OTIS Homework Problems

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A collection of solutions for Geometry, Inequalities and others, meant to be sent to Evan Chen, in the OTIS Application Homework.

For some problems—such as the first geometry problem—I checked the official solution after solving them. If my solution was incorrect, I revised it; if it was correct, I sometimes drew inspiration from the official solution to improve my own.

Contents

1 Problems

1.1 Geometry

A.1. **(USAJMO 2012, P1)** Given a triangle ABC, let P and Q be points on segments AB and AC, respectively, such that $AP = AQ$. Let S and R be distinct points on segment BC such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

A.2. **(IMO 2019, P2)** Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of BP, CQ, PQ. Suppose that PQ is tangent to the circumcircle of $\triangle KLM$. Prove that $OP = OQ$.

A.3. **(USAMO 1993, P2)** Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

1.2 Inequalities

B.1. Suppose that $a^2 + b^2 + c^2 = 1$ for positive real numbers a, b, c. Find the minimum possible value of

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

B.2. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \le 4$. Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \ge 3.$$

B.3. Let a, b, c, d be positive reals with $(a + c)(b + d) = 1$. Prove that

$$\frac{a^3}{b + c + d} + \frac{b^3}{c + d + a} + \frac{c^3}{d + a + b} + \frac{d^3}{a + b + c} \ge \frac{1}{3}.$$

1.3 Additional

C.1. Write a computer program to find the number of ordered pairs of prime numbers (p, q) such that when

$$N = p^2 + q^3$$

is written in decimal (without leading zeros), each digit from 0 to 9 appears exactly once. For example, $(109, 1163)$ is one such pair because $109^2 + 1163^3 = 1573049628$.

C.2. **(Balkan MO 1997, P4)** Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f\big(xf(x) + f(y)\big) = f(x)^2 + y$$

holds for all real numbers x and y.

C.3. **(USAMO 2014, P1)** Let a, b, c, d be real numbers such that $b - d \ge 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

C.4. **(USA TSTST 2017, P2)** Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

2 Geometry Solutions

2.1 Problem A.1

Problem statement

Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

By the *Alternate Segment Theorem*, $\overline{AC}$ is tangent to (QRS) and $\overline{AB}$ is tangent to (PRS). Assume for the sake of contradiction that (QRS) and (PRS) are distinct. In that case, $A \in \overline{BC}$ since $\overline{BC}$ is the radical axis and $\text{Pow}_{(QRS)}(A) = \text{Pow}_{(PRS)}(A)$. This leads to a contradiction, as $A \notin \overline{BC}$. Therefore, P, Q, R and S are concyclic.

2.2 Problem A.2

Problem statement

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of BP, CQ, PQ. Suppose that PQ is tangent to the circumcircle of $\triangle KLM$. Prove that $OP = OQ$.

2.3 Problem A.3

Problem statement

Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

3 Inequalities Solutions

3.1 Problem B.1

Problem statement

Suppose that $a^2 + b^2 + c^2 = 1$ for positive real numbers a, b, c. Find the minimum possible value of

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

The minimum possible value is $3\sqrt[3]{abc}$.

The most important thing to notice is that

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = \frac{a^2b^2 + b^2c^2 + c^2a^2}{abc}$$

can be achieved by rearranging the *AM-GM inequality*, as you can see:

$$\frac{a^2b^2 + b^2c^2 + c^2a^2}{3} \geq abc\sqrt[3]{abc} \implies \frac{a^2b^2 + b^2c^2 + c^2a^2}{abc} \geq 3\sqrt[3]{abc}.$$

3.2 Problem B.2

Problem statement

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a+b+c)^2 \le 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge 3.$$

3.3 Problem B.3

Problem statement

Let a, b, c, d be positive reals with $(a+c)(b+d) = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

4 Additional Solutions

4.1 Problem C.1

Problem statement

Write a computer program to find the number of ordered pairs of prime numbers (p, q) such that when

$$N = p^2 + q^3$$

is written in decimal (without leading zeros), each digit from 0 to 9 appears exactly once. For example, $(109, 1163)$ is one such pair because $109^2 + 1163^3 = 1573049628$.

The answer is 1270 ordered pairs. I have no idea what I'm doing...

```
from bisect import bisect_left, bisect_right
from math import isqrt

def sieve(n: int) -> list[int]:
  b = bytearray(b"\x01") * (n + 1)
  b[0:2] = b"\x00\x00"
  r = isqrt(n)
  for p in range(2, r + 1):
    if b[p]:
      start = p * p
      b[start:n + 1:p] = b"\x00" * (((n - start) // p) + 1)
  return [i for i, v in enumerate(b) if v]

def is_pandigital_0_9(x: int) -> bool:
  if x < 10**9 or x >= 10**10:
    return False
  s = str(x)
  return len(s) == 10 and set(s) == set("0123456789")

def main() -> None:
  primes = sieve(100_000)
  primes_q = [q for q in primes if q <= 2154]
  p2 = [p * p for p in primes]
  total = 0
  for q in primes_q:
    q3 = q * q * q
    lo = 10**9 - q3
    hi = 10**10 - 1 - q3
    i0 = bisect_left(p2, max(0, lo))
    i1 = bisect_right(p2, hi)
    for i in range(i0, i1):
      if is_pandigital_0_9(p2[i] + q3):
        total += 1
  print(total)

if __name__ == "__main__":
  main()
```

4.2 Problem C.2

Problem statement

Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f\Big(xf(x) + f(y)\Big) = f(x)^2 + y$$

holds for all real numbers x and y.

4.3 Problem C.3

Problem statement

Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

4.4 Problem C.4

Problem statement

Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

5 References

This document was made possible thanks to the help and inspiration of the following resource:

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