OTIS Homework Problems

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A collection of solutions for Geometry, Inequalities and others, meant to be sent to Evan Chen, in the OTIS Application Homework.

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1 Problems

1.1 Geometry

1. **(#2.28, JMO 2012)** Given a triangle ABC, let P and Q be points on segments AB and AC, respectively, such that $AP = AQ$. Let S and R be distinct points on segment BC such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

2. **(#2.35, IMO 2009)** Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of BP, CQ, PQ. Suppose that PQ is tangent to the circumcircle of $\triangle KLM$. Prove that $OP = OQ$.

3. **(#3.25, USAMO 1993)** Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

1.2 Inequalities

1. Suppose that $a^2 + b^2 + c^2 = 1$ for positive real numbers a, b, c. Find the minimum possible value of
$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

2. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \le 4$. Prove that
$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \ge 3.$$

3. Let a, b, c, d be positive reals with $(a + c)(b + d) = 1$. Prove that
$$\frac{a^3}{b + c + d} + \frac{b^3}{c + d + a} + \frac{c^3}{d + a + b} + \frac{d^3}{a + b + c} \ge \frac{1}{3}.$$

1.3 Additional

1. Write a computer program to find the number of ordered pairs of prime numbers (p, q) such that when
$$N = p^2 + q^3$$
is written in decimal (without leading zeros), each digit from 0 to 9 appears exactly once. For example, $(109, 1163)$ is one such pair because $109^2 + 1163^3 = 1573049628$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which
$$f\Big(xf(x) + f(y)\Big) = f(x)^2 + y$$
holds for all real numbers x and y.

3. Let a, b, c, d be real numbers such that $b - d \ge 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

4. Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

2 Geometry Solutions

2.1 Problem 1.

Enunciado do problema

Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

By the *Alternate Segment Theorem*, $\overline{AC}$ is tangent to (QRS) and $\overline{AB}$ is tangent to (PRS). Assume for the sake of contradiction that (QRS) and (PRS) are distinct. In that case, $A \in \overline{BC}$ since $\overline{BC}$ is the radical axis and $\text{Pow}_{(QRS)}(A) = \text{Pow}_{(PRS)}(A)$. This leads to a contradiction, as $A \notin \overline{BC}$. Therefore, P, Q, R and S are concyclic.

2.2 Problem 2.

Enunciado do problema

2.3 Problem 3.

Enunciado do problema

3 Inequalities Solutions

3.1 Problem 1.

3.2 Problem 2.

3.3 Problem 3.

4 Additional Solutions

4.1 Problem 1.

4.2 Problem 2.

4.3 Problem 3.

4.4 Problem 4.

5 References

This document was made possible thanks to the help and inspiration of the following resource:

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