UNIT 9 LINE AND SURFACE INTEGRALS

Unit Structure

9.0 Overview
9.1 Learning Objectives
9.2 Line Integrals
9.2.1 Line Integrals Along Curves Defined Parametrically
9.2.2 Lateral Surface Area of a Generalised Cylinder
9.2.3 Application of Line Integrals to Vector Fields
9.3 Green's Theorem in the Plane
9.4 Properties of Line Integrals
9.4.1 Independence of Path of Line Integrals in the Plane
9.5 Surface Integrals
9.6 The Gauss Divergence Theorem
9.7 Stokes' Theorem
9.8 Supplementary exercises
9.9 Summary
9.10 Answers to Activities and Supplementary exercises

9 .0 OVERVIEW

In this unit 9, integrals over curves and surfaces are studied. Some important results that relate integrals over a region to an integral over the boundary of the region have been established.

9.1 LEARNING OBJECTIVES

When you have successfully completed this Unit, you should be able to do the following.

- Evaluate line integral of a function over a curve.
- Evaluate line integral involving vectors functions.
- Explain and use Green's theorem in a plane.
- Explain the conditions under which line integrals are independent of path.
- Integrate a function over a surface.
- Explain and use Gauss Divergence Theorem.
- Explain and use Stokes' Theorem
- Evaluate line integrals of a function over a curve in three-dimensional space.

9.2 LINE INTEGRALS

Instead of integrating a function over an interval [a, b], we will now integrate over a curve C in space. Such an integral is called a **line integral**.

Line integrals are used to measure sums of physical quantities as the values vary along points on a curve. An example of a physical application of line integral is in finding the total mass of a wire which lies along a piecewise smooth curve C of finite length and that has the mass density (mass per unit length) function of the wire at any point (x, y, z) on C is $f(x, y, z)$.

Let C be a curve found in a plane or in the three-dimensional space and let f be a function defined at each point of C. We partition C into small pieces ΔC of length ΔS and form the sum $\sum f\Delta S$ where f is evaluated at a point in ΔC (see Figure 9.1).

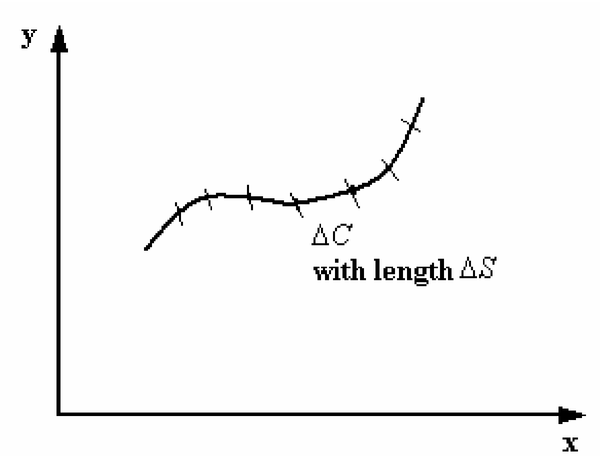


Figure 9.1

If the curve is 'smooth enough' and f is continuous on C, theory guarantees that the sums converge to a <u>single number</u> $\int_c \boldsymbol{f\, ds}$ as the lengths ΔS approach zero.

Note:

- The number $\int_c \boldsymbol{f\, ds}$ is called the <u>line integral of f over curve C</u>.
- $\int_c \boldsymbol{ds}$ <u>is the arc length of C</u>

where $ds^2 = dx^2 + dy^2$ or $ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$

Example 1

Find the line integral of $f(x)=(x+1)^4$ over curve C where C is the line $y = 2x$ from (0, 0) to (1, 2).

Solution

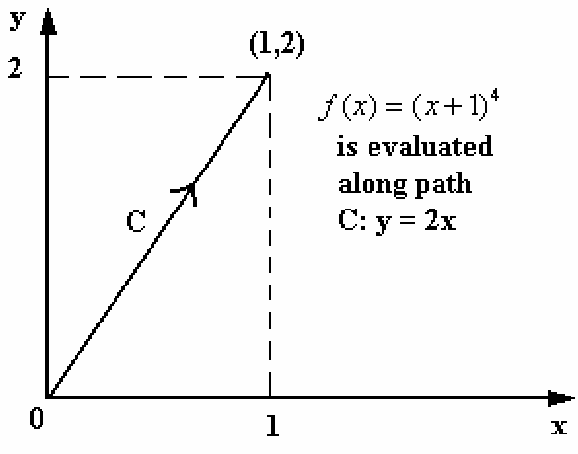


Figure 9.2

$$\int_c f\,ds = \int_{(0,0)}^{(1,2)} (x+1)^4\,ds$$

Since $y = 2x$ (from C), $\dfrac{dy}{dx} = 2$

$$\therefore\ ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{5}\,dx$$

$$\therefore\ \int_c f\,ds = \int_{x=0}^{x=1} (x+1)^4 \times \sqrt{5}\,dx = \sqrt{5}\left[\frac{(x+1)^5}{5}\right]_0^1 = \frac{31\sqrt{5}}{5}$$

■

Example 2

Find the arc length of the curve

(i) $y = 5\text{-}3x$, from x = 0 to x = 5,

(ii) $y=\sqrt{r^2-x^2}$ from (-r, 0) to (r, 0).

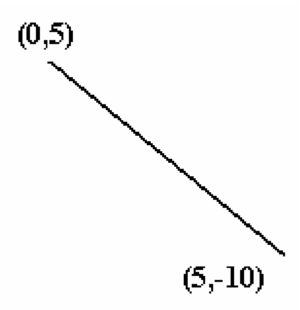
Solution

(i) $\int_c ds = \int_{x=0}^{x=5} ds$

Since y = 5-3x, $\frac{dy}{dx} = -3$

$\therefore\ ds = \sqrt{1+(-3)^2}\,dx = \sqrt{10}\,dx$

$\therefore\ Arclength = s = \int_{x=0}^{x=5} \sqrt{10}\,dx = 5\sqrt{10}$



The line segment has in fact a length of $s = \sqrt{(-10-5)^2 + (5-0)^2} = \sqrt{250} = 5\sqrt{10}$ between x = 0 and x = 5 thus confirms our answer obtained using line integral.

■

(ii) Since the curve C is $y = \sqrt{r^2 - x^2}$, $\frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}}$.

Hence $ds = \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\,dx = \frac{r}{\sqrt{r^2 - x^2}}dx$

Therefore the arc length is given by $s = \int_{(-r,0)}^{(r,0)} ds = \int_{x=-r}^{x=r} \frac{r}{\sqrt{r^2 - x^2}}dx$.

We next apply the substitution $x = r\sin(\theta)$ so that $\frac{dx}{d\theta} = r\cos(\theta)$ and $s = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} r\,d\theta = \pi r$.

By this time you should have remarked that it is the arc length of a semicircle with radius 1 unit (see figure 9.3) which is in deed πr .

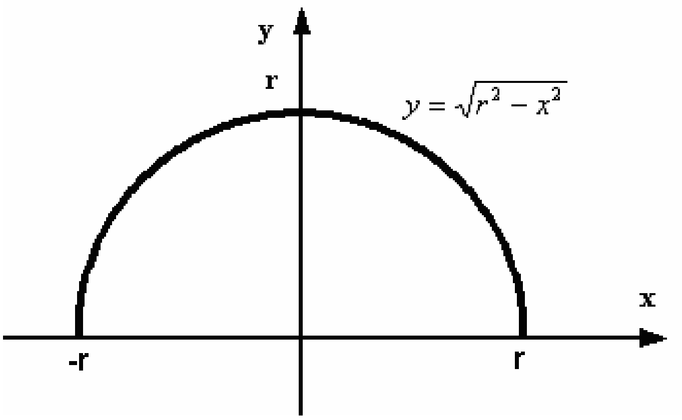


Figure 9.3

■

9.2.1 LINE INTEGRAL ALONG CURVE DEFINED PARAMETRICALLY

Suppose a smooth curve *C* is given in parametric form and that we can express the line integral of a continuous function *f* over *C* as an integral with respect to the parameter.

For instance, let us consider the curve in the plane that is given by the parametric equations:

$$C:\ x = x(t), \quad y = y(t), \qquad a \le t \le b,$$

where $x(t)$ and $y(t)$ have continuous first partial derivatives and $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \neq 0.$

If [a ,b] is divided into subintervals of length Δt, the points on the curve that correspond to the points of subdivision divide *C* into pieces of length Δs, then

$$\int_c f\, ds = \int_{t=a}^{t=b} f\left(x(t), y(t)\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\ .$$

This means that we can evaluate a line integral by substituting parametric values and then evaluating the resulting integral with respect to the real valued parameter.

Example 3

Evaluate

$$I = \int_c 2x^2 + 2y^2 \, ds,$$

where C is the curve given by $x = 3\cos(\theta)$ and $y = 3\sin(\theta)$, $0 \le \theta \le 2\pi$.

Solution

$x = 3\cos(\theta)$ implies that $\dfrac{dx}{d\theta} = -3\sin(\theta)$ and $y = 3\sin(\theta)$ implies that $\dfrac{dy}{d\theta} = 3\cos(\theta)$ so that

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3$$

and

$$\int_c f\,ds = \int_{\theta=0}^{\theta=2\pi} 2\left((3\cos(\theta))^2 + (3\sin(\theta))^2\right) 3d\theta = \int_{\theta=0}^{\theta=2\pi} 54 d\theta = 108\pi .$$

■

9.2.2 LATERAL SURFACE AREA OF A GENERALISED CYLINDER

The graph in three-dimensional space of an equation $F(x, y) = 0$ is a generalized cylinder with generating lines parallel to the z-axis. The lateral surface area of that part of the cylinder between the surfaces $Z = h_1(x, y)$ and $Z = h_2(x, y)$ is

$$\int_c \left[h_2 \left(x, y\right) - h_1 \left(x, y\right) \right] ds ,$$

where C is the curve given by the graph $F(x, y) = 0$ in the xy plane.

Example 4

Find the lateral surface area of that part of cylinder $x^2 + y^2 = r^2$ between the planes $z = 0$ and $z = \text{h}$.

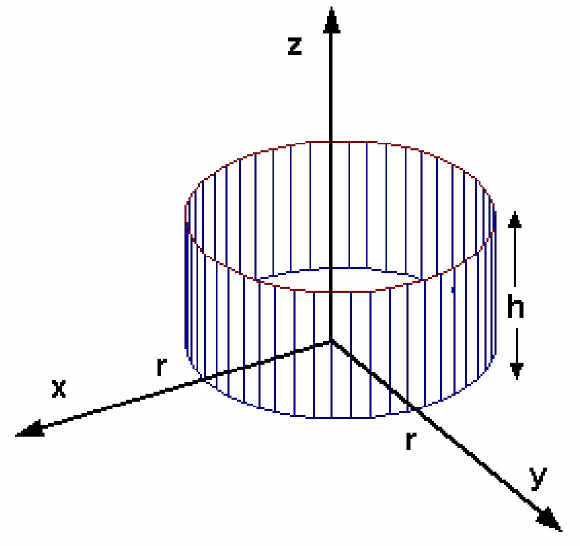


Figure 9.4

Solution

We need the parametric equation of the circle $x^2 + y^2 = r^2$ in the xy plane: $x = r\cos\theta$, $y = r\sin\theta$, $0 \leq \theta \leq 2\pi$ (**Note**: parameter θ is used instead of t).

$$\frac{dx}{d\theta} = -r\sin\theta, \quad \frac{dy}{d\theta} = r\cos\theta$$

$$\therefore ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\, d\theta = r d\theta$$

The lateral surface area is

$$S = \int_c (h-0)ds = \int_0^{2\pi} hr d\theta = rh\int_0^{2\pi} d\theta = 2\pi rh\ .$$

(It is a known result that the <u>curved surface area</u> of a cylinder of height h and base radius r is $2\pi rh$).

■

Example 5

Find the lateral surface area of that part of the cylinder $x^2 + y^2 = 4$ between the planes $z = 0$ and $x + 2y + z = 6$.

Solution

Parametric representation of circle $x^2 + y^2 = 4$ in xy plane is $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \leq \theta \leq 2\pi$.

See figure 9.4 where you can consider r = 2 for a sketch of the cylinder and this cylinder is bounded above by z = 6-x-2y (here $h_2(x, y) = 6 - x - 2y$) and below by z = 0 ($h_1(x, y) = 0$).

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = 2d\theta$$

$$\therefore Lateral\, Surface\, Area = \int_C (6 - x - 2y) - 0\, ds$$

$$= \int_0^{2\pi} [6 - 2\cos\theta - 2(2\sin\theta))] 2d\theta$$

$$= [12\theta - 4\sin\theta + 8\cos\theta]_0^{2\pi} = 24\pi$$

■

Activity 1

Evaluate the following line integrals:

(a) $\oint_C \left(x^2 - y^2\right) ds$

where C is the circle $x^2 + y^2 = 4$.

(b) $\int_{(0,0)}^{(1,1)} x ds$, where C is the line $y = x$.

9.2.3 APPLICATION OF LINE INTEGRALS TO VECTOR FIELDS

One of the most important applications of line integrals involve vectors and integrals of the form

$$\int_c \vec{F} \cdot \hat{T}\, ds,$$

where $\vec{F}$ is a vector field and $\hat{T}$ is the unit tangent vector of C.

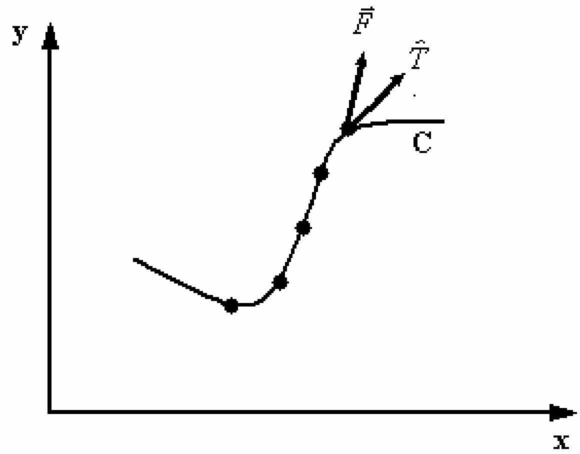


Figure 9.5

Note:

1. A vector-valued function $\vec{F}$ that is defined on a region R is called a vector-field on R. If each vector value of $\vec{F}$ represents a force, the vector field is also called a force-field.

2. If a function ϕ is defined and has partial derivatives in region R, then the vectors $\vec{F} = \vec{\nabla}\phi$ form a vector field on R. Such vector fields are called gradient fields or conservative fields.

Definition

Let $\vec{F}$ be a continuous vector field defined on a smooth oriented curve C. Then the **line integral** of $\vec{F}$ over C, denoted $\int_C \vec{F}.d\mathbf{r}$ is defined by

$$\int_C \vec{F}.d\mathbf{r} = \int_C \vec{F}.\hat{T}\, ds,$$

where $\hat{T}$ is the tangent vector at (x, y, z) for the given orientation of C, parametrized by a smooth vector-valued function $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

.

The line integral can be performed without calculating the unit tangent vector and the differential arc length of *C* and present a much simpler alternative formula as described below.

Suppose that

$\vec{F}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$ and the curve C defined parametrically as

$C : x = x(t),\; y = y(t),\; a \le t \le b$

The vector form of the curve is then

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}, a \le t \le b$

$\vec{r}(t)$ is in fact the position vector a point on C

and

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{i} \quad and \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ .$$

Note $\frac{d\vec{r}}{dt}$ has same direction as that of the <u>tangent to the curve</u> *C* at *t*.

Therefore, the unit tangent vector is

$$\hat{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

$$\vec{F}\cdot\hat{T} = \left(P\hat{i} + Q\hat{i}\right)\cdot\frac{\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{i}\right)}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} = \frac{P\frac{dx}{dt} + Q\frac{dy}{dt}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}\ .$$

From section 9.2, for the curve *C* defined parametrically as $x = x(t),\ y = y(t),\ a \le t \le b,$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt\ .$$

Thus

$$\int_c \vec{F}\cdot\hat{T}\,ds = \int \frac{P\dfrac{dx}{dt} + Q\dfrac{dy}{dt}}{\sqrt{\left(\dfrac{dx^2}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\int_c \vec{F}\cdot\hat{T}\,ds = \int_{t=a}^{t=b} \left(P\frac{dx}{dt} + Q\frac{dy}{dt} \right) dt.$$

Since

$$dx = \frac{dx}{dt}dt \quad and \quad dy = \frac{dy}{dt}dt\ ,$$

$$\boxed{\int_c \vec{F}\cdot\hat{T}\ ds = \int_c P\,dx + Q\,dy}.$$

We now consider some methods of evaluation of such line integrals.

Example 6

Evaluate

$$I = \int_{C\,(1,3)}^{(3,9)} \left(2x - 3y^2\right)dx + 4xy\,dy\,,$$

where C is the curve $y = 3x$.
Solution: Direct Substitution

$y = 3x \ \Rightarrow\ dy = 3dx$

Substituting $y = x$ and $dy = 3dx$ into I, we have

$$I = \int_{x=1}^{3} (2x - 3(3x))^2\,dx + 4x(3x)(3dx)$$
$$= \int_{x=1}^{3} \left(85x^2\right)dx = \frac{2210}{3}$$

■

Note

a) Integrals of the form $\int_C Pdx+Qdy$ are usually evaluated by substitution of parametric values. The values of the integrals do not depend on the particular smooth parametrization of C used to evaluate the integrals, except for direction.

b) The counter-clockwise direction will be termed positive direction; the clockwise direction will be termed a negative direction.

$$\int_{-C} \vec{F}\cdot\hat{T}\,ds = -\left(\int_C \vec{F}\cdot\hat{T}\,ds\right)$$

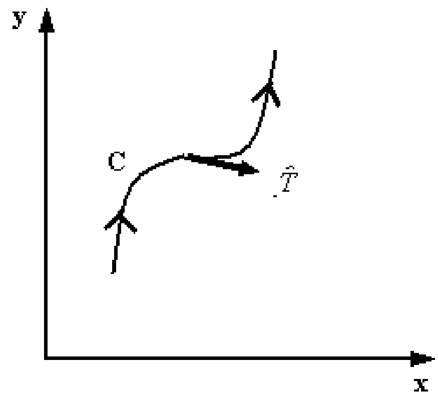


Figure 9.6(i)

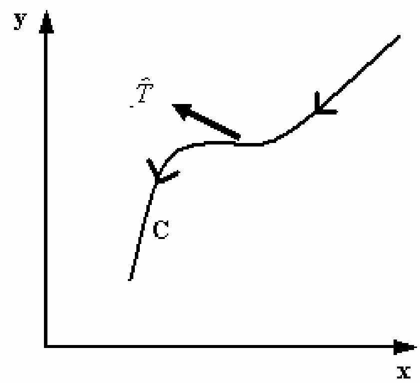


Figure 9.6 (ii)

so

$$\int_{-C} Pdx+Qdy = -\left(\int_C Pdx+Qdy\right)$$

Example 7

Evaluate

$$I = \int_{(2,3)}^{(6,27)} \left(x^2 - xy + 1\right)dx + \left(x^2 - 3y^3\right)dy$$

where C is given by $x = 2t,\; y = 3t^2$.

Solution: Substituting $x = 2t$, $y = 3t^2$

$dx = 2dt$, and $dy = 6tdt$,

$$I = \int_1^3 \left(4t^2 - 6t^3 + 1\right)2dt + \left(4t^2 - 3(3t^2)^3\right)6tdt$$
$$= \int_1^3 \left(-486t^7 + 12t^3 + 8t^2 + 2\right)dt = \frac{-1194620}{3}$$

■

Example 8

Evaluate

$$I = \int_{C}{}_{(1,0)}^{(-1,0)} (x^3 - y^3)dy$$

where C is the semicircle $y = \sqrt{1-x^2}$

Solution

We sketch the curve C: since $y = \sqrt{1-x^2}$, $y \geq 0$ (i.e C lies above x-axis); the initial (starting) point is (1,0) and the end point is (-1,0)

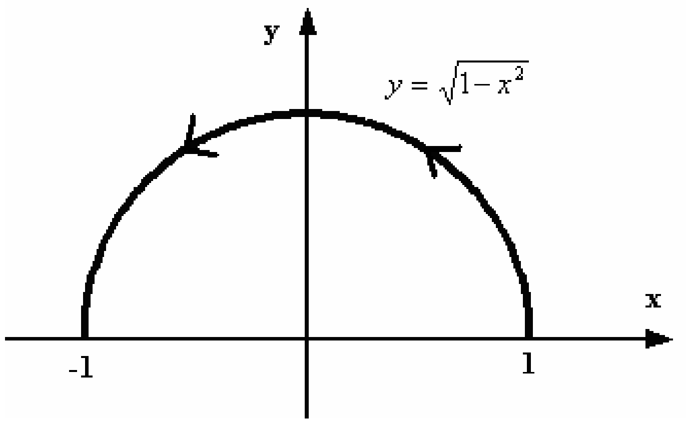


Figure 9.7

Using direct substitution:

$$y = \sqrt{1-x^2} \Rightarrow dy = \frac{-x}{\sqrt{1-x^2}}dx$$

$$\text{and} \qquad y^3 = \left(1-x^2\right)^{\frac{3}{2}}$$

$$\therefore \qquad I = \int_{x=1}^{x=-1}\left[x^3 - \left(1-x^2\right)^{\frac{3}{2}}\right]\left(\frac{-x}{\sqrt{1-x^2}}\right)dx$$

which looks quite difficult to integrate. So we use a parametric form of the semi-circle: $x = \cos(\theta)$, $y = \sin\theta$, $0 \le \theta \le \pi$.

Thus we have

$$\begin{aligned} I \qquad &= \int_0^{\pi}\left(\cos^3\theta - \sin^3\theta\right)\cos\theta d\theta \\ &= \int_0^{\pi}\cos^4 d\theta - \int_0^{\pi}\sin^3\theta d(\sin\theta) \\ &= \int_0^{\pi}\left(\frac{1+\cos 2\theta}{4}\right)^2 d\theta - \left[\frac{(\sin\theta)^4}{4}\right]_0^{\pi} \\ &= \int_0^{\pi}\left(1+\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta - 0 \\ &= \frac{3\pi}{8} \end{aligned}$$

■

If C is a piece-wise smooth curve defined as C: $x(t)$, $y(t)$, $a \le t \le b$, and if $P(x,y)$ and $Q(x,y)$ are continuous along C, then

$$\int_c Pdx + Qdy = \int_{c_1} Pdx + Qdy + \int_{c_2} Pdx + Qdy + \ldots\ldots + \int_{c_{n+1}} Pdx + Qdy$$

where

$C_1 : x(t),\ y(t),\ a \le t \le t_1$
$C_2 : x(t),\ y(t),\ t_1 \le t \le t_2$
.
.
.
$C_{n+1} : x(t),\ y(t),\ t_n \le t \le b$

If the path of integration is a simple closed curve (traced just once), then we use $\oint$ or $\oint_c$ instead of $\int$. .

Example 9

Evaluate

$$I = \oint (y+1)^2 dx + (y - x^2)\, dy$$

when C is a triangle with vertices (2,0), (2,2), (0,0).

Solution

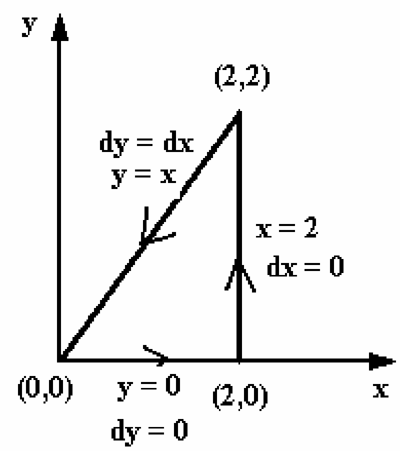


Figure 9.8

Now,

$$\oint_C = \int_{\substack{(0,0)\\ c_1}}^{(2,0)} + \int_{\substack{(2,0)\\ c_2}}^{(2,2)} + \int_{\substack{(2,2)\\ c_3}}^{(0,0)}$$

Since C_1 is the curve $y = 0 \Rightarrow dy = 0$ and substituting $y = 0$ and $dy = 0$, we have

$$\int_{C_1} (y+1)^2 dx + (y - x^2)\, dy = \int_{x=0}^{x=2} 1\, dx = 2$$

Since C_2 is the curve $x = 2 \Rightarrow dx = 0$ and substituting $x = 2$ and $dx = 0$, we have

$$\int_{C_2} (y+1)^2 dx + (y - x^2)\, dy = \int_{y=0}^{y=2} (y-4)\, dy = -6$$

Since C_3 is given by $y = x$ or $dy = dx$ and applying substitution, we have

$$\int_{C_3} (y+1)^2 dx + (y - x^2)\, dy = \int_{x=2}^{x=0} (x+1)^2\, dx + (x - x^2) dx = \int_{x=2}^{x=0} (3x+1)\, dx = -8$$

Hence,

$$I = \oint (y+1)^2 dx + (y - x^2)\, dy = 2 + -6 + -8 = -12$$

■

Note:

1. If $\vec{F}$ represents the force acting on a particle moving along arc AB, then the total work associated with $\vec{F}$ is

$$\int_A^B \vec{F} \cdot \hat{T}\, ds = \int_A^B P\, dx + Q dy$$

where $\vec{F} = P(x,y)\, \hat{i} + Q(x,y)\, \hat{j}$

2. If $\vec{F}$ represents the velocity of a liquid then $\oint_c \vec{F} \cdot \hat{T}\, ds$ is called the circulation of $\vec{F}$ around the simple closed curve C.

If $\int_c \vec{F} \cdot \hat{T}\, ds = 0$, then $\vec{F}$ is called irrotational.

Example 10

If a force F=$(3y+2x)\hat{i}$+$(x^2+2yx)\hat{j}$ displaces a particle in the *xy*-plane from (0,3) to (2,7) along the curve $y=x^2+3$. Find the work done.

Solution

$$\text{F}=Pi+Qj=(3y+2x)\hat{i}+(x^2+2yx)\hat{j}$$

$$P=(3y+2x) \text{ and } Q=(x^2+2yx)$$

$\therefore$ Work done

$$\int_{c\ (0,3)}^{(2,7)} \vec{F}\cdot\hat{T}ds=\int_{c\ (0,3)}^{(2,7)} Pdx+Qdy=\int_{c\ (0,3)}^{(2,7)}(3y+2x)dx+(x^2+2xy)dy$$

Using direct substitution:

$y=x^2+3$ and $dy=2xdx$, we then obtain

$$\text{Work done} \quad =\int_{x=0}^{x=2} 4x^4+2x^3+15x^2+2x+9dx=\frac{478}{5}$$

■

Example 11

If $\vec{F}=\dfrac{-3y\hat{i}+2x\hat{j}}{x^2+y^2}$ and C is the circle $x^2+y^2=4$ traversed counter-clockwise, find the circulation of $\vec{F}$ around C.

Solution

Given $\vec{F}=\dfrac{-3y}{x^2+y^2}\hat{i}+\dfrac{2x}{x^2+y^2}\hat{j}$, then

$$P=\frac{-3y}{x^2+y^2} \text{ and } Q=\frac{2x}{x^2+y^2}.$$

Parametric form of C is $x = 2\cos\,\theta, y = 2\sin\theta,\ 0 \leq \theta \leq 2\pi$

$\therefore$ Circulation =

$$\oint_C \vec{F}\cdot\hat{T}\,ds = \oint_C P\,dx + Q\,dy = \oint_C \frac{-3y}{x^2+y^2}dx + \frac{2x}{x^2+y^2}dy$$

$$= \oint \frac{-3\sin\theta}{2}(-2\sin\theta)\,d\theta + \frac{\cos\theta}{1}(2\cos\theta)\,d\theta \quad .$$

$$= \int_0^{2\pi}(\sin^2\theta + 2)\,d\theta = \int_0^{2\pi}\left(-\frac{\cos(2\theta)}{2} + \frac{5}{2}\right)d\theta = 5\pi$$

■

Consider a fluid flow model with fluid flowing along the *xy*-plane with a velocity vector $\vec{F}$. If C is a simple closed curve, then the unit tangent vector of C is

$$\hat{T} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}\,.$$

The vector obtained by a clockwise rotation of $\hat{T}$ by $\pi/2$ is

$$\hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\,.$$

Note that $\hat{T}.\hat{n} = 0$ showing that they are indeed perpendicular each other.

If the region enclosed by C is on the left as C is traced, then $\hat{n}$ will point outward: $\hat{n}$ is called the outer unit normal of the region enclosed by C.

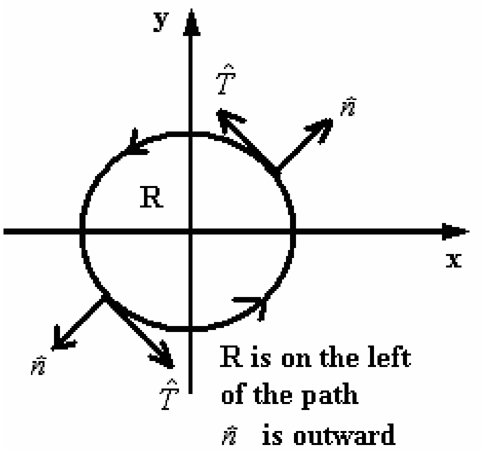


Figure 9.9

Since the dot product of $\vec{F}\bullet\hat{n}$ is the component of velocity vector $\vec{F}$ in the direction $\hat{n}$, the amount of fluid per unit time that flows in the direction of $\hat{n}$ across a small piece ΔC of the curve C with length ds is $\vec{F}\bullet\hat{n}\,ds$. The net flow across C is called the <u>outward flux</u> of $\vec{F}$ across C.

If $\vec{F}= P\hat{i}+Q\hat{j}$, then

$$\vec{F}\cdot\hat{n}=\left(P\hat{i}+Q\hat{j}\right)\cdot\left(\frac{dy}{ds}\hat{i}-\frac{dx}{ds}\hat{j}\right)$$
$$=P\frac{dy}{ds}-Q\frac{dx}{ds}\ and$$

The <u>outward flux</u> is given by

$$\boxed{\int_c \vec{F}\cdot\hat{n}\,ds = \int_c Pdy - Qdx}.$$

<u>Example 12</u>

Find the outward flux of $\vec{F}=(x-3y)\hat{i}+(x+4y)\hat{j}$ in terms of a across the curve

$C:\ x = a\cos t,\ y = a\sin t,\ 0 \le t \le 2\pi$

where $a > 0$.

<u>Solution</u>

C is a circle centre (0,0) and radius a. Figure 9.10 indicates an outward flow across C at each point of C, so we should expect the flux to be positive.

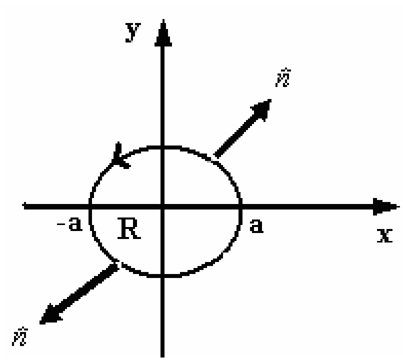


Figure 9.10

The flux is $\int_C \vec{F}\cdot\hat{n}ds$ where

$$\int_C \vec{F}\cdot\hat{n}ds = \int_C Pdy - Qdx = \int_C (x-3y)dy - (x+4y)dx \cdot$$

$$\vec{F} = (x-3y)\hat{i} + (x+4y)\hat{j}$$

Given that the parametric equations of C are $x = a\cos(t)$, $y = a\sin(t)$, $0 \le t \le 2\pi$, we have $dx = -a\sin(t)\, dt$ and $dy = a\cos(t)\, dt$.

Outward flux =

$$\int_0^{2\pi} [(a\cos(t) - 3a\sin(t))a\cos(t) - (a\cos(t) + 4a\sin(t))(-a\sin(t))]dt$$

$$= \int_0^{2\pi} a^2[1 + 3\sin^2(t) - 2\sin(t)\cos(t)]dt$$

$$= \int_0^{2\pi} a^2[1 + 3\left(\frac{1-\cos(2t)}{2}\right) - \sin(2t)]dt$$

$$= \int_0^{2\pi} a^2[\frac{5}{2} + \frac{3\cos(2t)}{2} - \sin(2t)]dt = 5\pi a^2$$

■

The line integral of $\vec{F}$ over a curve C in three dimensional space is considered in Section 9.8.

Activity 2

Evaluate the following integrals

(a) $\int_{C\,(0,-1)}^{(0,1)} y^2dx + x^2dy$,

where C is the semi-circle $x = \sqrt{1-y^2}$.

(b) $\int_{C\,(0,0)}^{(2,4)} ydx + xdy$,

where C is the parabola $y = x^{2.}$

Activity 3

Evaluate the following line integrals

(a) $\oint_C y^2 dx + xy dy$,

where C is the square with vertices (1,1), (-1,1), (-1,-1), (1,-1).

(b) $\oint y \; dx - x dy$,

where C is the circle $x^2 + y^2 = 1$.

9.3 GREEN'S THEOREM IN THE PLANE

If $P(x,y)$, $Q(x,y)$, $\dfrac{\partial P}{\partial y}$ and $\dfrac{\partial Q}{\partial x}$ are continuous, single-valued functions over a closed region R, bounded by the curve C such that direction of each part of C is chosen to have R on the left as the boundary is traced, then

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

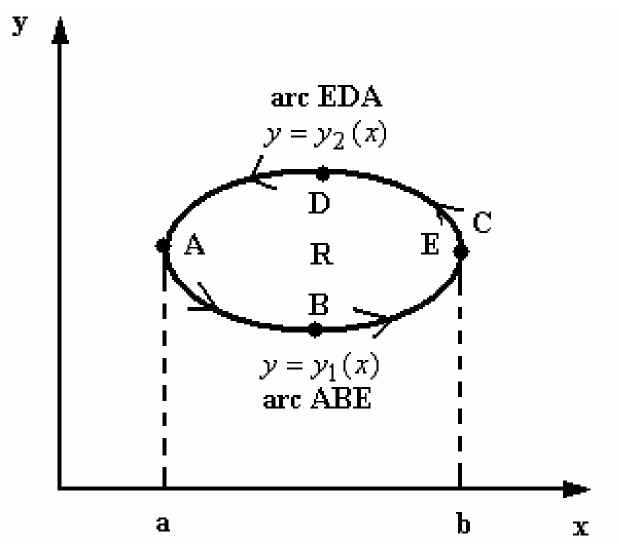


Figure 9.11

Proof

Let the curve C be divided into two curves $C_1(ABE)$ and $C_2(EDA)$ with equations $y = y_1(x)$ and $y = y_2(x)$ respectively.

Then using double integral:

$$\iint_R \frac{\partial P}{\partial y} dxdy = \int_{x=a}^{x=b} \int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial P}{\partial y} dydx$$

since region R can be described as:

For x fixed, y varies from $y = y_1(x)$ to $y = y_2(x)$
x varies from $x = a$ to $x = b$.

$$\begin{aligned}
\iint_R \frac{\partial P}{\partial y} dxdy &= \int_{x=a}^{b} \left[P(x,y)\right]_{y_1(x)}^{y_2(x)} dx = \int_{x=a}^{x=b} \left[P(x,y_2) - P(x,y_1)\right] dx \\
&= -\int_{x=b}^{x=a} P(x,y_2)dx - \int_{x=a}^{b} P(x,y_1)dx \\
&= -\left[\int_b^a P(x,y)dx + \int_a^b P(x,y)dx\right] \\
&= -\left[\int_{C_2(ABE)} P(x,y)dx + \int_{C_1(EDA)} P(x,y)dx\right] \\
&= -\oint_C P(x,y)dx
\end{aligned}$$

Thus,

$$\iint_R \frac{\partial P}{\partial y} dxdy = \oint_C P(x,y)dx \qquad (1)$$

Similarly, we can show that

$$\iint_R \frac{\partial Q}{\partial x} dxdy = -\oint_C P(x,y)dy \qquad (2)$$

Thus from (1) and (2)

$$\oint Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Example 13

Using Green's theorem, evaluate $\oint_C \left(\frac{1}{2} x^3 y^2 dx + (x^2 + y)dy \right)$ where C is the boundary, described counter clockwise, of the triangle with vertices (0,0), (2,0), (2,6).

Solution

Step 1:

Sketch the curve C in order to obtain and describe the region R (see figure 9.12).

The region R can be described as:

For x fixed, y varies from $y = 0$ to $y = x$
x varies from $x = 0$ to $x = 1$

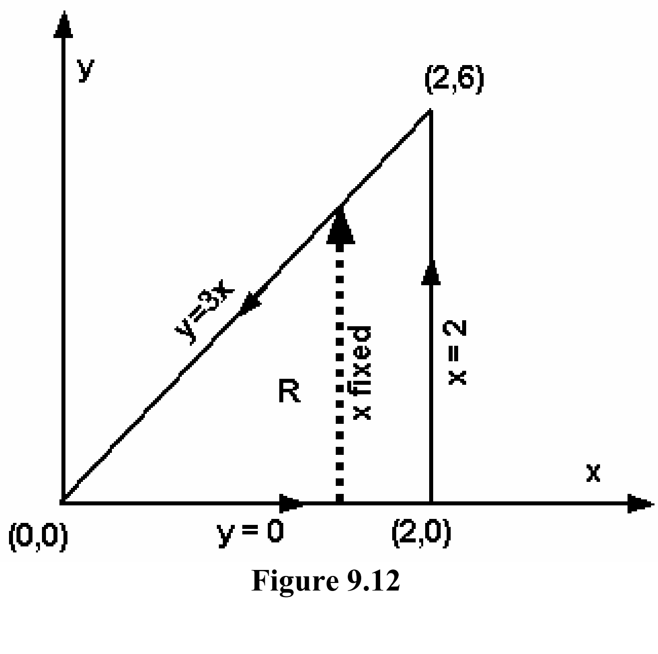


Figure 9.12

Step 2:

Here $P(x,y)=\frac{1}{2}x^3y^2$ and $Q(x,y)=x^2+y$

$$\frac{\partial P}{\partial y}=x^3y,\quad \frac{\partial Q}{\partial x}=2x$$

$$\therefore \int_C \frac{1}{2}x^3y^2\,dx+(x^2+y)dy=\iint\limits_R\left(2x-x^3y\right)dxdy$$

$$=\int_{x=0}^{2}\int_{y=0}^{y=3x}2x-x^2dydx$$

$$=\int_0^2(6x^2-3x^3)dx=4$$

■

<u>Example 14</u>

Use Green's theorem to evaluate

$$\oint_c\left(\sin(x)+x^2y\right)dx+\left(3x^{\frac{3}{2}}+2\cos(y)\right)dy$$

where C is the square formed by the lines

$$y=\pm 2, x=0, x=4$$

<u>Solution</u>

Step 1:

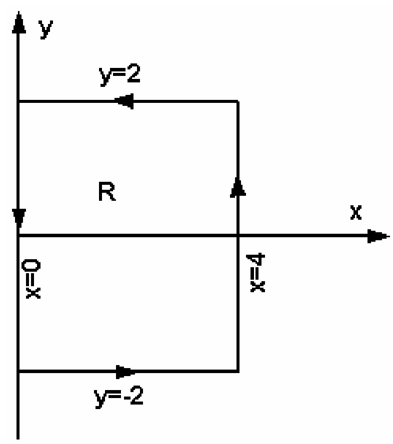


Figure 9.13

The region R can be described as:

For x fixed, y varies from $y = -2$ to $y = 2$

x varies from $x = 0$ to $x = 4$

Step 2:

$$\oint_C \left(\sin(x) + x^2 y\right)dx + \left(3x^{\frac{3}{2}} + 2\cos(y)\right)dy = \oint_C Pdx + Qdy$$

On comparing we have

$$P = \sin(x) + x^2 y \ , \ \ Q = 3x^{\frac{3}{2}} + 2\cos(y)$$

and

$$\frac{\partial P}{\partial y} = x^2 \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{9}{2}x^{\frac{1}{2}} \ .$$

By Green's theorem,

$$\oint_C \left(\sin(x) + x^2 y\right)dx + \left(3x^{\frac{3}{2}} + 2\cos(y)\right)dy = \iint_R \left(\frac{9}{2}x^{\frac{1}{2}} - x^2\right)dydx$$

$$\int_{x=0}^{4}\int_{y=-2}^{2}\left(\frac{9}{2}x^{\frac{1}{2}} - x^2\right)dydx = \int_{x=0}^{4}\left(\frac{9}{2}x^{\frac{1}{2}} - x^2\right)dx \int_{y=-2}^{2} dy$$

$$= \frac{32}{3}$$

■

<u>Example 15</u>

Verify Green's Theorem in the plane for

$$I = \oint_C \left(x^3 - x^2 y\right)dx + xy^2 dy$$

where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$.

Solution

$P = x^3 - x^2 y \quad , \quad Q = xy^2 .$

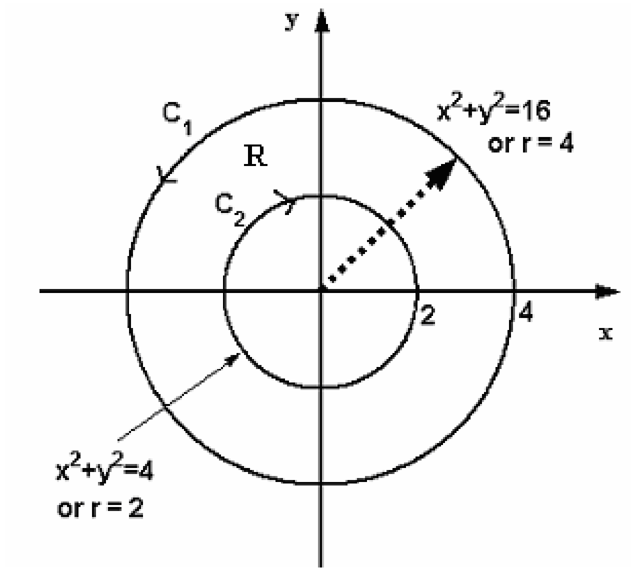


Figure 9.14

By Green's theorem,

$$I = \iint_R \left(y^2 + x^2\right) dx\, dy$$

Using polar coordinates, the region R enclosed between the two circles can be described as (An arrow shot from the origin enters the region at r = 2 and leaves it at r = 4, see figure 9.14):

For θ fixed, r varies from $r = 2$ to $r = 4$.
θ varies from $\theta = 0$ to $\theta = 2\pi$

$$\therefore I = \int_{\theta=0}^{2\pi} \int_{r=2}^{4} r^2 \left(r dr d\theta\right) = \int_0^{2\pi} \int_2^4 r^3 dr d\theta$$

$$= 2\pi \left[\frac{r^4}{4}\right]_2^4 = 120\pi$$

Let us now evaluate I directly (i.e. without using Green's theorem):

In polar coordinates:

C_1 becomes $x = 4\cos\theta$, $\quad y = 4\sin\theta, 0 \le \theta \le 2\pi$

C_2 becomes $x = 2\cos\theta$, $\quad y = 2\sin\theta$, $\quad -2\pi \le \theta \le 0$

$$I = \int_{C_1\ 0}^{2\pi} + \int_{C_2\ 2\pi}^{0}$$

$$= I_1 + I_2$$

$$C_1 = x^3 - x^2 y = (4\cos\theta)^3 - (4\cos\theta)^2(4\sin\theta)$$
$$= 4^3\left[\cos^3\theta - \cos^2\theta\sin\theta\right]$$
$$xy^2 = (4\cos\theta)\cdot(4\sin\theta)^2 = 4^3\cos\theta\sin^2\theta$$
$$dx = -4\sin\theta\, d\theta, \quad dy = 4\cos\theta\, d\theta$$

$$I_1 = \int_0^{2\pi}\left\{4^3\left[\cos^3\theta - \cos^2\theta\sin\theta\right]\left[-4\sin\theta\right] + \left[4^3\cos\theta\sin^2\theta\right]\left[4\cos\theta\right]\right\}d\theta$$

$$= 4^4\int_0^{2\pi}\left[-\sin\theta\cos^3\theta + \sin^2\theta\cos^2\theta + \sin^2\theta\cos^2\theta\right]d\theta$$

$$= 4^4\int_0^{2\pi}\cos^3\theta d(\cos\theta) + 4^4\int_0^{2\pi}\frac{\sin^2 2\theta}{2}d\theta$$

$$= 4^4\left[\frac{\cos^4\theta}{4}\right]_0^{2\pi} + 4^4\int_0^{2\pi}\frac{1-\cos 4\theta}{(2)(2)}d\theta$$

$$= 0 + \frac{4^4\pi}{2}$$

$$I_2 = -\int_0^{2\pi} 2^4[\text{ same as above }]d\theta = \frac{-2^4\pi}{2}$$

$$\therefore I = \left[4^4 - 2^4\right]\frac{\pi}{2} = 120\pi$$

■

Green's Theorem can be used to simplify the evaluation of certain integrals which otherwise would be difficult to compute. An example is given below.

Example 16

Use Green's theorem to evaluate $\oint_C e^x \sin y dx + e^x \cos y dy$, where C is the closed curve bounding the semicircle $y = \sqrt{4 - x^2}$ and the interval [-2,2].

Solution

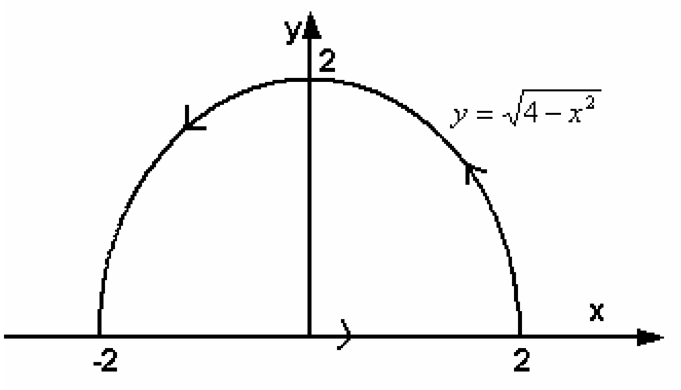


Figure 9.13

Here $P = e^x \sin y$ and $Q = e^x \cos y$.

$\therefore \dfrac{\partial P}{\partial y} = e^x \cos y$ and $\dfrac{\partial Q}{\partial x} = e^x \cos y$.

We find that by Green's theorem

$$\oint_C e^x \sin y dx + e^x \cos y dy = \iint_R (e^x \cos y - e^x \cos y) dx dy = 0.$$

It should be noted that the direct computation of this integral is very tedious.

9.4 PROPERTIES OF LINE INTEGRALS

We now investigate conditions under which the value of the line integral depends only on the initial and terminal points of a curve C and not on the path between these points.

Theorem 1

Let P and Q be two functions of x and y, such that $P,\ Q, \dfrac{\partial P}{\partial y}, \dfrac{\partial Q}{\partial x}$ are continuous and single-valued *at every point* of a simply connected region in R. The necessary and sufficient condition that $\oint_C P(x,y)dx + Q(x,y)dy = 0$ for every closed curve C drawn in R is that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ for every point in R.

Let us look at two examples which seem to contradict Theorem 1.

Example 17

Find by Green's Theorem the value of

$$I = \oint_C \left[x^3 e^x + xy\right]dx + \left[y^2 \sin y + x^2\right]dy$$

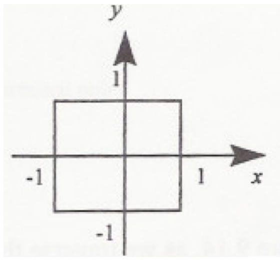


Figure 9.14

where C is the square formed by the lines $x = \pm 1,\quad y = \pm 1$.

Solution

Since $\dfrac{\partial P}{\partial y} = x$ and $\dfrac{\partial Q}{\partial x} = 2x \quad \therefore\ I = \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy = 0$

Question: Why is Theorem 1 not contradicted by this result? (Check every word in the theorem carefully and find out which one is violated).

Example 19

Consider $I = \oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$ and we might immediately expect $I = 0$.

Note: $P, \quad Q, \quad \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous everywhere except at (0, 0).

Hence Theorem 1 fails for any region including the origin.

Let us change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

$$I = \oint_C \frac{1}{r^2} \left[-r\sin\theta(-r\sin\theta\, d\theta + \cos\theta\, dr) + r\cos\theta\,(r\cos\theta\, d\theta + \sin\theta\, dr) \right]$$

$$= \oint_C \left[\sin^2\theta + \cos^2\theta\right] d\theta = \oint_C d\theta$$

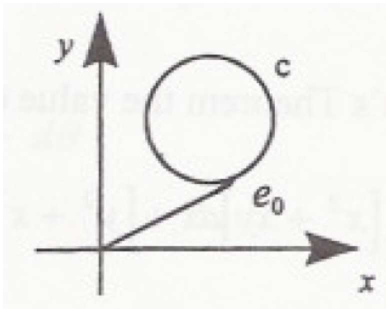


Figure 9.15

Now in Figure 9.15, as we traverse the path *C*,

$$I = \int_{\theta_0}^{\theta_0} d\theta = \theta_0 - \theta_0 = 0 \ .$$

In Figure 9.16 as we traverse the path C_1

$$I = \int_{\theta_0}^{\theta_0 + 2\pi} d\theta = 2\pi$$

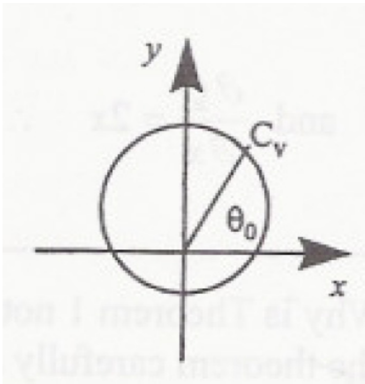


Figure 9.16

9.4.1 INDEPENDENCE OF PATH OF LINE INTEGRALS IN THE PLANE

Theorem 2

Let *P* and *Q* satisfy the conditions of Theorem 1. The necessary and sufficient condition that

$$\int_{(x_1,y_1)}^{(x_2,y_2)} P(x,y)dx + Q(x,y)dy$$

be independent of the curve connecting (x_1, y_1) and (x_2, y_2) is that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ at all points of the region *R*. In this case, the line integral is a function of the end points only.

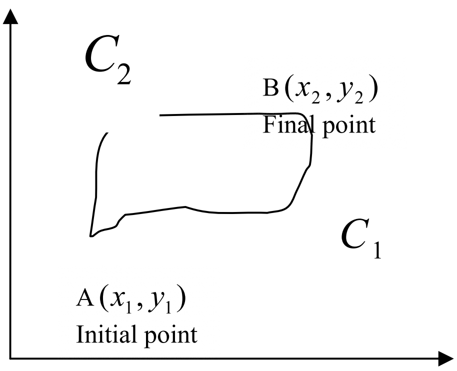


Figure 9.17

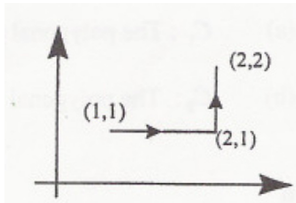
Consider the following example.

Example 18

Evaluate $I = \int_{(1,1)}^{(2,2)} \frac{10+y^2}{x^3} dx - \frac{1+x^2}{x^2} y dy$

Solution

$$\frac{\partial P}{\partial y} = \frac{2y}{x^3}, \quad \frac{\partial Q}{\partial x} = \frac{2y}{x^3}$$



and both functions are continuous, except at $x = 0$.
Hence, the line integral is independent of path
so long as it does not cross the y-axis.

Figure 9.18

Choose the path $y = 1$ from (1,1) to (2,1) and the path $x = 2$ from (2,1) to (2,2). Then,

$$I = \int_{(1,1)}^{(2,1)} \frac{2}{x^3} dx - 0 + \int_{(2,1)}^{(2,2)} 0 - \frac{5}{4} y \, dy = \left. \frac{1}{x^2} \right|_1^2 - \left. \frac{5}{8} y^2 \right|_1^2 = -\frac{9}{8}$$

Theorem 3

Let $P(x,y)$ and $Q(x,y)$ satisfy the conditions of Theorem 1. The necessary and sufficient condition that there exists a function $\phi(x,y)$ such that $\frac{\partial \phi}{\partial x} = P$ and $\frac{\partial \phi}{\partial y} = Q$ is that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ at all points of the region R.

Note If $\nabla \phi = P\,\hat{i} + Q\,\hat{j} = \vec{F}$, then $\vec{F}$ is a conservative vector field. The negative of ϕ is called the potential of $\vec{F}$.

Thus if there exists such a function ϕ such that $\nabla\phi = P\,\hat{i} + Q\,\hat{j}$ in R, then $\int_C P\,dx + Q\,dy$ is independent of path in R.

When a line integral is independent of path in a region, then it may be easier to evaluate the integral by replacing the given curve with a more convenient path that has the same initial and terminal points.

Example 19

Evaluate $I = \int_C (x^2 - 9y + 10)dx - (9x + y^3 + 4)dy$ over the curves

(a) C_1 : The polygonal path from (0,-1) to (1,0) to (0,1)

(b) C_2 : The polygonal path from (0,-1) to (1,0) to (0,1) to (-1,0) to (0,-1).

Solution

(a) Here $P = x^2 - 9y + 10$ and $Q = -9x - y^3 - 4$

Therefore $\dfrac{\partial P}{\partial y} = -9 = \dfrac{\partial Q}{\partial x}$ for all (x, y).

P, Q, $\dfrac{\partial P}{\partial y}$ and $\dfrac{\partial Q}{\partial x}$ are continuous at all points on xy plane. Thus, we can use a path $C_1^{/}$ from (0, -1) to (0,1) along y-axis.

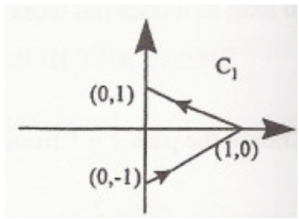


Figure 9.19

(**Note** C_1 and $C_1^{/}$ have same initial and terminal points).

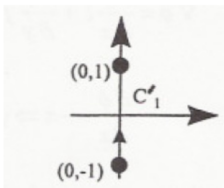


Figure 9.20

We use direct substitution to evaluate I on $C_1^{/}$.

On $C_1^{/}$: $x = 0$, $dx = 0$, $-1 \le y \le 1$

$$I = \int_{y=-1}^{y=1} (0 - 9y + 10)(0) - (0 + y^3 + 4)dy$$

$$= \int_{y=-1}^{1} - y^3 - 4\, dy = \left[-\frac{1}{4} y^4 - 4y \right]_{-1}^{1} = \left(-\frac{1}{4} - 4 \right) - \left(-\frac{1}{4} + 4 \right) = -8$$

(b) C_2 is a closed curve, so

$$\int_{C_2} \left(x^2 - 9y + 10\right) dx - \left(9x + y^3 + 4\right) dy = 0$$

Example 20

Determine whether there is a function ϕ such that $\nabla \phi = -x\,\hat{i} + y\,\hat{j}$ for all (x, y). If there is, find such a function.

Solution

Here $\nabla \phi = P\,\hat{i} + Q\,\hat{j} = -x\,\hat{i} + y\,\hat{j}$

$\therefore P = -x$ and $Q = y$

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$$

Since $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ for all (x, y) such a function ϕ exists.

To find ϕ:

$$\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} = -x\,\hat{i} + y\,\hat{j} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = -x \Rightarrow \phi = -\frac{x^2}{2} + h(y) \tag{2}$$

Differentiating (2) with respect to y gives

$$\frac{\partial \phi}{\partial y} = 0 + h'(y) = h'(y) \tag{3}$$

Compare (3) with (1) (from (1) $\frac{\partial \phi}{\partial y} = y$)

$$\therefore \quad \frac{\partial \phi}{\partial y} = h'(y) = y \Rightarrow \quad h(y) = \int y\, dy = \frac{y^2}{2} + C$$

Hence from (2) , $\phi(x, y) = \frac{-x^2}{2} + \frac{y^2}{2} + C$

■

Activity 4

Find by Green's theorem, the value of

(a) $\oint_C \left(x^2 + y\right) dx + \left(x + y^2\right) dy$

along the curve C formed by $y^3 = x^2$ and $y = x$ between (0,0) and (1,1).

(b) $\oint_C \frac{1}{y} dx + \frac{1}{x} dy$

along the closed path formed by $y = 1, \quad x = 4,$ and $y = \sqrt{x}$.

(c) $\oint_C xy\, dx + x^2 dy$, where C denotes the triangle with endpoints (1,1), (2,1), and (2,2)

(d) $\oint_C (3xy + y^2)dx + (2xy + 5x^2)dy$ along the path $C : (x-1)^2 + (y+2)^2 = 1$.

Activity 5

Show that

(a) $\int_{(0,1)}^{(1,2)} \left(x^2 + y^2\right) dx + 2xy\, dy$

(b) $\displaystyle\int_{(0,0)}^{(1,1)} \frac{1-y^2}{(1+x)^3}\,dx + \frac{y}{(1+x)^2}\,dy$

are independent of path and determine their values.

9.5 SURFACE INTEGRALS

In Section 8.4 of this manual, we considered the use of double integral to obtain the surface integral of a function f over a surface S in three-dimensional space and this was denoted by

$$\iint_S f \;\; d\sigma$$

where $d\sigma$ denotes the differential area (Also $\iint_S \; d\sigma$ is the surface area of S).

In the last part of Section 9.2.1, we used line integrals to measure the flux of a two-dimensional vector field across a plane curve. In this Section, we will see how to use surface integrals to measure the flux of a three-dimensional vector field across a surface in three-dimensional space. We will measure the flux of a vector field $\vec{F}$ across a surface S in the direction of a unit normal value $\vec{n}$.

At each point on the surface, there will be two unit normal vectors: $\hat{n}$ and $-\hat{n}$. We choose one of these normals to identify the positive normal direction at each point.

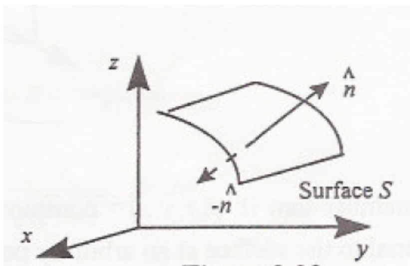


Figure 9.21

For example, if S is a closed surface, such as a sphere, we always choose the outward pointing normal vector as the positive one. Once the normal $\hat{n}$ is chosen, we say that we have oriented the surface, and we call the surface together with its normal field an oriented surface.

Definition

Suppose that $\vec{F}$ is a continuous vector field defined over an oriented surface S and that $\hat{n}$ is the chosen unit normal field on the surface.

The Flux of the three-dimensional vector field $\vec{F}$ across an oriented surface S in the direction of $\hat{n}$ is

$$\boxed{\text{Flux} = \iint_S \vec{F}.\hat{n}\ d\sigma}$$

Example 21

Find the flux of the field $\vec{F} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

Solution

The surface S is shown

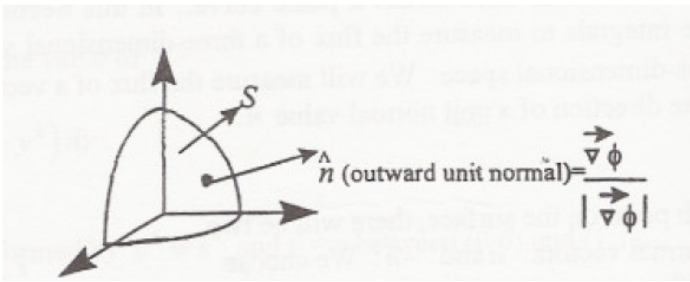


Figure 9.22

Remember that if $\phi(x,y,z) =$ constant is the equation of a surface *S*, then $\nabla\vec{\phi}$ is a normal to the surface at an arbitrary point (*x,y,z*).

Let $\quad \phi(x,y,z) = x^2 + y^2 + z^2 - a^2$

then

$$\begin{aligned}\nabla\vec{\phi} &= \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} \\ &= 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}\end{aligned}$$

then

$$\hat{n} = \frac{\nabla \vec{\phi}}{\left|\nabla \vec{\phi}\right|} = \frac{2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}}{2\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\,\vec{i} + y\,\vec{j} + z\,\vec{k}}{a}$$

thus

$$\text{Flux} = \iint_S \quad \vec{F}.\hat{n}\ d\sigma$$

$$= \iint_S \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \frac{1}{a} \begin{pmatrix} x \\ y \\ z \end{pmatrix} d\sigma$$

$$= \iint_S \quad \frac{x^2 + y^2 + z^2}{a}\, d\sigma = \iint \frac{a^2}{a}\, d\sigma$$

$$= \iint a\ d\sigma$$

$$= \iint_\Omega \iint a\,.\frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy$$

where Ω is the projection of S onto the *xy*-plane

$$= a^2 \int_0^{\frac{\pi}{2}} \int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{2}} -\left[\sqrt{a^2 - r^2}\right]_0^a d\theta$$

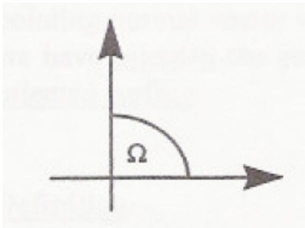


Figure 9.23

$$0 \le r \le a \quad 0 \le \theta \le \frac{\pi}{2} \qquad\qquad = a^2 \int_0^{\frac{\pi}{2}} -\left(0-a\right) d\theta = \frac{\pi}{2} a^3$$

■

<u>Activity 6</u>

(a) Evaluate $\iint_s \vec{F} \cdot \hat{n} ds$ where

$\vec{F} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, S is that part of the paraboloid $z = 4 - x^2 - y^2$ with $z \ge 3$, and $\hat{n}$ is directed upward.

(b) Evaluate the flux $\vec{F} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ across that part of the surface $z = 3$ with $x^2 + y^2 \le 1$, in the direction of the downward unit normal vector.

9.6 THE GAUSS DIVERGENCE THEOREM

The flux of a vector field $\vec{F} = f_1(x,y,z)\vec{i} + f_2(x,y,z)\vec{j} + f_3(x,y,z)\vec{k}$ across a closed smooth oriented surface S in the direction of the surface's outward unit normal field $\hat{n}$ equals the integral of $\vec{\nabla} . \vec{F}$ over the closed bounded region D, completely enclosed by the surface, that is,

$$\boxed{\oiint_S \vec{F} . \hat{n}\, d\sigma = \iiint_D \vec{\nabla} . \vec{F}\, dv}$$

where dv = element of volume.

We assume that $\vec{F}$ has continuous first partial derivatives in D and on the boundary of D.

Example 22

We solve Example 21 by using the Divergence Theorem but in this case: S is the sphere $x^2 + y^2 + z^2 = a^2$.

$$\text{Flux} = \oiint_S \vec{F} . \hat{n} \; d\sigma$$

$$\equiv \iiint_D div\, F \;\; dx\, dy\, dz \qquad (dv = dx\, dy\, dz)$$

$$div\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

thus

$$\text{Flux} = \iiint_D 3\, dx\, dy\, dz = 3 \underbrace{\iiint_D dx\, dy\, dz}$$

$$= 3 \times \text{ volume of sphere}$$

$$= 3 \times \frac{4}{3} \pi\, a^3$$

$$= 4\, \pi\, a^3$$

■

Example 23

Use the Gauss Divergence Theorem to evaluate $\oiint \vec{F} . \vec{n}\; d\sigma$

where $\vec{F} = (xy^2 + z)\vec{i} + \left(x^2 y + z\right)\vec{j} + \left(\frac{1}{3} z^3 + x\right)\vec{k}$ and S: entire surface of the hemispherical region : $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$.

Solution

$$div\vec{F} = \frac{\partial}{\partial x}\left(xy^2 + z\right) + \frac{\partial}{\partial y}\left(x^2 y + z\right) + \frac{\partial}{\partial z}\left(\frac{1}{3} z^3 + x\right)$$
$$= y^2 + x^2 + z^2$$
$$= a^2$$

Thus

$$\oiint \vec{F} \cdot \hat{n}\, d\sigma = \iiint_v div\, \vec{F} \;\; dv$$

$$= \iiint_v a^2 \;\; dx\, dy\, dz$$

$$= a^2 \text{ x volume } v$$

$$= a^2 \cdot \frac{2}{3}\pi\, a^3 = \frac{2\pi}{3} a^5$$

<u>Example 24</u>

Verify the Divergence Theorem by calculating directly

$\oiint_S \vec{F} \cdot \hat{n} \;\, d\sigma$ where $\vec{F} = x^3\, \vec{i} + x^2 y\, \vec{j} + x^2 z\, \vec{k}$ and S is the surface bounded by $z = 0$, $z = b$ and $x^2 + y^2 = a^2$.

(i) **<u>Direct Calculation</u>**

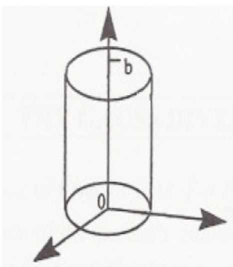


Figure 9.24

$S = S_1 + S_2 + S_3$

S_1 : Plane $z = 0$

S_2 : Curved surface $x^2 + y^2 = a^2$

S_3 : plane $z = b$

thus $\oiint_S \vec{F} \cdot \hat{n}\, d\sigma$

$$= \iint_{S_1} \vec{F} \cdot \vec{\hat{n}}\, d\sigma + \iint_{S_2} \vec{F} \cdot \vec{\hat{n}}\, d\sigma + \iint_{S_3} \vec{F} \cdot \vec{\hat{n}}\, d\sigma \,.$$

We calculate $\iint_{S_1} \vec{F} \cdot \vec{\hat{n}} \;\, d\sigma$

the outward normal to surface S_1 is $\vec{n} = -\vec{k}$
thus

$$\iint_{S_1} \vec{F} \, . \vec{\hat{n}} \, d\sigma$$

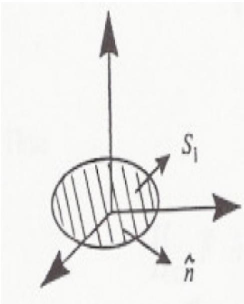


Figure 9.25

$$= \iint_{\Omega} \begin{pmatrix} x^3 \\ x^2 y \\ x^2 z \end{pmatrix} . \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} d\sigma$$

$$= -\iint_{S} x^2 z \, d\sigma$$

$$= -\iint_{\Omega} x^2 z \sqrt{1^2 + 2x^2 + 2y^2} \, dx \, dy$$

where $\Omega =$ circle $x^2 + y^2 = a^2$.

As $z = 0$, $\iint_{S_1} \vec{F} \, . \vec{\hat{n}} \; d\sigma = 0$.

An outward normal to the surface $z =$ h is $\vec{n} = \vec{k}$.

Thus

$$\iint_{S_3} \vec{F} \, . \, \vec{n} \; d\sigma = \iint_{S_3} x^2 z \; d\sigma$$

$$= \iint_{\Omega} x^2 b \, . \sqrt{1 + 0 + 0} \; dx \, dy$$

$$= b \iint_{\Omega} x^2 \; dx \, dy$$

$$= b \int_0^{2\pi} \int_0^{a} r^2 \cos^2 \theta \, . \, r \, dr \, d\theta$$

$$= b \left(\int_0^{2\pi} \cos^2 \theta \, d\theta \right) \int_0^{a} r^3 \, dr$$

$$= b \, \frac{1}{2} \int_0^{2\pi} 1 + \cos \, 2\,\theta \, d\theta . \quad \frac{a^4}{4}$$

$$= \frac{b \, a^4}{8} (2\,\pi) = \frac{\pi \, a^4 b}{4}$$

■

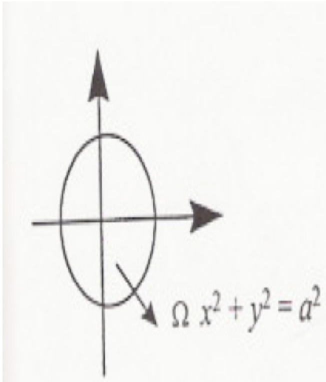
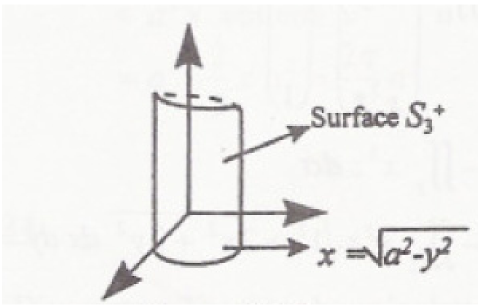


Figure 9.26

To evaluate $\iint_{S_2} \vec{F} \,.\, \hat{n} \; d\sigma$

We write the equation of the curved surface $x^2 + y^2 = a^2$ in the form $x = \pm\sqrt{a^2 - y^2}$ and we project onto the *yz*-plane.



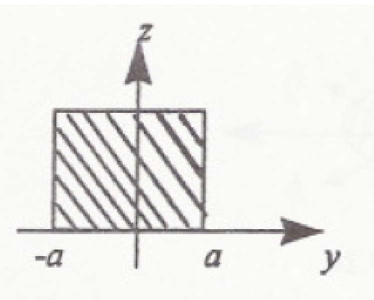


Figure 9.27 **Figure 9.28**

Let $S_3 = S_3^+ \;\; a \; S_3^-$

$$x = \sqrt{a^2 - y^2}$$

$$\phi(x,y,z) = x^2 + y^2 - a^2$$

$$\nabla \vec{\phi} = 2x\,\vec{i} + 2y\,\vec{j} \qquad \text{and} \qquad \hat{n} = \frac{x\,\vec{i} + y\,\vec{j}}{a}$$

$$\iint_{S_3^+} \vec{F}\,.\,\hat{n}\;d\sigma = \iint_{\Omega yz} \begin{pmatrix} x^3 \\ x^2 y \\ x^2 z \end{pmatrix} . \frac{1}{n} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \sqrt{1 + x_y^2 + x_z^2}\;dy\,dz$$

$$= \iint_{\Omega yz} \frac{1}{a}\left(x^4 + x^2 y^2\right) \sqrt{1 + x_y^2 + x_z^2}\;dy\,dz$$

$$x^2 = a^2 - y^2$$

$$2x\,\frac{\partial x}{\partial y} = -zy \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x}$$

$$\frac{\partial x}{\partial z} = 0$$

$$\sqrt{1 + x_y{}^2 + x_z^2} = \sqrt{1 + \frac{y^2}{x^2} + 0} = \sqrt{\frac{x^2 + y^2}{x}} = \frac{a}{x}$$

Thus

$$\iint_{S_3^+} \vec{F}\,.\,\hat{n}\;d\sigma = \iint_{\Omega yz} \frac{1}{a}\left(x^4 + x^2 y^2\right)\frac{a}{x}\,dy\,dz$$

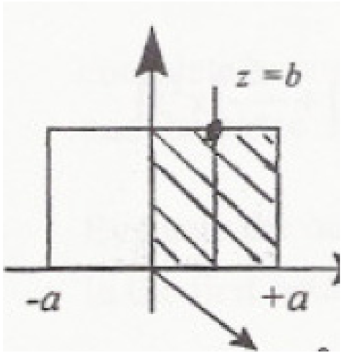
$$= \iint_{\Omega yz} \frac{1}{a}\left(x^3 + xyz\right) a\;\;dy\,dz$$

$$= \iint_{\Omega y\,z} \left(x^3 + xyz\right) dy\,dz$$

$$= \iint_{\Omega yz} x\left(a^2 - y^2\right) + xy^2 dy\,dz$$

$$= a^2 \iint_{\Omega yz} x \quad dy\,dz$$

$$= a^2 \quad \iint_{\Omega yz} \sqrt{a^2 - y^2}\;dy\,dz$$



$$= a^2 \int_{-a}^{+a} \left[\int_0^b \sqrt{a^2 - y^2}\ dz \right] dy$$

$$= a^2 \int_{-a}^{+a} \sqrt{a^2 - y^2}\ .[2]_0^b\ dy$$

$$= a^2 \int_{-a}^{+a} \sqrt{a^2 - y^2}\ b\ dy$$

$$= 2\,a^2\, b \int_0^a \sqrt{a^2 - y^2}\ dy$$

Figure 9.29

Put $y = a \sin\theta$

$$= 2\,a^2\, b\ \frac{\pi\, a^2}{4} = \frac{\pi\, a^4}{2} b\ .$$

Similarly,

$$\iint_{S_3^-} \vec{F} . \hat{n}\ d\sigma = \frac{\pi\, a^4}{2} b\,.$$

Adding the three integrals,

$$\oiint_S \vec{F} . \hat{n}\ d\sigma = \iint_{S_1} \vec{F} . \hat{n}\, d\sigma + \iint_{S_2} \vec{F} . \hat{n}\, d\sigma + \iint_{S_3} \vec{F} . \hat{n}\, d\sigma$$

$$= 0 + \left(\frac{\pi\, a^4 b}{2} \right)\ \text{x}\ 2 + \frac{\pi\, a^4 b}{4}$$

$$= \frac{5\pi\, a^4\ b}{4}$$

(iii) We evaluate $\oiint_S \vec{F} . \hat{n}\ d\sigma$ using the Divergence Theorem.

$$div\, \vec{F} = \frac{\partial}{\partial x} = \left(x^3\right) + \frac{\partial}{\partial y}\left(x^2 y\right) + \frac{\partial}{\partial z}\left(x^2 z\right)$$

$$= 3x^2 + x^2 + x^2$$

$$= 5x^2\,.$$

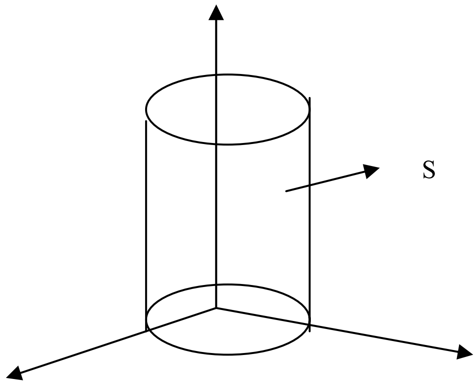


Figure 9.30

$$\oiint_S \vec{F}\,.\vec{n}\;\; d\sigma = \iiint_V \; div\; F \;\; dV$$

$$= \iiint_V \;\; 5x^2 \;\; dV\;.$$

We use cylindrical coordinates to evaluate this integral 9.30

$$x = r\cos\theta\,,\quad y = r\sin\theta\,,\qquad z = 2$$

A parametrization of the region V is

$$0 \le r \le a \qquad 0 \le \theta \le 2\,\pi \qquad 0 \le 2 \le b$$

Thus,

$$\iiint_V \; 5x^2 \;\; dV = \int_0^b \int_0^{2\pi} \int_0^a 5\,r^2\;\cos^2\,\theta\, r\, dr\, d\theta\, dz$$

$$= \left(\int_0^b dz\right)\left(\int_0^{2\pi} \;\cos^2\,\theta\, d\theta\right)\int_0^a 5r^3\;\, dr$$

$$= b\left(\,\pi\,\right)\left(\frac{5}{4}\,a^4\right)$$

$$= \frac{5}{4}\,\pi\, a^4\;\, b$$

■

Activity 7

(a) Using divergence theorem, evaluate $\iint\limits_S \vec{F}.\hat{n}\, ds$, $\vec{F} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, S is the complete boundary of the region D bounded by the paraboloid $z = 4 - x^2 - y^2$ and the plane $z = 3$, and $\hat{n}$ is outward.

(b) Evaluate the outward flux of $\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$ across the boundary of the region in the first octant inside the sphere $x^2 + y^2 + z^2 = 1$.

(c) Use the Gauss Divergence theorem to evaluate the outward flux of $\vec{F} = x\hat{i} + 2y^2\hat{j} + 3z^2\hat{k}$ across the solid defined by $0 \le x^2 + y^2 \le 16$ and $0 \le z \le 2$.

9.7 STOKES' THEOREM

We recall Green's theorem in the plane.

If Ω is a plane region bounded by a closed path C and $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is a vector field, then

$$\oint_C P\ dx + Q\, dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy\,.$$

We have a similar theorem in three-dimensional space. That is, where S is a surface in three-dimensional space and the surface is bounded by a curve Γ.

Stokes' Theorem

Let S be an oriented surface bounded by a simple closed curve Γ and let $\vec{F}$ be a vector field.

Then

$$\boxed{\oint \vec{F}\,.\,d\,\vec{r} = \iint_S \nabla \wedge \vec{F}\,.\,\hat{n}\ d\sigma}$$

or

$$\oint_c \vec{F}\,.\,d\,\vec{r} = \iint_S curl\ \ \vec{F}\,.\,\hat{n}\ \ d\sigma$$

where $\hat{n}$ = outward unit normal to surface S.

Example 25

Verify Stokes' theorem when

$\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ and $S: \quad z = 9 - x^2 - y^2 \qquad z \geq 0$

Solution

The surface S is shown below

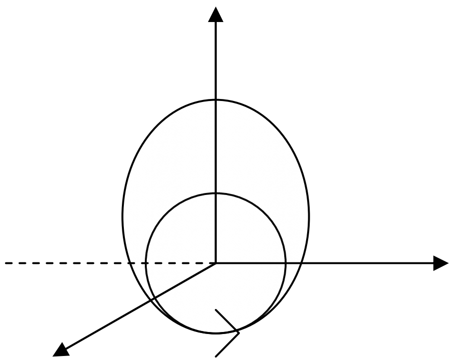


Figure 9.31

Here the boundary Γ is the circle $x^2 + y^2 = 9$ (put $z = 0$)

We parametrize Γ as $\vec{r} = (3\cos t)\vec{i} + (3\sin t)\vec{j}$

$$d\vec{r} = (-3\sin t\ \vec{i} + 3\cos t\ \vec{j})dt$$

$$0 \leq t \leq 2\pi$$

then

$$\oint_\Gamma \vec{F}.d\vec{r} = \int_0^{2\pi} \begin{pmatrix} z \\ x \\ y \end{pmatrix} . \begin{pmatrix} -3\sin t \\ 3\cos t \\ 0 \end{pmatrix} dt$$

$$= \int_0^{2\pi} -3z\sin t + 3x\cos t \quad dt$$

on Γ, $z = 0$

thus $\oint_\Gamma \vec{F}.d\vec{r} = \int_0^{2\pi} 0 + 3\cdot 3\cos t \cdot \cos t \ dt$

$$= 9\int_0^{2\pi} \cos^2 t \ dt = \frac{9}{2}\int_0^{2\pi} 1+\cos 2t \ dt$$
$$= 9\pi$$

We now calculate $\iint_s curl \ \vec{F}.\hat{n} \ d\sigma$

$$curl \ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \vec{i}\left[\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(x)\right] - \vec{j}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{yz}(z)\right] + \vec{k}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(z)\right]$$

$$= \vec{i}[1-0] - \vec{j}[0-1] + \vec{k}[1-0]$$

$$= \vec{i} + \vec{j} + \vec{k}$$

Surface $z = 9 - x^2 - y^2 \Rightarrow \phi(x,y,z) = z - 9 + x^2 + y^2$

$\nabla\vec{\phi} = 2x\vec{i} + 2y\vec{i} + \vec{k}$

$\hat{n} = \dfrac{\nabla\vec{\phi}}{|\nabla\vec{\phi}|} = \dfrac{2x\vec{i} + 2y\vec{j} + \vec{k}}{\sqrt{1+4x^2+4y^2}}$

Thus,

$$\iint_s curl \ \vec{F}.\hat{n} \ d\sigma = \iint_s \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix} \frac{1}{\sqrt{1+4x^2+4y^2}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} d\sigma$$
$$= \iint_s \frac{2x+2y+1}{\sqrt{1+4x^2+4y^2}} \ d\sigma$$
$$= \iint_\Omega \frac{2x+2y+1}{\sqrt{1+4x^2+4y^2}} \sqrt{1+z_x^2+z_y^2} \ dx\,dy$$

where Ω : = projection of S onto *xy*-plane

: = circle $x^2 + y^2 = 9$

$z = 9 - x^2 - y^2$

$z_x = -2x$

$z_y = -2y$

$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 4x^2 + 4y^2}$

hence $\iint_s curl\ \vec{F}\ .\ \hat{n}\ d\sigma = \iint_\Omega 1 + 2x + 2y\ \ dxdy$

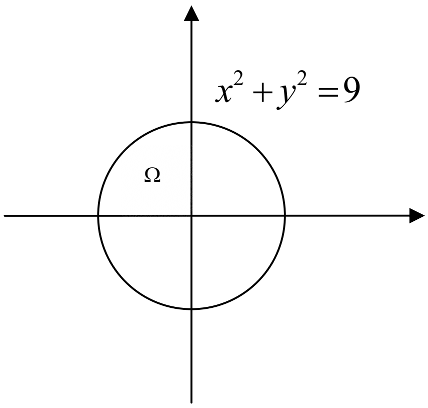


Figure 9.32

In polar coordinates,

$x = r\cos\theta \qquad y = r\sin\theta$

$0 \le r \le 3\ , \qquad 0 \le \theta \le 2\pi$

$$\iint_\Omega 1 + 2x + 2y\ \ dxdy = \int_0^{2\pi}\int_0^3 \left(1 + 2r\cos\theta + 2r\sin\theta\right)\ r\ drd\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2} + \frac{2r^3}{3}\cos\theta + \frac{2r^3}{3}\sin\theta\right]_\theta^3 d\theta$$

$$= \int_\theta^{2\pi} \frac{9}{2} + 18\cos\theta + 18\sin\theta\ d\theta$$

$$= 9\pi$$

Hence Stokes' Theorem is verified.

Example 26

Use Stokes' Theorem to evaluate

$\int_{\Gamma} \vec{F}\,.d\vec{r}$ where $\vec{F} = 5z\ \vec{i} - 4x\vec{j} + y\vec{k}$ and Γ is the triangular path from (0,0,0) to (3,2,2) to(2,3,3) to (0,0,0)

Solution

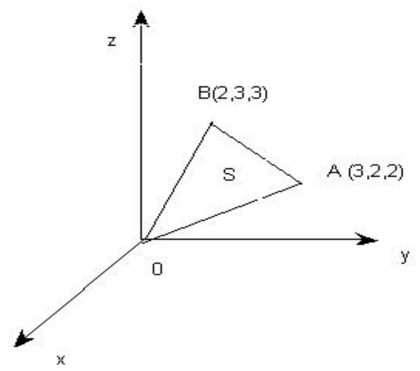


Figure 9.33

Surface S is a plane OAB .

We find a normal to the plane

A normal to the plane is $\vec{n} = \overrightarrow{OA} \times \overrightarrow{OB}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 2 \\ 2 & 3 & 3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(9-4) + \vec{k}(9-4)$$

$$= -5\vec{j} + 5\vec{k} = +5\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

a unit normal is $\hat{n} = \dfrac{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{2}}$

$$curl\ \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 5z & -4x & y \end{vmatrix} = \hat{i}\left[\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(-4x)\right] - \vec{j}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial z}(5z)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial y}(5z)\right]$$

$$= \vec{i}\,[(1-0)] - \vec{j}\,[0-5] + \vec{k}\,[-4-0]$$
$$= \vec{i} + 5\vec{j} - 4\vec{k}$$

Using Stokes' theorem

$$\int_{\Gamma} \vec{F}\cdot d\vec{r} = \iint_s curl\ \vec{F}\cdot\hat{n}\ d\sigma$$

$$= \iint_S \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} d\sigma$$

$$= \iint_S \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -5 & -4 \end{pmatrix} d\sigma$$

$$= \frac{-9}{\sqrt{2}} \iint_S d\sigma$$

$$= \frac{-9}{\sqrt{2}}\left(Area\ of\ surface\ S\right)$$

$$= \frac{-9}{\sqrt{2}}\ \frac{1}{2}\left|\vec{OA}\times\vec{OB}\right|$$

$$= \frac{-9}{\sqrt{2}} \ \frac{1}{2} \left| -5\vec{j} + 5\vec{k} \right|$$

$$= \frac{-9}{\sqrt{2}} \sqrt{25 + 25} = \frac{-9}{\sqrt{2}} \ \frac{1}{2} 5\sqrt{2} = -\frac{45}{2}$$

Example 27

Using Stokes' Theorem, evaluate

$$\oint_{\Gamma} \vec{F} \cdot d\vec{r}$$

where $\vec{F} = r^{-3}\vec{r}$, $\quad \vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ and Γ is the intersection of the hemisphere

$z = \sqrt{9 - x^2 - y^2}$ with the *xy*-plane.

Solution

$$\oint_{\Gamma} \frac{\vec{r}}{r^3} \cdot d\vec{r} = \iint_S \nabla \times \frac{\vec{r}}{r^3} \cdot \hat{n} \ dS$$

$$\nabla \times \frac{\vec{r}}{r^3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} = \hat{i}\left[z\frac{\partial}{\partial y} \ \frac{1}{r^3} - y\frac{\partial}{\partial z} \ \frac{1}{r^3} \right] + \hat{j}\left[\quad \right] + \hat{k}\left[\quad \right]$$

$$= \hat{i}\left[\frac{-3z}{r^4} \ \cdot \frac{y}{r} - \frac{-3y}{r^4} \ \frac{z}{r} \right] + \hat{j}(\ ) + \hat{k}(\ )$$

$$= \hat{i}0 + \hat{j}0 + \hat{k}0$$

$$= 0$$

$$\therefore \oint_{\Gamma} \frac{\vec{r}}{r^3} \cdot d\vec{r} = 0 .$$

Activity 8

Problem 1 Compute $\oint \boldsymbol{F}.d\boldsymbol{r}$ around the circle $(x-1)^2 + y^2 = 1, \quad z = 3$ when

$\boldsymbol{F} = -y\hat{i} + x\hat{j} + z\hat{k}$.

Problem 2 Evaluate, using Stokes' theorem

$$\oint_C 2xy^2z \ dx + 2x^2yz \ dy + \left[x^2y^2 - 2z\right] \ dz$$

where C is the curve $x = \cos t, \quad y = \sin t, \quad z = \sin t, 0 \le t \le 2\pi$,

directed with t increasing.

Example 28 Verify Stokes' theorem for $\vec{F} = 3y\hat{i} - xz\hat{j} + yz^2\hat{k}$ where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and Γ is its boundary.

Solution

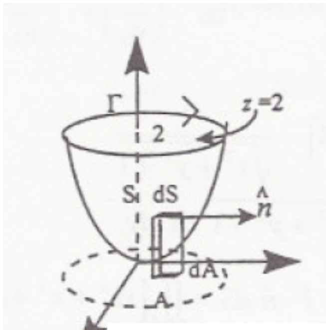


Figure 9.34

$$\oint_\Gamma \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n} \quad dS$$

$$\oint_\Gamma \vec{F}\cdot d\vec{r} = \oint_\Gamma 3y \ dx - xz \ dy + yz^2 \ dz$$

where Γ is the circle $x^2 + y^2 = 4$ at $z = 2$.

in parametric form $x = 2\cos\theta$, $y = 2\sin\theta$, $z = 2$

$$\oint_\Gamma \vec{F}\cdot d\vec{r} = \int_{2\pi}^{0}\left[3\cdot 2\sin\theta(-2\sin\theta) - 4\cos\theta.2\cos\theta + 8\sin\theta\cdot 0\right] d\theta$$

$$= \int_0^{2\pi}\left[12\sin^2\theta + 8\cos^2\theta\right] d\theta = 20\pi$$

We now need $\hat{n}$. The equation of S in $\phi(x,y,z) = x^2 + y^2 - 2z = 0$.

$\vec{\nabla}\phi = \hat{\imath}2x + \hat{\jmath}2y - \hat{k}2$

and $\hat{n} = \dfrac{\overrightarrow{\nabla\phi}}{\left|\overrightarrow{\nabla\phi}\right|} = \dfrac{2\left(\hat{i}x + \hat{j}y - \hat{k}\right)}{2\sqrt{x^2 + y^2 + 1}}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix} = \hat{i}\left(z^2 + x\right) + \hat{j}(0 - 0) + \hat{k}(-z - 3)$$

$$\vec{\nabla} \times \vec{F} \cdot \hat{n} = \frac{xz^2 + x^2 + z + 3}{\sqrt{x^2 + y^2 + 1}}$$

$$\iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\ dS = \iint_S \frac{xz^2 + x^2 + z + 3}{\sqrt{x^2 + y^2 + 1}} dS$$

Let us project the surface S onto the xy-plane.

Then $|\cos y|\ dS = dx\ dy$ (We want the magnitude only, hence

or $dS = dx\ dy|\sec y|$ the absolute value signs.)

$$\cos y = \left|\hat{n} \cdot \hat{k}\right| = \frac{+1}{\sqrt{x^2 + y^2 + 1}}$$

$$\therefore dS = \sqrt{x^2 + y^2 + 1}\ dx\ dy$$

and $\iint_S \nabla \times \vec{F} \cdot \hat{n}\ dS = \iint_A \left[xz^2 + x^2 + z + 3\right]\ dx\ dy$

$$= \iint_A \left\{ x\left[\frac{x^2 + y^2}{2}\right]^2 + x^2 + \frac{x^2 + y^2}{2} + 3 \right\} dx\ dy$$

Changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\ dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left[r\cos\theta \frac{r^4}{4} + r^2 \cos^2\theta + \frac{r^2}{2} + 3 \right] r\ dr\ d\theta$$

$= 20\pi$.

9.8 Supplementary Exercises

1. Find the arc length of the curve $y = x^2$, from (0,0) to (1,1)

2. Evaluate

$$I = \int_c \frac{3y}{x}\, ds,$$

   where C is the curve given by $x = t$, $y = t^2$, $1 \leq t \leq 2$.

3. Evaluate

$$I = \int_{(1,1)}^{(2,4)} \left(2x^2 + 4xy\right) dx + \left(2x^2 - y^2\right) dy$$

   where C is given by $x = t$, $y = t^2$.

4. If $\vec{F} = \dfrac{y\hat{i} - x\hat{j}}{x^2 + y^2}$ and C is the circle $x^2 + y^2 = 1$ traversed counter-clockwise, find the circulation of $\vec{F}$ around C.

5. Use Green's theorem to find the integral

$$\oint_C (e^{x^2} + y^2)dx + (2x + \sin y)dy$$

   where C denotes the curve

   a. The square with vertices (0,0),(3,0),(3,3) and (0,3)
   b. The circle of radius 2 centered at origin.

6. Use Gauss Divergence theorem to evaluate

   c. $\oiint_S \vec{F}.\hat{n}\, dS$, where $\vec{F} = x^2\,\hat{i} + xz\,\hat{j} - 4z\,\hat{k}$ and S is the solid sphere $x^2 + y^2 + z^2 \leq 9$.

7. Use Stokes' Theorem to find $\int_\Gamma \vec{F}.d\vec{r}$ where $\vec{F} = x^2\hat{i} + 2x\hat{j} + z^2\hat{k}$ and $\Gamma : 4x^2 + y^2 = 16$

9.8 SUMMARY

In this unit, you have studied the important concepts of line and surface integrals together with useful theorems of Gauss and Stokes.

KEYPOINTS:

1. (a) $\int_C f \; ds$ is the line integral of *f* over the curve *C*.

   (b) $\int_C ds$ is the arc length of *C* where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx$

   (c) Line integrals can be evaluated either by the method of direct substitution or parametrically as follows:

$$\int_C f(x,y)\, ds = \int_{t=a}^{t=b} f\big(x(t)\,,\, y(t)\big)\, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

   where $C : x = x(t)\,,\; y = y(t)$

2. $$\int_C \vec{F} \times \hat{T}\, ds = \int_C P\, dx + Q\, dy \qquad (1)$$

   where $\vec{F} = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$

   (1) is again evaluated either by direct substitution or parametrically.

3. Outward flux = $\int_C \vec{F} \,.\, \hat{n}\, ds = \int_C P\, dy - Q\, dx$.

4. **Green's theorem for the plane:**

$$\oint_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

5. Conditions under which a line integral is independent of path - Theorem 1 and Theorem 2 of Section 9.4.

6. Conservative vector fields and potential functions - Theorem 3 of Section 9.4.

7. Flux of the three-dimensional vector field across an oriented surface, S in the direction of $\hat{n}$ is

$$\iint_S \vec{F} \cdot \hat{n} \ d\sigma$$

8. **Gauss Divergence Theorem:**

$$\oiint_S \vec{F} \,.\, \hat{n} \, d\sigma = \iiint_D \vec{\nabla} \,.\, \vec{F} \, dv$$

where $dv = dx \, dy \, dx$ (in Cartesian coordinates).

9. **Stokes' Theorem:**

$$\oint \vec{F} \,.\, d\,\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \,.\, \hat{n} \, d\sigma$$

9.10 ANSWERS TO ACTIVITIES AND SUPPLEMENTARY EXERCISES

Activity 1

(a) 0 (b) $\frac{1}{\sqrt{2}}$

Activity 2

(a) $\frac{4}{3}$ (b) 8

Activity 3

(a) 0 (b) -2π

Activity 4

(a) 0 (b) $\frac{3}{4}$ (c) $\frac{5}{6}$ (d) 7π

Activity 5

(a) $\frac{13}{3}$ (b) $\frac{4}{3}\pi$ (c) 224π

Activity 6

(a) $\frac{9\pi}{2}$ (b) $\frac{3\pi}{2}$

Activity 7

(a) $\frac{3\pi}{2}$ (b) $\frac{\pi}{6}$

Activity 8

(a) π (b) 0

Answers to Supplementary Exercises

1. $\frac{\sqrt{5}}{2}+\frac{1}{4}\ln\left(2+\sqrt{5}\right)$

2. $\frac{1}{4}(17)^{\frac{3}{2}}-\frac{1}{4}(5)^{\frac{3}{2}}$

3. 41/3

4. -2π

5.(a) -9 (b) 8π

6. -144π

7. 16π