USAJMO 2012 Solutions

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1. Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

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Problem 1

Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Solution

By the *Alternate Segment Theorem,* $\overline{AC}$ is tangent to (QRS) and $\overline{AB}$ is tangent to (PRS). Assume for the sake of contradiction that (QRS) and (PRS) are distinct. In that case, $A \in \overline{BC}$ since $\overline{BC}$ is the radical axis and $\text{Pow}_{(QRS)}(A) = \text{Pow}_{(PRS)}(A)$. This leads to a contradiction, as $A \notin \overline{BC}$. Therefore, P, Q, R and S are concyclic.