UNIT 4 INFINITE SERIES

Unit Structure

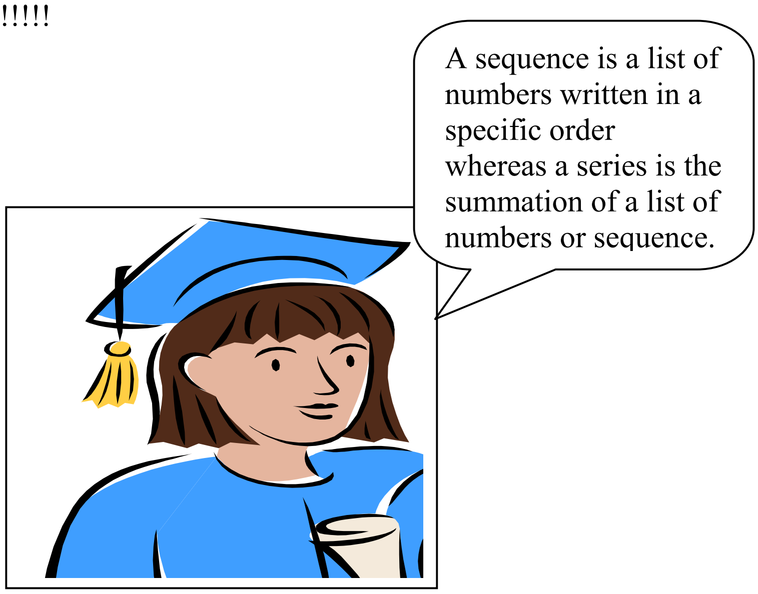
4.0 OVERVIEW

In this chapter we study sequences and infinite series. Series play an important role in the field of ordinary differential equations and without series large portions of the field of partial differential equations would not be possible.

4.1 DISTINGUISH BETWEEN SEQUENCES AND SERIES

Sequences? Series? Avoid the mess !!!!!!!!

Wrong !



Right!

Sequences
General sequence terms are denoted as follows,

$$\{a_1, a_2, a_3, \ldots, a_k, \ldots\}$$

In the notation above we must be cautious with the subscripts. The subscript of n+1 denotes the next term in the sequence and NOT one plus the n^{th} term! In other words,

$$a_{n+1} \neq a_n + 1$$

So, when writing subscripts to make sure that the “+1” does not migrate out of the subscript!

Series

Consider a sequence

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots, a_k, \ldots\}$$

and then add up all the terms of the sequence, so as to get a series :

$$a_1 + a_2 + a_3 + \ldots + a_k + \ldots = \sum_{n=1}^{\infty} a_n \ .$$

Thus, as mentioned previously, a series is the summation of a list of numbers or sequence. Since we started out with an infinite sequence we sum up an infinite list of numbers. Because of this the series above is sometimes called an infinite series. The n is often called an **index of summation** or just **index** for short.

Examples of sequences are :

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}, \ldots\right\},$$

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{-1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$$

and

$$\{b_n\}_{n=1}^{\infty}, \text{ where } b_n = n^{th} \text{ digit of } \pi$$

Remark :

in the first two series, to get the first few terms, we need to plug in values of n into the corresponding formula. This sequence is different from the first two in the sense that it does not have a specific formula for each term. However, it does tell us what each term should be. Each n^{th} term should be the n^{th}digit of π . We know that $\pi = 3.14159265359\ldots$ The sequence is then

$$\{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 9, \ldots\}$$

4.2 SEQUENCES

Definitions

1. We say that

$$\lim_{n\to\infty} a_n = L$$

if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of the a_n's approach L as n approaches infinity.

2. We say that

$$\lim_{n\to\infty} a_n = \infty$$

if we can make a_n as large as we want for all sufficiently large n. That is, the value of the a_n's get larger and larger without bound as n approaches infinity.

3. We say that

$$\lim_{n\to\infty} a_n = -\infty$$

if we can make a_n as negative as we want for all sufficiently large n. That is, the value of the a_n's get negative and larger without bound as n approaches infinity.

4. If $\lim_{n\to\infty} a_n$ exists and is finite we say that the sequence is **convergent**. If $\lim_{n\to\infty} a_n$ does not exist or is infinite we say the sequence **diverges**.

Next we investigate how to find the limits of sequences. We will use the following theorems and facts.

Theorem 1

Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.

This theorem basically tells us that we take the limits of sequences much like we take the limit of functions.

Theorem 2

If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$

This theorem is convenient for sequences that alternate in signs and note that it will only work if the sequence has a limit of zero.

Theorem 3

The sequence $\{r^n\}_{n=1}^{\infty}$ converges if $-1 < r \le 1$ and diverges for all other value of r. Also,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Note that the sequence in this theorem will converge for $r = 1$ and diverge for $r = -1$

Theorem 4 (also known as the squeeze theorem)

Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences such that

1. $a_n \le b_n \le c_n$ for all n greater than some positive integer N,

   and

2. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$

   then

$$\lim_{n\to\infty} b_n = L$$

Facts

Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and let c be a real number. Then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n\, b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}, \text{ provided } \lim_{n\to\infty} b_n \neq 0$$

Example 1.

Determine if the following sequences converge or diverge. If the sequence converges then determine its limit.

(a) $\left\{\dfrac{3n^2 - 1}{10n^2 + 5n^2}\right\}_{n=2}^{\infty}$

(b) $\left\{\dfrac{e^{2n}}{n}\right\}_{n=1}^{\infty}$

(c) $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

Solution

(a) We factorize the largest power of n from the numerator and denominator and then take the limit. Thus, we get

$$\lim_{n\to\infty}\frac{3n^2-1}{10n+5n^2}$$

$$=\lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)}=\lim_{n\to\infty}\frac{3-\frac{1}{n^2}}{\frac{10}{n}+5}$$

$$=\frac{3}{5}$$

(b) Using Theorem 1,

$$\lim_{n\to\infty}\frac{e^{2n}}{n}=\lim_{x\to\infty}\frac{e^{2x}}{x},$$

Applying L'Hopital's rule, we get

$$\lim_{x\to\infty}\frac{e^{2x}}{x}=\lim_{x\to\infty}\frac{2e^{2x}}{1}=\infty$$

So, the sequence diverges.

(c) Using Theorem 2,

$$\lim_{n\to\infty}\frac{(-1)^n}{n}=\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right|$$

and the right hand side of the latter equation can be written as $\lim_{n\to\infty}\frac{1}{n}$. This limit is obviously zero. Therefore, $\lim_{n\to\infty}\frac{(-1)^n}{n}=0$ and hence the corresponding sequence converges to zero.

Example 2

Find the following limits

(a) $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}}$

(b) $\lim_{n\to\infty} \frac{\cos(n)}{\sqrt{n}}$

(c) $\{(-1)^n\}_{n=1}^{\infty}$

Solution

(a) This sequence converges to zero by Theorem 2.

(b) Recall that

$$-1 \leq \cos(n) \leq 1,$$

for all integers n. Hence,

$$\frac{-1}{\sqrt{n}} \leq \cos(n) \leq \frac{1}{\sqrt{n}}.$$

Since $\lim_{n\to\infty} \frac{-1}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, applying Theorem 4, $\lim_{n\to\infty} \cos(n) = 0$

(c) By Theorem 3, this sequence converges to 1 .

Activity 1

Find the limit of the following converging sequence :

(a) $\left\{\left(1+\frac{\alpha}{n}\right)^{\beta n}\right\}_{n=1}^{\infty}$ ($\alpha \neq 0$ and $\beta \neq 0$)

(b) $\left\{\left(\frac{n-2}{n}\right)^{n}\right\}_{n=1}^{\infty}$

4.3 SERIES

4.3.1 CONVERGENCE AND DIVERGENCE

4.3.1.1 THE PARTIAL SUM CRITERION

Consider the following sequence $\{S_j\}_{j=1}^{\infty} = \{S_1, S_2, S_3, \ldots, S_k, \ldots, \}$,

where $S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \ldots + a_k$.

The sequence $\{S_j\}_{j=1}^{\infty}$ is said to be a sequence of partial sums of the infinite series :

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_k + \ldots$$

If the sequence of partial sums is convergent and if we define, $\lim_{k \to \infty} S_k = S$, then

$$\sum_{n=1}^{\infty} a_n = S$$

If the sequence of partial sums is divergent (*i.e.* either the limit does not exist or is infinite)
then we call the series **divergent**.

4.3.1.2 GEOMETRIC SERIES

A geometric series may be written in the form $\sum_{n=0}^{\infty} a\,r^n$ and its partial sum can be written $S_n = \dfrac{a(1-r^n)}{1-r}$.

We now discuss the convergence of a geometric series based on the magnitude of the common ratio r. The series will converge provided the partial sums form a convergent sequence. We therefore take the limit of the partial sum, which can be written as follows :

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a\,r^n}{1-r},$$

so that ,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a}{1-r} - \lim_{n\to\infty} \frac{a\,r^n}{1-r}$$

$$= \frac{a}{1-r} - \frac{a}{1-r} \lim_{n\to\infty} r^n .$$

Now, from Theorem 3 we know that the above limit will exist and be finite provided $-1 < r < 1$. (r can not be 1 since this leads to division by zero.)Therefore,

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}.$$

Hence, a geometric series will converge if $-1 < r < 1$ or $|r| < 1$

<u>Example 3</u>

Determine the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2^n + 7^n}{9^n} \right)$

(b) $\displaystyle\sum_{n=0}^{\infty} 4^n\, e^{-2n}$

(c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{6}{4n-1} - \frac{6}{4n+3} \right)$

Solution

(a) We express the series as the sum of two geometric series which are convergent. Hence, the series is convergent. (The sum of two convergent series is also a convergent one.)

$$\sum_{n=0}^{\infty}\left(\frac{2^n+7^n}{9^n}\right)=\sum_{n=0}^{\infty}\left(\frac{2}{9}\right)^n+\sum_{n=0}^{\infty}\left(\frac{7}{9}\right)^n$$

$$=\frac{1}{1-\frac{2}{9}}+\frac{1}{1-\frac{7}{9}}$$

(b) $\displaystyle\sum_{n=0}^{\infty}4^n\,e^{-2n}=\sum_{n=0}^{\infty}\left(\frac{4}{e^2}\right)^n=\frac{1}{1-\frac{4}{e^2}}$, which is a convergent geometric series.

(c) $\displaystyle S_n=\sum_{k=0}^{n}\left(\frac{6}{4k-1}-\frac{6}{4k+3}\right)$ is series is a telescoping series with partial sums:

$$S_n=\left(\frac{6}{3}-\cancel{\frac{6}{7}}\right)+\left(\cancel{\frac{6}{7}}-\cancel{\frac{6}{11}}\right)+\left(\cancel{\frac{6}{11}}-\cancel{\frac{6}{16}}\right)+\ldots+\left(\cancel{\frac{6}{4n-1}}-\cancel{\frac{6}{4n+3}}\right)+\left(\cancel{\frac{6}{4(n+1)-1}}-\frac{6}{4(n+1)+3}\right)$$

Thus, every term except the first and last term canceled out. This is the origin of the name **telescoping series**.

$$S_n=2-\frac{6}{4n+3}$$

We determine the convergence of this series by taking the limit of the partial sums:

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(2-\frac{6}{4n+3}\right)=2$$

The sequence of partial sums is convergent and so the series is convergent and has a value of 2. That is,

$$\sum_{k=0}^{\infty}\left(\frac{6}{4k-1}-\frac{6}{4k+3}\right)=2$$

Activity 2

Determine the convergence of the following series.

(a) $$\sum_{n=0}^{\infty}\left(\frac{1}{n^2+3n+2}\right)$$

(b) $$\sum_{n=0}^{\infty}\left(\frac{1}{n^2+4n+3}\right)$$

(c) $$\sum_{n=0}^{\infty}\left(\frac{1}{n^2+4n+3} - 7^n e^{-5n}\right)$$

4.3.1.3 DIVERGENT SERIES

Geometric series with $|r| \geq 1$ are not the only series to diverge.

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is known as the Harmonic Series and this series diverges to ∞.

We show graphically that the series is divergent.
Consider the graph of $y = \frac{1}{x}$ on the interval $[1, \infty)$ as shown in Fig. 1.

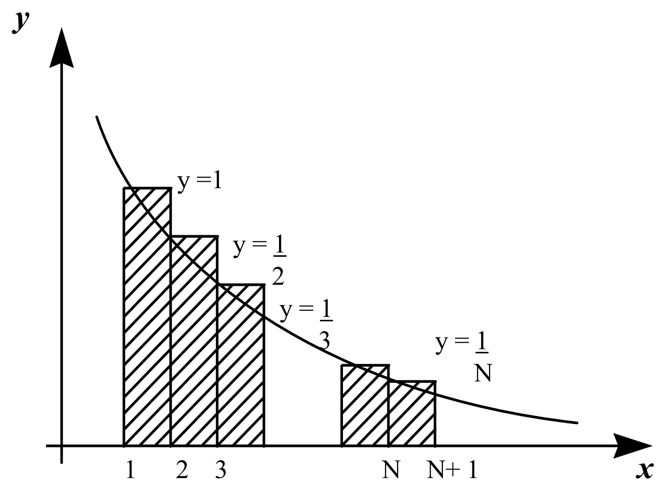


Figure 1

The partial sum

$$S_N = \sum_{n=1}^{N} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N}$$

$$= \text{sum of areas of shaded rectangles}$$

$$\geq \text{ area under curve } y = \frac{1}{x} \text{ from 1 to N+1}$$

$$= \int_1^{N+1} \frac{1}{x} dx = [\ln x]_1^{N+1}$$

$$= \ln(N+1) - \ln 1 = \ln(N+1).$$

We find that $S_N \geq \ln(N+1)$

Since $\ln(N+1) \to +\infty$ as $N \to +\infty$, we find that

$$\lim_{N\to+\infty} S_N = \sum_{n=1}^{\infty} \frac{1}{n} \geq \lim_{N\to+\infty} \ln(N+1) = +\infty$$

Since the sequence of partial sums $\{S_N\}$ diverges, the Harmonic Series is a divergent series.

4.3.1.4 THE NTH - TERM TEST FOR DIVERGENCE

If $\lim_{n\to\infty} u_n \neq 0$ or if $\lim_{n\to\infty} u_n$ fails to exist, then $\sum_{n=1}^{\infty} u_n$ diverges.

This test only says that a series is guaranteed to diverge if the series terms do not go to zero in the limit. If the series terms do happen to go to zero the series may or may not converge!

The condition that if $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n = 0$ is thus a <u>Necessary</u>

condition for convergence and <u>not sufficient</u> condition for the series $\sum_{n=1}^{\infty} u_n$ to converge.

To see that $\lim_{n\to+\infty} u_n = 0$ is not a sufficient condition for convergence, we consider the

Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$.

We have $\lim_{n\to+\infty} \frac{1}{n} = 0$, but the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example 4
Determine the convergence of the following series.

(a) $\sum_{n=1}^{\infty} \frac{1-2^n}{1+2^n}$

(b) $\sum_{n=1}^{\infty} \frac{n}{\alpha n+1}$, for $\alpha \neq 0$

(c) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

Solution

(a) Since $\lim\limits_{n\to\infty}\left(\dfrac{1-2^n}{1+2^n}\right)=\lim\limits_{n\to\infty}\left(\dfrac{\frac{1}{2^n}-1}{\frac{1}{2^n}+1}\right)=-1$, the series diverges by the n^{th} term test.

(b) By division, we get

$$\begin{array}{r|l} & \frac{1}{\alpha} \\ \hline \alpha n+1 & n \\ & n+\frac{1}{\alpha} \\ \hline & -\frac{1}{\alpha} \end{array}$$

Therefore

$$\lim_{n\to\infty}\frac{n}{\alpha n+1}=\lim_{n\to\infty}\left(\frac{1}{\alpha}-\frac{\frac{1}{\alpha}}{\alpha n+1}\right)$$

$$=\frac{1}{\alpha}\quad\text{since}\quad\frac{1}{\alpha n+1}\to 0\ \text{ as }\ n\to\infty.$$

Using the nth - Term Test for divergence, $\sum\limits_{n=1}^{\infty}\dfrac{n}{\alpha n+1}$ diverges since $\lim\limits_{n\to\infty}\dfrac{n}{\alpha n+1}\neq 0$.

(c) $\lim\limits_{n\to\infty} n\sin\left(\dfrac{1}{n}\right)=\lim\limits_{n\to\infty}\dfrac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$,

applying L'Hopital's rule, we obtain

$$\lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\cancel{\frac{-1}{n^2}}\cos\left(\frac{1}{n}\right)}{\cancel{\frac{-1}{n^2}}}=1$$

Hence, the series diverges.

Activity 3

(a) Determine the convergence of the following series.

$$\sum_{n=1}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}$$

(b) Consider the following series :

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+3} \right)^2 x^n$$

(i) Using the nth-term test for divergence, show that the series diverges when $x = \pm 1$.

(ii) Can you use the same test to conclude about the nature of the series with $|x| < 1$? Justify your answer.

(c) $$\sum_{n=1}^{\infty} \frac{5n}{\sqrt{7n^2 + 6}}$$

(d) $$\sum_{n=1}^{\infty} \sqrt[n]{7}$$

4.3.1.5 THE *P*-SERIES TEST

Theorem 5 If p is a real constant, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$.

4.3.1.6 COMPARISON TEST FOR CONVERGENCE

Let $\sum_{n=1}^{\infty} u_n$ be a series such that $u_n > 0$ for all n .

(a) **Test for Convergence**

Suppose $0 < u_n \le v_n$ for all $n > n_0$ where n_0 is some positive integer. If $\sum_{n=1}^{\infty} v_n$ converges, then $\sum_{n=1}^{\infty} u_n$ converges.

(b) **Test for Divergence**

Suppose $0 < v_n \le u_n$ for all $n > n_0$. If $\sum_{n=1}^{\infty} v_n$ diverges, then $\sum_{n=1}^{\infty} u_n$ diverges.

In other words, we have two series of positive terms and the terms of one of the series is always larger than the terms of the other series. If the larger series is convergent then the smaller series must also be convergent. Likewise, if the smaller series is divergent then the larger series must also be divergent.

Do not misuse this test. Just because the smaller of the two series converges does not say anything about the larger series. The larger series may still diverge. Likewise, just because we know that the larger of two series diverges we can not say that the smaller series will also diverge!

Example 5

Determine if the following series converge or diverge.

(a) $$\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2(n)}$$

(b) $$\sum_{n=1}^{\infty} \frac{n^2 + 3}{n^4 + 5}$$

Solution

(a) For n large, $\frac{n}{n^2-\cos^2(n)} \to \frac{1}{n}$ and also, $\frac{n}{n^2-\cos^2(n)} > \frac{1}{n}$.

Further, $\sum_{n=1}^{\infty}\frac{1}{n}$ is a harmonic series and is therefore divergent. Hence, by the comparison test, $\sum_{n=1}^{\infty}\frac{n}{n^2-\cos^2(n)}$ diverges (or by theorem 5 , the p-series test).

(b) Observe that $\frac{n^2+3}{n^4+5} < \frac{n^2+3}{n^4}$ and $\sum_{n=1}^{\infty}\frac{n^2+3}{n^4} = \sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{3}{n^4}$.

Thus, we can write the series as a sum of two series and both of these series are convergent by the *p*-series test. Since $\sum_{n=1}^{\infty}\frac{n^2+3}{n^4}$ is the sum of two convergent series, it is therefore convergent.
Further, the terms of this series are larger than the terms of the original series. By the Comparison Test, we conclude that the original series must also be convergent.

Activity 4

Use the Comparison Test to establish the convergence or divergence of each of the following series.

(a) $\frac{1}{3}+\frac{1}{5}+\frac{1}{9}+\frac{1}{17}+\frac{1}{33}+\dots$

(b) $\sum_{n=1}^{\infty}\frac{1}{n^3+4n^2+5}$

4.3.1.7 Limit Comparison Test

Suppose that $u_n > 0$ and $v_n > 0$ for all $n > 0$ and that $\lim_{n\to\infty} \frac{u_n}{v_n} = L$.

(i) If $0 < L < \infty$, then the two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ either both converge or both diverge.

(ii) If L = 0 and $\sum_{n=1}^{\infty} v_n$ converges, then $\sum_{n=1}^{\infty} u_n$ also converges.

(iii) If L= ∞ and $\sum_{n=1}^{\infty} v_n$ diverges, then $\sum_{n=1}^{\infty} u_n$ also diverges.

No conclusion can be reached otherwise.

Example 6

Use the limit Comparison Test to establish the convergence or divergence of each of the following series.

(a) $$\sum_{n=1}^{\infty} \frac{5^n}{7^n - n}$$

(b) $$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

(c) $$\sum_{n=1}^{\infty} \frac{n^3}{n^4 + n}$$

(d) $$\sum_{n=1}^{\infty} \frac{n + \sqrt{n}}{1 + n^2\sqrt{n}}$$

(a) We compare the series with $\sum_{n=1}^{\infty}\frac{5^n}{7^n}$. Note that $\lim_{n\to\infty}\frac{5^n/(7^n - n)}{5^n/7^n} = \lim_{n\to\infty}\frac{7^n}{7^n - n}$.

The latter limit is clearly an indeterminate case and we therefore use L'Hopital's rule to simplify it.

$$\lim_{n\to\infty}\frac{7^n}{7^n - n} = \lim_{n\to\infty}\frac{7^n \ln 7}{7^n \ln 7 - 1}.$$

Since we get another indeterminate case, we apply L'Hopital's rule again to obtain

$$\lim_{n\to\infty}\frac{7^n \ln 7}{7^n \ln 7 - 1} = \lim_{n\to\infty}\frac{7^n (\ln 7)^2}{7^n (\ln 7)^2} = 1.$$

So that,

$$\lim_{n\to\infty}\frac{5^n/(7^n - n)}{5^n/7^n} = 1.$$

$\sum_{n=1}^{\infty}\frac{5^n}{7^n}$ is a convergent geometric series and therefore, by the limit comparison test, $\sum_{n=1}^{\infty}\frac{5^n}{7^n - n}$ is convergent.

(b) We compare the series with $\sum_{n=2}^{\infty}\frac{1}{n}$. The ratio of the general terms for large n is given by

$$\lim_{n\to\infty}\frac{1/\ln n}{1/n} = \lim_{n\to\infty}\frac{n}{\ln n}.$$

Using L'Hopital's rule, we get $\lim_{n\to\infty}\frac{1/\ln n}{1/n} = \lim_{n\to\infty}\frac{n}{\ln n} = \lim_{n\to\infty} n = \infty$.

(c) For large n, $u_n = \frac{n^3}{n^4 + n}$ behaves like $v_n = \frac{1}{n}$. $\sum_{n=1}^{\infty} v_n$ diverges and $\frac{u_n}{v_n} = \frac{n^3}{n^4 + n}\Big/\frac{1}{n} = \frac{n^4}{n^4 + n} = \frac{1}{1 + \frac{1}{n^3}} \to 1 > 0$ as $n \to \infty$. Hence, $\sum_{n=1}^{\infty}\frac{n^3}{n^4 + n}$ diverges.

(d) For large n, $u_n = \dfrac{n+\sqrt{n}}{1+n^2\sqrt{n}}$ behaves as $v_n = \dfrac{1}{n^{3/2}}$. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges (p-series test0 . $\dfrac{u_n}{v_n} = \dfrac{1+\dfrac{1}{n^{1/2}}}{1+\dfrac{1}{n^{5/2}}} \to 1 > 0$ as $n \to \infty$. Hence, $\displaystyle\sum_{n=1}^{\infty}\frac{n+\sqrt{n}}{1+n^2\sqrt{n}}$ converges.

Activity 5

Use the limit comparison test to determine the convergence of the series

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2+1}}$

(b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}}\right)$

Hint : Compare it with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

(c) $\displaystyle\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}} - 1\right)$

(Hint : compare it with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$.

Indeed, for large n, the MacLaurin series of $e^{\frac{1}{n}} - 1 \approx \dfrac{1}{n} + \dfrac{1}{2n^2}$. We neglect the term $\dfrac{1}{2n^2}$)

(d) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$

4.3.1.8 THE RATIO TEST

Let $\sum_{n=1}^{\infty} u_n$ be a series with $u_n > 0$ and suppose that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$.

Then the series

(a) converges if L < 1
(b) diverges if L > 1.
(c) may converge or it may diverge if L = 1.

Example 7

(a) Consider the series $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$.

Solution

Here

$$u_n = \frac{3^n}{n^2} \quad \text{and} \quad u_{n+1} = \frac{3^{n+1}}{(n+1)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{3^{n+1}}{(n+1)^2}}{\frac{3^n}{n^2}} = \frac{3^{n+1}}{(n+1)^2 . 3^n} n^2 = \frac{3n^2}{(n+1)^2}$$

$$= \frac{3n^2}{n^2 + 2n + 1}$$

$$= \frac{3}{1 + \frac{2}{n} + \frac{1}{n^2}}$$

Now,

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{3}{\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)} = 3$. Since L = 3 > 1, the series $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$ diverges.

(b) Next, consider the series $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$

Solution

We have $u_n = \frac{(n!)^2}{(2n)!}$ and we simplify U_{n+1} to obtain

$$u_{n+1} = \frac{((n+1)!)^2}{(2(n+1))!} = \frac{(n+1)!\ (n+1)!}{(2n+2)!}$$

$$= \frac{n!(n+1)n!(n+1)}{(2n+2)(2n+1)(2n)!}$$

$$= \frac{(n\ !)^2}{(2n)!}\frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= u_n\frac{(n+1)^2}{2(n+1)(2n+1)}$$

Thus $\frac{u_{n+1}}{u_n} = \frac{(n+1)}{2(2n+1)}$ after simplification.

We then find that

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n+1}{4n+2} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{4+\frac{2}{n}}$$

$$= \frac{1}{4}.$$

Here $L = \frac{1}{4} < 1$, thus $\sum_{n=1}^{\infty}\frac{(n\ !)^2}{(2n)!}$ converges.

Next, consider the series $\sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

We apply the Ratio-Test to both series:

for $\sum_{n=1}^{\infty} \frac{1}{n}$, we have $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}}$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= 1 .$$

No conclusion can be reached by the Ratio-Test. However, we know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

For $\sum_{n=1}^{\infty} \frac{1}{n^2}$, we have $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

The Ratio-Test provides no information in this case also. But, we have seen that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent *p*-series with $p = 2$.

Thus, we have shown that when $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, the series $\sum_{n=1}^{\infty} u_n$ may either converge or diverge.

Activity 6

Do the following series converge or diverge?

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n n!}$ (b) $\sum_{n=1}^{\infty} \frac{n^4}{2^n}$ (c) $\sum_{n=1}^{\infty} n^2 e^{-n}$ (d) $\sum_{n=1}^{\infty} \frac{n^3}{n!}$

4.4 STRATEGIES FOR SERIES

Here is a general set of guidelines to help you determine the convergence of a series. Note that these are a general set of guidelines and because some series can have more than one test applied to them, we will get a different result depending on the path that we take through this set of guidelines.

1. With a quick glance does it look like the series terms do not converge to zero in the limit, *i.e.* does $\lim_{n\to\infty} u_n \neq 0$? If so, use the Divergence Test. Note that you should only do the divergence test if a quick glance suggests that the series terms may not converge to zero in the limit.

2. Is it a p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ or a geometric series $\sum_{n=0}^{\infty} a\, r^n$? If so, then p-series converge for p > 1 and a geometric series converges if $|\, r \,| < 1$.

3. Does the series behave as a p-series or a geometric series for large values of n? If so, use the comparison test.

4. Is the series a rational expression involving only polynomials or polynomials under radicals (*i.e.* a fraction involving only polynomials or polynomials under radicals)? If so, try the Comparison Test or the Limit Comparison Test.

5. Does the series contain factorials or constants raised to powers involving n? If so, then the Ratio Test may work. Note that if the series term contains a factorial then very often, the most suitable test is the Ratio Test.

4.5 POWER SERIES

When studying infinite series, we have given some tests to determine whether a given series converges or not. In this section, we study infinite polynomials called *Power Series.*

Definition

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots + c_n(x-a)^n + \ldots$$

in which the center *a* and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Example 11

The geometric power series with center 0 is given by

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

The power series converges to $\dfrac{1}{1-x}$ when $|x| < 1$.

Example 12

The power series $1 - \frac{1}{3}(x-1) + \frac{1}{9}(x-1)^2 + \ldots + \left(-\frac{1}{3}\right)^n (x-1)^n + \ldots$

has center a = 1 and $c_n = \left(-\frac{1}{3}\right)^n$.

Expressing $1-\frac{1}{3}(x-1)+\frac{1}{9}(x-1)^2+...+\left(-\frac{1}{3}\right)^n(x-1)^n+...$ in the form $\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n(x-1)^n$, we find that for a fixed value x, the series is a geometric series.

Thus the series converges provided the common ratio r satisfies $|r| < 1$.

Here $r=-\frac{1}{3}(x-1)$, and hence there is convergence when

$\left|-\frac{1}{3}(x-1)\right| < 1$, that is, for $-2 < x < 4$,

and the sum is given by

$$\frac{1}{1-\frac{1}{3}(x-1)}=\frac{3}{3+(x-1)}=\frac{3}{x+2}$$

Thus we can write

$$\frac{3}{2+x}=1-\frac{1}{3}(x-1)+\frac{1}{9}(x-1)^2+...+\left(-\frac{1}{3}\right)^n(x-1)^n+... \quad \text{for } -2 < x < 4 .$$

We find that the function $\frac{3}{2+x}$ can be expressed as a power series for $-2 < x < 4$.

4.6 TAYLOR SERIES AND MACLAURIN SERIES

One of the most important theorems in Calculus is Taylor's Theorem. This theorem provides an estimate for the error involved when a function $f(x)$ is approximated near the point $x = a$ by the polynomial of degree n in (x - a) which best describes the behavior of the function $f(x)$ near that point.

4.6.1 TAYLOR'S THEOREM

Let f have continuous derivatives up to, and including order $n+1$ on some interval containing the point $x = a$. Then, if x is any other point in the interval, $f(x)$ can be expressed in the form

$$f(x) = P_n(x) + R_{n+1}(x)$$

where

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a)$$

and $R_{n+1}(x) = \dfrac{(x-a)^{n+1}}{(n+1)!}f^{n+1}(\xi)$ where $a < \xi < x$

$P_n(x)$ is called the Taylor polynomial of order n generated by f at a and $R_{n+1}(x)$ is the remainder term.

Example 13

Find the Taylor polynomial of order 3 generated by $f(x) = \dfrac{1}{x}$ about $x = 1$. Write down the remainder term.

Solution

Here $a = 1$ and the Taylor polynomial of order 3 is given by

$$P_3(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f^3(1)$$

We compute $f(1)$, $f'(1)$, $f''(1)$ and $f^3(1)$. We find that $f(1) = 1$ and

$$f'(x) = \frac{-1}{x^2} = -x^{-2} \qquad f'(1) = -1$$

$$f''(x) = 2x^{-3} \qquad f''(1) = 2$$

$$f^3(x) = -6x^{-4} \qquad f^3(1) = -6$$

$$f^4(x) = 24x^{-5} \qquad f^4(1) = 24$$

We then have $P_3(x) = 1 + (x-1)(-1) + \frac{(x-1)^2}{2!}(2) + \frac{(x-1)^3}{3!}(-6)$.

Thus, the cubic approximation to the function $\frac{1}{x}$ about $x = 1$ is given by

$$P_\xi(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3.$$

The remainder term is $R_4(x) = \frac{(x-1)^4}{4!}f^4(\xi)$ where $1 < \xi < x$.

Thus $$R_4(x) = \frac{(x-1)^4}{24}\cdot\frac{24}{\xi^5} = \frac{(x-1)^4}{\xi^5}$$

Thus we can write

$$f(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \frac{(x-1)^4}{\xi^5}.$$

Example 14

Find the Taylor polynomial of order 4 generated by $f(x) = e^x$ at $a = 0$. Write down the remainder term.

Solution

$f(x) = e^x \qquad\qquad f(0) = e^0 = 1$

$f'(x) = e^x \qquad\qquad f'(0) = e^0 = 1$

$f''(x) = e^x \qquad\qquad f''(0) = e^0 = 1$

$f^3(x) = e^x \qquad\qquad f^3(0) = e^0 = 1$

$f^4(x) = e^x \qquad\qquad f^4(0) = e^0 = 1$

The Taylor polynomial of order 4 is

$$P_4(x) = f(0) + (x-0)f'(0) + \frac{(x-0)^2}{2!}f''(0) + \frac{(x-0)^3}{3!}f^3(0) + \frac{(x-0)^4}{4!}f^4(0)$$

$$= 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

The remainder term $R_5(x)$ is given by $R_5(x) = \dfrac{(x-0)^5}{5!}f^5(\xi)$ where $a < \xi < x$

Since $f^5(x) = e^x$, we obtain $R_5(x) = \dfrac{x^5}{5!}e^{\xi}$.

Thus we can write

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}e^{\xi} \quad \text{where } 0 < \xi < x.$$

Activity 7

Find the Taylor polynomial of order 4 generated by $f(x) = \cos x$ at $a = \dfrac{\pi}{3}$.

The Taylor series corresponding to a = 0 is called a *Maclaurin series*.

4.6.2 TAYLOR'S FORMULA FOR FUNCTIONS OF TWO VARIABLES

In this section, we extend Taylor's formula introduced in Section 4.6, to functions of two variables.

Taylor's Formula for $f(x,y)$ at the point (*a,b*)

Suppose $f(x,y)$ and its partial derivatives are continuous throughout an open rectangular region R centred at a point (*a,b*). Then throughout R

$$f(a+h,b+k) = f(a,b) + h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b) +$$

$$\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2hk\frac{\partial^2 f}{\partial x\partial y}(a,b) + k^2\frac{\partial^2 f}{\partial y^2}(a,b)\right)$$

$$+\frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right)_{(a,b)} +\ldots$$

<u>or</u>

for $(x,y) \in$R

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) +$$

$$\frac{1}{2!}\left((x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right) +\ldots$$

Example 15

Use Taylor's formula to find a quadratic polynomial that approximates $f(x, y) = \sinh x \sinh y$ near the origin.

Solution

The required quadratic polynomial

$$f(x,y) \approx f(0,0) + (x-0)\frac{\partial f}{\partial x}(0,0) + (y-0)\frac{\partial f}{\partial y}(0,0)$$

$$+\frac{1}{2!}\left[(x-0)^2\frac{\partial^2 f}{\partial x^2}(0,0) + 2xy\frac{\partial^2 f}{\partial x \partial y}(0,0) + y^2\frac{\partial^2 f}{\partial y^2}(0,0)\right].$$

$f(x,y) = \sinh x \sinh y \qquad f(0,0) = \sinh 0 \sinh 0 = 0$

$\dfrac{\partial f}{\partial x} = \cosh x \sinh y \qquad \dfrac{\partial f}{\partial x}(0,0) = \cosh 0 \sinh 0 = 0$

$\dfrac{\partial f}{\partial y} = \sinh x \cosh y \qquad \dfrac{\partial f}{\partial y}(0,0) = \sinh 0 \cosh 0 = 0$

$\dfrac{\partial^2 f}{\partial x^2} = \sin x \sinh y \qquad \dfrac{\partial^2 f}{\partial x^2}(0,0) = 0$

$\dfrac{\partial^2 f}{\partial x \partial y} = \cosh x \cosh y \qquad \dfrac{\partial^2 f}{\partial x \partial y}(0,0) = \cosh 0 \cosh 0 = 1$

$\dfrac{\partial^2 f}{\partial y^2} = \sinh x \cosh y \qquad \dfrac{\partial^2 f}{\partial y^2}(0,0) = 0$

Thus

$$f(x,y) \approx 0 + (x-0).0 + (y-0).0 + \frac{1}{2!}\left[x^2.0 + 2\,xy.1 + y^2.0\right] \approx xy$$

The quadratic polynomial is *xy*.

Example 16

Use Taylor's formula to find a quadratic polynomial that approximates $e^x \ln(1+y)$ near the origin.

Solution

Here

$$f(x,y) = e^x \ln(1+y) \qquad f(0,0) = e^0 \ln(1+0) = 1\ln 1 = 0$$

$$\frac{\partial f}{\partial x} = e^x \ln(1+y) \qquad \frac{\partial f}{\partial x}(0,0) = 0$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{1+y} \qquad \frac{\partial f}{\partial y}(0,0) = \frac{e^0}{1+0} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \ln(1+y) \qquad \frac{\partial^2 f}{\partial x^2}(0,0) = e^0 \ln 1 = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{e^x}{1+y} \qquad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-e^x}{(1+y)^2} \qquad \frac{\partial^2 f}{\partial y^2}(0,0) = \frac{-e^0}{(1+0)^2} = -1$$

The required quadratic polynomial is

$$f(x,y) \approx 0 + (x-0).0 + (y-0).1 + \frac{1}{2!}\left[(x-0)^2.0 + 2xy.1 + (y-0)^2.-1\right]$$

$$\approx y + \frac{1}{2}\left(2xy - y^2\right)$$

$$\approx y + xy - \frac{1}{2}y^2$$

Activity 8

Use Taylor's formula for $f(x,y)$ at the origin to find a quadratic polynomial approximation of f near the origin :

(i) $f(x,y) = e^{2x}\cos 3y$

(ii) $f(x,y) = \cos\,(x + y)$

(iii) $f(x,y) = \dfrac{1}{1 - x - y}$

4.7 SUMMARY

In this unit, you have learnt the following:

1. An ***infinite series*** is an expression of the form

$$u_1 + u_2 + \ldots + u_n + \ldots = \sum_{n=1}^{\infty} u_n \; .$$

The sum given by $S_N = \sum_{n=1}^{N} u_n$ gives a sequence of partial sums $\{S_N\}$.

2. An infinite series is said to be convergent to the limit L if the sequence of partial sums converges to the limit L. Otherwise, it is said to be divergent.

3. A geometric series is convergent if $|r| < 1$ and divergent if $|r| \geq 1$

4. ***The nth Term Test for Divergence***:

   The series $\sum_{n=1}^{\infty} u_n$ is divergent if for the nth term u_n, $\lim_{n \to +\infty} u_n$ is non zero or does not exist.

5. ***Comparison Test for Convergence***

Let $\sum_{n=1}^{\infty} u_n$ be a series such that $U_n > 0$ for all n.

(i) Suppose $0 < u_n \leq v_n$ for all $n > n_0$, where n_0 is some positive integer. If $\sum_{n=1}^{\infty} v_n$ converges, then $\sum_{n=1}^{\infty} u_n$ converges.

(ii) Suppose $0 < v_n \leq u_n$ for all $n > n_0$. If $\sum_{n=1}^{\infty} v_n$ diverges, then $\sum_{n=1}^{\infty} u_n$ diverges.

- The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$

6. ***Limit Comparison Test***

Suppose that $u_n > 0$ and $v_n > 0$ for all $n > 0$ and that $\lim_{n \to +\infty} \frac{u_n}{v_n} = L$.

(iv) If $0 < L < \infty$, then the two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ either both converge or both diverge.

(v) If $L = 0$ and $\sum_{n=1}^{\infty} v_n$ converges, then $\sum_{n=1}^{\infty} u_n$ also converges.

(vi) If $L = \infty$ and $\sum_{n=1}^{\infty} v_n$ diverges, then $\sum_{n=1}^{\infty} u_n$ also diverges.

(vii) No conclusion can be reached otherwise.

7. ***Ratio Test***

Let $\sum_{n=1}^{\infty} u_n$ be a series with $u_n > 0$ and suppose $\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = L$, then

(i) The series converges if $L < 1$.

(ii) The series diverges if $L > 1$.

(iii) The series may converge or diverge if $L = 1$.

8. ***Taylor's Theorem***

Let f have continuous derivatives up to, and including order $n+1$ on some interval containing the point $x = a$. Then, if x is any other point in the interval,

$$f(x) = P_n(x) + R_{n+1}(x)$$

where the Taylor polynomial is given by

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a)$$

and the remainder term is given by

$$R_{n+1}(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{n+1}(\xi) \quad \text{for} \quad a < \xi < x\,.$$

9. Taylor's formula a function f (x,y) of two variables *x* and *y* at the point (a,b).

Let f (x,y) and its partial derivatives be continuous throughout a region *R* centered at a point (a,b). Then for (x,y) $\in R$,

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) +$$

$$\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right] + \ldots$$

4.8 ANSWERS TO ACTIVITIES

Activity 1

(a) Note that

$$\lim_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^{\beta n} = \lim_{n\to\infty} e^{\beta n \ln\left(1+\frac{\alpha}{n}\right)},$$

by the continuity of the exponential function,

$$\lim_{n\to\infty} e^{\beta n \ln\left(1+\frac{\alpha}{n}\right)} = e^{\lim_{n\to\infty} \beta n \ln\left(1+\frac{\alpha}{n}\right)}.$$

Let $x = \frac{1}{n}$, we then get $x \to 0$ as $n \to \infty$ and
Therefore,

$$e^{\lim_{n\to\infty} \beta n \ln\left(1+\frac{\alpha}{n}\right)} = e^{\lim_{x\to 0} \frac{\beta}{x} \ln(1+\alpha x)}.$$

Applying L'Hopital's rule, we hence obtain

$$e^{\lim_{x\to 0} \beta \frac{\ln(1+\alpha x)}{x}} = e^{\lim_{x\to 0} \beta \frac{\alpha}{1+\alpha x}} = e^{\alpha\beta}$$

(b) e^{-2}

Activity 2

(a) Write the term of the series as partial fractions :

$$\frac{1}{n^2+3n+2}=\frac{1}{n+1}-\frac{1}{n+2}$$

$n+1$ We write the terms of the general partial sum for this series using the partial fraction form to get

$$S_n=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\ldots+\left(\cancel{\frac{1}{n}}-\cancel{\frac{1}{n+1}}\right)+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+2}\right)$$

This is a telecosping sum and the partial sum is given by :

$$S_n=1-\frac{1}{n+2}$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+2}\right)=1$$

The sequence of partial sums is convergent and so the series is converges to 1.

(b) same procedure as in (a). convergent and limit is $\frac{5}{12}$.

(c) $$\sum_{n=0}^{\infty}\left(\frac{1}{n^2+4n+3}-7^n e^{-5n}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{n^2+4n+3}\right)-\sum_{n=0}^{\infty}7^n e^{-5n}$$

and this is the sum of two convergent sequences. Hence it is a convergent sequence.

Activity 3

(a) $\lim\limits_{n\to\infty}\frac{4n^2-n^3}{10+2n^3}=-\frac{1}{2}$. The limit of the series terms is not zero and so by nth-term test for divergence the series diverges.

(b)(i) I When $|x|=1, |u_n|=\left(\frac{n+1}{2n+3}\right)^2\to\frac{1}{4}$ as $n\to\infty$. Therefore, $u_n \not\to 0$ as $n\to\infty$

(ii) In that case, $u_n\to 0$ as $n\to\infty$ and we can not conclude anything about the convergence/divergence of the series. This is neither a sufficient condition for convergence nor does it indicate that the series diverges. We need to revert to other tests.

(c) $\lim\limits_{n\to\infty}\frac{5n}{\sqrt{7n^2+6}}=\lim\limits_{n\to\infty}\frac{5}{\sqrt{7+\frac{6}{n^2}}}=\frac{5}{\sqrt{7}}$.

Hence the series diverges by the nth term test for divergence.

(d) $\lim_{n\to\infty} \sqrt[n]{7} = \lim_{n\to\infty} 7^{1/n} = 1$

Hence the series diverges by the nth term test for divergence.

Activity 4

(a) $0 \le u_n = \frac{1}{2^n + 1} \le \frac{1}{2^n}$, for all $n \in N$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (it is a geometric series with common ratio r = $\frac{1}{2}$), it follows, by the comparison test $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$ converges

(b) For all integers n > 0, $\frac{1}{n^3 + 4n^2 + 5} < \frac{1}{n^3}$. Also, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent series by the p-series test. Therefore, by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^3 + 4n^2 + 5}$ is convergent.

Activity 5

(a) We compare this series with the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the series is also divergent

(b) Divergent

(c) Divergent

(d) For large n, $\sin\frac{1}{n^2}$ is approximately equal to $\frac{1}{n^2}$ as $\frac{1}{n^2}$ is small.

We take $u_n = \sin\frac{1}{n^2}$ and $v_n = \frac{1}{n^2}$.

Then

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{x\to 0}\frac{\sin x}{x} \qquad \text{by putting } x = \frac{1}{n^2}$$

$$= \lim_{x\to 0}\frac{\cos x}{1} \qquad \text{by l'Hôpital's Rule}$$

$$= \frac{\cos 0}{1} = 1 \qquad (0 < L < \infty)$$

The series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (this is a p-series with $p = 2 > 1$).

Therefore $\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$ also converges.

Activity 6

(a) Convergent (b) Convergent (c) Divergent

(d) Convergent (by the comparison test) as $\frac{\sin^2 n}{n^3} \le \frac{1}{n^3}$ since $\sin^2 n \le 1$. As $\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges, $\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^3}$ converges.

Activity 7

$$\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{4}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{48}\left(x-\frac{\pi}{3}\right)^4$$

Activity 8

(i) $1+2x+2x^2-\dfrac{9y^2}{2}$

(ii) $1-\dfrac{1}{2}x^2-xy-\dfrac{1}{2}y^2$

(iii) $1+x+y+x^2+2xy+y^2$