ but above $z = 1$ in the first octant.

   b) below $x + y + z = 5$ and above $z = 2$, in the first octant.

   c) outside $x^2 + y^2 = 16$ and inside $x^2 + y^2 = 25$ and below z = 5 in the first octant.

7. Sketch and use cylindrical coordinates to describe the regions:

   a) inside $x^2 + y^2 = 4$, below $z = \sqrt{x^2 + y^2}$, above $z = 1$.

   b) inside $x^2 + y^2 + z^2 = 25$, $x + y = 5$ ($x + y \leq 5$) in 1st octant.

   c) $x^2 + y^2 = 2x+2y-1$, $z = 5$, $z = 1$.

8. Sketch and use spherical coordinates to describe the regions:

   a) inside $x^2 + y^2 + z^2 = 4$, in the first octant

   b) inside $x^2 + y^2 = 36$, below $z=2\sqrt{x^2+ y^2}$ in the first octant

   c) inside $z=3\sqrt{x^2+ y^2}$, below $z = 2$ -y

7.9 SUMMARY

In this Unit, you have studied the sketching of curves and surfaces

KEYPOINTS:

1. The steps to sketch the surface are given in Section 7.4.

2. General equation of a plane in three dimensional space is

$$ax + by + cz + d = 0$$

where $(a, b, c) \neq (0, 0, 0)$

3. Recognise equation of a generalised cylinder.

4. **Quadric surfaces**: (a) $ax^2 + by^2 + cz^2 + dx + ey + fz + g = 0$

(b) Most of these surfaces can be sketched by drawing only one coordinate trace and a typical trace in a perpendicular plane.

5. Describe region in plane as:

For x fixed, y varies from $y = f_1(x)$ to $y = f_2(x)$
x varies from $x = a$ to $x = b$.

or

For y fixed, x varies from $x = g_1(y)$ to $x = g_2(y)$
y varies from $y = a$ to $y = b$.

6. Describe region in plane using polar coordinates:

For θ fixed, r varies from $r = r_1(\theta)$ to $r = r_2(\theta)$

θ varies from $\theta = \theta_1$ to $\theta = \theta_2$.

Also, $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$.

7. Describe regions in three-dimensional space as:

For x and y fixed, z varies from $z = h_1(x, y)$ to $z = h_2(x, y)$.

For x fixed, y varies from $y = g_1(x)$ to $y = g_2(x)$

x varies from $x = a$ to $x = b$.

8. Steps to sketch and describe regions in three dimensional space are given in Section 7.6.1.

9. (a) Cylindrical coordinates (r, θ , z) are given by

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

(and $x^2 + y^2 = r^2$).

(b) Describe region in three dimensional space using cylindrical coordinates as:

For r and θ fixed, z varies from $z = h_1(r,\theta)$ to $z = h_2(r,\theta)$

For θ fixed, r varies from $r = g_1(\theta)$ to $r = g_2(\theta)$

θ varies from $\theta = \theta_1$ to $\theta = \theta_2$.

10. (a) Spherical coordinates (ρ , θ , ϕ) are given by

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

(and $r = \sqrt{x^2 + y^2} = \rho\sin\phi$).

(b) Describe regions in three dimensional space using spherical coordinates as:

For θ and ϕ fixed, ρ varies from $\rho = f_1(\theta, \phi)$ to $\rho = f_2(\theta, \phi)$

For θ fixed, ϕ varies from $\phi = g_1(\theta)$ to $\phi = g_2(\theta)$ θ varies from $\theta = \theta_1$ to $\theta = \theta_2$.

11. For the transformation of coordinates

$$x = \phi_1(u, v)$$
$$y = \phi_2(u, v)$$

the Jacobian is given by

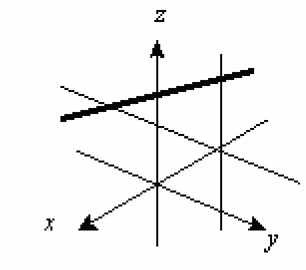
$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$dx\, dy = J\, du\, dv$.

7.10 ANSWERS TO ACTIVITIES AND SUPPLEMENTARY EXERCISES

Activity 1

(a) (b)



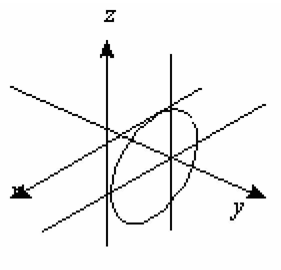


Figure 7.44 **Figure 7.45**

Activity 2

(a) (b)

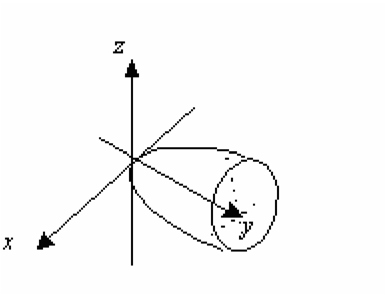


Figure 7.46

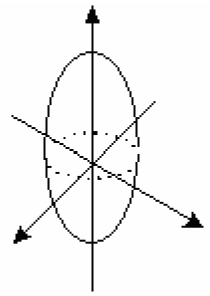


Figure 7.47

Activity 3

(a) For x fixed, y varies from $y = x$ to $y = x^2 + 1$,

x varies from $x = 0$ to $x = 1$

(b) For x fixed, y varies from $y = 0$ to $y = (2 - x) / 2$

x varies from $x = 0$ to $x = 2$

or

For y fixed, x varies from $x = 0$ to $x = 2 - 2y$

y varies from $y = 0$ to $y = 1$

Activity 4

(a) For x and y fixed, z varies from $z = x$ to $z = 3 - x - y$

For x fixed, y varies from $y = 0$ to $y = 3 - 2x$

x varies from $x = 0$ to $x = \dfrac{3}{2}$

(b) For x and y fixed, z varies from

$$z = -\sqrt{4x^2 + 4y^2} \quad \text{to} \quad z = -\sqrt{4x^2 + 4y^2}$$

For x fixed, y varies from $y = -\sqrt{1-x^2}$ to $y = \sqrt{1-x^2}$

x varies from $x = -1$ to $x = 1$

Activity 5

(a) For r and θ fixed, z varies from

$$z = 0 \text{ to } z = 2\,r\cos\theta + 2\,r\sin\theta$$

For θ fixed, r varies from $r = 0$ to $r = 2$

θ varies from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

(b) For r and θ fixed, z varies from $z = 0$ to $z = 4 - r^2$

For θ fixed, r varies from $r = 0$ to $r = 2$

θ varies from $\theta = 0$ to $\theta = 2\pi$

Activity 6

(a) For θ and ϕ fixed, ρ varies from $\rho = 3\sec\phi$ to $\rho = 5$

For θ fixed, ϕ varies from $\phi = 0$ to $\phi = \cos^{-1}\left(\frac{3}{5}\right)$

θ varies from $\theta = 0$ to $\theta = 2\pi$

(b) For θ and ϕ fixed, ρ varies from $\rho = 0$ to $\rho = 3$

For θ fixed, ϕ varies from $\phi = 0$ to $\phi = \dfrac{\pi}{2}$

θ varies from $\theta = 0$ to $\theta = 2\pi$

Activity 7

1. From $x = u + v$ and $y = u - v$, we obtain $x + y = 2u$ and $x - y = 2v$

The family u = constant are straight lines parallel to the line $y = -x$

The family v = constant are straight lines parallel to $y = x$

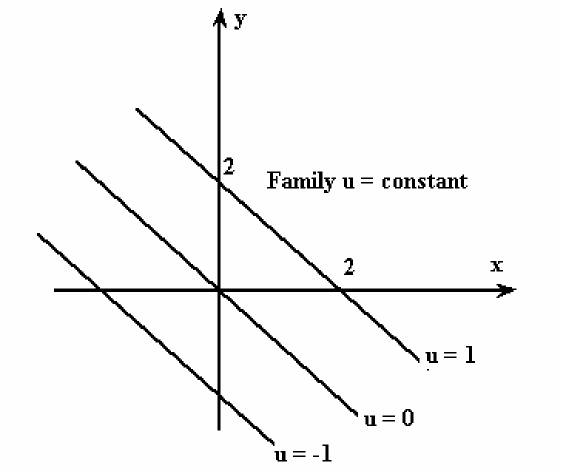


Figure 7.48

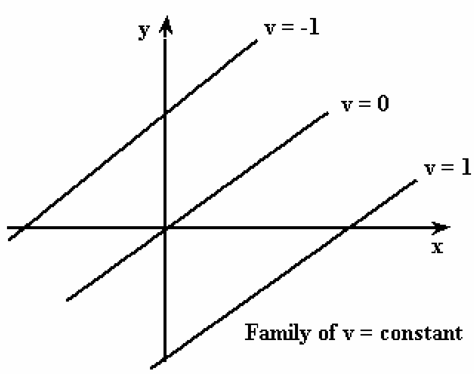


Figure 7.49

The square $x = 1$, $x = 2$, $y = 1$, $y = 2$ is mapped onto the square with vertices (1,0) $\left(\frac{3}{2},\frac{1}{2}\right)$, (2,0) and $\left(\frac{3}{2},-\frac{1}{2}\right)$

Activity 8

1. (a) $x = au$ $\qquad$ $y = bv$

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is mapped onto $\frac{(au)^2}{a^2}+\frac{(bv)^2}{b^2}=1$

That is, $u^2 + v^2 = 1$ which is the unit circle in the (u,v) plane.

(b) Jacobian $= \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & o \\ o & b \end{vmatrix} = ab$

Geometrical significance : The <u>unit circle</u> in the u-v plane, which has area π, is mapped onto an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the x-y plane, and from the relation $dA=Jdu\,dv$, we find that the area of the ellipse is πab.

2. $u = \dfrac{x}{1-x-y}$

(a) $\dfrac{\partial u}{\partial x} = \dfrac{(1-x-y)(11)-(x)(-1)}{(1-x-y)^2} = \dfrac{1-x-y+x}{(1-x-y)^2}$ and $\dfrac{\partial u}{\partial y} = \dfrac{(1-x-y)(0)-(x)(-1)}{(1-x-y)^2}$

$$\frac{1-y}{(1-x-y)^2} \qquad\qquad = \frac{x}{(1-x-y)^2}$$

Similarly,

$\dfrac{\partial v}{\partial x} = \dfrac{y}{(1-x-y)^2}$ and $\dfrac{\partial v}{\partial y} = \dfrac{1-x}{(1-x-y)^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{1-y}{(1-x-y)^2} & \dfrac{x}{(1-x-y)^2} \\ \dfrac{y}{(1-x-y)^2} & \dfrac{1-x}{(1-x-y)^2} \end{vmatrix}$$

$$= \frac{(1-x)(1-y)-xy}{(1-x-y)^2} = \frac{1-x-y+xy-xy}{(1-x-y)^2}$$

$$= \frac{1-x-y}{(1-x-y)^2} = \frac{1}{(1-x-y)}$$

From $u = \dfrac{x}{1-x-y}$ and $v = \dfrac{x}{1-x-y}$, we have $\dfrac{u}{v} = \dfrac{x}{y}$

$$uy = xv$$

$$y = \frac{xv}{u}$$

We replace $y = \dfrac{xv}{u}$ into $u = \dfrac{x}{1-x-y}$ to obtain $x = \dfrac{u}{1+u+v}$ and $y = \dfrac{xv}{u} = \dfrac{v}{1+u+v}$

(b) Calculate $\dfrac{\partial(x,y)}{\partial(u,v)}$ directly thus $\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

$$= = \begin{vmatrix} \dfrac{1+v}{(1+u+v)^2} & \dfrac{-u}{(1+u+v)^2} \\ \dfrac{-v}{(1+u+v)^2} & \dfrac{1+u}{(1+u+v)^2} \end{vmatrix} = \frac{1}{1+u+v}$$

or

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = 1-x-y = 1-\frac{u}{1+u+v}-\frac{v}{1+u+v} = \frac{1}{1+u+v}$$

(c) For u curves, put $\dfrac{x}{1-x-y}$ = constant = c, which is the same as $\left(1+\dfrac{1}{c}\right)x+y=1$ or the equation of the straight line. Similarly the *v*-curves is a family of straight lines with equations $\left(\dfrac{1}{c}+1\right)y+x=1$

The square in the *uv*-plane is shown below :

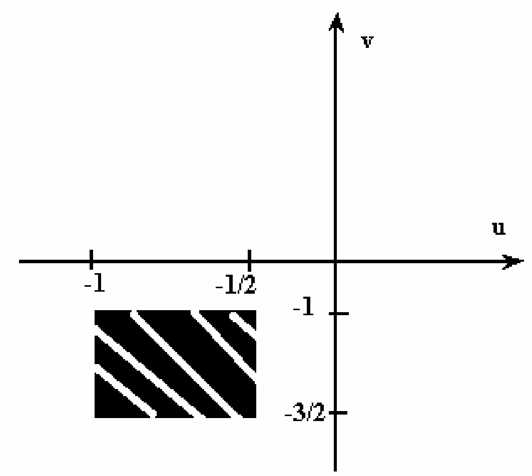


Figure 7.50

When $u = -½$ the u-curve in the x-y plane is $-x + y = 1$, by putting $c = -½$.

Similarly line $u = -1$ is mapped onto line $y = 1$

When $u = -1$, the u-curve is $x = 1$ and when $v = \frac{-3}{2}$, the v-curve is $\frac{1}{3}y + x = 1$

(d) The Region R in the xy-plane is shaded below

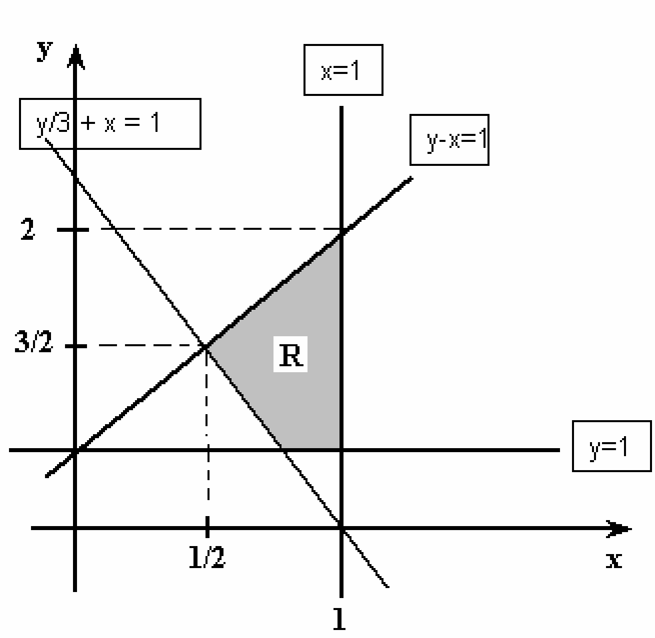


Figure 7.51

3. The Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$$= \begin{vmatrix} + & - & + \\ \sin\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

$$= \sin\phi\cos\theta \begin{vmatrix} r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ 0 & -r\sin\phi \end{vmatrix} - -r\sin\phi\sin\theta \begin{vmatrix} \sin\phi\sin\theta & r\cos\phi\sin\theta \\ \cos\phi & -r\sin\phi \end{vmatrix}$$

$$+r\cos\phi\cos\theta \begin{vmatrix} \sin\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & 0 \end{vmatrix}$$

$$= \sin\phi\cos\theta\left(-r^2\sin^2\phi\cos\theta\right) + r\sin\phi\sin\theta\left(-r\sin^2\phi\sin\theta - r\cos^2\phi\sin\theta\right)$$
$$+r\cos\phi\cos\theta\left[0 - r\sin\phi\cos\phi\cos\theta\right]$$

$$= -r^2\sin^3\phi\cos^2\theta + r\sin\phi\sin\theta - r\sin\phi\left(\sin^2\phi + \cos^2\phi\right) - r^2\sin\phi\cos^2\phi\cos^2\theta$$

$$= -r^2\sin\phi\cos^2\theta\left(\sin^2\phi + \cos^2\phi\right) - r^2\sin\phi\sin^2\theta(1)$$

$$= -r^2\sin\phi\left(\cos^2\theta + \sin^2\theta\right)$$

$$= -r^2\sin\phi$$

Using result $dx\ dy\ dz = \left|\dfrac{\partial(x, y, z)}{\partial(u, v, \omega)}\right| du \quad dv \quad d\omega$

we have

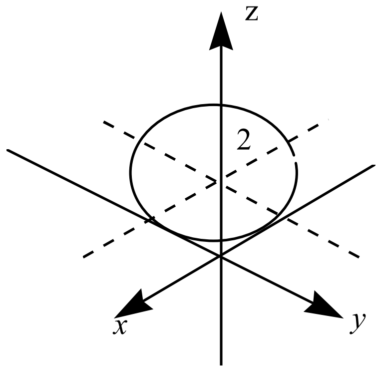
$dx\ dy\ dz = r^2\sin\phi\ dr\ d\theta\ d\phi$

4. Solution in the same way as in Activity 8 (3).

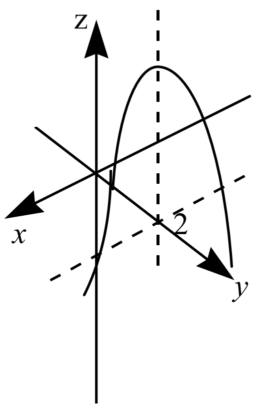
Answers to Supplementary exercises

1.

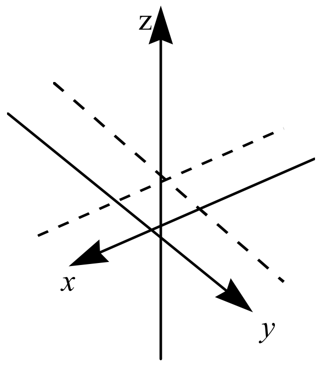
a)



b)

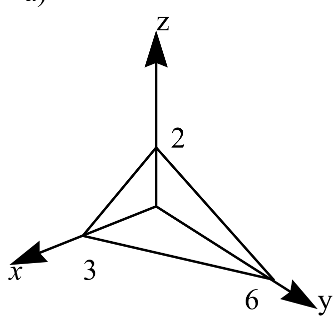


c)

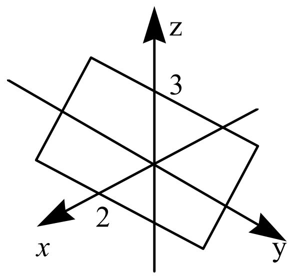


2.

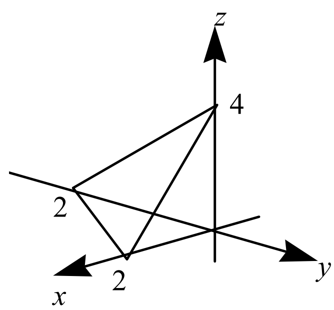
a)



b)

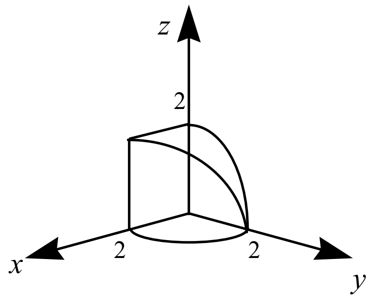


c)



3.

a)

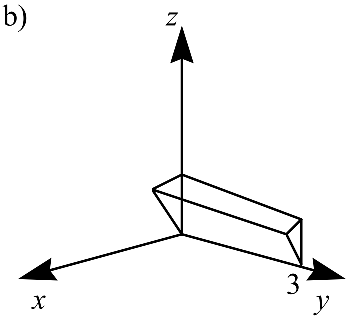


For x and y fixed, z varies from $z = 0$ to $z = \sqrt{4-y^2}$

For x fixed, y varies from $y = 0$ to $y = \sqrt{4-x^2}$

x varies from $x = 0$ to $x = 2$

b)

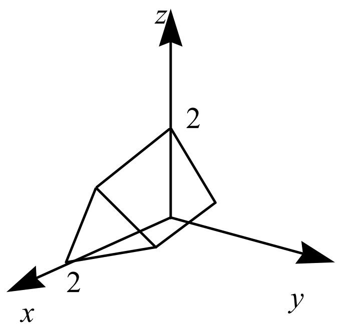


For x and y fixed, z varies from $z = 2x$ to $z = 2$

For x fixed, y varies from $y = 0$ to $y = 3$

x varies from $x = 0$ to $x = 1$

c)



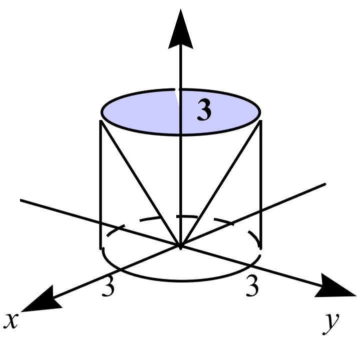
For x and y fixed, z varies from $z = y/2$ to $z = (4 - y)/2$

For x fixed, y varies from $y = 0$ to $y = 2$

x varies from $x = 0$ to $x = 2$

4.

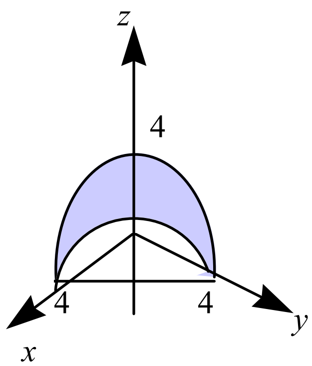
a)



For r and θ fixed, z varies from $z = 0$ to $z = r$

For θ fixed, r varies from $r = 0$ to $r = 3$

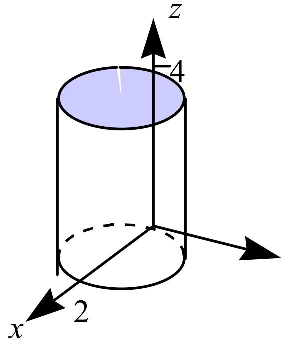
b)



For r and θ fixed, z varies from $z = 0$ to $z = \sqrt{16-r^2}$

For θ fixed, r varies from $r = 0$ to $r = 4/(\cos\theta + \sin\theta)$

c)



For r and θ fixed, z varies from $z = 0$ to $z = 4$

For θ fixed, r varies from $r = 0$ to $r = 2\cos\theta$

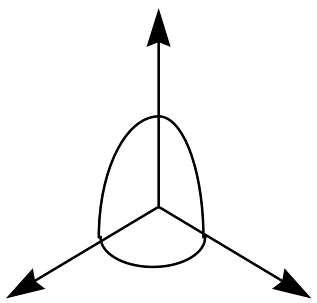
θ varies from $\theta = 0$ to $\theta = 2\pi$

θ varies from $\theta = 0$ to $\theta = \pi/2$

θ varies from $\theta = -\pi/2$ to $\theta = \pi/2$

5.

a)

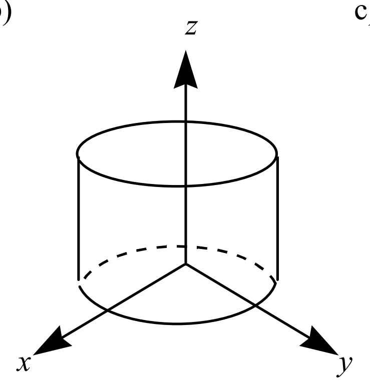


For θ and ϕ fixed, ρ varies from $\rho = 0$ to $\rho = 1$

For θ fixed, ϕ varies from $\phi = 0$ to $\phi = \pi/2$

θ varies from $\theta = 0$ to $\theta = \pi/2$

b)

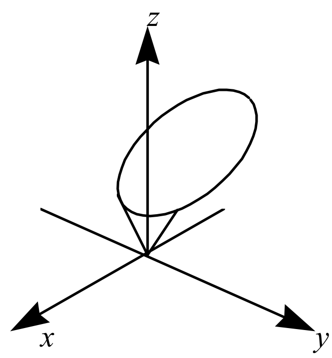


For θ and ϕ fixed, ρ varies from $\rho = 0$ to $\rho = 4\text{cosec}\,\phi$

For θ fixed, ϕ varies from $\phi = \pi/4$ to $\phi = \pi/2$

θ varies from $\theta = 0$ to $\theta = 2\pi$

c)



For θ and ϕ fixed, ρ varies from $\rho = 0$ to $\rho = 2/(\sin\phi\cos\theta + \cos\phi)$

For θ fixed, ϕ varies from $\phi = 0$ to $\phi = \tan^{-1}\left(\frac{1}{2}\right)$

θ varies from $\theta = 0$ to $\theta = 2\pi$

6. (a) For x and y fixed, z varies from z = 1 to $z = \sqrt{16 - x^2 - y^2}$
For x fixed, y varies from 0 to $y = \sqrt{16 - x^2}$
x varies from 0 to 4

(b) For x and y fixed, z varies from z = 2 to z = 5-x-y
For x fixed, y varies from y = 0 to y = -x+3
x varies from 0 to 3

(c) For x and y fixed, z varies from z = 0 to z = 5
For x fixed, y varies from $y = \sqrt{16 - x^2}$ to $y = \sqrt{25 - x^2}$
x varies from 0 to 5

7. (a) For r and θ fixed, z varies from $z = 1$ to $z = r$.

For θ fixed, r varies from $r = 1$ to $r = 2$.

θ varies from $\theta = 0$ to $\theta = 2\pi$

(b) For r and θ fixed, z varies from $z = 0$ to $z = \sqrt{25 - x^2 - y^2}$.

For θ fixed, r varies from $r = 0$ to $r = \dfrac{5}{\sin(\theta) - \cos(\theta)}$.

θ varies from $\theta = 0$ to $\theta = \pi/2$

(c) For r and θ fixed, z varies from $z = 1$ to $z = 5$.

For θ fixed, r varies from $r = 0$ to $r = 2(\cos(\theta) + \sin(\theta))$.

θ varies from $\theta = 0$ to $\theta = \pi/2$

8. (a) For θ and ϕ fixed, ρ varies from $\rho = 0$ to $\rho = 2$

For θ fixed, ϕ varies form $\phi = 0$ to $\phi = \pi/2$

θ varies from $\theta = 0$ to $\theta = \pi/2$

(b) For θ and ϕ fixed, ρ varies from $\rho = \sec\phi$ to $\rho = 6\operatorname{cosec}(\phi)$

For θ fixed, ϕ varies form $\phi = \tan^{-1}\left(\sqrt{\dfrac{1}{8}}\right)$ to $\phi = \pi/2 - \tan^{-1}\left(\dfrac{1}{6}\right)$

θ varies from $\theta = 0$ to $\theta = 2\pi$

(c) For θ and ϕ fixed, ρ varies from $\rho = 0$ to $\rho = \dfrac{2}{(\cos(\phi) + \sin(\phi)\sin(\theta))}$

For θ fixed, ϕ varies form $\phi = 0$ to $\phi = \tan^{-1}\left(\dfrac{1}{3}\right)$

θ varies from $\theta = 0$ to $\theta = 2\pi$