UNIT 8 MULTIPLE INTEGRALS

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8.0 OVERVIEW

In this Unit, you will study integrals over certain types of regions in the plane and in three dimensional space.

8.1 LEARNING OBJECTIVES

When you have successfully completed this Unit, you will be able to do the following:

- Evaluate double integrals over regions in the plane using Cartesian and polar coordinates.
- Change order of integration in double integrals.
- Evaluate double integrals using general curvilinear coordinates.
- Evaluate triple integrals over regions in three dimensional space using Cartesian, cylindrical and spherical coordinates.
- Obtain area of a surface.

8.2 DOUBLE INTEGRALS

8.2.1 DOUBLE INTERGRALS - CARTESIAN COORDINATES

In this section, we study how to integrate a continuous function over a bounded region. Double integrals are usually used to calculate the area of a region, volume of solids or surface area among others.

Consider the double integral given as

$$I_1 = \iint_R f(x,y)\, dR = \iint_R f(x,y)dxdy. \qquad (1)$$

I_1 denotes the integration of a continuous function *f(x,y)* over a bounded simple region *R* in the *xy*-plane.

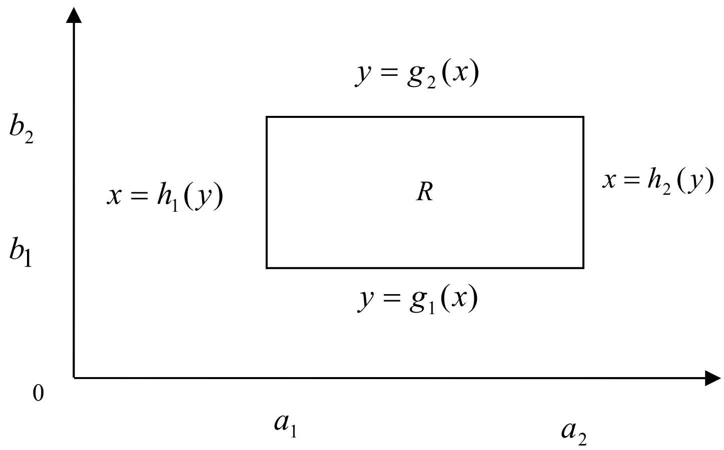

Figure 8.1

This region of integration can also be described in two ways:

<u>Description 1:</u>

For x fixed, y varies from $y = g_1(x)$ to $y = g_2(x)$, and x varies from $x = a_1$ to $x = a_2$.

We say that the region R is vertically simple.

Description 2:

For y fixed, x varies from $x = h_1(y)$ to $x = h_2(y)$, and y varies from $y = b_1$ to $y = b_2$. In this case, the region R is said to be horizontally simple.

We see that a rectangular region of integration can be described by both Description 1 and Description 2. This is usually not the case for a general shaped region. The way we describe the region of integration is important since it sets the **limits** and the **order** of the integration.

Suppose we evaluate (1) by using Description 1. Then we must first find the integral of *f(x,y)* with respect to *y* while treating *x* as a constant. Denoting

$$F(x, y) = \int f(x, y)dy,$$

we easily find that

$$\int_{y=g_1(x)}^{y=g_2(x)} f(x, y)dy = \left[F(x, y)\right]_{y=g_1(x)}^{y=g_2(x)} = F(x, g_2(x)) - F(x, g_1(x)).$$

Now, letting

$$R(x) = F(x, g_2(x)) - F(x, g_1(x)),$$

and

$$T(x) = \int R(x)dx,$$

we obtain

$$I_1 = \int_{a_1}^{a_2} R(x)dx = \left[T(x)\right]_{a_1}^{a_2} = T(a_2) - T(a_1).$$

Similarly, if we choose to evaluate (1) by using Description 2, we find that

$$I_1 = \iint_R f(x, y)dR = \int_{y=b_1}^{y=b_2} \left[\int_{x=h_1(x)}^{x=h_2(x)} f(x, y)dx\right]dy$$

We note that in this case, we first integrate with respect to *x* treating y as a constant and then integrate the resulting function with respect to *y*.

Let us next consider the double integration of a function over a rectangular region.

Example 1

Find $\iint\limits_R f(x,y)dR$, where $f(x,y) = x + y - 3$ and R denotes the rectangular region

$$R: 1 \le x \le 4, 1 \le \mathrm{y} \le 5.$$

Solution

We describe R as follows:

For x fixed, y varies from $y = 1$ to $y = 5$ and x varies from $x = 1$ to $x = 4$.

Hence we can write the integral as

$$\iint\limits_R f(x,y)dR = \int_{x=1}^{x=4}\int_{y=1}^{y=5} x + y - 3\,dy\,dx.$$

Now

$$\begin{aligned}
\int_{x=1}^{x=4}\int_{y=1}^{y=5} x + y - 3\,dy\,dx &= \int_{x=1}^{x=4}\left(\int_{y=1}^{y=5} x + y - 3\,dy\right)dx \\
&= \int_{x=1}^{x=4}\left[x\,y + \frac{y^2}{2} - 3y\right]_{y=1}^{y=5} dx \\
&= \int_{\mathrm{x}=1}^{\mathrm{x}=4}\left[(5x + \frac{25}{2} - 15) - (x + \frac{1}{2} - 3)\right]dx \\
&= \int_{x=1}^{x=4} 4x dx \\
&= \left[2x^2\right]_{x=1}^{x=4} \\
&= 2(16) - 2(1) \\
&= 30.
\end{aligned}$$

In this example the double integration represents the volume of a solid prism which is bounded above by the surface $z = x + y - 3$ and bounded below by the region R. In general, an integration of the form (1) represents the volume bounded above by the surface $z = f(x,y)$ and below by the region R.

Furthermore, $\int_c^d\int_a^b f(x,y)\;dxdy$ is called a <u>Repeated</u> or <u>iterated</u> integral.

The following fundamental rules hold for double integrals:

1. $\iint_R \alpha\, f(x,y)dxdy = \alpha \iint_R f(x,y)dxdy$ where α is a constant.

2. $\iint_R [f(x,y)+g(x,y)]\, dxdy = \iint_R f(x,y)\, dxdy + \iint_R g(x,y)\, dxdy$

that is the integral of the sum of two functions is equal to the sum of their two integrals.

3. If the region R consists of two sub regions R_1 and R_2 that have at most one portion of the boundary in common, then

$$\iint_R f(x,y)dxdy = \iint_{R_1} f(x,y)dxdy + \iint_{R_2} f(x,y)dxdy$$

(i.e when regions are joined together, the corresponding integrals are added).

4. If $f(x,y) \geq 0$ in R, then

$$\iint_R f(x,y)dxdy \geq 0.$$

Similarly, if $f(x,y) \leq 0,$ then

$$\iint_R f(x,y) \leq 0.$$

5. If $f(x,y) \geq \phi(x,y)$, then

$$\iint_R f(x,y)dxdy \geq \iint \phi(x,y)dxdy.$$

Example 2

Evaluate $\int_0^1 \int_{\sqrt{x}}^1 12xy^2 dxdy$.

Solution

We begin by evaluating the inner integral. The variable x is considered to be constant while we integrate with respect to y.

$$\int_0^1 \int_{\sqrt{x}}^1 12xy^2 dydx = \int_{x=0}^{x=1}\left[\int_{y=\sqrt{x}}^{y=1} 12xy^2 dy\right]dx$$

$$= \int_0^1 \left[12\frac{xy^3}{3} \right]_{y=\sqrt{x}}^{y=1} dx$$

$$= \int_0^1 4x - 4x^{\frac{5}{2}} dx$$

$$= \left[2x^2 - \frac{8}{7}x^{\frac{7}{2}} \right]_0^1$$

$$= \frac{6}{7} \leftarrow$$

Example 3

Evaluate $\int_0^2 \int_0^x e^{y/x}\, dydx.$

$$\int_0^2 \int_0^x e^{y/x}\, dydx = \int_{x=0}^{x=2} \left[\int_{y=0}^{y=x} e^{y/x}\, dy \right] dx$$

$$= \int_0^2 \left[xe^{y/x} \right]_{y=0}^{y=x} dx$$

$$= \int_0^2 x(e-1)dx$$

$$= (e-1)\int_0^2 x\, dx = 2(e-1) \leftarrow$$

Example 4

Evaluate $\iint_R x^2 y\, dxdy$ where R is the region in the (positive) first quadrant with $x + y \le 1$.

Solution

First we sketch and describe the region

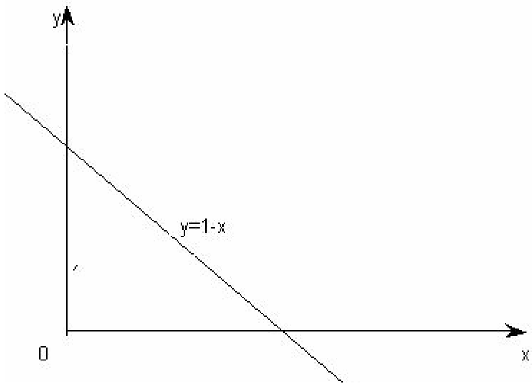

Figure 8.2

The region is described as:

For x fixed, y varies from $y = 0$ to $y = 1- x$,

x varies from $x = 0$ to $x = 1$.

$$\iint_R x^2 y \, dxdy = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} x^2 y \; dxdy = \int_{x=0}^{x=1}\left[\int_{y=0}^{y=1-x} x^2 y \; dy\right] dx$$

$$= \int_0^1 \left[x^2 \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{x^2(1-x)^2}{2} dx$$

$$= \frac{1}{2}\int_0^1 \left(x^4 - 2x^3 + x^2\right) dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{8}\right]_0^1$$

$$= \frac{1}{60} \leftarrow$$

8.2.1.1 Use Of Double Integral To Find The Area Of A Region

The area of a region R in the xy plane is given as

$$A = \iint_R dxdy .$$

We can also view A as the volume of a solid of height 1 over the region R.

Example 5

Use double integral to find the area of the region bounded by $x = y - y^2$ and $x + 3y = 0$.

Solution

We first sketch and describe the region

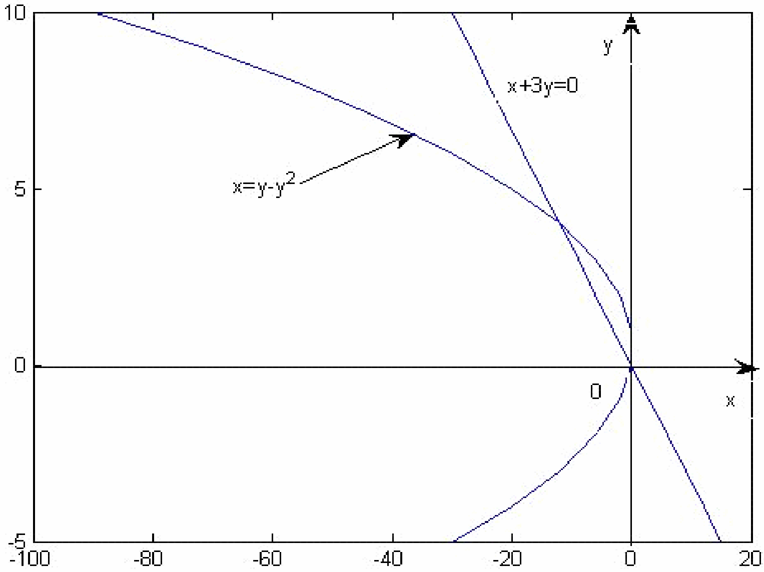

Figure 8.3

The region is horizontally simple and we describe it as :

For y fixed, x varies from $x = -3y$ to $x = y - y^2$

y varies from $y = 0$ to $y = 4$

Note : The two boundary curves intersect at $y = 0$ and $y = 4$

since $-3y = y - y^2 \Rightarrow y = 0, y = 4$

$$\text{Area} = \int_{y=0}^{4}\int_{x=-3y}^{y-y^2} dxdy = \int_{y=0}^{4}\left[\int_{x=-3y}^{x=y-y^2} dx\right]dy$$

$$= \int_0^4 [x]_{x=-3y}^{x=y-y^2}\, dy$$

$$= \int_0^4 4y - y^2 dy$$

$$= \left[2y^2 - \frac{y^3}{3} \right]_0^4$$

$$= \frac{32}{3} \leftarrow$$

8.2.1.2 Use of Double Integral to Find Volume of a Region in Three-Dimensional Space

Double integrals are very useful to evaluate many physical quantities, one of them being the volume of some solids. We proceed by first describing the region of integration.

Suppose a region *R* in three dimensional space can be described by the statements:

For *x* and *y* fixed, *z* varies from $z = h_1(x, y)$ to $z = h_2(x, y)$

For *x* fixed, *y* varies from $y = g_1(x)$ to $y = g_2(x)$,

x varies from $x = a_1$ to $x = a_2$.

The surfaces $z = h_1(x,y)$ and $z = h_2(x,y)$ form the bottom and top of the region.

The volume of the region R is then given as

$$\mathrm{V} = \int_{x=a_1}^{x=a_2} \int_{y=g_1(x)}^{y=g_2(x)} [z]_{h_1(x,y)}^{h_2(x,y)} dydx = \int_{x=a_1}^{x=a_2} \int_{y=g_1(x)}^{y=g_2(x)} [h_2(x,y) - h_1(x,y)] dydx$$

Example 6

Use double integral to find the volume of the region in the first octant bounded by $x + 2y + 4z = 8$.

Solution

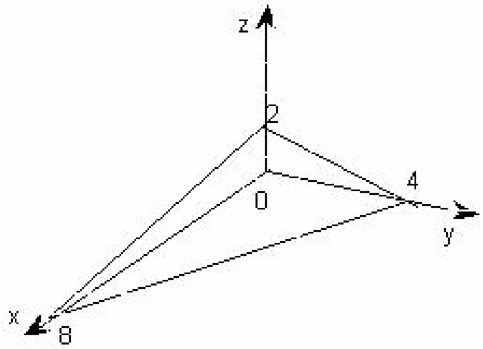

Figure 8.4

The region is described as :

For x and y fixed, z varies from $z=0$ to $z=\dfrac{8-x-2y}{4}$,

For x fixed, y varies from $y=0$ to $y=\dfrac{8-x}{2}$,

x varies from $x=0$ to $x=8$.

$$\text{Volume} \quad = \int_{x=0}^{x=8}\int_{y=0}^{y=\frac{8-x}{2}} [z]_0^{\frac{8-x-2y}{4}}\, dydx$$

$$= \int_{x=0}^{8}\int_{y=0}^{\frac{8-x}{2}} \frac{8-x-2y}{4}\ dydx$$

$$= \int_{x=0}^{8}\left[\frac{8y-xy-y^2}{4}\right]_0^{\frac{8-x}{2}} dx$$

$$= \frac{32}{3}.$$

Activity 1

(i) Evaluate the integrals:

(a) $\int_0^1 \int_0^y xy\, dx\, dy$ (b) $\int_0^{\pi/4} \int_0^y \sec x\ \tan x\, dxdy$

(ii) Use double integral to find the area bounded by the equations:

(a) $y=\cos x$, y = $\dfrac{1}{2}$ $\left(\dfrac{-\pi}{3} \le x \le \dfrac{\pi}{3}\right)$

(b) $y = x(x-2)(x+1),\ y = 0$

8.2.2 DOUBLE INTEGRAL - POLAR COORDINATES

The evaluation of certain type of integrals is sometimes easier when using polar coordinates rather than Cartesian coordinates. This transformation is useful when we deal with integrals of functions over regions such as circles and ellipses, among others.

The procedure of changing a Cartesian integral into polar integral is usually carried out in two steps.

Step 1 a) Express the integrand in terms of polar coordinates by applying the following substitutions: $x = r\cos\theta, y = r\sin\theta$ and $x^2 + y^2 = r^2$.

b) Replace $dx\,dy$ by $r\,dr\,d\theta$ by using the relation

$$dxdy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = rdrd\theta.$$

Step 2 Sketch and describe the region in polar coordinates by statements of the form,

For θ fixed, r varies from $r = g_1(\theta)$ to $r = g_2(\theta)$

θ varies from $\theta = \theta_1$ to $\theta = \theta_2$ where $\theta_1 < \theta_2$.

Here $g_1(\theta)$, $g_2(\theta)$, θ_1 and θ_2 represent the limits of the integration.

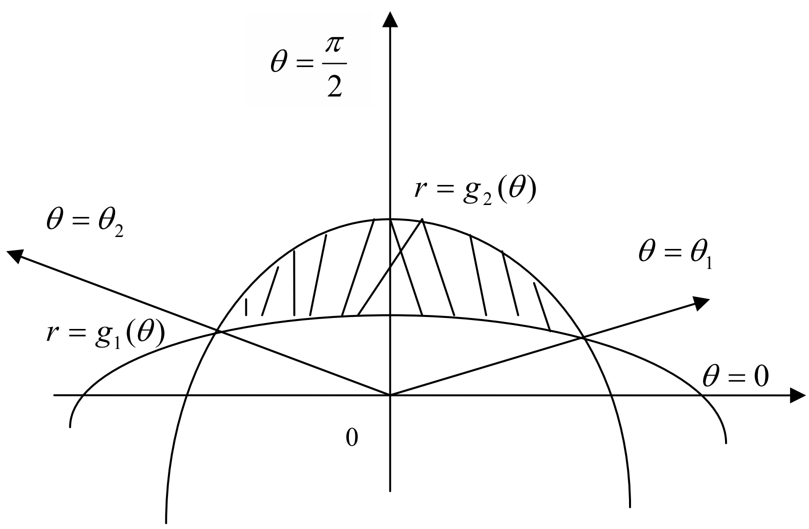

Figure 8.5

If an arrow is shot from the origin, it enters the region at the curve $r = g_1(\theta)$ and leaves the region at $r = g_2(\theta)$. The *limits of* θ are usually obtained by solving the equation $g_1(\theta) = g_2(\theta)$.

Example 7

Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)} dydx$

Solution

The region in the *xy*-plane can be described as (using the limits of integration):

For x fixed, y varies from $y = 0$ to $y = \sqrt{1-x^2}$

x varies from $x = 0$ to $x = 1$

(Since $\int_0^1 \int_0^{\sqrt{1-x^2}} dydx$ means $\int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} dydx$)

The region is shown in Figure 8.6.

It is the part of the circle with centre (0,0) and radius one unit in the first quadrant.

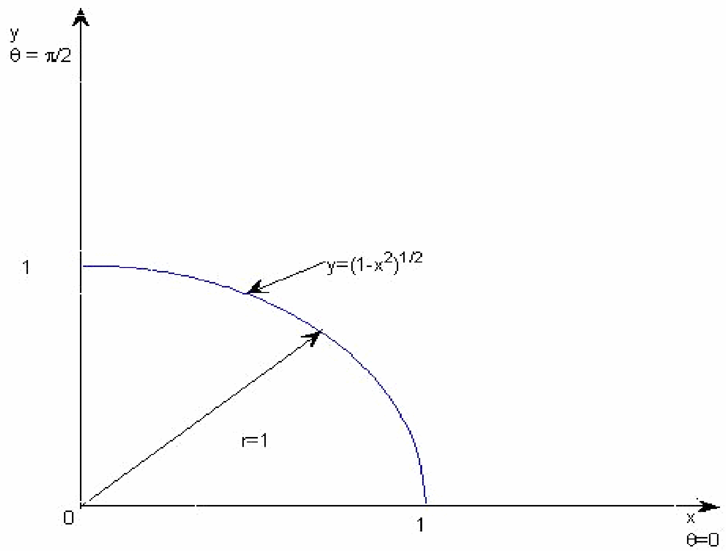

Figure 8.6

In polar coordinates, $x^2 + y^2 = 1$

$\Rightarrow r = 1$

The region can be described in polar coordinates as:

For θ fixed, r varies from $r = 0$ to $r = 1,$

θ varies from θ = 0 to θ = $\pi/2$.

Thus, $\displaystyle\int_{x=0}^{x=1}\int_{y=0}^{\sqrt{1-x^2}} e^{-\left(x^2+y^2\right)} dxdy = \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1} e^{-r^2} rdrd\theta$

$$= \int_{\theta=0}^{\theta=\pi/2}\left[\int_{r=0}^{r=1} re^{-r^2}\,dr\right]d\theta = \int_{\theta=0}^{\theta=\pi/2}\left[-\frac{1}{2}\int_{r=0}^{1} -2re^{-r^2}\,dr\right]d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} -\frac{1}{2}\left(e^{-1}-e^{0}\right)d\theta$$

$$= -\frac{1}{2}\left(e^{-1}-e^{0}\right)\int_{\theta=0}^{\pi/2} d\theta$$

$$= \pi/4\left(1-e^{-1}\right) \leftarrow$$

Note : $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} f(x,y)\ dydx$ means $\displaystyle\int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}} f(x,y)\ dydx$

and not $\displaystyle\int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{1-x^2}} f(x,y)\ dydx$

The limits of the integration describe the boundaries of the region of the integration.

Example 8

Evaluate $\displaystyle\int_0^2\int_0^{\sqrt{1-(x-1)^2}} \frac{x}{x^2+y^2}\,dydx$

Solution

$$\frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$dydx = rdrd\theta$

The region can be described as

For x fixed, y varies from $y = 0$ to $y = \sqrt{1-(x-1)^2}$ (or $y^2 + (x-1)^2 = 1$)

x varies from $x = 0$ to $x = 2$

(Since $\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} dydx$ means $\int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{1-(x-1)^2}} dydx$)

Thus the region is half of the circle with centre (1,0) and radius one unit, above the x-axis.

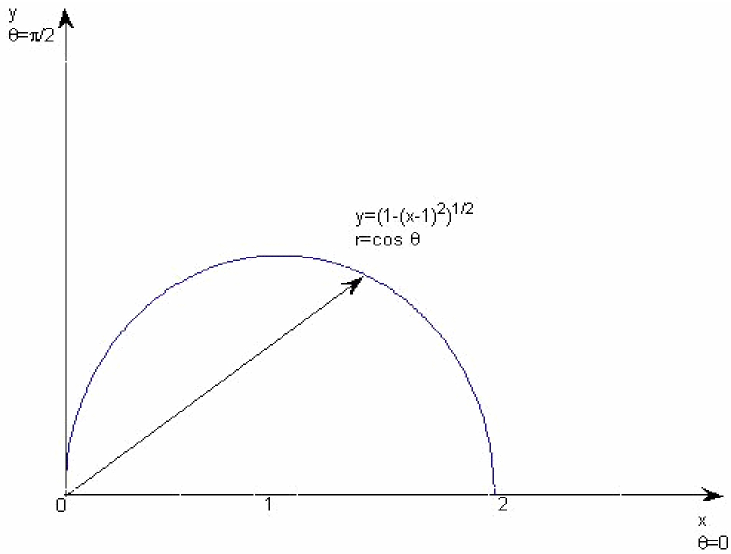

Figure 8.7

$$y^2 + (x-1)^2 = 1 \Rightarrow r^2 \sin^2\theta + (r\cos\theta - 1)^2 = 1$$

$$\text{or} \quad r = 2\cos\theta$$

Therefore, in polar coordinates, the region is described as:

For θ fixed, r varies from $r = 0$ to $r = 2 \cos \theta$

θ varies from θ = 0 to θ = $\pi/2$

Hence, $\int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{1-(x-1)^2}} \frac{x}{x^2+y^2} dydx = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2\cos\theta} \frac{\cos\theta}{r} r\, drd\theta$

$$= \int_{\theta=0}^{\theta=\pi/2} \left[\int_{r=0}^{r=2\cos\theta} \cos\theta\, dr \right] d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} [r(\cos\theta)]_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/2} 2(\cos\theta)\cos\theta\, d\theta = \frac{\pi}{2} \leftarrow$$

Example 9

Evaluate $\int_0^{\infty} e^{-x^2}\,dx$.

Solution

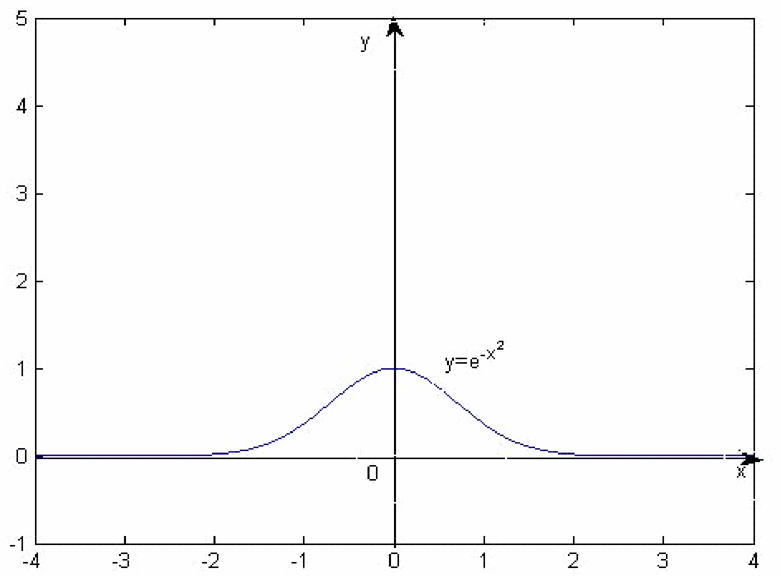

Figure 8.8

Let I = $\int_0^{\infty} e^{-x^2}\,dx$.

Since $\int_0^{\infty} e^{-x^2}\,dx = \int_0^{\infty} e^{-y^2}\,dy$,

$$I^2 = \int_0^{\infty} e^{-x^2}\,dx \int_0^{\infty} e^{-y^2}\,dy$$

$$= \int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)}\,dxdy$$

Now the **presence of the term ($x^2 + y^2$) in the integrand** suggests that we transform to polar coordinates.

Note that the region of integration is the first quadrant.

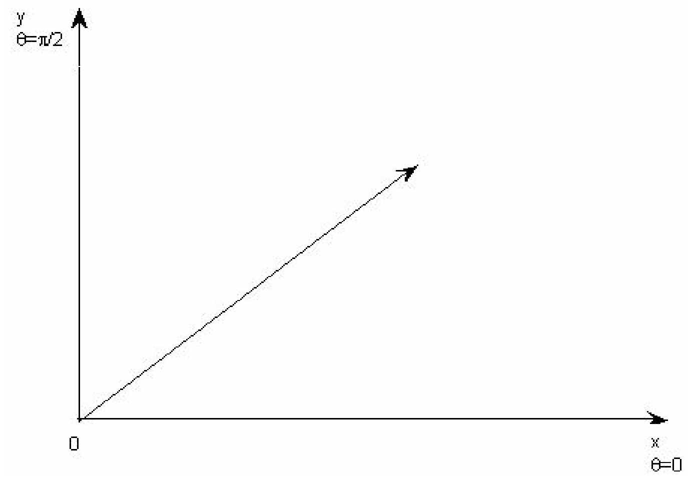

Figure 8.9

In polar coordinates, the region can be described as :

For θ fixed, r varies form $r = 0$ to $r = \propto$,

θ varies from θ=0 to $\theta = \dfrac{\pi}{2}$.

The integrand, $e^{-\left(x^2+y^2\right)} = e^{-r^2}$

Therefore, $I^2 = \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dxdy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, rdrd\theta$

$$= \int_0^{\pi/2} \left\{ \underset{\varepsilon \to \infty}{\ell im} \left[-\frac{1}{2} e^{-r^2} \right]_0^{\varepsilon} \right\} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\, d\theta = \pi/4$$

$$\therefore I = \frac{\sqrt{\pi}}{2} \leftarrow$$

Also by Symmetry, $\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}$

Activity 2

(i) Find the area of the regions bounded by

(a) $x^2 + y^2 = 4$, above y =1

(b) $x^2 + y^2$ =9, y = 2x, x- axis in the first quadrant

8.2.3 CHANGE OF ORDER OF INTEGRATION

In this section, we show that the evaluation of some double integrals can be simplified by a change in the order of integration. On changing the order of integration, the limits of integration also change as shown in Example 9.

Example 9

Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy \; dydx$, and hence evaluate it.

Solution

Since $I = \int_0^1 \int_{x^2}^{2-x} xy \; dydx$ means $\int_{x=0}^{x=1} \int_{y=x^2}^{y=2-x} xy \; dydx$, the region of integration can be described as:

For x fixed, y varies from $y = x^2$ to $y = 2 - x$,

x varies from $x = 0$ to $x = 1$.

This means that if a vertical arrow is shot, it enters the region at $y = x^2$ and leaves it at $y = 2 - x$.

Here we integrate w.r.t to y, **first** and **then** integrate w.r.t x.

Changing the order of integration ***means*** we integrate w.r.t x **first** and **then** integrate w.r.t y.

We sketch the region of integration

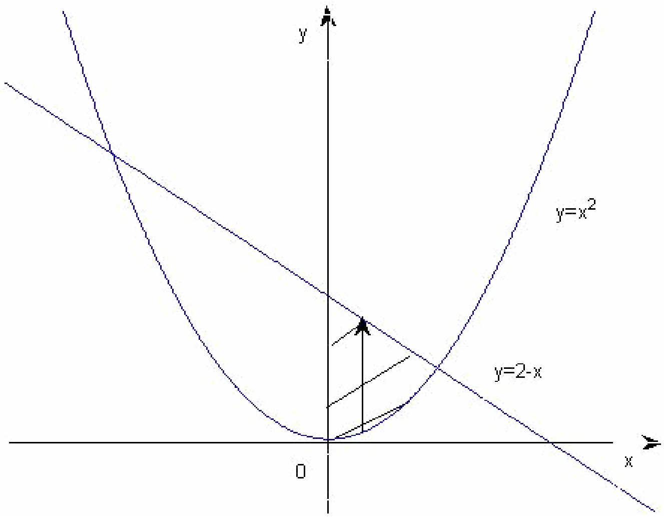

Figure 8.10

To change the order of integration, we shoot a horizontal arrow (y-fixed)

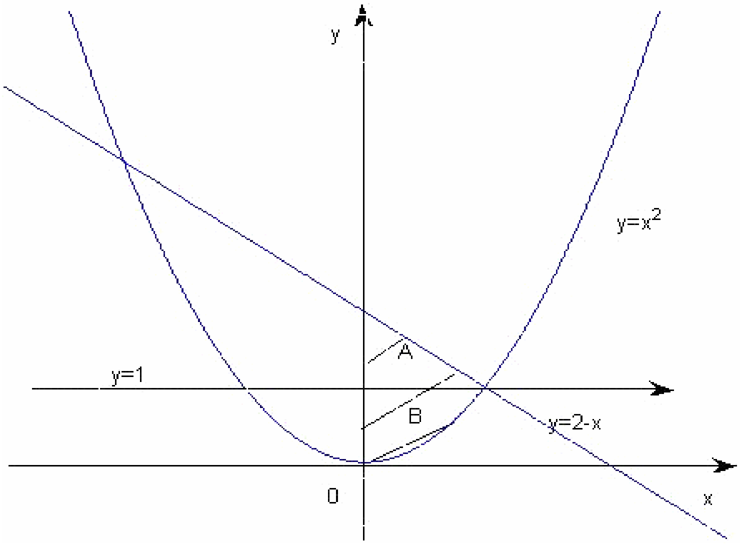

Figure 8.11

Here, to integrate w.r.t x first, we need to 'split' the region of integration into two parts as shown in Figure 8.11.

The regions can thus be described as :

Region A For y fixed, x varies form $x = 0$ to $x = 2\text{-}y$,

y varies from $y = 1$ to $y = 2$.

and

Region B For y fixed, x varies from $x = 0$ to $x = \sqrt{y}$,

y varies form $y = 0$ to $y = 1$.

Therefore,

$$\int_{x=0}^{x=1}\int_{y=x^2}^{y=2-x} xy\ dxdy = \int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{y}} xy\ dxdy + \int_{y=1}^{y=2}\int_{x=0}^{x=2-y} xy\ dxdy$$

$$= \int_{y=0}^{y=1} y\left[\int_{x=0}^{x=\sqrt{y}} xdx\right]dy + \int_{y=1}^{y=2} y\left[\int_{x=0}^{x=2-y} xdx\right]dy$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2 dy$$

$$= \frac{3}{8} \leftarrow$$

Example 11

Evaluate $\int_0^{\infty}\int_y^{\infty}\frac{e^{-x}}{x}dxdy$, after changing the order of integration.

Solution

$$\int_0^{\infty}\int_y^{\infty}\frac{e^{-x}}{x}dxdy = \int_{y=0}^{y=\infty}\int_{x=y}^{x=\infty}\frac{e^{-x}}{x}dxdy$$

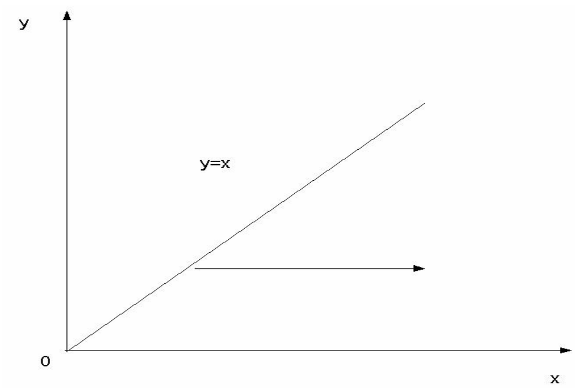

Figure 8.12

The region (Figure 8.12) of integration is described as :

For y fixed, x varies from $x = y$ to $x = \infty$,

y varies from $y = 0$ to $y = \infty$.

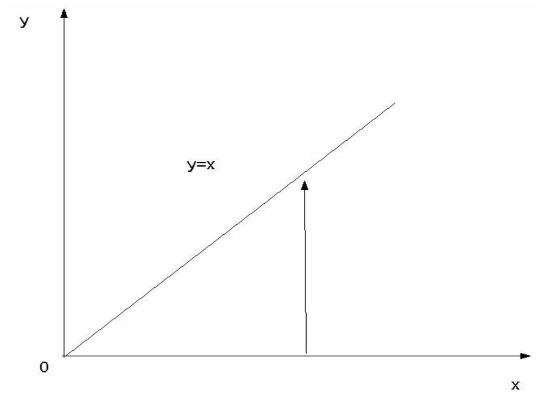

Figure 8.13

To change order of integration, (Figure 8.13) we describe the region of integration as:

For x fixed, y varies from $y = 0$ to $y = x$,

x varies form $x = 0$ to $x = \propto$.

$$\int_{y=0}^{\infty}\int_{x=y}^{\infty}\frac{e^{-x}}{x}\ dxdy = \int_{x=0}^{\infty}\int_{y=0}^{x}\frac{e^{-x}}{x}\ dydx$$

$$= \int_{x=0}^{\infty} \frac{e^{-x}}{x} [y]_{y=0}^{y=x} dx$$

$$= \int_{x=0}^{\infty} \frac{e^{-x}}{x} .x \ dx = \int_{x=0}^{\infty} e^{-x} dx = 1 \leftarrow$$

Example 12

Change the order of integration in the double integral

$\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y) \ dxdy$.

Solution

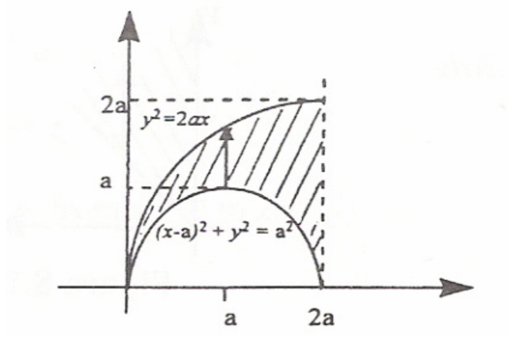

Figure 8.14

The region (Figure 8.14) is described as

For x fixed, y varies from $y = \sqrt{2ax - x^2}$ to $y = \sqrt{2ax}$

$$\downarrow \qquad\qquad \downarrow$$

$$y^2 + x^2 - 2ax = 0 \qquad y^2 = 2ax$$

or

$$y^2 + (x-a)^2 = a^2$$

x varies from $x = 0$ to $x = 2a$

To change the order of integration, that is to integrate w.r.t x first and then to integrate w.r.t y, we need to divide the region in three portions as follows:

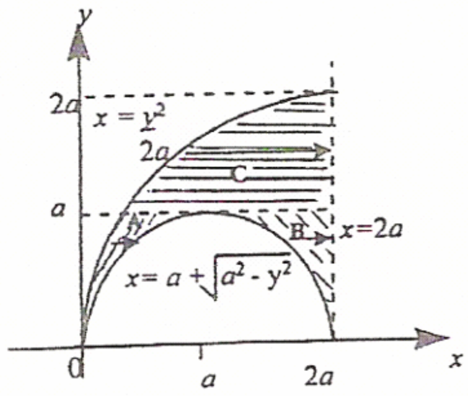

Figure 8.15

The regions of integration can be described as :

Region A :

For y fixed, x varies from $x = \dfrac{y^2}{2a}$ to $x = a + \sqrt{a^2 - y^2}$,

y varies from $y = 0$ to $y = a$.

Region B :

For y fixed, x varies from $x = a + \sqrt{a^2 - y^2}$ to $x = 2a$,

y varies from $y = 0$ to $y = a$.

Region C:

For y fixed, x varies from $x = \dfrac{y^2}{2a}$ to $x = 2a$,

y varies from $y = a$ to $y = 2a$.

Thus,

$$\int_{x=o}^{x=2a} \int_{y=\sqrt{2ax-x^2}}^{y=\sqrt{2ax}} f(x,y)\ dydx = \int_{y=0}^{y=a} \int_{x=\frac{y^2}{2a}}^{x=a+\sqrt{a^2-y^2}} f(x,y)\ dxdy +$$

$$\int_{y=0}^{y=a} \int_{x=a+\sqrt{a^2-y^2}}^{x=2a} f(x,y)\ dxdy + \int_{y=a}^{y=2a} \int_{x=\frac{y^2}{2a}}^{x=a+\sqrt{a^2-y^2}} f(x,y)\ dxdy$$

Activity 3

(i) Change the order of the integration, and hence evaluate the following:

(a) $\int_0^a \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\, dx\, dy$

(b) $\int_0^a \int_0^x \frac{\cos y}{\sqrt{(a-x)(a-y)}}\, dydx$.

8.2.4 GENERAL CHANGE OF VARIABLES IN MULTIPLE INTEGRAL

We have seen in previous sections that it is useful to express certain integrals in terms of polar coordinates. Using the concept of transformation of coordinates introduced in the previous unit, we extend these ideas to include general change of variables in multiple integrals.

Let us consider Cartesian coordinates (x,y) that are related to coordinates (u,v) by the equations

$$x = x(u,v) \text{ and } y = y(u,v). \qquad (2)$$

We assume that x and y have continuous first partial derivatives with respect to u and v, so that x and y are differentiable.

Equations (2) then imply

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv,$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv. \qquad (3)$$

We also assume that $\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \neq 0$. (4)

If condition (4) is satisfied, the system of equations in (3) can be solved for (du,dv) in terms of (dx,dy), so differential changes in x and y can be related to specific differential changes in u and v. It can then be proved that the system of equations (2) can be theoretically solved for u and v in terms of x and y, at least for small changes in the variables.

To evaluate

$\iint_R f(x,y)\,dxdy$ in *uv*-coordinates, we proceed as follows:

Step 1 Express integrand $f(x,y)$ in terms of u and v coordinates by substituting x and y in terms of u and v using the transformation equations(2).

Step2 Obtain the Jacobian, J,

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\ .$$

$dxdy = J\,dudv$

Thus, replace $dxdy$ by $J\,dudv$.

Step3 Obtain the boundary equations in *uv*-plane from those in the *xy*-plane.

Step 4 Describe the region in the *uv*-plane to obtain the limits of integration and hence evaluate the integral.

Example 12

Use the linear transformation

$x=2u+3v,\ y = 2u-v$

to evaluate $\iint_R (x+y)\,dxdy$ where R is the parallelogram bounded by $y = x - 3$,

$y = x + 1$, $x = 3 - 3y$, $x = 11 - 3y$.

Solution

Integrand : $x + y = 2u + 3v + 2u - v$

$= 4u + 2v$

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = -8$$

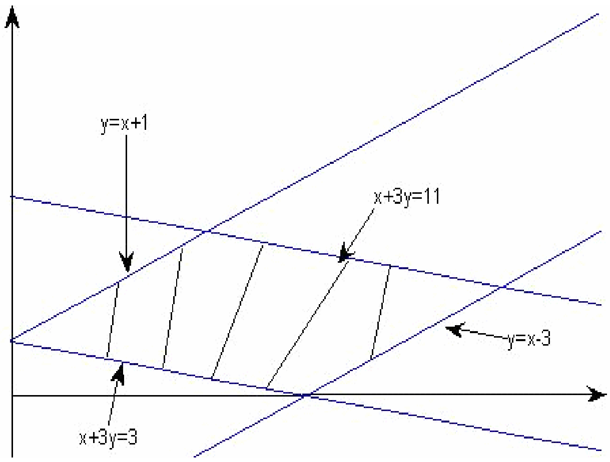

Figure 8.16

Boundary equations

i) $y = x - 3$

$\Rightarrow 2u - v = 2u + 3v - 3$

$\Rightarrow v = 3/4$

ii) $y = x + 1 \Rightarrow 2u - v = 2u + 3v + 1$

$\Rightarrow v = -\dfrac{1}{4}$

iii) $x + 3y = 3 \Rightarrow 2u + 3v + 6u - 3v = 3$

$\Rightarrow u = 3/8$

iv) $x + 3y = 11 \Rightarrow u = 11/8$

Therefore the boundary equations in the *uv* plane are $v = 3/4$, $v = -1/4$, $u = 3/8$, $u = 11/8$

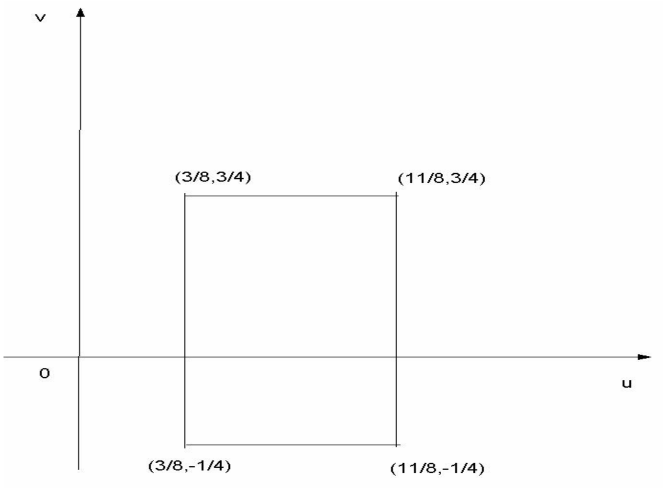

Figure 8.17

The region in *uv* plane can be described as

For *u* fixed, *v* varies from $v = -\frac{1}{4}$ to $v = \frac{3}{4}$

u varies from $u = \frac{3}{8}$ to u = $\frac{11}{8}$

Thus,

$$\iint_R (x+3y)\ dxdy = \int_{3/8}^{11/8}\int_{-1/4}^{3/4} 4u + 2v|-8|\ dudv$$

$$= 16\int_{3/8}^{11/8}\left[2uv + \frac{v^2}{2}\right]_{-1/4}^{3/4} du$$

$$= 92 \leftarrow$$

Example 14

Using the transformations

$x = u \sec v$, $y = u \tan v$, evaluate the integral $\iint_R e^{x^2 - y^2}\, dxdy$,

where R is the region bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 16$ and $x = 2y$ in the first quadrant.

Solution

$x^2 - y^2 = (u \sec v)^2 - (u \tan v)^2 = u^2$

$\therefore$ integrand $= e^{x^2 - y^2} = e^{u^2}$

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \sec v & u \tan v \sec v \\ \tan v & u \sec^2 v \end{vmatrix} = u \sec v$$

The boundary equations become :

$x^2 - y^2 = 16 \Rightarrow u^2 = 16$ or $u = 4$

$x^2 - y^2 = 1 \Rightarrow u^2 = 1$ or $u = 1$

$x = 2y \Rightarrow u \sec v = 2u \tan v \Rightarrow v = \pi/6$

$y = 0 \Rightarrow u \tan v = 0 \Rightarrow v = 0$

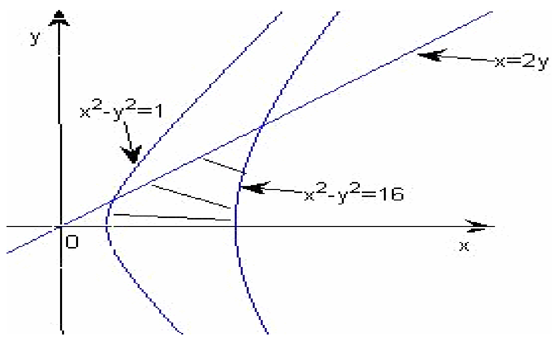

Figure 8.18

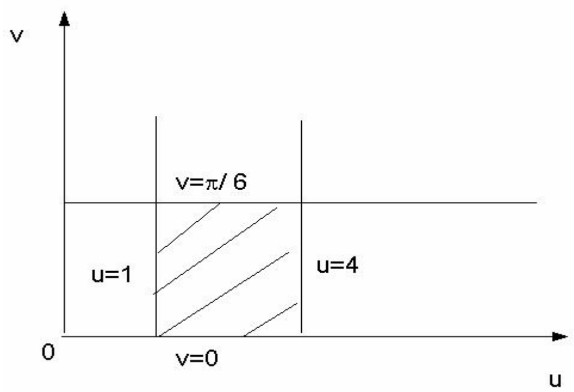

Figure 8.19

In uv-plane the region can be described as :

For v fixed, u varies from $u = 1$ to $u = 4$

v varies from $v = 0$ to $v = \pi/6$

$$\iint_R e^{x^2-y^2}\,dxdy = \int_{v=0}^{\pi/6}\int_{u=1}^{4} e^{u^2}\ u\sec v\ dudv$$

$$= \int_{v=0}^{\pi/6} \sec v\left[\frac{1}{2}\int_{u=1}^{4} 2ue^{u^2}\,du\right]dv$$

$$= \int_{v=0}^{\pi/6} \sec v\,\frac{1}{2}\left(e^{16}-e\right)dv$$

$$= \frac{1}{2}\left(e^{16}-e\right)\left[\ln\left|\sec v+\tan v\right|\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left(e^{16}-e\right)\ln\sqrt{3} \leftarrow$$

<u>Activity 4</u>

(i) (a) Evaluate the Jacobian that corresponds to the translation of coordinates $x = u + h,\ y = v + k,\ h$ and k constants.

(b) Use the linear transformations $x = 2u + v, y = u + 2v$ to express $\iint_R xy\ dxdy$ in terms of (u,v) coordinates where R is the region bounded by $y = 2x - 3,\ \ y = 2x,\ \ x = 2y - 3,\ \ x = 2y$ (Do not evaluate the integral).

8.3 TRIPLE INTEGRALS

8.3.1 TRIPLE INTEGRALS: CARTESIAN COORDINATES

Triple integrals involve the same ideas as double integrals. They are often used to evaluate the volume of solids or the flux of a vector field.

Consider a region in three dimensional space denoted by R. Let $f(x,y,z)$ be continuous at all points in R and the boundary of R, then the integration of f over R is denoted by

$$\iiint_R f(x,y,z)dxdydz.$$

If the region R can be described as:
For x and y fixed, z varies from $z = h_1(x,y)$ to $z= h_2(x,y)$
For x fixed, y varies from $y = g_1(x)$ to $y = g_2(x)$
x varies form $x = a$ to $x = b$, then

$$\iiint_R f(x,y,z)dxdydz = \int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)dxdydz \tag{5}$$

Note how **the limits of integration** corresponds to the **description of the region**.

8.3.1.1 Evaluation of Triple Integrals

The triple integral (5) is evaluated by first integrating the inner integral with respect to z while x and y are treated as constants. The next integration is carried out by treating x as a constant while we integrate with respect to y. We then integrate with respect to x.

Example 15

Evaluate $\int_0^1 \int_0^{x^2} \int_0^{xy} 32x^2 yz \, dzdydx$

Solution

$$\int_0^1 \int_0^{x^2} \int_0^{xy} 32x^2 yz \;\; dzdydx = \int_0^1 \int_0^{x^2} \left[\int_0^{xy} 32x^2 yzdz \right] dxdy$$

$$= \int_0^1 \int_0^{x^2} x^2 y \left[16z^2 \right]_0^{xy} dydx = \int_0^1 \int_0^{x^2} 16x^4 y^3 dydx$$

$$= \int_0^1 x^4 \left[4y^4 \right]_0^{x^2} dx$$

$$= \int_0^1 4x^{12} dx = \left[\frac{4}{13} x^{13} \right]_0^1$$

$$= \frac{4}{13} \leftarrow$$

Volume of a region

The volume V of the region R is given by

$$\boxed{V = \iiint_R dxdydz = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} \int_{z=h_1(x,y)}^{h_2(x,y)} dxdydz}$$

Example 16

Find the volume of the region bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes $x = 0, y = 0, z = 0$.

Solution

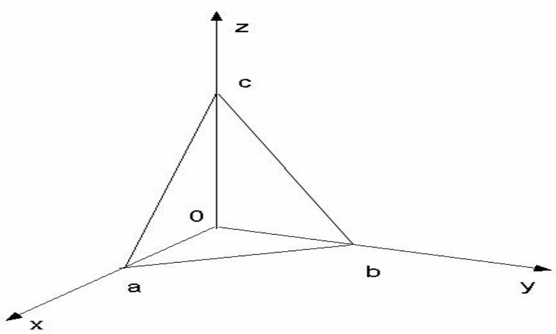

Figure 8.20

The region can be described as:

For x and y fixed, z varies from $z = 0$ to $z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

(**Note** that the top surface is $z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$ and the bottom surface is $z = 0$. The two surfaces intersect to form a side edge with equation

$c\left(1-\frac{x}{a}-\frac{y}{b}\right) = 0 \ \ or \ \ y = b\left(1-\frac{x}{a}\right)$

For x fixed, y varies from $y = 0$ to $y = b\left(1-\frac{x}{a}\right)$,

x varies from $x = 0$ to $x = a$

$$\text{Volume} = \int_{x=0}^{a}\int_{y=0}^{y=b\left(1-x/a\right)}\int_{z=0}^{z=c\left(1-x/a-y/b\right)} dzdxdy = \int_{x=0}^{a}\int_{y=0}^{y=b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dydx$$

$$= c\int_{x=0}^{x=a}\left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= \frac{bc}{2}\int_0^a\left(1-\frac{x}{a}\right)^2 dx = \frac{abc}{6} \leftarrow$$

Example 17

Evaluate $\iiint_R (x+y+z)dxdydz$

where $R : 0 \le x \le 1, \quad 0 \le y \le x^2, \qquad 0 \le z \le x + y$

Solution

$$\iiint_R (x+y+z)dxdydz$$
$$= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left[\int_{z=0}^{z=x+y}(x+y+z)dz\right]dxdy$$
$$= \int_0^1\int_0^{x^2}\left[\frac{(x+y+z)^2}{2}\right]_0^{x+y} dydx$$
$$= \int_0^1\left[\int_0^{x^2}\frac{3}{2}(x+y)^2\,dy\right]dx = \int_0^1 \frac{3}{2}x^4 + \frac{x^6}{2} + \frac{3}{2}x^5 dx$$
$$= \frac{87}{140} \leftarrow$$

Example 18

Calculate the volume of the solid bounded by the following surfaces $z = 0, x^2 + y^2 = 1,$ $x + y + z = 3$ in the first quadrant.

Solution

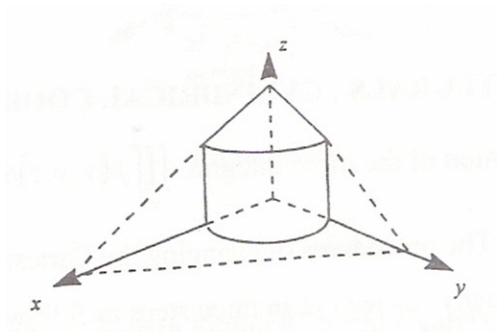

Figure 8.21

Note that the top and bottom surfaces are $z = 0$ and $z = 3 - x - y$

The region can be described as:

For x and y fixed, z varies from $z = 0$ to $z = 3 - x - y$,

For x fixed, y varies from $y = 0$ to $y = \sqrt{1 - x^2}$,

x varies from $x = 0$ to $x = 1$.

$$\text{Volume} = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{3-x-y} dzdydx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} 3-x-y\ dydx$$

$$= \int_0^1 \left[(3-x)y - \frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 (3-x)\sqrt{1-x^2} - \frac{\left(1-x^2\right)}{2} dx$$

$$= \left[\frac{3x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1}x + \frac{1}{3}\left(1-x^2\right)^{3/2} - \frac{1}{2}x + \frac{x^3}{6}\right]_0^1$$

$$= \frac{3\pi}{4} - \frac{2}{3} \leftarrow$$

8.3.2 TRIPLE INTEGRALS : CYLINDRICAL COORDINATES

We now show how we use cylindrical coordinates to simplify the evaluation of the triple integral $\iiint_R f\left(x,y,z\right)dxdydz$. The procedure of changing the Cartesian Integral into cylindrical integral is usually carried out in three steps as follows:

Step1 We express the integrand $f(x,y,z)$ in terms of cylindrical coordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $x^2 + y^2 = r^2$.

Step2 Replace $dxdydz$ by $rdrd\theta dz$, since

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| drd\theta dz$$
$$= rdrd\theta dz$$

Step3 Describe the region R in terms of cylindrical coordinates to obtain the limits of integration.

Example 19

Find the volume bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z + y = 3$

Solution

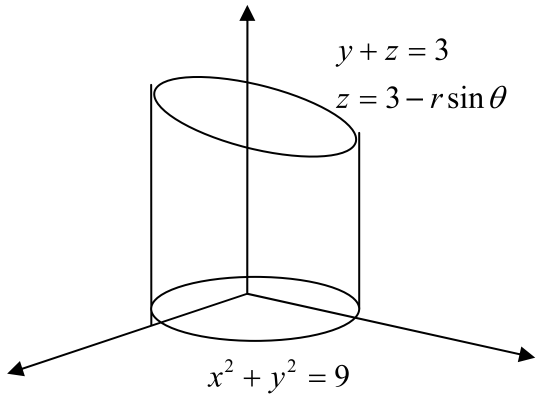

Figure 8.22

The region of integration is given in Figure 8.22 and it can be described as:

For θ and r fixed, z varies form $z = 0$ to $z = 3 - r\sin\theta$,

For θ fixed, r varies from $r = 0$ to $r = 3$,

θ varies form $\theta = 0$ to $\theta = 2\pi$.

Therefore, volume = $\displaystyle\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{z=3-r\sin\theta} r\, dzdrd\theta$

$$= \int_0^{2\pi}\int_0^3 3r - r^2\sin\theta\ drd\theta$$

$$= \int_0^{2\pi}\left[\frac{3}{2}r^2 - \frac{r^3}{3}\sin\theta\right]_0^3 d\theta$$

$$= \int_0^{2\pi}\frac{27}{2} - 9\sin\theta\ d\theta = \left[\frac{27}{2}\theta + 9\cos\theta\right]_0^{2\pi}$$

$$= 27\pi \leftarrow$$

Example 20

Find the volume enclosed by the sphere $x^2 + y^2 + z^2 = 9$ and enclosed by the cylinder is $x^2 + y^2 = 4$.

Solution

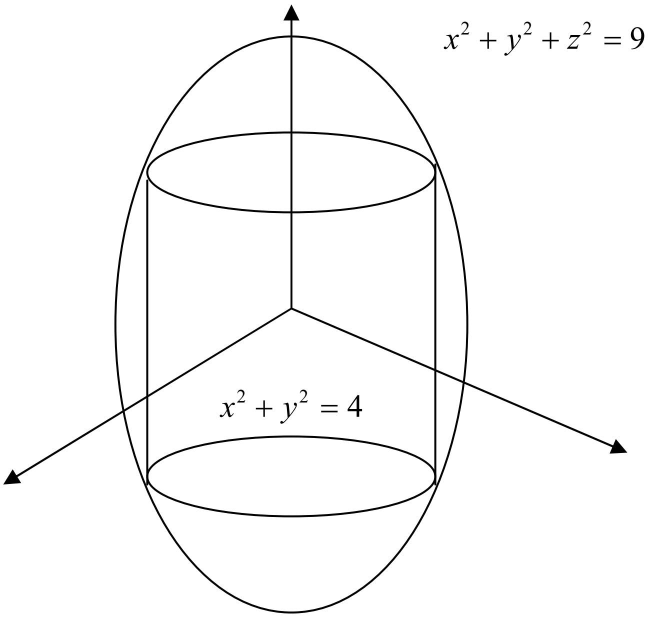

Figure 8.23

In cylindrical coordinates, $x^2 + y^2 + z^2 = 9$ becomes $z^2 = 9 - r^2$ or $z = \sqrt{9 - r^2}$ and $x^2 + y^2 = 4$ becomes $r^2 = 4$ or $r = 2$.

Using symmetry, the required volume is equal to 8 times the volume in the first octant.

The region in the first quadrant can be described as :

For θ and r fixed, z varies from $z = 0$ to $z = \sqrt{9 - r^2}$

For θ fixed, r varies from $r = 0$ to $r = 2$

θ varies form $\theta = 0$ to $\theta = \pi/2$

$$\text{Volume} = 8\int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\int_{z=0}^{z=\sqrt{9-r^2}} r\,dz\,dr\,d\theta = 8\int_{\theta=0}^{\pi/2}\int_{r=0}^{2} r\left(9-r^2\right)^{1/2} dr\,d\theta$$

$$= 8\int_{\theta=0}^{\pi/2}\left[-\frac{1}{3}\left(9-r^2\right)^{3/2}\right]_0^2 d\theta = 8\int_{\theta=0}^{\pi/2}\left[9-\frac{5}{3}\sqrt{5}\right]d\theta$$

$$= 4\pi\left[9-\frac{5}{3}\sqrt{5}\right] \leftarrow$$

8.3.3 TRIPLE INTEGRALS : SPHERICAL COORDINATES

We now consider evaluation of triple integral over the region *R* which can be more easily described by spherical coordinates.

The procedure of changing a Cartesian integral into spherical coordinates can be carried out in three steps as follows:

Step 1 We express the integrand $f(x, y, z)$ in terms of spherical coordinates by substituting $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$ and $x^2 + y^2 + z^2 = \rho^2$.

Step 2 We replace $dxdydz$ by $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ since

$$dxdydz = \left|\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}\right| d\rho\, d\phi\, d\theta$$

$$= \rho^2 \sin\phi\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\theta.$$

Step 3 Describe the region in terms of spherical coordinates to obtain the limits of integration.

Example 21

Find the volume bounded by the sphere $x^2 + y^2 + z^2 = a^2$ and inside the cone $z^2 = x^2 + y^2$.

Solution

In spherical coordinates, $x^2 + y^2 + z^2 = a^2$ becomes $\rho^2 = a^2$ or $\rho = a$, and $z^2 = x^2 + y^2$ becomes $(\rho\cos\phi)^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$

or $\cos^2\phi = \sin^2\phi$

or $\tan\phi = 1$

or $\phi = \pi/4$

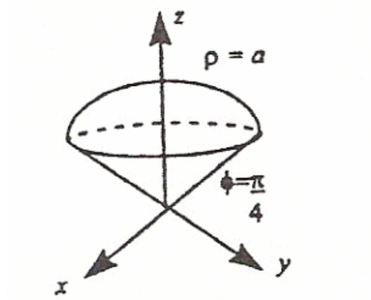

Figure 8.24

The region can be described as :

For θ and ϕ fixed, ρ varies from $\rho = 0$ to $\rho = a$,

(**Note** : an arrow shot from the origin leaves the region at $\rho = a$)

For θ fixed, ϕ varies from $\phi = 0$ to $\phi = \pi/4$,

θ varies from $\theta = 0$ to $\theta = 2\pi$.

$$\text{Volume} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \frac{a^3}{3} \sin\phi \, d\phi \, d\theta = \frac{a^3}{3} \int_{\theta=0}^{2\pi} \left[-\cos\phi\right]_0^{\pi/4} d\theta$$

$$= \frac{a^3}{3} . 2\pi \left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi \ a^3}{3} \left(2 - \sqrt{2}\right) \leftarrow$$

Example 22

Evaluate $\iiint_R \left(x^2 + y^2 + z^2\right) \, dxdydz$

where R is the region enclosed by the sphere $x^2 + y^2 + z^2 = 1$

Solution

The integrand $f(x,y,z) = x^2 + y^2 + z^2 = \rho^2$ in spherical coordinates

$dxdydz = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

The spherical region can be described as:

For θ and ϕ fixed, ρ varies from $\rho = 0$ to $\rho = 1$

For θ fixed, ϕ varies from $\phi = 0$ to $\phi = \pi$

θ varies from $\theta = 0$ to $\theta = 2\pi$

$$\text{Volume} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{\rho=1} \rho^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \left[\frac{\rho^5}{5} \right]_0^1 \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{5} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{5} \int_{\theta=0}^{2\pi} [-\cos\phi]_0^{\pi} \, d\theta = \frac{1}{5} .2\pi.(2) = \frac{4\pi}{5}$$

Activity 5

Find the volume of the regions described by

(a) inside both $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 = 4$

(b) inside $x^2 + y^2 + z^2 = 4$, above $z = 1$

8.4 SURFACES

There are three common analytical representations of surfaces:

a) $z = f(x,y)$ where f is a single valued continuous function defined on a region R of the xy-plane.

Examples are :

$z = x^2 + y^2$ which is the surface of a cone.

$z = \sqrt{4 - x^2 - y^2}$ which is the upper surface of a hemisphere.

b) $F(x,y,z) = 0.$

Examples are :

1. A sphere of radius 2: $x^2 + y^2 + z^2 = 4$.
2. An ellipsoid: $\dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$.

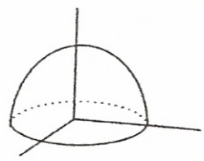

Figure 8.25(i)

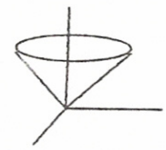

Figure 8.25(ii)

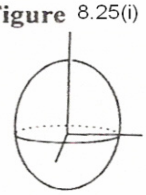

Figure 8.25(iii)

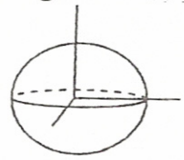

Figure 8.25(iv)

c) $x = f(u,v)$, $y = g(u,v)$, $z = h(u,v)$ where f, g, and h are continuous functions defined on a connected region R of the uv-plane, is called a parametric surface.

Examples are :

$x = a \sin\phi \cos\theta$ This is the parametric form

$y = a \sin\phi \sin\theta$ for the surface of a sphere

$z = a \cos\phi$ of radius a.

$x = (a + b \cos\phi) \cos\theta$

$y = (a + b \cos\phi) \sin\theta$

$z = b \sin\phi$

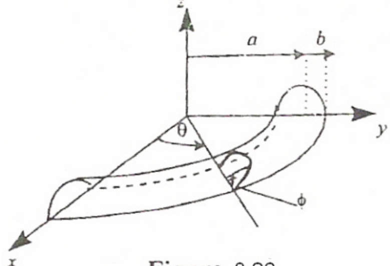

Figure 8.26

Torus or doughnut.

We assume that the surfaces considered in this section have two well-defined sides. If one draws two oppositely directed normals PN and PN' at any point P of a two-sided surface and allows the point P to move along any path which does not cross the edge of a surface, then the direction of the moving normal PN can never be made to coincide with that of PN'.

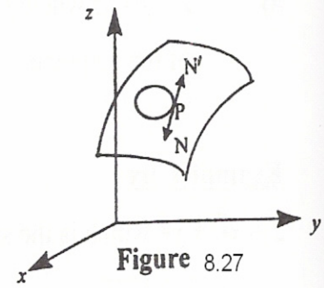

Figure 8.27

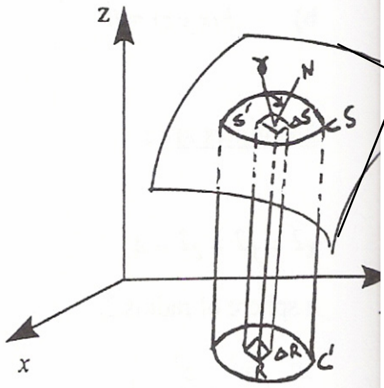

Figure 8.28

Let $z = f(x,y)$ be the equation of the surface S. The function $z = f(x,y)$ is assumed to have continuous first partial derivatives with respect to x and y. This implies that a continuously turning tangent plane is uniquely defined at every point of the surface S.

Let S' be a portion of S bounded by a closed curve C, and such that any line parallel to the z-axis cuts S' in only one point. If C' is the projection of C on the xy-plane, let the region R, of which C' is the boundary, be subdivided by lines parallel to the axes into sub-regions ΔR_i. Through these subdividing lines pass planes to the z-axis. These planes cut from S' small regions $\Delta S'_i$ of area $\Delta\sigma_i$. Let ΔA_i be the area of ΔR_i. Then, except for infinitesimals of higher order,

$$\Delta A_i = \cos\gamma_i \,\Delta\sigma_i$$

where $\cos\alpha_i$, $\cos\beta_i$, $\cos\gamma_i$ represent the direction cosines of the normal to S at any point (x_i,y_i,z_i) of $\Delta S'_i$.

The direction numbers of a normal to the surface $\phi(x,y,z) = c$ are given by

$\phi_x : \phi_y : \phi_z$

Now we have been talking about a surface $z = f(x,y)$

Let $\phi(x,y,z) = f(x,y) - z = 0$

Then $\phi_x : \phi_y : \phi_z = f_x : f_y : -1$

And $\therefore$ $\boxed{\text{the direction \#'s of a normal to } z = f(x,y) \text{ are } f_x : f_y : -1}$

and $\cos \alpha_i = \dfrac{(f_x)_i}{\pm\sqrt{f_{x_i}^2 + f_{y_i}^2 + 1}}$; $\cos\beta_i = \dfrac{(f_y)_i}{\pm\sqrt{f_{x_i}^2 + f_{y_i}^2 + 1}}$

$$\cos\gamma_i = \frac{-1}{\pm\sqrt{f_{x_i}^2 + f_{y_i}^2 + 1}}$$

Thus, using the positive value for $\cos y_i$

$$\Delta\sigma_i = \sec\gamma_i \Delta A_i = \sqrt{(f_x)_i^2 + (f_y)_i^2 + 1}\ \Delta A_i$$

Then

$$\ell im_{n\to\infty} \sum_{k=1}^{n} \sqrt{(f_x)_i^2 + (f_y)_i^2 + 1}\ \Delta A_i$$

is defined as the area of the surface $S^{/}$. Since this limit is

$$\int_R\int \sqrt{f_x^2 + f_y^2 + 1}\ dA$$

the value of σ is given by

$$\boxed{\sigma = \int_R\int \sec\gamma dA = \int_R\int \sqrt{f_x^2 + f_y^2 + 1}\ dx\ dy}$$

Similarly, by projecting S on the other coordinate planes, it can be shown that

$$\sigma = \int_{R^{/}}\int \sec\alpha\ dA = \int_{R^{//}}\int \sec\beta\ dA$$

The integral of a funtion $\phi(x,y,z)$ over the surface $z = f(x,y)$ can now be defined by the equation

$$\int_{S'}\int \phi(x,y,z)d\sigma = \int_R\int \phi[x,y,f(x,y)]\sqrt{f_x^2+f_y^2+1}\ dx\ dy$$

It is, of course, necessary in this above definition that the function $\phi(x,y,z)$ be continuous and single-valued for all points of the surface S'.

Example 23

Find the area of that portion of the surface of the cylinder $x^2 + y^2 = a^2$ which lies in the first octant between the planes $z = 0$ and $z = mx$.

Solution

Since the cylinder $x^2 + y^2 = a^2$ has sides parallel to the z-axis, a projection on the xy-plane is not possible and we must therefore project the area onto the xz or the yz plane. (You can draw a rough picture to convince yourself of this).

Let us project on the xz-plane and to make the picture clearer we re-orient the axes as shown :

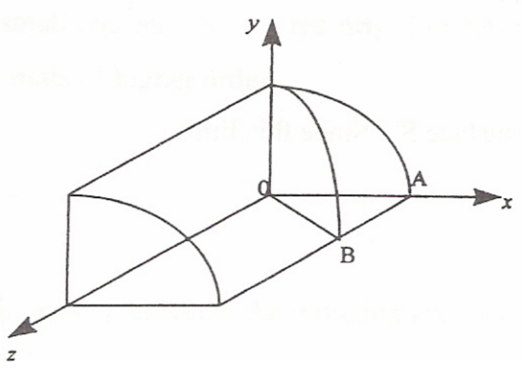

Figure 8.29

The projection on the xz-palne is the triangle AOB. The equation of the surface will be of the form $y = g(x,z)$ and thus the direction numbers of the normal to the surface are

$$\frac{\partial y}{\partial x} : -1 : \frac{\partial y}{\partial z}$$

and

$$\sec\beta = \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + 1 + \left(\frac{\partial y}{\partial z}\right)^2}$$

$$= \sqrt{\left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2 + 1 + 0}$$

$$= \frac{a}{\sqrt{a^2 - x^2}}$$

Hence,

$$\sigma = \int_0^a \int_0^{mx} \frac{a}{\sqrt{a^2 - x^2}}\ dz\ dx$$

$$= \int_0^a amx\ \left(a^2 - x^2\right)^{-\frac{1}{2}} dx = a^2 m$$

Example 24

Find the area of the surface of the sphere

$x^2 + y^2 + z^2 = a^2$

cut off by the cylinder

$x^2 - ax + y^2 = 0.$

Solution

$z = f(x,y) = \sqrt{a^2 - y^2 - x^2}$

$x^2 - ax + y^2 = \left(x - \frac{a}{2}\right)^2 + y^2 - \frac{a^2}{4}$

Circle center $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

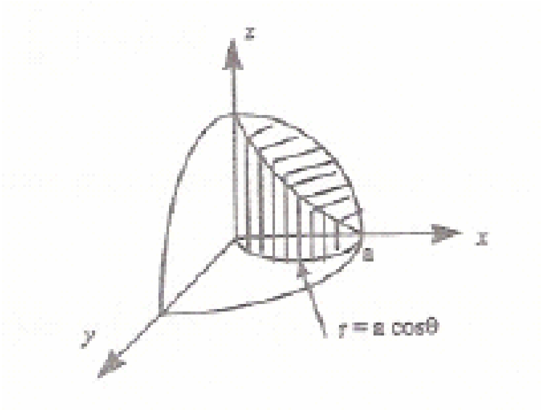

Figure 8.30

When you have the integral set up in Cartesian coordinates, you will likely find it simpler to integrate by changing to polar coordinates.

Answer : $4a^2\left(\dfrac{\pi}{2}-1\right)$

<u>**Example 25**</u>

Evaluate $I=\int_S\int\left(x^2+y^2\right)z\ dS$ where S is the top of the hemisphere

$x^2+y^2+z^2=1$

<u>**Solution**</u>

We will project dS onto the xy-plane.

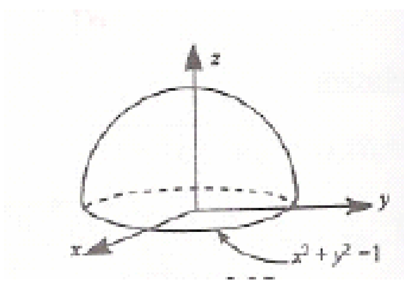

Figure 8.31

We have

$dS = sec\ \gamma\, dx\, dy$

The equation of the surface is

$$\phi(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$$
$$\therefore \nabla\vec{\phi} = 2\hat{i}x + 2\hat{j}y + 2\hat{k}z$$

and this vector is normal to the surface with direction numbers *x:y:z.*

$$\therefore \cos y = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{1} = z$$

so $dS = \frac{1}{z}\ dxdy$

$$\therefore I = \iint \left(x^2 + y^2\right) z . \frac{1}{z} dx\ dy$$

evaluated over the circle $x^2 + y^2 = 1$.

Changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$I = \int_0^{2\pi}\int_0^1 r^2 r dr d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \frac{2\pi}{4} = \frac{\pi}{2}$$

8.4.1 ADDITIONAL SOLVED PROBLEMS

USEFUL FORMULA

If $z = f(x,y)$ is the equation of the surface *S*, then the surface area is given by the integral

$\iint_s d\sigma = \iint_\Omega \sqrt{1 + z_x^2 + z_y^2}\, dxdy$ where Ω is the projection of *S* onto the *xy*-plane.

Example 26

Find the surface area of the Cone shown below

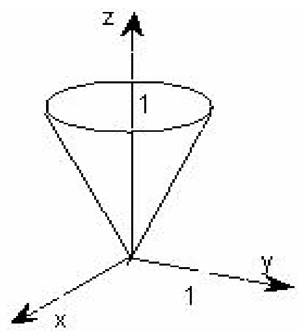

Figure 8.32

Solution

The surface of the cone is defined by the equation

$z^2 = y^2 \quad 0 \le y \le 1$

$z^2 = y^2 \quad \Rightarrow z = \pm y$, so that $z_y = \pm 1$ and $z_x = 0$

thus

$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 0 + 1^2} = \sqrt{2}$

The projection of S onto the xy-plane is the circle (disc) $x^2 + y^2 \le 1$

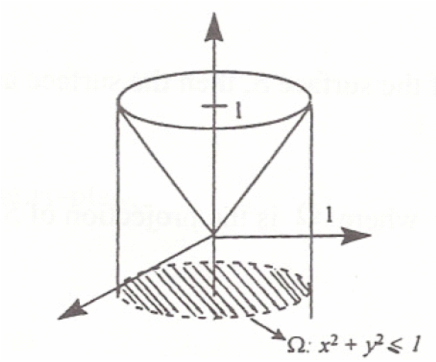

Figure 8.33

Thus, the surface area is given by

$$SA = \iint_S d\sigma = \iint_{x^2+y^2\le 1} \sqrt{2}\ dxdy$$

$$= \sqrt{2}\iint_{x^2+y^2\le 1} dxdy$$

$$= \sqrt{2} \times \text{area of circle } x^2 + y^2 = 1$$

$$= \sqrt{2} \times \pi \times 1^2 = \sqrt{2}\,\pi$$

Example 27

Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 1$ inside the upper part of the cone $x^2 + y^2 = z^2$

Solution

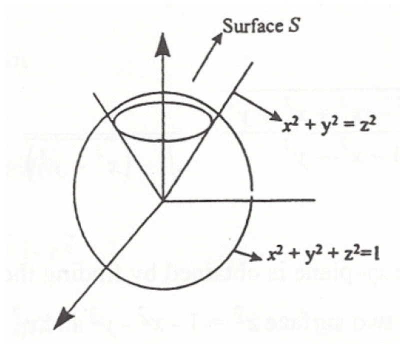

Figure 8.34

The surface S is shown

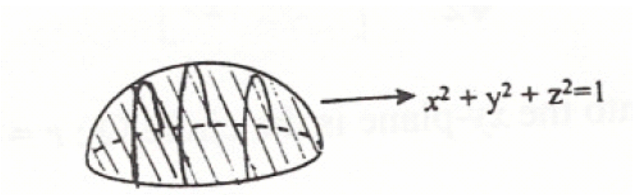

Figure 8.35

Now $x^2 + y^2 + z^2 = 1 \Rightarrow z^2 = 1 - x^2 - y^2$

$$z = \pm\sqrt{1 - x^2 - y^2}$$

Since we are considering the upper part of the sphere, we have $z \geq 0$, thus

$$z = \sqrt{1 - x^2 - y^2} = \left(1 - x^2 - y^2\right)^{1/2}$$

$$\frac{dz}{dx} = \frac{1}{2}\left(1 - x^2 - y^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{1 - x^2 - y^2}}$$

and similarly, $\dfrac{dz}{dy} = \dfrac{-y}{\sqrt{1 - x^2 - y^2}}$

Thus $\sqrt{1+z_x^2+z_y^2} = \sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}$

$$= \sqrt{\frac{1-x^2-y^2+x^2+y^2}{1-x^2-y^2}} = \frac{1}{\sqrt{1-\left(x^2+y^2\right)}}$$

The projection of S onto the *xy*-plane is obtained by finding the *x,y* coordinates of the points of intersection of the two surface $z^2 = 1 - x^2 - y^2$ and $z^2 = x^2 + y^2$.

Thus $1 - x^2 - y^2 = x^2 + y^2$

$1 = 2\,(x^2 + y^2)$

In polar coordinates, $2r^2 = 1 \Rightarrow r = \frac{1}{\sqrt{2}}$

We see that the projection onto the *xy*-plane is the circle $\Omega : r = \frac{1}{\sqrt{2}}$

The *SA* is given by

$$\iint_s d\sigma = \iint_\Omega \sqrt{1+z_x^2+z_y^2}\ dxdy$$

$$= \iint_\Omega \frac{1}{\sqrt{1-\left(x^2+y^2\right)}}\ dxdy \quad \text{where } \Omega \text{ is the interior of circle } x^2+y^2 = \frac{1}{2}$$

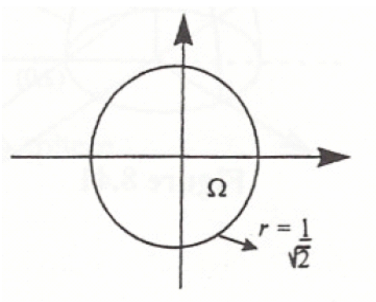

Figure 8.36

We transform into polar coordinates to obtain

$$\iint_{\Omega} \frac{1}{\sqrt{1-\left(x^2+y^2\right)}}\ dxdy$$

$$= \int_0^{2\pi} \int_{r=0}^{r=\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-r^2}}\ rdrd\theta$$

Put $u = 1 - r^2$

$$= \int_0^{2\pi} \left[\int_{u=1}^{u=\frac{1}{2}} \frac{1}{\sqrt{u}} r \frac{du}{-2r} \right] d\theta$$

$$= \int_0^{2\pi} d\theta \left(\frac{-1}{2} \int_1^{1/2} u^{-\frac{1}{2}} du \right)$$

$$= 2\pi \quad \frac{-1}{2} \left[\frac{u^{1/2}}{1/2} \right]_1^{1/2} = 2\pi \times -1 \left[\left(\frac{1}{2}\right)^{1/2} - 1 \right]$$

$$= -2\pi \left[\frac{1}{\sqrt{2}} - 1 \right]$$

$$= \pi\left(2 - \sqrt{2}\right)$$

Example 28

Find the area of the portion of the parabolic cylinder $z = 9 - y^2$ that is in the first octant and between the yz-plane and the plane $x = 3y$.

Solution

We first sketch the surface S

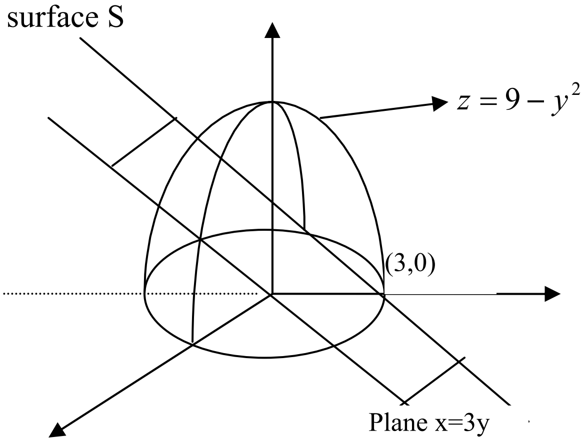

Figure 8.37

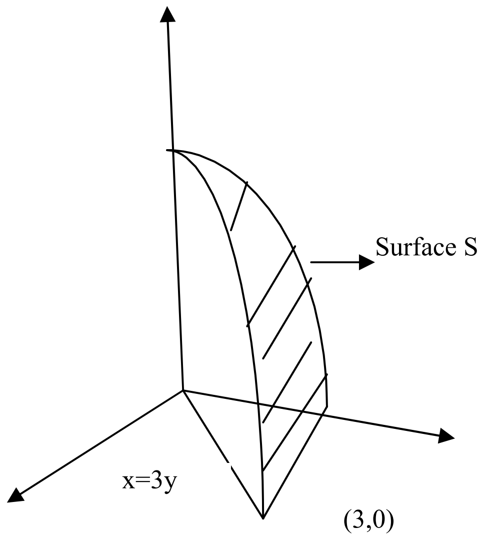

Figure 8.38

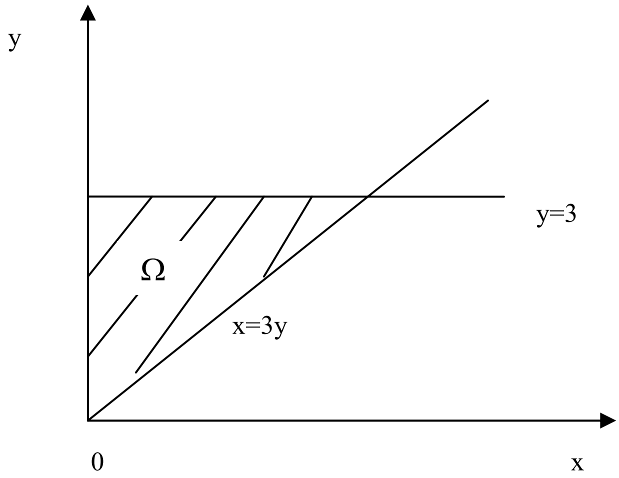

Figure 8.39

Consider $z = 9 - y^2$. When $z=0$, $y^2 = 9 \Rightarrow y = \pm 3$.

The projection onto the xy plane is $\Omega : x = 3y, 0 \le y \le 3$.

$z_x = 0$, $z_y = -2y$, thus $\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 0 + 4y^2}$

The surface area is equal to

$$\iint_s d\sigma = \iint_\Omega \sqrt{1 + z_x^2 + z_y^2}\, dxdy$$

$$= \iint_\Omega \sqrt{1+4y^2}\ \ dxdy$$

$$= \int_{y=0}^{3} \left[\int_{x=0}^{x=3y} \sqrt{1+4y^2}\ \ dx \right] dy$$

$$= \int_0^3 \left[\sqrt{1+4y^2}\, x \right]_{x=0}^{x=3y} dy$$

$$= \int_0^3 \sqrt{1+4y^2}\ \ 3y\ \ dy$$

Put $t = 1 + 4y^2$

$$\frac{dt}{dy} = 8y \Rightarrow dy = \frac{dt}{8y}$$

$$\therefore \iint_s d\sigma = \int_{t=1}^{37} \sqrt{t}\ \ 3y\ \ \frac{dt}{8y}$$

$$= \frac{3}{8}\ \ \frac{2}{3}\left[t^{3/2}\right]_1^{37}$$

$$= \frac{1}{4}\left[37\sqrt{37} - 1\right]$$

8.5 SUPPLEMENTARY EXERCISES

1. Find the surface area of a sphere

   $x^2 + y^2 + z^2 = 16$

2. Find the surface area of the portion of the cylinder $x^2 + y^2 = 4y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$

3. Find the area of the region which is on the right of the circle $x^2 + y^2 = 4$ and on the left of the curve $y^2 = 4 - x$.
4. Evaluate the double integral

$$\int_0^a \int_0^{a-x} \frac{dy\, dx}{1 + (x+y)^2}$$

   by using the transformations $u = x + y$ and $v = -x + y$.
5. Evaluate the double integral

$$\int_0^1 \int_y^{2-y} \frac{x+y}{x^2} e^{x+y} dx\, dy$$

   by using the transformations $u = x + y$ and $v = \frac{y}{x}$.

6. Evaluate the surface area

$$\iint_S (x^2 + y^2) z^2 dS$$

   where S is the upper hemisphere of $x^2 + y^2 + z^2 = 1$.

8.6 SUMMARY

In this unit, you have been introduced to the important concept of multiple integrals.

KEYPOINTS:

1. $$\int_{y=a}^{y=b} \int_{x=g_1(y)}^{x=g_2(y)} f(x,y)\,dx\,dy = \int_{y=a}^{y=b} \left[\int_{x=g_1(y)}^{x=g_2(y)} f(x,y)\,dx \right] dy$$

2. Area of a region R in xy plane $= \iint_R dx\,dy$.

3. Volume of a region R in three dimensional space $= \int_{x=a}^{b} \int_{y=g_1(x)}^{y=g_2(x)} \left[h_2(x,y) - h_1(x,y) \right] dy\,dx$

4. Double integrals in polar coordinates:

$$dx\,dy = r\,dr\,d\theta$$

$$\iint f(x,y)\,dx\,dy = \iint f\,(r\cos\,\theta,\, r\sin\,\theta)\,r\,dr\,d\theta\ .$$

5. Steps to evaluate $\iint_R f(x,y)\,dx\,dy$ in uv-coordinates are given in Section 8.2.4.

6. **Triple Integrals:**

(a) $$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)\,dxdydz$$

$$= \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \left[\int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)\,dz \right] dxdy$$

$$= \int_{x=a}^{x=b} \left[\int_{y=g_1(x)}^{y=g_2(x)} F(x,y)dy \right] dx$$

(b) **Cylindrical coordinates (*r*, θ , *z*):**

(i) $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

$x^2 + y^2 = r^2$.

(ii) $dx\, dy\, dz = r\, dr\, d\theta\, dz$

(iii) $\iiint f(x,y,z)\, dx\, dy\, dz$

$= \iiint f\,(r\cos\theta\ , r\sin\theta, z)\, r\, dr\, d\theta\ dz$

(c) **Spherical coordinates** (ρ, ϕ, θ)

(i) $z = \rho\cos\phi$, $x = \rho\sin\phi\ \cos\theta$,

$\mathrm{y} = \rho\sin\phi\ \sin\theta$, $x^2 + y^2 + z^2 = \rho^2$

(ii) $dx\, dy\, dz = \rho^2\ \sin\phi\ d\rho\ d\phi\ d\theta$

(iii) $\iiint f(x,y,z)\, dx\, dy\, dz$

$= \iiint f\,(\rho\sin\phi\ \cos\theta\ , \rho\ \sin\phi\ \ \sin\theta,\ \rho\ \cos\phi\,)\rho^2\ \sin\phi\ d\rho\ d\phi\ \ d\theta$

7. **Surfaces**:

(a) Let $z = f(x,y)$ be the equation of the surface, *S*.

Surface area = $\iint_S d\sigma$

$$= \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\ dxdy,$$

where *R* is the projection of *S* on the *xy* plane.

If the projection of the surface is done on the *xz* plane, then

$$\text{the surface area} = \iint_S d\sigma$$

$$= \iint_{R'} \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + 1 + \left(\frac{\partial f}{\partial z}\right)^2}\, dxdz$$

(b) Integral of a function $\phi(x, y, z)$ over the surface $z = f(x, y)$ is given by

$$\iint_S \phi(x, y, z)\, d\sigma = \iint_R \phi(x, y, f(x, y)) \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dxdy \,.$$

8.7 ANSWERS TO ACTIVITIES AND SUPPLEMENTARY EXERCISES

Activity 1

(i) (a) 1/8 (b) $ln\left(\sqrt{2}+1\right) - \frac{\pi}{4}$

(ii) (a) $\sqrt{3} - \frac{\pi}{3}$ (b) $\frac{37}{12}$

Activity 2

(i) (a) $\frac{4\pi}{3} - \sqrt{3}$ (b) $\frac{9}{2}\tan^{-1} 2$

Activity 3

(i) (a) $\int_0^a x\, dx \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\, dy = \frac{a^2 b^2}{8}$.

(b) $\int_0^a \int_y^a \frac{\cos y}{\sqrt{(a-x)(a-y)}}\, dxdy = 2\sin a$.

Activity 4

(i) (a) 1 (b) $\int_0^1 \int_0^1 (2u+v)(u+2v)3\,du\,dv$.

Activity 5

(a) $\dfrac{32\pi\left(8-3^{\frac{3}{2}}\right)}{3}$ (b) $\dfrac{5\pi}{3}$

Answers for supplementary exercises

1. 64π
2. 64
3. $5\sqrt{3}-\dfrac{4\pi}{3}$
4. $\dfrac{1}{2}\ln(1+a^2)$
5. e^2-1
6. $\dfrac{4\pi}{15}$