EECS4214

Digital Communications

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Week 5

Week 4 - Lecture 9

Review of the Last Lecture

Review

- Quantization.
  - Scalar versus Vector

    Compressed.
  - Uniform versus Non-uniform.

    Contd.

Will review in the lecture!

Uniform Quantization

N-Level Uniform Quantizer Analysis

- The mid-point of each interval is selected as the quantized output of each interval. At this point the quantization error is zero
- Quantization error varies from $-\Delta/2$ and $+\Delta/2$
- Quantization error continues to grow at the extreme ends

Signal-to-Quantization Noise Ratio (SQNR)

We consider

- X = Input signal is a random variable and follows a PDF of $f_X(x)$
- $\hat{X}_q$ = Quantized output signal
- $n_q = X - \hat{X}_q$ = Quantization noise error

$$\boxed{\text{SQNR} = \frac{\mathbb{E}[X^2]}{\mathbb{E}[n_q^2]}}$$

- $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$
- $\mathbb{E}[n_q^2] = \int_{-\infty}^{\infty} (X - \hat{X}_q)^2 f_X(x) dx = \frac{\Delta^2}{12}$

We will prove it!

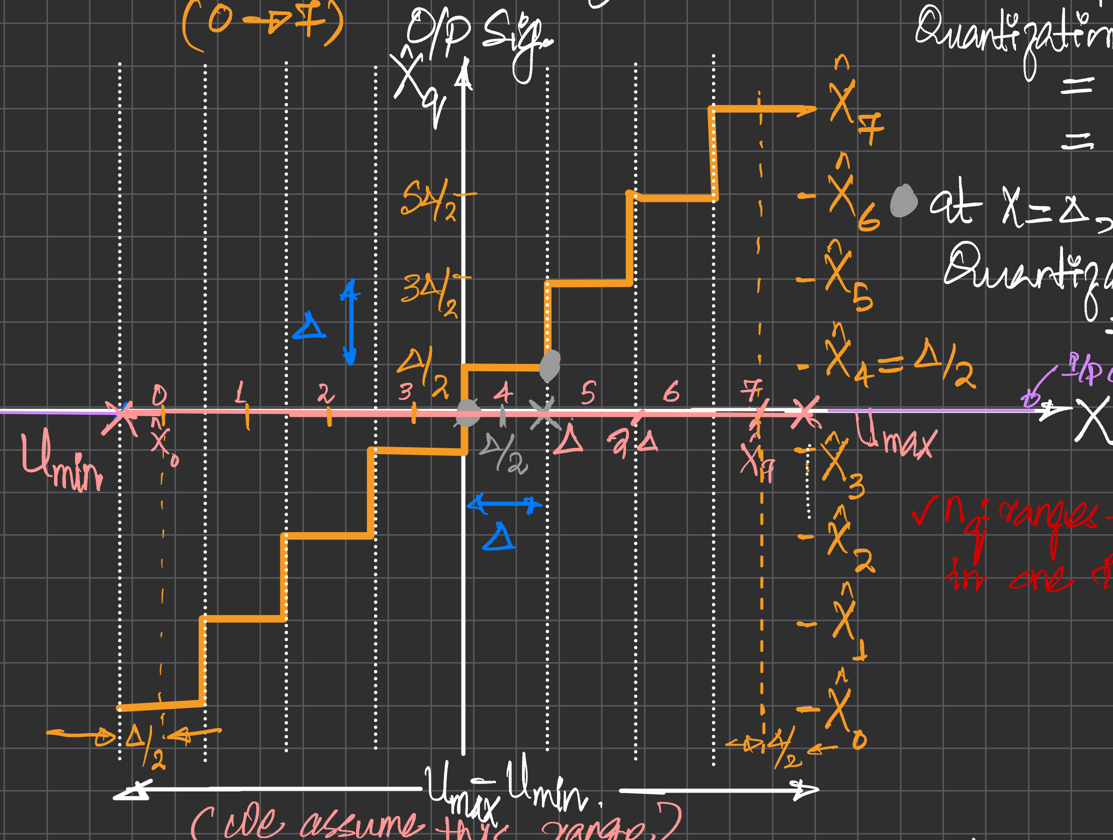
SQNR for Uniform Quantization

See the next page.

The SQNR for uniform quantization we will derive in class!

Transfer Function of an 8-level Quantizer

(0 → 7)



$-\infty$

X can take this value as I/p is random!

I/p can take this value.

- at $X=0$, $\hat{X}_q = \Delta/2$.
  Quantization Noise $= X - \hat{X}_q = -\Delta/2$ max^m
- at $X=\Delta$, $\hat{X}_q = \Delta/2$.
  Quantization Noise $= \Delta/2$. max^m

✓ n_q ranges from $-\Delta/2$ to $\Delta/2$ in one interval Δ

*. The midpoint of each interval is selected as the quantized O/p of each interval.

For finding SQNR, lets find $E[n_q^2]$ first:

$$E[n_q^2] = \int_{-\infty}^{\infty} (x-\hat{x}_q)^2 f_{X}(x)\,dx$$

$$= \int_{-\infty}^{\hat{x}_0-\Delta/2} (x-\hat{x}_0)^2 f_X(x)\,dx + \sum_{i=0}^{N-1} \int_{\hat{x}_i-\frac{\Delta}{2}}^{\hat{x}_i+\Delta/2} \overbrace{(x-\hat{x}_i)}^{\text{Quantization Error}}{}^2 f_X(x)\,dx$$

$$+ \int_{\hat{x}_{N-1}+\Delta/2}^{\infty} (x-\hat{x}_{N-1})^2 f_X(x)\,dx$$

0 to (8−1) = 0 to 7. in the 8-level Quantizer.

range for each ith interval (i: 0 → 7)

$\hat{x}_i - \frac{\Delta}{2} \rightarrow \hat{x}_i + \frac{\Delta}{2}$

Consider the simplifying assumptions as in the next slide!

SQNR for Uniform Quantization

To derive $\mathbb{E}[n_q^2]$, we consider the following simplifying assumptions:

- The input signal amplitude does not exceed the maximum limit of the quantizer, i.e.,

$$\hat{X}_0 - \frac{\Delta}{2} < X < \hat{X}_{N-1} + \frac{\Delta}{2}$$

  and

$$f_X(x) = 0 \quad \text{for} \quad X < \hat{X}_0 - \frac{\Delta}{2}, \quad X > \hat{X}_{N-1} + \frac{\Delta}{2}.$$

- The probability of having a signal in a specific interval does not vary, i.e., if $\Delta \to 0$, then

$$f_X(x) \approx f_X(\hat{x}_i). \qquad \text{if} \quad \Delta \to 0$$

Using the assumptions, we have:

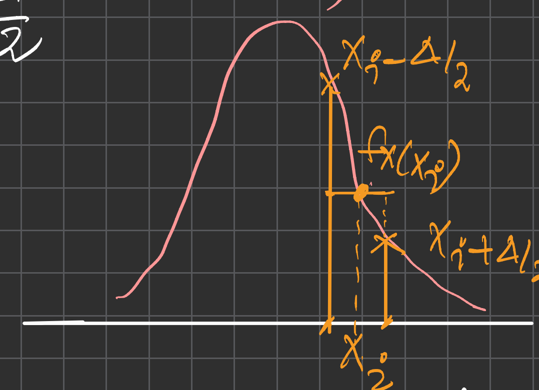
$$E[n_q^2] = \sum_{i=0}^{N-1} f_X(\hat{x}_i) \int_{\hat{x}_i-\Delta/2}^{\hat{x}_i+\Delta/2} (x-x_i)^2 \, dx$$

$$= \sum_{i=0}^{N-1} f_X(\hat{x}_i) \left. \frac{(x-\hat{x}_i)^3}{3} \right|_{\hat{x}_i-\frac{\Delta}{2}}^{\hat{x}_i+\frac{\Delta}{2}}$$

$$= \sum_{i=0}^{N-1} f_X(\hat{x}_i) \frac{\Delta^3/8 + \Delta^3/8}{3}$$

$$= \underbrace{\sum_{i=0}^{N-1} f_X(\hat{x}_i) \, \Delta}_{=1} \left(\Delta^2/12\right)$$

$$= \Delta^2/12$$

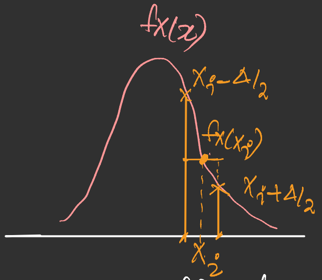


for Δ small; $\int f_X(x)\, dx = 1$

SQNR for Uniform Quantization

- The probability of all possible outcomes is one:

$$\int f_X(x)dx = \sum_{i=0}^{q-1} f_X(x_i)\Delta = 1.$$



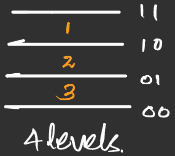
for Δ small; $\int f_{X(x)} dx = 1$

SQNR for Uniform Quantization

$$\text{SQNR} = \frac{E[x^2]}{E[n_q^2]} \leftarrow \Delta^2/2$$

With these assumptions, we finally derived:

$$\boxed{\text{SQNR} = \frac{12}{\Delta^2}\mathbb{E}[X^2] = \frac{12}{\Delta^2}\int x^2 f_X(x)dx,}$$



where the quantization step size, Δ, is given by:

$$\Delta = \frac{U_{max} - U_{min}}{q-1} = \frac{2U_0}{q-1} \approx \frac{2U_0}{q} = \frac{U_0}{2^n}, \quad \text{assuming } U_{max} = U_{min} = U_0$$

Symmetric.

where $q = 2^n$, that is the number of quantization levels, each having n bits (n bits/level)

SQNR for Uniform Quantization

$\boxed{q = 2^n}$

$\frac{U_0^2}{12q^2}$

Thus, the quantization noise power is:

$$\mathbb{E}[n_q^2] = \frac{\Delta^2}{12} = \frac{U_0^2}{12 \cdot 2^{2n}}.$$

$$\text{SQNR [dB]} = 10\log_{10}\frac{\mathbb{E}[X^2]}{U_0^2} + 6n + 10.8$$

taken as 4.8 empirically

Note that as $\Delta \to \infty$, SQNR tends to zero.

Example: Uniformly Distributed Signal

Consider a uniformly distributed signal passed through a uniform quantizer. What will be the SQNR?

$$f_X(x) = \frac{1}{2U_0}, \quad |x| < U_0$$

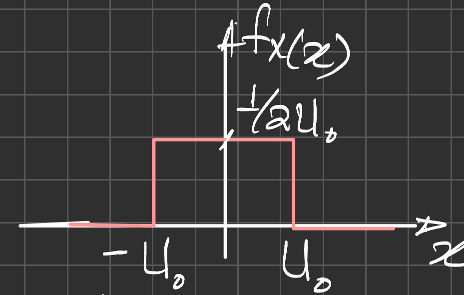
$$\text{SQNR} = (q-1)^2$$

$$\text{SQNR [dB]} = 20\log_{10}(q-1)$$

$$\text{SQNR [dB]} = 48 \quad (\text{when } q = 256)$$

Example: Uniformly Distributed I/P Signal

$$f_X(x) = \begin{cases} \dfrac{1}{2U_0} & |x| < U_0 \\ 0 & \text{otherwise} \end{cases}$$



a) SQNR for any 'q'

$$SQNR = \frac{12}{\Delta^2}\int_{-U_0}^{U_0} x^2 f_X(x)\,dx = \frac{\cancel{12}^{\,3}}{\cancel{4}U_0^2/(q-1)^2}\int_{-U_0}^{U_0}\frac{1}{2U_0}x^2\,dx$$

$$\Delta = \frac{U_0-(-U_0)}{q-1} = \frac{2U_0}{q-1}$$

$$= \frac{3(q-1)^2}{U_0^2}\cdot\frac{1}{2U_0}\cdot\frac{x^3}{3}\Bigg|_{-U_0}^{U_0}$$

$$= \frac{\cancel{3}(q-1)^2}{\cancel{2U_0^3}}\;\frac{\cancel{U_0^3+U_0^3}}{\cancel{3}} = \frac{(q-1)^2\,\cancel{U_0^3}}{\cancel{U_0^3}} = (q-1)^2.$$

b) For $q = 256$;

$$SQNR = 10\log_{10}(q-1)^2 = 20\log_{10}255 = 48\,dB.$$

Example: Uniformly Distributed Signal

SQNR ≃ $20\log_{10} q^2 = 20\log_{10} (2^n)^2$

For $q \gg 1$ and $q = 2^n$

$$\text{SQNR [dB]} = 20n\log_{10} 2 = 6n$$

$$\text{Bit Rate (bits/sec)} = F_s \text{ (samples/sec)} \times \underbrace{\frac{\text{SQNR (dB)}}{6}}_{} \text{ (bits/sample)}$$

"n" bits/sample.
OR bits/level.

For Scalar Quantization

Example 1 - Uniformly Distributed Signal

An analog signal is sampled at the Nyquist rate $F_s = 20$KHz and quantized into q=1024 levels. Find bit-rate and the time duration T_b of one bit of the binary encoded signal.

$$(a) R_b = n.F_s = F_s \log_2 q, \qquad (b) T_b = \frac{1}{R_b}$$

(a) $R_b = ?$ bits/sec

$= \underbrace{\text{samples/sec}}_{f_s} \times \underbrace{\text{bits/sample}}_{n}$.

(b) $T_b = \frac{1}{R_b}$.

$\checkmark$ Using $E[x^2] = \frac{1}{T}\int_0^T x^2_{(t)}\,dt$

Example 2 - Uniformly Distributed Signal

A PCM system uses a uniform quantizer followed by a 8-bit binary encoder. The bit rate of the system is 56Mbps. Find the output SQNR when a sinusoidal wave of 1MHz frequency and peak-to-peak amplitude (1V) is applied to the input.

$\boxed{A = 0.5}$

$$(a)\ R_b/n = F_s \quad (b) SQNR = \frac{0.5x^2_{\max}}{\Delta^2/12} = \frac{E[x^2]}{\Delta^2/12} = \frac{A^2/2}{\Delta^2/12}$$

Ratio (not dB).

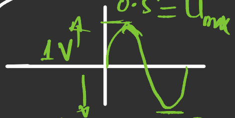
(a) $R_b = 56$ mbps

$n = 8$

$f_s = \frac{R_b}{n}$

(b) $\Delta = \frac{2U_0}{L-1}$

$= \frac{1}{256-1}$



$U_{max} - U_{min} = 1$ $\quad U_{min} = -0.5$

Example: Gaussian Distributed Signal

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{x^2}{2\sigma^2})$, $\sigma = U_0/4$ = standard deviation See the next page!

$$\mathrm{SQNR} = 3\frac{(q-1)^2}{16} \quad \mathrm{SQNR}[dB] = 10\log_{10}3 + 20\log_{10}(q-1) - 10\log_{10}16$$

$$\mathrm{SQNR}[dB] = 48 - 7.25 \text{ (when } q = 256\text{)}$$

Comparing the SQNR for the case when the number of quantization levels is $q = 256$, we can see that the uniform quantizer underperforms for Gaussian distributed signal.

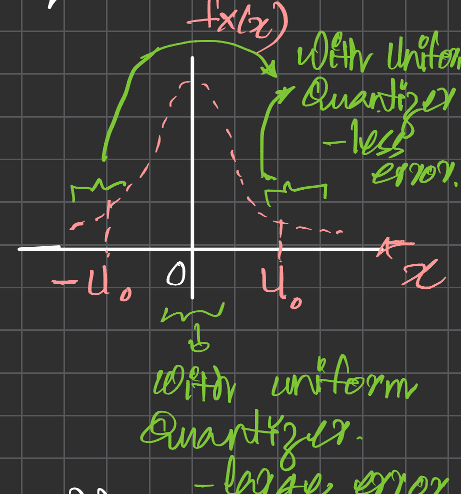
Gaussian Distributed Sig - Example

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2} \qquad |x| \le U_0$$

$\sigma = \frac{U_0}{2}$, SQNR = ? **(a) for any q** **(b) for $q = 256$**

Solution

$$\text{SQNR} = \frac{12}{\Delta^2} \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2}\, dx \quad -(1)$$



However, for zero-mean Gaussian we have:

$$E[x^2] - \cancel{(E[x])^2}_{0} = \sigma^2 = E[x^2] = \left(\frac{U_0}{4}\right)^2 = \frac{U_0^2}{16}$$

&

$$E[n_q^2] = \frac{\Delta^2}{12} = \frac{(2U_0)^2}{12(q-1)^2} = \frac{U_0^2}{3(q-1)^2}$$

$$\therefore \quad \text{SQNR} = \frac{E[x^2]}{E[n_q^2]} = \frac{U_0^2/16}{U_0^2/3(q-1)^2} = \frac{3(q-1)^2}{16}$$

(a) $\text{SQNR} = 10\log_{10}(3) + 20\log_{10}(q-1) - 10\log_{10}(16)$

(b) $\overbrace{48 - \underbrace{7.25}_{\text{loss!}}\ \text{dB}}^{\text{as compared to uniform } X}$

Thank You
Happy Learning

☺