Notes on Olympiad Algebra

Samuel de Araújo Brandão

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Contents

1 Inequalities

1.1 Replacements

1.1.1 abc=1

If $abc = 1$, replacing a, b, c with $\frac{x}{y}$, $\frac{y}{z}$, $\frac{z}{x}$, respectively, usually homogenizes the inequality.

1.1.2 Ravi substitution

If a,b,c are said to be the sides of a triangle, replace them with $y + z$, $z + x$,$x + y$.

1.2 Schur's Inequality

Let a,b,c be nonnegative real numbers and $r > 0$ be positive real number. Then

$$\sum_{\text{cyc}} a^r(a^2 + bc) \geq \sum_{\text{cyc}} a^{r+1}(b + c).$$

Equality occurs if $a = b = c$ or $a = b$ and $c = 0$ and permutations.

For example, when $r = 1$:

$$\sum_{\text{cyc}}(a^3 + abc) \geq \sum_{\text{cyc}} a^2 b.$$

An easier way to write the Schur's Inequality is the following:

$$\sum_{\text{cyc}} a^r(a - b)(a - c) \geq 0.$$

By the way, using this form of the Schur's Inequality, we can even prove it. Say WLOG, that $a \geq b \geq c$. It can be done because it is a symmetrical sum over a,b,c. Therefore, $(a - b)$ is positive, as is $(a - c)$. $\square$

Example 1.1 — Prove

$$a^2 + b^2 + c^2 + (abc)^{\frac{1}{3}}\sqrt{\frac{ab + bc + ca}{3}} \geq \frac{4}{3}(ab + bc + ca).$$

Solution.

$$\begin{aligned} a^2 + b^2 + c^2 + (abc)^{\frac{1}{3}}\sqrt{\frac{ab + bc + ca}{3}} &\geq \frac{a^2 + b^2 + c^2}{3} + (abc)^{\frac{2}{3}} \\ &\geq \frac{1}{3}\sum_{\text{sym}} a^{\frac{4}{3}} b^{\frac{2}{3}} \\ &\geq \frac{2}{3}\sum_{\text{cyc}} ab. \end{aligned}$$

Moreover,

$$\frac{2}{3}(a^2 + b^2 + c^2) + \frac{2}{3}(ab + bc + ca) \geq \frac{2}{3}(ab + bc + ca) + \frac{2}{3}(ab + bc + ca)$$

💡 **Insight** — The hardest and most important step here is dividing $a^2 + b^2 + c^2$ by 3, and later, using the Schur's Inequality. In order to notice it, you needed to see that $(abc)^{\frac{1}{3}}$ by itself is weak and doesn't help much. Therefore, using a fraction of $a^2 + b^2 + c^2$ to "complete" it, would be really helpful. However, the Schur step is harder to notice without a previous pattern recognition. Nevertheless, it is still possible to see that $a^2 = (a^{\frac{2}{3}})^3$ and $a^{\frac{4}{3}} b^{\frac{2}{3}} = (a^2 b)^{\frac{2}{3}}$

1.3 Weighted Power Mean

Let $a_1, \dots, a_n$ and $w_1, \dots, w_n$ be positive real numbers, where $w_1 + \cdots + w_n = 1$. Besides that, let r be any real number. Therefore,

$$P(r) = \begin{cases} (w_1 a_1^r + \cdots + w_n a_n^r)^{\frac{1}{r}} \iff r \neq 0 \\ a_1^{w_1} \dots a_n^{w_n} \iff r = 0. \end{cases}$$

If $s > r$, $P(s) > P(r)$.

Example 1.2: Taiwan TST 2014 — Let a,b,$c > 0$. Prove that

$$3(a+b+c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

Solution. First, divide everything by 9 and raise everything to the power of 3. Therefore, we gotta prove

$$\left(\frac{1}{3}(a+b+c)\right)^3 \geq \left(\frac{8}{9}(abc)^{\frac{1}{3}} + \frac{1}{9}\left(\frac{a^3+b^3+c^3}{3}\right)^{\frac{1}{3}}\right)^3.$$

However, by the Weighted Power Mean,

$$\left(\frac{8}{9}(abc)^{\frac{1}{3}} + \frac{1}{9}\left(\frac{a^3+b^3+c^3}{3}\right)^{\frac{1}{3}}\right)^3 \leq \frac{8}{9}(abc) + \frac{1}{9}\left(\frac{a^3+b^3+c^3}{3}\right).$$

Hence, we must prove that

$$\left(\frac{(a+b+c)}{3}\right)^3 \geq \frac{24abc + a^3+b^3+c^3}{27} \iff (a+b+c)^3 \geq 24abc + a^3 + b^3 + c^3. \quad \square$$

1.4 Problems

1.4.1 IMO Shortlist 2004 A5

Problem Statement

If a, b ,c are three positive real numbers such that $ab+bc+ca=1$, prove that

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a}\le\frac{1}{abc}.$$

First solution After homogenizing the inequality, we must prove that

$$\sum_{\text{cyc}}\sqrt[3]{\frac{(ab+bc+ca)^2}{a}+6b(ab+bc+ca)} =$$
$$\sum_{\text{cyc}}\sqrt[3]{\frac{(ab+bc+ca)(7ab+bc+ca)}{a}} \le$$
$$\frac{(ab+bc+ca)^2}{abc}.$$

By the Hölder's inequality,

$$\left(\sum_{\text{cyc}}\sqrt[3]{\frac{(ab+bc+ca)(7ab+bc+ca)}{a}}\right)^3\le\left(\sum_{\text{cyc}}ab+bc+ca\right)\left(\sum_{\text{cyc}}7ab+bc+ca\right)\left(\sum_{\text{cyc}}\frac{1}{a}\right).$$

Therefore, it suffices to prove the following.

$$\left(\sum_{\text{cyc}}ab+bc+ca\right)\left(\sum_{\text{cyc}}7ab+bc+ca\right)\left(\sum_{\text{cyc}}\frac{1}{a}\right)\le\frac{(ab+bc+ca)^6}{(abc)^3}. \iff$$
$$\frac{24(ab+bc+ca)^3}{abc}\le\frac{(ab+bc+ca)^6}{(abc)^3} \iff$$
$$24(abc)^2\le(ab+bc+ca)^3 \iff$$
$$2\sqrt[3]{3}(abc)^{\frac{2}{3}}\le ab+bc+ca.$$

By the AM-GM inequality, $ab+bc+ca\ge 3(abc)^{\frac{2}{3}}$. Hence, we must prove that $3\ge 2\sqrt[3]{3}$. Which is true, since $27>24$.

Consequently,

$$\frac{1}{abc}\ge\sum_{\text{cyc}}\sqrt[3]{\frac{1}{a}+6b}.$$

Second solution, found by DottedCalculator By the Hölder's inequality,

$$\left(\sum_{\text{cyc}}\left(\frac{1}{a}+6b\right)\right)\left(\sum_{\text{cyc}}1\right)^2\ge\left(\sum_{\text{cyc}}\sqrt[3]{\frac{1}{a}+6b}\right)^3.$$

Therefore, it suffices to prove that

$$\frac{1}{(abc)^3}\ge 54(a+b+c)+9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\iff 1\ge 54(abc)^3(a+b+c)+9(abc)^2 =$$
$$9(abc)^2(6abc(a+b+c)+1)$$

since $ab+bc+ca=1$.

By the AM-GM inequality, $\frac{1}{27} \geq (abc)^2$, and by the Titu's Lemma, $a^2b^2+b^2c^2+c^2a^2 \geq \frac{(ab+bc+ca)^2}{3} = \frac{1}{3}$. Hence, $9(abc)^2 \leq \frac{1}{3}$ and $6abc(a+b+c)+1 \leq 3$. It happens because

$$6abc(a+b+c)+1 = 3(ab+bc+ca)^2 - 3(a^2b^2+b^2c^2+c^2a^2)+1 = 4-3(a^2b^2+b^2c^2+c^2a^2).$$

Thus,

$$1 \geq 9(abc)^2(6abc(a+b+c)+1).$$

Consequently,

$$\frac{1}{abc} \geq \sum_{\text{cyc}} \sqrt[3]{\frac{1}{a}+6b}.$$

1.4.2 IMO 2000 P2

Problem Statement

Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1.$$

First solution Since $abc=1$, let $a=\frac{x}{y}$, $b=\frac{y}{z}$ and $c=\frac{z}{x}$. Therefore, it suffices to prove that

$$\prod_{\text{cyc}}\left(\frac{x}{y}-1+\frac{z}{y}\right) = \prod_{\text{cyc}}\left(\frac{x-y+z}{y}\right) \leq 1 \iff \prod_{\text{cyc}}(x-y+z) \leq xyz.$$

Let $u=x-y+z$,$v=y-z+x$ and $w=z-x+y$. Hence, it is enough to show that

$$uvw \leq \frac{(u+v)(v+w)(w+u)}{8},$$

which is true by AM-GM. Thus,

$$\prod_{\text{cyc}}\left(a-1+\frac{1}{b}\right) \leq 1.$$

Second solution Since $abc=1$, let $a=\frac{x}{y}$, $b=\frac{y}{z}$ and $c=\frac{z}{x}$. Therefore, it suffices to prove that

$$\prod_{\text{cyc}}\left(\frac{x}{y}-1+\frac{z}{y}\right) = \prod_{\text{cyc}}\left(\frac{x-y+z}{y}\right) \leq 1 \iff \prod_{\text{cyc}}(x-y+z) \leq xyz.$$

Now, notice that

$$\prod_{\text{cyc}}(x-y+z) = \sum_{\text{sym}} x^2y - 2xyz - x^3 - y^3 - z^3.$$

Hence, it is enough to show that

$$\sum_{\text{sym}} x^2y - x^3 - y^3 - z^3 \leq 3xyz,$$

which is true by the Schur's inequality. Thus,

$$\prod_{\text{cyc}}\left(a-1+\frac{1}{b}\right) \leq 1.$$

💡 **Insight** — $a = \frac{x}{y}$, $b = \frac{y}{z}$ and $c = \frac{z}{x}$ is the best possible strategy when $abc = 1$, always remember that.
Both the Schur's and the AM-GM methods didn't come to my mind, even though they were really easy. I feel that the Schur's method didn't come to my mind only because I was lazy, because it was so easy. However, the AM-GM method was harder. I have been noticing a pattern that follows like this: an equality must obligatorily be true in order for the question to be true. Hence, if it easy, basically every ineq you know should be tested, just because those inequalities seems to be easy.

1.4.3 Iran 1998 P5

Problem Statement

When $x(\geq 1)$, $y(\geq 1)$, $z(\geq 1)$ satisfy $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, prove in equality.

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{\text{cyc}} x\right)\left(\sum_{\text{cyc}} 1 - \frac{1}{x}\right) \geq \left(\sum_{\text{cyc}} \sqrt{x-1}\right)^2.$$

However, since $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sum_{\text{cyc}} \left(1 - \frac{1}{x}\right) = 1.$$

Therefore,

$$\sum_{\text{cyc}} x \geq \left(\sum_{\text{cyc}} \sqrt{x-1}\right)^2 \implies \sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$