Mod Arith 1: Proofs

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This document collects my solutions to the OTIS problem sets from the **Mod Arith 1: Proofs** unit, written during my preparation for mathematical olympiads.

The solutions reflect my understanding and problem-solving approach at the time of writing. Some arguments were informed by discussions, official notes, or published sources; when so, attribution is provided (see section 3).

If you find errors or have suggestions, please contact me at samuelbaraujo19@gmail.com .

Contents

1 Practice Problems

PENH61 [9♣] **(PEN H61)** Find all positive integer solutions to $2^x - 5 = 11^y$.

12JM05 [5♣] **(USAJMO 2012 P5)** For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \le k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

09IMO1 [3♣] **(IMO 2009 P1)** Let n be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ $(k \ge 2)$ be distinct integers in the set $1, 2, \ldots, n$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

25AMO1 [9♣] **(USAMO 2025 P1)** Let k and d be positive integers. Prove that there exists a positive integer N such that for every odd integer $n > N$, the digits in the base-$2n$ representation of n^k are all greater than d.

17TSTST4 [5♣] **(USA TSTST 2017 P4)** Find all nonnegative integer solutions to $2^a + 3^b + 5^c = n!$.

21USATST1 [5♣] **(USA TST 2021 P1)** Determine all integers $s \ge 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

25JMO6 [5♣] **(USAJMO 2025 P6)** Let S be a set of integers with the following properties:

- $\{1, 2, \ldots, 2025\} \subseteq S$.
- If $a, b \in S$ and $\gcd(a, b) = 1$, then $ab \in S$.
- If for some $s \in S$, $s + 1$ is composite, then all positive divisors of $s + 1$ are in S.

Prove that S contains all positive integers.

20IBERO2 [3♣] **(Iberoamerican 2020 P2)** Let T_n denotes the least natural such that

$$n \mid 1 + 2 + 3 + \cdots + T_n = \sum_{i=1}^{T_n} i$$

Find all naturals m such that $m \ge T_m$.

04SLN6 [3♣] **(Shortlist 2004 N6)** Given an integer $n > 1$, denote by P_n the product of all positive integers x less than n and such that n divides $x^2 - 1$. For each $n > 1$, find the remainder of P_n on division by n.

14POL5 [2♣] **(Polish MO 2014 P5)** Find all pairs (x, y) of positive integers that satisfy

$$2^x + 17 = y^4$$

02SLN1 [2♣] **(Shortlist 2002 N1)** What is the smallest positive integer t such that there exist integers $x_1, x_2, \ldots, x_t$ with

$$x_1^3 + x_2^3 + \ldots + x_t^3 = 2002^{2002}\,?$$

06IMO4 [2♣] **(IMO 2006 P4)** Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

22JBMO4 [2♣] **(JBMO 2022 P3)** Find all quadruples of positive integers (p, q, a, b), where p and q are prime numbers and $a > 1$, such that

$$p^a = 1 + 5q^b.$$

22JMO1 [2♣] **(USAJMO 2022 P1)** For which positive integers m does there exist an infinite arithmetic sequence of integers $a_1, a_2, \ldots$ and an infinite geometric sequence of integers $g_1, g_2, \ldots$ satisfying the following properties?

- $a_n - g_n$ is divisible by m for all integers $n \geq 1$;
- $a_2 - a_1$ is not divisible by m.

12CGMO3, 16ELMO4, 20JBMO4

2 Solutions

2.1 Lecture Notes

2.2 Mandatory

2.2.1 PEN H61

Problem Statement

Find all positive integer solutions to $2^x - 5 = 11^y$.

2.2.2 USAJMO 2012 P5

Problem Statement

For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \le k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

2.3 Not mandatory

2.3.1 USAMO 2025 P1

Problem Statement

Let k and d be positive integers. Prove that there exists a positive integer N such that for every odd integer $n > N$, the digits in the base-$2n$ representation of n^k are all greater than d.

2.3.2 USA TSTST 2017 P4

Problem Statement

Find all nonnegative integer solutions to $2^a + 3^b + 5^c = n!$.

2.3.3 IMO 2006 P4

Problem Statement

Find all nonnegative integer solutions to $2^a + 3^b + 5^c = n!$.

2.3.4 Iberoamerican 2020 P2

Problem Statement

Let T_n denotes the least natural such that

$$n \mid 1 + 2 + 3 + \cdots + T_n = \sum_{i=1}^{T_n} i$$

Find all naturals m such that $m \geq T_m$.

2.3.5 Shortlist 2004 N6

Problem Statement

Given an integer $n > 1$, denote by P_n the product of all positive integers x less than n and such that n divides $x^2 - 1$. For each $n > 1$, find the remainder of P_n on division by n.

2.3.6 USAJMO 2025 P6

Problem Statement

Let S be a set of integers with the following properties:

- $\{1, 2, \ldots, 2025\} \subseteq S$.
- If $a, b \in S$ and $\gcd(a, b) = 1$, then $ab \in S$.
- If for some $s \in S$, $s + 1$ is composite, then all positive divisors of $s + 1$ are in S.

Prove that S contains all positive integers.

3 References