1 Geometry

1. **(#2.35, IMO 2009)** Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of BP, CQ, PQ. Suppose that PQ is tangent to the circumcircle of $\triangle KLM$. Prove that $OP = OQ$.

2. **(#3.25, USAMO 1993)** Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

2 Inequalities

1. Suppose that $a^2+b^2+c^2 = 1$ for positive real numbers a, b, c. Find the minimum possible value of
$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

2. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that
$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$

3. Let a, b, c, d be positive reals with $(a + c)(b + d) = 1$. Prove that
$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

3 Additional

1. Write a computer program to find the number of ordered pairs of prime numbers (p, q) such that when
$$N = p^2 + q^3$$
is written in decimal (without leading zeros), each digit from 0 to 9 appears exactly once. For example, $(109, 1163)$ is one such pair because $109^2 + 1163^3 = 1573049628$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which
$$f(xf(x) + f(y)) = f(x)^2 + y$$
holds for all real numbers x and y.

3. Let a, b, c, d be real numbers such that $b-d \geq 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

4. Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?