

# Homework for Unit 2

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## Problem 3

a) circumference :  $L = a_1 + a_2 + b + c$

$$b = \sqrt{a_2^2 + h^2}$$

$$c = \sqrt{a_1^2 + h^2}$$

$$\begin{aligned} L &= a_1 + a_2 + \sqrt{a_2^2 + h^2} + \sqrt{a_1^2 + h^2} \\ &= L(a_1, a_2, h) \end{aligned}$$

$$\frac{\partial L}{\partial a_1} = 1 + \frac{a_1}{\sqrt{a_1^2 + h^2}}$$

analog formula for  $a_2$

$$\frac{\partial L}{\partial h} = \frac{h}{\sqrt{a_2^2 + h^2}} + \frac{h}{\sqrt{a_1^2 + h^2}}$$

uncertainty variance :

$$\sigma_L^2 = \left( \frac{\partial L}{\partial a_1} \right)^2 \Delta a_1^2 + \left( \frac{\partial L}{\partial a_2} \right)^2 \Delta a_2^2 + \left( \frac{\partial L}{\partial h} \right)^2 \Delta h^2$$

$$= \sqrt{\Delta a_1^2 + \Delta a_2^2 + \Delta h^2}$$

$$\begin{aligned} &= \left( 1 + \frac{a_1}{\sqrt{a_1^2 + h^2}} \right)^2 \Delta a_1^2 + \left( 1 + \frac{a_2}{\sqrt{a_2^2 + h^2}} \right)^2 \Delta a_2^2 \\ &\quad + \left( \frac{h}{\sqrt{a_1^2 + h^2}} + \frac{h}{\sqrt{a_2^2 + h^2}} \right)^2 \Delta h^2 \end{aligned}$$

b, area

$$A = \frac{1}{2} (a_1 + a_2) \cdot h$$
$$= A(a_1, a_2, h)$$

$$\frac{\partial A}{\partial a_1} = \frac{\partial A}{\partial a_2} = \frac{h}{2}$$

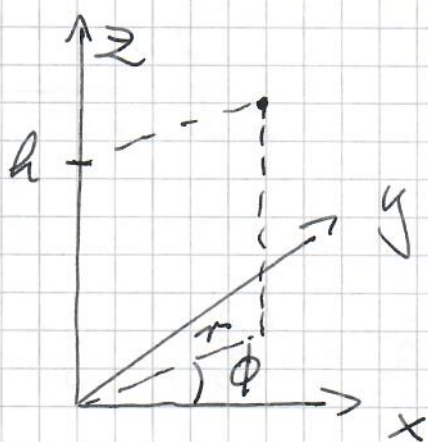
$$\frac{\partial A}{\partial h} = \frac{1}{2} (a_1 + a_2)$$

uncertainty variance:

$$\sigma_A^2 = \left( \frac{\partial A}{\partial a_1} \right)^2 \Delta a_1^2 + \left( \frac{\partial A}{\partial a_2} \right)^2 \Delta a_2^2 + \left( \frac{\partial A}{\partial h} \right)^2 \Delta h^2$$
$$= \frac{h^2}{4} (\Delta a_1^2 + \Delta a_2^2) + \frac{1}{4} (a_1 + a_2)^2 \Delta h^2$$

#### Problem 4

a)



express old (cartesian) variables in terms of new (cylindrical) ones:

$$x = r \cdot \cos(\phi)$$

$$y = r \cdot \sin(\phi)$$

$$z = h$$

b) Jacobian matrix:

$$J_{x,y,z} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial h} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial h} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial h} \end{pmatrix}$$

$$\frac{\partial x}{\partial r} = \cos(\phi)$$

$$\frac{\partial y}{\partial r} = \sin(\phi)$$

$$\frac{\partial z}{\partial r} = 0$$

$$\frac{\partial x}{\partial \phi} = -r \cdot \sin(\phi)$$

$$\frac{\partial y}{\partial \phi} = r \cdot \cos(\phi)$$

$$\frac{\partial z}{\partial \phi} = 0$$

$$\frac{\partial x}{\partial h} = \frac{\partial y}{\partial h} = 0$$

$$\frac{\partial z}{\partial h} = 1$$

$$J_{x,y,z} = \begin{pmatrix} \cos(\phi) & -r \cdot \sin(\phi) & 0 \\ \sin(\phi) & r \cdot \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) Jacobian determinant:

$$\begin{aligned} \det(J_{x,y,z}) &= r \cdot \cos^2(\phi) + r \cdot \sin^2(\phi) \\ &= r \end{aligned}$$

the 3D volume element transforms like

$$dx \cdot dy \cdot dz = r \cdot dr \cdot d\phi \cdot dh$$

### Problem 5:

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a) 1 mile = 1609 m

$$v = 30 \frac{\text{miles}}{\text{h}} = \underline{\underline{48.27 \frac{\text{km}}{\text{h}}}}$$

$$v = 60 \frac{\text{miles}}{\text{h}} = \underline{\underline{96.54 \frac{\text{km}}{\text{h}}}}$$

$$v = 70 \frac{\text{miles}}{\text{h}} = \underline{\underline{112.63 \frac{\text{km}}{\text{h}}}}$$

b) 1 foot = 0.3048 m  
1 acre = 4045 m<sup>2</sup>

• North America:

$$A = 200 \text{ feet} \cdot 85 \text{ feet} = 17 \cdot 10^3 \text{ feet}^2$$

$$A = 17 \cdot 10^3 \text{ feet}^2 = \overset{17000}{\text{feet}} \cdot \left(0.3048 \frac{\text{m}}{\text{foot}}\right)^2$$
$$= 1579 \text{ m}^2$$

$$A = 1579 \text{ m}^2 = 1579 \text{ m}^2 \cdot \frac{1 \text{ m}^2}{4045 \text{ Acre}}$$
$$= \underline{\underline{0.390 \text{ Acres}}}$$

• Europe:

$$A = 60 \text{ m} \cdot 30 \text{ m} = 1800 \text{ m}^2$$

$$A = 1800 \text{ m}^2 \cdot \frac{1 \text{ m}^2}{4045 \text{ Acre}} = \underline{\underline{0.445 \text{ Acres}}}$$

c)  $T_F = T_C \cdot \frac{9^\circ\text{F}}{5^\circ\text{C}} + 32^\circ\text{F}$

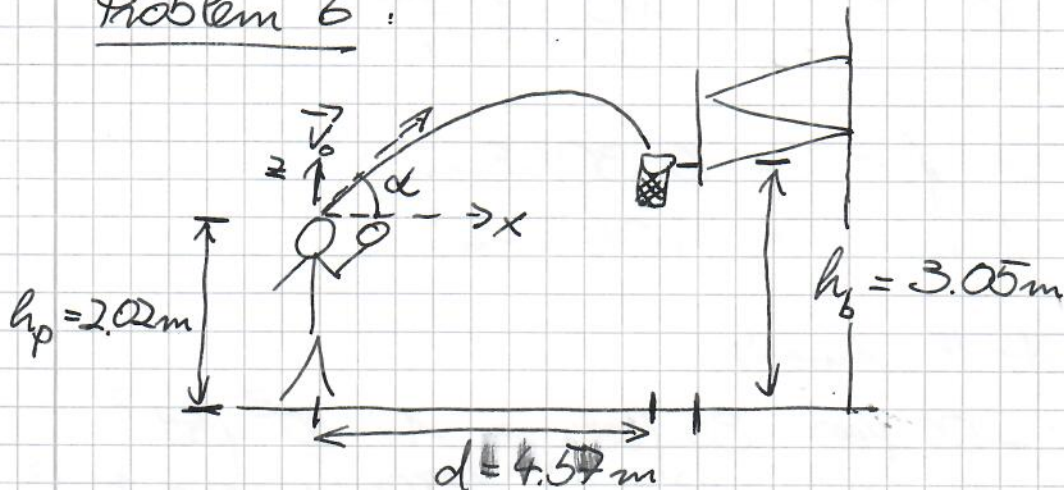
$$T_C = (T_F - 32^\circ\text{F}) \cdot \frac{5}{9} \frac{^\circ\text{C}}{^\circ\text{F}}$$

$$T_F = 103^\circ F$$

$$T_C = 71^\circ F \cdot \frac{5^\circ C}{9^\circ F} = \underline{\underline{39.4^\circ C}}$$

The patient has fever and should take medication.

Problem 6:



Put the origin of the coordinate system to the initial position of the ball.

The  $x$ -axis is oriented towards the basket and the  $z$ -axis towards the sky.

Without losing generality we can set  $y(t) = 0$  for all times  $t > 0$ .

distance ball - basket:

$$\Delta x = 4.57 \text{ m}$$

$$\Delta z = 3.05 \text{ m} - 2.02 \text{ m} = 1.03 \text{ m}$$

gravitational force directed towards the center of the earth:

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

no forces in  $x$  and  $y$ -directions

initial position:

$$\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

initial velocity:

$$\vec{v}_0 = v_0 \begin{pmatrix} \cos(\alpha) \\ 0 \\ \sin(\alpha) \end{pmatrix}$$

$$v_0 = 6 \cdot 10 \frac{\text{m}}{\text{s}}$$

constant acceleration:

$$\vec{a} = \frac{\vec{F}}{m} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

position at time  $t > 0$ :

$$\vec{x}(t) = \vec{x}_0 + \vec{v}_0 \cdot t + \frac{1}{2} \vec{a} t^2$$

$$= \begin{pmatrix} v_0 \cdot t \cdot \cos(\alpha) \\ 0 \\ v_0 \cdot t \cdot \sin(\alpha) - \frac{g}{2} t^2 \end{pmatrix}$$

plotting  $z(t)$  against  $x(t)$ , this is a parabola!

b) target point  $\vec{x}_b$  must be on this trajectory at an arbitrary time  $t > 0$

two equations:

$$(I) \quad v_0 \cdot t \cdot \cos(\alpha) = \Delta x$$

$$(II) \quad v_0 \cdot t \cdot \sin(\alpha) - \frac{g}{2} t^2 = \Delta z$$

solve (I) for  $t$ .  $t = \frac{\Delta x}{v_0 \cdot \cos(\alpha)}$

and plug into (II) =

$$\frac{v_0 \cdot \sin(\alpha)}{v_0 \cdot \cos(\alpha)} \Delta x - \frac{g}{2} \frac{\Delta x^2}{v_0^2 \cos^2(\alpha)} = \Delta z$$

multiply with  $\cancel{v_0} \cdot \cos^2(\alpha)$ :

$$\cancel{v_0} (\Delta x \cdot \sin(\alpha) \cdot \cos(\alpha) - \Delta z \cdot \cos^2(\alpha)) = \frac{g \Delta x^2}{2 v_0^2}$$

this can be solved graphically by plotting

$$f(\alpha) = \Delta x \cdot \sin(\alpha) \cos(\alpha) - \Delta z \cos^2(\alpha)$$

as a function of  $\alpha \in [0, 360^\circ]$   
and looking for intersections with

$$C = \frac{g \Delta x^2}{2 v_0^2} = \frac{9.81 \frac{\text{m}}{\text{s}^2} \cdot (4.57 \text{ m})^2}{2 \cdot (6.1 \frac{\text{m}}{\text{s}})^2} = 2.753 \text{ m}$$

There is no angle  $\alpha$ , under which the basket can be hit!

$C$  is too large!  $g$  and  $\Delta x$  cannot be changed but the initial velocity  $v_0$  can be increased.

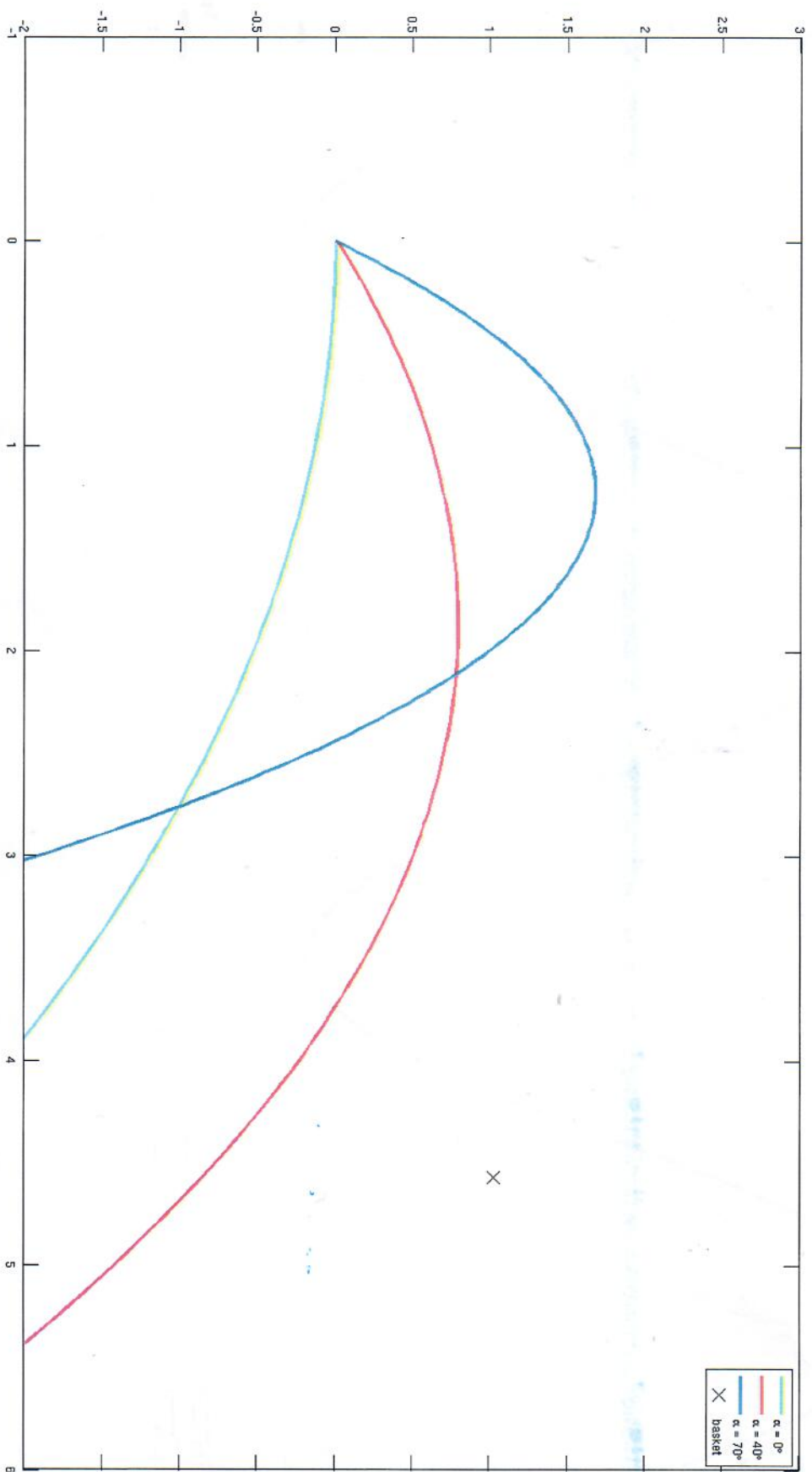
How much does it need to be increased?

$C$  must become smaller than the maxima of  $f(\alpha)$ .

→ either differentiate  $f(\alpha)$  or read the value off the  $y$ -axis of the plot:

$$C \lesssim 1.83 \text{ m} = C_0$$

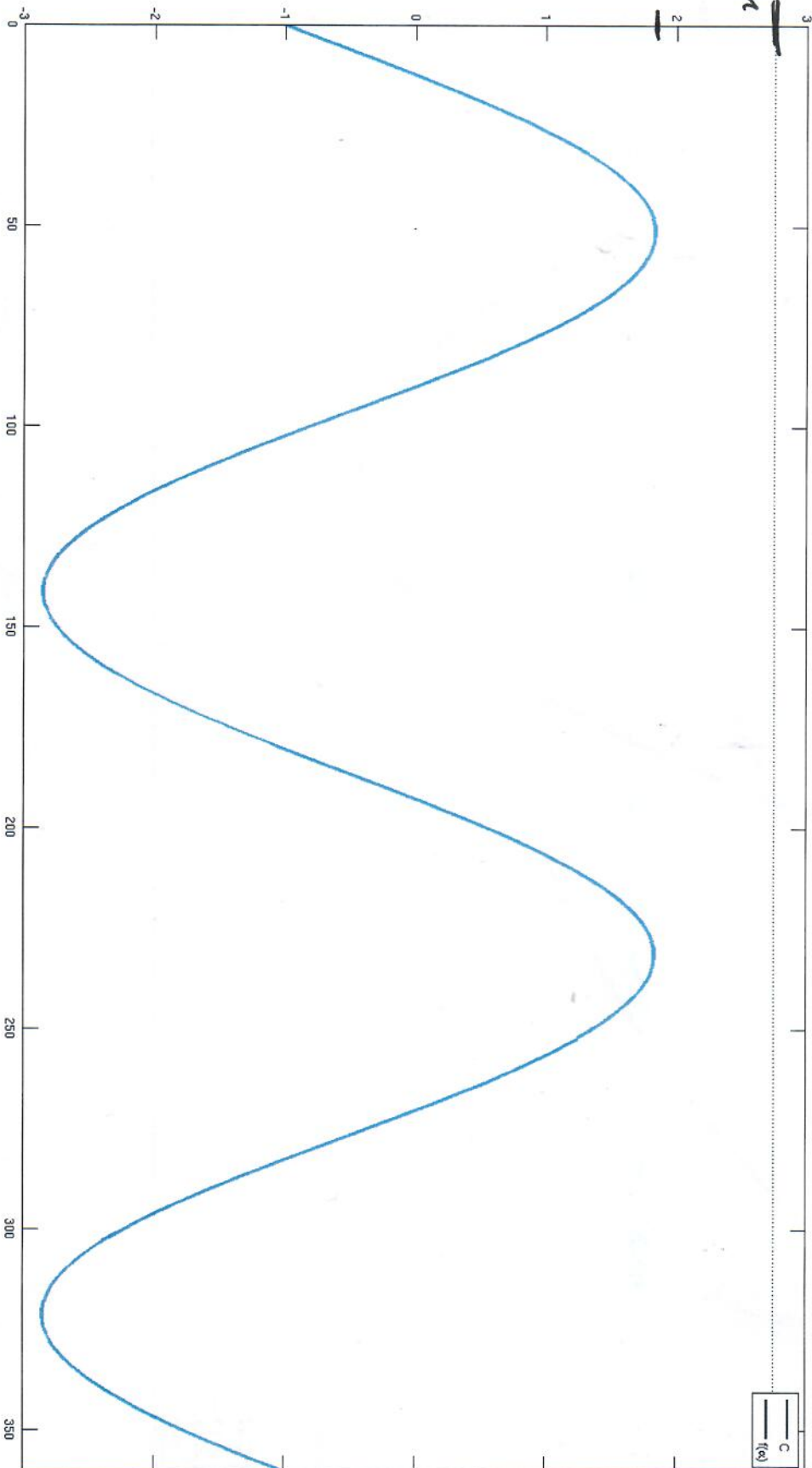
8- Problem 6a.



# Problem 66

$$C = 2.753m$$

$$C_0 = 1.83m$$



for the velocity this implies:

$$v_0 = \sqrt{\frac{g x^2}{2c}} \approx \sqrt{\frac{g x^2}{2c_0}}$$

$$v_0 \approx \sqrt{\frac{9.81 \frac{m}{s^2} \cdot (4.57 m)^2}{2 \cdot 1.83 m}} = \underline{\underline{7.48 \frac{m}{s}}}$$

The initial velocity was too slow.  
The ball must ~~be~~ thrown with at  
least  $7.48 \frac{m}{s}$ !

### Problem 7

$$m = 10 \cdot 10^3 \text{ kg}$$

$$v = 80 \frac{\text{km}}{\text{h}} = 80 \cdot \frac{1000 \text{ m}}{3600 \text{ s}} = 22.22 \frac{m}{s}$$

a) The total energy will be conserved.  
Kinetic energy will be transformed  
into potential energy:

$$E_{\text{kin}} = \frac{m}{2} v^2 = mgh = E_{\text{pot}}$$

$$h = \frac{v^2}{2g}$$

$$h = \frac{\left(22.22 \frac{m}{s}\right)^2}{2 \cdot 9.81 \frac{m}{s^2}} = \underline{\underline{25.17 m}}$$

Without friction, the lorry will  
continue to climb  $25.17 m$ !

b) Now friction forces consume half of the energy:

$$E_{\text{kin}} = \frac{m}{2} v^2 = E_{\text{pot}} + E_{\text{int}} \quad \#$$

$$E_{\text{int}} = E_{\text{pot}} = mgh'$$

$$\frac{m}{2} v^2 = 2 \cdot mgh'$$

$$h' = \frac{v^2}{4g} = \frac{h}{2}$$

Then the lony climbs only half as high: 12.58 m

c) Now  $v' = \frac{1}{2} v = 11.1 \frac{\text{m}}{\text{s}}$

Reuse old formula:

$$\frac{m}{2} v'^2 = \frac{m}{8} v^2 = mgh''$$

$$h'' = \frac{v^2}{8g} = \frac{h}{4}$$

With half the speed, the lony climbs only  $\frac{1}{4}$  as high: 6.29 m