Master of Power Engineering – 1st Year

Steady State and Transient Conduction

Heat and Mass Transfer

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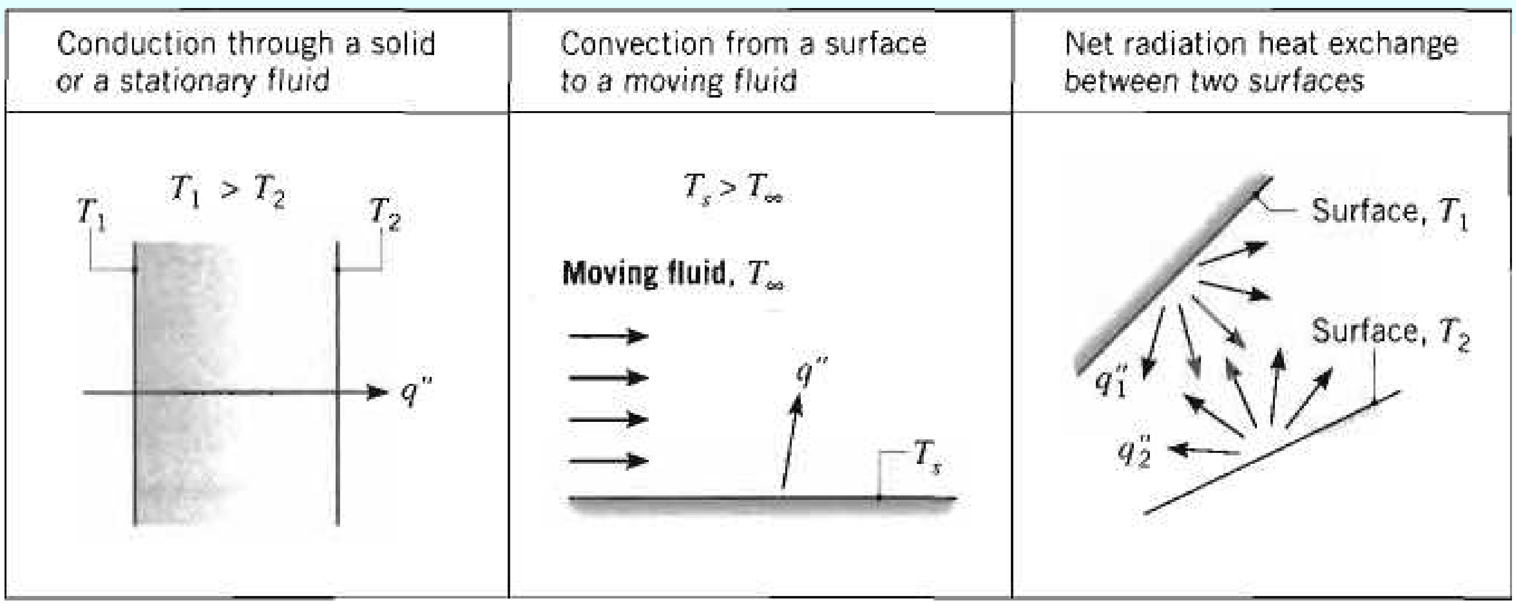
Contents?

- **Steady State Conduction**, Analysis of fins, Critical thickness of insulation, Systems with internal heat generation.
- **Transient conduction analysis:** Application of numerical methods to conduction problems.
- **Theory of heat convection.** Conservation equation of energy, mass & momentum and their analogies.
- **Significance of various dimensionless numbers**, laminar & turbulent boundary layer concept, thermal boundary layer, forced convection inside tubes and ducts, Forced convection over external bodies, Natural convection.
- **Boiling and Condensation**
- **Radiation** properties and laws, Radiation exchange among black and gray bodies, Electrical analogy, Radiation through participating gases.
- **Mass transfer** by convection and molecular diffusion, Fick's laws, Calculation of mass transfer coefficient, Interface mass transfer.

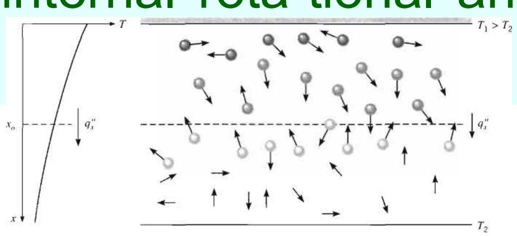
Introduction

- *What is heat transfer? How is heat transferred? Why is it important?*
- Heat transfer (or heat) is thermal energy in transit due to a spatial temperature difference.
- When a temperature gradient exists in a stationary medium, which may be a solid or a fluid, we use the term ***conduction*** to refer to the heat transfer that will occur across the medium. In contrast, the term ***convection*** refers to heat trans-fer that will occur between a surface and a moving fluid when they are at different temperatures. The third mode of heat transfer is termed ***thermal radiation***.



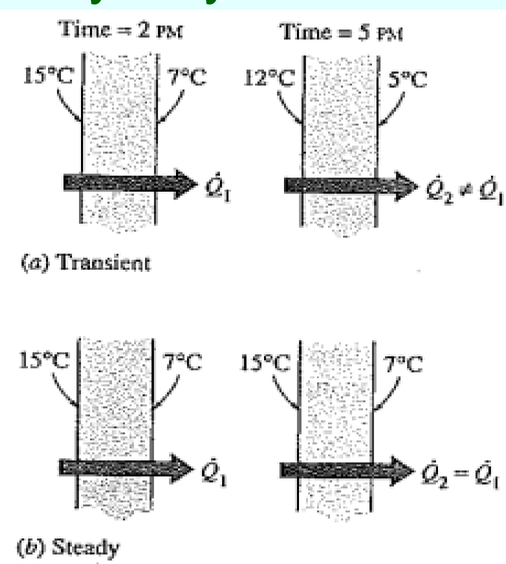
Introduction

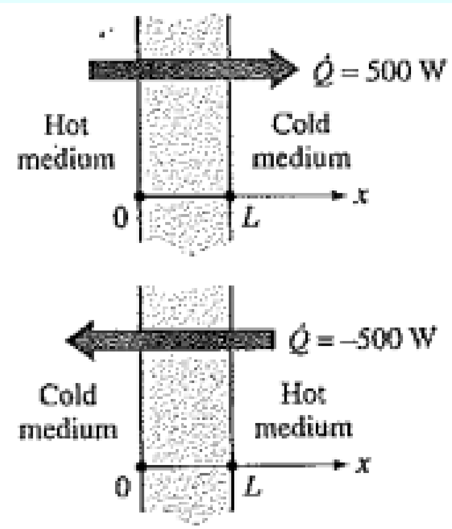
- The driving force for any form of heat transfer is the ***temperature***, and the larger the temperature difference, larger the heat transfer rate.
- The word ***conduction***, we should immediately conjure up concepts of ***atomic*** and ***molecular activity***, for it is processes at these levels that sustain this mode of heat transfer. Conduction may be viewed as the transfer of energy from the more energetic to the less energetic particles of a substance due to interactions be-tween the particles.
- The conduction can take place in liquid and gases as well as in solid provided that there is no bulk motion involved.
- Consider a gas in which there exists a temperature gradient and assume that there is *no bulk, or macroscopic, motion.* The gas may occupy the space between two surfaces that are maintained at different temperatures. We associate the temperature at any point with the energy of gas molecules in proximity to the point. This energy is related to the random translational motion, as well as to the internal rota-tional and vibrational motions, of the molecules.



Introduction

- Higher temperatures are associated with higher molecular energies, and when neighboring molecules collide, as they are constantly doing, a transfer of energy from the more energetic to the less energetic molecules must occur.
- The heat transfer problem are often classified as ***steady state*** and ***transient*** (or unsteady). The term steady implies no change with time at any point within the medium, while transient implies variation with time of time dependence. Therefore, temperature or heatflux remains unchanged with time during steady state heat transfer through any medium at any location, although both quantities may vary from one location to another.





Introduction

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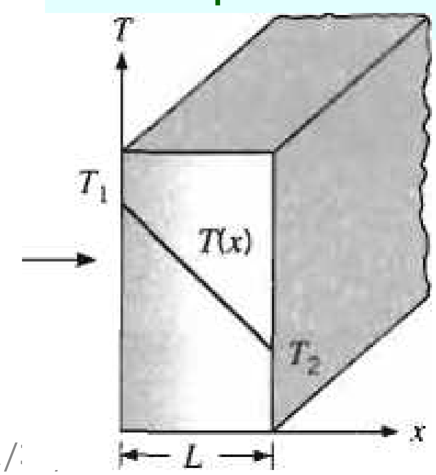
- For example, heat transfer through walls of a house is steady when the conditions inside the house and the outdoors remain constant for several hours.
- The cooling of an apple in a refrigerator, on the other hand, is a transient heat transfer process since the temperature at any fixed point within the apple will change with time during cooling.

Introduction

- During transient heat transfer, the temperature normally varies with time as well as position.
- In a special case of variation with time but not with position, the temperature of medium changes ***uniformly*** with time. Such heat transfer systems are call ***lumped systems***. A small metal object such as a thermocouple junction or a thin copper wire, for example, can be analyzed as a lumped system during a heating or cooling process.
- Although heat transfer and temperature are closely related, they are of a different nature. Unlike temperature, heat transfer has direction as well as magnitude, and thus it is a vector quantity.
- Thus, we must specify both direction and magnitude in order to describe heat transfer completely at a point.
- In general, accepted convention is that heat transfer in the positive direction of a coordinate axis is positive and in the opposite direction it is negative. Thus, a positive quantity indicates heat transfer in positive direction and a negative quantity indicates heat transfer in the negative direction.

Introduction

- **Conduction heat transfer:** The exposed end of a metal spoon suddenly immersed in a cup of hot coffee will eventually be warmed due to the conduction of energy through the spoon. On a winter day there is significant energy loss from a heated room to the outside air. This loss is principally due to conduction heat transfer through the wall that separates the room air from the out-side air.
- It is possible to quantify heat transfer processes in terms of appropriate *rate equations.* These equations may be used to compute the amount of energy being transferred per unit time. For heat conduction, the rate equation is known as ***Fourier's law.*** For the one-dimensional plane wall, having a temperature distribution ***T(x)***, the rate equation is expressed as.

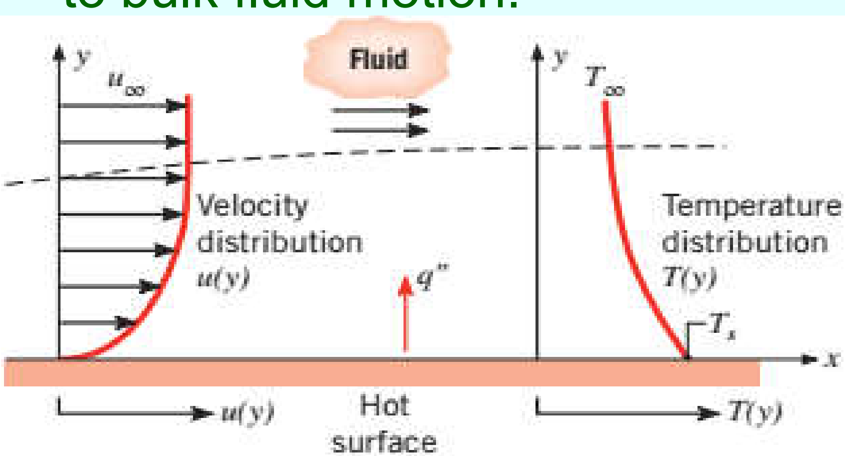


$$q_x'' = -k\frac{dT}{dx}$$

The ***heat flux q"*** **(W/m^2)** is the heat transfer rate in the ***x*** direction *per* unit area *per-pendicular* to the direction of transfer, and it is proportional to the ***temperature gradient, dT/dx,*** in this direction. The parameter ***k*** is a *transport* property known as the ***thermal conductivity*** (W/m•K) and is a characteristic of the wall material. The minus sign is a consequence of the fact that heat is transferred in the direction of de-creasing temperature.

Introduction

- **Convection**: The convection heat transfer *mode* is comprised of *two mechanisms.* In addition to energy transfer due to *random molecular motion (diffusion*), energy is also trans-ferred by the *bulk,* or *macroscopic, motion* of the fluid. This fluid motion is associ-ated with the fact that, at any instant, large numbers of molecules are moving col-lectively or as aggregates. Such motion, in the presence of a temperature gradient, contributes to heat transfer. Because the molecules in the aggregate retain their ran-dom motion, the total heat transfer is then due to a superposition of energy transport by the random motion of the molecules and by the bulk motion of the fluid. It is customary to use the term *convection* when referring to this cumulative transport and the term *advection* when referring to transport due to bulk fluid motion.



We are especially interested in convection heat transfer, which occurs between a fluid in motion and a bounding surface when the two are at different temperatures

Introduction

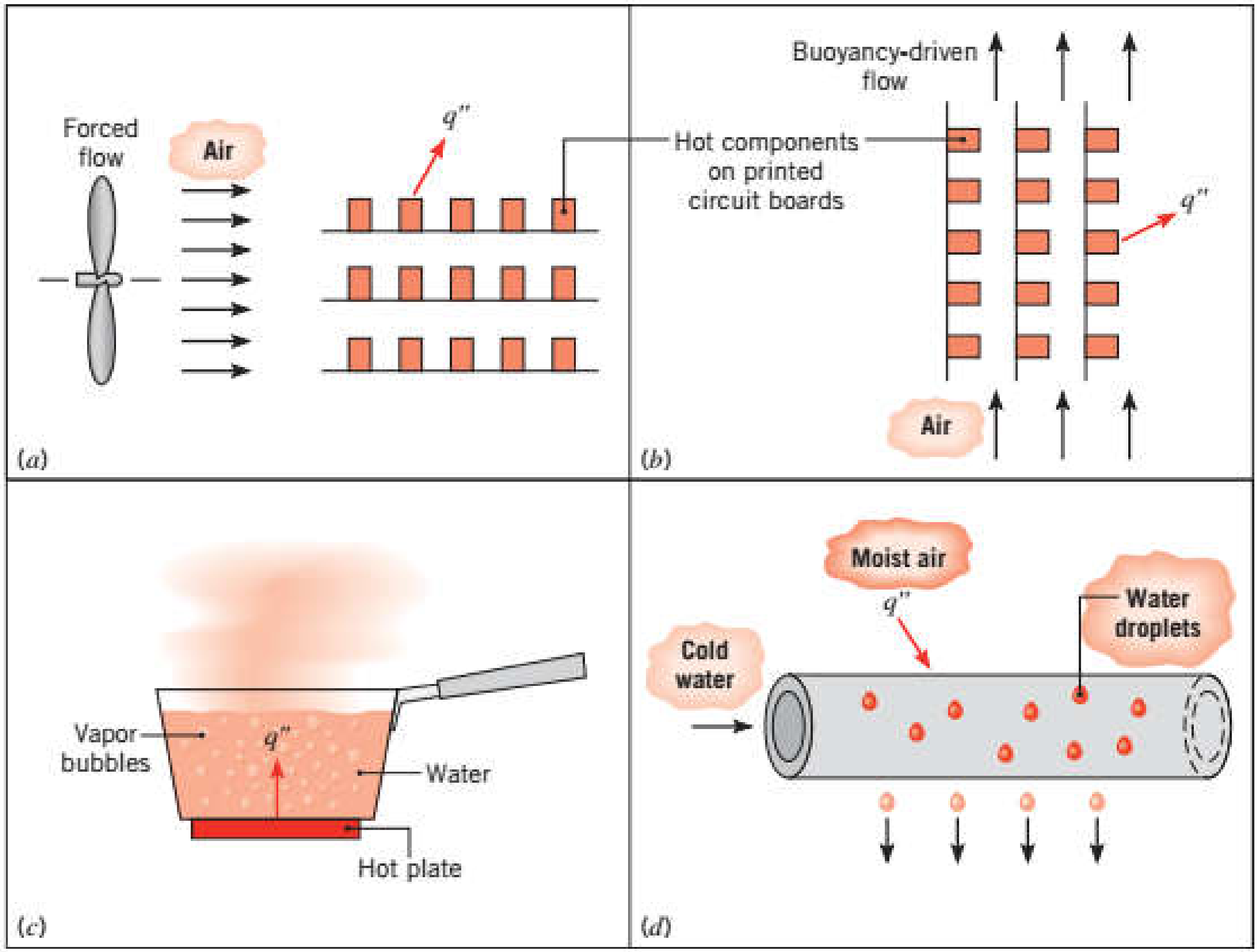
- A consequence of the fluid-surface interaction is the development of a region in the fluid through which the velocity varies from zero at the surface to a finite value u_∞ associated with the flow. This region of the fluid is known as the ***hydrodynamic, or velocity, boundary layer.*** Moreover, if the surface and flow temperatures differ, there will be a region of the fluid through which the temperature varies from T_s at $y = 0$ to T_∞ in the outer flow. This region, called the ***thermal boundary layer***, may be smaller, larger, or the same size as that through which the velocity varies. In any case, if $T_s > T_\infty$, convec-tion heat transfer will occur from the surface to the outer flow.
- The convection heat transfer mode is sustained both by random molecular mo-tion and by the bulk motion of the fluid within the boundary layer. The contribution due to random molecular motion (diffusion) dominates near the surface where the fluid velocity is low. In fact, at the interface between the surface and the fluid ($y = 0$), the fluid velocity is zero and heat is transferred by this mechanism only.

Introduction

- Convection heat transfer may be classified according to the nature of the flow. We speak of *forced convection* when the flow is caused by external means, such as by a fan, a pump, or atmospheric winds.
- As an example, consider the use of a fan to provide ***forced*** convection air cooling of hot electrical components on a stack of printed circuit boards. In contrast, for ***free (or natural) convection*** the flow is induced by buoyancy forces, which are due to density differences caused by temperature variations in the fluid.
- An example is the ***free convection*** heat transfer that occurs from hot components on a vertical array of circuit boards in air. Air that makes contact with the components experiences an increase in tem-perature and hence a reduction in density. Since it is now lighter than the surround-ing air, buoyancy forces induce a vertical motion for which warm air ascending from the boards is replaced by an inflow of cooler ambient air.

Introduction

- **Convection heat transfer processes. (a) Forced convection. (b) Natural convection. (c) Boiling. (d) Condensation**



Introduction

- While we have presumed *pure* forced convection in Figure a and *pure* nat-ural convection in Figure b, conditions corresponding to *mixed* (*combined) forced* and *natural convection* may exist.
- For example, if velocities associated with the flow of **Figure a** are small and/or buoyancy forces are large, a secondary flow that is comparable to the imposed forced flow could be induced. In this case, the buoyancy-induced flow would be normal to the forced flow and could have a significant effect on convection heat transfer from the components.
- In **Figure b**, mixed convection would result if a fan were used to force air upward between the circuit boards, thereby assisting the buoyancy flow, or downward, thereby opposing the buoyancy flow.

Introduction

- The convection heat transfer mode as energy transfer occur-ring within a fluid due to the combined effects of conduction and bulk fluid motion. Typically, the energy that is being transferred is the ***sensible***, or internal thermal, energy of the fluid. However, there are convection processes for which there is, in addition, ***latent* heat** exchange. This latent heat exchange is generally associated with a phase change between the liquid and vapor states of the fluid. Two special cases of interest are ***boiling*** & ***condensation***.
- For ex., convection heat transfer results from fluid motion induced by vapor bubbles generated at the bottom of a pan of boiling water (Fig. c) or by the condensation of water vapor on the outer surface of cold water pipe (Fig. *d).*
- Regardless of the particular nature of the convection heat transfer process, the appropriate rate equation is of the form $q'' = h(T_s - T_\infty)$

  where ***q"***, the convective *heat flux* (W/m^2), is proportional to the difference between the surface and fluid temperatures, T_s and T_∞, respectively. This expression is known as ***Newton's law of cooling***, and the parameter h (W/m^2•K) is termed the *convection heat transfer coefficient.* It depends on conditions in the boundary layer, which are influenced by surface geometry, the nature of the fluid motion, and an as-sortment of fluid thermodynamic and transport properties.

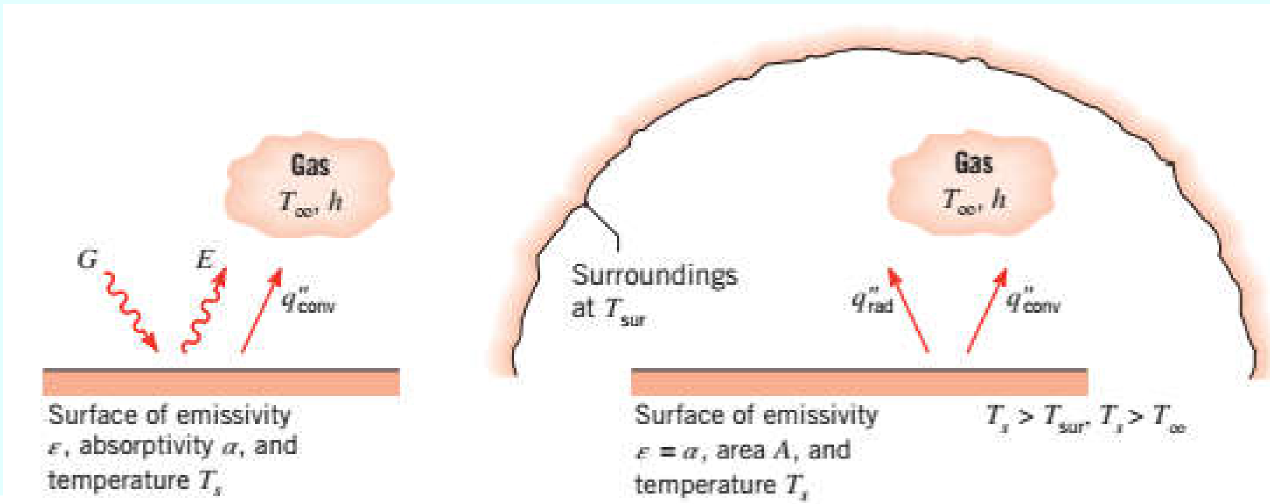
Introduction

- When Equation $q'' = h(T_s - T_\infty)$ is used, the convection heat flux is presumed to be ***positive*** if heat is transferred *from* the surface ($T_s > T_\infty$) and *negative* if heat is transferred *to* the surface ($T_\infty > T_s$). However, if $T_\infty > T_s$, there is nothing to preclude us from ex-pressing Newton's law of cooling as. $q'' = h(T_\infty - T_s)$ in which case heat transfer is positive if it is to the surface.
- **Radiation:** Thermal radiation is energy *emitted* by matter that is at a nonzero temperature. Al-though we will focus on radiation from solid surfaces, emission may also occur from liquids and gases. Regardless of the form of matter, the emission may be at-tributed to changes in the electron configurations of the constituent atoms or mole-cules. The energy of the radiation field is transported by electromagnetic waves (or alternatively, photons). While the transfer of energy by conduction or convection requires the presence of a material medium, radiation does not. In fact, radiation transfer occurs most efficiently in a vacuum.
- Radiation that is *emitted* by the surface originates from the thermal energy of matter bounded by the surface, and the rate at which energy is released per unit area (W/m^2) is termed the surface *emissive power E.*

Introduction

- There is an upper limit to the emissive power, which is prescribed by the *Stefan-Boltzmann law* $E_b = \sigma T_s^4$

  where T_s is the *absolute temperature* (K) of the surface and σ is the ***Stefan-Boltzmann constant*** (σ = 5.67 X 10^{-8} W/m^2 • K^4). Such a surface is called an ideal radiator or blackbody.



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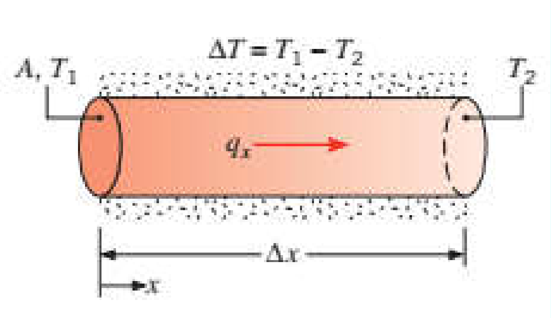
Conduction Rate Equation

- ***Conduction Rate Equation:*** Fourier's law is phenomenological; that is, it is developed from observed phenomena rather than being derived from first principles.
- A cylindrical rod of known material is insulated on its lateral surface, while its end faces are maintained at different temperatures, with $T_1 > T_2$.



- We are able to measure the heat transfer rate $\boldsymbol{q_x}$, and we seek to determine how $\boldsymbol{q_x}$ depends on the following variables: $\boldsymbol{\Delta T}$, the temperature difference; $\boldsymbol{\Delta x}$, the rod length; and A, the cross-sectional area.
- Imagine first holding $\boldsymbol{\Delta T}$ and $\boldsymbol{\Delta x}$ constant and varying $\boldsymbol{A}$. So, $\boldsymbol{q_x}$ is directly proportional to A. Similarly, holding $\boldsymbol{\Delta T}$ and A constant, we observe that $\boldsymbol{q_x}$ varies inversely with $\boldsymbol{\Delta x}$. Finally, holding A and $\boldsymbol{\Delta x}$ constant, we find that $\boldsymbol{q_x}$ is directly proportional to $\boldsymbol{\Delta T}$. The collective effect is then

$$q_x \propto A \frac{\Delta T}{\Delta x}$$

Conduction Rate Equation

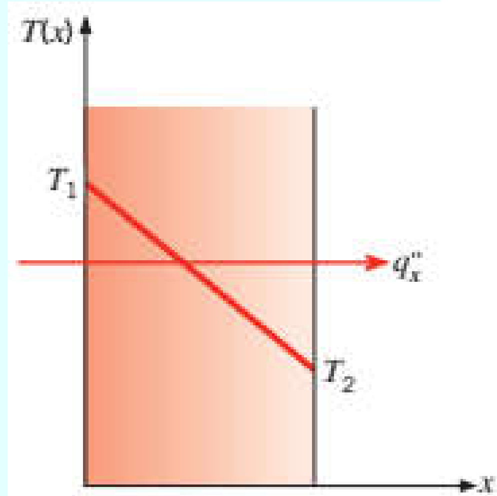
- The proportionality may be converted to an equality by introducing a coefficient that is a measure of the material behavior. Hence, we write.

$$q_x = kA\frac{\Delta T}{\Delta x}$$ the heat *rate* $$q_x = -kA\frac{dT}{dx}$$ Or the heat *flux* $$q''_x = \frac{q_x}{A} = -k\frac{dT}{dx}$$

where ***k***, the *thermal conductivity* (W/m K), is an important *property* of the material. Evaluating this expression in the limit as ***Δx*** → 0, we obtain for the heat *rate*.

- Recall that the ***minus sign*** is necessary because heat is always transferred in the direction of ***decreasing temperature***.

- ***Fourier's law***, *implies that the heat flux is a directional quantity*. In particular, the direction of q_x'' is *normal* to the cross-sectional area ***A***. Or, more generally, the direction of heat flow will always be normal to a surface of constant temperature, called an *isothermal* surface. Figure illustrates the direction of heat flow q_x'' in a plane wall for which the *temperature gradient* ***dT/dx*** is negative. From Equation, it follows that q_x'' is positive. Note that the isothermal surfaces are planes normal to the ***x***-direction.



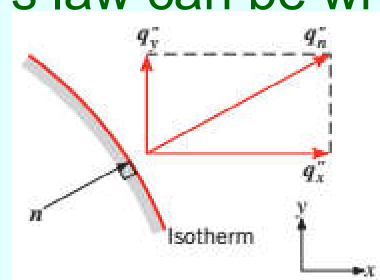
Conduction Rate Equation

- Recognizing that the heat flux is a vector quantity, we can write a more general statement of the conduction rate equation (*Fourier's law*) as follows:

$$\boldsymbol{q}'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial x} + \boldsymbol{j}\frac{\partial T}{\partial y} + \boldsymbol{k}\frac{\partial T}{\partial z}\right)$$

where ∇ is the three-dimensional del operator, *i*, *j*, and *k* are the unit vectors in the *x*, *y*, and *z* directions, and $T(x, y, z)$ is the scalar temperature field. It is implicit in above Equation that the heat flux vector is in a direction perpendicular to the isothermal surfaces. Thus, Fourier's law can be written

$$\boldsymbol{q}'' = q''_n \boldsymbol{n} = -k\frac{\partial T}{\partial n}\boldsymbol{n}$$



where $\boldsymbol{q_n''}$ is the heat flux in a direction $\boldsymbol{n}$, which is normal to an *isotherm*, and $\boldsymbol{n}$ is the unit normal vector in that direction. This is illustrated for the 2-D case in Figure. The heat transfer is sustained by a temperature gradient along $\boldsymbol{n}$. Note also that the heat flux vector can be resolved into components such that, in Cartesian coordinates, expression for $\boldsymbol{q''}$

$$\boldsymbol{q}'' = \boldsymbol{i}q''_x + \boldsymbol{j}q''_y + \boldsymbol{k}q''_z$$

$$q''_x = -k\frac{\partial T}{\partial x} \qquad q''_y = -k\frac{\partial t}{\partial y} \qquad q''_z = -k\frac{\partial T}{\partial z}$$

Conduction Rate Equation

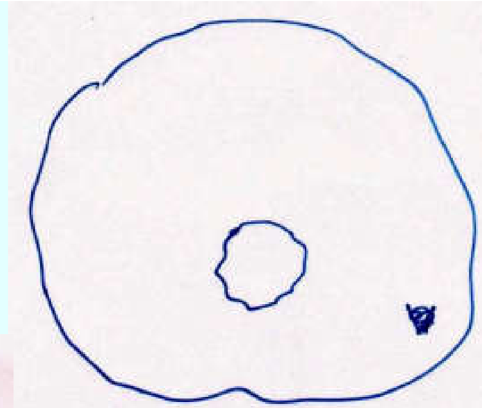
- **Thermodynamic properties:** The product ρc_p (J/m^3 K), commonly termed the *volumetric heat capacity*, measures the ability of a material to store thermal energy. Because substances of large density are typically characterized by small specific heats, many solids and liquids, which are very good energy storage media, have comparable heat capacities (ρc_p > 1 MJ/m^3 K).
- In heat transfer analysis, the ratio of the thermal conductivity to the heat capacity is an important property termed the ***thermal diffusivity α***, which has units of m^2/s:

$$\alpha = \frac{k}{\rho c_p}$$

It measures the ability of a material to conduct thermal energy relative to its ability to store thermal energy. Materials of large α will respond quickly to changes in their thermal environment, whereas materials of small α will respond more sluggishly, taking longer to reach a new equilibrium condition.

Generalized conduction equation

- Let us consider a 3-D body in which temperature varies in three coordinate direction within the body. Also assume there are uniform heat generation within the body. Considering arbitrary volume **V**, we can write first of thermodynamic for the control volume



3-D body

$$\frac{dU}{dt} = \dot{Q}$$ [U is the internal energy or stored energy manifested by temp. or internal thermal energy]

Here $\dot{Q}$ is the having two components:

i) heat transfer rate across the boundary surface of V.

ii) volumetric heat generation.

$$\dot{Q}_1 = -\int_{A_s} \bar{\dot{q}}'' \cdot d\bar{A}_s = -\int_{A_s} \bar{\dot{q}}'' \cdot \tilde{n}\, dA_s = -\int_V \nabla \cdot \bar{\dot{q}}''\, dV$$ [Gauss's divergence theorem]

$$\dot{Q}_2 = \int_V \dot{q}_g'''\, dV$$

$$\frac{dU}{dt} = \int_V \rho c \frac{\partial T}{\partial t}\, dV.$$

Generalized conduction equation

$$\therefore \int_V \rho c \frac{\partial T}{\partial t}\, dV = -\int_V \nabla\cdot\bar{\dot{q}}''\, dV + \int_V \dot{q}_g'''\, dV$$

$$\Rightarrow \int_V \left[\rho c \frac{\partial T}{\partial t} + \nabla\cdot\bar{\dot{q}}'' - \dot{q}_g'''\right] dV = 0$$

Since the volume V is arbitrary, the integrand must vanish identically. Thus, we get

$$\rho c \frac{\partial T}{\partial t} + \nabla\cdot\bar{\dot{q}}'' - \dot{q}_g''' = 0$$

From Fourier's law $\bar{\dot{q}}'' = -k\nabla T$

$$\therefore \boxed{\rho c \frac{\partial T}{\partial t} = \nabla\cdot(k\nabla T) + \dot{q}_g'''}$$

Generalized conduction equation

Cartesian coordinate,

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}_g'''$$

Considering k as constant,

$$\rho c \frac{\partial T}{\partial t} = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial y^2}\right] + \dot{q}_g'''$$

$$\Rightarrow \frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}_g'''}{k}$$

$\alpha \Rightarrow$ thermal diffusivity

Thermal diffusivity is a measure of how fast heat diffuses through a body.

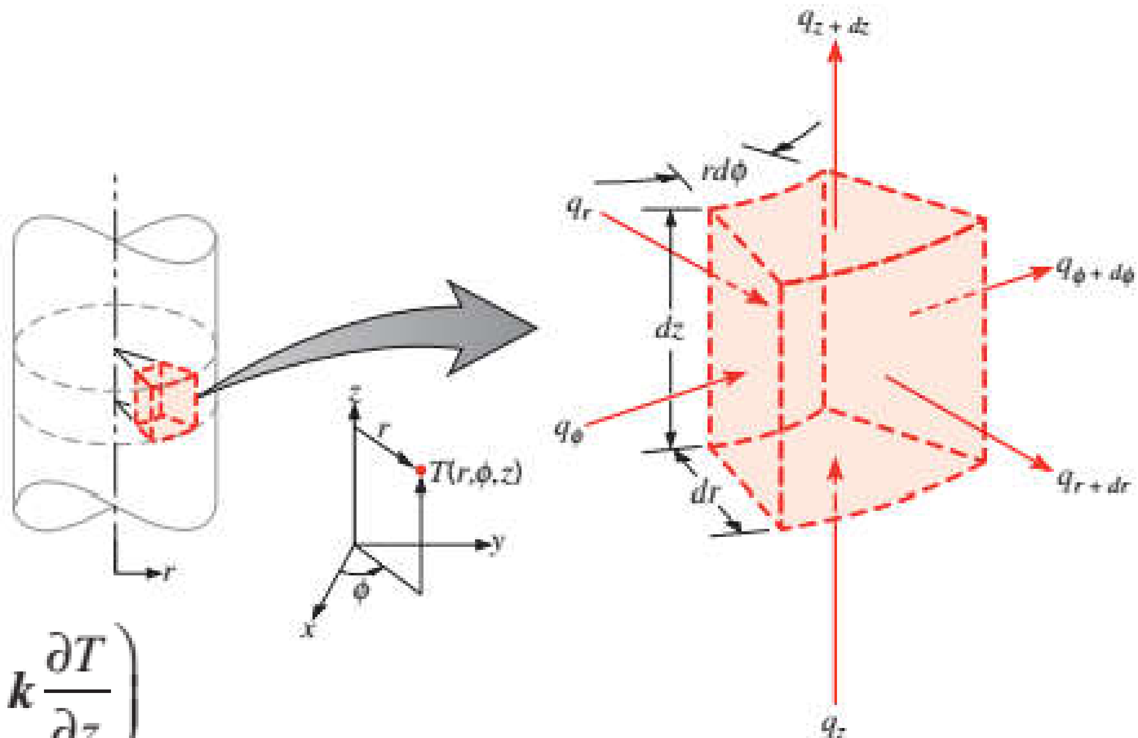
$\Downarrow$

question of time

The equation, therefore states that *at any point in the medium the net rate of energy transfer by conduction into a unit volume plus the volumetric rate of thermal energy generation must equal the rate of change of thermal energy stored within the volume.*

Generalized conduction equation

- **Cylindrical Coordinates:** When the del operator ∇ of Equation is expressed in cylindrical coordinates, with *i*, *j*, and *k* representing the unit vectors in the *r*, φ, and *z* directions, the general form of the heat flux vector and hence of Fourier's law is



$$q'' = -k\nabla T = -k\left(i\frac{\partial T}{\partial r} + j\frac{1}{r}\frac{\partial T}{\partial \phi} + k\frac{\partial T}{\partial z}\right)$$

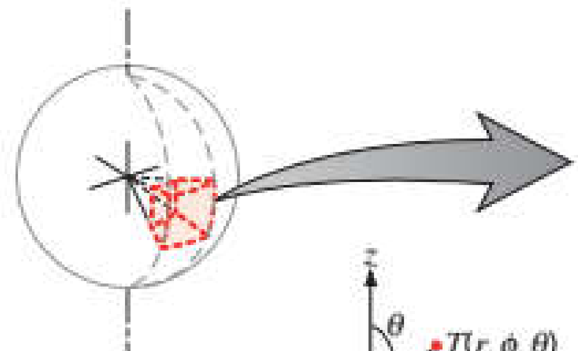
where $\quad q''_r = -k\frac{\partial T}{\partial r} \qquad q''_\phi = -\frac{k}{r}\frac{\partial T}{\partial \phi} \qquad q''_z = -k\frac{\partial T}{\partial z}$

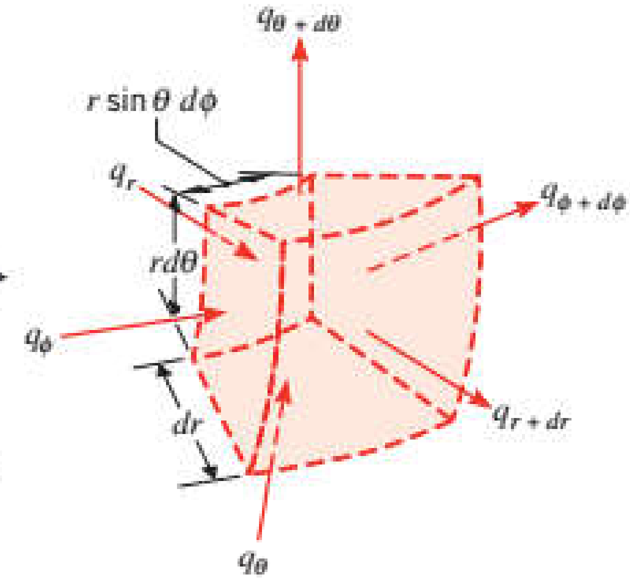
Applying an energy balance to the differential control volume, the general form of the heat equation is:

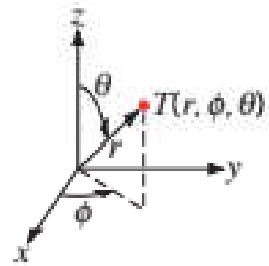
$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t}$$

Generalized conduction equation

- **Spherical Coordinates:** In spherical coordinates, with *i*, *j*, and *k* representing the unit vectors in the *r*, θ, and φ directions, the general form of the heat flux vector and Fourier's law is







$$\boldsymbol{q''} = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial r} + \boldsymbol{j}\frac{1}{r}\frac{\partial T}{\partial \theta} + \boldsymbol{k}\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)$$

where $q_r'' = -k\dfrac{\partial T}{\partial r} \qquad q_\theta'' = -\dfrac{k}{r}\dfrac{\partial T}{\partial \theta} \qquad q_\phi'' = -\dfrac{k}{r\sin\theta}\dfrac{\partial T}{\partial \phi}$

Applying an energy balance to the differential control volume, the general form of the heat equation is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

Conduction Rate Equation

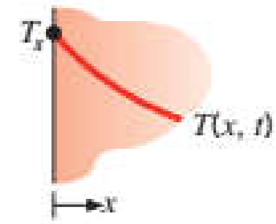
- **Boundary and Initial Conditions:** To determine the temperature distribution in a medium, it is necessary to solve the appropriate form of the heat equation. Such a solution depends on the physical conditions existing at the *boundaries* of the medium and, if the situation is time dependent, on conditions existing in the medium at some *initial time.*
- With regard to the *boundary conditions,* there are several common possibilities that are simply expressed in mathematical form. Because the heat equation is second order in the spatial coordinates, two boundary conditions must be expressed for each coordinate needed to describe the system. Because the equation is first order in time, however, only one condition, termed the *initial condition*, must be specified.
- *Three kinds of boundary conditions* commonly encountered in heat transfer are summarized in Table. The conditions are specified at the surface $x = 0$ for a one dimensional system. Heat transfer is in the positive x-direction with the temperature distribution, which may be time dependent, designated as $T(x, t)$.

Conduction Rate Equation

- **Table : Boundary conditions for the heat diffusion equation at the surface (x = 0):**

1. Constant surface temperature

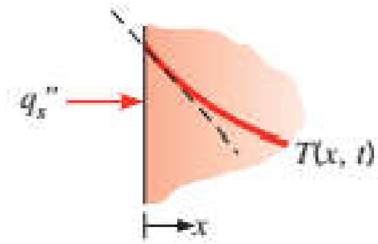
$$T(0, t) = T_s$$



2. Constant surface heat flux

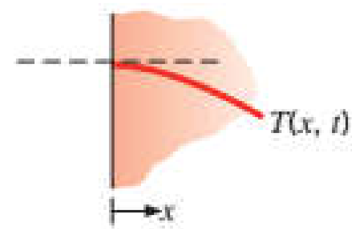
(a) Finite heat flux

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = q_s''$$



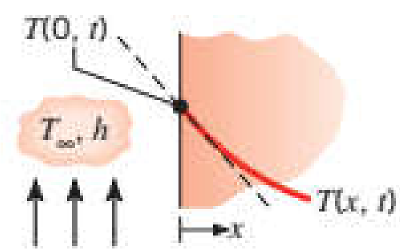
(b) Adiabatic or insulated surface

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$



3. Convection surface condition

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0, t)]$$



Conduction Rate Equation

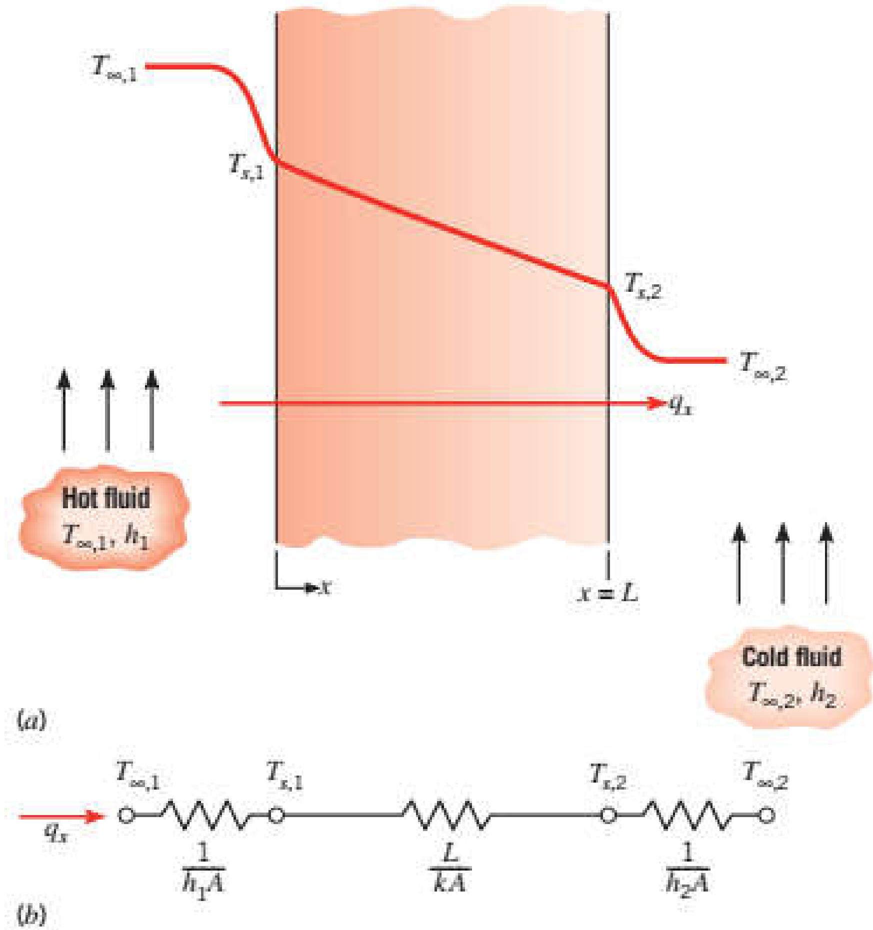
- The first condition corresponds to a situation for which the surface is maintained at a fixed temperature T_s. It is commonly termed a ***Dirichlet condition***, or a boundary condition of the ***first kind***. It is closely approximated, for example, when the surface is in contact with a melting solid or a boiling liquid. In both cases, there is heat transfer at the surface, while the surface remains at the temperature of the phase change process.
- The second condition corresponds to the existence of a fixed or constant heat flux q_s'' at the surface. This heat flux is related to the temperature gradient at the surface by Fourier's law, which may be expressed as

$$q_x''(0) = -k \frac{\partial T}{\partial x}\bigg|_{x=0} = q_s''$$

- It is termed a ***Neumann condition***, or a boundary condition of the ***second kind***, and may be realized by bonding a thin film electric heater to the surface. A special case of this condition corresponds to the *perfectly insulated*, or *adiabatic*, surface. The boundary condition of the ***third kind*** corresponds to the existence of convection heating (or cooling) at the surface and is obtained from the surface energy balance

Temperature Distribution

- **Heat Transfer through Plane Wall:** The temperature distribution in the wall can be determined by solving the heat equation with the proper boundary conditions.
- For *one-dimensional, steady-state conduction in a plane wall with no heat generation, the heat flux is a constant, independent of x.* If the thermal conductivity of the wall material is assumed to be constant, the equation may be integrated twice to obtain the *general solution* $T(x) = C_1 x + C_2$
- To obtain the constants of integration, C_1 and C_2, boundary conditions must be introduced. We choose to apply conditions of the first kind at $x = 0$ and $x = L$, in which case



$$T(0) = T_{s,1} \quad \text{and} \quad T(L) = T_{s,2}$$

Temperature Distribution

- Applying the condition at *x* = 0 to the general solution, it follows that $T_{s,1}=C_2$
- Similarly, at x = L, $T_{s,2} = C_1L + C_2 = C_1L + T_{s,1}$
- in which case $\frac{T_{s,2} - T_{s,1}}{L} = C_1$
- Substituting into the general solution, the temperature distribution is then

$$T(x) = (T_{s,2} - T_{s,1})\frac{x}{L} + T_{s,1}$$

- From this result it is evident that, *for one-dimensional, steady-state conduction in a plane wall with no heat generation and constant thermal conductivity, the temperature varies linearly with x.*
- The temperature distribution, we may use Fourier's law, to determine the conduction heat transfer rate. That is, $q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,1} - T_{s,2})$
- Note that *A* is the area of the wall *normal* to the direction of heat transfer and, for the plane wall, it is a constant independent of *x*. The heat flux is then

$$q_x'' = \frac{q_x}{A} = \frac{k}{L}(T_{s,1} - T_{s,2})$$

Temperature Distribution

- **Thermal Resistance:** An analogy exists between the diffusion of heat and electrical charge. Just as an electrical resistance is associated with the conduction of electricity, a thermal resistance may be associated with the conduction of heat. Defining resistance as the ratio of a driving potential to the corresponding transfer rate, it follows from Equation that the *thermal resistance for conduction* in a plane wall is $$R_{t,\text{cond}} \equiv \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA}$$
- Similarly, for electrical conduction in the same system, Ohm's law provides an electrical resistance of the form $$R_e = \frac{E_{s,1} - E_{s,2}}{I} = \frac{L}{\sigma A}$$
- A thermal resistance may also be associated with heat transfer by convection at a surface. From Newton's law of cooling, $q = hA(T_s - T_\infty)$
- The *thermal resistance for convection* is then $$R_{t,\text{conv}} \equiv \frac{T_s - T_\infty}{q} = \frac{1}{hA}$$

Temperature Distribution

- The *equivalent thermal circuit* for the plane wall with convection surface conditions is shown in Figure *b (page 15)*. The heat transfer rate may be determined from separate consideration of each element in the network. Since *qx* is constant throughout the network, it follows that

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

- In terms of the *overall temperature difference*, $T_{\infty,1}$ $T_{\infty,2}$, and the *total thermal resistance*, R_{tot}, the heat transfer rate may also be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} \qquad R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

- Because the conduction and convection resistances are in series and may be summed.
- Radiation exchange between the surface and surroundings may also be important if the convection heat transfer coefficient is small (as it often is for natural convection in a gas). A *thermal resistance for radiation* may be defined by $R_{t,rad} = \frac{T_s - T_{sur}}{q_{rad}} = \frac{1}{h_r A}$

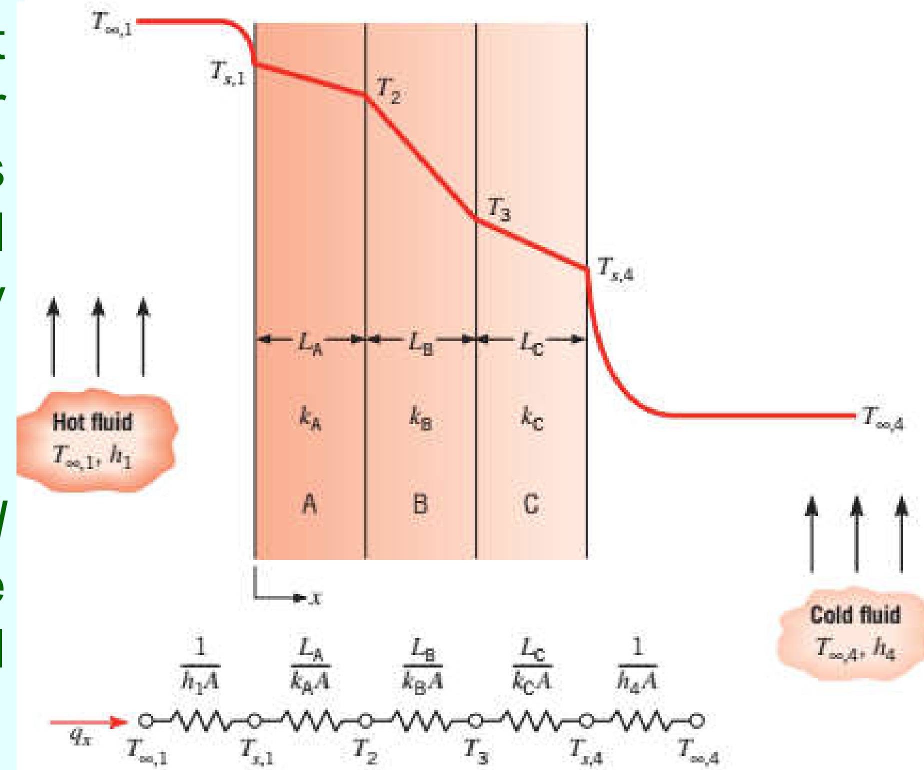
Temperature Distribution

- **The Composite Wall:** Equivalent thermal circuits may also be used for more complex systems, such as *composite walls.* The one-dimensional heat transfer rate for this system may be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\Sigma R_t}$$

- where $T_{\infty,1}$ $T_{\infty,4}$ is the *overall* temperature difference, and the summation includes all thermal resistances. Hence

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{[(1/h_1 A) + (L_A/k_A A) + (L_B/k_B A) + (L_C/k_C A) + (1/h_4 A)]}$$



- Alternatively, the heat transfer rate can be related to the temperature difference and resistance associated with each element. For example,

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{(1/h_1 A)} = \frac{T_{s,1} - T_2}{(L_A/k_A A)} = \frac{T_2 - T_3}{(L_B/k_B A)} = \cdots$$

Temperature Distribution

- With composite systems, it is often convenient to work with an *overall heat transfer coefficient U*, which is defined by an expression analogous to Newton's law of cooling. Accordingly, $q_x \equiv UA\,\Delta T$

  where ***ΔT*** is the overall temperature difference. The overall heat transfer coefficient is related to the total thermal resistance, and from Equations we see that ***UA* =1/*R*$_{tot}$**. Hence, for the composite wall,

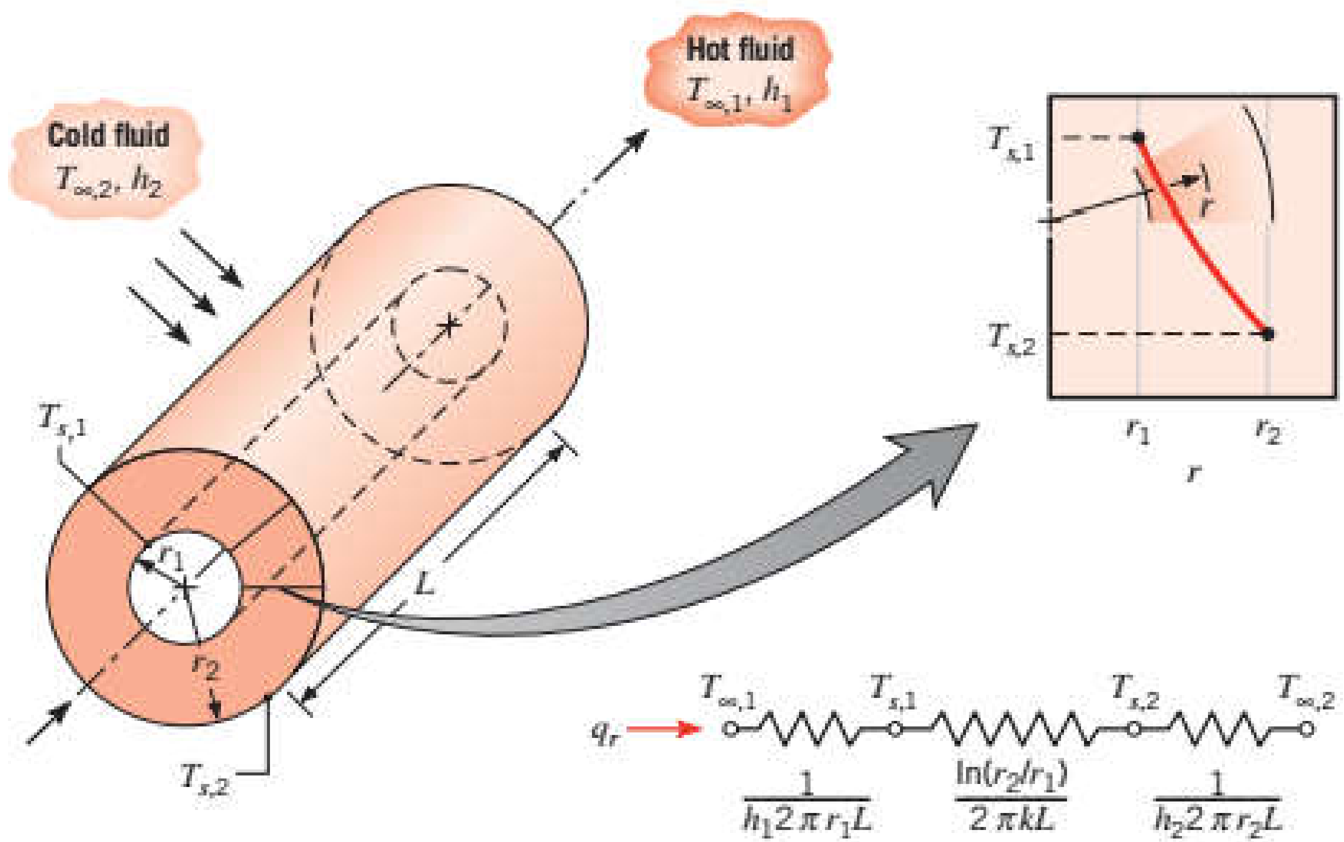
$$U = \frac{1}{R_{tot}A} = \frac{1}{[(1/h_1) + (L_A/k_A) + (L_B/k_B) + (L_C/k_C) + (1/h_4)]}$$

Temperature Distribution

- **Cylindrical system:** A common example is the hollow cylinder whose inner and outer surfaces are exposed to fluids at different temperatures. For steady-state conditions with no heat generation, the appropriate form of the heat equation, Equation, is

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

where, for the moment, *k* is treated as a variable. The physical significance of this result becomes evident if we also consider the appropriate form of Fourier's law.



Temperature Distribution

- The rate at which energy is conducted across any cylindrical surface in the solid may be expressed as $$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr}$$ where $A = 2\pi rL$ is the area normal to the direction of heat transfer. Since Equation dictates that the quantity $kr(dT/dr)$ is independent of r, it follows from Equation that the conduction *heat transfer rate* q_r (*not* the heat flux q''_r) is a *constant in the radial direction*
- We may determine the temperature distribution in the cylinder by solving Equation and applying appropriate boundary conditions. Assuming the value of k to be constant, Equation may be integrated twice to obtain the general solution $T(r) = C_1 \ln r + C_2$
- To obtain the constants of integration C_1 and C_2, we introduce the following boundary conditions: $T(r_1) = T_{s,1} \quad \text{and} \quad T(r_2) = T_{s,2}$
- Applying these conditions to the general solution, we then obtain $$T_{s,1} = C_1 \ln r_1 + C_2 \quad \text{and} \quad T_{s,2} = C_1 \ln r_2 + C_2$$
- Solving for C_1 and C_2 and substituting we obtain $$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

Temperature Distribution

- Recalling how we treated the composite plane wall and neglecting the interfacial contact resistances, the heat transfer rate may be expressed as

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln(r_2/r_1)}{2\pi k_A L} + \frac{\ln(r_3/r_2)}{2\pi k_B L} + \frac{\ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}}$$

- The foregoing result may also be expressed in terms of an overall heat transfer coefficient. That is,

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{\text{tot}}} = UA(T_{\infty,1} - T_{\infty,4})$$

- If U is defined in terms of the inside area, $A_1 = 2\pi r1L$, above Equations may be equated to yield

$$U_1 = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A}\ln\frac{r_2}{r_1} + \frac{r_1}{k_B}\ln\frac{r_3}{r_2} + \frac{r_1}{k_C}\ln\frac{r_4}{r_3} + \frac{r_1}{r_4}\frac{1}{h_4}}$$

Assignment: Find out Critical Insulation Thickness in a Cylindrical system

Steady State and Transient Conduction

Heat and Mass Transfer

Department of Power Engineering, Jadavpur University, Kolkata–700106, India

Contents?

- **Steady State Conduction**, Analysis of fins, Critical thickness of insulation, Systems with internal heat generation.
- **Transient conduction analysis:** Application of numerical methods to conduction problems.
- **Theory of heat convection.** Conservation equation of energy, mass & momentum and their analogies.
- **Significance of various dimensionless numbers**, laminar & turbulent boundary layer concept, thermal boundary layer, forced convection inside tubes and ducts, Forced convection over external bodies, Natural convection.
- **Boiling and Condensation**
- **Radiation** properties and laws, Radiation exchange among black and gray bodies, Electrical analogy, Radiation through participating gases.
- **Mass transfer** by convection and molecular diffusion, Fick's laws, Calculation of mass transfer coefficient, Interface mass transfer.

One-dimensional, steady-state solutions to the heat equation with no generation

	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a]The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.

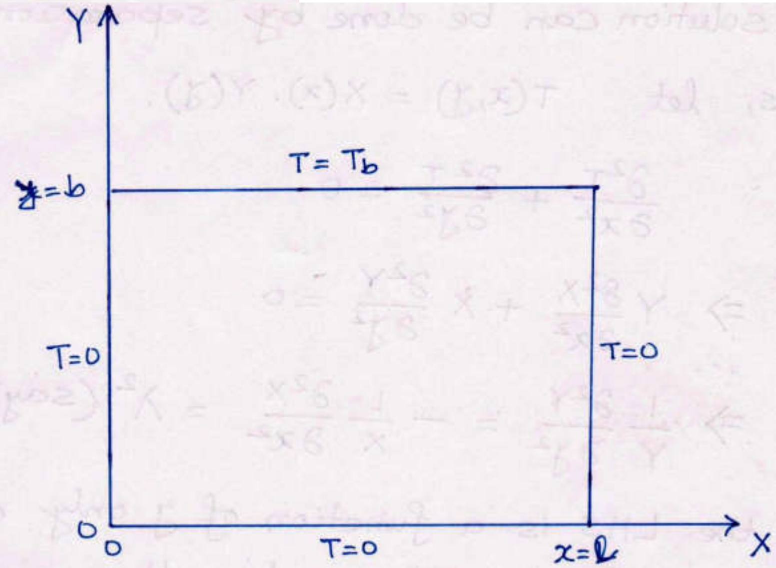
Analytical Solution of 2D Heat Conduction in Rectangular Plate

Case I:

The differential equation is $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$.

Boundary conditions:

at $y=0$, $T=0$
at $x=0$, $T=0$
at $x=l$, $T=0$
at $y=b$, $T=T_b$



The governing equation is a linear, homogeneous partial differential equation of second order.

Analytical Solution of 2D Heat Conduction in Rectangular Plate

The solution can be done by separation of variables.

Thus, let $T(x,y) = X(x) \cdot Y(y)$.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\Rightarrow Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

$$\Rightarrow \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \lambda^2 \text{ (say)}.$$

$$\therefore \frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0$$

$$\frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y = 0$$

[As the BC in x is zero both at $x=0$ & l $\Rightarrow$ the const λ^2 is to be chosen such that the equations generate as shown.]

Analytical Solution of 2D Heat Conduction in Rectangular Plate

The solution of the above equations give

$$X = (C_1 \cos \lambda x + C_2 \sin \lambda x)$$

$$Y = C_3 e^{-\lambda y} + C_4 e^{\lambda y}$$

Thus, $T(x,y) = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y})$.

The constants are to be derived by applying the boundary conditions.

i) at $y = 0$ and all x, $T = 0$

$$\therefore 0 = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 + C_4).$$

So, $C_3 + C_4 = 0 \Rightarrow C_3 = -C_4$.

$$\therefore T = (C_1 \cos \lambda x + C_2 \sin \lambda x)\, C_4 (e^{\lambda y} - e^{-\lambda y}).$$

Analytical Solution of 2D Heat Conduction in Rectangular Plate

ii) at $x=0$ and all y, $T=0$.

$$0 = (C_1 \cos 0 + C_2 \sin 0)\, C_4 (e^{\lambda y} - e^{-\lambda y}).$$

$$\Rightarrow \quad 0 = C_1 C_4 (e^{\lambda y} - e^{-\lambda y}).$$

$C_4 \neq 0$ so, $C_1 = 0$.

$$\therefore \quad T = C_2 C_4 \sin \lambda x \,(e^{\lambda y} - e^{-\lambda y})$$

$$= C \sin \lambda x \,(e^{\lambda y} - e^{-\lambda y}).$$

Analytical Solution of 2D Heat Conduction in Rectangular Plate

iii) at $x = l$, and all y, $T = 0$.

$$\therefore \quad 0 = C \sin \lambda l \,(e^{\lambda y} - e^{-\lambda y}).$$

$C \neq 0$, So, $\sin \lambda l = 0$

$$\Rightarrow \lambda l = n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{l}, \qquad n = 1, 2, 3 \ldots$$

$$\therefore \quad T = C_n \sin \frac{n\pi x}{l} \left(e^{\lambda y} - e^{-\lambda y}\right)$$

$$= 2 C_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l} = C_n' \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l}$$

iv) at $y = b$, and all x $T = T_b$.

$$T_b = C_n' \sin \frac{n\pi x}{l} \sinh \frac{n\pi b}{l}.$$

Analytical Solution of 2D Heat Conduction in Rectangular Plate

$$\Rightarrow \quad T_b \sin\frac{n\pi x}{l} = C_n' \sin^2\frac{n\pi x}{l} \sinh\frac{n\pi b}{l}$$

As the above equation is true over all x from $x=0$ to $x=l$,

$$\int_0^l T_b \sin\frac{n\pi x}{l}\,dx = \frac{C_n'}{2}\sinh\frac{n\pi b}{l}\int_0^l\left(1-\cos\frac{2n\pi x}{l}\right)dx.$$

$$\Rightarrow \quad T_b.\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l = \frac{C_n'}{2}\sinh\frac{n\pi b}{l}\left[x-\frac{\sin\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right]_0^l$$

$$\Rightarrow \quad \frac{l}{n\pi}T_b\left[\underset{1}{\underset{\downarrow}{\cos 0}} - \underset{(-1)^n}{\underset{\downarrow}{\cos n\pi}}\right] = \frac{C_n'}{2}\sinh\frac{n\pi b}{l}\left[l-\frac{l}{2n\pi}\left(\underset{0}{\underset{\downarrow}{\sin 2n\pi}}-\underset{0}{\underset{\downarrow}{\sin 0}}\right)\right]$$

$$\Rightarrow \quad \frac{\cancel{l}}{n\pi}T_b\left[1-(-1)^n\right] = \frac{C_n'}{2}\sinh\frac{n\pi b}{l}\cdot\cancel{l}.$$

$$\Rightarrow \quad C_n' = \frac{2T_b}{n\pi}\left[\frac{1-(-1)^n}{\sinh\frac{n\pi b}{l}}\right]$$

Analytical Solution of 2D Heat Conduction in Rectangular Plate

$$\therefore \quad T = \frac{2T_b}{n\pi}\left[\frac{1-(-1)^n}{\sinh\frac{n\pi b}{l}}\right]\sin\frac{n\pi x}{l}\,\sinh\frac{n\pi y}{l}$$

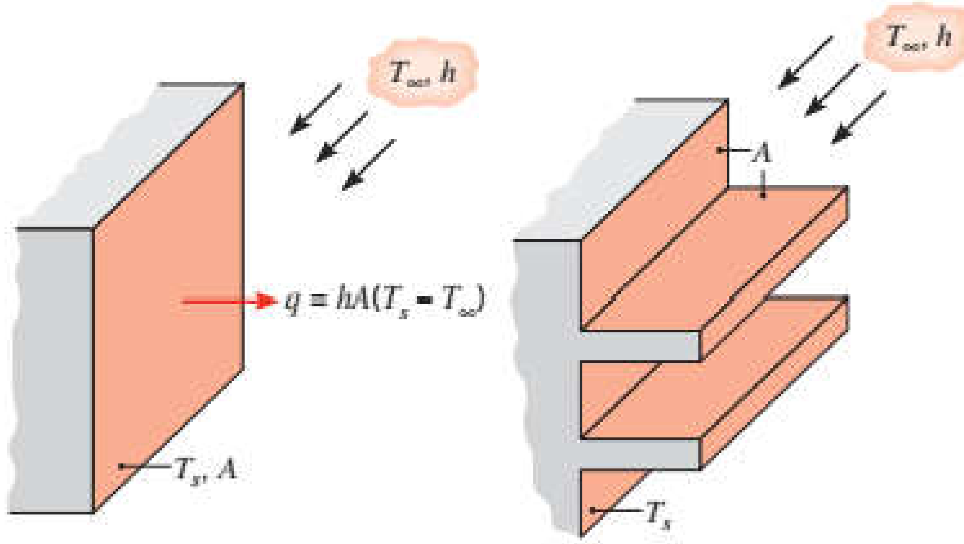
$$\Rightarrow T = \frac{2T_b\left[1-(-1)^n\right]}{n\pi}\,\sin\frac{n\pi x}{l}\,\frac{\sinh\frac{n\pi y}{l}}{\sinh\frac{n\pi b}{l}} \quad \text{for } n = 1, 2, 3\ldots$$

As, the equation is linear, the generalised solution is,

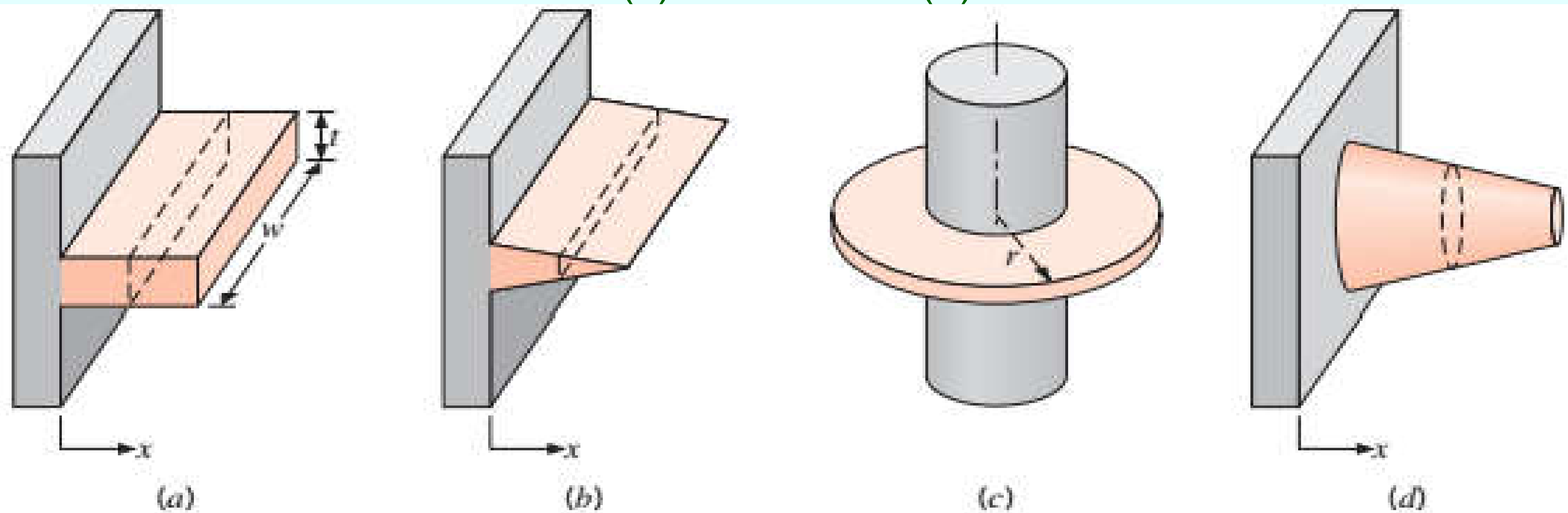
$$\boxed{T = \sum_{n=1}^{\infty}\frac{2T_b\left[1-(-1)^n\right]}{n\pi}\,\frac{\sinh\frac{n\pi y}{l}}{\sinh\frac{n\pi b}{l}}\,\sin\frac{n\pi x}{l}.}$$

Fins and Extended Surfaces

- ***Fin/ Extended surface:*** The term *extended surface* is commonly used to depict an important special case involving heat transfer by conduction within a solid and heat transfer by convection from the boundaries of the solid.
- Frequent application is one in which an extended surface is used specifically to *enhance* heat transfer between a solid and an adjoining fluid. Such an extended surface is termed a *fin.*
- A ***straight fin*** is any extended surface that is attached to a *plane wall.* It may be of uniform cross-sectional area, or its cross-sectional area may vary with the distance *x* from the wall.



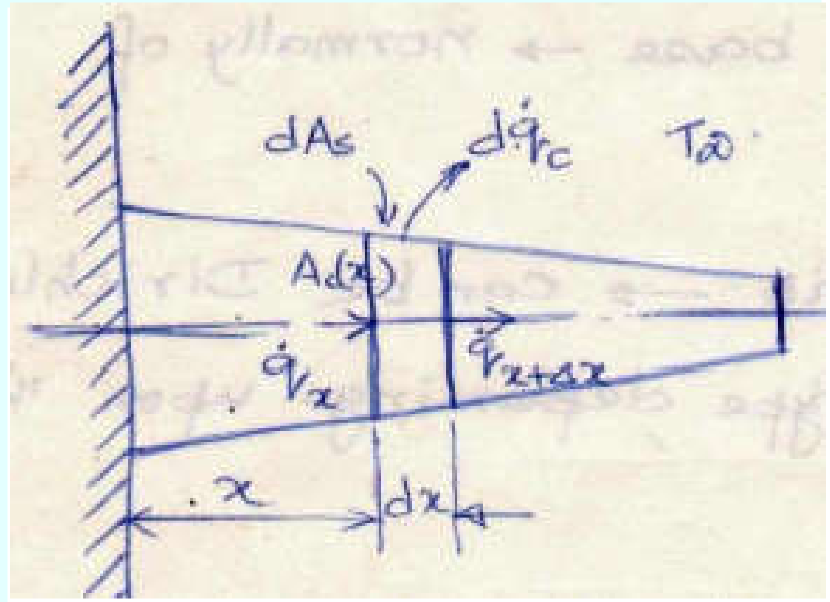
- **Fin configurations:** (*a*) Straight fin of uniform cross section. (*b*) Straight fin of nonuniform cross section. (*c*) Annular fin. (*d*) Pin fin.



- An ***annular fin*** is one that is circumferentially attached to a cylinder, and its cross section varies with radius from the wall of the cylinder. The foregoing fin types have rectangular cross sections, whose area may be expressed as a product of the fin thickness t and the width w for straight fins or the circumference $2\pi r$ for annular fins. In contrast a ***pin fin*, or *spine***, is an extended surface of circular cross section. Pin fins may be of uniform or nonuniform cross section.

- **Extended surface:** To determine the heat transfer rate associated with a fin, we must first obtain the temperature distribution along the fin.

- **Assumptions:** Assume *one-dimensional* conditions in the longitudinal (x-) direction, even though conduction within the fin is actually two-dimensional.



In practice the *fin is thin*, and temperature changes in the transverse direction within the fin are small compared with the temperature difference between the fin and the environment.

Hence, we may assume that the *temperature is uniform across the fin thickness*, that is, it is only a function of x.

Consider *steady-state* conditions and also assume that the *thermal conductivity is constant*, that *radiation* from the surface is negligible, that *heat generation effects* are absent, and that the *convection heat transfer coefficient h is uniform* over the surface.

Generalized Conduction Analysis

- Applying the conservation of energy requirement, to the differential element, we obtain $\dot{q}_x = \dot{q}_{x+dx} + d\dot{q}_c$ and from Fourier's law we know that

$$\dot{q}_x = -kA_c\frac{dT}{dx}, \quad \dot{q}_{x+dx} = \dot{q}_x + \frac{d\dot{q}_x}{dx}\cdot dx, \quad d\dot{q}_c = h\cdot dA_s(T-T_\infty)$$

$$\therefore \quad -\frac{d\dot{q}_x}{dx}\cdot dx = d\dot{q}_c$$

$$\Rightarrow \frac{d}{dx}\left(kA_c\frac{dT}{dx}\right) = h\frac{dA_s}{dx}(T-T_\infty).$$

$$\Rightarrow \frac{d}{dx}\left(A_c\frac{dT}{dx}\right) = \frac{h}{k}\frac{dA_s}{dx}(T-T_\infty)$$

$$\Rightarrow \boxed{\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\cdot\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\cdot\frac{h}{k}\frac{dA_s}{dx}\right)(T-T_\infty) = 0}$$

where, where $\boldsymbol{A_c}$ is the *cross-sectional* area, which may vary with $\boldsymbol{x}$. $\boldsymbol{dA_s}$ is the *surface* area of the differential element.

- Above equation is a second order ordinary differential equation and provides a general form of the energy equation for an extended surface. Its solution for appropriate boundary conditions provides the temperature distribution, which may be used to calculate the conduction rate at any *x*.

Generalized Conduction Analysis

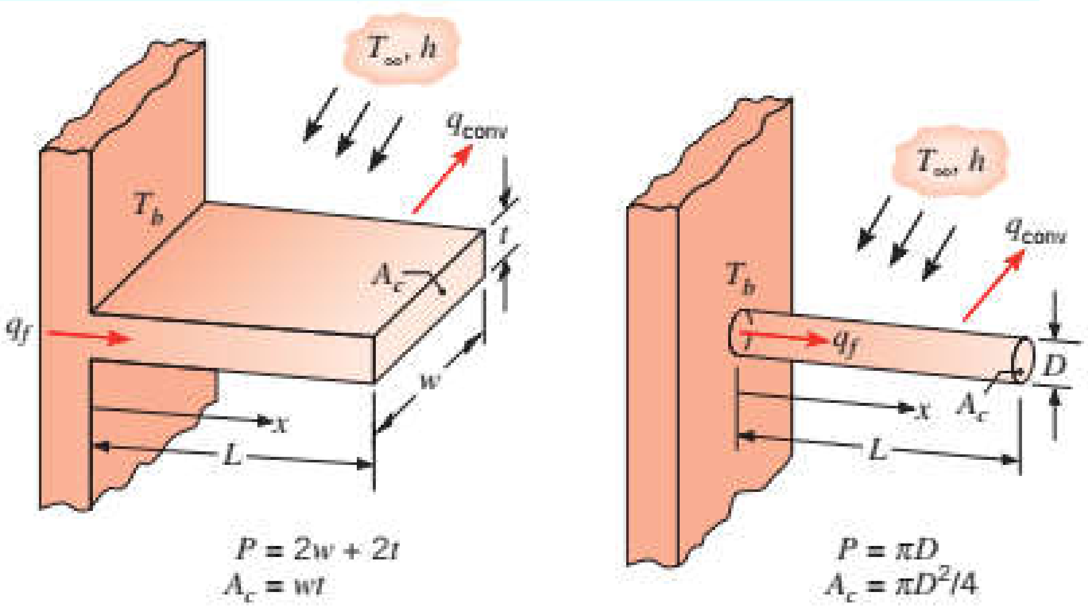
- The solution required two boundary conditions:

i) Boundary condn at the base → normally of Dirichlet type.

ii) Boundary condn at the tip → can be Dirichlet, Neumann or Convective type depending upon the particular case.

- ***Fins of Uniform Cross-Sectional Area:***

Take a simplest case of straight rectangular and pin fins of uniform cross section. Each fin is attached to a base surface of temperature $T(0) = T_b$ and extends into a fluid of temperature T_∞.



Generalized Conduction Analysis

- For the prescribed fins, $\boldsymbol{A_c}$ and P are constant. Accordingly,

For uniform c/s area fin $\frac{dA_c}{dx} = 0$

$$\frac{dA_s}{dx} = P \text{ (perimeter)}.$$

So the governing equ^n becomes:

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

Let $T - T_\infty = \theta$ and $\frac{hP}{kA_c} = m^2$.

$$\therefore \boxed{\frac{d^2\theta}{dx^2} - m^2\theta = 0}$$

Above equation is a linear, homogeneous, second-order differential equation with constant coefficients.

Generalized Conduction Analysis

- The earlier equation has generalized solution is of the form

$$\theta = C_1 e^{mx} + C_2 e^{-mx} \qquad \left[\begin{aligned} \sinh x &= \frac{e^{x} - e^{-x}}{2} \\ \cosh x &= \frac{e^{x} + e^{-x}}{2} \end{aligned} \right]$$

$$\theta = C_3 \cosh mx + C_4 \sinh mx$$

$$\theta = C_5 \cosh m(L - x) + C_6 \sinh m(L - x), \quad L \text{ being the length}$$

- A suitable generalized solution may be chosen for the convenient solution. it is necessary to specify appropriate boundary conditions. One such condition may be specified in terms of the temperature at the *base* of the fin ($x = 0$)
- **Case I: *Long Fin***
- **Case II: *Fin with Insulated Tip***
- **Case III: *Fin with Convective Tip***
- **Case IV: *Fin with Prescribed Temperature at the Tip***

Case I: *Long Fin*

Boundary Conditions are, at $x=0$, $T=T_0$ & $\theta=\theta_0$

at $x=L$, $T\rightarrow T_\infty$ & $\theta\rightarrow 0$ and $L\rightarrow\infty$

Solution, $\theta = C_1 e^{mx} + C_2 e^{-mx}$

$$0 = C_1 e^{mL} + C_2 e^{-mL}, \quad \text{as } L\rightarrow\infty,\ e^{-mL}\rightarrow 0$$

$$\therefore C_1 e^{mL} \rightarrow 0 \text{ or } C_1 = 0$$

Then, $\theta = C_2 e^{-mx}$

$$\theta_0 = C_2 e^0 = C_2$$

$$\therefore \theta = \theta_0 e^{-mx} \quad \text{or,} \quad \frac{T-T_\infty}{T_0-T_\infty} = e^{-mx}$$

$$\theta = C_1 \cosh m(L-x) + C_2 \sinh m(L-x)$$

$x=L, \theta=0 \rightarrow C_1=0$

$x=0, \theta=\theta_0 \rightarrow C_2=\dfrac{\theta_0}{\sinh mL}$

$$\frac{\theta}{\theta_0} = \frac{\sinh m(L-x)}{\sinh mL}$$

Generalized Conduction Analysis

Rate of heat transfer from the fin;

$$\dot{q} = -kA_c \frac{dT}{dx}\bigg|_{x=0} \quad \text{or} \quad \dot{q} = \int_0^L h(T - T_\infty)\, dA_s$$

$$\frac{dT}{dx} = \frac{d\theta}{dx} = -\theta_0 m e^{-mx} \Rightarrow \frac{d\theta}{dx}\bigg|_{x=0} = -m\theta_0$$

$$\therefore \quad \dot{q} = kA_c m\theta_0 = kA_c\sqrt{\frac{hP}{kA_c}}\,\theta_0 = \sqrt{hPkA_c}\,\theta_0$$

Alternately, $\dot{q} = \int_0^L h\theta_0 e^{-mx} P\,dx = hP\theta_0 \int_0^L e^{-mx}\,dx$

$$= -\frac{hP\theta_0}{m}\left(e^{-mL} - e^0\right), \quad \boxed{\text{as } L \to \infty \quad e^{-mL} \to 0}$$

$$= \frac{hP\theta_0}{\sqrt{\frac{hP}{kA_c}}} = \sqrt{hPkA_c}\,\theta_0$$

Case II: *Fin with Insulated Tip*

at $x=0$, $\theta=\theta_0$ and $x=L$, $\frac{d\theta}{dx}=0$

$$\theta = C_1 \cosh m(L-x) + C_2 \sinh m(L-x)$$

$$\frac{d\theta}{dx} = -mC_1 \sinh m(L-x) - mC_2 \cosh m(L-x)$$

$$\left.\frac{d\theta}{dx}\right|_{x=L} = -mC_1 \sinh 0 - mC_2 \cosh 0$$

$$\Rightarrow \quad mC_2 = 0 \Rightarrow C_2 = 0.$$

$$\theta = C_1 \cosh m(L-x) \Rightarrow \theta_0 = C_1 \cosh mL \Rightarrow C_1 = \frac{\theta_0}{\cosh mL}$$

$$\therefore \quad \theta = \theta_0 \frac{\cosh m(L-x)}{\cosh mL}$$

$$\Rightarrow \frac{\theta}{\theta_0} = \frac{T-T_\infty}{T_0-T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

- Rate of heat transfer from the fin

$$\dot{q} = -kA_c \frac{d\theta}{dx}\bigg|_{x=0} = -kA_c \frac{\theta_0}{\cosh mL} \cdot \left[-m \sinh m(L-x)\right]_{x=0}$$

$$= kA_c m \tanh mL \cdot \theta_0 = \sqrt{hPkA_c}\ \theta_0 \tanh mL$$

- **Case III:** ***Fin with Convective Tip***

$$\theta = C_1 \cosh m(L-x) + C_2 \sinh m(L-x)$$

with b.c. at $x=0$, $\theta = \theta_0$

at $x = L$, $-k\cancel{A_c}\frac{dT}{dx} = h\cancel{A_c}(T - T_\infty)$

$$\Rightarrow -k\frac{d\theta}{dx}\bigg|_{x=L} = h\theta\big|_{x=L}$$

$$\frac{d\theta}{dx} = -m\left[C_1 \sinh m(L-x) + C_2 \cosh m(L-x)\right]$$

$$\frac{d\theta}{dx}\bigg|_{x=L} = -m\left[C_1 \sinh 0 + C_2 \cosh 0\right]$$

$$= -mC_2.$$

Generalized Conduction Analysis

$$\theta\Big|_{x=L} = C_1 \cosh 0 + C_2 \sinh 0 = C_1$$

$$\therefore \quad -k C_2 m = h C_1 \Rightarrow C_1 = \frac{km}{h} C_2 .$$

$$\theta = C_2 \left\{ \frac{km}{h} \cosh m(L-x) + \sinh m(L-x) \right\}$$

$$\Rightarrow \quad \theta_0 = C_2 \left\{ \frac{km}{h} \cosh mL + \sinh mL \right\}$$

$$\Rightarrow \quad C_2 = \frac{\theta_0}{\frac{km}{h} \cosh mL + \sinh mL}$$

So,

$$\frac{\theta}{\theta_0} = \frac{\frac{km}{h} \cosh m(L-x) + \sinh m(L-x)}{\frac{km}{h} \cosh mL + \sinh mL}$$

Generalized Conduction Analysis

$$\Rightarrow \quad \frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh m(L-x) + \frac{h}{km}\sinh m(L-x)}{\cosh mL + \frac{h}{km}\sinh mL}$$

- Rate of heat transfer from the fin

$$\dot{q} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \theta_0 \sqrt{hPkA_c}\left[\frac{\sinh mL + \frac{h}{km}\cosh mL}{\cosh mL + \frac{h}{km}\sinh mL}\right]$$

- **Case IV:** ***Fin with Prescribed Temperature at the Tip***

$$x = 0 \longrightarrow \theta = \theta_0$$

$$x = L \longrightarrow \theta = \theta_L$$

$$\theta = C_1 \cosh m(L-x) + C_2 \sinh m(L-x)$$

$$\theta_L = C_1 \quad \therefore \quad \theta = \theta_L \cosh m(L-x) + C_2 \sinh m(L-x)$$

$$\theta_0 = \theta_L \cosh mL + C_2 \sinh mL$$

Generalized Conduction Analysis

$$\Rightarrow C_2 = \frac{\theta_0 - \theta_L \cosh mL}{\sinh mL}$$

$$\therefore \theta = \theta_L \cosh m(L-x) + \frac{\theta_0 - \theta_L \cosh mL}{\sinh mL} \sinh m(L-x)$$

$$\Rightarrow \frac{\theta}{\theta_0} = \frac{\theta_L}{\theta_0} \cosh m(L-x) + \frac{\theta_0 - \theta_L \cosh mL}{\theta_0 \sinh mL} \sinh m(L-x)$$

$$= \frac{\theta_L \left[\cosh m(L-x) \sinh mL - \cosh mL \sinh m(L-x)\right] + \theta_0 \sinh m(L-x)}{\theta_0 \sinh mL}$$

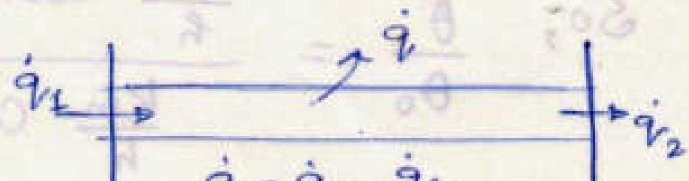
$$= \frac{\theta_L \sinh \{mL - m(L-x)\} + \theta_0 \sinh m(L-x)}{\theta_0 \sinh mL}$$

$$= \frac{\theta_L \sinh mx + \theta_0 \sinh m(L-x)}{\theta_0 \sinh mL}$$

Generalized Conduction Analysis

- Rate of heat transfer from the fin

$$\dot{q} = -kA_c \frac{d\theta}{dx}\Big|_{x=0} - \left(-kA_c \frac{d\theta}{dx}\Big|_{x=L}\right).$$



$$\dot{q} = \dot{q}_1 - \dot{q}_2$$

$$\frac{d\theta}{dx} = \frac{m\theta_L \cosh mx - m\theta_0 \cosh m(L-x)}{\sinh mL}$$

$$\frac{d\theta}{dx}\Big|_{x=0} = \frac{m\theta_L - m\theta_0 \cosh mL}{\sinh mL}, \quad \frac{d\theta}{dx}\Big|_{x=L} = \frac{m\theta_L \cosh mL - m\theta_0}{\sinh mL}$$

$$\therefore \dot{q} = kA_c\left[\frac{m\theta_0 \cosh mL - m\theta_L + m\theta_L \cosh mL - m\theta_0}{\sinh mL}\right] = \frac{kA_c m}{\sinh mL}(\cosh mL - 1)(\theta_0 + \theta_L)$$

Generalized Conduction Analysis

$$\dot{q} = \int_0^L h(T - T_a)\, dA_s = \int_0^L hP\theta\, dx$$

$$= hP \int_0^L \frac{\theta_L \sinh mx + \theta_0 \sinh m(L-x)}{\sinh mL}\, dx$$

$$= \frac{hP}{\sinh mL}\left[\theta_L \frac{\cosh mx}{m} - \frac{\theta_0 \cosh m(L-x)}{m}\right]_0^L$$

$$= \frac{hP}{m \sinh mL}\left[\theta_L \cosh mx - \theta_0 \cosh m(L-x)\right]_0^L$$

$$= \frac{\sqrt{hPkA_c}}{\sinh mL}\left[\theta_L \cosh mL - \theta_L \cosh 0 - \theta_0 \cosh 0 + \theta_0 \cosh mL\right]$$

$$= \frac{\sqrt{hPkA_c}}{\sinh mL}\left[(\theta_L + \theta_0)\cosh mL - (\theta_L + \theta_0)\right]$$

$$= \sqrt{hPkA_c}\left(\frac{\cosh mL - 1}{\sinh mL}\right)(\theta_L + \theta_0)$$

Generalized Conduction Analysis

$$\frac{\theta}{\theta_0} = \frac{\theta_L \sinh mx + \theta_0 \sinh m(L-x)}{\theta_0 \cdot \sinh mL}$$

$$\frac{d\theta}{dx} = \frac{m\theta_L \cosh mx - m\theta_0 \cosh m(L-x)}{\sinh mL}$$

At $\min^{m}$ θ, $\frac{d\theta}{dx} = 0$

$$\therefore \cancel{m}\theta_L \cosh mx = \cancel{m}\theta_0 \cosh m(L-x)$$

$$\Rightarrow \frac{\cosh m(L-x)}{\cosh mx} = \frac{\theta_L}{\theta_0}$$

$$\Rightarrow \frac{e^{m(L-x)} + e^{-m(L-x)}}{e^{mx} + e^{-mx}} = \frac{\theta_L}{\theta_0}.$$

Temperature distributions and heat rates for fins of uniform cross section

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b	Fin Heat Transfer Rate q_f
A	Convection: $h\theta(L) = -kd\theta/dx\vert_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$
B	Adiabatic: $d\theta/dx\vert_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	$M \tanh mL$
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$
D	Infinite fin ($L \to \infty$): $\theta(L) = 0$	e^{-mx}	M

$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\,\theta_b$

Fin Performance Parameters

- **Fin Effectiveness and Fin Resistance:** Fins are used to enhance the heat transfer from a surface by increasing the effective surface area. However, the fin itself poses a conduction resistance. For this reason, there is no assurance that the heat transfer rate will increase through the use of fins.
- One assessment of fin performance may be made by evaluating the fin effectiveness, ε_f. It is defined, for a surface of constant base temperature, as *the ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin*. Therefore

$$\varepsilon_f = \frac{q_f}{hA_{c,b}\theta_b}$$

  where $A_{c,b}$ is the fin cross-sectional area at the base. In general, the use of fins may rarely be justified unless $\varepsilon_f \geq 2$.
- For example, for an infinitely long fin and assuming the convection coefficient of the finned surface is the same as that of the unfinned base, it follows that

$$\boxed{\varepsilon_f = \left(\frac{kP}{hA_c}\right)^{1/2}}$$

Fin Performance Parameters

- Equation suggests that the use of fins can be better justified under conditions for which the convection coefficient ***h*** is small.
- It is evident that the need for fins is stronger when the fluid is a gas rather than a liquid. If fins are to be used on a surface separating a gas and a liquid, they are generally placed on the gas side.
- A common example is the tubing in an automobile radiator. Fins are applied to the outer tube surface, over which there is flow of ambient air (small *h*), and not to the inner surface, through which there is flow of water (large ***h***). Note that, if $\varepsilon_f > 2$ is used as a criterion to justify the implementation of fins, Equation yields the requirement that $(kP/hA_c) > 4$.
- Equation provides an upper limit to ε_f, which is reached as ***L*** approaches infinity. However, it is certainly not necessary to use very long fins to achieve near maximum heat transfer enhancement.
- Fin performance may also be quantified in terms of a thermal resistance. Treating the difference between the base and fluid temperatures as the driving potential, a *fin resistance* may be defined as

$$R_{t,f} = \frac{\theta_b}{q_f}$$

Fin Performance Parameters

- Combining the expression for the thermal resistance due to convection at the exposed base, $R_{t,b} = \dfrac{1}{hA_{c,b}}$ So, the fin effectiveness $\varepsilon_f = \dfrac{R_{t,b}}{R_{t,f}}$
- The fin effectiveness may be interpreted as a ***ratio of thermal resistances***, and to increase ε_f it is necessary ***to reduce the conduction/convection resistance of the fin***. If the fin is to enhance heat transfer, its resistance must not exceed that of the exposed base.
- ***Fin Efficiency:*** Fin performance is provided by the fin efficiency, η_f. A logical definition of the fin efficiency is the actual fin heat transfer rate, q_f, divided by the maximum possible heat transfer rate. The maximum heat transfer rate from the fin would occur *if* it were entirely at the base temperature, T_b. Therefore

$$n_f \equiv \frac{q_f}{q_{max}} = \frac{q_f}{hA_f(T_b - T_\infty)} = \frac{q_f}{hA_f\theta_b}$$

  where A_f is the surface area of the fin.
- For a straight fin of uniform cross section and an adiabatic tip

$$\eta_f = \frac{M \tanh mL}{hPL\theta_b} = \frac{\tanh mL}{mL}$$

Straight Fin with Triangular Profile

- A straight fin with triangular profile may end up to a point or line when it is called tapered fin; or it may end up to a definite thickness when it is called trapezoidal fin
- Let 't' be the thickness of a tapered fin at its base and 'L' is the length of the fin. Assuming the thickness of the fin to be sufficiently small, the heat conduction may be taken as 1-D. the cross-sectional area to heat conduction is changing with length. Let the width of the fin is 'b'.

$\therefore$ Cross-sectional area at x, $A = \frac{t \cdot x}{L} b$

Considering the thickness to be much smaller than the width,

Perimeter $P = 2\left[b + \frac{tx}{L}\right] \approx 2b$ (as $b \gg t$)

Governing eqn:

$$\frac{d}{dx}\left(kA_c \frac{dT}{dx}\right) - h\frac{dA_s}{dx}(T - T_\infty) = 0$$

Straight Fin with Triangular Profile

Let $\theta = T - T_\infty$, & $\dfrac{dA_s}{dx} = P$ (taken as const $= 2b$)

$$\therefore \frac{d^2\theta}{dx^2} + \frac{1}{A}\frac{dA}{dx}\frac{d\theta}{dx} - \frac{hP}{kA}\theta = 0$$

$$\Rightarrow \frac{d^2\theta}{dx^2} + \frac{L}{txb}\frac{d}{dx}\left(\frac{txb}{L}\right)\frac{d\theta}{dx} - \frac{h.2b}{k.\frac{txb}{L}}\theta = 0$$

$$\Rightarrow \frac{d^2\theta}{dx^2} + \frac{1}{x}\frac{d\theta}{dx} - \frac{2hL}{kt}\frac{1}{x}\theta = 0$$

Let, $\boxed{\dfrac{2hL}{kt} = B^2}$

$$\therefore \frac{d^2\theta}{dx^2} + \frac{1}{x}\frac{d\theta}{dx} - \frac{B^2}{x}\theta = 0$$

$$\Rightarrow \boxed{x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - B^2 x\theta = 0}$$

Straight Fin with Triangular Profile

Let, $\boxed{Z = 2B\sqrt{x}} \Rightarrow \dfrac{dZ}{dx} = Bx^{-\frac{1}{2}}$

$$\frac{d\theta}{dx} = \frac{d\theta}{dZ} \cdot \frac{dZ}{dx} = Bx^{-\frac{1}{2}} \frac{d\theta}{dZ}$$

$$\frac{d^2\theta}{dx^2} = \frac{d}{dx}\left[Bx^{-\frac{1}{2}} \frac{d\theta}{dZ}\right]$$

$$= B\frac{d\theta}{dZ} \frac{d}{dx}\left(x^{-\frac{1}{2}}\right) + Bx^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{d\theta}{dZ}\right)$$

$$= -\frac{1}{2} B\frac{d\theta}{dZ} x^{-\frac{3}{2}} + Bx^{-\frac{1}{2}} \frac{d^2\theta}{dZ^2} \cdot \frac{dZ}{dx}$$

$$= -\frac{1}{2} Bx^{-\frac{3}{2}} \frac{d\theta}{dZ} + Bx^{-\frac{1}{2}} \frac{d^2\theta}{dZ^2} \cdot Bx^{-\frac{1}{2}}$$

$$= -\frac{1}{2} Bx^{-\frac{3}{2}} \frac{d\theta}{dZ} + B^2 x^{-1} \frac{d^2\theta}{dZ^2}$$

Straight Fin with Triangular Profile

Substituting,

$$B\left[-\frac{B}{2}x^{-\frac{3}{2}}\frac{d\theta}{dZ} + B^2 x^{-1}\frac{d^2\theta}{dZ^2}\right]x^2 + x\cdot B x^{-\frac{1}{2}}\frac{d\theta}{dZ} - B^2 x\theta = 0$$

$$\Rightarrow \quad B^2 x\frac{d^2\theta}{dZ^2} - \frac{1}{2}B x^{+\frac{1}{2}}\frac{d\theta}{dZ} + B x^{\frac{1}{2}}\frac{d\theta}{dZ} - B^2 x\theta = 0$$

$$\Rightarrow \quad B^2 x\frac{d^2\theta}{dZ^2} + \frac{1}{2}B x^{\frac{1}{2}}\frac{d\theta}{dZ} - B^2 x\theta = 0$$

$$\Rightarrow \quad Z^2\frac{d^2\theta}{dZ^2} + Z\frac{d\theta}{dZ} - Z^2\theta = 0 \qquad \Bigg| \; Z^2 = 4B^2 x$$

$$\Rightarrow \quad \frac{d^2\theta}{dZ^2} + \frac{1}{Z}\frac{d\theta}{dZ} - \theta = 0.$$

- The above equation is known as modified Bessel Equation of Zero Order.

Straight Fin with Triangular Profile

- The solution of the above equation is given by

$$\theta = C_1 I_0(z) + C_2 K_0(z)$$

or, $$\theta = C_1 I_0(2B\sqrt{x}) + C_2 K_0(2B\sqrt{x})$$

where, I_0 and K_0 are the Bessel function of zero order of first and second kind.

To evaluate the constants C_1 and C_2, the boundary conditions are required.

At $x=0$, (i.e. at the fin tip), temperature has to be finite and so θ is finite.

But $I_0(0) = 1$ and $K_0(0) = \infty$

So, C_2 must be zero. $\therefore \theta = C_1 I_0(2B\sqrt{x})$

At $x = L$, $T = T_b$, so $\theta = \theta_b = T_b - T_\infty$

$$\therefore \theta_b = C_1 I_0(2B\sqrt{L}) \Rightarrow C_1 = \frac{\theta_b}{I_0(2B\sqrt{L})}$$

Straight Fin with Triangular Profile

- Therefore the temperature of distribution along the fin is

$$\frac{\theta}{\theta_b} = \frac{T - T_\infty}{T_b - T_\infty} = \frac{I_0(2B\sqrt{x})}{I_0(2B\sqrt{L})}$$

- Rate of heat transfer from the fin

$$\dot{q} = kA \left.\frac{d\theta}{dx}\right|_{x=L}$$

$$\frac{d\theta}{dx} = \frac{\theta_b}{I_0(2B\sqrt{L})} \frac{d}{dx}\left[I_0(2B\sqrt{x})\right]$$

$$= \frac{\theta_b}{I_0(2B\sqrt{L})} I_1(2B\sqrt{x}) \cdot Bx^{-\frac{1}{2}}$$

$$\therefore \left.\frac{d\theta}{dx}\right|_{x=L} = \frac{\theta_b}{I_0(2B\sqrt{L})} I_1(2B\sqrt{L}) \cdot BL^{-\frac{1}{2}}$$

Straight Fin with Triangular Profile

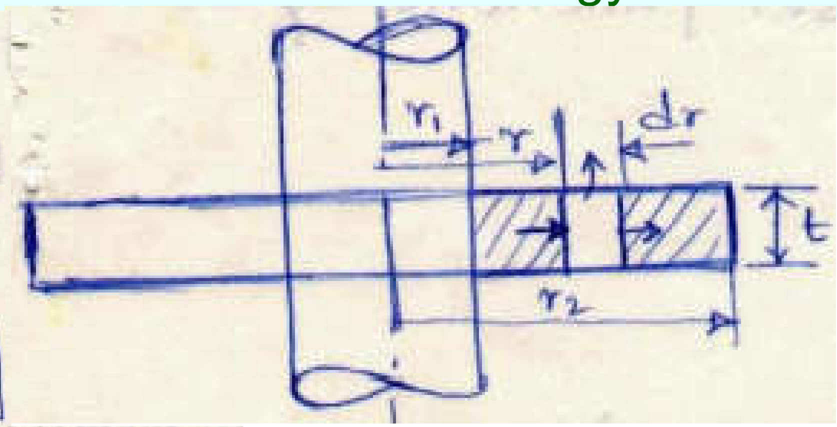
$$\text{So, } \dot{q} = k A_L \frac{\theta_b I_1(2B\sqrt{L})}{I_0(2B\sqrt{L})} B L^{-\frac{1}{2}}$$

$$= k \cdot tb \cdot \frac{\theta_b I_1(2B\sqrt{L})}{I_0(2B\sqrt{L})} \cdot \sqrt{\frac{2hL}{kt}} \cdot L^{-\frac{1}{2}}$$

$$= b \cdot \sqrt{2hkt}\; \theta_b \frac{I_1(2B\sqrt{L})}{I_0(2B\sqrt{L})}$$

Circular Fin/ Annular Fin

- The annular fin of rectangular profile is commonly used to enhance heat transfer to or from circular tubes. Applying the conservation of energy



$$\dot{q}_r = \dot{q}_{r+dr} + d\dot{q}_c$$

$$\dot{q}_r = -k A_c \frac{dT}{dr} = -k \cdot 2\pi r t \frac{dT}{dr}$$

$$\dot{q}_{r+dr} = \dot{q}_r + \frac{d\dot{q}_r}{dr} dr$$

$$d\dot{q}_c = h \cdot 2 \cdot 2\pi r \, dr (T - T_\infty) \quad \left[\text{Top \& bottom surfaces}\right]$$

$$\therefore \; -\frac{d\dot{q}_r}{dr} \cancel{dr} = h\, 2\, 2\pi r \cancel{dr} (T - T_\infty)$$

$$\Rightarrow \; k \cdot 2\pi \cancel{r} t \frac{d}{dr}\left(r \frac{dT}{dr}\right) - 2h \cdot 2\pi r (T - T_\infty) = 0$$

$$\Rightarrow \; \frac{d}{dr}\left(r \frac{dT}{dr}\right) - \frac{2hr}{kt}(T - T_\infty) = 0$$

$$\Rightarrow \; \frac{d}{dr}\left(r \frac{d\theta}{dr}\right) - \frac{2hr}{kt}\theta = 0 \quad \left[\theta = T - T_\infty\right]$$

Circular Fin/ Annular Fin

- The annular fin of rectangular profile is commonly used to enhance heat transfer to or from circular tubes. Applying the conservation of energy

$$\Rightarrow \frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \frac{2h}{kt}\theta = 0 \qquad \text{Let } \frac{2h}{kt} = m^2$$

$$\Rightarrow \frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - m^2\theta = 0.$$

Let $y = mr \Rightarrow \frac{dy}{dr} = m$

$$\frac{d\theta}{dr} = \frac{d\theta}{dy}\cdot\frac{dy}{dr} = m\frac{d\theta}{dy}$$

$$\frac{d^2\theta}{dr^2} = \frac{d}{dy}\left(m\frac{d\theta}{dy}\right)\cdot\frac{dy}{dr} = m^2\frac{d^2\theta}{dy^2}$$

$$\therefore m^2\frac{d^2\theta}{dy^2} + \frac{m}{y}m\frac{d\theta}{dy} - m^2\theta = 0$$

$$\Rightarrow \frac{d^2\theta}{dy^2} + \frac{1}{y}\frac{d\theta}{dy} - \theta = 0$$

Circular Fin/ Annular Fin

This is a modified form of Bessel equⁿ whose solution can be $\theta = C_1 I_0(\xi) + C_2 K_0(\xi)$

$$\Rightarrow \quad \theta = C_1 I_0(mr) + C_2 K_0(mr)$$

where I_0 and K_0 are the modified zero order Bessel functions of the first and second kind respectively.

B.C. $\Rightarrow r = r_1, \; \theta = \theta_0$

$\Rightarrow r = r_2, \; \dfrac{d\theta}{dr} = 0$ (for insulated tip say)

$$\therefore \quad \theta_0 = C_1 I_0(mr_1) + C_2 K_0(mr_1)$$

$$\frac{d\theta}{dr} = C_1 I_1(mr) \cdot m - C_2 K_1(mr) \cdot m$$

$$\left.\frac{d\theta}{dr}\right|_{r=r_2} = C_1 I_1(mr_2) \cdot m - C_2 K_1(mr_2) \cdot m = 0 \Rightarrow C_1 = \frac{C_2 K_1(mr_2)}{I_1(mr_2)}$$

Circular Fin/ Annular Fin

$$\theta_0 = \frac{C_2 K_1(mr_2)}{I_1(mr_2)} \cdot I_0(mr_1) + C_2 K_0(mr_1)$$

$$= C_2 \left[\frac{K_1(mr_2) \cdot I_0(mr_1) + K_0(mr_1) \cdot I_1(mr_2)}{I_1(mr_2)} \right]$$

$$\Rightarrow \quad C_2 = \frac{\theta_0 \, I_1(mr_2)}{K_1(mr_2)\, I_0(mr_1) + K_0(mr_1)\, I_1(mr_2)}$$

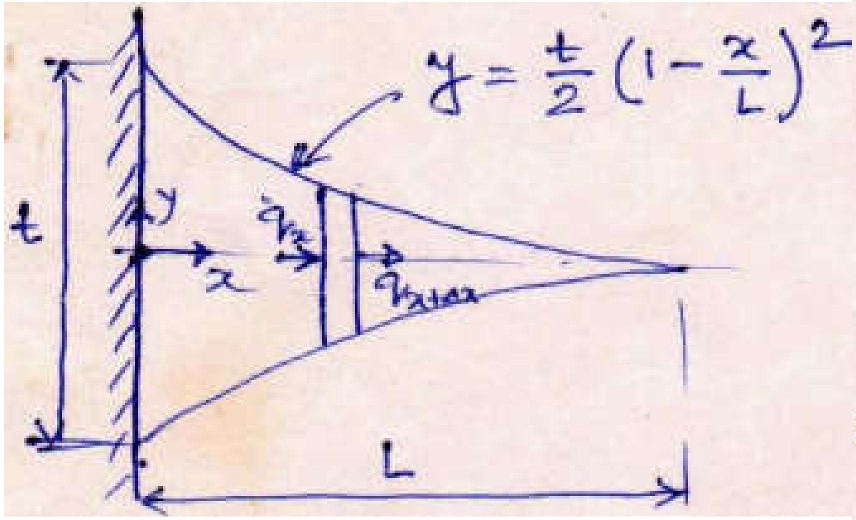
$$C_1 = \frac{\theta_0 \, K_1(mr_2)}{K_1(mr_2)\, I_0(mr_1) + K_0(mr_1)\, I_1(mr_2)}$$

$$\therefore \quad \frac{\theta}{\theta_0} = \frac{K_1(mr_2) \cdot I_0(mr) + I_1(mr_2)\, K_0(mr)}{K_1(mr_2)\, I_0(mr_1) + K_0(mr_1)\, I_1(mr_2)}$$

where $I_1(mr_0) = \frac{d}{d(mr)}\{I_0(mr)\}$, $K_1(mr) = -\frac{d}{d(mr)}\{K_0(mr)\}$

are the modified first order Bessel functions of first and second kind

Straight Parabolic Fin



Width of the fin = b.

$$A_c(x) = y(x).b$$
$$= 2\frac{t}{2}\left(1-\frac{x}{L}\right)^2.b = t\left(1-\frac{x}{L}\right)^2.b.$$

$$\frac{dA_s}{dx} = P = 2\left[b+2y(x)\right] \approx 2b$$

$$\frac{dA_c}{dx} = bt\,2\left(1-\frac{x}{L}\right).-\frac{1}{L} = -\frac{2bt}{L}\left(1-\frac{x}{L}\right)$$
$$= \frac{2bt}{L}\left(\frac{x}{L}-1\right)$$

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\left(T-T_\infty\right) = 0.$$

$$\Rightarrow \frac{d^2T}{dx^2} + \frac{1}{bt\left(1-\frac{x}{L}\right)^2}\cdot\frac{2bt}{L}\left(\frac{x}{L}-1\right)\frac{dT}{dx} - \frac{1}{bt\left(1-\frac{x}{L}\right)^2}\cdot\frac{h}{k}\cdot 2b\left(T-T_\infty\right) = 0$$

$$\Rightarrow \frac{d^2T}{dx^2} - \frac{2}{L\left(1-\frac{x}{L}\right)}\frac{dT}{dx} - \frac{2h}{kt}\frac{1}{\left(1-\frac{x}{L}\right)^2}\left(T-T_\infty\right) = 0$$

Straight Parabolic Fin

$$\Rightarrow \frac{d^2\theta}{dx^2} - \frac{2}{L\left(1-\frac{x}{L}\right)}\frac{d\theta}{dx} - \frac{2h}{kt}\frac{1}{\left(1-\frac{x}{L}\right)^2}\theta = 0$$

$$\Rightarrow \frac{d^2\theta}{dx^2} - \frac{2}{L-x}\frac{d\theta}{dx} - \frac{2hL^2}{kt}\frac{1}{(L-x)^2}\theta = 0$$

$$L - x = z.$$

$$\Rightarrow \frac{dz}{dx} = -1$$

$$\frac{d\theta}{dx} = \frac{d\theta}{dz}\cdot\frac{dz}{dx} = -\frac{d\theta}{dz}$$

$$\frac{d^2\theta}{dx^2} = \frac{d}{dz}\left(\frac{d\theta}{dx}\right)\cdot\frac{dz}{dx} = \frac{d}{dz}\left(\frac{d\theta}{dz}\cdot\frac{dz}{dx}\right)\frac{dz}{dx}$$

$$= \frac{d}{dz}\left(-\frac{d\theta}{dz}\right)\cdot -1 = \frac{d^2\theta}{dz^2}$$

$$\frac{d^2\theta}{dz^2} + \frac{2}{z}\frac{d\theta}{dz} - \frac{2hL^2}{kt}\frac{1}{z^2}\theta = 0$$

Straight Parabolic Fin

$$\Rightarrow \frac{d^2\theta}{dz^2} + \frac{2}{z}\frac{d\theta}{dz} - \frac{B^2}{z^2}\theta = 0 \quad , \quad B^2 = \frac{2hL^2}{kt}$$

$$z^2\frac{d^2\theta}{dz^2} + 2z\frac{d\theta}{dz} - B^2\theta = 0$$

$$\Rightarrow \frac{d}{dz}\left(z^2\frac{d\theta}{dz}\right) - B^2\theta = 0$$

- **Assignment:** Temperature Distribution in a Trapezoidal Fin

Efficiency of common fin shapes

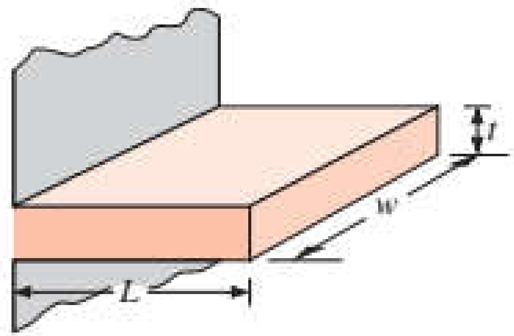
Straight Fins

Rectangular[a]

$A_f = 2wL_c$

$L_c = L + (t/2)$

$A_p = tL$

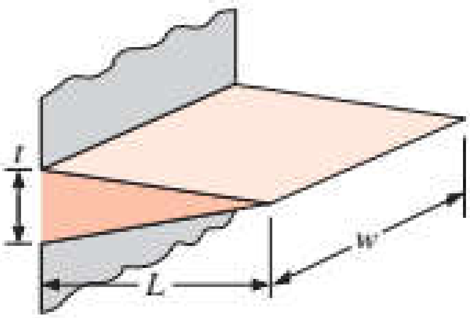


$$\eta_f = \frac{\tanh mL_c}{mL_c}$$

Triangular[a]

$A_f = 2w[L^2 + (t/2)^2]^{1/2}$

$A_p = (t/2)L$



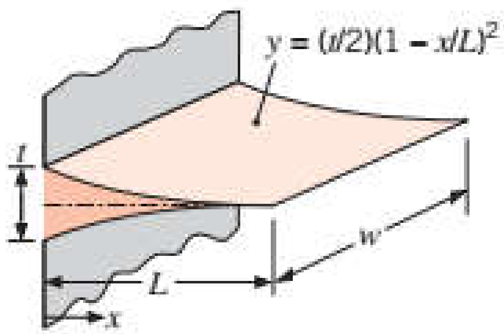
$$\eta_f = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)}$$

Parabolic[a]

$A_f = w[C_1L + (L^2/t)\ln(t/L + C_1)]$

$C_1 = [1 + (t/L)^2]^{1/2}$

$A_p = (t/3)L$



$$\eta_f = \frac{2}{[4(mL)^2 + 1]^{1/2} + 1}$$

Efficiency of common fin shapes

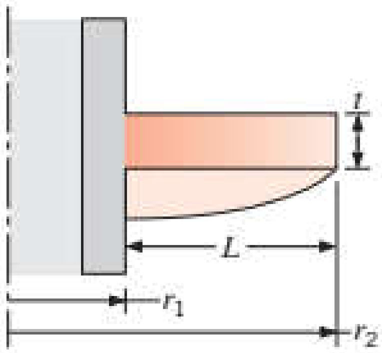
Annular Fin

Rectangular[a]

$A_f = 2\pi\,(r_{2c}^2 - r_1^2)$

$r_{2c} = r_2 + (t/2)$

$V = \pi\,(r_2^2 - r_1^2)t$



$$\eta_f = C_2 \frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}$$

$$C_2 = \frac{(2r_1/m)}{(r_{2c}^2 - r_1^2)}$$

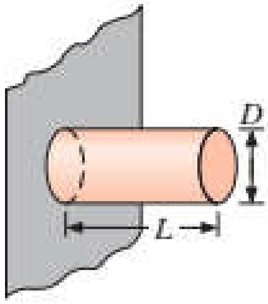
Pin Fins

Rectangular[b]

$A_f = \pi D L_c$

$L_c = L + (D/4)$

$V = (\pi D^2/4)L$



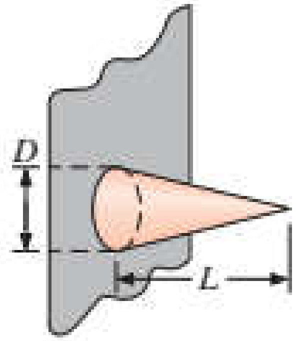
$$\eta_f = \frac{\tanh mL_c}{mL_c}$$

Efficiency of common fin shapes

Triangular[b]

$$A_f = \frac{\pi D}{2}[L^2 + (D/2)^2]^{1/2}$$

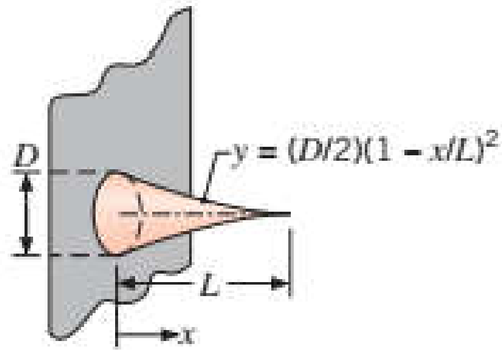
$$V = (\pi/12)D^2L$$



$$\eta_f = \frac{2}{mL}\frac{I_2(2mL)}{I_1(2mL)}$$

Parabolic[b]

$$A_f = \frac{\pi L^3}{8D}\left\{C_3C_4 - \frac{L}{2D}\ln[(2DC_4/L) + C_3]\right\}$$



$$\eta_f = \frac{2}{[4/9(mL)^2 + 1]^{1/2} + 1}$$

$$C_3 = 1 + 2(D/L)^2$$

$$C_4 = [1 + (D/L)^2]^{1/2}$$

$$V = (\pi/20)D^2L$$

[a] $m = (2h/kt)^{1/2}$.

[b] $m = (4h/kD)^{1/2}$.

Steady State and Transient Conduction

Heat and Mass Transfer

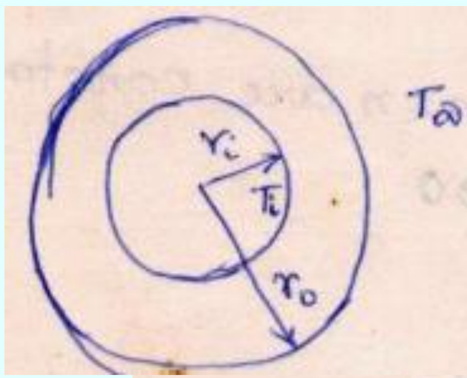
Department of Power Engineering, Jadavpur University, Kolkata–700106, India

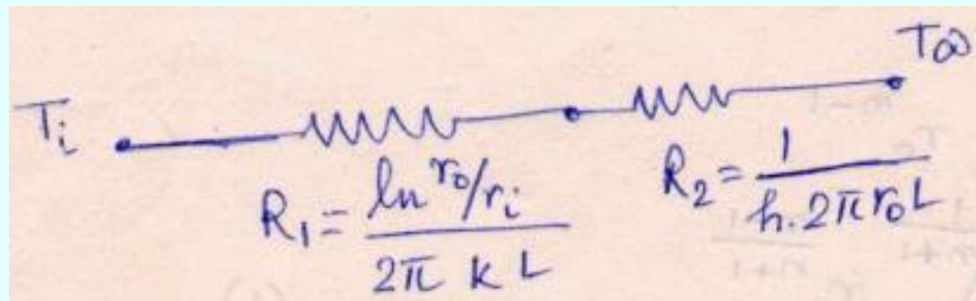
Contents?

- **Steady State Conduction**, Analysis of fins, Critical thickness of insulation, Systems with internal heat generation.
- **Transient conduction analysis:** Application of numerical methods to conduction problems.
- **Theory of heat convection.** Conservation equation of energy, mass & momentum and their analogies.
- **Significance of various dimensionless numbers**, laminar & turbulent boundary layer concept, thermal boundary layer, forced convection inside tubes and ducts, Forced convection over external bodies, Natural convection.
- **Boiling and Condensation**
- **Radiation** properties and laws, Radiation exchange among black and gray bodies, Electrical analogy, Radiation through participating gases.
- **Mass transfer** by convection and molecular diffusion, Fick's laws, Calculation of mass transfer coefficient, Interface mass transfer.

Critical Thickness of Insulation

- ***Cylinder:*** Let us consider a cylinder of outer radius r_i and temperature T_i (const.) kept in an atmosphere at temperature T_∞. Let the convective heat transfer coefficient from the cylinder to the atmosphere is h, which is constant. Now suppose an insulating layer is added to the cylinder such that the outer radius of the insulation is r_0. the conductivity of the insulating material is K. So, there are two resistance: conductive and convective resistance





- Total resistance

$$R = R_1 + R_2 = \frac{\ln \frac{r_o}{r_i}}{2\pi k L} + \frac{1}{2\pi h r_o L}$$

Critical Thickness of Insulation

$$\Rightarrow \frac{dR}{dr_0} = \frac{1}{2\pi k L r_0} - \frac{1}{2\pi h L r_0^2} = 0 \Rightarrow 2\pi h L r_0^2 = 2\pi k L r_0$$

$$\Rightarrow r_0 = \frac{k}{h}$$

$$\frac{d^2R}{dr_0^2} = -\frac{1}{2\pi k L r_0^2} + \frac{1}{\pi h L r_0^3}$$

Putting $r_0 = \frac{k}{h}$, $\frac{d^2R}{dr_0^2} = -\frac{h^2}{2\pi k^3 L} + \frac{h^2}{\pi k^3 L} = \frac{h^2}{2\pi k^3 L}$ which is +ve

- So, at **r_0=k/h** the total resistance to heat transfer is the minimum and therefore the associated heat transfer rate is the maximum, when the wall temperature difference is defined. This is thickness of insulation is called the critical insulation thickness. It implies, minimum resistance, resulting maximum heat transfer rate for a defined temperature or a minimum temperature for a defined $\dot{q}$. Here the heat transfer coefficient ***h*** is assumed to be constant.

Critical Thickness of Insulation

- When ***h*** is function of r_0 and the film temperature $(T_0 + T_\infty)/2$, where T_0 is the temperature on the outermost radius of insulation. Let consider

$$h = C r_0^{-m} (T_0 - T_\infty)^n, \text{ where } C, m \,\&\, n \text{ are constants } \,\&\, m, n \geq 0$$

$$\therefore \dot{q} = h \cdot 2\pi r_0 L (T_0 - T_\infty)$$

$$= 2\pi r_0 L \{C r_0^{-m} (T_0 - T_\infty)^n\} (T_0 - T_\infty)$$

$$= 2\pi C L r_0^{1-m} (T_0 - T_\infty)^{n+1}$$

$$\Rightarrow (T_0 - T_\infty)^{n+1} = \frac{\dot{q}}{2\pi C L} r_0^{m-1}$$

$$\Rightarrow T_0 - T_\infty = \left(\frac{\dot{q}}{2\pi C L}\right)^{\frac{1}{n+1}} r_0^{\frac{m-1}{n+1}} \quad \ldots\ldots (1)$$

Critical Thickness of Insulation

- For the steady state conduction across the insulation

$$T_i - T_o = \frac{\dot{q} \ln \frac{r_o}{r_i}}{2\pi k L}$$

$$\therefore \quad T_i - T_\infty = \frac{\dot{q}}{2\pi k L} \ln \frac{r_o}{r_i} + \left(\frac{\dot{q}}{2\pi C L}\right)^{\frac{1}{n+1}} r_o^{\frac{m-1}{n+1}}$$

Differentiating w.r.t r_o (as $T_i - T_\infty$ is ~~a constant~~ a minimum for hollow cylinder or const. for solid)

$$\frac{d(T_i - T_\infty)}{dr_o} = 0 = \frac{\dot{q}}{2\pi k L} \cdot \frac{1}{r_o} + \left(\frac{\dot{q}}{2\pi C L}\right)^{\frac{1}{n+1}} \frac{m-1}{n+1} r_o^{\frac{m-1}{n+1} - 1}$$

$$\Rightarrow \cancel{\frac{\dot{q}}{2\pi k L r_o}} (\dot{q})^{1 - \frac{1}{n+1}} + 2\pi k L r_o \cdot \left(\frac{1}{2\pi C L}\right)^{\frac{1}{n+1}} \cdot \frac{m-1}{n+1} r_o^{\frac{m-1}{n+1} - 1} = 0$$

$$\Rightarrow \dot{q}^{\frac{n}{n+1}} + k \cdot \frac{(2\pi L)^{n/n+1}}{C^{1/n+1}} \cdot \frac{m-1}{n+1} r_o^{\frac{m-1}{n+1}} = 0 \quad \ldots (2)$$

Critical Thickness of Insulation

Again, from eq.(1)

$$(\dot{q})^{\frac{1}{n+1}} = (2\pi C L)^{\frac{1}{n+1}} r_o^{\frac{-m+1}{n+1}} (T_o - T_\infty)$$

$$\Rightarrow (\dot{q})^{\frac{n}{n+1}} = (2\pi C L)^{\frac{n}{n+1}} r_o^{\frac{n(-m+1)}{n+1}} (T_o - T_\infty)^n$$

Putting the above in eq. (2)

$$(2\pi C L)^{\frac{n}{n+1}} r_o^{\frac{n(1-m)}{n+1}} (T_o - T_\infty)^n + k \cdot \frac{(2\pi L)^{\frac{n}{n+1}}}{C^{1/(n+1)}} \cdot \frac{m-1}{n+1} r_o^{\frac{m-1}{n+1}} = 0$$

$$\Rightarrow C^{\frac{n}{n+1}} r_o^{\frac{n(1-m)}{n+1}} (T_o - T_\infty)^n + \frac{k}{C^{1/(n+1)}} \cdot \frac{m-1}{n+1} r_o^{\frac{m-1}{n+1}} = 0$$

Critical Thickness of Insulation

$$\Rightarrow \quad c \, r_o^{\frac{n(1-m)-m+1}{m+1}} (T_o - T_\infty)^n + k \cdot \frac{m-1}{n+1} = 0$$

$$\Rightarrow \quad c \cdot r_o^{1-m} (T_o - T_\infty)^n + k \cdot \frac{m-1}{n+1} = 0$$

$$\Rightarrow \quad h \, r_o = k \cdot \frac{1-m}{n+1}$$

$$\Rightarrow \quad \boxed{r_o = \frac{k}{h} \cdot \frac{1-m}{n+1}}, \quad \text{for } m, n \geq 0$$

Critical Thickness of Insulation

- ***Sphere:*** *with constant* ***h***

$$R_1 = \text{Conductive resistance} = \frac{r_o - r_i}{4\pi k r_i r_o}$$

$$R_2 = \text{Convective resistance} = \frac{1}{4\pi r_o^2 h}$$

$$R = R_1 + R_2 = \frac{1}{4\pi k}\left(\frac{1}{r_i} - \frac{1}{r_o}\right) + \frac{1}{4\pi r_o^2 h}$$

$$\frac{dR}{dr_o} = \frac{1}{4\pi k r_o^2} - \frac{2}{4\pi h r_o^3}$$

for R being extremum, $\frac{dR}{dr_o} = 0 \Rightarrow 4\pi h r_o^3 = 2 \times 4\pi k r_o^2$

$$\Rightarrow \boxed{r_o = \frac{2k}{h}}$$

- at $r_o = \frac{2k}{h}$, $\frac{d^2R}{dr_o^2} = +\text{'ve}$, so the total resistance is minimum.

Critical Thickness of Insulation

Sphere: with variable *h*

Let $h = C r_o^{-m} (T_o - T_\infty)^n$

$$\dot{q} = 4\pi r_o^2 h (T_o - T_\infty) = 4\pi C r_o^{2-m} (T_o - T_\infty)^{n+1}$$

$$\therefore \quad T_o - T_\infty = \left(\frac{\dot{q}}{4\pi C}\right)^{\frac{1}{n+1}} r_o^{\frac{m-2}{n+1}} \quad \dots\dots\dots \quad (1)$$

From Conduction, $\quad T_i - T_o = \dfrac{\dot{q}}{4\pi k}\left(\dfrac{1}{r_i} - \dfrac{1}{r_o}\right) \quad \dots \quad (2)$

Adding (1) & (2),

$$T_i - T_\infty = \frac{\dot{q}}{4\pi k}\left(\frac{1}{r_i} - \frac{1}{r_o}\right) + \left(\frac{\dot{q}}{4\pi C}\right)^{\frac{1}{n+1}} r_o^{\frac{m-2}{n+1}}$$

Taking derivative w.r.t r_o,

$$0 = \frac{\dot{q}}{4\pi k} \cdot \frac{1}{r_o^2} + \left(\frac{\dot{q}}{4\pi C}\right)^{\frac{1}{n+1}} \cdot \frac{m-2}{n+1}\, r_o^{\frac{m-2}{n+1} - 1}$$

Critical Thickness of Insulation

$$\Rightarrow \frac{\dot{q}}{4\pi k} + \left(\frac{\dot{q}}{4\pi C}\right)^{\frac{1}{n+1}} \frac{m-2}{n+1} r_o^{\left(\frac{m-2}{n+1}+1\right)} = 0$$

$$\Rightarrow \dot{q}^{\frac{n}{n+1}} + \frac{k}{C^{1/n+1}} \cdot (4\pi)^{\frac{n}{n+1}} \cdot \frac{m-2}{n+1} r_o^{\left(\frac{m-2}{n+1}+1\right)} = 0$$

from eq. (1), $\dot{q} = 4\pi C r_o^{2-m} (T_o - T_\infty)^{n+1}$

$$\therefore \left\{4\pi C r_o^{2-m} (T_o - T_\infty)^{n+1}\right\}^{\frac{n}{n+1}} + \frac{k}{C^{\frac{1}{n+1}}} (4\pi)^{\frac{n}{n+1}} \frac{m-2}{n+1} r_o^{\left(\frac{m-2}{n+1}+1\right)} = 0$$

$$\Rightarrow C (T_o - T_\infty)^n r_o^{(2-m)\frac{n}{n+1}} + k \cdot \frac{m-2}{n+1} r_o^{\left(\frac{m-2}{n+1}+1\right)} = 0.$$

$$\Rightarrow C (T_o - T_\infty)^n r_o^{\frac{(2-m)n - m + 2}{n+1} - 1} + k \cdot \frac{m-2}{n+1} = 0$$

Critical Thickness of Insulation

$$\Rightarrow \quad c\,(T_0 - T_\infty)^{n}\, r_0^{\,1-m} + k\,\frac{m-2}{n+1} = 0$$

$$\Rightarrow \quad h\, r_0 = k\,\frac{2-m}{n+1}$$

$$\Rightarrow \quad \boxed{r_0 = \frac{k}{h}\,\frac{2-m}{n+1}}$$

Variable Conductivity

- In many heat conduction process, the material of the medium is not homogeneous and the conductivity varies across the medium. This would be as $k = k(x)$ or $k(r)$ for 1-D conduction in plane, cylindrical and spherical geometries. Also the imposed temperature conditions can often be sufficiently different so that the conductivity varies enough with temperature, $k = k(T)$, that the effect must also be taken into account.
- There can also be common circumstances when both the effects arise together as in an inhomogeneous material with highly temperature dependent thermal conductivity as $k(x,t)$ or $K(r,t)$.
- **Temperature dependent conductivity:**

Plane Region:

Governing Equ^{n} — $\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$ for steady state & without heat generation

$$\Rightarrow k\frac{dT}{dx} = C_1$$

This implies constant heat flux $\dot{q}''_x$ at all x across the medium as $\dot{q}''_x = -k\frac{dT}{dx}$. But as k is variable, $\frac{dT}{dx}$ (i.e. temperature gradient) is not constant.

Variable Conductivity

$$\int_{T_1}^{T_2} k\,dT = C_1 \int_0^L dx$$, when the boundary conditions are $x=0,\ T=T_1$ and $x=L,\ T=T_2$.

$$\Rightarrow \int_{T_1}^{T_2} k\,dT = C_1 L.$$

$$\Rightarrow C_1 = \frac{1}{L}\int_{T_1}^{T_2} k\,dT$$

Average or effective conductivity $k_m = \frac{1}{(T_2 - T_1)}\int_{T_1}^{T_2} k\,dT$

k_m is an integrated average value of $k(T)$ over the temperature range T_1 to T_2. This is in general not equal to $\frac{k(T_1)+k(T_2)}{2}$ for a nonlinear variation of $k(T)$.

Variable Conductivity

$$C_1 = \frac{1}{L}\int_{T_1}^{T_2} k\,dT = \frac{k_m(T_2 - T_1)}{L}$$

$$\dot{q}'' = -C_1 = \frac{k_m(T_1 - T_2)}{L} = \frac{T_1 - T_2}{\frac{L}{k_m}}$$

For the temperature distribution integrating over 0 to x

$$\int_{T_1}^{T} k\,dT = C_1\int_0^x dx = C_1 x = \frac{k_m(T_2 - T_1)}{L}\cdot x$$

Postulating the functional dependence of $k(T)$ first k_m is determined and then using the above integral the dependence of T on x can be evaluated.

Variable Conductivity

- **Similar variation for cylindrical system:**

$$\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = 0 \Rightarrow rk\frac{dT}{dr} = C_1$$

$$\Rightarrow k\,dT = C_1\frac{dr}{r}$$

$$\Rightarrow \int_{T_1}^{T_2} k\,dT = C_1\int_{r_1}^{r_2}\frac{dr}{r} = C_1 \ln\frac{r_2}{r_1} = -\frac{\dot{q}'}{2\pi}\ln\frac{r_2}{r_1}$$

$$\dot{q} = -2\pi r L k\frac{dT}{dr}$$

$$\Rightarrow \dot{q}' = \frac{\dot{q}}{L} = -2\pi r k\frac{dT}{dr}$$

$$\therefore C_1 = rk\frac{dT}{dr} = -\frac{\dot{q}'}{2\pi}$$

$$k_m = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} k\,dT$$

$$\therefore \dot{q}' = \frac{2\pi k_m (T_1 - T_2)}{\ln\frac{r_2}{r_1}}$$

$$\Rightarrow \dot{q} = \frac{T_1 - T_2}{\frac{\ln r_2/r_1}{2\pi k_m L}}$$

Again for temp. distr. $\int_{T_1}^{T} k\,dT = \frac{k_m(T_2 - T_1)}{\ln r_2/r_1}\ln\frac{r}{r_1}$

Variable Conductivity

- **Location Dependent Conductivity:** this condition commonly arises because of material inhomogeneity during fabrication or processing. Examples of such materials are heat-treated materials. Other common circumstances include a material which is a mixed composite of several spatial density embeded in another uniform material.
- **Plane Wall:** for wall with location dependent conductivity $k = k(x)$. For 1-D steady state conduction without any heat generation:

$$\frac{d}{dx}\left[k(x)\frac{dT}{dx}\right] = 0$$

with boundary conditions $T = T_1$ at $x = 0$ and $T = T_2$ at $x = L$.

$k(x)\frac{dT}{dx} = C_1$ indicates the local heat flux at any x, i.e. $\dot{q}''_x$, is a constant quantity. However, as k is a fn. of x, so $\frac{dT}{dx}$ is also not a constant. This means that the temperature distribution is ~~to~~ no longer a straight line.

Variable Conductivity

$$\int_{T_1}^{T_2} dT = \int_0^L \frac{C_1}{k(x)}\,dx$$

$$\Rightarrow T_2 - T_1 = C_1 \int_0^L \frac{dx}{k(x)} \quad \Rightarrow C_1 = -\dot{q}_x'' = \frac{T_2 - T_1}{\int_0^L \frac{dx}{k(x)}}$$

Example: Let for a plane wall of 30cm thick the thermal conductivity is given by $k = 200 - 50x$, W/mK, where x is in m. The two end walls of the plate are maintained at 450°C and 35°C. Find the steady, one dimensional heat flux through the wall and the temperature distribution.

$$\dot{q}_x'' = \frac{T_1 - T_2}{\int_0^{0.3} \frac{dx}{200 - 50x}}, \qquad 200 - 50x = p.$$

$$\cancel{\Rightarrow -500 \cdot 2x\,dx = dp}$$

$$\Rightarrow -50\,dx = dp$$

$$= -\frac{50(T_1 - T_2)}{\int_{200}^{185} \frac{dp}{p}} = \frac{-50(450 - 35)}{\ln \frac{185}{200}} = 266156.87 \text{ W/m}^2$$

$$= 266.16 \text{ KW/m}^2.$$

Variable Conductivity

Cylindrical system:

$$\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = 0 \quad , \qquad k = k(r).$$

$$\Rightarrow \quad r\,k(r)\frac{dT}{dr} = C_1$$

$$\Rightarrow \quad dT = \frac{C_1}{r\,k(r)}\,dr$$

$$\dot{q}_r'' = -2\pi r \cdot k(r)\frac{dT}{dr} \Rightarrow$$ heat transfer rate per unit length of the cylinder.

$$\therefore \quad \dot{q}_r' = -2\pi C_1$$

Variable Conductivity

Considering boundary conditions as $T = T_1$ at $r = r_1$ and $T = T_2$ at $r = r_2$,

$$T_2 - T_1 = C_1 \int_{r_1}^{r_2} \frac{dr}{r\,k(r)} \Rightarrow C_1 = \frac{T_2 - T_1}{\int_{r_1}^{r_2} \frac{dr}{r.k(r)}}$$

Therefore,

$$\dot{q}'_r = \frac{2\pi (T_1 - T_2)}{\int_{r_1}^{r_2} \frac{dr}{r.k(r)}}$$

- **Solve for Spherical system : Assignment**

Variable Conductivity

- **Temperature and Location Dependent Conductivity:** When the material is such that its conductivity is a function of both location and temperature – with location due to non homogenity of the material and with temperature due to the inherent characteristic of the material – the problem gets complicated. It can be simplified if the two dependence of the conductivity is independent of each other and therefore cab be decoupled.

$$k(T, x) = \phi(T) \cdot \psi(x).$$

Then $$\frac{d}{dx}\left[k(T,x)\frac{dT}{dx}\right] = 0$$

$$\Rightarrow \quad \phi(T) \cdot \psi(x)\frac{dT}{dx} = C_1$$

$$\Rightarrow \quad \phi(T)\, dT = \frac{C_1}{\psi(x)}\, dx$$

$$\Rightarrow \quad \int_{T_1}^{T_2} \phi(T)\, dT = \int_0^L \frac{C_1}{\psi(x)}\, dx = C_1 \int_0^L \frac{dx}{\psi(x)}$$

Variable Conductivity

$$C_1 = k(T,x)\frac{dT}{dx} = -\dot{q}_x''$$

$$\therefore \quad \dot{q}_x'' = \frac{-\int_{T_1}^{T_2}\phi(T)\,dT}{\int_0^L \frac{dx}{\psi(x)}} = \frac{\int_{T_2}^{T_1}\phi(T)\,dT}{\int_0^L \frac{dx}{\psi(x)}}$$

Heat Conduction with Spatially Varying Distributed Internal Energy Source

In an one-dimensional, steady heat transfer through a plane wall with internal energy generation, the governing eqn is

$$\frac{d^2T}{dx^2} + \frac{\dot{q}_g'''}{k} = 0$$

where, $\dot{q}_g'''$ is a constant rate of internal energy generation per unit volume. In a problem where the source arises as a surface load which penetrates the region, the flux level gradually diminishes in intensity by absorption. Such situation may arise in case of surface irradiation by fission products, microwave heating etc and the internal energy generation varies with location as $\dot{q}_g''' = \dot{q}_g'''(x)$

Heat Conduction with Spatially Varying Distributed Internal Energy Source

So, $$\frac{d^2T}{dx^2} = -\frac{\dot{q}_g'''(x)}{k}$$

$$\Rightarrow \quad T = -\iint \frac{\dot{q}_g'''(x)}{k}(dx)^2 + C_1 x + C_2.$$

Consider an incident heat flux $(\dot{q}_0'')$ on the wall which penetrates through the body and in the process gets attenuated by absorption.

The change in the heat flux is expressed as

$$\frac{d\dot{q}''}{dx} \propto \dot{q}'' \Rightarrow \frac{d\dot{q}''}{dx} = -\mu \dot{q}''$$ the negative sign indicates decay in $\dot{q}''$.

$$\Rightarrow \quad \dot{q}'' = + \dot{q}_0'' e^{-\mu x}$$

Heat Conduction with Spatially Varying Distributed Internal Energy Source

$$\dot{q}'''(x) = -\frac{d\dot{q}''}{dx} = \mu \dot{q}_0'' e^{-\mu x}$$

$$\therefore \frac{d^2T}{dx^2} = -\frac{\mu \dot{q}_0''}{k} e^{-\mu x}$$

$$\Rightarrow \frac{dT}{dx} = +\frac{\cancel{\mu}\, \dot{q}_0''}{k} e^{-\mu x} + C_1 \quad \Rightarrow T = -\frac{\cancel{\mu}\, \dot{q}_0''}{\mu k} e^{-\mu x} + C_1 x + C_2$$

Boundary condns. $\quad x = 0, \quad T = T_1$

$\quad x = L, \quad T = T_2$

$$T_1 = C_2 - \frac{\dot{q}_0''}{\mu k} \Rightarrow T = -\frac{\dot{q}_0''}{\mu k} e^{-\mu x} + C_1 x + T_1 + \frac{\dot{q}_0''}{\mu k}$$

$$T_2 = -\frac{\dot{q}_0''}{\mu k} e^{-\mu L} + C_1 L + T_1 + \frac{\dot{q}_0''}{\mu k}$$

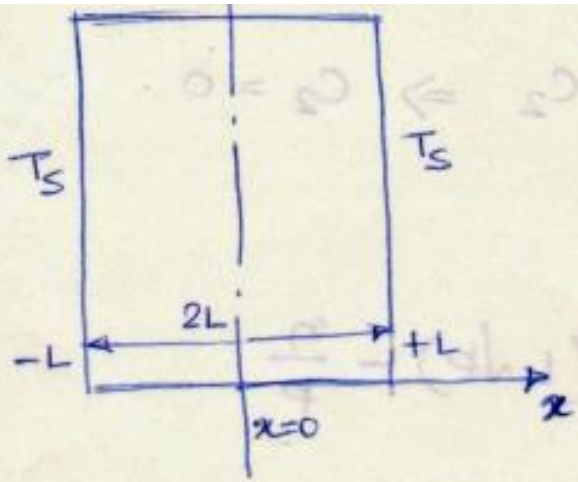
Heat Conduction with Spatially Varying Distributed Internal Energy Source

$$\Rightarrow C_1 = \frac{(T_2 - T_1)}{L} + \frac{\dot{q}_0''}{\mu k L} e^{-\mu L} - \frac{\dot{q}_0''}{\mu k L}$$

$$\therefore T = -\frac{\dot{q}_0''}{\mu k} e^{-\mu x} + (T_2 - T_1)\frac{x}{L} + \frac{\dot{q}_0''}{\mu k L} e^{-\mu L} x - \frac{\dot{q}_0''}{\mu k L} x + T_1 + \frac{\dot{q}_0''}{\mu k}$$

$$\Rightarrow T - T_1 = \frac{\dot{q}_0''}{\mu k}\left[\left(1 - e^{-\mu x}\right) - \frac{x}{L}\left(1 - e^{-\mu L}\right)\right] + (T_2 - T_1)\frac{x}{L}.$$

Temperature Dependent Energy Generation



Consider a plane wall, with temp. T_s at the two ends and having an internal generation given by,

$$\dot{q}_g'''(T) = \dot{q}_o''' \left[1+\beta(T-T_s)\right] = \dot{q}_o'''(1+\beta\theta)$$

where, $\theta = T - T_s$ and $\dot{q}_o'''$ is the local distributed source strength at the surface of the region.

Temperature Dependent Energy Generation

The governing equation under steady condition with one dimensional variation and for constant thermal conductivity may be written as

$$\frac{d^2T}{dx^2} + \frac{\dot{q}_g'''(T)}{k} = 0$$

$$\Rightarrow \frac{d^2\theta}{dx^2} + \frac{\dot{q}_0'''}{k}(1+\beta\theta) = 0$$

The equation has boundary conditions : $x = 0$, $\frac{dT}{dx} = \frac{d\theta}{dx} = 0$

$x = L$, $T = T_s \Rightarrow \theta = 0$

The governing eqn.:

$$\Rightarrow \frac{d^2\theta}{dx^2} + n + p\theta = 0, \text{ where } n = \frac{\dot{q}_0'''}{k}, \; p = \frac{\dot{q}_0'''\beta}{k}$$

Temperature Dependent Energy Generation

The above differential equation is linear in θ.

Putting $\phi = n + p\theta$, $\therefore \frac{d\phi}{dx} = p\frac{d\theta}{dx} \Rightarrow \frac{d^2\phi}{dx^2} = p\frac{d^2\theta}{dx^2}$

$$\therefore \frac{d^2\phi}{dx^2} + p.\phi = 0$$

The general solution of the equation is

$$\phi = C_1 \cos(x\sqrt{p}) + C_2 \sin(x\sqrt{p})$$

$$\Rightarrow \theta = C_1 \cos(x\sqrt{p}) + C_2 \sin(x\sqrt{p}) - \frac{n}{p}$$

$$\frac{d\theta}{dx} = -\sqrt{p}\, C_1 \sin(x\sqrt{p}) + \sqrt{p}\, C_2 \cos(x\sqrt{p})$$

at $x = 0$, $\frac{d\theta}{dx} = 0 \Rightarrow 0 = \sqrt{p}.C_2 \Rightarrow C_2 = 0$

$$\therefore \theta = C_1 \cos(x\sqrt{p}) - \frac{n}{p}$$

Temperature Dependent Energy Generation

at $x = L$, $\theta = 0 \Rightarrow 0 = C_1 \cos(L\sqrt{p}) - \frac{n}{p}$

$$\Rightarrow C_1 = \frac{n}{p\cos(L\sqrt{p})}$$

So,
$$\theta = \frac{n}{p\cos(L\sqrt{p})}\cos(x\sqrt{p}) - \frac{n}{p}$$

$$= \frac{n}{p}\left[\frac{\cos(x\sqrt{p})}{\cos(L\sqrt{p})} - 1\right] = \frac{1}{\beta}\left[\frac{\cos(x\sqrt{p})}{\cos(L\sqrt{p})} - 1\right]$$

$$\therefore T = T_s + \frac{1}{\beta}\left[\frac{\cos(x\sqrt{p})}{\cos(L\sqrt{p})} - 1\right]$$

Temperature Dependent Energy Generation

at $x = 0$, $T = T_0$ & $\theta = \theta_0 = \frac{1}{\beta}\left[\frac{1}{\cos(L\sqrt{p})} - 1\right]$

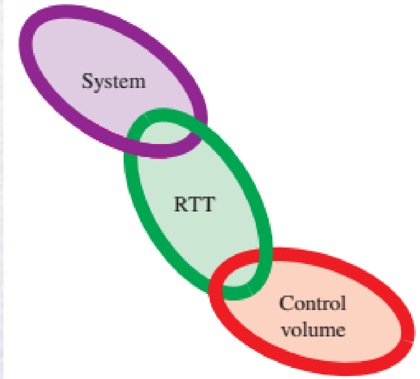
$$\therefore \boxed{\frac{\theta}{\theta_0} = \frac{T - T_s}{T_0 - T_s} = \frac{\cos(x\sqrt{p}) - \cos(L\sqrt{p})}{1 - \cos(L\sqrt{p})}}$$

$\frac{\theta}{\theta_0}$ varies between 1 (at $x=0$) to 0 (at $x=L$).

For $L\sqrt{p} = \pi/2$, $\frac{\theta}{\theta_0} = \cos(x\sqrt{p}) \Rightarrow \theta = \theta_0 \cos(x\sqrt{p})$

Reynolds Transport Theorem

A theorem which relates the variation of a property within a system with that in a control volume.



$$\left.\frac{dN}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV} \rho \eta \, d\forall + \int_{CS} \rho \eta (\vec{V}_r \cdot \hat{n})\, dA$$

N = extensive property, (mass, momentum, energy)

η = corresponding specific property (per unit mass)

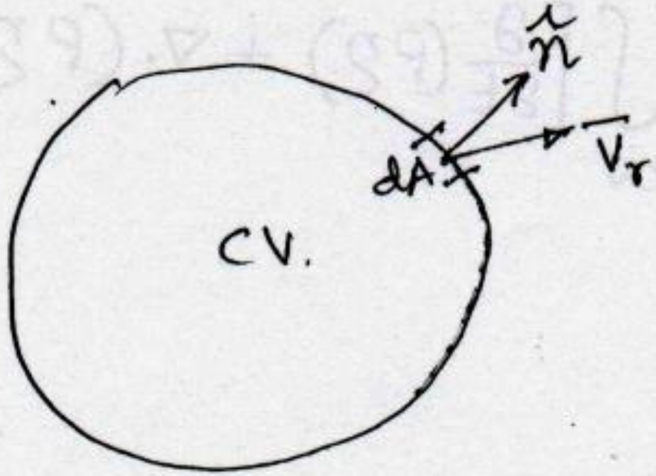
$= \dfrac{N}{m}$

ρ = fluid density

$\vec{V}_r$ = Relative velocity of fluid with respect to the control volume at the control surface

$\hat{n}$ = unit area vector at the control surface

Reynolds Transport Theorem



Assumptions:

1) Control volume is stationary. $\Rightarrow \bar{V}_r$ is equal to the absolute velocity of fluid $\bar{V}$

2) Control ~~surface~~ volume is ~~not moving~~ (non-deformable).

Thus the shape and size of the control volume are not changing with time.

Reynolds Transport Theorem

$$\therefore \left.\frac{dN}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV} \rho\eta \, d\forall + \int_{CS} \rho\eta(\vec{V}\cdot\hat{n})\,dA$$

Applying Gauss divergence theorem,

$$\int_{CS} \rho\eta(\vec{V}\cdot\hat{n})\,dA = \int_{CV} \nabla\cdot(\rho\eta\bar{V})\,d\forall$$

$$\therefore \left.\frac{dN}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV} \rho\eta \, d\forall + \int_{CV} \nabla\cdot(\rho\eta\bar{V})\,d\forall$$

As, the control volume is non-deformable

$$\frac{\partial}{\partial t}\int_{CV} \rho\eta \, d\forall = \int_{CV} \frac{\partial}{\partial t}(\rho\eta)\,d\forall$$

Conservation of Mass

$$\therefore \left.\frac{dN}{dt}\right|_{sys} = \int_{CV} \frac{\partial}{\partial t}(\rho\eta)\, d\forall + \int_{CV} \nabla\cdot(\rho\eta\overline{V})\, d\forall$$

$$\Rightarrow \boxed{\left.\frac{dN}{dt}\right|_{sys} = \int_{CV} \left[\frac{\partial}{\partial t}(\rho\eta) + \nabla\cdot(\rho\eta\overline{V})\right] d\forall}$$

- **Control Volume Approach:**

Using the relation between system approach and C.V. approach, we write,

$$\left.\frac{dN}{dt}\right|_{sys} = \int_{CV} \left[\frac{\partial}{\partial t}(\rho\eta) + \nabla\cdot(\rho\eta\overline{V})\right] d\forall .$$

Conservation of Mass

When, N = mass, m, $\eta = 1$.

So,

$$\left.\frac{dm}{dt}\right|_{sys} = \int_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{V}) \right] d\forall .$$

But, from the definition of a system,

$$\left.\frac{dm}{dt}\right|_{sys} = 0 .$$

So, $$\int_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{V}) \right] d\forall = 0 \Leftarrow \left\{ \text{This is the integral form of the conservation of mass equation} \right.$$

Thus,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{V}) = 0} \Rightarrow \text{conservation of mass equation in differential form.}$$

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \bar{V}\cdot\nabla\rho + \rho\,\nabla\cdot\bar{V} = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho\,\nabla\cdot\bar{V} = 0.$$

where $\frac{D}{Dt}$ is the substantial derivative
or
(total derivative).

such that,

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \bar{V}\cdot\nabla\phi$$

$$= \frac{\partial\phi}{\partial t} + v_x\frac{\partial\phi}{\partial x} + v_y\frac{\partial\phi}{\partial y} + v_z\frac{\partial\phi}{\partial z} \quad \left[\text{in Cartesian system}\right]$$

Conservation of Mass

Substantial derivative indicates the rate of change of a quantity ϕ (e.g. mass, ~~temp~~., conc. etc.) as experienced by an observer who is moving along with the flow. The observations made by a moving observer are affected by the stationary time rate of change of the property and also depends on where the observer goes as it floats along with the flow.

To elaborate, ϕ can be a function of t, x, y and z in the fluid continuum. If the change in observed ϕ by the observer after time Δt, when the observer shifts position by $\Delta x, \Delta y, \Delta z$ ~~can~~ is $\Delta\phi$, then

$$\Delta\phi = \frac{\partial\phi}{\partial t}\Delta t + \frac{\partial\phi}{\partial x}\Delta x + \frac{\partial\phi}{\partial y}\Delta y + \frac{\partial\phi}{\partial z}\Delta z$$

$$\Rightarrow \frac{\Delta\phi}{\Delta t} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial\phi}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial\phi}{\partial z}\frac{\Delta z}{\Delta t}$$

Conservation of Mass

as $\Delta t \to 0$,

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + v_x \frac{\partial \phi}{\partial x} + v_y \frac{\partial \phi}{\partial y} + v_z \frac{\partial \phi}{\partial z}$$

Substantial derivative is the rate of change of ϕ along a pathline.

- **Cartesian coordinate system:**

$$\frac{\partial \rho}{\partial t} + \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\rho v_x + \hat{j}\rho v_y + \hat{k}\rho v_z\right) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0.$$

Conservation of Mass

For, steady flow, $\frac{\partial \rho}{\partial t} = 0$.

$\therefore \quad \nabla \cdot (\rho \bar{V}) = 0$

or, in Cartesian coordinate system;

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

For incompressible fluid, $\rho =$ const. Therefore, for the flow of incompressible fluid, steady or unsteady,

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

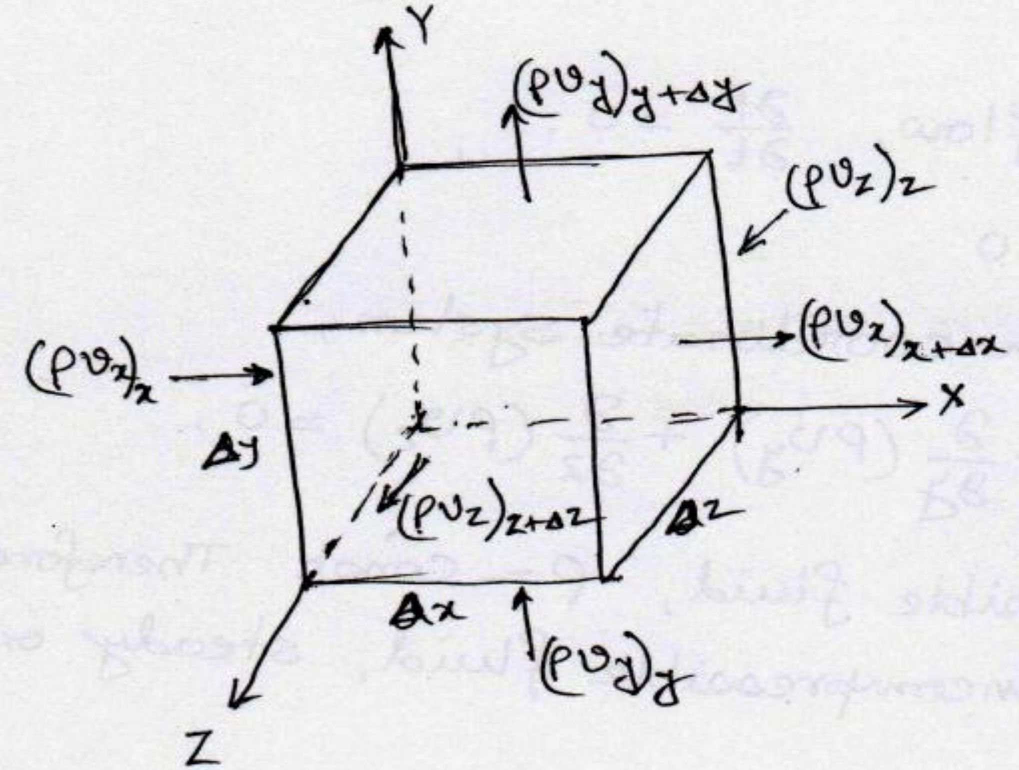
or, $\nabla \cdot \bar{V} = 0$.

Conservation of Mass

Alternative Approach:

(or C.V. fixed in space).

Let us consider an elemental system in Cartesian co-ordinate system fixed in space and having volume, $\Delta x\, \Delta y\, \Delta z$.



[The Control volume is fixed in space (i.e. stationary) and non-deformable (i.e. its volume is not changing).]

Conservation of Mass

$$\begin{bmatrix}\text{Rate of change of} \\ \text{mass within the} \\ \text{control volume}\end{bmatrix} = \begin{bmatrix}\text{Rate of mass} \\ \text{coming into} \\ \text{the C.V.}\end{bmatrix} - \begin{bmatrix}\text{Rate of mass} \\ \text{going out of} \\ \text{the C.V.}\end{bmatrix}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} \cdot \Delta x \Delta y \Delta z = \left[\rho v_x\big|_x \cdot \Delta y \Delta z + \rho v_y\big|_y \Delta z \Delta x + \rho v_z\big|_z \Delta x \Delta y \right] - \left[\rho v_x\big|_{x+\Delta x} \Delta y \Delta z + \rho v_y\big|_{y+\Delta y} \Delta z \Delta x + \rho v_z\big|_{z+\Delta z} \Delta x \Delta y \right]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\rho v_x\big|_{x+\Delta x} - \rho v_x\big|_x}{\Delta x} + \frac{\rho v_y\big|_{y+\Delta y} - \rho v_y\big|_y}{\Delta y} + \frac{\rho v_z\big|_{z+\Delta z} - \rho v_z\big|_z}{\Delta z} = 0$$

In the limit, as $\Delta x, \Delta y, \Delta z \longrightarrow 0$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0}$$

Conservation of Mass

In cylindrical coordinate system (r, θ, z),

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0}$$

In spherical coordinate system (r, θ, ϕ)

$\phi \rightarrow$ azimuthal

$\theta \rightarrow$ polar

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0}$$

Conservation of Momentum

Linear Momentum:

Again, from Reynolds transport theorem,

N = momentum = $m\overline{V}$ (a vector).

$\eta = \overline{V}$ (i.e. the velocity of fluid).

$$\left.\frac{dN}{dt}\right|_{sys} = \int_{CV} \left[\frac{\partial}{\partial t}(\rho\eta) + \nabla\cdot(\rho\eta\overline{V})\right] d\forall.$$

$$\Rightarrow \left.\frac{dN}{dt}\right|_{sys} = \int_{CV} \left[\frac{\partial}{\partial t}(\rho\overline{V}) + \nabla\cdot(\rho\overline{V}\,\overline{V})\right] d\forall$$

From Newton's law, $\left.\frac{dN}{dt}\right|_{sys} = \left.\frac{d}{dt}(m\overline{V})\right|_{sys}$

$$= \left.\Sigma\overline{F}\right|_{sys}.$$

Conservation of Momentum

In, The derivation of Reynolds transport theorem we consider the balance between the system and the control volume at a time when the control volume just overlaps the system. Thus the resultant force on the system is identical to the resultant force on the control volume.

So,
$$\int_{CV} \left[\frac{\partial}{\partial t}(\rho \bar{V}) + \nabla \cdot (\rho \bar{V}\,\bar{V})\right] d\forall = \sum \bar{F}_{cv}.$$

Force on the control volume can be of two types — body force and surface force. [in Continuum mechanics]

$$\therefore \sum \bar{F}_{CV} = \bar{F}_{b,cv} + \bar{F}_{s,cv}$$

Conservation of Momentum

Question of Incompressibility:

When is a given flow approximately incompressible?

Let us take the one dimensional spatial term of the conservation of mass equation

$$\frac{\partial}{\partial x}(\rho v_x) = \rho \frac{\partial v_x}{\partial x} + v_x \frac{\partial \rho}{\partial x} \approx \rho \frac{\partial v_x}{\partial x} \quad \left[\text{as if } \rho \text{ is a constant}\right]$$

This can be true when

$$\left| v_x \frac{\partial \rho}{\partial x} \right| << \left| \rho \frac{\partial v_x}{\partial x} \right|$$

$$\Rightarrow \left| \frac{\partial \rho}{\rho} \right| << \left| \frac{\partial v_x}{v_x} \right|$$

Conservation of Momentum

From the definition of speed of sound

$$a = \sqrt{\frac{\partial p}{\partial \rho}} \Rightarrow \delta p = a^2 \delta \rho$$

Again from Bernoullie's equation, neglecting change in elevation,

$$\delta p = -\rho v_x dv_x$$

$$\Rightarrow a^2 \delta \rho = -\rho v_x dv_x$$

$$\Rightarrow \frac{\delta \rho}{\rho} = -\frac{v_x^2}{a^2} \frac{dv_x}{v_x}$$

$$\Rightarrow \left| \frac{\delta \rho / \rho}{\delta v_x / v_x} \right| = \frac{v_x^2}{a^2} = M^2. \qquad [M = \text{Mach No.}]$$

Conservation of Momentum

So, the condition of incompressibility is valid when

$$M^2 << 1$$

Thus for low Mach no. the flow can be considered as incompressible. How low is low?

Commonly accepted limit is

$$M \leq 0.3$$

Thus for air at standard conditions, a flow can be considered as incompressible if the velocity is less than 100 m/s.