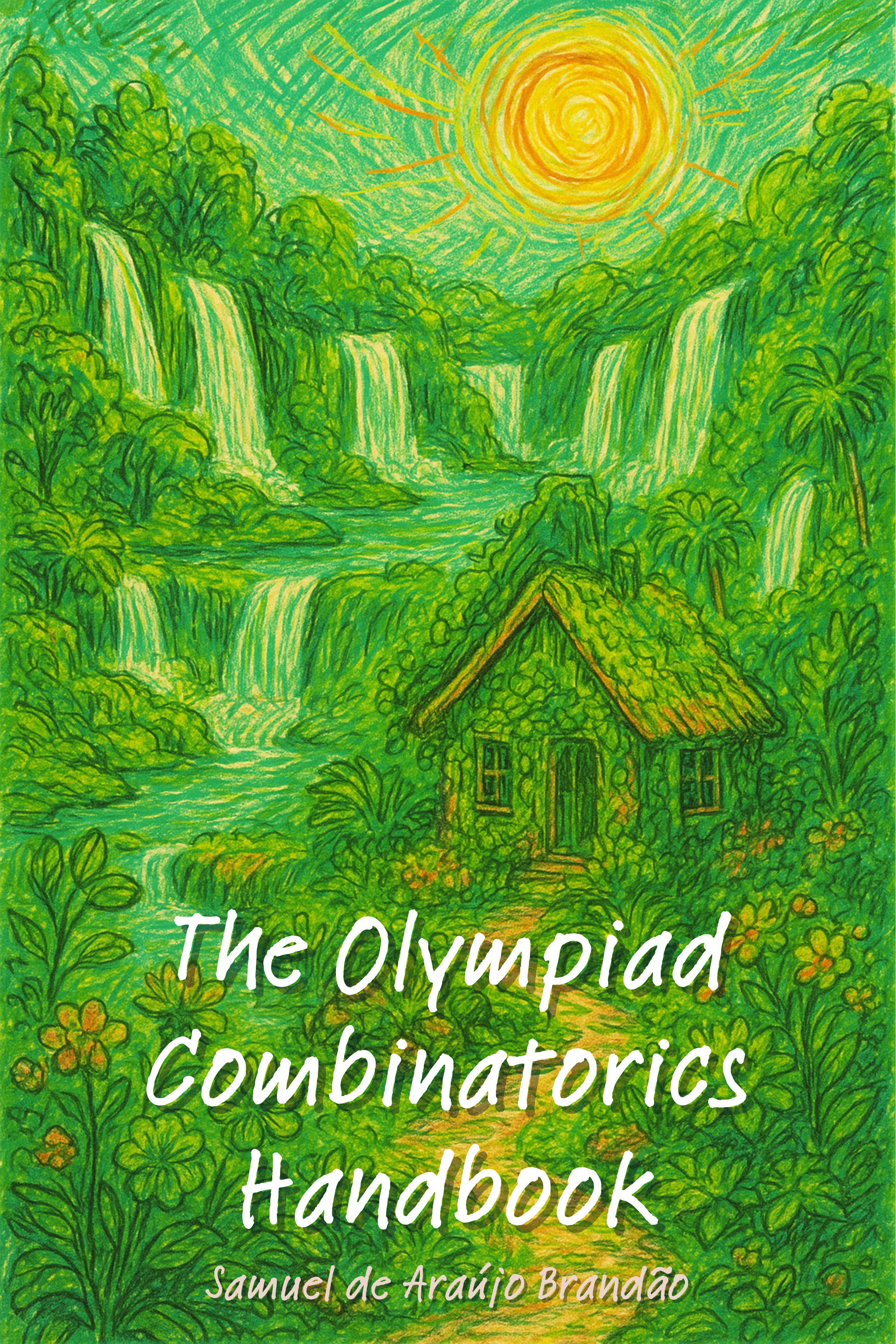

The Olympiad Combinatorics Handbook

Samuel de Araújo Brandão

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Preface

Acknowledgements

Introduction

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Part I

Fundamentals

Chapter 1

Counting Basics

1.1 Permutations

A permutation can be defined as the arrangement of objects where the order matters.

Example 1.1 How many ways can one arrange 4 math books, 3 chemistry books, 2 physics books, and 1 biology book on a bookshelf so that all the math books are together, all the chemistry books are together, and all the physics books are together?

Solution. Consider the math books first. There are 4 possibilities for the first book, 3 possibilities for the second, 2 possibilities for the third and 1 for the fourth.

By the *Multiplication Principle*, there are $4 \times 3 \times 2 \times 1 = 4! = 24$ ways to arrange the math books. The symbol "!" denotes factorial, for integers $n \geq 1$, $n! = n \cdot (n-1) \cdot \ldots \cdot 1$

Similarly, there are 3! ways to arrange the chemistry books, 2! ways to arrange the physics books and only 1! way to arrange the biology book. Besides that, there are 4! ways to arrange the four subject blocks. The order matters, since it is a permutation.

Again, by the Multiplication Principle, one can say that there are

$$4! \cdot (4! \cdot 3! \cdot 2! \cdot 1!) = 6912$$

ways to arrange the books on the bookshelf.

Note 1.2 Do not worry, later problems and solutions are going to be harder than this one, it is only an introduction.

Note 1.3 $0! = 1$. This might be a surprise for you, but makes sense, since there is exactly one way to arrange 0 elements: the empty arrangement.

Example 1.4 From the five letters a,b,c,d,e, how many different 3-letter arrangements without repetition can be formed?

This time, we can't use 5! or 3! directly, but the idea is the same. For the first letter, there are 5 possibilities, 4 possibilities for the second and 3 possibilities for the third. In total, there are $5 \cdot 4 \cdot 3 = 60$. This count is the number of k-permutations, as you can see in Definition 1.1.

Definition 1.1: Number of Permutations

The number of different groups of k objects chosen from a total of n objects with regard to order is equal to

$$P(n,k) = \frac{n!}{(n-k)!}$$

When $n, k \in \mathbb{N}$ and $0 \leq k \leq n$.

Note 1.5 You might see the *number of permutations* denoted by ${}_nP_k$ instead of $P(n,k)$ either.

1.2 Combinations

A combination can be defined as the arrangement of objects where the order does not matter.

1.2.1 Binomial Coefficient

Definition 1.2: Binomial Coefficient

The number of different groups of k objects chosen from a total of n objects without regard to order is equal to

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

when $n \in \mathbb{N}, k \in \mathbb{Z}$ and $0 \leq k \leq n$. $\binom{n}{k} = 0 \iff k < 0$ or $k > n$ and $\binom{n}{0} = \binom{n}{n} = 1$.

We derive it from $P(n,k)$ by dividing by $k!$, since order is ignored. We have to count abc and bca, for example, a repetition and divide $P(n,k)$ by $k!$, as we can see in the following.

Example 1.6 How many ways are there to choose 2 letters from the set $\{a, b, c\}$?

Solution. Let's first demonstrate it using the Definition 1.1. $P(3,2) = 6$ gives ab, ac, ba, bc, ca, cb. Each subset is counted 2! times. Divide by 2!. Hence $\binom{3}{2} = 3$. That is exactly what goes behind Definition 1.4!

Example 1.7 How many ways are there to divide 9 people into one committee of 3, one committee of 4 and one committee of 2?

Solution. For the first committee, we decide between 9 people, so there are $\binom{9}{3} = 84$ possibilities. Once it is done, there are 6 people left for the second committee, so there are $\binom{6}{4} = 15$ possibilities. After that, the remainder must go in the third committee. In total, there are $\frac{9!}{3!4!2!} = 84 \cdot 15 = 1260$ ways to divide the 9 people.

This example illustrates what is called the *multinomial coefficient*:

1.2.2 Multinomial Coefficient

Definition 1.3: Multinomial Coefficient

For nonnegative integers $k_1, k_2, \ldots, k_m$ with $k_1 + k_2 + \cdots + k_m = n$, the multinomial coefficient is defined as

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!k_2!\cdots k_m!}.$$

As you can see, the multinomial coefficient is a generalization of the binomial coefficient, where $k_m = n - k$ and $m = 2$.

It would have been easier to solve the Example 1.7 using the multinomial coefficient, as you can see below.

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{9}{3, 4, 2} = \frac{9!}{3!4!2!} = 1260.$$

Let's finish this section with a really well known example.

Example 1.8 In how many ways can the letters of the word `MISSISSIPPI` be arranged?

Let's use the multinomial coefficient to solve this one. There are

- 1 letter `M`,
- 4 letters `I`,
- 4 letters `S`,
- 2 letters `P`.

Therefore,

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{11}{1, 4, 4, 2} = \frac{11!}{1!4!4!2!} = 34650$$

1.2.3 Symmetry

Definition 1.4: Binomial Coefficient

Choosing k objects is the same as rejecting $n-k$ objects:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Example 1.9 In tossing a fair coin 10 times, how many sequences contain exactly 8 heads?

Solution. To solve this one, you could either count the times that exactly 8 heads appear or count the times that exaclty 2 tails appear, as you can see:

$$\binom{10}{8} = \binom{10}{2} = 45.$$

1.3 Pascal's Identity

Theorem 1.1: Pascal's Identity

For all $n \geq 1$ and $1 \leq k \leq n-1$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. It is well known that

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!} \quad \text{and} \quad \binom{n-1}{k} = \frac{(n-1)!}{k!(n-1-k)}.$$

Therefore,

$$\begin{aligned}\frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-1-k)}&=\frac{(n-1)!k+(n-1)!(n-k)}{k!(n-k)!}\\&=\frac{(n-1)!(k+(n-k))}{k!(n-k)!}\\&=\frac{n!}{k!(n-k)!}.\end{aligned}$$ □

Example 1.10 A club has n members. In how many ways can you choose a committee of size k if you separate the cases:

- one special member (say Alice) is included,
- or Alice is not included?

Solution.

- If Alice is included: choose the other $k-1$ from $n-1$ members, resulting in $\binom{n-1}{k-1}$.
- If Alice is not included: choose the k from $n-1$ members, resulting in $\binom{n-1}{k}$.

So total:

$$\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}.$$

That is exactly Pascal's Identity in story form.

1.4 The Binomial Theorem

Theorem 1.2: Binomial Theorem

For any nonnegative integer n and $x,y \in \mathbb{R}$ or $\mathbb{C}$

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

1.5 Sum of Row

1.6 Alternating Sum

1.7 Vandermonde's Identity

Corollary 1.3: Vandermonde's Identity

1.8 Hockey-Stick Identity

Corollary 1.4: Hockey-Stick Identity

1.9 Grid Paths and Recursion

Corollary 1.5: Grid Paths Recursion

1.10 Pigeonhole Principle

1.11 Problems