EECS4214

Digital Communications

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Week 5

Week 5 - Lecture 10

Review of the Last Lecture

Review

Uniform Quantization.

- Step size "Δ" - same
- Desired SQNR
  $SQNR_{[dB]}$ } for

1. Uniformly Distributed I/p Signal.
2. Gaussian Distributed I/p Signal.
   - SQNR is not good here. (−7.25 dB)

Motivation for non-uniform Q

Will review in the lecture!

Uniform Quantization

Summary: Uniform Quantization

white - Symmetric
Orange - Asymmetric.

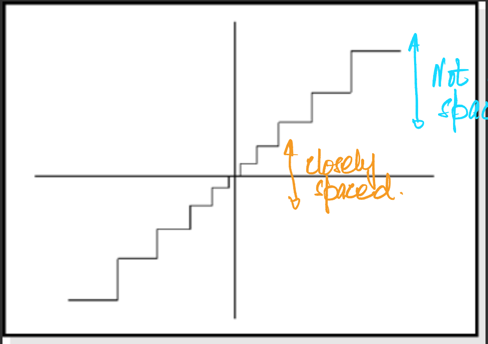
- Input signal amplitude range [-U_0 U_0] - Symmetric
  Asymmetric Signal [U_1 U_2]
- Number of quantization levels: $q = 2^n$
- Quantization step: $\Delta = \frac{2U_0}{q-1}$
  $\Delta = \frac{U_2-U_1}{(q-1)} \approx \frac{U_2-U_1}{2^n}$
- Quantization noise power: $\mathbb{E}[n_q^2] = \Delta^2/12$
- $SQNR = 10\log_{10}\frac{\mathbb{E}[X^2]}{U_0^2} + 6n + 4.8[dB]$
- Uniformly Distributed Signal: SQNR= $20\log_{10}(q-1)$ [dB]
- Uniformly Distributed Signal: $q >> 1$, SQNR= $6n$ [dB] → (More "n" bits/level OR more levels, higher SQNR)
- Gaussian Distributed Signal: SQNR= $20\log_{10}(q-1) - 7.25$dB

q - levels.
$q = 2^n$

Non-uniform Quantization

Non-uniform Quantization

- Nonuniform quantizers have unequally spaced levels
- The spacing can be chosen to optimize the Signal-to-Noise Ratio for a particular type of signal



Not closely spaced!

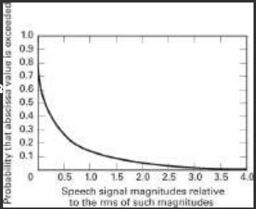
closely spaced.

Unequally spaced!

Non-uniform Quantization

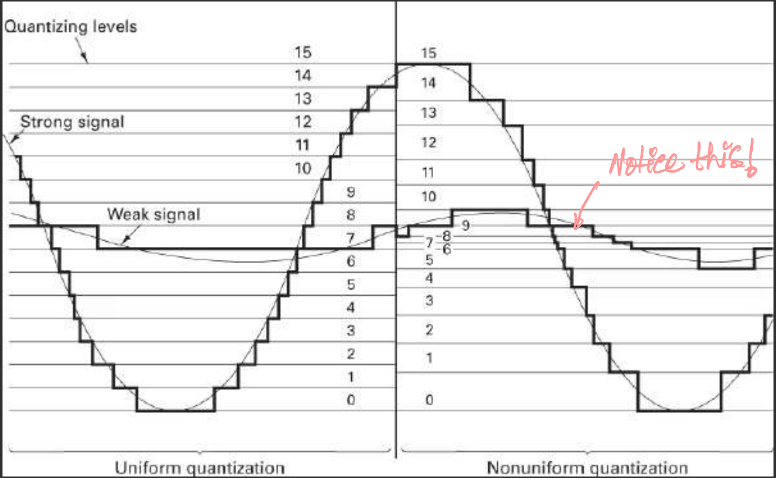
- Many signals such as speech have a nonuniform distribution

Low amplitude signals occur more frequently



- Use more levels at regions with large probability density function (pdf)
- Concentrate quantization levels in areas of largest pdf
- OR use fine quantization (small step size) for weak signals and coarse quantization (large step size) for strong signals

Uniform and Non-uniform Quantization of Signals



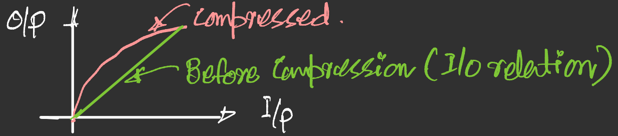
Notice this!

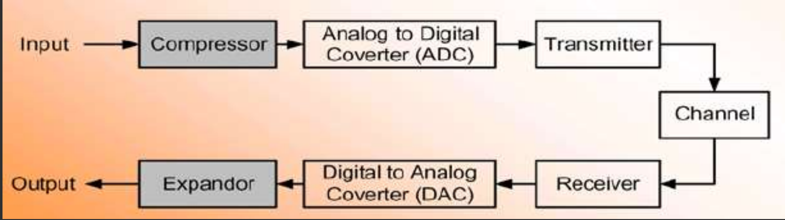
Note
mostly low amp. signals occur in practice!

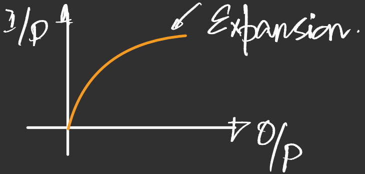
Non-uniform Quantization using Companding

- Companding improves the overall SQNR by reducing the quantization noise for the predominant weak signals, at the expense of an increase in noise for the rarely occurring strong signals
- Thus, it makes the SQNR a constant for all the signal values within the input range
- Here, first the baseband signal is passed through a **compressor** and then applying the compressed signal to a uniform quantizer
  - In compression, the SQNR of the weak signal values is enhanced by decreasing the quantization step size or increasing the quantizer resolution
  - Conversely, strong signals can be efficiently represented with minimal degradation in SQNR by increasing the quantization step size or decreasing the quantization resolution
- The resultant signal is then transmitted
- At the receiver, a device called **Expander** is used to restore the signal samples to their correct relative level

Companding

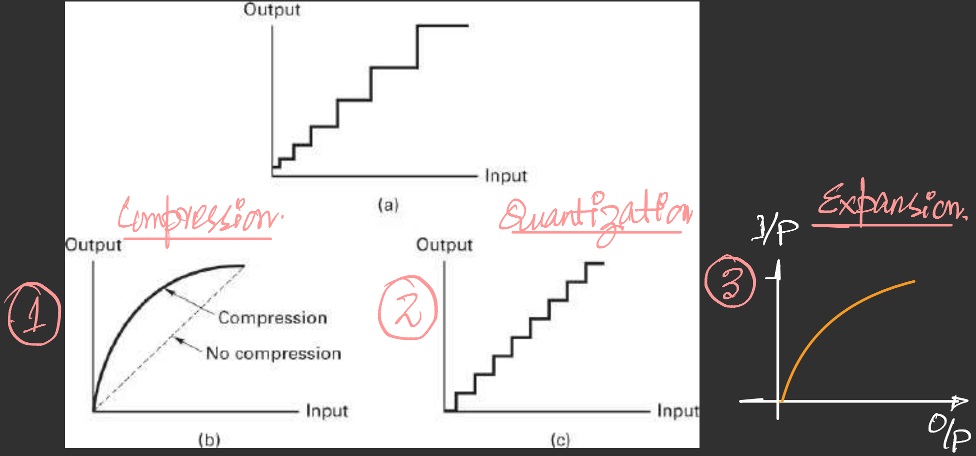






(Check the 'x' & 'y' axis here!)

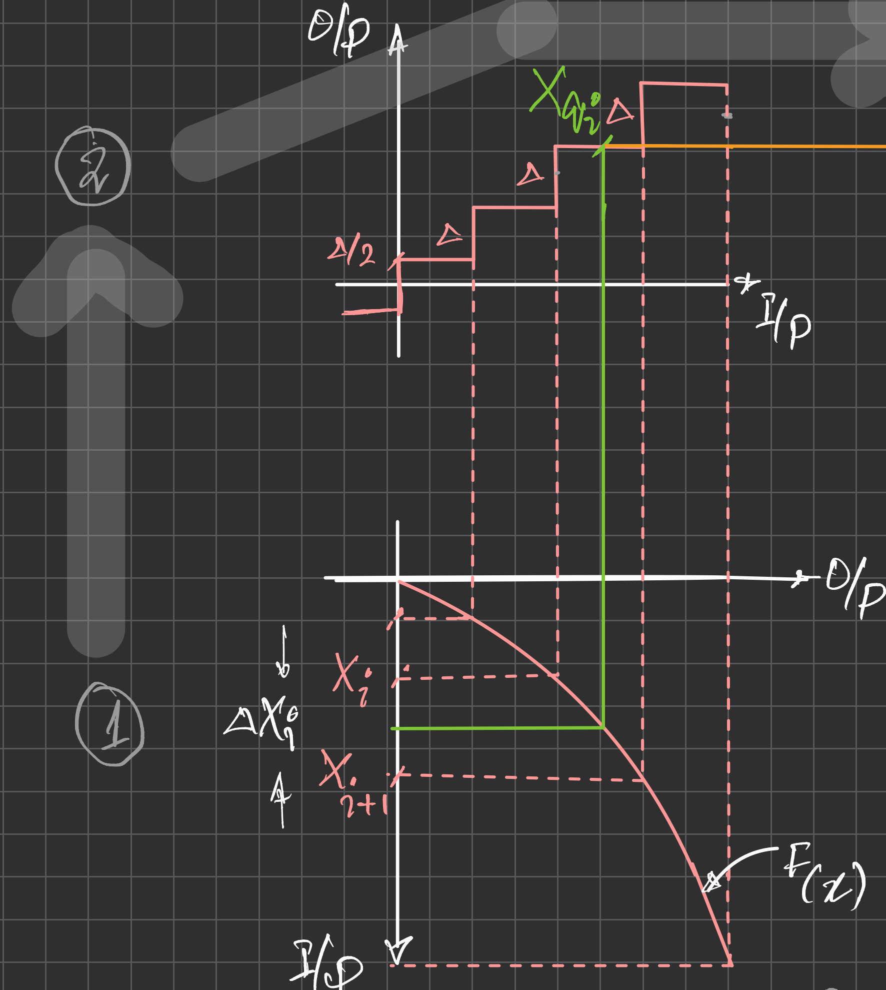
Companding

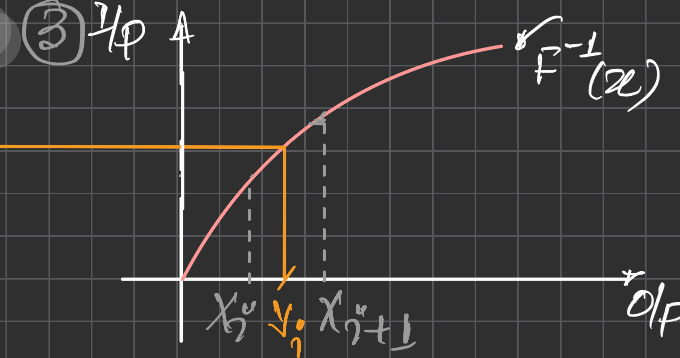


(a) the nonuniform quantifier characteristic, (b) the compression characteristic, (c) the uniform quantizer characteristic

SQNR for Non-uniform Quantization

The SQNR for non-uniform quantization we will derive in class! See next!





X_{q_i} : Quantized o/p of uniform quantizer.

$\Delta X_i : X_{i+1} - X_i$

Width of Quantization interval 'i'

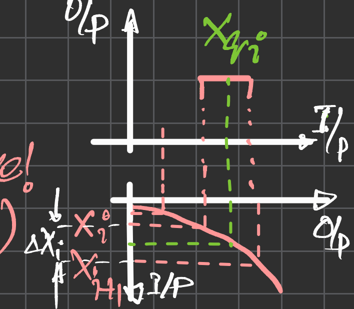
$\boxed{\text{Motive is to find SQNR.}}$

① Compression → ② Uniform Quantization → ③ Expansion.

Consider the interval $[X_i \;\; X_{i+1}]$ at the I/P of Compressor

Quantization noise power at level i can be given as:

$$E[n^2_{q_i}] = \int_{X_i}^{X_{i+1}} (x - y_i)^2 f_X(x)\, dx \quad — \text{(A)}$$



Inverse transformation here!
(Expansion at O/P)

where

$$y_i = F^{-1}(x_{q_i})$$

X_{q_i} : Quantized O/P of uniform Quantizer at level i

$\Delta X_i = X_{i+1} - X_i$ width of Quantization interval i

Considering two simplifying assumptions:

① $y_i = \dfrac{x_i + x_{i+1}}{2}$ or $\begin{cases} x_i = y_i - \dfrac{\Delta x_i}{2} \\ x_{i+1} = y_i + \dfrac{\Delta x_i}{2} \end{cases}$

② $f_X(x) \simeq f_X(y_i)$ (prob. dist. of I/p & O/p are the same) as $\Delta \to 0$

Applying 1 & 2 in A

$$E[n_{q_i}^2] = \int_{y_i - \frac{\Delta x_i}{2}}^{y_i + \frac{\Delta x_i}{2}} (x - y_i)^2 f_{X}(y_i)\, dx$$

$$= f_X(y_i) \int_{y_i - \frac{\Delta x_i}{2}}^{y_i + \frac{\Delta x_i}{2}} (x - y_i)^2 dx = f_X(y_i) \left. \frac{(x - y_i)^3}{3} \right|_{y_i - \frac{\Delta x_i}{2}}^{y_i + \frac{\Delta x_i}{2}}$$

$$= f_X(y_i) \frac{2(\Delta x_i^3 / 8)}{3} = f_X(y_i) \cdot \frac{\Delta x_i^3}{12} \quad \text{— } B$$

Assume $u = F_{(x)}$, that is, 'u' is the transformed version of 'x'

$$du = F'_{(x)} dx$$

compression applied to 'x' $\rightarrow F_{(x)} = u$

For small Δ ; $\quad \Delta u = F'_{(y_i)} \Delta x_i \quad \leftarrow$ ($x_i = y_i$ for small Δ)

$$\Delta x_i = \Delta u / F'_{(y_i)}$$

Expansion applied to x_{q_i} $\rightarrow F'(x_{q_i})$

Therefore eqn (B) $\Rightarrow$

$$E[n_{q_i}^2] = f_X(y_i) \cdot \underbrace{\frac{\Delta u^2}{12\,[F'(y_i)]^2}}_{\Delta x_i^2} \cdot \Delta x_i$$

Keep it separate!

Total average quantization noise power:

$$E[n_q^2] = \sum_{i=0}^{q-1} E[n_{q_i}^2]$$

all the quantization intervals!

$$= \frac{\Delta u^2}{12} \sum_{i=0}^{q-1} \frac{f_X(y_i)\,\Delta x_i}{F'(y_i)^2}$$

Again as $\Delta \to 0$ $\therefore$

① $y_i \to x_i$

② $\Sigma \to \int$

③ $x_i \to x$.

$$= \frac{\Delta u^2}{12} \int_{X_{min}}^{X_{max}} \frac{f_X(x)}{[F'(x)]^2}\,dx$$

Notes:

For small Δ

$\Delta \to 0$

$y_i \to x_i$

And as $\Delta \to 0$

$x_i \to x$

$\Delta x \to dx$

$\boxed{\text{SQNR}}$ can then be given as: $10 \log_{10} \frac{E[X^2]}{E[n_q^2]}$.

SQNR for Non-uniform Quantization

The SQNR is given by: $SQNR = \frac{\mathbb{E}[X^2]}{\mathbb{E}[n_q^2]}$ where

$$\mathbb{E}[n_q^2] = \frac{\Delta^2}{12} \int_x \frac{f_X(x)}{[F'(x)]^2} dx$$

What happens when $F(x) = x$?

linear (no compression)

$$E[n_q^2] = \frac{\Delta^2}{12} \int_x \frac{f_X(x)}{(1)^2} dx = \frac{\Delta^2}{12}(1)$$

When logarithmic → compression!

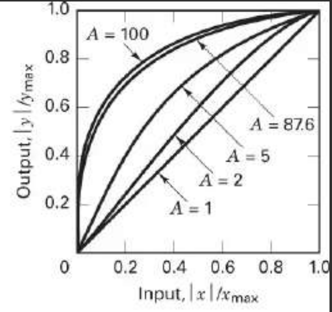
Non-Uniform Quantization using Companding

- There are in fact two standard logarithms based companding techniques
  - European standard called **A-law companding**
  - US standard called μ**-law companding**

Compression Laws: A- Law

- Commonly used logarithmic compression law
- The A-law is used for PCM telephone systems in the Europe
- A practical value for A is 100

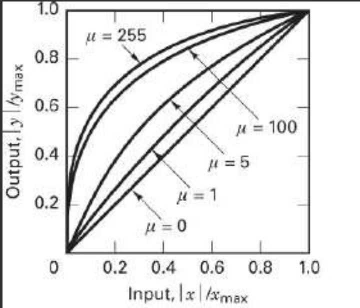
$$F(x) = \begin{cases} \frac{1+\ln(A|x|)}{1+\ln(A)} \cdot \operatorname{sgn}(x), & \frac{1}{A} \leq |x| \leq 1 \\ \frac{A|x|}{1+\ln(A)} \cdot \operatorname{sgn}(x), & |x| \leq \frac{1}{A} \end{cases}$$



Compression Laws: μ - Law

- Commonly used logarithmic compression law
- The μ-law is used for PCM telephone systems in the USA, Canada, and Japan
- A practical value for μ is 255

$$F(x) = \frac{\ln(1+\mu|x|)}{\ln(1+\mu)}\operatorname{sgn}(x), \quad -1 \leq x \leq 1$$



SQNR for μ - Law

The SQNR is given by: $SQNR = \frac{\mathbb{E}[X^2]}{\mathbb{E}[n_q^2]}$ where

Proof on next slide!

$$\mathbb{E}[n_q^2] = \frac{\Delta^2}{12}\frac{\ln^2(1+\mu)}{\mu^2}\int_{-1}^{1}(1+\mu|x|)^2 f_X(x)dx$$

This is a general expression with μ law compressor function $F^{'}(x)$ and thus can be used in conjunction with any type of signal distributions $f_X(x)$

In μ-law

$$F_{\mu}(x) = \frac{\ln(1+\mu|x|)}{\ln(1+\mu)} \operatorname{sgn}(x), \quad x \in \{-1, 1\}$$

Quantization noise power = ?

$$E[n_q^2] = \frac{\Delta U^2}{12} \int_{-1}^{1} \frac{f_X(x)}{[F'(x)]^2} dx$$

NOTE

ΔU here is the step size of the transformed variable U

We have: $F(x) = \dfrac{\ln(1+\mu|x|)}{\ln(1+\mu)}$

$$\therefore F'(x) = \frac{1}{\ln(1+\mu)} \cdot \frac{\mu}{(1+\mu|x|)}$$

$$E[n_q^2] = \frac{\Delta U^2}{2} \int_{-1}^{1} \frac{\ln^2(1+\mu)(1+\mu|x|)^2}{\mu^2} f_X(x) dx$$

"We may replace it with Δ to be consistent"

$$\boxed{E[n_q^2] = \frac{\Delta^2}{12} \frac{\ln^2(1+\mu)}{\mu^2} \int_{-1}^{1} (1+\mu|x|)^2 f_X(x) dx.}$$

Summary: Non-Uniform Quantization

- Symmetric Input signal amplitude range [-U_0 U_0]
  Asymmetric Signal [U_1 U_2]
- Number of quantization levels: $q = 2^n$ uniform stepsize.
- Quantization step: $\Delta_i = \frac{2U_0}{(q-1)F'(x_i)} = \frac{\Delta}{F'(x)} \approx \frac{U_0}{2^{n-1}F'(x_i)}$
  $\Delta_i = \frac{U_2-U_1}{(q-1)F'(x_i)} = \frac{\Delta}{F'(x)} \approx \frac{U_2-U_1}{2^n F'(x_i)}$ Assymetric.
- Quantization noise power: $\mathbb{E}[n_q^2] = \frac{\Delta^2}{12}\int_x \frac{f_X(x)}{[F'(x)]^2}dx$

✓ $\Delta = \frac{2U_0}{q-1}$

✓ $\Delta \approx \frac{2U_0}{q}$

✓ $\approx \frac{U_0}{q2^{-1}} = \frac{U_0}{2^{n-1}}$

$$\mathbb{E}[n_q^2] = \frac{\Delta^2}{12}\frac{\ln^2(1+\mu)}{\mu^2}\int_{-1}^{1}(1+\mu|x|)^2 f_X(x)dx$$

(for μ-law!)

Example: Uniform Distribution of Input Signal

If $|U_0| < 1$, integrate over U_0, else integrate from -1 to 1.

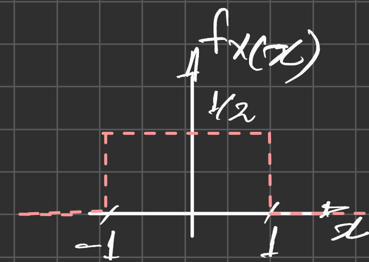
- $\mathbb{E}[X^2] = \frac{\Delta^2(q-1)^2}{12} = \frac{U_0^2}{3}$
- $\mathbb{E}[n_q^2] = \frac{\Delta^2}{12}\frac{\ln^2(1+\mu)}{\mu^2}\int_{-1}^{1}(1+\mu|x|)^2\frac{1}{2U_0}dx$
- $\mathbb{E}[n_q^2] = 2\frac{\Delta^2}{12}\frac{\ln^2(1+\mu)}{3\mu^3}[(1+\mu U_0)^3 - 1]\frac{1}{2U_0}$ (if $|U_0| < 1$)
- $\mathbb{E}[n_q^2] = \frac{\Delta^2}{12}\frac{\ln^2(1+\mu)}{3\mu^3}[(1+\mu)^3 - 1]\frac{1}{U_0}$ (if $|U_0| \geq 1$)
- SQNR $= \frac{q^2}{\frac{\ln^2(1+\mu)}{3\mu^3}[(1+\mu)^3-1]\frac{1}{U_0}}$ (if $|U_0| \geq 1$)

$$\text{SQNR} = 20\log_{10}(q-1) - 10\log_{10}\frac{\ln^2(1+\mu)}{3\mu^3}[(1+\mu)^3 - 1]\frac{1}{U_0}$$

What happens when $\mu \to 0$?

Derivations next!

Example: Uniform Distribution of I/P Signal



$$f_{X}(x) = 1/2 \qquad x \in \{-1, 1\}$$

$$E[n_q^2] = ?$$

we have $$E[n_q^2] = \frac{\Delta^2}{12} \frac{\ln^2(1+\mu)}{\mu^2} \int_{-1}^{1} (1+\mu|x|)^2 f_{X}(x)\, dx$$

$$= \frac{\Delta^2}{12} \frac{\ln^2(1+\mu)}{\mu^2} \cdot \frac{\cancel{1}\, 2}{2} \int_{0}^{1} (1+\mu x)^2 dx$$

Symmetric

$y = 1+\mu x$

$dy = \mu dx$

$$= \frac{\Delta^2}{12} \frac{\ln^2(1+\mu)}{\mu^2} \left. \frac{(1+\mu x)^3}{3\mu} \right|_{0}^{1}$$

$$= \frac{\Delta^2}{36\mu^3} \ln^2(1+\mu) \left[(1+\mu)^3 - 1\right]$$

& $$E[x^2] = \int_{-1}^{1} x^2 \frac{1}{2} dx = \frac{1}{2} \left. \frac{x^3}{3} \right|_{-1}^{1} = \frac{1}{6}(2) = \frac{1}{3}$$

$$\text{SQNR [dB]} = 10\log_{10} \frac{E[x^2]}{E[n_q^2]}$$

$$= 10 \log_{10} \frac{(1/3)}{\frac{\Delta^2}{36\mu^3} \ln^2(1+\mu)[(1+\mu)^3-1]}$$

$$= 10 \log_{10} \frac{12\mu^3}{\Delta^2 \ln^2(1+\mu)[(1+\mu)^3-1]}$$

If the uniform PDF is given as follows:

$$f_{X(x)} = \frac{1}{2U_0} \qquad \text{"Symmetric"}$$

$\text{SQNR [dB]} = ?$

$$\Delta = \frac{2U_0}{(q-1)} \quad , \quad E[x^2] = \int_{-U_0}^{U_0} x^2 \times \frac{1}{2U} \, dx$$

$$= \frac{1}{2U} \frac{x^3}{3} \Big|_{-U_0}^{U_0} = \frac{2U_0^3}{6U_0} = \frac{U_0}{3}$$

$$E[x^2] = \frac{u_0^2}{3} \; * \qquad * \; \Delta = \frac{2u_0}{q-1}$$

$$= \frac{(q-1)^2 \Delta^2}{2^2 \cdot 3} = \frac{(q-1)^2 \Delta^2}{12}.$$

$$SQNR[dB] = 10\log_{10} \frac{(q-1)^2 \Delta^2/\cancel{12}}{\ln^2(1+\mu)(\Delta^2/3\cancel{6})\mu^3[(1+\mu)^3-1]}$$

$$= 10\log_{10} \frac{3(q-1)^2 \mu^3}{\ln^2(1+\mu)[(1+\mu)^3-1]}$$

$$= 10\log_{10} \frac{q^2\mu^3}{\frac{\ln^2(1+\mu)}{3}[(1+\mu)^3-1]} \qquad (q-1) \simeq q$$

$$= 20\log_{10} q - 10\log_{10} \frac{\ln^2(1+\mu)[(1+\mu)^3-1]}{3\mu^3}$$

For the case $\mu \to 0$

$$SQNR[dB] = 20\log q - A$$

where,

$$A = \lim_{\mu \to 0} 10 \log_{10} \frac{\ln^2(1+\mu)\left[(1+\mu)^3 - 1\right]}{3\mu^3}$$

$$= \lim_{\mu \to 0} 10 \log_{10} \frac{\ln^2(1+\mu)}{3\mu^2} + \lim_{\mu \to 0} 10 \log_{10} 3(1+\mu)^2$$

Applying L'hospital rule:

$$A = \lim_{\mu \to 0} 10 \log_{10} \frac{2\ln(1+\mu)}{(1+\mu)\cdot 6\mu} + \lim_{\mu \to 0} 10 \log_{10} 3(1+\mu)^2$$

Applying L'hospital rule again:

$$A = 10 \log_{10} \frac{2}{(1+\mu)\left[6 + 12\mu\right]} + 10 \log_{10} 3$$

$$= 10 \log_{10} \frac{2}{6} \cdot 3 = 10 \log_{10} 1 = 0$$

$$SQNR_{[dB]} = 20 \log_{10} q$$ when $\mu \to 0$.

Example: Gaussian Distribution of Input Signal

- $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, |x| \leq 1/3$
- Find SQNR when σ =1

$$SQNR = \frac{\mathbb{E}[X^2]}{\frac{\Delta^2}{12}\frac{\ln^2(1+\mu)}{\mu^2}\int_{-1/3}^{1/3}(1+\mu|x|)^2\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx}$$

Example : Gaussian Distributed Signal

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \qquad |x| \leq 1/3$$

SQNR = ? when $\sigma = 1$

$$E[n_q^2] = \frac{\Delta^2}{12} \frac{\ln^2(1+\mu)}{\mu^2} \int_{-1/3}^{1/3} (1+\mu|x|)^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2} \cdot dx$$

$$E[n_q^2] = \frac{\Delta^2 F(\mu)}{12\sqrt{2\pi}}$$

(After numerical integration)

μ	$F(\mu)$
0	1
1	0.7
2	0.62
3	0.59
4	0.58
5	0.59
6	0.59
7	0.59

$\mu^* = 4$ to minimize Quant. Noise Power!

Thank You
Happy Learning

☺