EECS4214

Digital Communications

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Week 2

Week 2 - Lectures 3 and 4

Review of the last Lecture

Review

- Block Diagram of DCS
- Communication Channel
  - Wired: modeled as LPF
  - Wireless: modeled as BPF

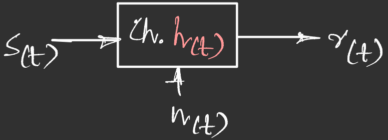
- Noise in the Ch./Txm/RCV
- Interference from other sig.
- Distortion. (Diffraction, Reflection, Refraction etc.)
- Attenuation.

Will review in the lecture!

makes the sig. difficult to be recovered at the RCV.

- Noise is random in nature.
- Mathematically, Ch. is modeled as:

$$r_{(t)} = As_{(t)} + n_{(t)} \quad \text{(ideal)}$$



$$r_{(t)} = A\underbrace{s_{(t)} * h_{(t)}}_{\text{time-invariant}} + n_{(t)}$$

- ISI effect. (Smearing of pulses...)

Review of Probability Concepts

Axioms and Counting Methods

Given the sample space S and an event A; a probability function $P(\cdot)$ associated to an event A is a real number such that

- $P(A) \geq 0$ for every event A
- $P(S) = 1$

cannot happen at the same time!

- For countable **mutually exclusive events** $A_1, A_2, \cdots, A_n$, we have $P(A_1 \cup A_2 \cup \cdots A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$.

Permutations and **Combinations** are two important counting methods

Methods of Counting

- Permutations - The arrangement of items in a specific order

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

total!

- Combinations - The selection of items without considering the order

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Properties, Conditional Probability, Baye's Rule

- **Range of Probability of an event A:** $0 \leq P(A) \leq 1$
- **Probability of impossible event:** Zero
- **Probability of an event A to not happen:** $P(\bar{A}) = 1 - P(A)$
- **Probability of two events:**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **Probability of two independent events:**

$$P(A \cup B) = P(A) + P(B) - \mathbf{P(A)P(B)}$$

Properties, Conditional Probability, Baye's Rule

- Conditional Probability (**Baye's Rule**)

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

- **Total Probability Theorem**

$$P(A) = P(A/A_1)P(A_1) + P(A/A_2)P(A_2) + \cdots + P(A/A_n)P(A_n)$$

Example 1

Urn A has 5 red balls and 2 white balls. Urn B has 3 red balls and 2 white balls. An urn is selected randomly and 2 white balls are drawn successively. Each urn is equally likely to be selected. Find the probability of 2 white balls that are taken out **without replacement**.

Solution

This problem can be solved using **conditional probability** and law of **total probability**:

$$
\begin{aligned}
P(2W) =& P(1W/A)P(1W/(1W \cap A))P(A) + P(1W/B)P(1W/(1W \cap B))P(B) \\
=& \frac{2}{7} \times \frac{1}{6} \times \frac{1}{2} + \frac{2}{5} \times \frac{1}{4} \times \frac{1}{2} \\
=& \frac{1}{42} + \frac{1}{20}
\end{aligned}
$$

Example 2

Given: Equally likely!

- Urn A: 5 Red, 6 Green, 2 White
- Urn B: 3 Red, 3 Green, 4 White
- Urn C: 6 Red, 2 Green, 1 White

Find:

a) $P(W/A) = ?$

b) $P(B/W) = ?$

$$P_{(W/A)} = \frac{2}{13}$$

$$P_{(B/W)} = \frac{P_{(B\cap W)}}{P_{(W)}}$$

$$P_{(W)} = \underbrace{P_{(W/A)} \cdot P_{(A)}}_{\frac{2}{13} \times \frac{1}{3} = \frac{2}{39}} + \underbrace{P_{(W/B)} \cdot P_{(B)}}_{\frac{4}{30}} + \underbrace{P_{(W/C)} \cdot P_{(C)}}_{\frac{1}{9} \times \frac{1}{3} = \frac{1}{18}}$$

$$= 0.2217$$

$$P_{(B\cap W)} = P_{(W/B)} \cdot P_{(B)}$$

$$= \frac{4}{10} \times \frac{1}{3} = \frac{4}{30}$$

$$P_{(B/W)} = \frac{4}{30} \times 0.2217.$$

Random Variables

Discrete Random Variable

If a random variable X can take values ~~on~~ from **a finite set of values**, X is said to be a discrete random variable

- $P(X_i) \geq 0, \forall i \in \mathbb{Z}$
- $\sum_{i=1}^{\infty} P(X_i) = 1$

The **Cumulative Distribution Function (CDF)** of X is given as:

$$F_X(x) = P(X_i \leq x), \quad \forall x \in \mathbb{Z}$$

Stair case fn.

The **Probability Density Function (PDF)** of X is given as:

Mass

$$f_X(x) = P(X = x) = F_X(x) - F_X(x^-)$$

bar graph

Example 3

Consider the experiment of rolling two dice. Let X represents the total number that shows up on the upper faces of two dice:

- Find $P(4 \leq X \leq 6)$
- Find $P(X \geq 5)$
- Sketch the PDF and CDF of X

Example 3

Solution

(i) $$P_{(4 \leq X \leq 6)} = \underbrace{P_{(X=4)}}_{(1,3)(3,1)(2,2)} + P_{(X=5)} + P_{(X=6)}$$

$$= \frac{3}{36} + \frac{4}{36} + \frac{5}{36} = \frac{12}{36}$$

(ii) $$P_{(X \geq 5)} = 1 - P_{(X \leq 4)}$$

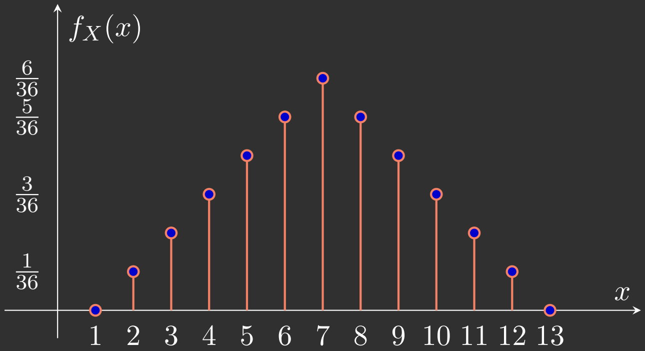
$$= 1 - P_{(X=2)} + P_{(X=3)} + P_{(X=4)}$$

$$= 1 - \left(\frac{1}{36} + \frac{2}{36} + \frac{3}{36}\right) = \frac{30}{36}$$

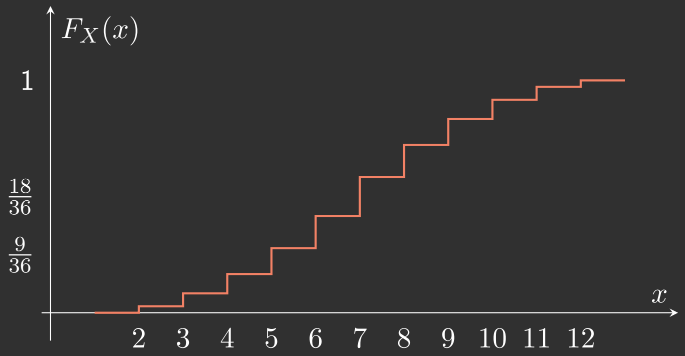
* Here, for $1 < n \leq 7$; $P = \frac{(n-1)}{36}$; $P_{(X=1)} = 0$

for $7 < n < 13$; $P = \frac{(13-n)}{36}$; $P_{(X=13)} = 0$

Example 3 - PDF



Example 3 - CDF



Example 3 - If $X \sim \text{Poisson}(\lambda)$

What happens if X becomes a Poisson distributed random variable? Find:

- $P(X = 5)$
- $P(X \geq 5)$

For a **Poisson random variable**

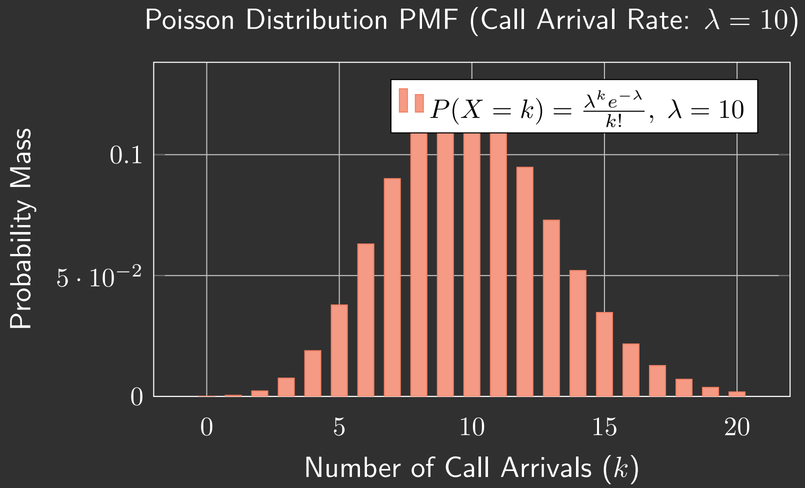
$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

λ = mean.

$$P(X = 5) = \frac{e^{-\lambda}\lambda^5}{5!}$$

$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - \sum_{i=0}^{4} \frac{e^{-\lambda}\lambda^i}{i!}$$

Poisson Distribution for Call Arrival Rates in Cellular Comm.



Continuous Random Variable - CDF $F_X(x)$ and PDF $f_X(x)$

If X is a continuous random variable

- $f_X(x) \geq 0$ positive.
- $\int_{-\infty}^{\infty} f_X(x) = 1$ How total prob. is distributed among possible outcomes of the random variable.
- $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx$ Likelihood of various events upto a certain given I/P.
  Cumulative.
- $f_X(x) = \frac{dF_X(x)}{dx}$

Continuous Random Variable - CDF $F_X(x)$

$F_X = \int_{-\infty}^{x} f_X(x)\,dx$

The CDF of X follows the following properties

- $0 \leq F_X(x) \leq 1$
- $F_X(\infty) = 1$
- $F_X(-\infty) = 0$
- $P(a \leq x \leq b) = F_X(b) - F_X(a)$

$P(X \leq b) - P(X \leq a) = \int_{-\infty}^{b} f_X(x)\,dx - \int_{-\infty}^{a} f_X(x)\,dx$

Any other Alternative?

$\underbrace{\int_{-\infty}^{b} \cdots + \int_{a}^{\infty} \cdots}_{a \to b}$

- $P(a \leq x \leq b) = \int_a^b f_X(x)\,dx$
- $P(X > a) = 1 - P(X \leq a) = 1 - F_X(a)$ **a.k.a Complementary CDF (CCDF)**

Physical Meaning of CDF and PDF

The CDF describes the likelihood of random variable X falling within specific intervals of X

$$F_X(x) = P(X \leq x)$$

- In fading models, the CDF can describe the probability that the received signal strength falls below a certain value - **outage probability**

The PDF describes the **likelihood** of a random variable X taking a particular value x

$$f_X(x)$$

- Modeling signal strength distributions (e.g., Rayleigh, Rician PDFs)

Example - PDF

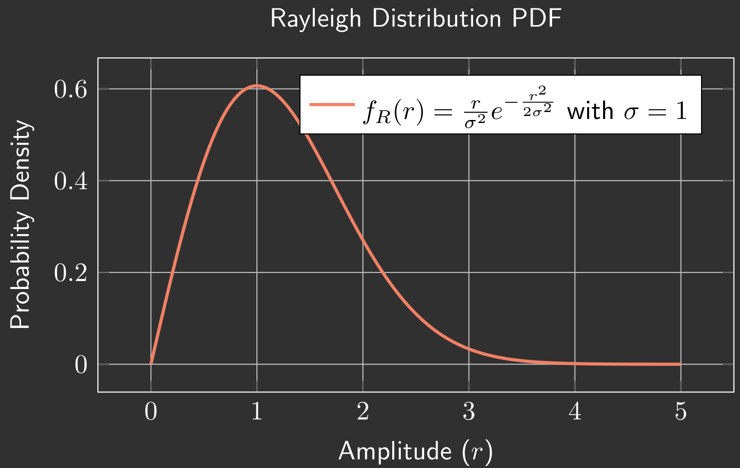


Figure: Rayleigh Distribution Probability Density Function (PDF)

Expectation and Moments

For a general function $g(X)$ of random variable X, its expectation can be derived as follows:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(X) f_X(x) dx$$

Special Cases - Moments

- $g(X) = c$ constant DC $E[c] = \int_{-\infty}^{\infty} c\, f_{X}(x) dx = c.$
- First moment of X, $g(X) = X$ $E[X] = \int_{-\infty}^{\infty} X f_{X}(x) dx$ ← mean
- Second moment of X, $g(X) = X^2$ $E[X^2] = \int_{-\infty}^{\infty} X^2 f_{X}(x) dx$
- nth moment of X, $g(X) = X^n$

Expectation and Moments

For a general function $g(X)$ of random variable X, its expectation can be derived as follows:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(X) f_X(x) dx$$

Special Cases - Central Moments

- nth central moment of X, $g(X) = (X - \mathbb{E}[X])^n$
- Second central moment of X, $g(X) = (X - \mathbb{E}[X])^2$ Also called *variance*, σ_X^2

$$\sigma_X^2 = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$= \int_{-\infty}^{\infty} (x - E[x])^2 f_X(x) dx$$

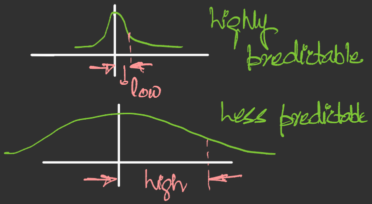
Variance and Standard Deviation

- *Variance* is a measure of "randomness" of a random variable
- By specifying the variance of a random variable, we are constraining the width of its probability density function
- The square root of the variance, σ_X, is called the *standard deviation* of X
- The variance and the mean square value are related by

$$\sigma_X^2 = \mathbb{E}\{X^2\} - (\mathbb{E}\{X\})^2$$

Significance of Variance and Standard Deviation

- *Standard deviation* indicates how far the values of X can be from the mean value
- For example:
  - 3 ± 0.1 - Highly predictable, i.e., less standard deviation
  - 3 ± 5 - Less predictable, i.e., high standard deviation



- Specific to wireless communications:
  - Standard deviation measures the variation in received signal strength, such as RSSI (Received Signal Strength Indicator)
    - A lower standard deviation indicates stable signal strength, which implies a more reliable connection
    - A higher standard deviation suggests fluctuating signal strength, which could lead to poor communication quality
  - In wireless channels, noise and interference affect signal quality. Standard deviation can quantify the variability of noise levels

Example 4

Consider $F_X(x) = 1 - 2e^{-2x}$, $x \geq 0$. Find:

- $P(X \leq 3)$
- $P(X \geq 1, X \leq 3)$

$$P(X \leq 3) = F_{X}(3) = 1-2e^{-2(3)}$$

$$P(X \geq 1, X \leq 3) = F_{X}(3) - F_{X}(1)$$

Remember

$$P(X \leq a) = F_{X}(a)$$

$$P(X \geq b) = 1 - F_{X}(b)$$

Example 5

Consider $f_X(x) = 2e^{-2x}$, $x > 0$. Find the mean and standard deviation?

Solution

$$E[X] = \int_0^\infty x f_{X(x)} dx = \int_0^\infty x \cdot 2 \cdot e^{-2x} dx$$

Using

$$\int_0^\infty t^n e^{-t} dt = n!$$

$2x = t$

$2dx = dt$, $t: 0 \to \infty$

$$E[X] = \int_0^\infty \frac{t}{2} \times 2 \cdot e^{-t} \frac{dt}{2} = \frac{1}{2} \int_0^\infty t \cdot e^{-t} dt$$

$$= \frac{1}{2} \times 1! = \frac{1}{2}$$

If mean and standard deviation are same, which random variable it could be?

Examples: Call time, email writing, Rayleigh fading (power becomes exponentially distributed in wireless communications) etc.

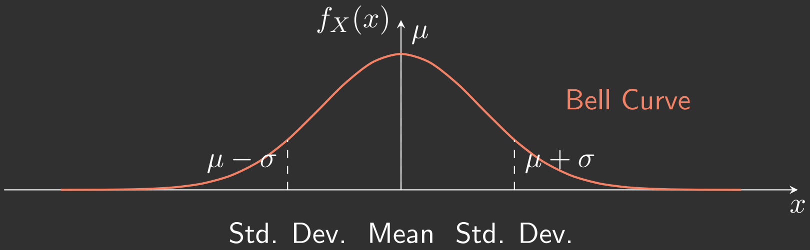
Gaussian Distribution

If X is a random variable distributed in a Gaussian manner, then its PDF is given by:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where:

- $\mu = \mathbb{E}[X]$ is the **mean**
- $\sigma^2 = \text{Var}[X]$ is the **variance**



Significance of Gaussian Distribution in DC

- The sum of a large number of random variables (of any type) tends to follow a Gaussian distribution. This phenomenon is known as the **Central Limit Theorem**
- In wireless communication systems, thermal noise generated by electronic components (e.g., resistors, amplifiers) is modeled as a Gaussian random variable
- The variations in the received signal strength due to fading and shadowing effects can often be approximated using a Gaussian distribution, particularly under weak scattering environments
- In **digital communication**, the errors introduced by additive noise are often modeled using a Gaussian distribution (Additive White Gaussian Noise, **AWGN**) **(later)**

Transformation of Random Variables

Consider the case, where the distribution of random variable:

- Y is unknown
- X is known

Then the following Single Variable Transformation methods can be used to find the distribution of the unknown random variable

1. Direct Method
2. CDF Method

The goal is to find the probability distribution function (PDF) of Y, $f_Y(y)$, based on the known PDF of X, $f_X(x)$, where $Y = g(X)$

Single Random Variable Transformation

1 Direct Method

If $Y = g(X)$, then given the PDF of X as $f_X(x)$ and the function $g(X)$ to be invertible, then:

$$f_Y(y) = \left. \frac{f_X(y)}{\left|\frac{dy}{dx}\right|} \right|_{y=g^{-1}(x)}$$

Example: $y = 2x,\ f_X(x) = \lambda e^{-\lambda x}$

Solution:

$\frac{dy}{dx} = 2$

$$\frac{f_X(y/2)}{2} = \left. \frac{\lambda e^{-\lambda x}}{2} \right|_{x=y/2} = \frac{\lambda e^{-\lambda y/2}}{2}$$

Single Random Variable Transformation

2 CDF Method

If $Y = g(X)$, then given the CDF of X as $F_X(x)$, you can find the CDF of Y, $F_Y(y)$, as:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$$

From the CDF of Y, the PDF is obtained by differentiation:

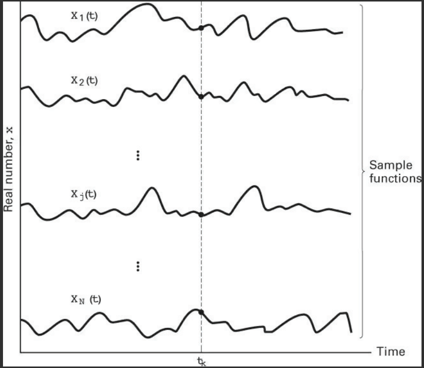
$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

Not in the scope of this course!

Random Processes

Random Processes

- A random process $X(A,t)$ can be viewed as a function of two variables: an event A and time t
- For a specific event A_j, we have a single time function $X(A_j,t) = X_i(t)$ (i.e., a **sample function**) at a specific time!
- The totality of all sample functions is called an *ensemble*
- For a specific time t_k, $X(A,t_k)$ is a **random variable** $X(t_k)$ whose value depends on the event
- For a specific event $A = A_j$ and a specific time $t = t_k$, $X(A_j,t_k)$ is simply a **number**



$X(A,t)$

We shall designate the random process by $X(t)$ and let the functional dependence upon A be implicit

Characterization of a Random Process

- **Amplitude Domain:** PDF and CDF (as studied)
- **Time Domain:** Autocorrelation function

  $\updownarrow$ FT Pair
- **Frequency Domain:** Power spectral density function

Statistical Averages of a Random Process

- Because the value of a random process at any future time is unknown (since the identity of the event A is unknown):
  - A random process whose distribution functions are continuous can be described statistically with a PDF
- In most situations, it is **not practical to determine empirically** the probability distribution of a random process
- A partial description consisting of the **mean** and **autocorrelation function** is often adequate for the needs of a **communication system**

Statistical Averages of a Random Process

- The **mean** of the random process $X(t)$

$$\mathbb{E}\{X(t_k)\} = \int_{-\infty}^{\infty} x\, f_{X_k}(x)\, dx$$

where $f_{X_k}(x)$ is the density over the *ensemble* of events at time t_k

- The **autocorrelation function** of the random process $X(t)$

$$R_X(t_1, t_2) = \mathbb{E}\{X(t_1)X(t_2)\}$$

where $X(t_1)$ and $X(t_2)$ are random variables obtained by observing $X(t)$ at times t_1 and t_2, respectively

  - The **autocorrelation function** is a measure of the degree to which two time samples of the same random process are related

Stationarity

- A random process $X(t)$ is said to be **stationary** in the **strict sense** if none of its statistics are affected by a shift in the time origin
- A random process is said to be **wide-sense stationary** (WSS) if two of its statistics, its mean and autocorrelation function, do not vary with a shift in the time origin, that is:

$$\mathbb{E}\{X(t)\} = m_X = \text{constant}$$

and

$$R_X(t_1, t_2) = R_X(t_1 - t_2)$$

This means the **autocorrelation function** depends only on the time difference $\tau = t_1 - t_2$. That is, all pairs of values of $X(t)$ at points in time separated by $\tau = t_1 - t_2$ have the same correlation value

Thank You
Happy Learning
☺