MATH 1211

TUTORIAL SHEETS

1-11

TUTORIAL SHEET 1

1. If $A=\begin{pmatrix}1 & 2 & 4\\ -1 & -2 & 3\\ 7 & 0 & 1\end{pmatrix}$, evaluate $\left|A^2\right|, |5A|, 5|A|, \left|\frac{1}{3}A^T\right|, \left|A^3A^T\right|$.

2. Solve the following equations:

(a) $\begin{vmatrix}x & 2 & 3\\ -2 & x & 4\\ -3 & -4 & x\end{vmatrix}=0$;

(b) $\begin{vmatrix}x & x & x\\ y & x & x\\ 0 & y & x\end{vmatrix}=0$, $x\neq 0$;

3. By using the properties of the determinant of a matrix, prove the following:

(i) $\begin{vmatrix}x+a & a & a\\ a & x+a & a\\ a & a & x+a\end{vmatrix}=x^2(x+3a)$

(ii) $\begin{vmatrix}x^2 & x & 1\\ y^2 & y & 1\\ z^2 & z & 1\end{vmatrix}=(x-y)(x-z)(y-z)$

4. Calculate the determinant of the five matrices and state those that are singular.

$$A=\begin{pmatrix}2 & 0 & -1\\ 1 & 2 & 2\\ 3 & 2 & 4\end{pmatrix},\ B=\begin{pmatrix}2 & 1 & 2\\ -1 & 1 & 5\\ 4 & 2 & 3\end{pmatrix},\ AB^2, A+B, AB+A^2.$$

5. Find the inverses of the following matrices by the adjoint method.

(a) $\begin{pmatrix}1 & 3 & 2\\ 1 & 1 & 1\\ 2 & -3 & -1\end{pmatrix}$; (b) $\begin{pmatrix}1 & -1 & 2\\ 2 & 1 & -1\\ -1 & 2 & 1\end{pmatrix}$

ANSWERS

1. $|A| = 98; |A^2| = (98)^2; |5A| = 5^3|A|; 5|A| = 5 \times 98; |\tfrac{1}{3}A^T| = \tfrac{1}{27}|A|; |A^3A^T| = (98)^4$

2. (a) $x = 0$; (b) $x = y$.

4. 12; -3; 108; 0(singular); 0(singular).

5. (a) $\begin{pmatrix} 2 & -3 & 1 \\ 3 & -5 & 1 \\ -5 & 9 & -2 \end{pmatrix}$ (b) $\dfrac{1}{14}\begin{bmatrix} 3 & 5 & -1 \\ -1 & 3 & 5 \\ 5 & -1 & 3 \end{bmatrix}$

TUTORIAL SHEET 2

1. Use Cramer's rule to solve the following systems of equations:

(a) $\begin{aligned} 2x + y - z &= 1 \\ x - y - z &= 0 \\ x + y - z &= 1 \end{aligned}$ (b) $\begin{aligned} -x + 3y - 2z &= 7 \\ 3x + 3z &= -3 \\ 2x + y + 2z &= -1 \end{aligned}$

2. Use Gauss Elimination Method to solve the following systems:

(i) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -2 & 4 & -1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -17 & 8 & -5 \\ 0 & -5 & -2 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 6 \\ 2 \\ 2 \\ 2 \end{bmatrix}$

3. Use LU-factorization method to solve the following systems:

(i) $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$; (ii) $\begin{bmatrix} 3 & 9 & 6 \\ 18 & 48 & 39 \\ 9 & -27 & 42 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 23 \\ 136 \\ 45 \end{bmatrix}$

4. Find the inverses of the following matrices by using row operations.

(a) $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & -3 & -1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$

Hence solve the following systems of equations:

(i) $\begin{aligned} x_1 + 3x_2 + 2x_3 &= -3 \\ x_1 + x_2 + x_3 &= 2 \\ 2x_1 - 3x_2 - x_3 &= -4 \end{aligned}$, (ii) $\begin{aligned} x_1 - x_2 + 2x_3 &= 16 \\ 2x_1 + x_2 - x_3 &= 1 \\ -x_1 + 2x_2 + x_3 &= -3 \end{aligned}$

5. Determine the ranks of the following matrices:

(i) $\begin{bmatrix} 5 & 7 & -3 \\ 3 & 4 & 1 \\ 4 & -1 & 5 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \\ 2 & 1 & 5 \end{bmatrix}$; (iii) $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 7 \\ 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 6 \end{bmatrix}$.

ANSWERS

1. (a) $x = 0$, $y = 1/2$, $z = -1/2$

   (b) $x = 2$, $y = 1$, $z = -3$

2. (i) $[4 \quad , \quad 2 \quad , \quad -5]^T$; (ii) $[4 \quad , \quad 0 \quad , \quad 2 \quad 6]^T$

3. (a) $[-1 \quad , \quad 3 \quad , \quad 2]^T$; (b) $[-1/3 \; , \; 4/3 \; , \; 2]^T$

4. (a) $\begin{pmatrix} 2 & -3 & 1 \\ 3 & -5 & 1 \\ -5 & 9 & -2 \end{pmatrix}$ (b) $\frac{1}{14}\begin{bmatrix} 3 & 5 & -1 \\ -1 & 3 & 5 \\ 5 & -1 & 3 \end{bmatrix}$

   (i) $x_1 = -16,\ x_2 = -23,\ x_3 = 41$; (ii) $x_1 = 4,\ x_2 = -2,\ x_3 = 5$.

5. (i) 3; (ii) 2; (iii) 3.

TUTORIAL SHEET 3

1. Find the general solution of the system of equations

$$\begin{aligned} x_1 + x_2 - \lambda x_3 &= \mu \\ 3x_1 - 2x_2 - x_3 &= 1 \\ 4x_1 - 3x_2 - x_3 &= 2 \end{aligned}$$

in each of the three cases (i) $\lambda = 1$, $\mu = 9$; (ii) $\lambda = 2$, $\mu = -3$; (iii) $\lambda = 2, \mu = 0$.

2. Find the eigenvalues and the corresponding eigenvectors of the matrices below.

(i) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$; (ii) $B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(iii) $C = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$; (iv) $D = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

3. (a) Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(b) Obtain a matrix Q such that $Q^{-1}AQ$ is diagonal, and hence find A^5.

(c) Write down the eigenvalues and eigenvectors of the matrices

$$A^3, (A+6I), (A-5I)^{-1}.$$

4. Given the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -2 & -1 \\ 2 & 3 & 2 \end{pmatrix}$, show that $A^3 - A = A^2 - I$. Hence find A^{-1}.

ANSWERS

1. (i) $x_1 = 11,\ x_2 = 10,\ x_3 = 12$

   (ii) $x_1 = t\ ,\ x_2 = t-1\ ,\ x_3 = t+1$

   (iii) No solution.

2. (i) $-1\,,\left[1\ ,-2\right]^T\ ;\quad 3\,,\left[1\ ,\ 2\right]^T\ .$

   (ii) $1, 1, 1, \left[1\ ,\ \ 0\ ,\ \ 0\right]^T\ ,\ \left[0\ ,\ \ -2\ ,\ \ 1\right]^T.$

   (iii) $-1\,,\left[1\ ,\ \ 0\ ,\ \ 1\right]^T;\ 2\,,\left[1\ ,\ \ 3\ ,\ \ 1\right]^T;\ 1\,,\left[3\ ,\ \ 2\ ,\ \ 1\right]^T.$

   (iv) $1\,,1\,,\left[2\ ,\ \ -1\ ,\ \ 0\right]^T$ and $\left[1,\ \ 0,\ \ -1\right]^{\mathrm{T}};5,\left[1,\ \ 2,\ \ 1\right]^{\mathrm{T}}.$

3. (a) $1\,,\left[-1\ ,\ \ 1\ ,\ \ 1\right]^T;\ 2\,,\left[0\ ,\ \ 1\ ,\ \ 1\right]^T;\ 0\,,\ \left[-1\ ,\ \ 1\ ,\ \ 0\right]^T\ .$

   (b) $Q = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix};\quad A^5 = \begin{pmatrix} 1 & 1 & -1 \\ 31 & 31 & 1 \\ 31 & 31 & 1 \end{pmatrix}$

   (c) 1, 8, 0, same eigenvectors; 7, 8, 6, same eigenvectors;
   $-\dfrac{1}{4}\,,\quad -\dfrac{1}{3}\,,\quad -\dfrac{1}{5}\,,$ same eigenvectors.

4. $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{pmatrix}.$

TUTORIAL SHEET 4

1. Find the sum of the series

$$\ln\frac{1}{2}+\ln\frac{2}{3}+\ln\frac{3}{4}+\ldots+\ln\frac{n}{n+1}.$$

Hence, determine whether or not the series $\sum_{r=1}^{\infty}\ln\frac{\mathrm{r}}{\mathrm{r}+1}$ converges.

2. Test the following series for convergence:

(i) $\sum_{n=1}^{\infty}\frac{1}{(n+1)^{\frac{3}{2}}}$ **; (ii)** $\sum_{n=1}^{\infty}\frac{n-5}{n^2}$ **; (iii)** $\sum_{r=1}^{\infty}\sqrt{r^2+1}+r$ **; (iv)** $\sum_{r=1}^{\infty}\frac{2r}{1+r^2}$ **;**

(v) $\sum_{n=1}^{\infty}\frac{3n+5}{n-7}$ **; (vi)** $\sum_{n=1}^{\infty}\frac{\cos^4 nx}{n^2}$ **; (vii)** $\sum_{n=1}^{\infty}\frac{(3n-5)2^n}{n!}$ **; (viii)** $\sum_{r=1}^{\infty}\frac{3^r+4^r}{4^r+5^r}$ **;**

(ix) $\sum_{r=1}^{\infty}\frac{r^r}{r!}$ **; (x)** $\sum_{n=1}^{\infty}\frac{1.2.3.\ldots.n}{4.7.10.\ldots.(3n+1)}$.

3. Use the Taylor series to find a quadratic approximation to each of the following functions at the specified points:

(i) $5x^3y-x^2+xy^2-3x+4y$ at $(-1,2)$ **; (ii)** $y\sin xy$ at $(\pi/2,1)$ **;**

ANSWERS

1. $S_n=\ln\frac{1}{n+1}$**.** The sum to infinity is divergent (Hint: write the expression as a telescoping sum and use the fact that if a sequence of partial sums does not converge, then the corresponding infinite series diverges.

2. Below, **C : convergent ; D : divergent**
(i) C ; (ii) D ; (iii) D ; (iv) D ; (v) D ; (vi) C ; (vii) C ; (viii) C ; (ix) D ; (x) C.

3. **(i)** $-4+33(x+1)-5(y-2)-31(x+1)^2+19(x+1)(y-2)-(y-2)^2$;

(ii) $1+(y-1)-\frac{1}{2}(x-\pi/2)^2-\frac{1}{2}\pi(x-\pi/2)(y-1)-\frac{1}{8}\pi^2(y-1)^2$.

TUTORIAL SHEET 5

1. If $\mathbf{a} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}, \mathbf{b} = -4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}, \mathbf{c} = \hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$, find

(i) $\mathbf{a} \wedge \mathbf{b}$ **; (ii)** $\mathbf{a}.\mathbf{b} \wedge \mathbf{c}$ **; (iii)** $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$ **; (iv)** $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$.

2. Prove the following vector identities

(i) $(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} - \mathbf{b}) = -2\mathbf{a} \wedge \mathbf{b}$

(ii) $(\mathbf{a} \wedge \mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}.\mathbf{b})^2$

3. Find the unit vectors which are perpendicular to both the vectors

$$3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 7\hat{\mathbf{k}}\ ,\ 4\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}\ .$$

Find also the sine of the angle between these two vectors.

4. Determine a unit vector normal to the plane of the vectors $\mathbf{a} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}},$ and $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$. Find the area of the triangle of which these two vectors form adjacent sides.

5. Find the vector **x** and the scalar λ which satisfy the equations

$$\mathbf{a} \wedge \mathbf{x} = \mathbf{b} + \lambda\,\mathbf{a}, \quad \mathbf{a}\,.\mathbf{x} = -3,$$

where $\mathbf{a} = -6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, and $\mathbf{b} = 3\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 7\hat{\mathbf{k}}$.

ANSWERS

1. **(i)** $\begin{pmatrix} -1 \\ 14 \\ 8 \end{pmatrix}$; **(ii)** -23; **(iii)** $\begin{pmatrix} 124 \\ 14 \\ -9 \end{pmatrix}$; **(iv)** $\begin{pmatrix} -47 \\ 63 \\ 19 \end{pmatrix}$.

3. $\pm\dfrac{1}{\sqrt{1302}}(17\hat{\mathbf{i}} + 22\hat{\mathbf{j}} + 23\hat{\mathbf{k}})$; $\sqrt{62/83}$

4. $\pm\dfrac{\sqrt{2}}{10}(4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}})$; $\dfrac{5\sqrt{2}}{2}$

5. $\mathbf{x} = \frac{1}{41}[31\,\hat{\mathbf{i}} + 33\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}]$; $\lambda = 27/41$.

TUTORIAL SHEET 6

1. Find grad ϕ for the following:
(a) $\phi = x^2 + y^2 - z^2$; ***(b)*** $\phi = 3xz^4 - x^2y^3z$; ***(c)*** $\phi = e^{xz} \sin yz$.

2. Find a unit normal vector to the surface at P:

(i) $2x + y - 3z = 10$; $P:\ (2,3,-1)$;

(ii) $x^2 + y^2 + 3z^2 = 28$; $P:\ (-1,0,3)$.

(iii) $x^3y - z\cos y + ye^{-2x} - e^x = \pi$; $P:\ (0,\pi,1)$.

3. Find the directional derivative of ϕ at point Q in the given direction:

(i) $\phi = 2x^2 - 4y^2 + z^2$; $Q:(0,1,2)$; $\mathbf{s} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$;

(ii) $\phi = \ln\sqrt{x^2 + y^2 + z^2}$; $Q:(a,b,c)$; towards the origin ;

(iii) $\phi = xe^{y^2z}$; $Q:(-1,\frac{1}{4},0)$; towards $(1,0,\frac{1}{2})$.

4. Find div **F** and curl **F** when

$$\mathbf{F} = (2xy^3 - z^2)\hat{\mathbf{i}} + (3x^2y^2 + z)\hat{\mathbf{j}} + (y - 2xz)\hat{\mathbf{k}}.$$

5. The temperature T of a heated circular plate at any of its point (x,y) is given by

$$T = \frac{64}{x^2 + y^2 + 2},$$

the origin being the centre of the plate. Find the rate of change of T at the point (1,2), in the direction $\theta = \dfrac{\pi}{3}$.

6. If $\varphi = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$, find $\nabla^2\varphi$.

ANSWERS

1. **(a)** $2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$;

(b) $(3z^4 - 2xy^3z)\hat{\mathbf{i}} - 3x^2y^2z\,\hat{\mathbf{j}} + (12xz^3 - x^2y^3)\hat{\mathbf{k}}$;

(c) $e^{xz}[(z\sin yz)\hat{\mathbf{i}} + (z\cos yz)\hat{\mathbf{j}} + (x\sin yz + y\cos yz)\hat{\mathbf{k}}]$.

2. **(i)** $(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}})/\sqrt{14}$; **(ii)** $(-\hat{\mathbf{i}} + 9\hat{\mathbf{k}})/\sqrt{82}$; **(iii)** $[(-2\pi - 1)\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}]/\sqrt{(1+2\pi)^2+2}$, ***or their negatives.***

3. **(i)** $2\sqrt{14}$ **; (ii)** $-(a^2 + b^2 + c^2)^{-1/2}$; **(iii)** $63/8\sqrt{69}$.

4. $2y^3 + 6x^2y - 2x$; **0.**

5. $\dfrac{-64(1+\sqrt{3})}{49}$.

6. $6z + 24xy - 2z^3 - 6y^2z$.

TUTORIAL SHEET 7

1. In each of the following,

(a) sketch the region of integration,

(b) evaluate the integral,

(c) write down the integral with the order of integration reversed,

(d) evaluate again and compare with **(b)**.

(i) $\int_0^4 \int_0^{4-x} dydx$; **(ii)** $\int_0^3 \int_0^x (x^2 + y^2)\, dydx$; **(iii)** $\int_0^1 \int_x^{\sqrt{x}} xy^2 \; dydx$.

2. Describe the region of integration and evaluate:

(a) $\int_0^1 \int_x^{2x} (2 + x^2 + y^2)\; dydx$; **(b)** $\int_0^{\pi} \int_0^{\sin x} y\, dydx$; **(c)** $\int_0^{\pi/2} \int_0^{\cos y} x^2 \sin y\, dxdy$ **.**

3. Find $\iint x\, dxdy$ over the first quadrant of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

4. Find the volume of the region in space that lies beneath $z = x^2 + y^2$ and above the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ in the $xy-$plane.

ANSWERS

1. **(i) 8 ; (ii)** 27 **; (iii)** 1/35 **.**

2. **(a)** 11/6 **; (b)** $\pi/4$ **; (c)** 1/12 **.**

3. 4.

4. 2/3**.**

TUTORIAL SHEET 8

1. Using the transformation $u = y,\ v = y^2 / x$, find the volume between the plane $z = 0$ and the surface $z = \exp[-x / y^2]$ bounded by the cylinder defined by $y = 1,\ y = 2,\ y^2 = x,\ y^2 = 2x$.

2. Evaluate $\int_0^\infty e^{-x}\sqrt{x}\,dx$ by integrating $e^{-x-y}\sqrt{xy}$ over the first quadrant with the change of variable $x = \frac{1}{2}u(1+v),\ y = \frac{1}{2}u(1-v)$.

3. Sketch the area over which the double integral

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \ln(x^2 + y^2)\, dxdy$$

is taken. By changing to polar coordinates, show that the value of the integral is $\pi(\ln 2 - \frac{1}{2})$.

4. Sketch the region over which the double integral is taken and evaluate it:

$$\int_0^4 dx \int_{\sqrt{4x-x^2}}^{\sqrt{16-x^2}} \frac{dy}{\sqrt{16 - x^2 - y^2}} .$$

ANSWERS

1. $\frac{7}{3}(e^{-1/2} - e^{-1})$

2. $\sqrt{\pi} / 2$.

4. *4*

TUTORIAL SHEET 9

1. Evaluate the following triple integrals:

(a) $\int_0^1 \int_1^2 \int_2^3 dzdxdy$; **(b)** $\int_0^1 \int_{x^2}^{x} \int_0^{xy} dzdydx$;

(c) $\int_0^6 \int_0^{12-2y} \int_0^{4-2y/3-x/3} x \ dzdxdy$; **(d)** $\int_0^{\pi/2} \int_0^4 \int_0^{\sqrt{16-z^2}} (16-r^2)^{1/2} rz \ drdzd\theta$.

2. Using triple integrals, find the volume of the tetrahedron bounded by the coordinate planes and the plane $6x+4y+3z=12$.

3. Use cylindrical coordinates to evaluate $\iiint\limits_{\Omega} f(x,y,z)\ dV$ where $f=z$ and Ω : the region above the cone $z^2=x^2+y^2$ and below the plane $z=2$.

4. Use spherical polar coordinates to evaluate $\iiint\limits_{\Omega} f(x,y,z)\ dV$:

f: $x^2+y^2+z^2$; Ω : the region above the cone $z^2=x^2+y^2$ and below the plane $z=1$.

ANSWERS

1. **(a)** 1; **(b)** 1/24 ; **(c)** 144 ; **(d)** 256π/5 .

2. 4.

3. 4π .

4. $3\pi/10$.

TUTORIAL SHEET 10

1. Evaluate the following line integrals:

(i) $\int_C (x^2 + 2y)\, dx$ from $(0,1)$ to $(2,3)$, where C is the line $y = x + 1$;

(ii) $\int_C x^2 y\, dx + (x^2 - y^2)\, dy$ from $(0,0)$ to $(1,4)$, where C is the curve $y = 4x^2$;

(iii) $\oint_C x\, dy - y\, dx$, where C is the curve $x = a\cos^3 t,\ y = a\sin^3 t,\ \ a$: constant .

2. For the given vector field **F** and the curve Γ, evaluate $\int_\Gamma \mathbf{F}.d\mathbf{r}$:

(a) $\mathbf{F}(x,y) = xy\hat{\mathbf{i}} + (2x - y)\hat{\mathbf{j}}$; Γ : the arc of $y = x^2$ from $(0,0)$ to $(1,1)$;

(b) $\mathbf{F}(x,y,z) = xy\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} - xz\,\hat{\mathbf{k}}$; Γ: $\mathbf{r}(t) = t\,\hat{\mathbf{i}} - 2t\,\hat{\mathbf{j}} - \ln t\,\hat{\mathbf{k}},\ \ 1 \le t \le 3$;

3. Show that $\int_C 2x\sin y\, dx + x^2\cos y\, dy$ is independent of the path C, and evaluate it from (0,0) to (1,π/2).

4. Verify Green's theorem for the following integrals:

(i) $\oint_C (x^2 + y)\, dx - xy^2\, dy$, where C is the square with vertices $(0,0), (1,0), (1,1)$ and (0,1) ;

(ii) $\oint_C (x - y)\, dx + (x + y)\, dy$, where C is the boundary of the finite area in the first quadrant between the curves $y = x^2$ and $y^2 = x$.

5. Use Green's theorem to evaluate $\oint_C x^2 y\, dx + y^3\, dy$, where C is the closed path formed by the graphs of $y^3 = x^2$ and $y = x$.

ANSWERS

1. **(i)** 32/3 **; (ii) -** 278/15 **; (iii)** $3\pi a^2 / 4$.

2. **(a)** 13/12 **; (b)** $-(254/3 + \ln 27)$.

3. 1.

4. **(i)** - 4/3 **; (ii)** 2/3 **.**

5. - 1/44.

TUTORIAL SHEET 11

1. Evaluate the surface integral $\iint_S \mathbf{F} . \hat{\mathbf{n}} \, dS$ by the divergence theorem, where

(i) $\mathbf{F} = x^3 \, \hat{\mathbf{i}} + z^3 \, \hat{\mathbf{k}}$, S: the surface of the cube $|x| \le 1, \; |y| \le 1, \; |z| \le 1$;

(ii) $\mathbf{F} = y^2 \, \hat{\mathbf{i}} + z^2 \, \hat{\mathbf{j}} + x^2 z \, \hat{\mathbf{k}}$, S: the surface of $x^2 + y^2 \le 4, \; \; x \ge 0, \; y \ge 0, \; |z| \le 1$;

2. Evaluate $\oint_C \mathbf{F} . d\mathbf{r}$ by Stokes' theorem, where

(a) $\mathbf{F} = -3y \, \hat{\mathbf{i}} + 3x \, \hat{\mathbf{j}} + z \, \hat{\mathbf{k}}$, C: the circle $x^2 + y^2 = 4, \; z = 1$;

(b) $\mathbf{F} = xyz \, \hat{\mathbf{j}}$, C: the boundary of the triangle with vertices (1,0,0),(0,1,0),(0,0,1);

3. Evaluate $\iint_S (\nabla \wedge \mathbf{F}) . d\mathbf{S}$ if $\mathbf{F} = (x + 2y) \, \hat{\mathbf{i}} - 3z \, \hat{\mathbf{j}} + x \, \hat{\mathbf{k}}$ and S is the surface of $2x + y + 2z = 6$ bounded by $x = 0, \; x = 1, \; y = 0$ and $y = 2$.

4. Evaluate $\iint_S (\nabla \wedge \mathbf{A}) . d\mathbf{S}$, where $\mathbf{A} = (x^2 + y - 4) \, \hat{\mathbf{i}} + 3xy \, \hat{\mathbf{j}} + (2xz + z^2) \, \hat{\mathbf{k}}$ and S is the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy plane.

ANSWERS

1. **(i)** 16 **; (ii)** 2π **.**

2. **(a)** 24π **; (b)** 0.

3. 1**.**

4. -16π**.**