UNIT 1 FURTHER DIFFERENTIATION AND INTEGRATION

Unit Structure

1.0 OVERVIEW

The objective of this Unit is to introduce further methods of differentiation and integration. The main contents are as follows:

- Logarithmic differentiation.
- Differentiation of inverse trigonometric functions.
- Parametric differentiation.
- Taylor and Maclaurin series.
- Integrals involving trigonometric functions.
- Integrals involving inverse trigonometric functions.
- Integration by parts.
- General properties of the definite integral.

1.1 LEARNING OBJECTIVES

By the end of this unit, you should be able to do the following:

1. Logarithmic differentiation.

2. Differentiation of inverse trigonometric functions.

3. Parametric differentiation.

4. Expansion of functions in power series

5. Integration of and involving inverse trigonometric functions.

6. Integration by parts.

1.2 FURTHER DIFFERENTIATION

1.2.1 Review

Let n be a constant, u and v be functions of x.

Table 1.1: Differentiation

1	$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$	
2	$y = (f(x))^n$	$\frac{dy}{dx} = n(f(x))^{n-1} f'(x)$	$f'(x) = \frac{df(x)}{dx} = \frac{df}{dx}$
3	$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$	*Product rule for* $\frac{dy}{dx}$
4	$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	*Quotient Rule*
5	$\frac{dy}{dx} = 1\Big/\frac{dx}{dy}$, $\frac{dx}{dy} = 1\Big/\frac{dy}{dx}$		*y is a function of x only*
6	$\frac{dy}{dx} = \frac{dy}{dz}\Big/\frac{dx}{dz} = \frac{dy}{dz} \times \frac{dz}{dx}$		*Chain Rule given that y and x are functions of z*
7	$\frac{d\ln x}{dx} = \frac{1}{x}$, $\frac{d\ln f(x)}{dx} = \frac{1}{f(x)} f'(x)$		*Natural log*
8	$\frac{d(e^x)}{dx} = e^x$, $\frac{de^{f(x)}}{dx} = e^{f(x)} f'(x)$, $\frac{d(a^x)}{dx} = a^x \ln a$		*Exponential of functions and a > 0*
9	$\frac{d\sin(x)}{dx} = \cos(x)$, $\frac{d\cos(x)}{dx} = -\sin(x)$, $\frac{d\tan(x)}{dx} = \sec^2(x)$		*Trigonometric functions*
10	$\frac{d\sin(f(x))}{dx} = \cos(f(x))f'(x)$, $\frac{d\cos(f(x))}{dx} = -\sin(f(x))f'(x)$, $\frac{d\tan(f(x))}{dx} = \sec^2(f(x))f'(x)$		
11	$\frac{d}{dx}g(y) = \frac{dg(y)}{dy}\frac{dy}{dx} = g'(y)\frac{dy}{dx}$		*Implicit functions*

Integration		
1	$\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1}+c, & n\neq -1 \\ \ln\lvert x\rvert + c, & n=-1 \end{cases}$ $\int \frac{1}{x}dx = \ln\lvert x\rvert + c$	
2	$\int (ax+b)^n dx = \begin{cases} \frac{(ax+b)^{n+1}}{(n+1)a}+c, & n\neq -1 \\ \frac{1}{a}\ln\lvert ax+b\rvert + c, & n=-1 \end{cases}$	*Can be applied to power of a linear expression in x only*
3	$\int v\left[\frac{du}{dx}\right]dx = uv - \int u\left[\frac{dv}{dx}\right]dx$	*Integration by parts*
4	$\int \frac{f'(x)}{f(x)}dx \equiv \int \frac{df(x)}{f(x)} = \ln\lvert f(x)\rvert + c$	*Function derivative in numerator*
5	$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + C$	*Function derivative in numerator*
6	$\int e^x dx = e^x + c$, $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$, $\int a^x\,dx = \frac{a^x}{\ln a} + c$	*Exponentials*
7	$\int \cos(x)dx = \sin(x) + c$, $\int \sin(x)dx = -\cos(x) + c$, $\int \sec^2(x)dx = \tan(x) + c$	*Trigonometry*
8	$\int \cos(ax+b)dx = \frac{1}{a}\sin(x) + c$, $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$, $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$	

Table 1.2

Tables 1.1-1.3 summarises all the basic formulae and rules that students need to know before going deep through advanced differentiation and integration.

Common Mistakes	
1	If $y = 3^x$, then $\frac{dy}{dx} \neq x3^{x-1}$
2	$\frac{d^2y}{dx^2} \neq u\frac{d^2v}{dx^2} + v\frac{d^2u}{dx^2}$.
3	$\frac{d^2x}{dy^2} = -\sin y$; but, $\frac{d^2y}{dx^2} \neq 1/-\sin y$
4	$\int 2^x \, dx \neq \frac{2^{x+1}}{x+1} + C$
5	$\int \sin x^2 \, dx \neq \frac{-\cos x^2}{2x} + C$. $\int e^{x^2} \, dx \neq \frac{e^{x^2}}{2x} + c$

Table 1.3

1.2.2 Logarithmic Differentiation

When a function consists of a number of factors it is often convenient to take logarithms before differentiating. This will transform the problem of differentiating a product into that of differentiating a sum.

A similar method can be applied to find the derivative of the function $y = u^v$, where *u* and *v* are functions of *x*.

Example 1

Find the derivative w.r.t. *x* of $\dfrac{(x^2+1)^{1/2}\,(x-1)^2}{(x+1)^{3/2}}$.

Step 1: Let $y = \dfrac{(x^2+1)^{1/2}\,(x-1)^2}{(x+1)^{3/2}}$.

Then $\ln y = \frac{1}{2}\ln(x^2+1) + 2\ln(x-1) - \frac{3}{2}\ln(x+1)$.

Step 2: Differentiating both sides w.r.t. *x*, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\,\frac{2x}{x^2+1} + \frac{2}{x-1} - \frac{3}{2(x-1)}$$

$$= \frac{3x^3+7x^2-x+7}{2(x^2+1)(x-1)(x+1)}$$

Step 3: Make $\dfrac{dy}{dx}$ subject of formula

$$\therefore \quad \frac{dy}{dx} = \frac{3x^3+7x^2-x+7}{2(x^2+1)(x-1)(x+1)} \cdot \frac{(x^2+1)^{1/2}\,(x-1)^2}{(x+1)^{3/2}}$$

$$= \frac{(3x^3+7x^2-x+7)(x-1)}{2(x^2+1)^{1/2}\,(x+1)^{5/2}}.$$

Example 2 Find $\dfrac{d}{dx}(a^x)$, $a > 0$.

Step 1: Let $y = a^x$.

Then on taking logs, we obtain $\ln y = x \ln a$.

Step 2: Now, differentiating w.r.t. *x*, we have

$$\frac{1}{y}\frac{dy}{dx} = \ln a$$

$$\boxed{\therefore \frac{dy}{dx} = a^x \ln a}$$

Note: (i) $\frac{d}{dx}2^x = 2^x \ln 2$

(ii) $\frac{d}{dt}19^{-t} = -19^{-t} \ln 19$.

<u>**Example 3**</u>

Differentiate $x^{\sin x} + (\ln x)^x$ w.r.t. *x*.

Note: We can't take logs directly since $\ln[x^{\sin x} + (\ln x)^x] \neq \ln x^{\sin x} + \ln[(\ln x)^x]$ **.**

We therefore do it in two parts, separately.

Step 1: Let $y_1 = x^{\sin x}$, $y_2 = (\ln x)^x$, so that $\frac{d}{dx}[x^{\sin x} + (\ln x)^x] = \frac{dy_1}{dx} + \frac{dy_2}{dx}$.

Step 2: Now, $\ln y_1 = \sin x . \ln x$ and $\ln y_2 = x . \ln(\ln x)$

Differentiating w.r.t. *x*, we have

$$\frac{1}{y_1}\frac{dy_1}{dx} = \sin x \cdot \frac{1}{x} + \ln x \cdot \cos x$$

$$\therefore \frac{dy_1}{dx} = x^{\sin x}\left(\sin x \cdot \frac{1}{x} + \ln x \cdot \cos x\right)$$

Also, $\frac{1}{y_2}\frac{dy_2}{dx} = x.\frac{1}{\ln x}.\frac{1}{x} + \ln(\ln x)$

$$\therefore \frac{dy_2}{dx} = (\ln x)^x \left(\frac{1}{\ln x} + \ln(\ln x)\right)$$

Step 3: Hence

$$\frac{d}{dx}[x^{\sin x} + (\ln x)^x] = x^{\sin x}\left(\frac{\sin x}{x} + \ln x.\cos x\right) + (\ln x)^x \left(\frac{1}{\ln x} + \ln(\ln x)\right).$$

<u>Activity 1</u>

1. Differentiate w.r.t. x each of the following:

   (i) $(x+2)^2\,(x+3)^3\,(x+4)^4$;

   (ii) $\sqrt{\frac{x-1}{x+1}}$;

   (iii) $\frac{\sqrt{x-1}\,(x+1)}{x^2+1}$;

   (iv) $a^{x^2+9}, \quad a > 0$;

   (v) $(\cot x)^{\sin x} + (\tan x)^{\cos x}$.

2. Using the fact that $\frac{d}{dx}a^x = a^x \ln a\ (a > 0)$, deduce that $\int a^x\,dx = \frac{a^x}{\ln a} + C$.

1.2.2 Differentiation of Inverse Trigonometric Functions

Firstly let us consider the differentiation of the following functions:

	Differentiation	Proof
1	$\frac{d[\cot(x)]}{dx} = -\operatorname{cosec}^2(x)$	$\frac{d[\cot(x)]}{dx} = \frac{d\left[\frac{1}{\tan(x)}\right]}{dx}$ $= \frac{-\sec^2 x}{\tan^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2(x)$
2	$\frac{d[\operatorname{cosec}(x)]}{dx} = -\operatorname{cosec}(x)\cot(x)$	$\frac{d[\operatorname{cosec}(x)]}{dx} = \frac{d\left[\frac{1}{\sin(x)}\right]}{dx}$ $= \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x}\frac{\cos x}{\sin x} = -\operatorname{cosec}(x)\cot(x)$
3	**Exercise:** Prove that $\frac{d[\sec(x)]}{dx} = \sec(x)\tan(x)$.	

Table 1.4

<u>**Principal values of inverse trigonometric functions**</u> $\sin^{-1}(x)$**,** $\cos^{-1}(x)$ **and** $\tan^{-1}(x)$**.**

Remark: Inverse trigonometric functions are ***not*** the reciprocals of the trig functions, i.e.,

$$\sin^{-1} x \neq \frac{1}{\sin x}, \quad \tan^{-1} x \neq \frac{1}{\tan x}, \text{ etc.}$$

The principal values of $\sin^{-1} x$ are values of $\theta = \sin^{-1} x$ for which the trigonometric function $x = \sin(\theta)$ is a one to one function such that it inverse exists. Similar interpretation can be made for the other trigonometric inverse functions.

		Principle values
1	$-1 \le x \le 1$	$-\dfrac{\pi}{2} \le \theta = \sin^{-1} x \le \dfrac{\pi}{2}$
2	$-1 \le x \le 1$	$0 \le \theta = \cos^{-1} x \le \pi$
3	$-\infty < x < \infty$	$-\dfrac{\pi}{2} < \theta = \tan^{-1} x < \dfrac{\pi}{2}$

Table 1.5

Figure 1.1-1.3 demonstrate relationship between trigonometry and inverse trigonometry for $\sin^{-1}(x)$**,** $\cos^{-1}(x)$ **and** $\tan^{-1}(x)$**.**

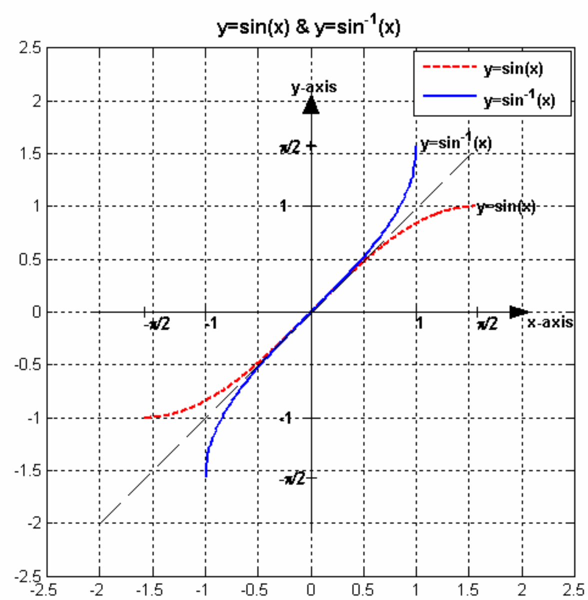

Figure 1.1

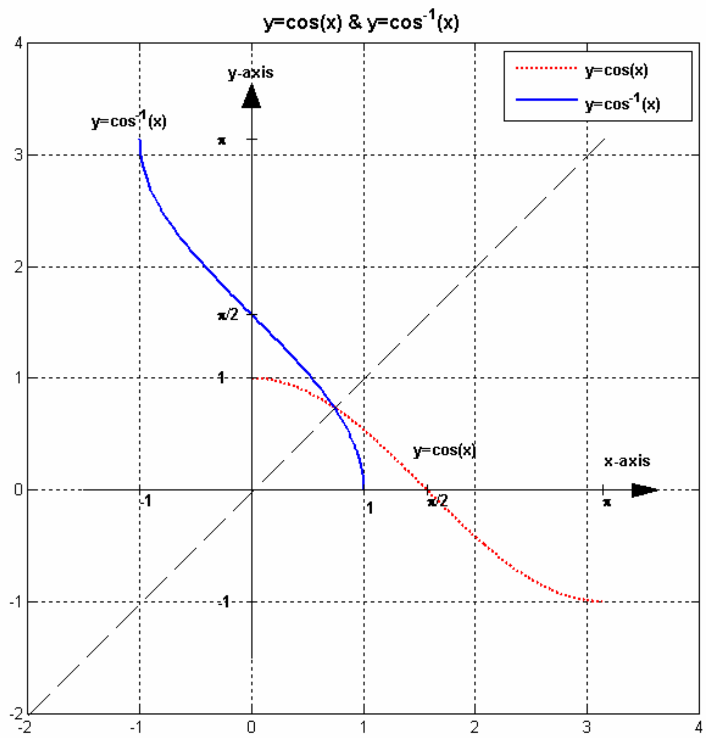

Figure 1.2

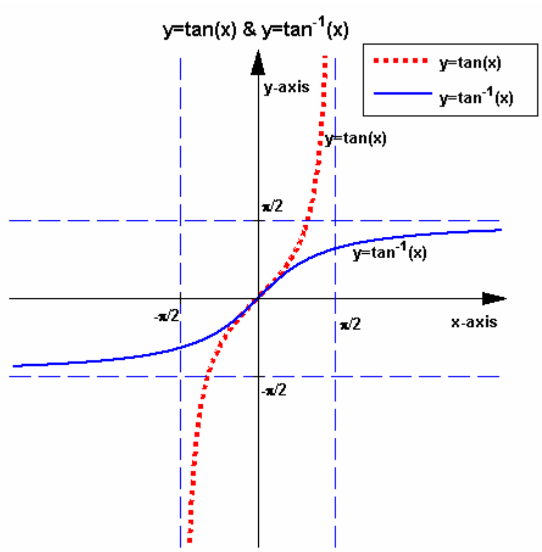

Figure 1.3

We next consider the differentiation of some important inverse trigonometry functions:

	Let a be a positive constant such that $\lvert x \rvert \neq a$	
(i)	$\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}}$,	$\lvert x \rvert < a$
(ii)	$\frac{d}{dx}\cos^{-1}\frac{x}{a} = \frac{-1}{\sqrt{a^2 - x^2}}$,	$\lvert x \rvert < a$
(iii)	$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{1}{a^2 + x^2}$	
(iv)	$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1 + x^2}$	
(v)	$\frac{d}{d}\text{se}^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$,	$\lvert x \rvert > 1$
(vi)	$\frac{d}{d}\text{cose}^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}}$,	$\lvert x \rvert > 1$

Table 1.6

Example 4: Prove that $\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}}$.

Proof:

Step 1: Let $y = \sin^{-1}\frac{x}{a}$, then $x = a\sin y$

Step 2: Differentiating x w.r.t to y, we have

$$\frac{dx}{dy} = a\cos y = a\sqrt{1 - \sin^2 y} = a\sqrt{1 - \left(\frac{x}{a}\right)^2}$$

Step 3: Using $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$, we have

$$\frac{dy}{dx} = a\frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = a\frac{1}{\sqrt{\frac{a^2 - x^2}{a^2}}} = \frac{1}{\sqrt{a^2 - x^2}} \text{ where } \lvert x \rvert = \lvert a\sin y \rvert < a \; .$$

Example 5: Prove that $\frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2 - x^2}$.

Proof:

Step 1: Let $y = \tan^{-1}\frac{x}{a}$, then $x = a\tan y$

Step 2: Differentiating x w.r.t to y, we have

$$\frac{dx}{dy} = a\sec^2 y = a(1 - \tan^2 y) = a(1 - \left(\frac{x}{a}\right)^2)$$

Step 3: Using $\frac{dy}{dx} = 1\Big/\frac{dx}{dy}$, we have

$$\frac{dy}{dx} = \frac{1}{a(1 - \left(\frac{x}{a}\right)^2)} = \frac{1}{a(\frac{a^2 - x^2}{a^2})} = \frac{a}{a^2 - x^2}.$$

Example 6: Prove that $\frac{d}{dx}\cos ec^{-1}x = \frac{-1}{x\sqrt{x^2 - 1}}, |x| > 1$.

Proof:

Step 1: Let $y = \cos ec^{-1}x$, then $x = \cos ec\, y$

Step 2: Differentiating x w.r.t to y, we have

$$\frac{dx}{dy} = -\cos ec\, y \cot y = -\cos ec\, y\sqrt{\cos ec^2 y - 1} = -x\sqrt{x^2 - 1}$$

Step 3: Using $\frac{dy}{dx} = 1\Big/\frac{dx}{dy}$, we have

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2 - 1}} \text{ where } |x| = |\cos ec\, y| > 1$$

<u>Example 7</u>

If $y = \sin^{-1}\left(\dfrac{4 - x^2)}{4 + x^2}\right)$, show that $\dfrac{dy}{dx} = \dfrac{-4}{4 + x^2}$.

Solution: Note: The function is quite complicate and it's preferable to use a substitution which allows us to apply the chain rule.

Step 1: Let $u = \left(\dfrac{4 - x^2)}{4 + x^2}\right)$, so that $y = \sin^{-1} u$, or $u = \sin y$.

Then,

$$\frac{du}{dy} = \cos y = \sqrt{1 - \sin^2 x} = \sqrt{1 - u^2}$$

$$= \sqrt{1 - \left(\frac{4 - x^2}{4 + x^2}\right)^2} = \frac{4x}{4 + x^2}$$

Step 2:

Also, $$\frac{du}{dx} = \frac{(4 + x^2)(-2x) - (4 - x^2)(2x)}{(4 + x^2)^2} = \frac{-16x}{(4 + x^2)^2}$$

Step 3:

Now, using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(1 \Big/ \frac{du}{dy}\right) \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-16x}{(4 + x^2)^2} \Big/ \frac{4x}{4 + x^2} = \frac{-4}{4 + x^2}$$

Example 8

If $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, show that $\dfrac{dy}{dx} = \dfrac{2}{1+x^2}$.

Since the function is rather complicated, we'll use a substitution and the chain rule.

Let, $u = \dfrac{1-x^2}{1+x^2}$, so that $y = \cos^{-1} u$, or $u = \cos y$.

Then,

$$\frac{du}{dy} = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-u^2}$$

$$= -\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2} = -\frac{2x}{1+x^2} \quad .$$

Also,

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{-4x}{(1+x^2)^2}.$$

Now, using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(1 \Big/ \frac{du}{dy}\right) \times \frac{du}{dx}$$

$$= -\frac{(1+x^2)}{2x} \times \frac{-4x}{(1+x^2)^2}$$

$$= \frac{2}{1+x^2} \quad .$$

Example 9

Show that if $y = \dfrac{\cos^{-1}\left(\dfrac{cx}{b}\right)}{\sqrt{b^2 - c^2x^2}}$, $|x| < 1$, c, b are constants, then $(b^2 - c^2x^2)y'' - 3c^2y'x - c^2y = 0$.

Note: It's preferable to rearrange the function in order to avoid a complex quotient rule.

Step 1: $y\sqrt{(b^2 - c^2x^2)} = \cos^{-1}\left(\dfrac{cx}{b}\right)$.

Step 2: Differentiate both sides w.r.t. x:

$$y\frac{-2c^2x}{2\sqrt{(b^2 - c^2x^2)}} + y'\sqrt{(b^2 - c^2x^2)} = \frac{-c}{b\sqrt{1 - \left(\dfrac{cx}{b}\right)^2}}$$

$$= \frac{-c}{\sqrt{b^2 - c^2x^2}}$$

Step 3: We now clear square roots before differentiating again. So, multiplying by $\sqrt{b^2 - c^2x^2}$ throughout, we have

$$-c^2yx + y'(b^2 - c^2x^2) = -c$$
$$y'(b^2 - c^2x^2) - c^2yx = -c$$

i.e., $(b^2 - c^2x^2)y' - c^2xy = -c$

Step 4: Differentiating w.r.t. x,

$$(b^2 - c^2x^2)y'' + y'(-2c^2x) - c^2xy' - c^2y = 0,$$

or, $(b^2 - c^2x^2)y'' - 3c^2y'x - c^2y = 0$.

Example 10

Show that if $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, $|x| < 1$, then $(1-x^2)\ y'' - 3x\,y' - y = 0$.

Here it's better to rearrange the function to avoid using the quotient rule.

Thus, we write $\quad y\ \sqrt{1-x^2} = \sin^{-1} x$.

Now differentiate both sides w.r.t. *x*:

$$y\ \frac{1}{2\sqrt{1-x^2}}\ (-2x) + \sqrt{1-x^2}\ \ y' = \frac{1}{\sqrt{1-x^2}}$$

We now clear square roots before differentiating again. So, multiplying by $\sqrt{1-x^2}$ throughout, we have

$$-xy + (1-x^2)\ y' = 1$$

i.e., $\quad (1-x^2)\ y' - xy = 1$

Differentiating w.r.t. *x*,

$$(1-x^2)y'' + y'(-2x) - (xy' + y) = 0\,,$$

or, $\quad (1-x^2)y'' - 3xy' - y = 0$.

Activity 2

1. Differentiate w.r.t. x:

   (i) $\sin^{-1}(3x - 4x^3)$;

   (ii) $\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$;

   (iii) $\tan^{-1}\left(\dfrac{\sin x}{\cos 2x}\right)$.

2. If $y = x\tan^{-1} x$ show that

$$(x + x^3)\,y' = y\,(1 + x^2) + x^2 ,$$

   and

$$(1 + x^2)\,y'' + 2\,x\,y' - 2\,(y + 1) = 0 .$$

1.2.4 Parametric Differentiation

Suppose that if x and y are given in terms of a parameter t, then the chain rule gives the derivative of y w.r.t. x as shown in the table below:

Derivative of y w.r.t x	Chain Rule
First (1^{st})	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\left.\frac{dy}{dt}\middle/\frac{dx}{dt}\right.$
	$\frac{d[\ldots]}{dx}=\frac{d[\ldots]}{dt}\times\frac{dt}{dx}=\left.\frac{d[\ldots]}{dt}\middle/\frac{dx}{dt}\right.$
Second (2^{nd})	$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d\left[\frac{dy}{dx}\right]}{dt}\times\frac{dt}{dx}==\left.\frac{d\left[\frac{dy}{dx}\right]}{dt}\middle/\frac{dx}{dt}\right.$
Third (3^{rd})	$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d\left[\frac{d^2y}{dx^2}\right]}{dt}\times\frac{dt}{dx}==\left.\frac{d\left[\frac{d^2y}{dx^2}\right]}{dt}\middle/\frac{dx}{dt}\right.$
$\vdots$	$\vdots$
n^{th}	$\frac{d^ny}{dx^n}=\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=\frac{d\left[\frac{d^{n-1}y}{dx^{n-1}}\right]}{dt}\times\frac{dt}{dx}==\left.\frac{d\left[\frac{d^{n-1}y}{dx^{n-1}}\right]}{dt}\middle/\frac{dx}{dt}\right.$

Table 1.7

Example 11: Let $x=\cos 5t,\ y=\sin 5t$. Then, since

$$\frac{dy}{dt}=5\cos 5t,\quad \frac{dx}{dt}=-5\sin 5t,$$

it follows that

$$\frac{dy}{dx}=\frac{5\cos 5t}{-5\sin 5t}=-\cot 5t.$$

Let's now find the second derivative, $\frac{d^2y}{dx^2}$. We use the chain rule again:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}.$$

Similarly, for the 3rd derivative we have

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) \times \frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) \Big/ \frac{dx}{dt}.$$

Proceed likewise for the other higher derivatives.

So, returning to our example, we find

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(-\cot 5t) \Big/ (-5\sin 5t)$$

$$= \frac{5\operatorname{cosec}^2 5t}{-5\sin 5t}$$

$$= -\operatorname{cosec}^3 5t\,;$$

and you can now easily show that

$$\frac{d^3y}{dx^3} = \frac{d}{dt}(-\operatorname{cosec}^3 5t) \Big/ (-5\sin 5t)$$

$$= -3\cot 5t \;\operatorname{cosec}^4 5t\,.$$

Activity 3

Find $\frac{d^2y}{dx^2}$ if

(i) $x = \sec 2t,\ y = \tan 2t\,;$

(ii) $x = 3\cos t - \cos 3t,\ y = 3\sin t - \sin 3t\,.$

1.2.5 Taylor and Maclaurin Series

We shall now briefly look at the representation or expansion of certain functions in power series. A power series in x is simply an infinite series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_rx^r + \cdots,$$

where the a_i's are constants not all zero.

We assume that our functions are continuous, single-valued and have continuous derivatives up to the *n*th order in a given interval.

Taylor Series

This is a representation of a function $f(x)$ by a power series in $(x-a)$; i.e., we are expanding $f(x)$ about the point $x = a$. Thus

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots \qquad \textbf{(†)}$$

An equivalent form of the series is obtained by putting $x = a + h$ in **(†)**

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots.$$

Maclaurin Series

This is a special case of Taylor's series obtained by putting $a = 0$ in **(†)**

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots,$$

i.e., we are now expanding the function about the origin.

It is clear that not all functions can have series expansions as they or their derivatives may not exist (i.e., they are infinite) at $x = a$ or at $x = 0$. Thus $1/x$, $\ln x$, $\cot x$ do not have Maclaurin series as they are all infinite at the origin. However, they can be expanded about some other point.

Finally, we note that both Taylor and Maclaurin series do not generally converge for all values of *x*, but only within a restricted range of values of *x*.

Let us first consider a few examples of Maclaurin series.

Example 12

$f(x) = e^x$

We have

$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f'(x) &= e^x & f'(0) &= 1 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= e^x & f^{(n)}(0) &= 1 \end{aligned}$$

Hence,

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots}$$

This is known as the **Exponential series** and is valid *for all* values of *x*.

Example 13

$f(x) = \sin x$

We have $f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

The values of the derivatives at $x = 0$ form cycles of $0, 1, 0, -1$. Hence

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} + \ldots}$$

This is the sine series and is valid *for all* values of *x* (in *radians*).

Similarly,

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^{n-1} x^{2n-2}}{(2n-2)\,!} + \ldots}.$$

This is the cosine series and is valid *for all* values of *x* (in *radians*).

Likewise we have

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots, \quad -\frac{\pi}{2} < x < \frac{\pi}{2};$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}x^n}{n} + \ldots, \quad -1 < x \le 1.$$

We shall now expand about some other point. Thus, we shall be finding the Taylor series of the function about the given point.

<u>Example 14</u>

Find $e^{x/5}$ in powers of $(x-5)$.

Here we are expanding the function $e^{x/5}$ about the point $x = 5$, i.e., finding its Taylor expansion about $x = 5$. We'll use **(†).**

So, $\quad f(x) = e^{x/5} \qquad\qquad f(5) = e$

$f'(x) = \frac{1}{5}e^{x/5} \qquad\qquad f'(5) = \frac{1}{5}e$

$f''(x) = \frac{1}{25}e^{x/5} \qquad\qquad f''(5) = \frac{1}{25}e$.

$$\therefore \quad e^{x/5} = e\left[1 + \frac{1}{5}(x-5) + \frac{1}{25}\frac{(x-5)^2}{2!} + \ldots + \frac{1}{5^{n-1}}\frac{(x-5)^{n-1}}{(n-1)!} + \ldots\right].$$

The series converges for all *x*.

Example 15

Find the Taylor expansion of $\ln x$ about $x = 3$ up to and including the term in x^4.

$f(x) = \ln x \qquad f(3) = \ln 3$

$f'(x) = 1/x \qquad f'(3) = 1/3$

$f''(x) = -1/x^2 \qquad f''(3) = -1/9$

$f'''(x) = 2/x^3 \qquad f'''(3) = 2/27$

$f^{iv}(x) = -6/x^4 \qquad f^{iv}(3) = -2/27$.

Hence, using **(†)**, we obtain

$$\ln x = \ln 3 + \frac{1}{3}(x-3) - \frac{1}{9}\frac{(x-3)^2}{2!} + \frac{2}{27}\frac{(x-3)^3}{3!} - \frac{2}{27}\frac{(x-3)^4}{4!} + \ldots$$

$$= \ln 3 + \frac{(x-3)}{3} - \frac{(x-3)^2}{18} + \frac{(x-3)^3}{81} - \frac{(x-3)^4}{324} + \ldots.$$

Example 16

If $y = (\sin^{-1} x)^2$**,** prove that

$$(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} - 2 = 0.$$

Hence using Maclaurin's expansion, prove that the first two non-zero terms in the expansion of $(\sin^{-1} x)^2$ are

$$x^2 + \frac{1}{3}x^4.$$

Solution:

$y = (\sin^{-1} x)^2$

Differentiating y w.r.t x

$$\frac{dy}{dx} = 2(\sin^{-1} x)\frac{1}{\sqrt{1-x^2}} \text{ or } \sqrt{1-x^2}\,\frac{dy}{dx} = 2\sqrt{y}$$

Again, differentiating w.r.t x

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{1}{2}(1-x^2)^{-1/2}(-2x) = y^{-1/2}\frac{dy}{dx} \text{ or}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = y^{-1/2}\sqrt{1-x^2}\,\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2 \text{ [Proven]}$$

Again, differentiating w.r.t x, we have

$$\left[(1-x^2)\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}(-2x)\right] - \left[x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right] = 0 \text{ or } (1-x^2)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$$

Again, differentiating w.r.t x, we have

$$\left[-2x\frac{d^3y}{dx^3} + (1-x^2)\frac{d^4y}{dx^4}\right] - \left[3\frac{d^2y}{dx^2} + 3x\frac{d^3y}{dx^3}\right] - \frac{d^2y}{dx^2} = 0 \text{ or}$$

$$-5x\frac{d^3y}{dx^3} + (1-x^2)\frac{d^4y}{dx^4} - 4\frac{d^2y}{dx^2} = 0$$

Evaluation of y and its derivatives:

When $x = 0$, then $y = 0$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 2$, $\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} = 8$

Using Maclaurin's Expansion,

$$y = (\sin^{-1} x)^2 = y(0) + xy'(0) + \frac{x^2}{2!}y'' + \frac{x^3}{3!}y''' + \frac{x^4}{4!}y'''' + \cdots$$

$$= 0 + 0 + \frac{x^2}{2}(2) + \frac{x^3}{6}(0) + \frac{x^4}{24}(8) + \cdots$$

$$= x^2 + \frac{x^4}{3} + \cdots \text{ [Proven]}$$

Example 17

By using the Maclaurin's expansion or otherwise show that the first three terms in the expansion of $\ln(\sec(2x)+\tan(2x))$ in powers of x are

$$2x+\frac{4}{3}x^3+\frac{4x^5}{3}.$$

Solution:

$y=\ln(\sec(2x)+\tan(2x))$, $y\big|_{x=0}=0$

$$\frac{\partial y}{\partial x}=\frac{2\sec(2x)\tan(2x)+2\sec^2(2x)}{(\sec(2x)+\tan(2x))}=2\sec(2x),\ \left.\frac{\partial y}{\partial x}\right|_{x=0}=2$$

$$\frac{\partial^2 y}{\partial x^2}=\frac{\partial(2\sec(2x))}{\partial x}=4\sec 2x\tan 2x=2\frac{dy}{dx}\tan 2x,\ \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0}=0$$

$$\frac{\partial^3 y}{\partial x^3}=\frac{\partial[2\frac{dy}{dx}\tan 2x]}{\partial x}=[2\frac{d^2y}{dx^2}\tan 2x]+[4\frac{dy}{dx}\sec^2 2x]=8\frac{\partial y}{\partial x}\tan^2 2x+4\frac{dy}{dx},\ \left.\frac{\partial^3 y}{\partial x^3}\right|_{x=0}=8$$

$$\begin{aligned}\frac{\partial^4 y}{\partial x^4}&=[8\frac{\partial^2 y}{\partial x^2}\tan^2 2x]+[32\frac{\partial y}{\partial x}\tan 2x\sec^2 2x]+4\frac{d^2y}{dx^2}\\&=[8\frac{\partial^2 y}{\partial x^2}\tan^2 2x]+[32\frac{\partial y}{\partial x}\tan 2x]+[32\frac{\partial y}{\partial x}\tan^3 2x]+4\frac{d^2y}{dx^2}\\&=[8\frac{\partial^2 y}{\partial x^2}\tan^2 2x]+[16\frac{\partial^2 y}{\partial x^2}]+[32\frac{\partial y}{\partial x}\tan^3 2x]+4\frac{d^2y}{dx^2}\end{aligned}$$

$$\left.\frac{\partial^4 y}{\partial x^4}\right|_{x=0}=0$$

$$\begin{aligned}\frac{\partial^5 y}{\partial x^5}&=[8\frac{\partial^3 y}{\partial x^3}\tan^2 2x]+[32\frac{\partial^2 y}{\partial x^2}\tan 2x\sec^2 2x]+16\frac{\partial^3 y}{\partial x^3}+\\&\quad[192\frac{\partial y}{\partial x}\tan^2 2x\sec^2 2x]+[32\frac{\partial^2 y}{\partial x^2}\tan^3 2x]+4\frac{d^3y}{dx^3}\end{aligned}$$

$$\left.\frac{\partial^5 y}{\partial x^5}\right|_{x=0}=160$$

Using Maclaurin's Expansion,

$y=\ln(\sec(2x)+\tan(2x))$

$$= y(0) + xy'(0) + \frac{x^2}{2!}y'' + \frac{x^3}{3!}y''' + \frac{x^4}{4!}y'''' + \frac{x^5}{5!}y'''' + \cdots$$

$$= 0 + 2x + 0 + 8\frac{x^3}{3!} + 0 + 160\frac{x^5}{5!} + \cdots$$

$$= 2x + \frac{4x^3}{3} + \frac{4x^5}{3} + \cdots \text{[Proven]}$$

Activity 4

1. Show that

(i) $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \ldots, \quad -\pi/2 < x < \pi/2$;

(ii) $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots, \quad -1 < x < 1$;

(iii) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots, \quad -1 \le x \le 1$.

2. Expand $\cos x$ about the point $x = \pi/3$.

1.3 FURTHER INTEGRATION

1.3.1 Integrals Involving Trigonometric Functions

1	$\int \sin x\, dx = -\cos x + C; \quad \int \cos x\, dx = \sin x + C;$
2	$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\ln \cos x + C = \ln \sec x + C$
3	$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \ln \sin x + C$
4	$\boxed{\int \operatorname{cosec} x\, dx = \ln \tan(x/2) + C = \ln\,[\operatorname{cosec} x - \cot x] + C}$
5	$\boxed{\int \sec x\, dx = \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C = \ln[\sec x + \tan x] + C}$

Table 1.8

Prove: $\boxed{\int \operatorname{cosec} x\, dx = \ln \tan(x/2) + C = \ln\,[\operatorname{cosec} x - \cot x] + C}$

Method: Use of the method of substitution to prove the first part

Step 1: Let $t = \tan\frac{x}{2}$. Then using the identity $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ and putting $A = \frac{x}{2}$, we get

$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}.$$

Step 2: We then deduce [construct the usual right-angled triangle] that

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \operatorname{cosec} x = \frac{1+t^2}{2t}, \quad \sec x = \frac{1-t^2}{1+t^2}.$$

Step 3: Also, $dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx = \frac{1}{2}(1+t^2)\, dx$, so that $dx = 2\,\frac{dt}{1+t^2}$. Hence

$$\int \text{cosec}\, x\, dx = \int \frac{1+t^2}{2t}\, 2\frac{dt}{1+t^2} = \int \frac{dt}{t} = \ln t + C$$

$$\therefore \qquad \int \text{cosec}\, x\, dx = \ln \tan(x/2) + C\,.$$

Method: To be able to prove the second part we need another form for this integral which is deduced by trigonometrical manipulation.

$$\ln \tan\frac{x}{2} = \ln\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \ln\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} \text{ [on multiplying top \& bottom by } 2\sin x/2\,]$$

$$= \ln\left(\frac{1-\cos x}{\sin x}\right)$$

$$= \ln[\text{cosec}\, x - \cot x]$$

Thus, we have shown that

$$\boxed{\int \text{cosec}\, x\, dx = \ln \tan(x/2) + C = \ln\,[\text{cosec}\, x - \cot x] + C}$$

Prove: $\boxed{\int \sec x\, dx = \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C = \ln[\sec x + \tan x] + C}$

Hint: This can be found by the same substitution as above, but it is perhaps more instructive to deduce it from

$$\int \text{cosec}\, x\, dx = \ln \tan\frac{x}{2} = \ln[\text{cosec}\, x - \cot x]\,,$$

by the substitution $x = \frac{\pi}{2} + \phi$.

Then $dx = d\phi,\quad \text{cosec}\, x = \sec\phi$ and $\cot x = -\tan\phi$

$$\therefore \qquad \int \sec\phi\, d\phi = \ln \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = \ln[\sec\phi + \tan\phi]\,,$$

or, $\boxed{\int \sec x\, dx = \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C = \ln[\sec x + \tan x] + C}$.

Integrals of Products of Sines and/or Cosines of Multiple Angles

Products of sines and/or cosines of multiple angles may be integrated by parts. It is however, easier to use the following trig identities, known as the **Factor Formulae**, to simplify the integrand before integrating.

Factor Formulae
$\cos ax \cos bx = \frac{1}{2}[\cos(a+b)x + \cos(a-b)x]$
$\sin ax \cos bx = \frac{1}{2}[\sin(a+b)x + \sin(a-b)x]$
$\sin ax \sin bx = \frac{1}{2}[\cos(a-b)x - \cos(a+b)x]$

Table 1.9

Example 18

(i) $\int \cos 6x \cos 2x \, dx = \frac{1}{2}\int (\cos 8x + \cos 4x)\, dx$

$$= \frac{\sin 8x}{16} + \frac{\sin 4x}{8} + C.$$

(ii) $\int \sin 5x \cos 2x \, dx = \frac{1}{2}\int (\sin 7x + \sin 3x)\, dx$

$$= -\frac{\cos 7x}{14} - \frac{\cos 3x}{6} + C$$

(iii) $\int \sin 3x \cos 7x \, dx = \frac{1}{2}\int [\sin 10x + \sin(-4x)]\, dx$

$$= -\frac{\cos 10x}{20} + \frac{\cos 4x}{8} + C \qquad [\text{Recall } \sin(-\alpha) = -\sin\alpha\,]$$

(iv) $\int \sin 3x \sin 4x \, dx = \frac{1}{2}\int (\cos x - \cos 7x)\, dx \qquad [\text{Recall } \cos(-\alpha) = \cos\alpha\,]$

$$= \tfrac{1}{2}[\sin x - \tfrac{1}{7}\sin 7x] + C.$$

Integrals of the form $\int \cos^m x \sin^n x \, dx$.

We'll consider those integrals where *m* and *n* are integers and **at least one of them is odd**. The case where *m* and *n* are both even will be dealt with later in Unit 4.

m	n	Substitution to be used
odd	even	$u = \sin x$
even	odd	$u = \cos x$
odd	odd	$u = \sin x$ or $u = \cos x$

Table 1.10

<u>Example 19</u>

(a) $\int \sin^3 x \, dx$

Here $m = 0, n = 3$. Since *n* is odd, we put $u = \cos x$, so that $du = -\sin x \, dx$.

Now,

$$\int \sin^3 x \, dx = \int (\sin^2 x) \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx = \int (1 - u^2)(-du)$$

$$= \tfrac{1}{3} u^3 - u + C$$
$$= \tfrac{1}{3} \cos^3 x - \cos x + C.$$

(b) $\int \cos^5 x \, dx$

Here $m = 5, n = 0$. Since *m* is odd, we put $u = \sin x$, so that $du = \cos x \, dx$.

Now,

$$\int \cos^5 x \, dx = \int (\cos^4 x) \cos x \, dx = \int (1 - \sin^2 x)^2 \cos x \, dx = \int (1 - u^2)^2 \, du$$

$$= \int (1 - 2u^2 + u^4) \, du$$
$$= u - \tfrac{2}{3} u^3 + \tfrac{1}{5} u^5 + C$$
$$= \sin x - \tfrac{2}{3} \sin^3 x + \tfrac{1}{5} \sin^5 x + C.$$

Example 20

$$\int \sin^5 x \cos^6 x \, dx$$

Here $m = 6, n = 5$. Since n is odd, we put $u = \cos x$, so that $du = -\sin x \, dx$.

$$\therefore \quad \int \sin^5 x \cos^6 x \, dx = -\int \sin^4 x \;\; u^6 \; du$$

$$= -\int (1-\cos^2 x)^2 u^6 \; du$$

$$= -\int (1-u^2)^2 u^6 \; du$$

$$= -\int (u^6 - 2u^8 + u^{10}) \, du$$

$$= -u^7/7 + 2u^9/9 - u^{11}/11 + C$$

$$= -\frac{\cos^7 x}{7} + \frac{2\cos^9 x}{9} - \frac{\cos^{11} x}{11} + C.$$

Example 21

$$\int \cos^3 x \sin^4 x \, dx$$

Here, $m = 3, n = 4$. Since m is odd, we put $u = \sin x$, so that $du = \cos x \, dx$.

$$\int \cos^3 x \sin^4 x \, dx = \int \cos^2 x \;\; u^4 \; du$$

$$= \int (1-u^2) u^4 \; du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

Activity 5

Find the following integrals:

(i) $\int \sin 3x \sin 2x \, dx$;

(ii) $\int \cos 4x \cos 2x \, dx$;

(iii) $\int \sin 3x \cos 5x \, dx$;

(iv) $\int \sin^2 x \cos^3 x \, dx$;

(v) $\int \sin^3 3x \cos^5 3x \, dx$.

1.3.2 Integrals Involving Inverse Trigonometric Functions

Consider the following standard integrals:

	Integration	**Substitution used**
standard integrals	$\boxed{\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + C}$	$x = a\sin\theta$
	$\boxed{\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\frac{x}{a} + C}$	$x = a\cos\theta$
	$\boxed{\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C}$	$x = a\tan\theta$
Hint	Presence of $\sqrt{a^2 - X^2}$ in the integrand	$X = a\sin\theta$
	Presence of $\sqrt{a^2 + X^2}$ in the integrand.	$X = a\tan\theta$

Table 1.11

We now apply these standard integrals in the following examples:

Example 22: **[**Direct use of the standard integrals for easier problems**]**

(a) $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{(3)^2 - x^2}} = \sin^{-1}\frac{x}{3} + C$;

(b) $\int \frac{dx}{\sqrt{5-x^2}} = \int \frac{dx}{\sqrt{(\sqrt{5})^2 - x^2}} = \sin^{-1}\frac{x}{\sqrt{5}} + C$.

(c) $\int \frac{dx}{4+x^2} = \int \frac{dx}{(2)^2 + x^2} = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$;

(d) $\int \frac{dx}{7+x^2} = \int \frac{dx}{(\sqrt{7})^2 + x^2} = \frac{1}{\sqrt{7}}\tan^{-1}\frac{x}{\sqrt{7}} + C$.

Example 23: **[**Considering harder problems**]**

Find $\int \frac{dx}{\sqrt{9-4x^2}}$.

Solution:

Step 1: We write $9-4x^2 = 4(\frac{9}{4} - x^2)$ and thus

$$\sqrt{9-4x^2} = \sqrt{4(\tfrac{9}{4} - x^2)} = 2\sqrt{\tfrac{9}{4} - x^2} \ .$$

Step 2: Then,

$$\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{2\sqrt{\frac{9}{4} - x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{9}{4} - x^2}} \ .$$

Step 3: Using the standard integral, with $a^2 = 9/4$, so that $a = 3/2$, we have

$$\int \frac{dx}{\sqrt{9-4x^2}} = = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{9}{4} - x^2}} = \frac{1}{2}\sin^{-1}\frac{x}{3/2} + C = \frac{1}{2}\sin^{-1}\frac{2x}{3} + C \ .$$

Example 24:

Evaluate $\displaystyle\int_0^{1/3} \frac{dx}{\sqrt{5-7x^2}}$.

Solution:

Step 1: $\displaystyle\int_0^{1/3} \frac{dx}{\sqrt{5-7x^2}} = \int_0^{1/3} \frac{dx}{\sqrt{7(5/7-x^2)}} = \int_0^{1/3} \frac{dx}{\sqrt{7}\sqrt{(5/7-x^2)}} = \frac{1}{\sqrt{7}}\int_0^{1/3} \frac{dx}{\sqrt{(5/7-x^2)}}$

Thus, if we consider $a^2 = 5/7$, then $a = \sqrt{5/7}$.

Step 2: Hence,

$$\int_0^{1/3} \frac{dx}{\sqrt{5-7x^2}} = \frac{1}{\sqrt{7}}\left[\sin^{-1}\frac{x}{\sqrt{5/7}}\right]_0^{1/3}$$

$$= \frac{1}{\sqrt{7}}\left[\sin^{-1}\left(\frac{1/3}{\sqrt{5/7}}\right) - \sin^{-1} 0\right]$$

$= 0.153235$ **[*RADIAN* mode]**

Example 25:

Evaluate the following integral: $\displaystyle\int_4^{11/2} \frac{1}{\sqrt{-4x^2+32x-55}}\,dx$

Solution:

Step 1: $\displaystyle\int_4^{11/2} \frac{1}{\sqrt{-4x^2+32x-55}}\,dx = \int_4^{11/2} \frac{1}{\sqrt{9-4(x-4)^2}}\,dx$

Step 2: Let $u = \dfrac{2}{3}(x-4)$, then $dx = \dfrac{3}{2}du$ and thus

$$\int_4^{11/2} \frac{1}{\sqrt{9-4(x-4)^2}} = \frac{1}{3}.\frac{3}{2}\int_0^1 \frac{1}{\sqrt{1-u^2}}du$$

Step 3: Hence, $\displaystyle\int_4^{11/2} \frac{1}{\sqrt{-4x^2+32x-55}}\,dx = \frac{1}{2}\sin^{-1}(u)\Big]_0^1 = \frac{\pi}{4}$

Example 26:

Find $\int \frac{dx}{4+9x^2}$.

Solution:
Step 1:

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{9(4/9+x^2)} = \frac{1}{9}\int \frac{dx}{(4/9+x^2)},$$

which is of the standard integral with $a = 2/3$.

Step 2:
Hence

$$\int \frac{dx}{4+9x^2} = \frac{1}{9}\int \frac{dx}{(4/9+x^2)} = \frac{1}{9}\,\frac{1}{2/3}\tan^{-1}\frac{x}{2/3} + C$$

$$= \frac{1}{6}\tan^{-1}\frac{3x}{2} + C.$$

Example 27:

Find $\int_0^2 \frac{dx}{3x^2-12x+17}$**,** giving your answer to 4 decimal places.

Solution:

Step 1: $\int_0^2 \frac{dx}{3x^2-12x+17} = \int_0^2 \frac{dx}{3[x^2-4x+17/3]} = \frac{1}{3}\int_0^2 \frac{dx}{[(x-2)^2+\left(\sqrt{\frac{5}{3}}\right)^2]}$

Step 2: $\int_0^2 \frac{dx}{3x^2-12x+17} = \frac{1}{3}\sqrt{\frac{3}{5}}\tan^{-1}\left.\left(\frac{x-2}{\sqrt{\frac{5}{3}}}\right)\right|_0^2 = 0.2576$

Activity 6

Find the following integrals, using both of the methods given:

(i) $\int \frac{dx}{\sqrt{4-5x^2}}$;

(ii) $\int_0^1 \frac{dt}{\sqrt{7-6t^2}}$;

(iii) $\int \frac{dx}{8+3x^2}$;

(iv) $\int_1^2 \frac{dz}{4z^2+5}$.

Example 28:

Find $\int \sqrt{4-x^2}\ dx$.

Solution:

Step 1: This contains a term of the form $\sqrt{a^2 - X^2}$, with $a = 2,\ X = x$

Let $x = 2\sin\theta$ such that $dx = 2\cos\theta\ d\theta$.

Step 2: Thus, $\int \sqrt{4-x^2}\ dx = \int \sqrt{4-4\sin^2\theta}\ 2\cos\theta\ d\theta$

$$= \int 2\cos\theta \;\; 2\cos\theta \; d\theta$$

$$= 4\int \cos^2\theta \; d\theta$$

$$= 2\int (1+\cos 2\theta) \; d\theta$$

$$= 2\theta + \sin 2\theta + C \,.$$

Step 3:

We need to revert back to terms of *x*. From our substitution $x = 2\sin\theta$, it follows that $\sin\theta = \dfrac{x}{2}$,

so that $\theta = \sin^{-1}\dfrac{x}{2}$, and $\cos\theta = \sqrt{1-x^2/4}$, and therefore

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\,(x/2)\,\sqrt{1-x^2/4} = x\sqrt{1-x^2/4}\,.$$

Hence,

$$\therefore \qquad \int \sqrt{4-x^2}\; dx = 2\sin^{-1}\frac{x}{2} + x\,\sqrt{1-x^2/4} + C\,.$$

Example 29:

Find $\int_{\sqrt{3}}^{3} \frac{dx}{x\sqrt{9+x^2}}$.

Solution:

Step 1: Here the presence of the form $\sqrt{a^2 + X^2}$ suggests that we put $x = 3\tan\theta$.

Step 2: Then $dx = 3\sec^2\theta \, d\theta$ and limits becomes

x	$\theta = \tan^{-1}(x/3)$
$\sqrt{3}$	$\pi/6$
3	$\pi/4$

Step 3:

$$\therefore \qquad \int_{\sqrt{3}}^{3} \frac{dx}{x\sqrt{9+x^2}} = \int_{\pi/6}^{\pi/4} \frac{3\sec^2\theta \, d\theta}{3\tan\theta \sqrt{9+9\tan^2\theta}} = \int_{\pi/6}^{\pi/4} \frac{\sec^2\theta}{\tan\theta \, 3\sec\theta} \, d\theta$$

$$= \frac{1}{3}\int_{\pi/6}^{\pi/4} \frac{\sec\theta}{\tan\theta} \, d\theta$$

$$= \frac{1}{3}\int_{\pi/6}^{\pi/4} \operatorname{cosec}\theta \, d\theta$$

$$= \frac{1}{3}\ln[\tan(\theta/2)]_{\pi/6}^{\pi/4}$$

$$= \frac{1}{3}[\ln\tan(\pi/8) - \ln\tan(\pi/12)]$$

$= 0.145195$. [Radian mode]

Example 30:

Find $\int \frac{\sqrt{9-4x^2}}{x}\,dx$.

Solution:

Step 1: $\sqrt{9-4x^2} = \sqrt{4(9/4-x^2)} = 2\sqrt{(\tfrac{3}{2})^2 - x^2}$, which now involves the form $\sqrt{a^2 - X^2}$; so we let $x = \tfrac{3}{2}\sin\theta$.

Step 2: $\therefore \int \frac{\sqrt{9-4x^2}}{x}\,dx = \int \frac{3\cos\theta}{\tfrac{3}{2}\sin\theta}\tfrac{3}{2}\cos\theta\,d\theta = 3\int \frac{\cos^2\theta}{\sin\theta}\,d\theta$

$$= 3\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = 3\int \operatorname{cosec}\theta\,d\theta - 3\int \sin\theta\,d\theta$$

$$= 3\ln[\operatorname{cosec}\theta - \cot\theta] + 3\cos\theta + C$$

$$= 3\ln\left(\frac{3-\sqrt{9-4x^2}}{x}\right) + \sqrt{9-4x^2} + C.$$

1.3.3 Integrals with a Quadratic in the Denominator

In this unit we shall consider only integrals of the form

	Type	Procedures
(i)	$\int \frac{dx}{ax^2+bx+c}$	If the quadratic is factorisable, then use partial fractions. If not, complete the square and use the appropriate standard integral result or the relevant substitution
(ii)	$\int \frac{dx}{\sqrt{ax^2+bx+c}}$	Whether the quadratic is factorisable or not, complete the square and use the appropriate standard integral result or the relevant substitution.

Table 1.12

Example 31:

(a) Find $\int \frac{dx}{6x^2 - 7x - 20}$.

Quadratic is factorisable; so we use partial fractions:

$$\frac{1}{6x^2 - 7x - 20} = \frac{1}{(2x-5)(3x+4)} = \frac{\frac{2}{23}}{2x-5} - \frac{\frac{3}{23}}{3x+4}.$$

$$\therefore \quad \int \frac{dx}{6x^2 - 7x - 20} = \int \frac{\frac{2}{23}}{2x-5} - \frac{\frac{3}{23}}{3x+4}\, dx = \tfrac{1}{23}\ln[(2x-5)/(3x+4)] + C$$

(b) Find $\int \frac{dx}{x^2 - 4x + 13}\, dx$

Quadratic is not factorisable; so we complete the square:

$$x^2 - 4x + 13 = (x-2)^2 + 9$$

$$\therefore \quad \int \frac{dx}{x^2 - 4x + 13}\, dx = \int \frac{dx}{9 + (x-2)^2} \quad \text{[which is of the form } \int \frac{dx}{a^2 + X^2}, \text{]}$$

$$= \frac{1}{3}\tan^{-1}\frac{x-2}{3} + C. \quad \text{[Alternatively, let } (x-2) = 3\tan\theta \text{]}$$

(c) Find $\int \frac{dx}{\sqrt{2+x-3x^2}}$.

Complete the square even though quadratic is factorisable:

$$2+x-3x^2 = 3\left(\frac{25}{36}-(x-\tfrac{1}{6})^2\right).$$

$$\therefore \quad \int \frac{dx}{\sqrt{2+x-3x^2}} = \int \frac{dx}{\sqrt{3\,[\frac{25}{36}-(x-\frac{1}{6})^2\,]}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\frac{25}{36}-(x-\frac{1}{6})^2}},$$

which is of the form $\int \frac{dx}{\sqrt{a^2-X^2}}$. Hence,

$$\int \frac{dx}{\sqrt{2+x-3x^2}} = \frac{1}{\sqrt{3}}\sin^{-1}\frac{x-1/6}{5/6}+C$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\frac{6x-1}{5}+C.$$

Alternatively, we could have used the substitution $(x-\frac{1}{6}) = \frac{5}{6}\sin\theta$.

We shall now consider some integrals which, though not of types **(i)** and **(ii)** above, can still be worked out by an appropriate trig substitution because they “contain” the standard integrals.

Example 32:

$$\int \frac{x}{\sqrt{5x-6-x^2}}\,dx$$

On completing the square we have

$$\int \frac{x}{\sqrt{5x-6-x^2}}\,dx = \int \frac{x}{\sqrt{\frac{1}{4}-(x-\frac{5}{2})^2}}\,dx\,,$$

in which we have the form $\sqrt{a^2 - X^2}$, $\quad a = \frac{1}{2}$, $\quad X = x - \frac{5}{2}$.

So, here we make the substitution

$$x - \tfrac{5}{2} = \tfrac{1}{2}\sin\theta$$

$$\therefore \quad x = \tfrac{5}{2} + \tfrac{1}{2}\sin\theta, \quad dx = \tfrac{1}{2}\cos\theta\, d\theta$$

$$\sqrt{\tfrac{1}{4}-(x-\tfrac{5}{2})^2} = \sqrt{\tfrac{1}{4}-\tfrac{1}{4}\sin^2\theta} = \tfrac{1}{2}\cos\theta\,.$$

$$\therefore \quad \int \frac{x}{\sqrt{5x-6-x^2}}\,dx = \int \frac{\frac{5}{2}+\frac{1}{2}\sin\theta}{\frac{1}{2}\cos\theta}\,\tfrac{1}{2}\cos\theta\, d\theta$$

$$= \frac{1}{2}\int (5+\sin\theta)\, d\theta$$

$$= \frac{1}{2}(5\theta - \cos\theta) + C\,.$$

The answer must be given in terms of *x*. From our substitution $x - \frac{5}{2} = \frac{1}{2}\sin\theta$, we have $\sin\theta = 2x - 5$, so that $\theta = \sin^{-1}(2x-5)$, and $\cos\theta = \sqrt{1-(2x-5)^2}$. The final answer is then

$$\int \frac{x}{\sqrt{5x-6-x^2}}\,dx = \frac{1}{2}[5\sin^{-1}(2x-5) - \sqrt{1-(2x-5)^2}\,] + C\,.$$

Example 33:

Find $\int \frac{dx}{\sqrt{x^2 - 4x + 13}}$.

$$\int \frac{dx}{\sqrt{x^2 - 4x + 13}}\, dx = \int \frac{dx}{\sqrt{9 + (x-2)^2}}$$. [On completing the square.]

Here we have the form $\sqrt{a^2 + X^2}$, $a = 3$, $X = x - 2$. So, let $x - 2 = 3\tan\theta$. The answer is

$$\ln[x - 2 + \sqrt{x^2 - 4x + 13}] + C .$$

However, we'll see a neater way of doing this integral when we study Hyperbolic Functions!

Activity 7

Find the following integrals:

(i) $\displaystyle\int \frac{dx}{2x^2 + x - 6}$;

(ii) $\displaystyle\int \frac{dx}{5x^2 + 7x + 8}$;

(iii) $\displaystyle\int \frac{dx}{\sqrt{2 - x - x^2}}$;

(iv) $\displaystyle\int \frac{dx}{\sqrt{(2-x)(4+3x)}}$;

(v) $\displaystyle\int_0^{1/2} \frac{dy}{\sqrt{y(1-y)}}$;

(vi) $\displaystyle\int_0^{1/2} \frac{du}{\sqrt{3 + 4u - 4u^2}}$;

(vii) $\displaystyle\int_{-2}^{-1/4} \frac{dz}{2z^2 + z + 1}$;

(viii) $\displaystyle\int_0^{a} \frac{x}{\sqrt{a^4 - x^4}}\, dx$, [Let $u = x^2$] ;

(ix) $\displaystyle\int_2^{2\sqrt{3}} \frac{dx}{x\sqrt{x^2 - 3}}$, [Let $u = 1/x$] ;

(x) $\displaystyle\int_2^{3} \frac{ds}{3s^2 - 2s - 1}$ **[Watch Out !]**

Integrals of the Form $\int \frac{dx}{a\sin mx + b}$ and $\int \frac{dx}{a\cos mx + b}$

Here we use the substitution $t = \tan(half\ \ angle)$, i.e., $t = \tan\frac{mx}{2}$.

Refer to the method for integrating cosec x.

Then $dt = \frac{m}{2}(1+t^2)\,dx$; $\sin mx = \frac{2t}{1+t^2}$, and $\cos mx = \frac{1-t^2}{1+t^2}$.

Example 34:

Find $\int \frac{dx}{5+4\cos 3x}$.

Here, $m = 3$. Let $t = \tan\frac{3x}{2}$. Then $dt = \frac{3}{2}(1+t^2)\,dx$, and $\cos 3x = \frac{1-t^2}{1+t^2}$.

$$\therefore \int \frac{dx}{5+4\cos 3x} = \int \frac{\frac{2}{3(1+t^2)}\,dt}{5+4\left(\frac{1-t^2}{1+t^2}\right)} = \frac{2}{3}\int \frac{dt}{9+t^2}$$

$$= \frac{2}{9}\tan^{-1}(t/3) + C$$

$$= \frac{2}{9}\tan^{-1}\left(\frac{1}{3}\tan\frac{3x}{2}\right) + C.$$

Note: If the integrand involves $\sin^2 x$, $\cos^2 x$ or $\tan^2 x$, we may use the simpler substitution $u = \tan x$, in which case

$$dx = du/(1+u^2), \quad \sin^2 x = u^2/(1+u^2), \quad \cos^2 x = 1/(1+u^2).$$

Example 35:

$$\int \frac{dx}{5\cos^2 x + 6\sin^2 x} = \int \frac{\sec^2 x}{5 + 6\tan^2 x}\, dx$$
$$= \int \frac{du}{5 + 6u^2} = \frac{1}{\sqrt{30}} \tan^{-1}\left(u\sqrt{6/5}\right) + C \quad [\text{Letting } u = \tan x\,]$$
$$= \frac{1}{\sqrt{30}} \tan^{-1}\left(\sqrt{6/5}\,\tan x\right) + C.$$

Activity 8

Find the following integrals:

(i) $\int \frac{dx}{3 + 2\cos 5x}$;

(ii) $\int \frac{d\theta}{3 + 5\cos\frac{1}{2}\theta}$;

(iii) $\int \frac{d\phi}{1 + \sin 3\phi}$;

(iv) $\int \frac{dx}{2 + 3\sin^2 x}$;

(v) $\int \frac{dx}{3 + 4\cos^2 x}$;

(vi) $\int \frac{\sin^2 x}{4 + 3\cos^2 x}\, dx$;

(vii) $\int_0^{\pi/4} \frac{3\cos^2 x + 2\sin^2 x}{5\cos^2 x + 4\sin^2 x}\, dx$;

(viii) $\int \frac{\tan^2 x}{2 + 3\cos^2 x}\, dx$;

(ix) $\int_0^{\pi/2} \frac{dx}{3\cos^2 x + 4\sin^2 x}$;

(x) $\int_0^{\pi/3} \frac{\sin^2 x}{1 + 2\sin^2 x}\, dx$.

1.3.4 Integration by Parts

Recall the product rule of differentiation

$$\frac{d}{dx}(UV) = U\frac{dV}{dx} + V\frac{dU}{dx}.$$

Now integrate both sides w.r.t. x

$$\int \frac{d}{dx}(UV)\,dx = \int U\frac{dV}{dx}\,dx + \int V\frac{dU}{dx}\,dx,$$

i.e., $$UV = \int U\frac{dV}{dx}\,dx + \int V\frac{dU}{dx}\,dx$$

or, $$\int U\frac{dV}{dx}\,dx = UV - \int V\frac{dU}{dx}\,dx,$$

or, $$\boxed{\int U\,dV = UV - \int V\,dU}$$

which is the well known formula used to integrate products of functions.

We'll now illustrate the method by a few examples.

Example 36:

Find $\int x^2 e^{3x}\,dx$

Here we choose $U = x^2$ and $dV = e^{3x}\,dx$.

Then $dU = 2x\,dx$ and $V = \frac{1}{3}e^{3x}$.

$$\therefore \quad \int x^2 e^{3x}\ dx = x^2\ \tfrac{1}{3}e^{3x} - \int\ \tfrac{1}{3}e^{3x}\ 2x\ dx$$

$$= \tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{3}\int x\, e^{3x}\ dx$$

We now need to find $\int x\, e^{3x}\ dx$.

This time we choose $U = x$ and $dV = e^{3x}\ dx$, so that $dU = dx$ and $V = \tfrac{1}{3}e^{3x}$.

$$\therefore \quad \int x\, e^{3x}\ dx = x\,\tfrac{1}{3}e^{3x} - \int \tfrac{1}{3}e^{3x}\ dx$$

$$= \tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x}.$$

Hence,

$$\int x^2\, e^{3x}\ dx = \tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{3}\left(\ \tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x}\ \right) + C$$

$$= \tfrac{1}{27}e^{3x}[9x^2 - 6x + 2] + C.$$

The above example shows that we sometimes have to use the method more than once to reach the result. In the next example we shall see that the integral we start out with appears again in the process.

Example 37:

Find $\int e^{2x}\cos 3x\ dx$.

Let $I = \int e^{2x}\cos 3x\ dx$.

Choose $U = \cos 3x$ and $dV = e^{2x}\ dx$, so that $dU = -3\sin 3x\ dx$ and $V = \tfrac{1}{2}e^{2x}$.

Then, $I = \cos 3x \, \tfrac{1}{2} e^{2x} - \int \tfrac{1}{2} e^{2x} \cdot -3\sin 3x \, dx$

$$= \tfrac{1}{2} e^{2x} \cos 3x + \tfrac{3}{2} \int e^{2x} \sin 3x \, dx$$

Now, with $U = \sin 3x$ and $dV = e^{2x} \, dx$ [$\therefore \quad dU = 3\cos 3x \, dx, \quad V = \tfrac{1}{2} e^{2x}$],

$$\int e^{2x} \sin 3x \, dx = \sin 3x \, \tfrac{1}{2} e^{2x} - \int \tfrac{1}{2} e^{2x} \cdot 3\cos 3x \, dx$$

$$= \tfrac{1}{2} e^{2x} \sin 3x - \tfrac{3}{2} \int e^{2x} \cos 3x \, dx \quad \text{[See, it appears again!]}$$

$$= \tfrac{1}{2} e^{2x} \sin 3x - \tfrac{3}{2} I$$

$$\therefore \quad I = \tfrac{1}{2} e^{2x} \cos 3x + \tfrac{3}{2} [\tfrac{1}{2} e^{2x} \sin 3x - \tfrac{3}{2} I]$$

We now collect all the I on the left

$$\therefore \quad \tfrac{13}{4} I = \tfrac{1}{2} e^{2x} \cos 3x + \tfrac{3}{4} e^{2x} \sin 3x$$

or, $\quad I = \tfrac{1}{13} e^{2x} [2\cos 3x + 3\sin 3x] + C$.

Hence,

$$\int e^{2x} \cos 3x \, dx = \tfrac{1}{13} e^{2x} [2\cos 3x + 3\sin 3x] + C.$$

Integrals of the Inverse Trigonometry

We shall integrate by parts.

Example 38:

$$\int \sin^{-1} x\, dx = \int \sin^{-1} x \cdot 1\, dx$$

Here we choose $U = \sin^{-1} x$ and $dV = 1 \cdot dx$, so that $dU = \dfrac{1}{\sqrt{1-x^2}} dx,\ V = x$.

$$\therefore \quad \int \sin^{-1} x\, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx = x \sin^{-1} x + \tfrac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\, dx$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + C.$$

Example 39:

$$\int \tan^{-1} x\, dx$$

Choose $U = \tan^{-1} x$ and $dV = 1 \cdot dx$, so that $dU = \dfrac{1}{1+x^2} dx,\ V = x$.

$$\therefore \quad \int \tan^{-1} x\, dx = x \tan^{-1} x - \int \frac{x}{1+x^2}\, dx$$

$$= x \tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) + C.$$

Activity 9

Find the following integrals:

(i) $\int e^{3x} \sin 2x \, dx$;

(ii) $\int x \sin^{-1} x^2 \, dx$;

(iii) $\int x^3 e^{2x} dx$;

(iv) $\int x \tan^{-1} x \, dx$;

(v) $\int \sin(\ln x) \, dx$ [**Hint**: Let $z = \ln x$, then integrate by parts.] **;**

(vi) $\int_1^2 \frac{\ln x}{\sqrt{x}} \, dx$.

1.4 SUMMARY

Having studied carefully this unit and done all the activities therein, you should now be familiar with all the techniques of differentiation and integration presented. Integration is an art and only lots of practice will enable you to be good at it. You should at a glance decide which method of integration is the most appropriate. The following supplementary exercises will help you consolidate what you have learnt so far.

1.5 SUPPLEMENTARY EXERCISES

1. Differentiate the following functions:

   (i) $(3x+1)^5/(2-x)^{10}$;

   (ii) $[x+\sqrt{1+x^2}]^n$;

   (iii) $a^x \tan x + \ln(1-\cos^3 x)$;

   (iv) $\sin^{-1}[(1+2\sin x)/(2+\sin x)]$;

   (v) $\tan^{-1}\sqrt{(1-x)/(1+x)}$;

   (vi) $\cot^{-1}(x/\ln x) + \cot^{-1}(\ln x/x)$;

   (vii) $(x+1/x)^x$.

2. If $x = 2y - \tan^{-1} y$, find the value of d^2y/dx^2 when $y = 1$.

3. Find dy/dx if $y = \sin[(x+y)^2]$.

4. Find d^2y/dx^2 when

   (i) $x = a\cos^3\theta,\ y = a\sin^3\theta$;

   (ii) $x = a[\ln\cot\frac{1}{2}\theta - \cos\theta],\ y = a\sin\theta$.

5. If $y = \sin^{-1}[\frac{1}{2}\cos x]$, prove that

$$y'' = \tan y\,[y'^2 - 1]$$

   If $-\pi/2 < y < \pi/2$, obtain the expansion of y in ascending powers of x as far as and including the term in x^2, and show that the coefficient of x^3 is zero.

6. Show that, for small values of x,

$$\sec^2(\tfrac{\pi}{4}+x) = 2 + 4x + ax^2 + bx^3 + cx^4 + \dots,$$

and determine a, b and c. Expand $\sqrt{2+\sec^2(\frac{\pi}{4}+x)}$ in ascending powers of x as far as the term involving x^3.

7. Find the following integrals:

(i) $\displaystyle\int \frac{4x+7}{4+(x+1)^2}\,dx$;

(ii) $\displaystyle\int \frac{d\phi}{1+\sin\phi}$;

(iii) $\displaystyle\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$;

(iv) $\displaystyle\int_0^1 \frac{dx}{(x+1)\sqrt{x^2+1}}$;

(v) $\displaystyle\int_b^a \frac{x\,dx}{\sqrt{(a-x)(x-b)}}$, $(a > x > b)$ [Let $a-x=(a-b)\cos^2\theta$] ;

(vi) $\displaystyle\int \frac{dx}{x\sqrt{3x^2+2x-1}}$ [Let $x=1/u$] ;

(vii) $\displaystyle\int \frac{e^{-1/x}}{x^4}\,dx$, [**Hint**: $\dfrac{e^{-1/x}}{x^4} = \dfrac{1}{x^2}\dfrac{e^{-1/x}}{x^2}$]**;**

(viii) $\displaystyle\int_0^{\pi/2} \cos\theta\,\ln(1+\sin\theta)\,d\theta$.

1.6 ANSWERS TO ACTIVITIES AND SUPPLEMENTARY EXERCISES

Activity 1

(i) $(x+2)(x+3)^2(x+4)^3(9x^2+52x+72)$;

(ii) $\dfrac{1}{(x-1)^{1/2}(x+1)^{3/2}}$;

(iii) $\dfrac{-x^3-x^2+7x-1}{2\sqrt{x-1}\,(x^2+1)^2}$;

(iv) $2x\,a^{x^2+9}\ln a$;

(v) $(\cot x)^{\sin x}\,[\cos x\ln\cot x-\sec x]+(\tan x)^{\cos x}\,[\operatorname{cosec} x-\sin x\ln\tan x]$.

Activity 2

1. (i) $\dfrac{3}{\sqrt{1-x^2}}$;

   (ii) $-\dfrac{2}{1+x^2}$;

   (iii) $\dfrac{3\cos x-\cos 3x}{2(1-\sin x\sin 3x)}$.

Activity 3

(i) $-\cot 2t$;

(ii) $\frac{1}{3}\sec^3 2t \operatorname{cosec} t$.

Activity 4

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2.2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2.3!}\left(x - \frac{\pi}{3}\right)^3 + \ldots + \frac{\cos\left(\frac{\pi}{3} + \frac{r\pi}{2}\right)}{r!}\left(x - \frac{\pi}{3}\right)^r + \ldots.$$

Activity 5

(i) $\frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$;

(ii) $\frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + C$;

(iii) $\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C$;

(iv) $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$;

(v) $-\frac{1}{18}\cos^6 3x + \frac{1}{24}\cos^8 3x + C$.

Activity 6

(i) $\frac{1}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}}{2}x\right)+C$;

(ii) 0.483039 ;

(iii) $\frac{1}{2\sqrt{6}}\tan^{-1}\left(\sqrt{3/8}\ x\right)+C$;

(iv) 0.0740874.

Activity 7

(i) $\frac{1}{7}\ln\left(\frac{2x-3}{x+2}\right)+C$;

(ii) $\frac{2}{\sqrt{111}}\tan^{-1}\left(\frac{10x+7}{\sqrt{111}}\right)+C$;

(iii) $\sin^{-1}\left(\frac{2x+1}{3}\right)+C$;

(iv) $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{5}\right)+C$;

(v) $\pi/2$;

(vi) $\pi/12$;

(vii) $\frac{2}{\sqrt{7}}\tan^{-1}\sqrt{7}$;

(viii) $\pi/4$; (ix) $\dfrac{\pi}{6\sqrt{3}}$;

(x) $\dfrac{1}{4}\ln(7/5)$.

Activity 8

(i) $\dfrac{2}{5\sqrt{5}}\tan^{-1}\left(\dfrac{1}{\sqrt{5}}\tan\dfrac{5x}{2}\right)+C$;

(ii) $\ln\dfrac{C\sqrt{2+\tan\frac{1}{4}x}}{\sqrt{2-\tan\frac{1}{4}x}}$;

(iii) $\dfrac{-2}{3[1+\tan\frac{3}{2}\phi]}+C$;

(iv) $\dfrac{1}{\sqrt{10}}\tan^{-1}\left(\sqrt{\dfrac{5}{2}}\tan x\right)+C$;

(v) $\dfrac{1}{\sqrt{21}}\tan^{-1}\left(\sqrt{\dfrac{3}{7}}\tan x\right)+C$;

(vi) $-\dfrac{x}{3}+\dfrac{\sqrt{7}}{6}\tan^{-1}\left(\dfrac{2\tan x}{\sqrt{7}}\right)+C$;

(vii) $\dfrac{\pi}{4}-\dfrac{1}{\sqrt{5}}\tan^{-1}\left(\dfrac{2}{\sqrt{5}}\right)$;

(viii) $-\dfrac{1}{2}\sqrt{\dfrac{5}{2}}\tan^{-1}\left(\sqrt{\dfrac{2}{5}}\tan x\right)+\dfrac{1}{2}\tan x+C$;

(ix) $\dfrac{\pi}{4\sqrt{3}}$;

(x) $\frac{1}{2}\left(\frac{\pi}{3}-\frac{1}{\sqrt{3}}\tan^{-1}3\right)$.

Activity 9

(i) $\frac{1}{13}e^{3x}(3\sin 2x-2\cos 2x)+C$;

(ii) $\frac{1}{2}[x^2\sin^{-1}x^2+\sqrt{1-x^4}]+C$;

(iii) $\frac{1}{8}e^{2x}[4x^3-6x^2+6x-3]+C$;

(iv) $\frac{1}{2}[(1+x^2)\tan^{-1}x-x]+C$;

(v) $\frac{1}{2}x\,[\sin\ln x-\cos\ln x]+C$;

(vi) $2\sqrt{2}\ln 2-4\sqrt{2}+4$.

Supplementary Exercises

1. (i) $5(3x+1)^4(8+3x)/(2-x)^{11}$;

(ii) $n[x+\sqrt{1+x^2}]^n/\sqrt{1+x^2}$;

(iii) $a^x[\sec^2 x+\ln a\tan x]+3\cos^2 x\sin x/(1-\cos^3 x)$;

(iv) $\sqrt{3}/(2+\sin x)$;

(v) $-1/2\sqrt{1-x^2}$;

(vi) 0;

(vii) $(x+1/x)^x\,[\ln(x+1/x)+(x^2-1)/(x^2+1)]$.

2. $-4/27$.

3. $\dfrac{2(x+y)\cos[(x+y)^2]}{1-2(x+y)\cos[(x+y)^2]}$.

4. (i) $y'' = \frac{1}{3a}\sec^4\theta\,\operatorname{cosec}\theta$;

(ii) $y'' = \frac{1}{a}\sec^4\theta\sin\theta$.

5. $y = \frac{1}{6}(\pi - \sqrt{3}\,x^2)$.

6. $a = 8,\ b = 40/3,\ c = 64/3;\quad 2 + x + \frac{7}{4}x^2 + \frac{59}{24}x^3$.

7. (i) $2\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}[(x+1)/2] + C$;

(ii) $-\dfrac{2}{1+\tan\phi/2} + C$;

(iii) $x - \sqrt{1-x^2}\,\sin^{-1}x + C$;

(iv) $2^{-1/2}\ln(1+\sqrt{2})$; (v) $(a+b)\pi/2$;

(vi) $\cos^{-1}\left(\dfrac{1-x}{2x}\right) + C$; (vii) $\left(\dfrac{1}{x^2}+\dfrac{2}{x}+2\right)e^{-1/x} + C$; (viii) $\ln 4 - 1$.