Computer Architecture

ICT 1019Y Week 05 Lecture

Karnaugh Maps

Introduction to Karnaugh Maps

- Simplification of Boolean functions is good…
  - Produces simpler (and usually faster) digital circuits
- … but also time-consuming and error-prone
  - Easy to mis-use identities

Introduction to Karnaugh Maps

- K-Maps are an easy, systematic method for reducing Boolean expressions
  - Named after Maurice Karnaugh (engineer at Bell Labs in 1950's)
  - Invented a graphical way of visualizing and then simplifying Boolean expressions

Introduction to Karnaugh Maps

- A Kmap is a matrix representing a Boolean function
  - Rows and column headers represent the input values
  - Cells represent corresponding output values
- Input values are formatted as *minterms*
  - Minterm is a product term that contains all of the function's variables exactly once, either complemented or not complemented

Minterms

- For example, the minterms for a function having the inputs x and y are: $\bar{x}\bar{y}, \bar{x}y, x\bar{y}$, and xy
- Consider the Boolean function,

$$F(x,y) = xy + x\bar{y}$$

- Its minterms are:

Minterm	X	Y
$\bar{X}\bar{Y}$	0	0
$\bar{X}Y$	0	1
$X\bar{Y}$	1	0
XY	1	1

Minterms

- Function with three inputs?
  - Minterms are similar…
  - Just imagine counting in binary to find all the minterms…

Minterm	X	Y	Z
$\bar{X}\bar{Y}\bar{Z}$	0	0	0
$\bar{X}\bar{Y}Z$	0	0	1
$\bar{X}Y\bar{Z}$	0	1	0
$\bar{X}YZ$	0	1	1
$X\bar{Y}\bar{Z}$	1	0	0
$X\bar{Y}Z$	1	0	1
$XY\bar{Z}$	1	1	0
XYZ	1	1	1

Introduction to Karnaugh Maps

- A Kmap has a cell for each minterm
  - Cell for each line for the truth table of a function
- The truth table for the function F(x,y) = xy is shown along with its corresponding Kmap

$F(X,Y) = XY$

X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

X \ Y	0	1
0	0	0
1	0	1

Introduction to Karnaugh Maps

- Truth table and Kmap for the function F(x,y) = x + y
- This function is equivalent to the OR of all of the minterms that have a value of 1

F(X,Y) = X+Y

X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

$$F(x,y) = x + y = \bar{x}y + x\bar{y} + xy$$

X \ Y	0	1
0	0	1
1	1	1

Introduction to Karnaugh Maps

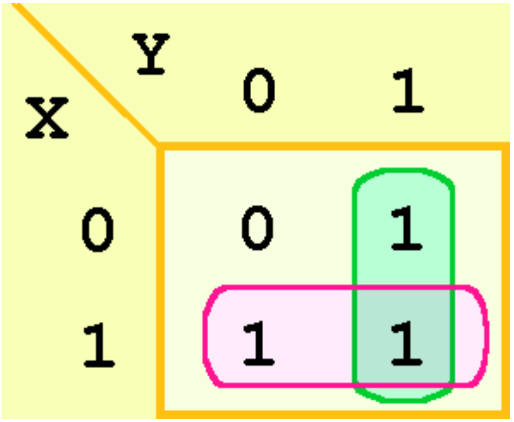
- Minterm function derived from Kmap was not in simplest terms
- Use Kmap to reduce expression to simplest terms
  - Find **adjacent 1's** in the Kmap that can be collected into groups that are **powers of two**

Two groups in this example:

X \ Y	0	1
0	0	1
1	1	1

Introduction to Karnaugh Maps

- Selected groups shown below
  - Groups are powers of two (# of elements)
  - Overlapping is OK!



Rules for Simplification

- Groupings can contain only 1's; no 0's
- Groups can be formed only at right angles
  - Diagonal groups are not allowed
- The number of 1's in a group must be a power of 2
  - A single 1 is OK then, but not three 1's!
- Groups must be made as large as possible
  - Otherwise simplification is incomplete
- Groups can overlap
- Groups can wrap around the sides of the Kmap

Kmap – Three Variables

- Extend to three variables? Easy!
- **Warning!** Note that the values for the yz combination at the top of the matrix form a pattern that is **not a normal binary sequence**
  - **Each position can only differ by 1 variable**

X \ YZ	00	01	11	10
0	$\overline{X}\,\overline{Y}\,\overline{Z}$	$\overline{X}\,\overline{Y}Z$	$\overline{X}YZ$	$\overline{X}Y\overline{Z}$
1	$X\overline{Y}\,\overline{Z}$	$X\overline{Y}Z$	XYZ	$XY\overline{Z}$

Kmap – Three Variables

- What do the values look like?
  - First row contains all minterms where x has a value of zero.
  - First column contains all minterms where y and z both have a value of zero

X \ YZ	00	01	11	10
0	$\overline{X}\,\overline{Y}\,\overline{Z}$	$\overline{X}\,\overline{Y}Z$	$\overline{X}YZ$	$\overline{X}Y\overline{Z}$
1	$X\overline{Y}\,\overline{Z}$	$X\overline{Y}Z$	XYZ	$XY\overline{Z}$

Kmap – Three Variables

- Example:

$$F(X,Y) = \overline{X}\,\overline{Y}Z + \overline{X}YZ + X\overline{Y}Z + XYZ$$

- Kmap:

X \ YZ	00	01	11	10
0	0	1	1	0
1	0	1	1	0

- What is the largest group of 1's that is a power of 2?

Kmap – Three Variables

- Look at the grouping closely
  - Changes in the variables x and y have no influence upon the value of the function
  - Thus, the function

$$F(X,Y) = \bar{X}\bar{Y}Z + \bar{X}YZ + X\bar{Y}Z + XYZ$$

  - **reduces to F(x) = z**

X \ YZ	00	01	11	10
0	0	1	1	0
1	0	1	1	0

You could verify this reduction with identities or a truth table

Kmap – Three Variables

- Example:

$$F(X,Y,Z) = \overline{X}\,\overline{Y}\,\overline{Z} + \overline{X}\,\overline{Y}Z + \overline{X}YZ + \overline{X}Y\overline{Z} + X\overline{Y}\,\overline{Z} + XY\overline{Z}$$

- Kmap:

X \ YZ	00	01	11	10
0	1	1	1	1
1	1	0	0	1

- **What are the largest groups of 1's that are a power of 2?**
  - **How many groups do you see?**

Kmap – Three Variables

- To make the **largest groups possible**, wrap around the sides
- **How do we interpret results?**
  - Green row?
  - Pink square?

X \ YZ	00	01	11	10
0	1	1	1	1
1	1	0	0	1

Kmap – Three Variables

- Green group – only the value of x is significant
  - Thus, $\overline{X}$
- Pink group – only the value of z is significant
- Our reduced function is: $F(X,Y,Z) = \overline{X} + \overline{Z}$

Recall that we had six minterms in our original function!

X \ YZ	00	01	11	10
0	1	1	1	1
1	1	0	0	1

Kmap – Four Variables

- Model can be extended to accommodate a four-input function
  - 16 minterms produced

WX \ YZ	00	01	11	10
00	$\overline{W}\overline{X}\overline{Y}\overline{Z}$	$\overline{W}\overline{X}\overline{Y}Z$	$\overline{W}\overline{X}YZ$	$\overline{W}\overline{X}Y\overline{Z}$
01	$\overline{W}X\overline{Y}\overline{Z}$	$\overline{W}X\overline{Y}Z$	$\overline{W}XYZ$	$\overline{W}XY\overline{Z}$
11	$WX\overline{Y}\overline{Z}$	$WX\overline{Y}Z$	$WXYZ$	$WXY\overline{Z}$
10	$W\overline{X}\overline{Y}\overline{Z}$	$W\overline{X}\overline{Y}Z$	$W\overline{X}YZ$	$W\overline{X}Y\overline{Z}$

Kmap – Four Variables

- Example: $F(W,X,Y,Z) = \overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z} + \overline{W}\,\overline{X}\,\overline{Y}Z + \overline{W}\,\overline{X}Y\overline{Z} + \overline{W}XY\overline{Z} + W\overline{X}\,\overline{Y}\,\overline{Z} + W\overline{X}\,\overline{Y}Z + W\overline{X}Y\overline{Z}$

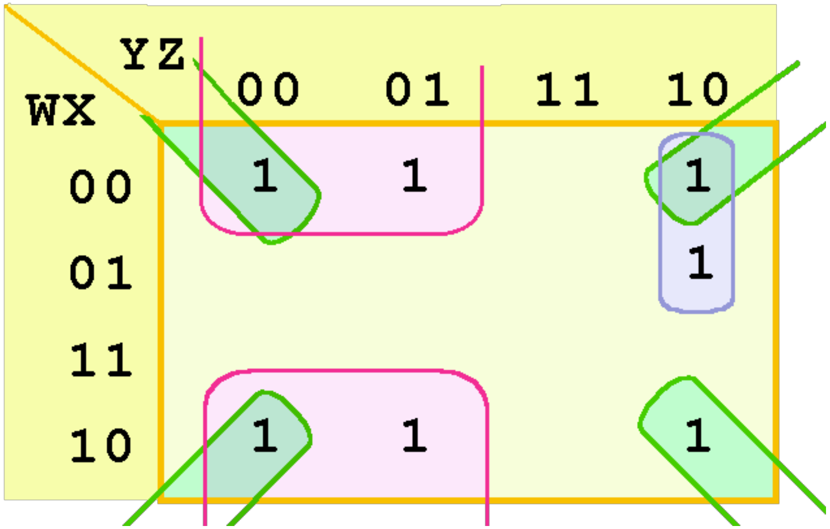
- Kmap (showing non-zero terms)

- **What largest groups should we select?**
  - Groups can overlap!
  - Groups can wrap!

WX \ YZ	00	01	11	10
00	1	1		1
01				1
11				
10	1	1		1

Kmap – Four Variables

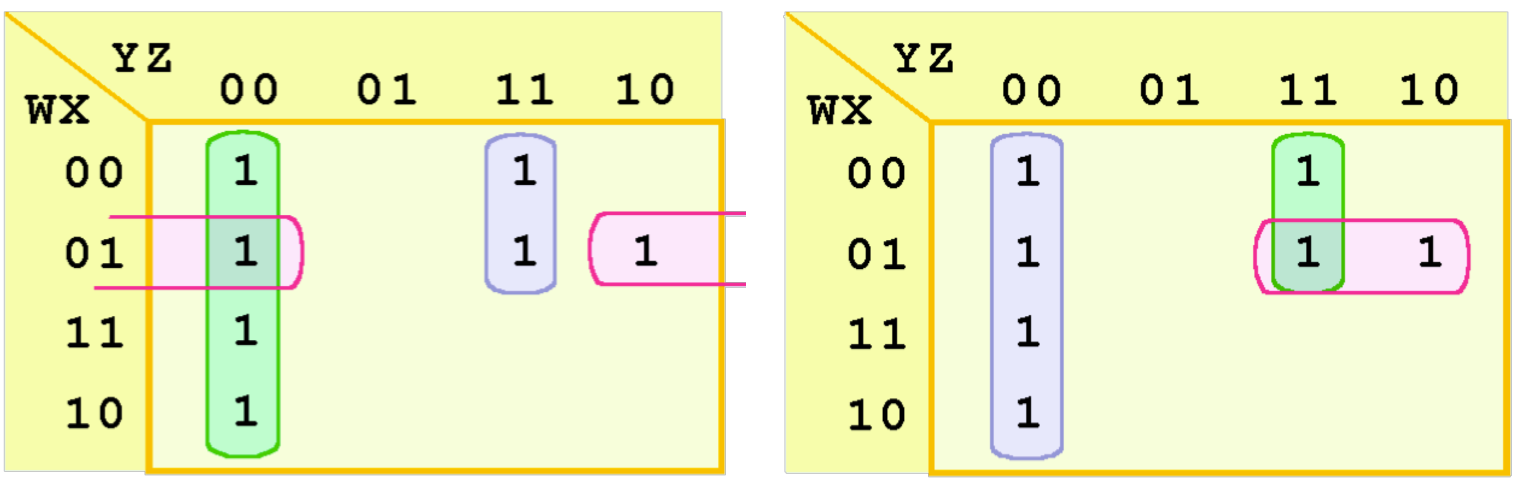
- Three groups
  1. Pink group that wraps top and bottom
  2. Green group that spans the corners
  3. Purple group entirely within the Kmap at the right



$$F(W,X,Y,Z) = \overline{X}\,\overline{Y} + \overline{X}\,\overline{Z} + \overline{W}Y\overline{Z}$$

Kmap – Four Variables

- Kmap simplification may not be unique
  - Possible to have different largest possible groups…
- The (different) functions that result from the groupings below are logically equivalent



Don’t Care Conditions

- Real circuits don’t always need to have an output defined for every possible input
  - Example: Calculator displays have 7-segment LEDs. These LEDs can display 2^7-1 patterns, but only ten of them are useful
- If a circuit is designed so that a particular set of inputs can never happen, we call this set of inputs a don’t care condition
  - Helpful for Kmap circuit simplification

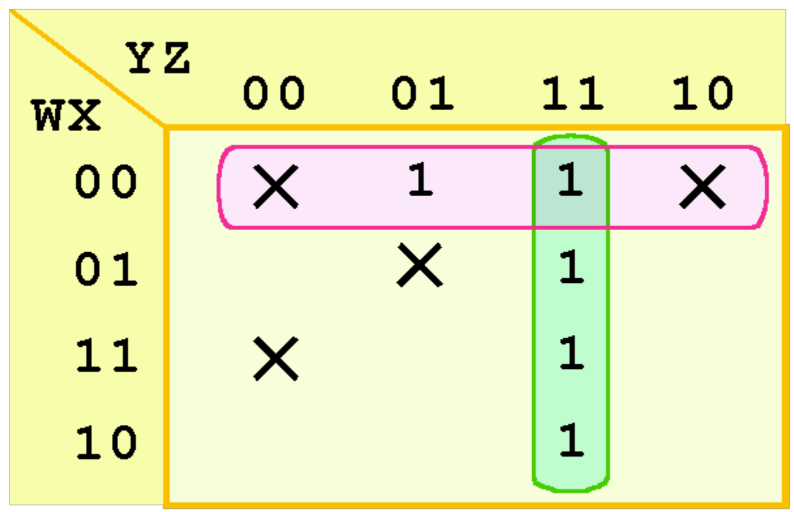
Don’t Care Conditions

- Represent a don’t care condition with an X
- Free to include or ignore the X’s when choosing groups

WX \ YZ	00	01	11	10
00	X	1	1	X
01		X	1	
11	X		1	
10			1	

Don’t Care Conditions

- Grouping option #1:



$$F(W,X,Y,Z) = \overline{W}\,\overline{X} + YZ$$

Don’t Care Conditions

- Grouping option #2:

WX \ YZ	00	01	11	10
00	×	1	1	×
01		×	1	
11	×		1	
10			1	

$$F(W,X,Y,Z) = \overline{W}Z + YZ$$

Don’t Care Conditions

- The truth table of

$$F(W,X,Y,Z)=\overline{W}\overline{X}+YZ$$

- differs from the truth table of

$$F(W,X,Y,Z)=\overline{W}Z+YZ$$

- However, the values for which they differ are the inputs for which we have don’t care conditions
  - **Either is an acceptable solution**