EECS4214

Digital Communications

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Week 4

Week 4 - Lecture 7

Review of the Last Lecture

Review

- Finished Topic 1: Probability, Random Variables, & $\underbrace{\text{Random Processes.}}$

WSS

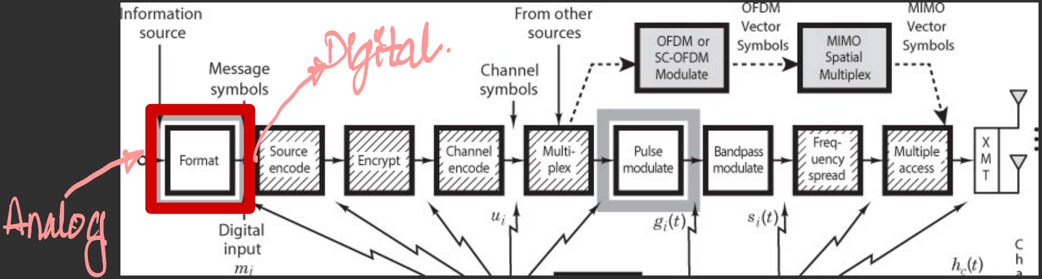
$\begin{cases} \text{mean. } E[X] \\ \text{Autocorrelation } R_X(\tau) \end{cases}$

- PSD & Autocorr. makes FT pairs
- $G_X(f)$ & $R_X(\tau)$ significance & their properties!
- RPs & Linear Systems.
- Examples.

Will review in the lecture!

Formatting

Formatting - the first step



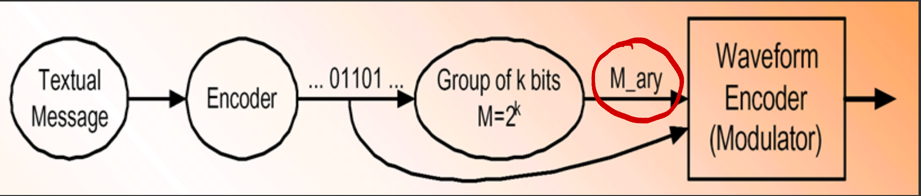
Formatting - the first step

- First essential signal processing step
- Ensures that the message (or source signal) is compatible with digital processing/communication
- Transformation from source information to digital information (electrical waveform of two or more levels)

M-Ary Communication.

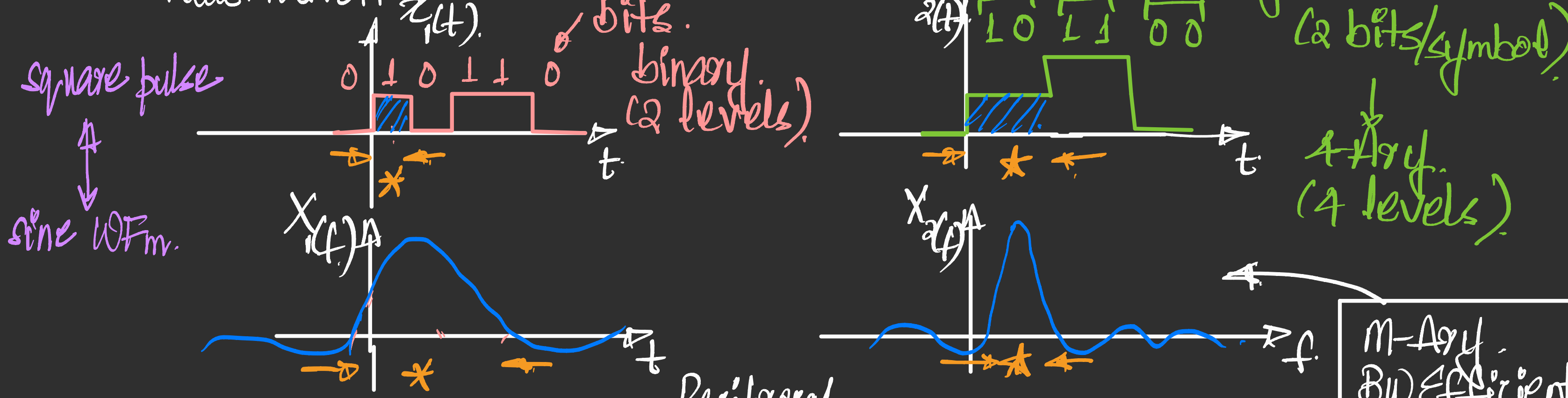
Formatting - Case I: Textual Information

- Character Coding: Alphanumeric and symbolic information are encoded into digital bits using one of several standard formats, e.g, ASCII, EBCDIC



M-Ary Communication.

- Here we transmit via more than two levels
- Transmitting symbols rather than bits.
- If M-levels, $\Rightarrow$ $2^K = M$; K bits/symbol.
- M-Ary Communication is more bandwidth efficient. Look at the following illustration



* Narrow in time-domain.
Broader in freq-domain.

* Broader in time-domain.
Narrow in freq-domain.

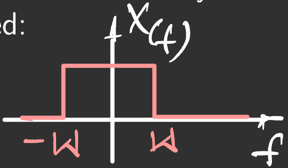
Formatting - Case II: Analog Information

Sampling → Quantization → PCM Coding

1. Sampling
   - The process of converting continuous-time analog signal, $x_a(t)$, into a discrete-time signal by taking the "samples" at discrete-time intervals
   - Here, we consider **band-limited signals**

Band-limited versus Time-limited Signals

- **Band-limited Signal:** A band-limited signal has a Fourier Transform strictly zero outside the bandwidth W. For example, if $x(t)$ is band-limited:



$$x(t) \xleftrightarrow{\mathcal{F}} X(f) \equiv 0, \quad |f| \geq W$$

where W is the message bandwidth

- **Time-limited Signal:** * A time-limited signal has infinite bandwidth, whereas a band-limited signal has infinite duration. In practice, all signals are time-limited and not band-limited. However, we use filters to limit the transmission bandwidth. Some examples are:
  - Voice signals $\approx 4\text{kHz} = W$
  - Music signals $\approx 15\text{kHz} = W$
  - Video signals $\approx 5\text{MHz} = W$

* Remember rectangular pulse & sinc FT pair!

Sampling Steps

If a signal $x(t)$ is sampled with a frequency greater than twice the bandwidth of the message signal $(2W)$, then the signal can be reconstructed back exactly without any distortion.

$$f_s = \text{Sampling Frequency} \geq 2W$$

Nyquist Criterion.

1. Original Signal $x(t)$
2. Original Signal Spectrum $X(f)$
3. Sampled Signal $x_s(t) = x(t)\delta_{T_s}(t)$

* Note, here we are using 'f' in Hz

* Remember the relation $\omega = 2\pi f$. where ω is in rad/sec.

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

this is using property of $\delta(t)$

$x(t)\delta(t-t_d) = x(t_d)\delta(t-t_d)$

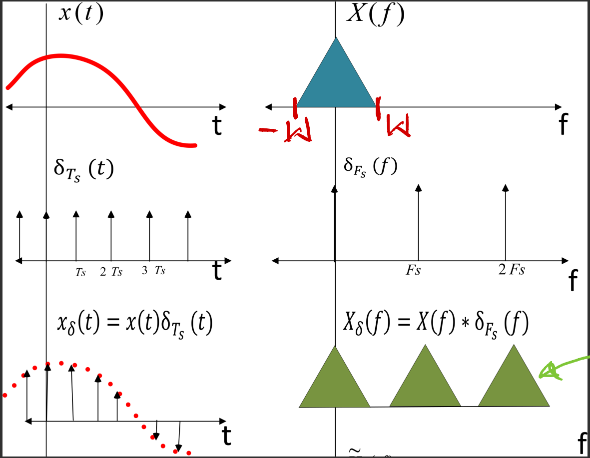
$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t - kT_s)$$

* We want to see that whether we can reconstruct the original sig from the samples!

Sampling Illustration

$$\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_s)$$

$$x_{\delta}(t) = x(t) \cdot s(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\,\delta(t-kT_s)$$



$$\delta_{F_s}(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(f-nf_s)$$

(proven fact)

Impulse train ↔ Impulse train
in time-domain in freq-dom

? Next slide!

Sampling Theorem

$$x_s(t) = x(t) \cdot \underbrace{\sum_{k=-\infty}^{\infty} \delta(t - kT_s)}$$

- Sampled Signal in frequency domain can be given as:

Remember again: $\sum_{k=-\infty}^{\infty} \delta(t-kT_s) \longleftrightarrow \frac{1}{T_s}\sum_{n=-\infty}^{\infty} \delta(f-nf_s)$

$$X_s(f) = X(f) * \overbrace{f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)}$$

Using the fact

- **Multiplication of signal in time-domain results in convolution in frequency-domain**

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

(look at the next page)

- The input spectra repeats periodically with period $f_s \geq 2W$

(look at the bottom right of the previous slide)

$$\boxed{X_s(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)}$$

Expanding Convolution.

$$= f_s \cdot \int_{-\infty}^{\infty} X(\nu) \cdot \sum_{n=-\infty}^{\infty} \delta(f - nf_s - \nu)\, d\nu$$

Impulse exists at $\nu = f - nf_s$.

$$= f_s \cdot \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

Applying sifting property of Impulse.

$$\int_{-\infty}^{\infty} x(t)\, \delta(t - t_d)\, dt = x(t_d).$$

Sampling Steps

- Spectrum Filtering with the LPF with the frequency response given as:

$$H_{\mathsf{LPF}}(f) = \begin{cases} 1, & -W \le f \le W \\ 0, & \text{otherwise} \end{cases}$$

- The corresponding impulse response $h_{\mathsf{LPF}}(t)$ is given as

$$h_{\mathsf{LPF}}(t) = 2W\,\mathrm{sinc}(2Wt)$$

- Remember, the square pulse and the sinc pulse makes up FT pair (refer Signals and Systems):

$$2W\,\mathrm{sinc}(2Wt) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{f}{2W}\right)$$

* Check the formula sheet.

Sampling Steps

- The filtered signal, $\tilde{X}(f)$ is given as:

$$\tilde{X}(f) = X_s(f) \cdot H_{\mathsf{LPF}}(f)$$

$Y(f) = H(f)\,X(f).$

- In time-domain (multiplication in frequency-domain is convolution in frequency-domain):

$$\tilde{x}(t) = x_\delta(t) * h_{\mathsf{LPF}}(t)$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\underbrace{\delta(t - kT_s) * h_{\mathsf{LPF}}(t)}$$

again using sifting property of an impulse

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\overbrace{2W \operatorname{sinc}(2W(t - kT_s))}$$

Sampling Steps

- For exact sampling, i.e., $f_s = 2W = \frac{1}{T_s}$

$$\tilde{x}(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} x(kT_s)\, \text{sinc}\left(\frac{t - kT_s}{T_s}\right)$$

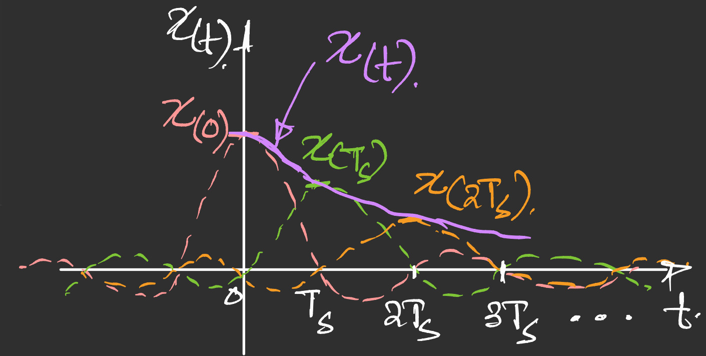
- We may check at various sampling instants $t = nT_S$, and it turns out *:

$$\tilde{x}(t) = x(t) \quad \text{(Ideally)}$$

See the Next Page!

*You may do it as an exercise!

$$\boxed{\tilde{x}(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(kT_s)\,\mathrm{sinc}\left(\frac{t-kT_s}{T_s}\right)}$$



The signal at sampling instants $t = nT_s$.

$$= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(kT_s)\,\mathrm{sinc}\left(\frac{nT_s-kT_s}{T_s}\right)$$

$$= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(kT_s)\,\mathrm{sinc}(n-k)$$

$= 1$ for $k = n$

$= 0$ for $k \neq n$.

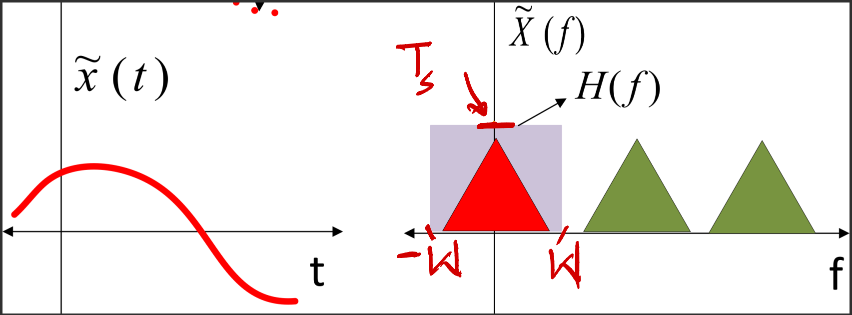
$$= \frac{1}{T_s}\, x(nT_s).$$

→ Can compensate in LPF.

T_s.

$$H_{\mathsf{LPF}}(f) = \begin{cases} 1, & -W \le f \le W \\ 0, & \text{otherwise} \end{cases}$$

Reconstructed Signal



Thank You

Happy Learning

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