UNIT 5 LIMITS

Unit Structure

5.0 OVERVIEW

In this Unit, we define the limit of a real valued function and study its different properties.

5.1 LEARNING OBJECTIVES

After having studied this Unit, you should be able to do the following:

1. Define limit.
2. Apply the properties of limit to the valuate limit of a function of a real variable.
3. Use L'Hôpital's rule to evaluate limits.

5.2 INTRODUCTION

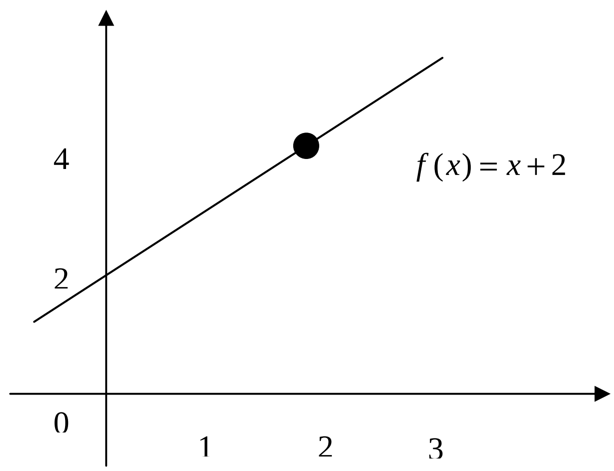
The concept of limits is very important to understand some properties of functions. In fact the condition for the continuity and differentiability of functions depends on the existence of their limit.

Usually we wish to find the limit of a function, say $f(x)$ at a point *c* or at extreme points such as $x = \infty$, or $x = -\infty$. Using mathematical notation, we write $\lim_{x \to c} f(x)$, $\lim_{x \to +\infty} f(x)$ or $\lim_{x \to -\infty} f(x)$ respectively.

5.3 Limits – Definition, Properties and Evaluation

Example 1

Consider the function $f(x) = x + 2$. How does $f(x)$ behave near $x = 2$?



We note that as *x* approaches 2 from the right, the function *f(x)* tends to the value of 4. We also find that when *x* approaches 2 from the left, the function *f(x)* tends to the value of 4 as well. Therefore we say that the limit of *f(x)* as *x* approaches 2 is 4, that is

$\lim_{x \to 2} f(x) = \lim_{x \to 2} x + 2 = 4$.

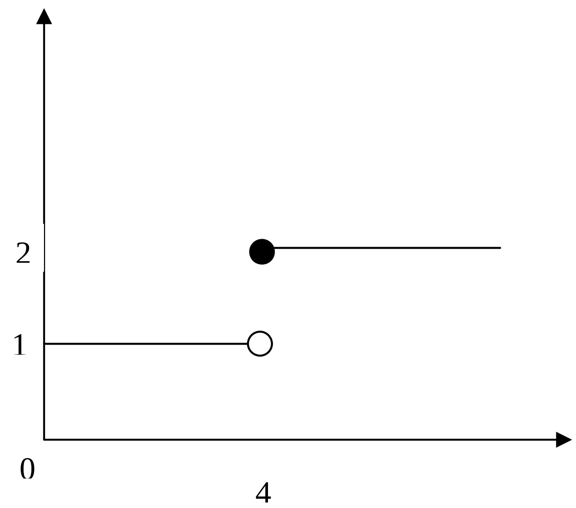
Definition

Let *f(x)* be defined on an open interval about x_0, except possibly at x_0 itself. If *f(x)* gets arbitrarily close to *L* for all *x* sufficiently close to x_0, we say that *f* approaches the limit *L* as *x* approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L.$$

Example 2

Consider the function $f(x) = \begin{cases} 2, \text{ if } x \geq 4 \\ 1, \text{ if } x < 4 \end{cases}$.



Let us evaluate the limit of *f(x)* at *x* = 4. When *x* approaches 4 from the right, the function f(*x*) tends to the value of 2, that is $\lim_{x \to 4^+} f(x) = 2$. However, when *x* approaches 4 from the left, f(*x*) tends to the value of 1, that is $\lim_{x \to 4^-} f(x) = 1$. Since $\lim_{x \to 4^+} f(x) \neq \lim_{x \to 4^-} f(x)$, $\lim_{x \to 4} f(x)$ does not exist.

Existence of limits

$$\lim_{x \to c} f(x) \text{ exists and is equal to } L \Leftrightarrow \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

Properties of Limits

Let *f(x)* and *g(x)* be two functions where $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = J$, $J \neq 0$.

Then

1. $\lim_{x \to c} [f(x) + g(x)] = L + J$

2. $\lim_{x \to c} [f(x) - g(x)] = L - J$

3. $\lim_{x \to c}[f(x).g(x)] = L\ J$

4. $\lim_{x \to c} kf(x) = kL$

5. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{J}$

6. If r and s are integers, then $\lim_{x \to c}[f(x)]^{\frac{r}{s}} = L^{\frac{r}{s}}$, provided $L^{\frac{r}{s}}$ is a real number.

Sandwich Theorem

Suppose $f(x) \le h(x) \le g(x)$ for all x in some open interval containing *d*, except possibly at *x* = *d* itself and also supposed that $\lim_{x \to d} f(x) = \lim_{x \to d} g(x) = L$. Then $\lim_{x \to d} h(x) = L$.

How do we find the limit of a function?

1. Substitute $x = c$ into the expression for $f(x)$. If $f(c)$ is real, check if $f(x) \to f(c)$ as $x \to c$.
2. In case $f(c)$ is not defined, obtain an equivalent expression for the function by algebraic simplifications and find the value of this expression at $x = c$.
3. In case step 1 and 2 does not work, then compute $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ using any of the above properties and calculate $f(x)$ for real numbers less than *c* and approaching *c*, and real numbers greater than *c* and approaching *c*.

Example 3

$$\lim_{x \to 2} x(x^3 + 2) = (\lim_{x \to 2} x)\ (\lim_{x \to 2} x^3 + \lim_{x \to 2} 2) = 2(8+2) = 20\ .$$

Example 4

To evaluate $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

Note that $\frac{\sqrt{x} - 2}{x - 4}$ is defined for all values of *x* except $x = 4$.

$$\frac{\sqrt{x}-2}{x-4}=\frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)}=\frac{1}{\sqrt{x}+2}.$$

Thus,

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{4}.$$

Example 5

$$\lim_{x\to\infty}\frac{x^3+10}{2x^3-2}=\lim_{x\to\infty}\frac{1+\frac{10}{x^3}}{2-\frac{2}{x^3}}=\frac{1}{2}.$$

Activity 1

1. Find the following limits:

(a) $\lim_{x\to -5}\frac{x^2}{5-x}$

(b) $\lim_{x\to -3}(5-x)^{\frac{4}{3}}$

(c) $\lim_{x\to -4}\frac{x^2-16}{x+4}$

(d) $\lim_{x\to 5}\frac{x-25}{\sqrt{x}-5}$;

(e) $\lim_{x\to\infty}\frac{21-3x^7}{5x^7+11}$

(f) $\lim_{x\to -7}(2x+5)$

(g) $\lim_{x\to 2}\frac{x+3}{x+6}$

(h) $\lim_{x\to 0}\frac{3}{\sqrt{3x+1}+1}$

2. Find the following limits:

(a) $\lim_{x\to 5}\frac{x-5}{x^2-25}$

(b) $\lim_{x\to 1}\dfrac{x^2+x-2}{x^2-1}$

(c) $\lim_{x\to 1}\dfrac{u^4-1}{u^3-1}$

(d) $\lim_{x\to 1}\dfrac{x-1}{\sqrt{x+3}-2}$

(e) $\lim_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$

Note:

If $\lim_{x\to c} f(x)$ yields $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then we have to use L'Hôpital's Rule.

5.4 L'HÔPITAL'S RULE

Theorem

Consider the function $\dfrac{f(x)}{g(x)}$. We can find the limit of this function for the following two cases.

Case 1

If $\lim_{x\to c} f(x)=0$ and $\lim_{x\to c} g(x)=0$, then $\lim_{x\to c}\dfrac{f(x)}{g(x)}=\lim_{x\to c}\dfrac{f'(x)}{g'(x)}=L$

Case 2

If $\lim_{x\to c} f(x)=\infty$ and $\lim_{x\to c} g(x)=\infty$, then $\lim_{x\to c}\dfrac{f(x)}{g(x)}=\lim_{x\to c}\dfrac{f'(x)}{g'(x)}=L$

In case (1) the limit is said to be of indeterminate form $\left(\dfrac{0}{0}\right)$, and in case (2) it is of indeterminate form $\left(\dfrac{\infty}{\infty}\right)$.

Note that **L'Hôpital's Rule** only applies to quotients of one of these two forms. The symbols ∞ *and* $-\infty$ are not real numbers.

Example 6

Use L'Hôpital's Rule to evaluate

(a) $\lim_{x\to 0}\frac{\sin 2x}{x}$;

(b) $\lim_{x\to 0}\frac{1-e^x}{x}$;

(c) $\lim_{x\to\infty}\frac{x^2}{e^x}$;

(d) $\lim_{x\to 0} x\ln x$.

Solution:

(a) $\lim_{x\to 0}\frac{\sin 2x}{x} \quad \left(\frac{\sin 0}{0}=\frac{0}{0}\right)$

$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{2\cos 2x}{1}=\frac{2\cos 0}{1}=2$ (by differentiating numerator and denominator).

(b) $\lim_{x\to 0}\frac{1-e^x}{x} \qquad \left(\frac{1-e^0}{0}=\frac{0}{0}\right)$

$=\lim_{x\to 0}\frac{-e^x}{1}=\frac{-e^0}{1}=-1.$

(c) Consider $\frac{x^2}{e^x}$ We find that as $x\to\infty$, we have $\left(\frac{\infty}{e^\infty}=\frac{\infty}{\infty}\right)$

$$\therefore \lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}\quad\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to\infty}\frac{2}{e^x}=\lim_{x\to\infty}2e^{-x}=0.$$

(d) $\lim\limits_{x\to 0}(x\ln x) \quad (= 0\cdot\infty)$

$$= \lim_{x\to 0}\frac{\ln x}{\frac{1}{x}} \quad \left(= \frac{-\infty}{\infty}\right)$$

$$= \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} \quad \text{(L'Hôpital's rule)}$$

$$= \lim_{x\to 0}(-x)$$

$$= 0.$$

Example 7

Determine the values of a and b such that

$$\lim_{x\to 0}\frac{a\cos x + (b+2)\sin x - a}{x^2} = 1.$$

Solution:

We find that when $x = 0$,

$$\frac{a\cos x + (b+2)\sin x - a}{x^2} = \frac{0}{0}.$$

Hence we apply L'Hôpital Rule to obtain

$$\lim_{x\to 0}\frac{a\cos x + (b+2)\sin x - a}{x^2} = \lim_{x\to 0}\frac{-a\sin x + (b+2)\cos x}{2x}$$

To use L'Hôpital Rule again, we require that (b+2) = 0. Putting $b = -2$, the limit becomes

$$\lim_{x\to 0}\frac{-a\sin x}{2x} = \lim_{x\to 0}\frac{-a\cos x}{2} = -\frac{a}{2}$$

Since we are given that the limit is equal to 1, then it follows that $a = -2$.

Hence we have $a = -2$ and $b = -2$.

Activity 2

1. Evaluate

(a) $\lim\limits_{x\to 0}\dfrac{\sin x}{x + x^2}$

(b) $\lim_{x\to\infty} \dfrac{x}{\ln x}$

(c) $\lim_{x\to\frac{1}{2}} \dfrac{\ln 2x}{2x-1}$

(d) $\lim_{x\to 0} \dfrac{e^{2x}-1}{x}$

(e) $\lim_{x\to 0} \dfrac{e^{x}-1}{\sin 2x}$

(f) $\lim_{x\to\frac{\pi}{2}} \dfrac{\tan 3x}{\tan x}$

(g) $\lim_{x\to 3\pi} \dfrac{1+\tan\frac{x}{4}}{\cos\frac{x}{2}}$

(h) $\lim_{x\to -2} \dfrac{x^2-4}{x^2+3x+2}$

(i) $\lim_{x\to 0^+} \dfrac{1-\sec x}{x^3}$

(j) $\lim_{x\to 0^-} \dfrac{1-\sec x}{x^3}$

2. Show that

(a) $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx} = \dfrac{a}{b}$

(b) $\lim_{x\to 0} \dfrac{\tan x}{x} = 1$

(c) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

(d) $\lim_{x\to 0} \dfrac{3\sin^{-1} x}{4x} = \dfrac{3}{4}$

(e) $\lim_{x\to 0} \dfrac{\sin(1+x)-\sin(1-x)}{x} = 2\cos 1$

5.5 SUMMARY

In this Unit you have studied the definition of limit for functions of one real variable and some techniques for evaluating the limit of a function at a point.

5.6 ANSWERS TO ACTIVITIES

Activity 1

1.
   (a) 5/2
   (b) 16
   (c) -8
   (d) -20/(-5+$\sqrt{5}$)
   (e) -3/5
   (f) -9
   (g) 5/8
   (h) 3/2
2.
   (a) 1/10
   (b) 3/2
   (c) 4/3
   (d) 4
   (e) 1/6

Activity 2

1.
   (a) 1
   (b) ∞
   (c) 1
   (d) 2
   (e) 1/2
   (f) 1/3
   (g) 1
   (h) 4
   (i) $-\infty$
   (j) ∞