EECS4214

Digital Communications

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January 20 and 22, 2025
Week 3

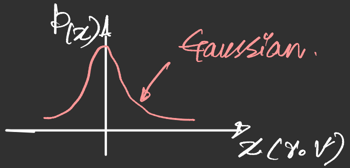
Week 3 - Lectures 5 and 6

1. Review of the Last Lecture
2. Random Processes (contd.)
3. Random Processes and Linear Systems + Examples!
4. Formatting - Next Week!

Review of the Last Lecture

Review

- Continuous r.v
  - Defined by
    - PDF } significance in DCS (PDFs
    - CDF }
      1. Poisson.
      2. Exponential.
      3. Gaussian (noise)

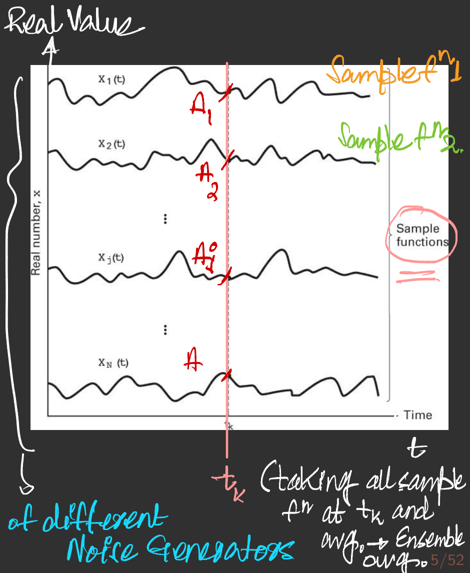


- Random Processes.

Will review in the lecture!

Random Processes

- A random process $X(A,t)$ can be viewed as a function of two variables: an event A and time t
- For a specific event A_j, we have a single time function $X(A_j,t) = X_i(t)$ (i.e., a **sample function**)
- The totality of all sample functions is called an *ensemble*
- For a specific time t_k, $X(A,t_k)$ is a **random variable** $X(t_k)$ whose value depends on the event
- For a specific event $A = A_j$ and a specific time $t = t_k$, $X(A_j,t_k)$ is simply a **number**



Random Processes

We shall designate the random process by $X(t)$ and let the functional dependence upon A be implicit

Characterization of a Random Process

- **Amplitude Domain:** PDF and CDF
- **Time Domain:** Autocorrelation function
- **Frequency Domain:** Power spectral density function

Statistical Averages of a Random Process

- Because the value of a random process at any future time is unknown (since the identity of the event A is unknown):
  - A random process whose distribution functions are continuous can be described statistically with a PDF

- In most situations, it is **not practical to determine empirically** the probability distribution of a random process

- A partial description consisting of the **mean** and **autocorrelation function** is often adequate for the needs of a **communication system**

Statistical Averages of a Random Process

- The **mean** of the random process $X(t)$ freezing the time. $t=t_k$

$$\mathbb{E}\{X(t_k)\} = \int_{-\infty}^{\infty} x \underbrace{f_{X_k}(x)}_{\text{at } t=t_k} dx$$

where $f_{X_k}(x)$ is the density over the *ensemble* of events at time t_k

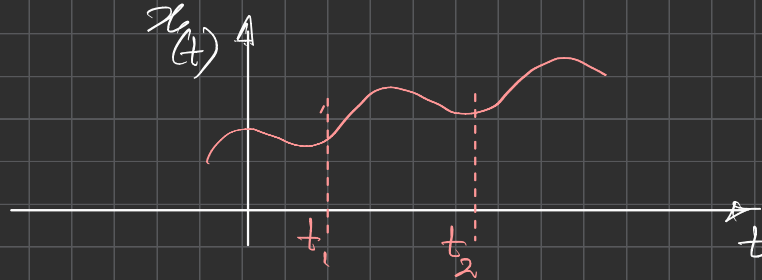
- The **autocorrelation function** of the random process $X(t)$

$$R_X(t_1, t_2) = \mathbb{E}\{X(t_1)X(t_2)\}$$

where $X(t_1)$ and $X(t_2)$ are random variables obtained by observing $X(t)$ at times t_1 and t_2, respectively

  - The **autocorrelation function** is a measure of the degree to which two time samples of the same random process are related (see next Page)

Autocorrelation



how these two samples at t_1 & t_2 correlate?

$E\{X_{(t_1)} X_{(t_2)}\}$ – Correlation high
(Similarity high)

– Correlation low
(Similarity low)

Random Processes (contd.)

Time Averaging and Ergodicity

- To compute $m_X = \mathbb{E}\{X(t_k)\}$ and $R_X(t_1, t_2)$ by ensemble averaging, we would have to average across all the sample functions of the process

mean = $X(t_k)$

Autocorr. = $X(t_1)\, X(t_2)$

$$\mathbb{E}[g(X(t))]\Big|_{t=t_k} = \int_{-\infty}^{\infty} g(x(t_k)) \boxed{f_X(x, t_k)}\, dx$$

?

We would also need to have complete knowledge of the first- and second-order joint probability density functions. **Such knowledge is generally not available!**

- When a random process belongs to a special class, known as an **ergodic process**, its **time averages equal its ensemble averages**, and the statistical properties of the process can be determined by time averaging over a single sample function of the process

- For a random process to be **ergodic, it must be stationary** in the **strict sense**

What is Stationarity?

Strict-sense/ Wide-sense?

Stationarity

- A random process $X(t)$ is said to be **stationary** in the **strict sense** if none of its statistics are affected by a shift in the time origin PDF/CDF remains the same!
- A random process is said to be **wide-sense stationary** (WSS) if two of its statistics, its mean and autocorrelation function, do not vary with a shift in the time origin, that is:

$$\mathbb{E}\{X(t)\} = m_X = \text{constant}$$

with the shift in time.

and

$$R_X(t_1, t_2) = \boxed{R_X(t_1 - t_2)}$$

$t_1 - t_2 = \tau$

$= R_X(\tau)$

This means the **autocorrelation function** depends only on the time difference $\tau = t_1 - t_2$. That is, all pairs of values of $X(t)$ at points in time separated by $\tau = t_1 - t_2$ have the same correlation value

Same $R_X(\tau)$

$t_1 - t_2 = \tau$, $t_3 - t_4 = \tau$, $t_5 - t_6 = \tau$ etc.

Autocorrelation of a Wide-Sense Stationary Random Process

$X(t_1) \; X(t_2)$

- For a **WSS** process, the autocorrelation function is only a function of the time difference $\tau = t_1 - t_2$

$$R_X(\tau) = \mathbb{E}\{X(t)X(t-\tau\} = \mathbb{E}\{X(t)X(t+\tau\} \quad \text{for} \quad -\infty < \tau < \infty$$

- For a **zero mean WSS** process, $R_X(\tau)$ indicates the extent to which the random values of the process separated by τ seconds in time are statistically correlated

Time Averaging and Ergodicity

- **Ensemble Averaging:** By freezing the time, we can average over all possible realizations

PDF over ensemble.

$$\mathbb{E}[g(X(t))]\Big|_{t=t_k} = \int_{-\infty}^{\infty} g(x(t_k)) f_X(x, t_k)\, dx$$

- **Time Averaging:** Averaging over time

$$\langle g(X(t)) \rangle = \frac{1}{T_0} \int_{T_0} g(x,t)\, dt = \lim_{T_0 \to \infty} \frac{1}{T_0} \int g(x,t)\, dt$$

$= x(t)$

✓ mean.

- For **Ergodic Process**

$$\mathbb{E}[g(X(t))]\Big|_{t=t_k} = \langle g(X(t)) \rangle$$

For auto corr.

$g(x,t) = X(t)X(t-\tau)$

Ergodic Process

- A random process is ergodic in the mean if:

$$m_X = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\, dt \quad ✓$$

- A random process is ergodic in the autocorrelation function if:

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)X(t+\tau)\, dt$$

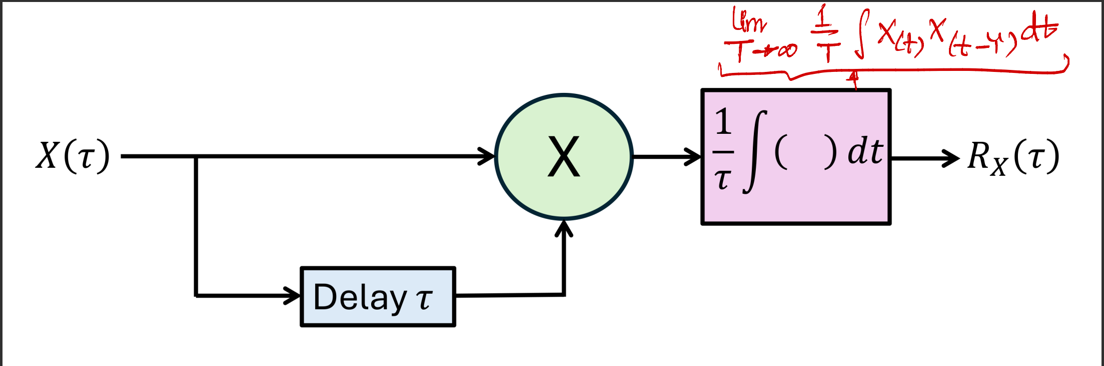
A reasonable assumption in the analysis of most communication signals (in the absence of transient effects) is that the random waveforms are ergodic in the mean and the autocorrelation function

Ergodic Process

Process Type	Stationary	Ergodic	Example in CS
Thermal noise	✓ WSS	✓	White Gaussian noise in a receiver
Random bit stream	✓ WSS	✓	Binary transmission over a channel
Speech signals	✗(over long periods)	✓ (over short periods)	Voice feature extraction
Multipath fading	✓ WSS	✗ (fast variations)	Wireless communication models
Internet traffic	✗(bursty and non-stationary)	✗(long-range dependencies)	Network congestion modeling

Table: Comparison of different processes in CS based on stationarity and ergodicity

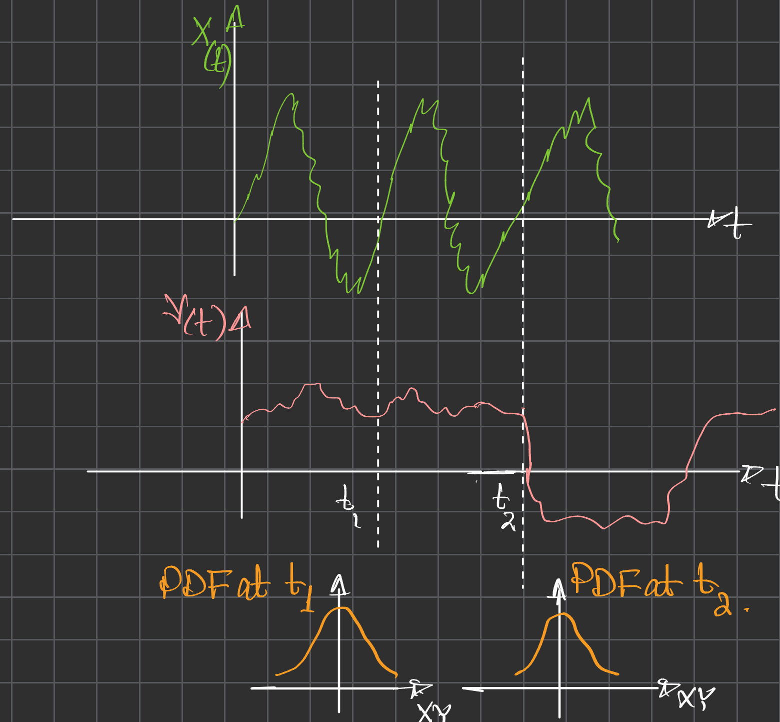
Schematic Representation of Autocorrelation Function



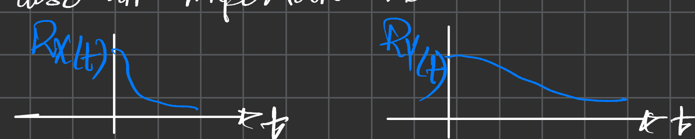
What is the significance of Autocorrelation Function?

Significance of Autocorrelation Function

- $R_X(\tau)$ gives us an idea of the frequency response that is associated with a random process
- If $R_X(\tau)$ changes slowly as τ increases from zero to some value $\implies X(t)$ contains low frequencies and vice versa



- The PDF of the two processes is the same at t_1 & t_2. However, $x(t)$ is rapidly varying than $y(t)$
- Thus Autocorrelation is also an important indicator to distinguish random processes.



Autocorrelation of a Wide-Sense Stationary Random Process

4 $R_X(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x(t)\,x(t)\,dt = \lim_{T\to\infty} \frac{1}{T} \int x^2(t)\,dt \leftarrow R_X(0) = E\{x^2(t)\}$

- Properties of the autocorrelation function of a real-valued WSS process are

S. No.	Expression	Property
1	$R_X(\tau) = R_X(-\tau)$	Symmetrical in τ about zero
2	$\lvert R_X(\tau)\rvert \leq R_X(0)$ for all τ	Maximum value occurs at origin
3	$R_X(\tau) \leftrightarrow G_X(f)$	Autocorrelation and power spectral density form a Fourier transform pair
4	$R_X(0) = \mathbb{E}\{X^2(t)\}$	Value at the origin is equal to the average power of the signal

"Weiner Khinchin Theorem"

Power Spectral Density

A random process $X(t)$ can generally be classified as a power signal having a power spectral density (PSD) $G_X(f)$ of the form:

$$G_X(f) = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2$$

where $X_T(f)$ is the Fourier transform of a truncated version (observing it only in the interval $(-T/2, T/2)$) of the non-periodic power signal with **power** given as:

$$P_X = \lim_{T \to \infty} \frac{1}{T} \int_T |x(t)|^2 \, dt$$

Recap: Energy of a Signal

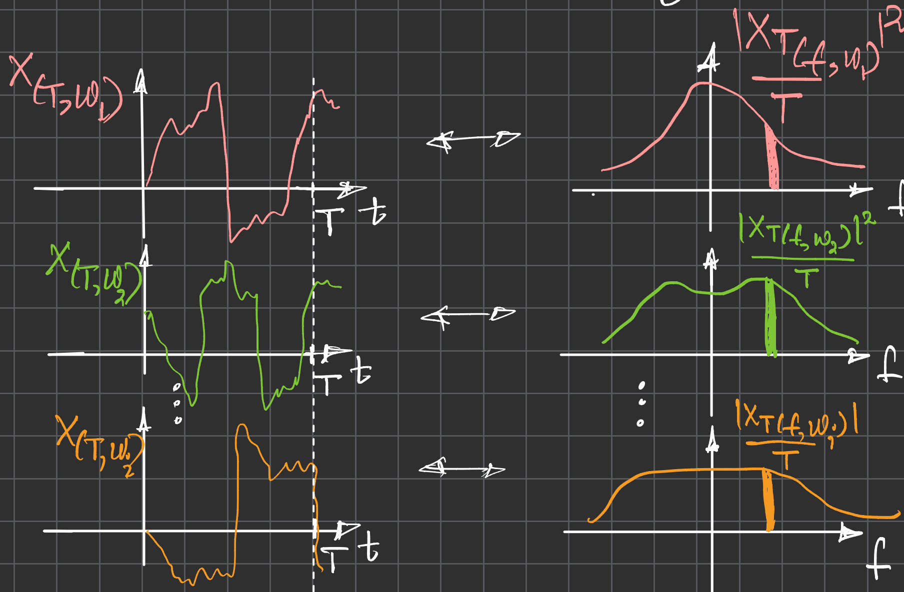
$$E_X = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Power Spectral Density of Random Process

Let $x(t)$ is a random process

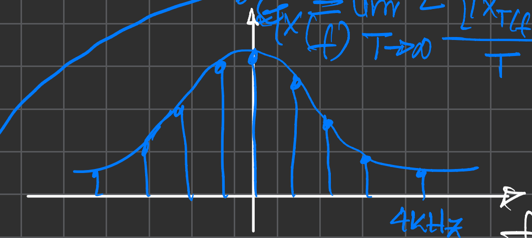
$x(t, w_i)$ is an ith realization of $x(t)$

$x_T(t, w_i)$ truncated realization over T



- If i average all FT realizations, I get power spectral density.

$$G_X(f) = \lim_{T\to\infty} \frac{E\left[|X_{T(f,w_i)}|^2\right]}{T}$$



w_i represents ith realization of r.p.

- We can get following information from Power Spectral Density.

  + How broad the sig. is in freq. domain.

  + What portion of the sig. BW is really important.

Here

$$X_{T(f,w_i)} = \int_T x_{T}(t, w_i) e^{-j2\pi f t} dt$$ [CTFT of ith realization of r.p. $x(t)$]

Significance of Power Spectral Density

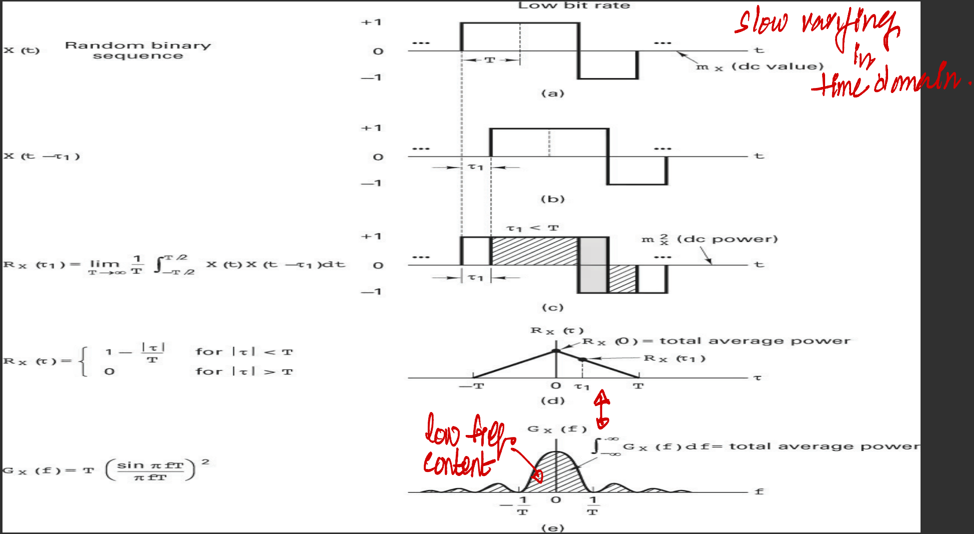
- $G_X(f)$ describes the distribution of a signal's power in the frequency domain
- Enables us to evaluate the signal power that will pass through a network having known frequency characteristics
- *One Slide Later - Figure*

Principle Features of PSD

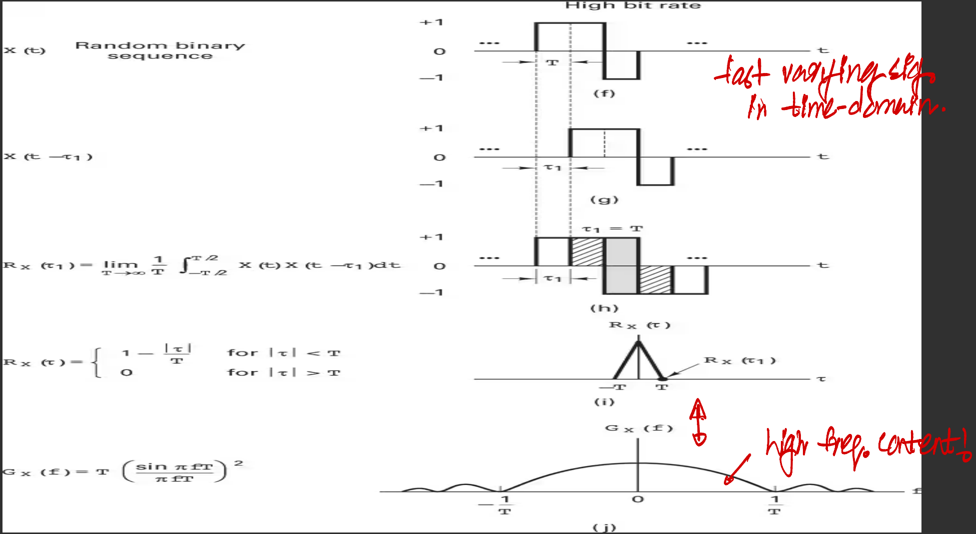
Principle Features of PSD

S. No.	Expression	Property
1	$G_X(f) \geq 0$	Non-negative and is always real
2	$G_X(f) = G_X(-f)$	Symmetric in f about zero for $X(t)$real valued
3	$G_X(f) \leftrightarrow R_X(\tau)$	PSD and Autocorrelation form a Fourier transform pair
4	$P_X = \int_{-\infty}^{\infty} G_X(f)df$	Relationship between average normalized power and PSD

Significance of PSD



Significance of PSD



Example 6

Given that $R_X(\tau) = \exp\left(-\frac{|\tau|}{T}\right)$, $\forall\tau$. Determine the PSD for the following cases:

- $T = 1$
- $T = 10$
- $T = \infty$
- $T = 0$

Solution

$$G_X(f) \leftrightarrow R_X(\tau)$$

$$G_X(f) = \int_{-\infty}^{\infty} \exp\left(-\frac{|\tau|}{T}\right) e^{-j2\pi f\tau}\, d\tau$$

product of even & odd f^n is odd; evaluates to zero.

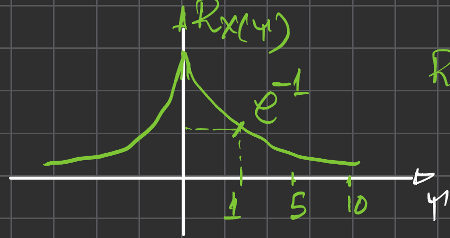
$$= \int_{-\infty}^{\infty} \underbrace{\exp\left(-\frac{|\tau|}{T}\right)}_{\text{`even'}} [\cos 2\pi f\tau - \underbrace{j\sin 2\pi f\tau}_{\text{`odd'}}]\, d\tau$$

$$= 2\int_{0}^{\infty} \underbrace{\exp\left(-\frac{|\tau|}{T}\right)\cos 2\pi f\tau}_{\text{Even}}\, d\tau$$

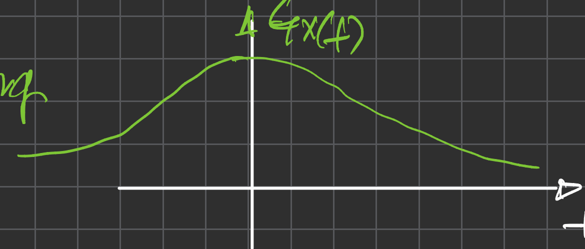
formula sheet!

$$= 2\int_{0}^{\infty} \exp\left(-\frac{\tau}{T}\right)\cos 2\pi f\tau\, d\tau = \underbrace{\frac{2\, 1/T}{1/T^2 + (2\pi f)^2}}_{} = \frac{2T}{1+(2\pi f T)^2}$$

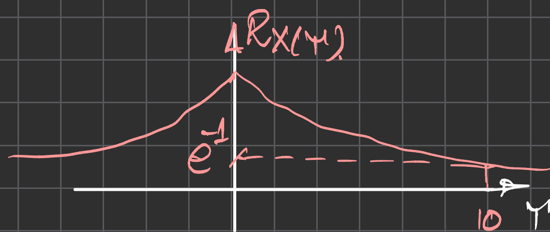
(a) $T=1$



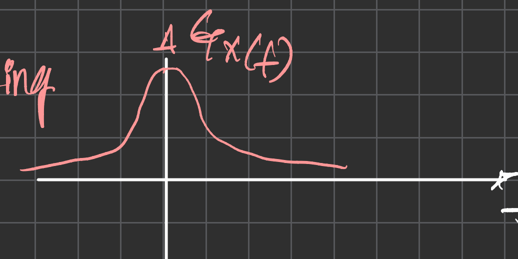
Rapidly varying signal



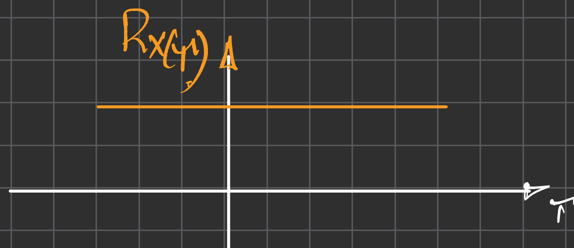
(b) $T=10$



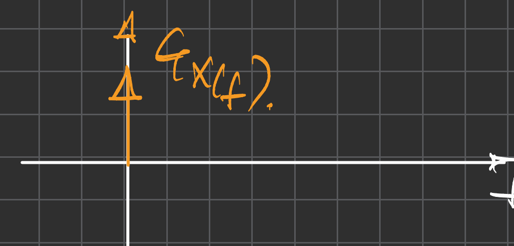
Slowly varying signal



(c) $T=\infty$



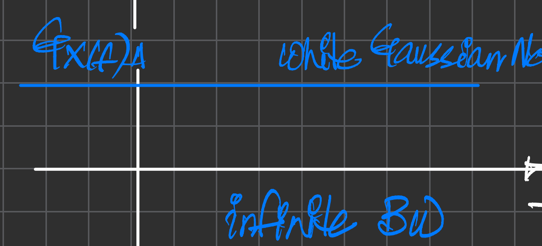
DC Signal



(d) $T=0$

White Noise

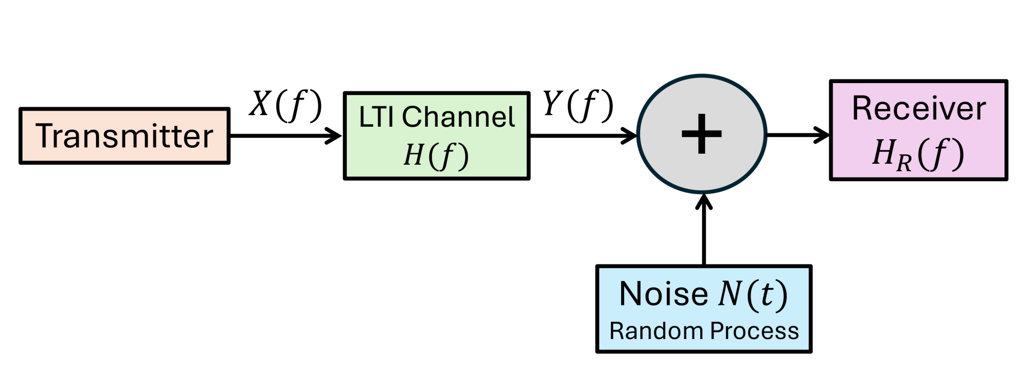
White Gaussian Noise



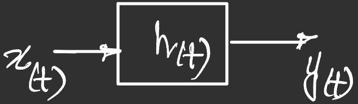
infinite BW

Random Processes and Linear Systems

Random Process and Linear Systems



Random Process and Linear Systems



- The LTI system is characterized by:
  - Time Domain

$$y(t) = x(t) * h(t)$$

  - Frequency Domain

Freq. Response of the LTI Sys.

$$Y(f) = H(f)X(f)$$

However, it is useful only for one realization of $x(t)$!!!

Random Process and Linear Systems

- **To describe the input random process as a whole:**
  - Autocorrelation function: $R_X(\tau)$
  - Power spectral density (PSD): $G_X(f)$

What about the output?

- If a random process forms the input to an LTI system, the output will also be a random process
- Each sample function of the input process yields a sample function of the output process

Random Process and Linear Systems

Transmission of a random process through linear systems can be characterized in time and frequency domain as follows:

- Time Domain

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

- Frequency Domain

$$G_Y(f) = G_X(f)|H(f)|^2$$

Random Process and Linear Systems

- For a **linear system**: If $x(t)$ is stationary/ergodic process, then $y(t)$ is also a stationary/ergodic process
- **Linear systems do not change the statistical behaviour of the process**

Example 7

Consider a random process with PSD $\overbrace{G_N(f) = N_0/2,}$ $\forall f$. Calculate the power of the random process after it is passed through a filter with frequency response given as follows:

$$H(f) = \frac{1}{1 + j(f/f_0)}$$

What type of system is this?

Solution

$$G_Y(f) = G_X(f)\,|H(f)|^2$$

$$= \frac{N_0}{2} \times \frac{1}{1 + (f/f_0)^2}; \quad \left(\frac{1}{\sqrt{1+(f/f_0)^2}}\right)^2$$

$$= \frac{N_0}{2} \times \frac{f_0^2}{f_0^2 + f^2}$$

$$P = \int_{-\infty}^{\infty} G_Y(f)\,df$$

$$= \frac{N_0}{\cancel{2}}\, \cancel{2} \int_{\cancel{-\infty}\;0}^{\infty} \frac{f_0^2}{f_0^2 + f}\,df$$

Using formula sheet.

$$\int \frac{1}{t^2 + a^2}\,dt = \frac{1}{a}\tan^{-1}\left(\frac{t}{a}\right) + c$$

$$P_Y = \frac{N_0 f_0^2}{f_0} \tan^{-1}\frac{f}{f_0}\Big|_0^{\infty}$$

$$P_Y = \frac{N_0 f_0 \pi}{2}$$

Example 8

Consider a random process with PSD $x(t) = A\cos(2\pi f t)$, where A is a random variable; check whether the process is stationary in wide-sense.

Solution

$$E[x(t)] = E[A\cos 2\pi t] = \overbrace{E[A]}^{\int_{-\infty}^{\infty} A f_A(a)\,da} \underbrace{\cos 2\pi t} \leftarrow \text{Depends on time!}$$

$$R_X(\tau) = E[x(t)x(t-\tau)]$$
$$= E[A^2](\cos 2\pi t)(\cos 2\pi(t-\tau))$$
$$= E[A^2]\cos 2\pi t \cos 2\pi(t-\tau) \leftarrow \text{Depends on time!}$$

The process is not WSS!

Example 9

Consider a random process with PSD $x(t) = \cos(\omega t + \theta)$, where θ is a uniformly distributed random variable with PDF given by $f_\theta(\theta) = \frac{1}{2\pi}$, $\theta \in [-\pi \quad \pi]$; check whether the process is stationary in wide-sense.

Solution

$$E[x(t)] = E\{\cos(\omega t + \theta)\} = \int_{-\pi}^{\pi} \cos(\omega t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{\sin(\omega t + \theta)}{2\pi} \Big|_{-\pi}^{\pi} = 0 \leftarrow \text{Independent of time `}t\text{'}$$

$$R_x(t,\tau) = E[x(t)\, x(t-\tau)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta) \cos(\omega t - \omega\tau + \theta) d\theta$$

$$2\cos a \cos b = \cos(a+b) + \cos(a-b)$$

$$R_x(t,\tau) = \frac{1}{4\pi} \left[\int_{-\pi}^{\pi} \cos(2\omega t + 2\theta - \omega\tau) + \cos\omega\tau \right) d\theta$$

$$= \frac{1}{4\pi} [0 + \cos\omega\tau(2\pi)] = \frac{\cos\omega\tau}{2} \leftarrow \text{Independent of time `}t\text{'}$$

Thus the process is WSS

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Example 10

Consider a random process with $R_X(\tau) = 1 - |\tau|, -1 \le \tau \le 1$. Determine the following:

$= 0$; otherwise

a PSD

b $\mathbb{E}[X^2(t)]$

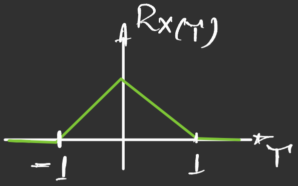
Solution

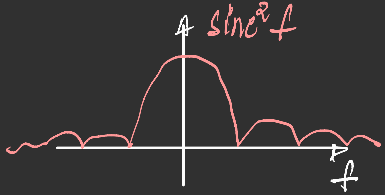
$R_X(\tau) = \mathrm{rect}(\tau) * \mathrm{rect}(\tau) = \mathrm{tri}(\tau)$

$\Updownarrow$

$G_X(f) = \mathrm{sinc}(f)\, \mathrm{sinc}(f) = \mathrm{sinc}^2 f$

$E[X^2(t)] = R_X(0) = 1$





Example 11

Consider a zero mean WSS random process, $X(t)$, with $R_X(\tau) = e^{-|\tau|}$, is passed through the channel with frequency response, $|H(f)| = \sqrt{1 + 4\pi^2 f^2}$, $|f| < 2$. Determine the following:

a PSD of $Y(t)$

b $\mathbb{E}[Y^2(t)]$

Solution

Example 6 $\left\{ \begin{array}{l} R_{X(\tau)} = e^{-|\tau|/T} \\ \updownarrow \\ G_{X(f)} = \dfrac{2 \times 1}{1+(2\pi f)^2} \end{array} \right.$

apply here!

(a) $G_{X(f)} = \dfrac{2}{1+(2\pi f)^2}$

$G_{Y(f)} = |H_{(f)}|^2 G_{X(f)} = 2.$

(b) $G_{Y(f)} = 8 \operatorname{sinc} 4\tau$

$= E[Y^2(t)] = 8$

Thank You
Happy Learning
☺