TUTORIAL 3

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1. Solve the following differential equations:

(i) $\sqrt{3+2x-x^2}\,\dfrac{dy}{dx}=(1+y^2)$;

(ii) $\dfrac{dy}{dx}=\dfrac{e^{x+y}}{y-1}$;

(iii) $\dfrac{dy}{dx}=\dfrac{y(\ln y-\ln x+1)}{x}$;

(iv) $\dfrac{dy}{dx}=\sqrt{(x-1)(4-y^2)},\qquad y(1)=2$;

(v) $(y-2x-1)\,dx+(x+y-4)\,dy=0$;

(vi) $\dfrac{dy}{dx}=\dfrac{\sec^2 y}{1+x^2}$;

(vii) $\sin x\dfrac{dy}{dx}+(\sin x+\cos x)y=2+\sin(2x)$;

(viii) $\sin x\dfrac{dy}{dx}+y\cos x=x\sin x$;

(ix) $(1+\sin x)\dfrac{dy}{dx}-y\cos x=y^2\tan x$;

(x) $\dfrac{dy}{dx}+xy^3+\dfrac{y}{x}=0,\qquad y(1)=1$.

ANSWERS

1. (i) $y = \tan\left(\sin^{-1}\left(\frac{x-1}{2}\right) + A\right)$;

(ii) $-ye^{-y} = e^{x} + A$;

(iii) $y = xe^{Ax}$;

(iv) $y = 2\cos\left[\frac{2}{3}(x-1)^{\frac{3}{2}}\right]$;

(v) $\frac{1}{\sqrt{2(x-1)^2 - 2(x-1)(y-3) - (y-3)^2}} = A$;

(vi) $\frac{y}{2} + \frac{\sin(2y)}{4} = \tan^{-1} x + A$;

(vii) $y = \frac{2 + \frac{1}{5}\sin(2x) - \frac{2}{5}\cos(2x) + Ae^{-x}}{\sin x}$;

(viii) $y = 1 - x\cot x + A\cos ecx$;

(ix) $y = \frac{1+\sin x}{\ln(\cos x) + A}$;

(x) $\frac{1}{y^2} = 2x^2 \ln x + x^2$.

TUTORIAL 4

1. Given $z_1 = 3+4i,\ z_2 = -2-4i,\ z_3 = 5-4i$, evaluate the following:

(i) $\dfrac{\bar{z}_1 + z_2}{z_3}$; **(ii)** $z_1^{\,2} + z_2^{\,2}$; **(iii)** $\arg(\bar{z}_2 - z_3)$; **(iv)** $\arg(z_1^{\,2} z_3^{\,3})$;

(v) $\mathrm{Im}(z_1 / z_2)$; **(vi)** $\mathrm{Re}(z_1 z_2 / z_3)$; **(vii)** $\left| z_1^{\,2} z_2^{\,3} / z_3^{\,4} \right|$.

2. **(a)** Express the following complex numbers in polar form and exponential form:

(i) $z = (1+i)(1+\sqrt{3}\,i)(\sqrt{3}-i)$;

(ii) $\zeta = \dfrac{(1+i)^5(1-\sqrt{3}\,i)^5}{(\sqrt{3}+i)^4}$.

(b) If $z = 2\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ and $w = 3\,(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$, find the polar form of

(i) $z\,w$; **(ii)** z / w; **(iii)** w / z; **(iv)** z^5 / w^2.

3. **(a)** Solve the following equations:

(i) $i\,z + (7-8i)z = 3z - 2i$;

(ii) $\begin{array}{c} (1+i)\,z + (2-i)\,w = -3i \\ (1+2i)\,z + (3+i)\,w = 2+2i \end{array}$;

(iii) $z^2 = -8 - 6i$;

(iv) $z^2 - (3+i)\,z + 4 + 3i = 0$.

(b) Solve for x and y if $(x^2 - 2x - y) + i(2x - y - 3) = 0$.

4. Given that one root of the equation

$$2x^4 + x^3 + 5x^2 + 4x - 12 = 0,$$

is a purely imaginary number, solve the equation.

5. Prove that if both $z_1 + z_2$ and $z_1 z_2$ are real, then either z_1 and z_2 are both real or $z_1 = \bar{z}_2$.

6. **(a)** If $w = (3z + i)/(i - z)$, show that $\operatorname{Re} z \geq 0$ implies $\operatorname{Im} w \leq 0$.

(b) If $w = i\,(1 - z)/(1 + z)$, show that $|z| < 1$ implies $\operatorname{Im} w > 0$.

(c) If $|z| = a$, show that $\operatorname{Im}(z + a^2/z) = 0$.

7. Evaluate the following:

(i) $(3 - 4i)^{10}/(-5 + 6i)^7$; **(ii)** 2^{3-2i}; **(iii)** $(-3)^{-i}$; **(iv)** $(3 + 2i)^i$;

(v) $(-1 + 3i)^{2-i}$; **(vi)** $2^{-i} + 5^i$; **(vii)** $(-7i)^{-i}$; **(viii)** $i^{\pi - i}$.

8. If $\zeta = \cos\theta + i \sin\theta$, show that

$$2\cos n\theta = \zeta^n + \zeta^{-n},$$

$$2i \sin n\theta = \zeta^n - \zeta^{-n}.$$

Hence, or otherwise, show that

$$\sum_{r=0}^{\infty} \frac{\cos r\theta}{2^r} = \frac{4-2\cos\theta}{5-4\cos\theta}, \qquad \sum_{r=0}^{\infty} \frac{\sin r\theta}{2^r} = \frac{2\sin\theta}{5-4\cos\theta}.$$

9. Solve the following equations:

(i) $z^3 = -8i$;

(ii) $z^4 = 2 - 2i$;

(iii) $z^6 + z^4 + z^2 + 1 = 0$;

(iv) $(z+1)^6 = 64(z-1)^6$;

(v) $(5+z)^5 - (5-z)^5 = 0$;

(vi) . $(z - \sqrt{3} + 2i)^6 + 64 = 0$.

ANSWERS

1. **(i)** $\frac{37}{41} - \frac{36}{41} i$ **;** **(ii)** $-19 + 40i$ **;** **(iii)** 2.28963 **;** **(iv)** -0.169632 **;**

(v) $1/5$ **;** **(vi)** $130/41$ **;** **(vii)** 1.3302 **.**

2. **(a)** **(i)** $z = 4\sqrt{2}\ (\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$; $4\sqrt{2}\exp(5\pi i/12)$;

(ii) $\zeta = 2^{7/2}\ (\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$; $2^{7/2}\exp(11\pi i/12)$.

(b) **(i)** $6\ (\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$;

(ii) $\frac{2}{3}\ (\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$;

(iii) $\frac{3}{2}\ (\cos\frac{\pi}{12} - i\sin\frac{\pi}{12})$;

(iv) $\frac{32}{9}\ (\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$.

3. **(a)** **(i)** $z = 14/65 - 8/65\, i$;

(ii) $z = -1 + 5i$; $w = 19/5 - 8/5\, i$;

(iii) $z = \pm(1 - 3i)$;

(iv) $z = 2 - i, 1 + 2i$.

(b) $x = 3,\ y = 3$; $x = 1,\ y = -1$ **.**

4. $x = 1, -3/2, \pm 2i$ **.**

7. **(i)** $5.50865 + 0.00576661i$;

(ii) $1.46766 - 7.86422i$;

(iii) $10.5251 - 20.6086i$;

(iv) $0.157935 + 0.532509i$;

(v) $-57.9889 + 32.2687i$;

(vi) $0.730607 + 0.360292\,i$;

(vii) $-0.0761626 - 0.193425\,i$;

(viii) $1.06111 - 4.69199\,i$.

9. **(i)** $2i,\ \pm\sqrt{3} - i$;

(ii) $i^k 2^{3/8}(\cos\frac{\pi}{16} - i\sin\frac{\pi}{16}), \quad k = 0, 1, 2, 3$;

(iii) $\pm i, \quad \cos[\frac{1}{4}\pi(2r+1)] + i\sin[\frac{1}{4}\pi(2r+1)], \quad r = 0, 1, 2, 3$;

(iv) $\dfrac{3 - 4i\sin\frac{1}{3}r\pi}{5 - 4\cos\frac{1}{3}r\pi}, \quad r = 0, 1, \cdots, 5$;

(v) $5i\tan\frac{1}{5}r\pi, \quad r = 0, \pm 1, \pm 2$;

(vi) $2\sqrt{3} - i,\ 2\sqrt{3} - 3i,\ \sqrt{3},\ \sqrt{3} - 4i,\ -3i,\ -i$.

TUTORIAL 5

1. Evaluate the following limits

   (a) $\lim_{x \to 0} \ln(x+1)\operatorname{cosec} x$

   (b) $\lim_{x \to \frac{\pi}{2}} \left(\frac{2x}{\pi} \sec x - \tan x \right)$

   (c) $\lim_{x \to 0} \frac{\sin x \sinh^{-1} x}{x^2}$

   (d) $\lim_{x \to 1} \frac{1 - x + \ln x}{1 + \cos \pi x}$

   (e) $\lim_{x \to +\infty} \tanh x$

   (f) $\lim_{x \to \infty} x^3 e^{-x}$

   (g) $\lim_{x \to 0} \frac{x^2 \tan x}{\tan x - x}$

   (h) $\lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^{bx}$

   (i) $\lim_{x \to \infty} \frac{x^2 + \cos x}{x^2}$

   (j) $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$

2. Determine the values of a, b, c and d such that

$$\lim_{x \to 0} \frac{a \cos x + b \sin x + c e^{-x} + d}{x^3} = \frac{1}{5}.$$

3. Evaluate the following limits

   (a) $\lim_{y \to 0} y^y$

   (b) $\lim_{n \to \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$

   (c) $\lim_{x \to \infty} (1+x)^{\frac{1}{x}}$

ANSWERS

1. (a) 1 (b) $-\dfrac{2}{\pi}$ (c) 1 (d) 0 (e) 1

   (f) 0 (g) 3 (h) 1 (i) 1 (j) 0

2. $a = b = c = -3/5;\ d = 6/5$

3. (a) 1 (b) 1 (c) 1

TUTORIAL 6

1. b Prove that

(a) $\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y=\dfrac{1}{2}(\cosh 2y-\cos 2x)$

(b) $\sin^{-1}(\tanh x)=\tan^{-1}(\sinh x)$

(c) $\dfrac{1+\tanh x}{1-\tanh x}=\cosh 2x+\sinh 2x$

(d) $\cosh^{-1}(\sec^2\theta+\tan^2\theta)=2\cosh^{-1}(\sec\theta)$

2. Given that $\coth x=-4$, find without using calculators the values of the following

(a) $\operatorname{cosech} x$

(b) $\operatorname{sech} x$

(c) $\sinh 2x$

(d) $\cosh 2x$

(e) $\tanh 2x$

3. If $\sinh^{-1} x=2\cosh^{-1} y$, prove that $x^2=4y^2(y^2-1)$.

4. Solve the following equations

(a) $13\sinh x+5\cosh x=24$

(b) $\cosh(x+1)+\cosh(x-1)=6$

(c) $3\sinh^2 x-4=5\cosh x$

(d) $5\sinh 2x\cosh 2x=-3$

(e) $\begin{aligned}\cosh x&=2\sinh y\\ 2\sinh x&=3-4\cosh y\end{aligned}$

(f) $\begin{aligned}\sinh x\cosh y&=3\\ \cosh x\sinh y&=-1\end{aligned}$

5. If $y=(\cosh^{-1} x)^2$, prove that $(x^2-1)y''+xy'=2$.

6. Differentiate the following functions with respect to x

(a) $\tan^{-1}\left(\tanh\left(\dfrac{x}{2}\right)\right)$

(b) $\sinh^{-1}(\tan 2x)$

(c) $\tanh 3x \cosh 2x$

(d) $\coth x \operatorname{cosech} 2x$

(e) $\tanh^{-1}(\sin 2x)$

7. Find the following integrals

(a) $\int \cosh 5x \sinh 3x \, dx$

(b) $\int \sinh^2 x \cosh^3 x \, dx$

(c) $\int \dfrac{dx}{\sqrt{3x^2 - 5x + 7}}$

(d) $\int \dfrac{dx}{\sqrt{x^2 - 3x + 5}}$

(e) $\int e^{-2x} \tanh x \, dx$

(f) $\int_0^1 x^2 \cosh x \, dx$

(g) $\int_0^1 \dfrac{x}{\sqrt{2x^2 - 2x + 1}} \, dx$

(h) $\int \dfrac{dx}{(1+e^x)(1-e^{-x})}$

(i) $\int_3^4 \dfrac{dx}{\sqrt{3x^2 - 6x + 1}}$

(j) $\int \tanh^{-1} x \, dx$

ANSWERS

2. **(a)** $\sqrt{15}$ **(b)** $\dfrac{\sqrt{15}}{4}$ **(c)** $\dfrac{8}{15}$ **(d)** $\dfrac{17}{15}$ **(e)** $\dfrac{8}{17}$

4 **(a)** 1.03817 **(b)** ± 1.2841 **(c)** ± 1.59828 **(d)** -0.253993

(e) $x = \frac{1}{2}\left(\sinh^{-1} 2 + \sinh^{-1} 4\right),\ y = \frac{1}{2}\left(\sinh^{-1} 2 - \sinh^{-1} 4\right)$

(f) $x = \sinh^{-1}\left(-\frac{11}{12}\right),\ y = \cosh^{-1}\frac{29}{24}$

6 **(a)** $\dfrac{\sec^2\frac{x}{2}}{2\left(1+\tan^2\frac{x}{2}\right)}$ **(b)** $\dfrac{2\sec^2 2x}{\sqrt{1+\tan^2 2x}}$

(c) $3\cosh 2x\,\mathrm{sech}^2 3x + 2\sinh 2x \tanh 3x$

(d) $-2\coth x \coth 2x\,\mathrm{cosech}\,2x - \mathrm{cosech}^2 x\,\mathrm{cosech}2x$

(e) $\dfrac{2\cos 2x}{1-\sin^2 2x}$

7 **(a)** $-\frac{1}{2}\cosh^2 x + \frac{1}{16}\cosh 8x$ **(b)** $-\frac{1}{8}\sinh x + \frac{1}{48}\sinh 3x + \frac{1}{80}\sinh 5x$

(c) $\dfrac{1}{\sqrt{3}}\sinh^{-1}\left(\dfrac{6x-5}{\sqrt{59}}\right)$ **(d)** $\sinh^{-1}\left(\dfrac{2x-3}{\sqrt{11}}\right)$ **(e)** $\dfrac{e^{-2x}}{2} - \ln(1+e^{2x})$

(f) $-2\cosh 1 + 3\sinh 1$ **(g)** $\dfrac{\sinh^{-1} 1}{\sqrt{2}}$ **(h)** $\frac{1}{2}\ln\left(\tanh\frac{x}{2}\right)$

(i) $\dfrac{1}{\sqrt{3}}\ln\left(\dfrac{15+\sqrt{219}}{12+\sqrt{138}}\right)$ **(j)** $x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2)$

TUTORIAL 7

1. If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u.$

2. If $f(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$, where $x, y, z \neq 0$, show that $f(x, y, z)$ is a harmonic function.

3. If $u = \ln\left(x^3 + y^3 - x^2 y - xy^2\right)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = -4(x + y)^{-2}.$

4. Let $F(x, y, z) = 2^x y - 2\sin y + x\tan^{-1} z$. Find $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}.$

   Hence find the total differential of F.

5. Find the total differential of the function $f(x, y, z)$ if

   $f(x, y, z) = z^2 \ln\dfrac{x}{y} - 3e^{xy}\coth z.$

6. If $u = x\cos y$ with $x = \xi - 2\eta$, $y = 3\xi + \eta$, use the chain rule to find $\dfrac{\partial^2 u}{\partial \xi\,\partial \eta}$. Leave your answer in terms of x and y.

7. Let $w = f(u, v)$ where $u = \dfrac{x}{z}$ and $v = \dfrac{y}{z}$. Use the chain rule to show that

   $x\,w_x + y\,w_y + z\,w_z = 0.$

8. Given that $z = f(x + ay) + g(x - ay)$, prove that $\dfrac{\partial^2 z}{\partial y^2} = a^2\dfrac{\partial^2 z}{\partial x^2}.$

9. If $H = f(y - z, z - x, x - y)$, prove that $\dfrac{\partial H}{\partial x} + \dfrac{\partial H}{\partial y} + \dfrac{\partial H}{\partial z} = 0.$

10. Given that $a^2 x^2 + b^2 y^2 - c^2 z^2 = 0$, show that $\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2} + \dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{c^2 z}.$

ANSWERS

4. $$\frac{\partial F}{\partial x} = \left(2^x \ln 2\right) y + \tan^{-1} z.$$

$$\frac{\partial F}{\partial y} = 2^x - 2\cos y.$$

$$\frac{\partial F}{\partial z} = \frac{x}{1+z^2}.$$

$$dF = \left(\left(2^x \ln 2\right) y + \tan^{-1} z\right) dx + \left(2^x - 2\cos y\right) dy + \left(\frac{x}{1+z^2}\right) dz.$$

5. $$df = \left(\frac{z^2}{x} - 3y\, e^{xy} \coth z\right) dx - \left(\frac{z^2}{y} + 3x\, e^{xy} \coth z\right) dy + \left(2z \ln\left(\frac{x}{y}\right) - 3e^{xy} \sinh z\right) dz.$$

6. $5\sin y - 3x\cos y.$

TUTORIAL 8

1. Find the differential equation whose general solution is given by:

(i) $y = (A + Bx)e^{-3x}$;

(ii) $y = e^{2x}(A\cos 2x + B\sin 2x)$;

(iii) $y = (A + Bx)e^{-2x} + 1$;

(iv) $y = Ae^{-x} + Be^{-3x} + x + 4$;

(v) $y = A\sin 3x + B\cos 3x + x/3$.

2. Find the general solution of each of the following differential equations:

(i) $y'' = 9$;

(ii) $y'' = 9x$;

(iii) $y'' = 9y$;

(iv) $y'' = 9y'$;

(v) $2y'' - 5\sqrt{3}y' + 6y = 0$;

(vi) $y'' + y' + 4y = 0$;

(vii) $2y'' - 2\sqrt{2}y' + y = 0$;

(viii) $y'' - 3y' + 4y = 0$;

(ix) $\ddot{x} + 3\dot{x} - 4x = 0$; [Dot on top means $\frac{d}{dt}$]

(x) $\ddot{\theta} + \omega^2\theta = 0$.

3. Find the solutions of the given differential equations:

(i) $y'' - 8y' + 16y = 0; \quad y(0) = \frac{1}{2},\ y'(0) = -\frac{1}{3}$;

(ii) $y'' + 9y = 0; \ \ y(0) = 4,\ y(\pi/6) = 5$;

(iii) $y'' - \sqrt{2}y' + y = 0; \quad y(0) = \sqrt{2},\ y'(0) = 0$;

(iv) $y'' - 12y' + 36y = 0; \quad y(0) = 1,\ y(1) = 0$;

(v) $y'' = y; \quad y(0) = y'(0) = 1$;

(vi) $y'' + y = 0; \quad y(0) = 0,\ y(\pi) = 1$;

(vii) $y'' + \pi^2 y = 0; \quad y(0) + y(1) = 0,\ \ y'(0) + y'(1) = 0$.

ANSWERS

1. **(i)** $y''+6y'+9y=0$;

(ii) $y''-4y'+8y=0$;

(iii) $y''+4y'+4y=4$;

(iv) $y''+4y'+3y=3x+16$;

(v) $y''+9y=3x$.

2. **(i)** $y=A+Bx+\frac{9}{2}x^2$;

(ii) $y=A+Bx+\frac{3}{2}x^3$;

(iii) $y=Ae^{-3x}+Be^{3x}$, **OR** $y=C_1\cosh 3x+C_2\sinh 3x$;

(iv) $y=Ae^{9x}+B$;

(v) $y=Ae^{2\sqrt{3}\,x}+Be^{\sqrt{3}\,x/2}$;

(vi) $y=e^{-x/2}\,[A\cos\frac{\sqrt{15}}{2}x+B\sin\frac{\sqrt{15}}{2}x]$;

(vii) $y=(A+Bx)e^{x/\sqrt{2}}$;

(viii) $y=e^{3x/2}\,[A\cos\frac{\sqrt{7}}{2}x+B\sin\frac{\sqrt{7}}{2}x]$

(ix) $x=Ae^{t}+Be^{-4t}$;

(x) $\theta=A\cos\omega t+B\sin\omega t$.

3. **(i)** $y = (\frac{1}{2} - \frac{7}{3}x)e^{4x}$;

(ii) $y = 4\cos 3x + 5\sin 3x$;

(iii) $y = \sqrt{2}\, e^{x/\sqrt{2}}(\cos\frac{x}{\sqrt{2}} - \sin\frac{x}{\sqrt{2}})$;

(iv) $y = (1 - x)e^{6x}$;

(v) $y = e^{x}$;

(vi) No solution!

(vii) $y = A\cos \pi x + B\sin \pi x$.

TUTORIAL 9

1. Find the general solution of each of the following differential equations:

(i) $3y'' - 7y' = 5$;

(ii) $y'' + 4y = e^{2x}/2, \quad y(0) = y'(0) = 0$;

(iii) $y'' + 4y' + 4y = xe^{2x}$;

(iv) $y'' - y' - 2y = e^{-x}\cos x$;

(v) $4y'' + 4y' + y = xe^{-x/2}\sin x, \quad y(0) = 0,\ y'(0) = 1$;

(vi) $2y'' - 3y' + 2y = x^3 - 5x + 2$;

(vii) $y'' - 2y' + 2y = e^{2x}\sin x$;

(viii) $y'' + 4y' + 3y = \sinh x, \quad y(0) = y'(0) = 0$;

(ix) $(D+3)^2 y = (x+5)e^x$;

(x) $y'' - 5y' + 6y = e^{2x} - 2e^{3x}, \quad y(0) = 1,\ y'(0) = 0$.

ANSWERS

1. **(i)** $y = A + Be^{-2x/3} - 5x/7$;

(ii) $y = \frac{1}{16}[e^{2x} - \cos 2x - \sin 2x]$;

(iii) $y = (A + Bx)e^{-2x} + \frac{1}{32}e^{2x}(2x - 1)$;

(iv) $y = Ae^{-x} + Be^{2x} - \frac{1}{10}e^{-x}(3\sin x + \cos x)$;

(v) $y = \frac{1}{4}e^{-x/2}(2 + 4x - 2\cos x - x\sin x)$;

(vi) $y = e^{3x/4}(A\cos\frac{\sqrt{7}}{4}x + B\sin\frac{\sqrt{7}}{4}x) + \frac{1}{8}(4x^3 + 18x^2 + 10x - 13)$;

(vii) $y = e^{x}(A\cos x + B\sin x) + \frac{1}{5}e^{2x}(\sin x - 2\cos x)$;

(viii) $y = \frac{1}{16}(e^{x} - 4xe^{-x} - e^{-3x})$;

(ix) $y = (Ax + B)e^{-3x} + \frac{1}{32}e^{x}(9 + 2x)$;

(x) $y = e^{3x}(1 - 2x) - xe^{2x}$.