

Homework for Unit 2

-1-

Problem 3

a) circumference : $L = a_1 + a_2 + b + c$

$$b = \sqrt{a_2^2 + h^2}$$

$$c = \sqrt{a_1^2 + h^2}$$

$$\begin{aligned} L &= a_1 + a_2 + \sqrt{a_2^2 + h^2} + \sqrt{a_1^2 + h^2} \\ &= L(a_1, a_2, h) \end{aligned}$$

$$\frac{\partial L}{\partial a_1} = 1 + \frac{\cancel{2a_1}}{\cancel{2}\sqrt{a_1^2 + h^2}}$$

analog formula for a_2

$$\frac{\partial L}{\partial h} = \frac{\cancel{rh}}{\cancel{2}\sqrt{a_2^2 + h^2}} + \frac{\cancel{rh}}{\cancel{2}\sqrt{a_1^2 + h^2}}$$

uncertainty variance :

$$\sigma_L^2 = \left(\frac{\partial L}{\partial a_1} \right)^2 \Delta a_1^2 + \left(\frac{\partial L}{\partial a_2} \right)^2 \Delta a_2^2 + \left(\frac{\partial L}{\partial h} \right)^2 \Delta h^2$$

$$= \cancel{\Delta a_1^2} + \cancel{\Delta a_2^2} + \cancel{\Delta h^2}$$

$$\begin{aligned} &= \left(1 + \frac{a_1}{\sqrt{a_1^2 + h^2}} \right)^2 \Delta a_1^2 + \left(1 + \frac{a_2}{\sqrt{a_2^2 + h^2}} \right)^2 \Delta a_2^2 \\ &\quad + \left(\frac{h}{\sqrt{a_1^2 + h^2}} + \frac{h}{\sqrt{a_2^2 + h^2}} \right)^2 \Delta h^2 \end{aligned}$$

b) area $A = \frac{1}{2} (a_1 + a_2) \cdot h$

$$= A(a_1, a_2, h)$$

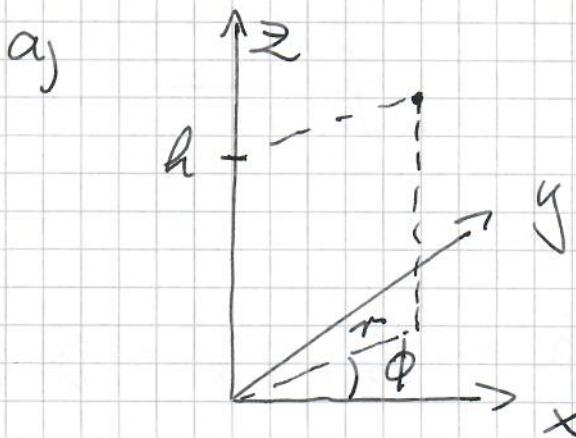
$$\frac{\partial A}{\partial a_1} = \frac{\partial A}{\partial a_2} = \frac{h}{2}$$

$$\frac{\partial A}{\partial h} = \frac{1}{2} (a_1 + a_2)$$

uncertainty variance:

$$\begin{aligned}\sigma_A^2 &= \left(\frac{\partial A}{\partial a_1}\right)^2 \Delta a_1^2 + \left(\frac{\partial A}{\partial a_2}\right)^2 \Delta a_2^2 + \left(\frac{\partial A}{\partial h}\right)^2 \Delta h^2 \\ &= \frac{h^2}{4} (\Delta a_1^2 + \Delta a_2^2) + \frac{1}{4} (a_1 + a_2)^2 \Delta h^2\end{aligned}$$

Problem 4



Express old (cartesian) variables in terms of new (cylindrical) ones:

$$x = r \cdot \cos(\phi)$$

$$y = r \cdot \sin(\phi)$$

$$z = h$$

b) Jacobian matrix:

$$\mathcal{J}_{x,y,z} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial h} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial h} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial h} \end{pmatrix}$$

$$\frac{\partial x}{\partial r} = \cos(\phi)$$

$$\frac{\partial y}{\partial r} = \sin(\phi)$$

$$\frac{\partial z}{\partial r} = 0$$

$$\frac{\partial x}{\partial \phi} = -r \cdot \sin(\phi)$$

$$\frac{\partial y}{\partial \phi} = r \cdot \cos(\phi)$$

$$\frac{\partial z}{\partial \phi} = 0$$

$$\frac{\partial x}{\partial h} = \frac{\partial y}{\partial h} = 0$$

$$\frac{\partial z}{\partial h} = 1$$

$$\mathcal{J}_{x,y,z} = \begin{pmatrix} \cos(\phi) & -r \cdot \sin(\phi) & 0 \\ \sin(\phi) & r \cdot \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) Jacobian determinant:

$$\det(\mathcal{J}_{x,y,z}) = r \cdot \cos^2(\phi) + r \cdot \sin^2(\phi) \\ = r$$

the 3D volume element transforms like

$$dx \cdot dy \cdot dz = r \cdot dr \cdot d\phi \cdot dh$$

Problem 5:

a) 1 mile = 1609 m

$$v = 30 \frac{\text{miles}}{\text{h}} = 48.27 \frac{\text{km}}{\text{h}}$$

$$v = 60 \frac{\text{miles}}{\text{h}} = 96.54 \frac{\text{km}}{\text{h}}$$

$$v = 70 \frac{\text{miles}}{\text{h}} = 112.63 \frac{\text{km}}{\text{h}}$$

b) 1 foot = 0.3048 m

$$1 \text{ acre} = 4045 \text{ m}^2$$

- North America:

$$A = 200 \text{ feet} \cdot 85 \text{ feet} = 17 \cdot 10^3 \text{ feet}^2$$

$$A = 17 \cdot 10^3 \text{ feet}^2 = 17000 \cdot \left(0.3048 \frac{\text{m}}{\text{foot}}\right)^2$$

$$= 1579 \text{ m}^2$$

$$A = 1579 \text{ m}^2 = 1579 \text{ m}^2 \cdot \frac{1 \text{ m}^2}{4045 \text{ Acre}}$$

$$= \underline{\underline{0.390 \text{ Acre}}}$$

- Europe:

$$A = 60 \text{ m} \cdot 30 \text{ m} = 1800 \text{ m}^2$$

$$A = 1800 \text{ m}^2 \cdot \frac{1 \text{ m}^2}{4045 \text{ Acre}} = \underline{\underline{0.445 \text{ Acre}}}$$

c) $T_F = T_C \cdot \frac{9}{5}^\circ C + 32^\circ F$

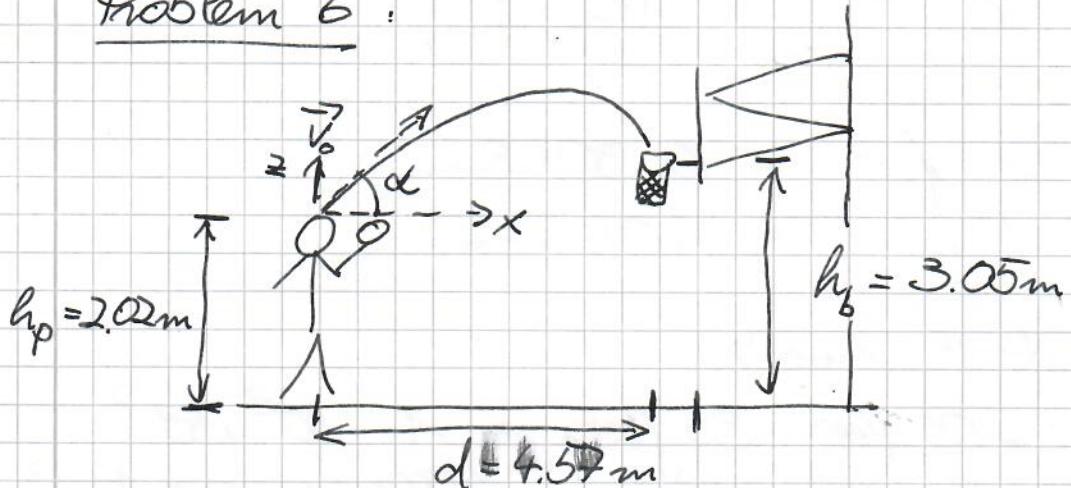
$$T_C = (T_F - 32^\circ F) \cdot \frac{5}{9} \frac{^\circ C}{^\circ F}$$

$$T_F = 103^\circ F$$

$$T_C = 71^\circ F \cdot \frac{5^\circ C}{9^\circ F} = \underline{\underline{39.4^\circ C}}$$

The patient has fever and should take medication.

Problem 6 :



Put the origin of the coordinate system to the initial position of the ball.

The x -axis is oriented towards the basket and the z -axis towards the sky.

Without lossing generality we can set $y(t) = 0$ for all times $t > 0$.

distance ball - basket :

$$\Delta x = 4.57\text{ m}$$

$$\Delta z = 3.05\text{ m} - 2.02\text{ m} = 1.03\text{ m}$$

gravitational force directed towards the center of the earth :

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

no forces in x and y -directions

initial position:

$$\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

initial velocity:

$$\vec{v}_0 = v_0 \begin{pmatrix} \cos(\alpha) \\ 0 \\ \sin(\alpha) \end{pmatrix}$$

$$v_0 = 6 \cdot 10 \frac{\text{m}}{\text{s}}$$

constant acceleration:

$$\vec{a} = \frac{\vec{F}}{m} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

position at time $t > 0$:

$$\begin{aligned} \vec{x}(t) &= \vec{x}_0 + \vec{v}_0 \cdot t + \frac{1}{2} \vec{a} t^2 \\ &= \begin{pmatrix} v_0 \cdot t \cdot \cos(\alpha) \\ 0 \\ v_0 \cdot t \cdot \sin(\alpha) - \frac{g}{2} t^2 \end{pmatrix} \end{aligned}$$

plotting $\vec{x}(t)$ against $x(t)$, this is a parabola!

b) target point \vec{x}_b must be on this trajectory at an arbitrary time $t > 0$

two equations:

$$(I) \quad v_0 \cdot t \cdot \cos(\alpha) = \Delta x$$

$$(II) \quad v_0 \cdot t \cdot \sin(\alpha) - \frac{g}{2} t^2 = \Delta y$$

$$\text{solve (I) for } t. \quad t = \frac{\Delta x}{v_0 \cdot \cos(\alpha)}$$

and plug into (II) =

$$\frac{\cancel{v_0} \cdot \sin(\alpha)}{\cancel{v_0} \cdot \cos(\alpha)} \Delta x - \frac{g}{2} \frac{\Delta x^2}{v_0^2 \cos^2(\alpha)} = \cancel{12}$$

multiply with ~~$v_0 \cdot \cos^2(\alpha)$~~ :

$$\cancel{v_0} (4x \cdot \sin(\alpha) \cdot \cos(\alpha) - 12 \cos^2(\alpha)) = \frac{g \Delta x^2}{2 v_0^2}$$

this can be solved graphically by plotting

$$f(\alpha) = 4x \cdot \sin(\alpha) \cos(\alpha) - 12 \cos^2(\alpha)$$

as a function of $\alpha \in [0, 360^\circ]$

and looking for intersections with

$$C = \frac{g \Delta x^2}{2 v_0^2} = \frac{9.81 \frac{m}{s^2} \cdot (4.57 m)^2}{2 \cdot (6.1 \frac{m}{s})^2} = 2.753 \text{ m}$$

There is no angle α , under which the basket can be hit!

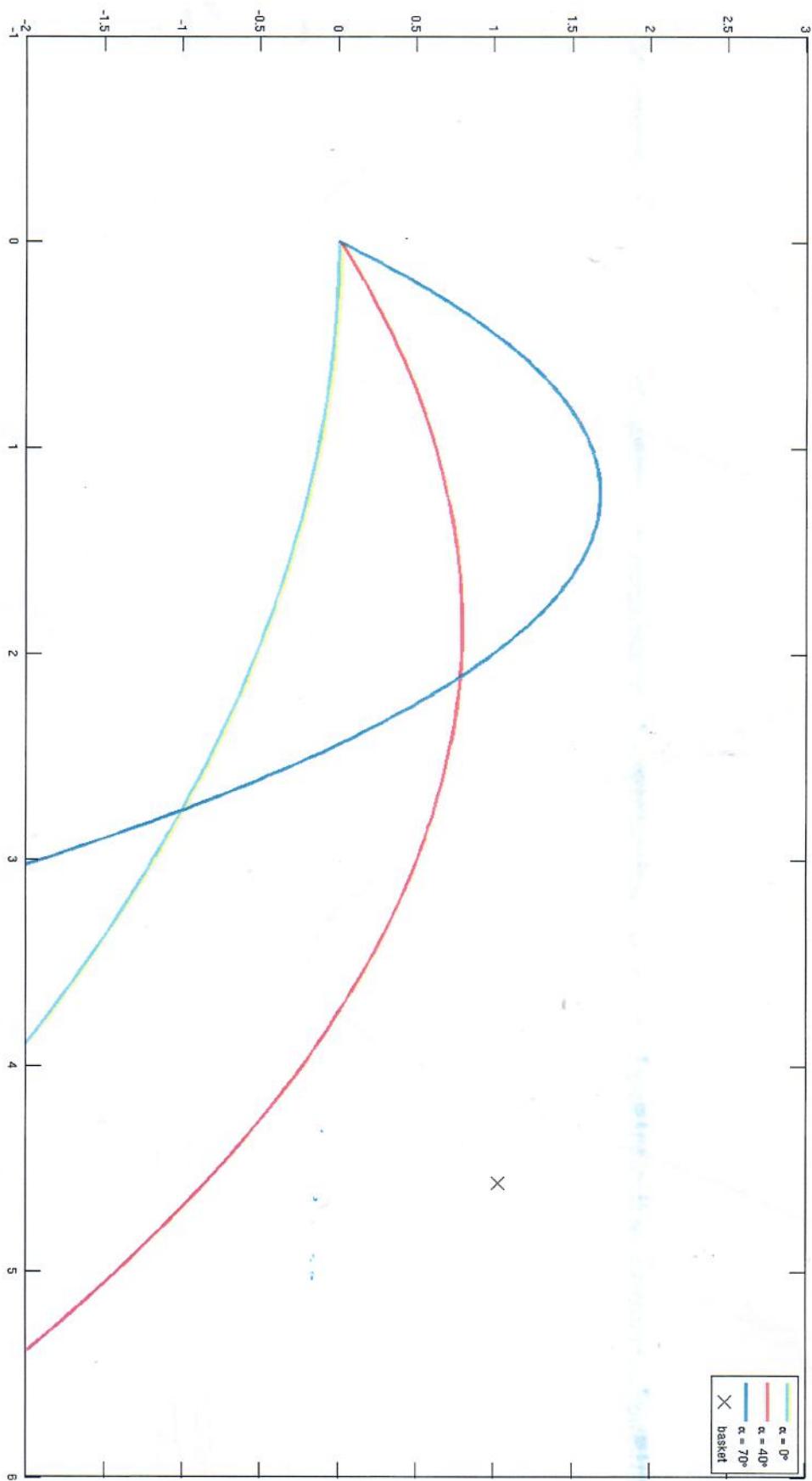
C is too large! g and Δx cannot be changed but the initial velocity v_0 can be increased.

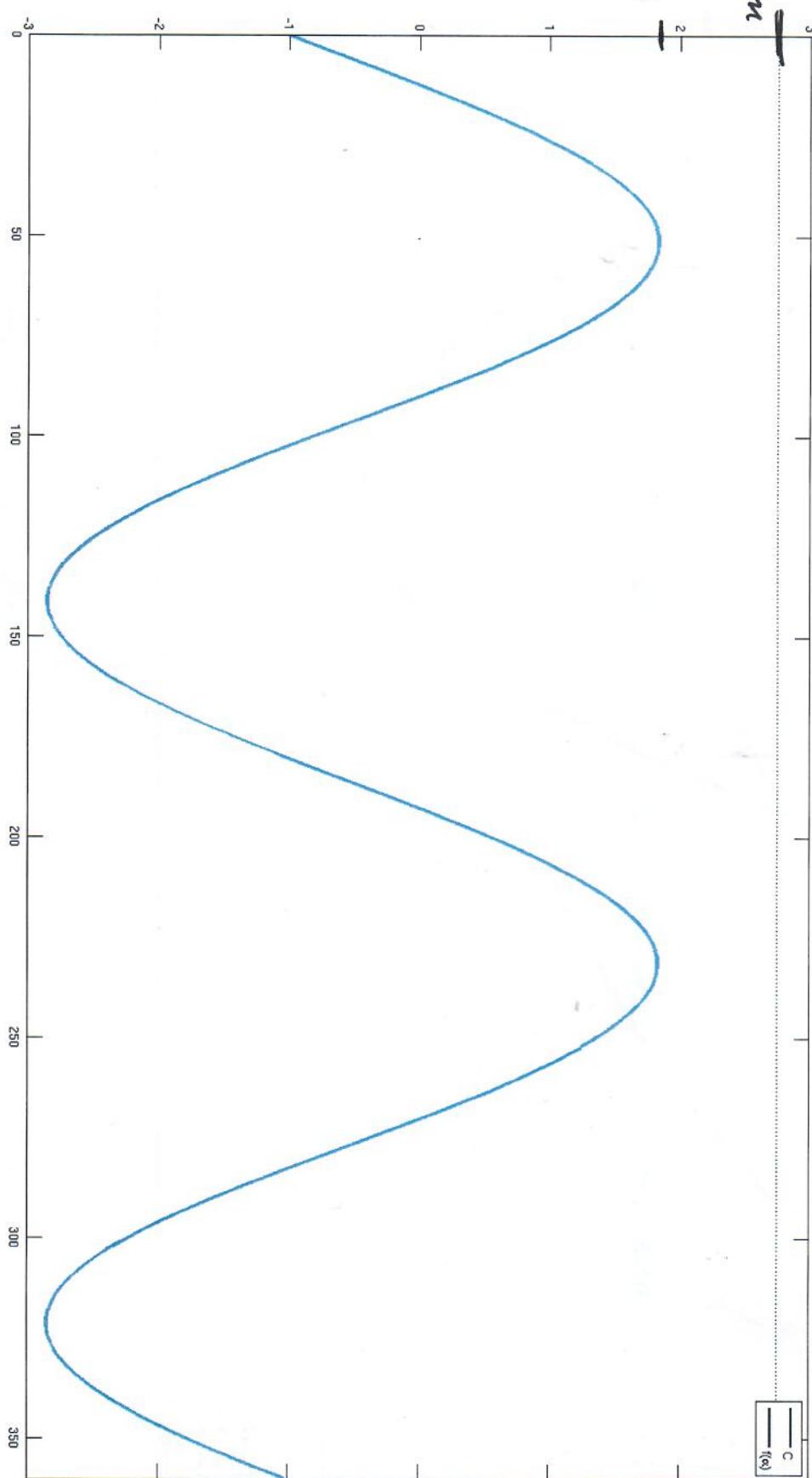
How much does it need to be increased?

C must become smaller than the maximum of $f(\alpha)$.

→ either differentiate $f(\alpha)$ or read the value off the y -axis of the plot:

$$C \approx 1.83 \text{ m} = C_0$$

Problem 6c

Problem 66

for the velocity this implies:

$$v_0 = \sqrt{\frac{g \cdot 4x^2}{2c}} \geq \sqrt{\frac{g \cdot 4x^2}{2c_0}}$$

$$v_0 \geq \sqrt{\frac{9.81 \frac{m}{s^2} \cdot (4.57 \text{ m})^2}{2 \cdot 1.83 \text{ m}}} = \underline{\underline{7.48 \frac{m}{s}}}$$

The initial velocity was too slow.

The ball must ~~be~~ be thrown with at least $7.48 \frac{m}{s}$!

Problem 7

$$m = 10 \cdot 10^3 \text{ kg}$$

$$v = 80 \frac{\text{km}}{\text{h}} = 80 \cdot \frac{1000 \text{ m}}{3600 \text{ s}} = 22.22 \frac{\text{m}}{\text{s}}$$

a) The total energy will be conserved.

Kinetic energy will be transformed into potential energy:

$$E_{\text{kin}} = \frac{m}{2} v^2 = mgh = E_{\text{pot}}$$

$$h = \frac{v^2}{2g}$$

$$h = \frac{(22.22 \frac{\text{m}}{\text{s}})^2}{2 \cdot 9.81 \frac{\text{m}}{\text{s}^2}} = \underline{\underline{25.17 \text{ m}}}$$

Without friction, the pony will continue to climb 25.17 m !

b) Now friction forces consume half of the energy:

$$E_{kin} = \frac{m}{2} v^2 = E_{pot} + E_{int} \quad \#$$

$$E_{int} = E_{pot} = mgh'$$

$$\frac{m}{2} v^2 = 2 \cdot mgh'$$

$$h' = \frac{v^2}{4g} = \frac{h}{2}$$

Then the pony climbs only half as high: ~~12.58 m~~

c) Now $v' = \frac{1}{2}v = 11.11 \frac{m}{s}$

Reuse old formula:

$$\frac{m}{2} v'^2 = \frac{m}{8} v^2 = mgh''$$

$$h'' = \frac{v^2}{8g} = \frac{h}{4}$$

With half the speed, the pony climbs only $\frac{1}{4}$ as high: 6.29 m