

Solutions IMO Shortlist 1998

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This document contains solutions to the Solutions IMO Shortlist 1998 problems, written by me during my preparation for the International Mathematical Olympiad.

The content reflects my own understanding and problem-solving process. Some solutions may have been inspired by the work of others or required external help, in which case proper attribution is given (see [section 3](#)).

If you notice any errors or have suggestions for improvement, I would greatly appreciate hearing from you at samuelbaraujo19@gmail.com.

Contents

1	Problems	2
2	Solutions	3
2.1	Problem 2	3
3	References	4

1 Problems

1. Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

2 Solutions

2.1 Problem 2.

Problem Statement

Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

First solution By Cauchy-Schwarz,

$$\begin{aligned} \left(\sum_{\text{cyc}} \frac{x^3}{(1+y)(1+z)} \right) \left(\sum_{\text{cyc}} (1+y)(1+z) \right) &\geq (x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}})^2 \\ &= x^3 + y^3 + z^3 + 2((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}). \end{aligned}$$

Therefore, it suffices to prove that

$$\begin{aligned} x^3 + y^3 + z^3 + 2((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}) &\geq \frac{3}{4} (3 + 2(x+y+z) + xy + yz + zx) \iff \\ 2 \sum_{\text{sym}} x^3 + 4 \sum_{\text{sym}} (x^{11}y^{11}z^5)^{\frac{1}{9}} &\geq \frac{3}{2} \sum_{\text{sym}} abc + \sum_{\text{sym}} (x^5y^2z^2)^{\frac{1}{3}} + \sum_{\text{cyc}} (x^4y^4z)^{\frac{1}{3}} \end{aligned}$$

which is true, since

- $\sum_{\text{sym}} x^3 \geq \sum_{\text{sym}} (x^5y^2z^2)^{\frac{1}{3}},$
- $\sum_{\text{sym}} x^3 \geq \sum_{\text{cyc}} (x^4y^4z)^{\frac{1}{3}},$
- $4 \sum_{\text{sym}} (x^{11}y^{11}z^5)^{\frac{1}{9}} \geq \frac{3}{2} \sum_{\text{sym}} abc,$

are all true by the Muirhead's inequality

Second solution By Titu's lemma,

$$\sum_{\text{cyc}} \frac{x^4}{x(1+y)(1+z)} \geq \frac{(x^2 + y^2 + z^2)^2}{3xyz + 2(xy + yz + za) + x + y + z} \geq \frac{(x^2 + y^2 + z^2)^2}{4(x^2 + y^2 + z^2)} \geq \frac{3}{4}$$

3 References