Notes on Olympiad Number Theory

Samuel de Araújo Brandão

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0.1 Problems

0.1.1 USAJMO 2019 P1

Problem Statement

There are $a+b$ bowls arranged in a row, numbered 1 through $a+b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear. A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$, provided that the difference $i-j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

We claim that, in order to reach the desired configuration, a and b cannot both be odd; equivalently, ab must be even. We will first show that an even product ab always allows us to reach the goal. Then we will prove that the desired configuration is impossible when ab is odd.

Let i' be the apple originally in bowl i and j' be the pear in j.

Claim — If $i-j$ is even, then i' can always be swapped with j'.

Proof. WLOG assume $i<j$. After n legal moves, i' and j' are in bowls $i+n$ and $j-n$, respectively. Since $i-j$ is even, $n=\frac{j-i}{2}$ is an integer. Therefore, i' and j' meet at

$$i+n=j-n=\frac{i+j}{2}.$$

After that, i' can move to $\frac{i+j}{2}+n$ to reach j and j' to $\frac{i+j}{2}-n$ to reach i. □

Now suppose ab is even.

- If $\min(a,b)=1$, say $a=1$, then b must be even for ab to be even. Therefore the bowls 1 and $a+b$ can be swapped, because $(a+b)-1$ is even.
- If $\min(a,b)\geq 2$
  - and $a+b$ is odd, $(a+b)-1$ is even, so the first and the last bowls can be swapped. Therefore, ab can be reduced to $(a-1)(b-1)$ since the bowls 1 and $a+b$ already have the desired fruits, which is still even. Thus, by complete induction, $a+b$ odd works, since $(a-1)+(b-1)$ is smaller than $a+b$.
  - and $a+b$ is even, $(a+b)-2$ and $(a+b-1)-1$ are even. a and b are both even numbers, because if they were both odd numbers, ab wouldn't be even. So the second and the last bowls can be swapped, as the first and penultimate bowls. Therefore, we can reduce ab to $(a-1)(b-1)$, which is still even, since the bowls 1, 2, $a+b-1$ and $a+b$ already have the desired fruits. Thus, by complete induction, $a+b$ even works, since $(a-2)+(b-2)$ is smaller than $a+b$.

Hence, when ab is even, we can reach the desired goal.

Now, by the sake of contradiction, say ab is odd. Let A be the amount of apples in the first odd-numbered bowls and B of pears in the last odd-numbered b bowls. We already know that i and j must have parity in order to $i-j$ be even. Therefore A and B are invariants, because after a legal move, i' and j' must keep having parity, so A and B are always going to each decrease by 1 or increase by 1.

Since $A-B$ is invariant, the remainder must always be equal, regardless of the fruits' order (apples first, then pears, or pears first, then apples). Since a is odd, but $a+b$ is even, $A=\frac{a+1}{2}$ and $B=\frac{b-1}{2}$

originally, but after the swaps, $A = \frac{a-1}{2}$ and $B = \frac{b+1}{2}$. Consequently,

$$\frac{a+1}{2} - \frac{b-1}{2} = \frac{a-1}{2} - \frac{b+1}{2} \iff \frac{a-b+2}{2} = \frac{a-b-2}{2} \iff 2 = -2$$

Contradiction! Hence, we can reach the desired goal if and only if ab is even.

💡 **Insight —** I am proud of myself because this is the first serious combinatorics problem I solved, even though there were a lot of mistakes in my original solution.

- Lack of rigorosity. I understand now that writing more to make the solution more comprehensible is good. However, time matters a lot in an olympiad context. Therefore, solving all the problems before actually writing the solutions seems to be the best thing to do. This way, I can keep track of my time better, understand if I can or not be that rigorous. Another thing about writing solutions is being more ”mathematician”. What I mean is that writing solutions mathematically is often faster and more understandable.
- Learned about induction, an awesome tool to use when an iteration or anything related must be demonstrated. In this problem, we needed to show that an specific configuration could only be obtained when ab were even. However, notice how many iterations and steps there are here!
- Learned about invariants either. Can be used in a similar way. Usually when something changes a lot, there is another thing that actually never changes, which can be useful to find contradictions.

0.1.2 USAJMO 2016 P4

Problem Statement

Find, with proof, the least integer N such that if any 2016 elements are removed from the set $1, 2, ..., N$, one can still find 2016 distinct numbers among the remaining elements with sum N.

The least integer N that satisfies the statement is

$$\sum_{i=2017}^{4032} i = 6049 \cdot 1008 = 6097392.$$

Notice that if we form pairs of numbers from the set with equal sum, there will be at least 3024 such pairs. Even if at most 2016 pairs are destroyed, there will still remain at least 1008 pairs with equal sum, i.e., 2016 numbers. Each pair consists of the xth number from the left and the xth number from the right. This way, each pair has sum $6048+1$ as in $(1, 6048), (2, 6047), \ldots, (3024, 3025)$. $6049 \cdot 1008 = 6097392$.

N can't be less than $6,097,392$ because the least possible sum is

$$\sum_{i=1}^{2016} i = 2033136.$$

However, the first 2016 numbers of the set can be removed, changing the least possible sum to

$$\sum_{i=2017}^{4032} i = 6097392.$$

💡 **Insight** — The most important idea that didn't came to my mind was forming pairs with equal sum, since I had already thought about lower bound $N \geq \sum_{i=1}^{2016} i$ (not entirely correct, but almost there). It feels like the biggest problem was focusing more on doing something instead of thinking about what to do. Therefore, from now on it is really important to think better about what I am going to do before just doing.