UNIT 3 FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Unit Structure

3.1 OVERVIEW

Differential equations are of fundamental importance as many physical laws and relations appear mathematically in the form of such equations. In this Unit, we shall study important standard methods to solve first-order differential equations.

3.2 LEARNING OBJECTIVES

By the end of this Unit, you should be able to do the following:

1. Explain the concepts of ordinary differential equations, order and degree, and, general and particular solutions.
2. Use the method of separation of variables.
3. Recognise homogeneous differential equations and find their solution.
4. Solve first-order linear differential equations.
5. Recognise Bernoulli's equation and find its solution.

3.3 DEFINITION OF A DIFFERENTIAL EQUATION

A differential equation is an equation involving derivatives of an unknown function of one or more variables.
Some examples of differential equations are:

$$\frac{dy}{dx} = x + y^2, \qquad \textbf{(1)}$$

$$4x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} - 3y = \sin x, \qquad \textbf{(2)}$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 5x = \ln t, \qquad \textbf{(3)}$$

$$\frac{d^3y}{dx^3} + 6\sqrt{\left(\frac{dy}{dx}\right)^2 + y^2} = 0. \qquad \textbf{(4)}$$

The variable on top in the derivative is called the *dependent variable*, and the one at the bottom is *the independent variable*. Thus, in Eqns (1), (2) and (4), y is the dependent variable and x is the independent variable; while in Eqn. (3), x is the dependent variable and t the independent variable.

A differential equation involving derivatives with respect to a single independent variable is said to be an ordinary differential equation (abbreviated as **ODE**).

The **order** of a differential equation is the order of the highest derivative contained in it. Thus, Eqn (1) is of the first order, Eqns. (2) and (3) are of the second order, while Eqn (4) is of the third order.

The **degree** of the differential equation is the power to which the highest-order derivative is raised when the equation is rationalized (i.e., fractional powers are removed). For example, Eqns (1), (2), (3) are all of the first degree, but Eqn (4) is of the second degree since when written in rationalized form it becomes

$$\left(\frac{d^3y}{dx^3}\right)^2 = 36\left[\left(\frac{dy}{dx}\right)^2 + y^2\right], \qquad \textbf{(5)}$$

the highest derivative d^3y/dx^3 now appearing to the second power.

A **linear** n^{th} order ODE is of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \qquad \textbf{(6)}$$

All ODEs not in the above form are said to be **non-linear**. If the coefficients a_i are all constants, we say it's a *constant-coefficient* ODE; else it's a *variable-coefficient* ODE.

The ODEs (2) and (3) are linear; while (1), (4) and (5) are non-linear. Eqn. (2) is a variable-coefficient ODE.

By the *solution* of an ODE, we mean a relationship between the dependent and the independent variables which when substituted into the differential equation, reduces it to an identity.

A **general solution** of an n^{th}-order differential equation is one involving n (essential) arbitrary constants.

For example:

Since $y = x^2 + C_1 x + C_2$ has two arbitrary constants and satisfies the second-order differential equation $y'' = 2$, it is the general solution of the differential equation.

A **particular solution** is a solution obtained from the general solution by assigning specific values to the arbitrary constants.

For example:

(a) $y = x^2 - 3x + 2$ is a particular solution of $y'' = 2$ and is obtained from the general solution $y = x^2 + C_1 x + C_2$ by imposing the conditions $y(0) = 2,\ y'(0) = -3$ which give $C_1 = -3$ and $C_2 = 2$. [**Note**: $y(0) = 2$ means $y = 2$ when $x = 0$.]

(b) $y = \sqrt{1 - x^2}$ is a solution of the ODE $y\dfrac{dy}{dx} + x = 0.$

(c) $y = Ae^x$, for any arbitrary real constant *A*, is a solution of the ODE $\dfrac{dy}{dx} = y$.

(d) $y = A \sin x + B \cos x$, for arbitrary real constants *A* and *B*, is a solution of the ODE $\dfrac{d^2y}{dx^2} + y = 0.$

FORMATION OF DIFFERENTIAL EQUATIONS

Suppose we are given an equation involving n arbitrary real constants. Differentiate it n times, with respect to the independent variable, to obtain n more equations. Eliminate the n arbitrary constants from these $n+1$ equations. The equation so obtained is a differential equation of n^{th} order.

So,

1 arbitrary constant $\rightarrow$ Differentiate once only $\rightarrow 1^{\text{st}}$ –order ODE.

2 arbitrary constants $\rightarrow$ Differentiate twice only $\rightarrow 2^{\text{nd}}$-order ODE.

And so on.

<u>Example 1</u>

Find the differential equation satisfied by the circles

$$(x-C)^2 + y^2 = 4\,, \tag{7}$$

where C is an arbitrary real constant.

<u>Solution</u>

Since equation (7) involves only 1 arbitrary real constant, we differentiate (7) w.r.t. x, only once. This gives

$$2(x-C) + 2y\frac{dy}{dx} = 0\,,$$

or,

$$C = x + y\frac{dy}{dx}\,. \tag{8}$$

Replacing (8) in (7), we get

$$\left(x - \left(x + y\frac{dy}{dx}\right)\right)^2 + y^2 = 4\,,$$

or, $$\left(-y\frac{dy}{dx}\right)^2 = 4 - y^2,$$

or, $$y\frac{dy}{dx} + \sqrt{4 - y^2} = 0,$$

which is a 1st – order ODE.

Example 2

Find the differential equation satisfied by

$$y = Ae^{x} + Be^{-x} + Ce^{2x}, \qquad \textbf{(9)}$$

where *A*, *B* and *C* are arbitrary real constants.

Solution

Since there are 3 constants in (9), we differentiate thrice only w.r.t. x. This gives

$$\frac{dy}{dx} = Ae^{x} - Be^{-x} + 2Ce^{2x}, \qquad \textbf{(10)}$$

$$\frac{d^2y}{dx^2} = Ae^{x} + Be^{-x} + 4Ce^{2x}, \qquad \textbf{(11)}$$

$$\frac{d^3y}{dx^3} = Ae^{x} - Be^{-x} + 8Ce^{2x}. \qquad \textbf{(12)}$$

We can eliminate the constants by solving three of the equations for *A*, *B*, and *C* and then substitute in the fourth equation.

Subtracting (10) from (12), we obtain

$$C = \frac{1}{6}e^{-2x}\frac{d^3y}{dx^3} - \frac{1}{6}e^{-2x}\frac{dy}{dx}. \qquad \textbf{(13)}$$

We replace equation (13) in (10) and (11) and solve for *A* and *B*:

$$A = -\frac{1}{2}e^{-x}\frac{d^3y}{dx^3} + \frac{1}{2}e^{-x}\frac{d^2y}{dx^2} + e^{-x}\frac{dy}{dx}. \qquad \textbf{(14)}$$

$$B = -\frac{1}{6}e^{x}\frac{d^3y}{dx^3} + \frac{1}{2}e^{x}\frac{d^2y}{dx^2} - \frac{1}{3}e^{x}\frac{dy}{dx}. \qquad \textbf{(15)}$$

By substituting *A*, *B* and *C* in equation (9), we get

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0,$$

which is a 3rd - order ODE.

Let us now consider the (variable-coefficient, 1st-order) ODE:

$$x^3 \frac{dy}{dx} + 3x^2 y + 2x = 0 \ . \qquad \textbf{(16)}$$

In (16), *x* is the independent variable and *y* the dependent variable. Let us write (16) in the form

$$(3x^2 y + 2x)\ dx + x^3 dy = 0 \, . \qquad \textbf{(17)}$$

Is (17) a differential equation? Yes, it is! But now, *x* is no longer an independent variable and *y* is no longer a dependent variable. Indeed, (17) is of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \ . \qquad \textbf{(18)}$$

Notice that (18) reduces to (17) when $M(x, y) = 3x^2 y + 2x$ and $N(x, y) = x^3$.

3.4 METHOD OF SEPARATION OF VARIABLES

If the differential equation (18) is such that $M(x, y)$ is independent of *y* and $N(x, y)$ is independent of *x*, then it is said to be an ODE in which the *variables are separable*. So, we can group all functions in *x* on one side and all functions in *y* on the other side.

Example 3

Consider the differential equation

$$(x^2+1)\frac{dy}{dx} = xy\,. \qquad \textbf{(19)}$$

We divide both sides by $y(x^2+1)$ so that (19) can be written in the form:

$$-\frac{x}{x^2+1}dx + \frac{1}{y}dy = 0\,. \qquad \textbf{(20)}$$

Since, $M(x,y) = -\frac{x}{x^2+1}$ depends on x only and $N(x,y) = \frac{1}{y}$ depends on y only, the differential equation (20) is separable.

From (20),

$$\int \frac{1}{y}dy = \int \frac{x}{x^2+1}dx\,,$$

or, $\ln y = \frac{1}{2}\ln(x^2+1) + C$, C: an arbitrary real constant

or, $\ln y = \ln\sqrt{x^2+1} + \ln A$, [Putting $C = \ln A$]

or, $y = A\sqrt{x^2+1}\,.$

Question: How come we can integrate w.r.t. y on one side but w.r.t. x on the other side?

<u>Example 4</u>

Consider the differential equation

$$x\cos y\;dx + (x+1)\sin y\;dy = 0\,. \qquad \textbf{(21)}$$

We can separate the variables by dividing by $(x+1)\cos y$ throughout. So, (21) can be rewritten as

$$\frac{x}{x+1}\,dx + \tan y\;dy = 0\,. \qquad \textbf{(22)}$$

From (22),

$$\int \tan y\,dy = -\int \frac{x}{x+1}\,dx\;\;,$$

or, $$\ln\sec y = -((x+1) - \ln(x+1)) + C\,,$$

or, $$\begin{aligned} x+1 &= \ln[(x+1)\cos y] + \ln A \\ &= \ln[A(x+1)\cos y] \end{aligned}\quad , \qquad [\text{Putting } C = \ln A\,]$$

or, $$B(x+1)\cos y = e^{x}\,. \qquad [\text{Putting } B = Ae^{-1}\,]$$

<u>Activity 1</u>

1. Form differential equations by eliminating the constants A and B:

   (i) $y^2 = A(A+\sqrt{2}x)$;

   (ii) $y = A\sin x$;

   (iii) $y = Ax^2 + A^2$;

   (iv) $y = 4x^2 + A\cos x$;

   (v) $y = A\ln x + Bx$;

   (vi) $y = A\cos 2x + B\sin 2x$.

2. Solve the following differential equations:

(i) $2x^3 y\,dx - (x^4 + 1)\,dy = 0$;

(ii) $\dfrac{dy}{dx} = \dfrac{4x^3 + 6x^2 + 1}{e^{-y} - \sin y}$;

(iii) $4x\,dy - y\,dx = x^2\,dy$;

(iv) $(9\cos^2 x + 4\sin^2 x)\dfrac{dy}{dx} = 6$;

(v) $x^2 \dfrac{dy}{dx} - 5 = 2\left(y + x\dfrac{dy}{dx}\right), \qquad y(-1) = 2$;

(vi) $(1 - \sin 2x)\dfrac{dy}{dx} = 3, \qquad y(\pi) = 4$.

3.5 HOMOGENEOUS EQUATIONS

An ODE of the form (18) is said to be a homogeneous equation if it can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right). \qquad \textbf{(23)}$$

To solve such a differential equation, we put $\frac{y}{x} = v$, that is, $y = vx$. Then the product rule (***v* is not a constant**!) gives $\frac{dy}{dx} = v + x\frac{dv}{dx}$. So, (23) reduces to

$$v + x\frac{dv}{dx} = f(v),$$

which can be written in the form

$$\frac{dx}{x} = \frac{1}{f(v)-v}\,dv, \quad \text{[We **always** get } \frac{dx}{x} \text{ on one side.]}$$

so that

$$\ln x + C = \int \frac{1}{f(v)-v}\,dv.$$

After carrying out the integration, replace v by $\frac{y}{x}$.

Example 5

Consider the differential equation

$$(4x^2 + 4xy + 3y^2)\,dx - (2xy - x^2)\,dy = 0. \qquad \textbf{(24)}$$

Equation (24) can be written in the form

$$\frac{dy}{dx} = \frac{4x^2 + 4xy + 3y^2}{2xy - x^2}$$

$$= \frac{x^2\left(4 + 4\left(\frac{y}{x}\right) + 3\left(\frac{y^2}{x^2}\right)\right)}{x^2\left(2\left(\frac{y}{x}\right) - 1\right)},$$

or, $$\frac{dy}{dx} = \frac{4 + 4\left(\frac{y}{x}\right) + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right) - 1} \quad . \qquad \textbf{(25)}$$

Since (24) can be expressed as a function of the ratio $\frac{y}{x}$, (24) is a homogeneous equation.

Let $y = vx$. Then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad . \qquad \textbf{(26)}$$

We substitute (26) into (25) to obtain

$$v + x\frac{dv}{dx} = \frac{4 + 4v + 3v^2}{2v - 1} \quad .$$

$$x\frac{dv}{dx} = \frac{4 + 4v + 3v^2}{2v - 1} - v$$

$$= \frac{4 + 5v + v^2}{2v - 1},$$

or, $$x\frac{dv}{dx} = \frac{(v+1)(v+4)}{2v - 1} . \qquad \textbf{(27)}$$

Equation (27) is separable and thus

$$\frac{dx}{x} = \frac{2v-1}{(v+1)(v+4)}dv\,, \quad \text{[See! Only } \frac{dx}{x} \text{ on the l.h.s.]}$$

or, $$\int\frac{dx}{x} = \int\left(\frac{-1}{(v+1)} + \frac{3}{(v+4)}\right)dv\,, \quad \text{[Partial Fractions]}$$

or, $$\ln x + C = -\ln(v+1) + 3\ln(v+4)\,,$$

or, $$\ln x + \ln A = -\ln(v+1) + 3\ln(v+4)\,, \text{ [Put } C = \ln A\text{]}$$

or, $$\ln[Ax] = \ln\left[\frac{(v+4)^3}{(v+1)}\right],$$

or, $$Ax = \frac{(v+4)^3}{(v+1)},$$

or, $$Ax = \frac{\left(\frac{y}{x}+4\right)^3}{\left(\frac{y}{x}+1\right)},$$

or, $$Ax^3 = \frac{(y+4x)^3}{(y+x)}.$$

<u>Example 6</u>

Let us consider the differential equation

$$(x+y+1)\,dx + (-x+y-5)\,dy = 0\,, \qquad \textbf{(28)}$$

which can be written in the form

$$\frac{dy}{dx} = \frac{x+y+1}{x-y+5}. \qquad \textbf{(29)}$$

Now, $\dfrac{x+y+1}{x-y+5} = \dfrac{1+\left(\frac{y}{x}\right)+\left(\frac{1}{x}\right)}{1-\left(\frac{y}{x}\right)+\left(\frac{5}{x}\right)}$, which is not a function of $\dfrac{y}{x}$ only. Hence, (29) is not a homogeneous equation. So, we seek a translation of axes of the form

$$x = X + h \quad \text{and} \quad y = Y + k,$$

where h and k are constants to be suitably determined.

Because $dx = dX$ and $dy = dY$, we have

$$\frac{dY}{dx} = \frac{dy}{dx}.$$

Thus, equation (29) becomes

$$\frac{dY}{dX} = \frac{X+Y+(h+k+1)}{X-Y+(h-k+5)}.$$

Some elementary algebra shows that such a transformation exists if h and k satisfy the system

$$\begin{aligned} h+k+1 &= 0 \\ h-k+5 &= 0 \end{aligned}.$$

Solving the above system for h and k gives

$$h = -3 \text{ and } k = 2.$$

So we let $x = X - 3$ and $y = Y + 2$.

Now, we have

$$\frac{dY}{dX} = \frac{X+Y}{X-Y} = \frac{1+\left(\frac{Y}{X}\right)}{1-\left(\frac{Y}{X}\right)}. \qquad \textbf{(30)}$$

Since (30) is homogeneous, we let $\frac{Y}{X} = v$. Then, $\frac{dY}{dX} = v + X\frac{dv}{dX}$ and (30) becomes

$$v + X\frac{dv}{dX} = \frac{1+v}{1-v},$$

or, $$X\frac{dv}{dX} = \frac{1+v^2}{1-v}.$$

Separating variables gives

$$\int \frac{1-v}{1+v^2}dv = \int \frac{1}{X}dX,$$

or, $$\int \frac{1}{1+v^2} - \frac{v}{1+v^2}dv = \int \frac{1}{X}dX,$$

Integrating both sides of the latter equation, we get

$$\tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln X + C,$$

or, $$\tan^{-1} v = \ln X + \ln A + \ln(\sqrt{1+v^2}),$$

putting $C = \ln A$ and simplifying, we get

$$\tan^{-1} v = \ln(AX\sqrt{1+v^2}).$$

Thus,

$$v = \tan\{\ln(AX\sqrt{1+v^2})\}$$

or

$$\frac{Y}{X} = \tan\left\{\ln\left(AX\sqrt{1+\frac{Y^2}{X^2}}\right)\right\},$$

giving

$$Y = X\tan\{\ln(A\sqrt{X^2+Y^2})\}$$

and hence

$$(y-2) = (x+3)\tan\{\ln(A\sqrt{(x+3)^2+(y-2)^2})\}.$$

Notice that we have used the fact that $v = \dfrac{Y}{X}$ and $X = x+3,\ Y = y-2.$. The solution must be in terms of the original variables.

<u>Activity 2</u>

1. Solve the following differential equations:

   (i) $(x^3 + y^2\sqrt{x^2+y^2})\,dx - xy\sqrt{x^2+y^2}\,dy = 0$;

   (ii) $2xy\,dx + (x^2 - y^2)\,dy = 0$;

   (iii) $\dfrac{dy}{dx} = \dfrac{x\tan\left(\dfrac{y}{x}\right) + y}{x}$;

   (iv) $\dfrac{dy}{dx} = (9x + y - 2)^2$; [Let $v = 9x + y - 2$.]

   (v) $(3x - y + 1)\,dx + (x + y + 3)\,dy = 0$;

   (vi) $(2x - y)\,dx + (4x + y - 3)\,dy = 0$.

3.6 FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

We shall now consider first-order linear ODEs. These are of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \textbf{(31)}$$

and are solved by first multiplying throughout by a function of x, called an **Integrating Factor**, $\mu(x)$, given by

$$\mu(x) = e^{\int P(x)\,dx}. \qquad \textbf{(32)}$$

Eq. (31) then reduces to the form

$$\frac{d}{dx}(\mu\, y) = \mu\, Q. \qquad \textbf{(33)}$$

We now integrate w.r.t. x on both sides, and then divide by μ throughout to obtain y.

<u>**Note**</u>**: Do not forget the constant of integration.**

<u>**Example 7**</u>

Consider the differential equation

$$\frac{dy}{dx} + \frac{2}{x} y = e^x . \qquad \textbf{(34)}$$

Here $P(x) = \dfrac{2}{x}, \quad Q(x) = e^x$. So, the integrating factor is, from (32),

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2 .$$

There is no need to include the constant of integration at this stage as it will cancel out afterwards. Try putting it in and see what happens, just to convince yourself.

Making use of (33), we have

$$\frac{d}{dx}(x^2 y) = x^2 e^x .$$

We now integrate both sides w.r.t x and solve for y.

$$\begin{aligned} x^2 y &= \int x^2 \, e^x dx \\ &= x^2 e^x - 2\int xe^x dx \qquad \text{[Integration by Parts].} \\ &= x^2 e^x - 2\left(xe^x - \int e^x dx\right) \end{aligned}$$

Hence,

$$x^2 y = x^2 e^x - 2xe^x + 2e^x + C ,$$

or

$$y = \frac{(2 - 2x + x^2)e^x + C}{x^2} .$$

Example 8

Consider the differential equation

$$\frac{dy}{dx} + y\sec x = \sec x\,.$$

Here, $P(x) = \sec x$, $Q(x) = \sec x$. So, the integrating factor is given by

$$\mu(x) = e^{\int \sec x\,dx} = e^{\ln[\sec x + \tan x]} = \sec x + \tan x\,.$$

Making use of (33), we have

$$\frac{d}{dx}((\sec x + \tan x)y) = (\sec x + \tan x)\sec x$$

$$= \sec^2 x + \frac{\sin x}{\cos^2 x}.$$

Integrating both sides w.r.t. x

$$(\sec x + \tan x)y = \int \sec^2 x + \frac{\sin x}{\cos^2 x}\,dx$$

$$= \tan x + \sec x + C$$

Hence,

$$y = 1 + \frac{C}{\sec x + \tan x}\,.$$

Example 9

Consider the differential equation

$$\frac{dy}{dx} - \frac{\tan y}{(1+x)} = (1+x)e^{x} \sec y \,. \qquad \textbf{(35)}$$

Multiplying (35) by $\cos y$ yields

$$\cos y \frac{dy}{dx} - \frac{\sin y}{(1+x)} = (1+x)e^{x} \,.$$

Let $\sin y = z$. Then, $\cos y \dfrac{dy}{dx} = \dfrac{dz}{dx}$

Consequently,

$$\frac{dz}{dx} - \frac{z}{(1+x)} = (1+x)e^{x} \,.$$

We see that $P(x) = \dfrac{-1}{(1+x)}$ and $Q(x) = (1+x)e^{x}$. The integrating factor is given by

$$\mu(x) = e^{\int \frac{-1}{(1+x)}dx} = e^{-\ln(1+x)} = e^{\ln(1+x)^{-1}} = \frac{1}{1+x} \,.$$

Making use of (33), we have

$$\frac{d}{dx}\frac{1}{1+x} z = \frac{1}{1+x}(1+x)e^{x}$$

$$= e^{x}.$$

We integrate w.r.t. x to get

$$\frac{1}{1+x} z = e^{x} + C \,,$$

or,

$$z = \sin y = (1+x)e^x + C(1+x),$$

that is,

$$y = \sin^{-1}\{(1+x)e^x + C(1+x)\}.$$

<u>Activity 3</u>

1. Solve the following differential equations:

(i) $\dfrac{dy}{dx} + \dfrac{y}{x} = e^{x^2}$;

(ii) $\dfrac{dy}{dx} + y\tan x = \sin x$;

(iii) $\dfrac{dy}{dx} + 4y = 2x + 3$;

(iv) $(x^3+1)\dfrac{dy}{dx} = 8x^3 + 2x + 1 - 3x^2 y$;

(v) $\cos x\dfrac{dy}{dx} + y\sin x = 2x^3\cos^2 x$;

(vi) $\sin y\dfrac{dy}{dx} = \cos y(1 - x\cos y)$ [Hint: Divide by $\cos^2 y$ and let $\sec y = z$];

3.7 BERNOULLI'S EQUATION

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \ , \qquad \textbf{(36)}$$

with $n \neq 0$ and $n \neq 1$, n being a real number, is known as a Bernoulli differential equation. It's a non-linear equation.

Divide both sides of (36) by y^n to obtain

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \, . \qquad \textbf{(37)}$$

Put $z = y^{1-n}$. Then $\dfrac{dz}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$. So, $y^{-n}\dfrac{dy}{dx} = \dfrac{1}{1-n}\dfrac{dz}{dx}$. Now, (37) reduces to

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x),$$

or,

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x) \, . \qquad \textbf{(38)}$$

Equation (38) is a first-order linear differential equation. It can be solved for z by the method of the last section.

Replace z by y^{1-n} to get the required solution of (36).

<u>Example 10</u>

Consider the differential equation

$$\frac{dy}{dx}+\frac{1}{x-2}y=5(x-2)y^{\frac{1}{2}} \ . \qquad \textbf{(39)}$$

This is a Bernoulli equation with $n=\frac{1}{2}$, $P(x)=\frac{1}{x-2}$ and $Q(x)=5(x-2)$.

To transform (39) into a linear equation, we first divide by $y^{\frac{1}{2}}$ to obtain

$$y^{-\frac{1}{2}}\frac{dy}{dx}+\frac{1}{x-2}y^{\frac{1}{2}}=5(x-2). \qquad \textbf{(40)}$$

Then, we put $z=y^{\frac{1}{2}}$. Since, $\frac{dz}{dx}=\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}$, (40) becomes

$$\frac{dz}{dx}+\frac{1}{2(x-2)}z=\frac{5(x-2)}{2}. \qquad \textbf{(41)}$$

Equation (41) is linear. The integrating factor is given by

$$\mu(x)=e^{\int\frac{1}{2(x-2)}dx}=e^{\frac{1}{2}\ln(x-2)}=(x-2)^{\frac{1}{2}}.$$

So,

$$\frac{d}{dx}((x-2)^{\frac{1}{2}}z)=\frac{5}{2}(x-2)(x-2)^{\frac{1}{2}},$$

$$\Rightarrow (x-2)^{\frac{1}{2}}z=\frac{5}{2}\int(x-2)^{\frac{3}{2}}\ dx$$

$$=(x-2)^{\frac{5}{2}}+C.$$

Hence,

$$z=(x-2)^2+C(x-2)^{-\frac{1}{2}}.$$

By substituting $z=y^{\frac{1}{2}}$, we obtain

$$\sqrt{y}=(x-2)^2+\frac{C}{\sqrt{(x-2)}}.$$

<u>Example 11</u>

Consider the differential equation

$$\cos x\frac{dy}{dx}+y\sin x=(2x\cos^2 x)y^3. \qquad \textbf{(42)}$$

Equation (42) can be rewritten as

$$\frac{dy}{dx} + y\cot x = (2x\cos x)y^3 \qquad \textbf{(43)}$$

This is a Bernoulli equation with $n = 3$, $P(x) = \cot x$ and $Q(x) = 2x\cos x$.

We divide both sides by y^3 to obtain

$$y^{-3}\frac{dy}{dx} + y^{-2}\cot x = 2x\cos x. \qquad \textbf{(44)}$$

Let $z = y^{-2}$. Then, $\dfrac{dz}{dx} = -2y^{-3}\dfrac{dy}{dx}$.

Substituting in (44), we obtain

$$\frac{dz}{dx} + (-2\cot x)\,z = -4x\cos x.$$

The integrating factor is given by

$$e^{\int -2\cot x\,dx} = e^{-2\ln\sin x} = \text{cosec}^2\ x.$$

Therefore,

$$\frac{d}{dx}[(\text{cosec}^2\ x)\ z] = \text{cosec}^2\ x\ (-4x\cos x) = -4x\cos x\cos ec^2 x.$$

Integrating w.r.t. *x*, we obtain

$$(\text{cosec}^2\ x)z = \int -4x\cos x\cos ec^2 x\ dx$$

Using the following substitution, $u = -4x$ and $\dfrac{dv}{dx} = \cos x\cos ec^2 x$, we get

$$du = -4\,dx \text{ and } v = -\cos ec\, x.$$

Hence,

$$(\cos ec^2 x)z = 4x\cos ec\, x - 4\int \cos ec\, x\, dx$$

$$= 4x\cos ec\, x - 4\ln(\cos ec\, x - \cot x) + C$$

Putting $z = y^{-2}$, we have the general solution

$$y^{-2} = 4x\sin x - 4\sin^2 x\ln(\operatorname{cosec} x - \cot x) + C\sin^2 x .$$

Activity 4

1. Solve the following differential equations:

(i) $\dfrac{dy}{dx} + y^3 x + y = 0$;

(ii) $\dfrac{dy}{dx} + y = e^x y^{-2}$;

(iii) $\dfrac{dy}{dx} + \dfrac{y}{x} = -4xy^{-2}$;

(iv) $\dfrac{dy}{dx} - xy = -e^{-x^2} y^3$;

(v) $y^2\,dx - (x^2 - xy)\,dy = 0$, [Hint: Treat x as dept. var.]

(vi) $\dfrac{dy}{dx} + 2y\tan x = y^2\tan^2 x, \quad y(0) = 1$;

3.8 SUMMARY

After successfully completing this Unit, you should be able to identify first-order differential equations and use an appropriate method to find the solutions.

3.9 SUPPLEMENTARY EXERCISES

1. Solve the following differential equations:

(i) $(-x - y - 4)\,dx + (x - 4y - 6)\,dy = 0$;

(ii) $\dfrac{dy}{dx} = \dfrac{2x + 7y}{2x - 2y}$ $\quad y(1) = 2$;

(iii) $(x^3 + 1)\dfrac{dy}{dx} + 6x^2 y = 6x^2$;

(iv) $\dfrac{dy}{dx} + y\tan x + \sin x = 0$;

(v) $\dfrac{dy}{dx} = \dfrac{2y}{x} - x^2 y^2$

3.10 ANSWERS TO ACTIVITIES & SUPPLEMENTARY EXERCISES

Activity 1

1. (i) $y = 2y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx}$;

(ii) $\frac{dy}{dx} = y\cot x$;

(iii) $y = \frac{x}{2}\frac{dy}{dx} + \frac{1}{4x^2}\left(\frac{dy}{dx}\right)^2$;

(iv) $\frac{dy}{dx} + y\tan x = 8x + 4x^2\tan x$;

(v) $y = x^2(1 - \ln x)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$;

(vi) $\frac{d^2y}{dx^2} + 4y = 0$.

2. (i) $y = A\sqrt{x^4 + 1}$;

(ii) $\cos y - e^{-y} = x^4 + 2x^3 + x + A$;

(iii) $y = A\left(\frac{x}{4 - x}\right)^{1/4}$;

(iv) $y = \tan^{-1}\left(\frac{2}{3}\tan x\right) + A$;

(v) $2y = 3\left(\frac{x - 2}{x}\right) - 5$;

(vi) $y = 1 - \frac{3}{\tan x - 1}$.

Activity 2

1. (i) $\dfrac{1}{3}\left(1+\left(\dfrac{y}{x}\right)^2\right)^{\frac{3}{2}} = \ln Ax$;

   (ii) $(y^3 - 3x^2 y)^{-\frac{1}{3}} = A$;

   (iii) $y = x\sin^{-1}(Ax)$;

   (iv) $y = 2 - 9x + 3\tan[3(x+A)]$;

   (v) $(y+2) = \sqrt{3}(x+1)\tan\left\{-\sqrt{3}\ln\left[A\sqrt{3(x+1)^2 + (y+2)^2}\right]\right\}$;

   (vi) $(y+2x-2)^2 = A\left(y+x-\dfrac{3}{2}\right)^3$.

Activity 3

1. (i) $y = \dfrac{1}{2x}e^{x^2} + \dfrac{A}{x}$;

   (ii) $y = \cos x \ln[\sec x] + A\cos x$;

   (iii) $y = \dfrac{x}{2} + \dfrac{5}{8} + Ae^{-4x}$;

   (iv) $y = \dfrac{2x^4 + x^2 + x + A}{x^3 + 1}$;

   (v) $y = \dfrac{1}{2}x^4\cos x + A\cos x$;

   (vi) $y = \cos^{-1}\left(\dfrac{1}{x+1+Ae^x}\right)$.

Activity 4

1. (i) $y = \dfrac{1}{\sqrt{-x - \frac{1}{2} + Ae^{2x}}}$;

(ii) $y^3 = \frac{3}{4}e^x + Ae^{-3x}$;

(iii) $y^3 = -\frac{12}{5}x^2 + Ax^{-3}$;

(iv) $y^{-2} = (2x + A)e^{-x^2}$;

(v) $x^{-1} = Ay + 1/(2y)$;

(vi) $2y^{-1} = \tan x + (2 - x)\sec^2 x$.

Supplementary Exercises

1. (i) $2(y+2) = (x+2)\tan\{2\ln[A\sqrt{4(y+2)^2 + (x+2)^2}]\}$;

(ii) $\dfrac{2y + x}{(y+2x)^2} = \dfrac{5}{16}$;

(iii) $y = \dfrac{x^6 + 2x^3 + A}{(x^3+1)^2}$;

(iv) $y = A\cos x + \cos x \ln[\cos x]$;

(v) $y = \dfrac{5x^2}{x^5 + 5A}$.