OTIS Homework Problems

Samuel de Araújo Brandão

August 30, 2025

A collection of solutions for Geometry, Inequalities and others, meant to be sent to Evan Chen, in the OTIS Application Homework.

For some problems—such as the first geometry problem—I checked the official solution after solving them. If my solution was incorrect, I revised it; if it was correct, I sometimes drew inspiration from the official solution to improve my own.

Important: Some problems are marked with ✓and others with ✗, meaning solved and unsolved, respectively.

Contents

1 Problems

1.1 Geometry

A.1. (USAJMO 2012, P1) ✓ Given a triangle ABC, let P and Q be points on segments AB and AC, respectively, such that $AP = AQ$. Let S and R be distinct points on segment BC such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

A.2. (IMO 2009, P2) ✓ Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of BP, CQ, PQ. Suppose that PQ is tangent to the circumcircle of $\triangle KLM$. Prove that $OP = OQ$.

A.3. (USAMO 1993, P2) ✗ Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

1.2 Inequalities

B.1. ✓ Suppose that $a^2+b^2+c^2 = 1$ for positive real numbers a, b, c. Find the minimum possible value of

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

B.2. (USAMO 2011, P1) ✓ Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a+b+c)^2 \le 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge 3.$$

B.3. ✓ Let a, b, c, d be positive reals with $(a+c)(b+d) = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \ge \frac{1}{3}.$$

1.3 Additional

C.1. ✗ Write a computer program to find the number of ordered pairs of prime numbers (p, q) such that when

$$N = p^2 + q^3$$

is written in decimal (without leading zeros), each digit from 0 to 9 appears exactly once. For example, $(109, 1163)$ is one such pair because $109^2 + 1163^3 = 1573049628$.

C.2. (Balkan MO 1997, P4) ✗ Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f\big(xf(x) + f(y)\big) = f(x)^2 + y$$

holds for all real numbers x and y.

C.3. **(USAMO 2014, P1)** ✗ Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

C.4. **(USA TSTST 2017, P2)** ✗ Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

2 Geometry Solutions

2.1 Problem A.1 ✓

Problem statement

Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

This answer was written some months ago with the help of [1], [2] and [3].

By the *Alternate Segment Theorem*, $\overline{AC}$ is tangent to (QRS) and $\overline{AB}$ is tangent to (PRS). Assume for the sake of contradiction that (QRS) and (PRS) are distinct. In that case, $A \in \overline{BC}$ since $\overline{BC}$ is the radical axis and $\mathrm{Pow}_{(QRS)}(A) = \mathrm{Pow}_{(PRS)}(A)$. This leads to a contradiction, as $A \notin \overline{BC}$. Therefore, P, Q, R and S are concyclic.

2.2 Problem A.2 ✓

Problem statement

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L, M be the midpoints of BP, CQ, PQ. Suppose that the line PQ is tangent to the circumcircle of $\triangle KLM$. Prove that $OP = OQ$.

Let $\measuredangle$ denote directed angles.

By the *Power of a point*, we can say that $OP = OQ \iff \text{Pow}_{(ABC)}(P) = \text{Pow}_{(ABC)}(Q) \iff PA \cdot PC = QA \cdot QB$, which can be proved knowing that $ML \parallel AC$ and $MK \parallel AB$ by the *Midpoint Theorem*. As we can see:

$$\measuredangle APQ = \measuredangle LMP = \measuredangle LKM \quad \text{and} \quad \measuredangle PQA = \measuredangle QMK = \measuredangle MLK.$$

Therefore, $\triangle APQ \sim \triangle KLM$.

Again, by the *Midpoint Theorem*, we have:

$$\frac{AP}{AQ} = \frac{MK}{ML} = \frac{2MK}{2ML} = \frac{QB}{PC}.$$

This solutions was only possible with the help of [5].

2.3 Problem A.3 ✗

Problem statement

Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

3 Inequalities Solutions

3.1 Problem B.1 ✓

Problem statement

Suppose that $a^2 + b^2 + c^2 = 1$ for positive real numbers a, b, c. Find the minimum possible value of

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

The minimum possible value is $\sqrt{3}$.

Let $x = \frac{ab}{c}$, $y = \frac{bc}{a}$ and $z = \frac{ca}{b}$. The most important thing to notice is that

$$(x + y + z)^2 \geq 3(zx + xy + yz)$$

And since $zx + xy + yz = a^2 + b^2 + c^2$, $x + y + z \geq \sqrt{3}$.

3.2 Problem B.2 ✓

Problem statement

Let a, b, c be positive real numbers such that $a^2+b^2+c^2+(a+b+c)^2 \le 4$. Prove that

$$\frac{ab+1}{(a+b)^2}+\frac{bc+1}{(b+c)^2}+\frac{ca+1}{(c+a)^2} \ge 3.$$

This answer was inspired in [4].

It is well-known that $a^2+b^2+c^2+(a+b+c)^2 \le 4 \iff a^2+b^2+c^2+ab+bc+ca \le 2$. Therefore, we can multiply both sides by 2 and state the following:

$$\sum_{\text{cyc}} \frac{ab+1}{(a+b)^2} \ge \sum_{\text{cyc}} \frac{2ab+(a+b)^2+(b+c)^2+(c+a)^2}{(a+b)^2}.$$

The most important step here is to notice that it is equal to

$$\sum_{\text{cyc}} \frac{(a+b)^2+(b+c)(c+a)}{(a+b)^2} = 3+\sum_{\text{cyc}} \frac{(b+c)(c+a)}{(a+b)^2},$$

making it easy to use the *AM-GM inequality*. It follows that

$$3+\sum_{\text{cyc}} \frac{\frac{(b+c)(c+a)}{(a+b)^2}}{3} \ge 3+\sqrt[3]{\prod_{cyc} \frac{(b+c)(c+a)}{(a+b)^2}}.$$

Since $\prod_{cyc} \frac{(b+c)(c+a)}{(a+b)^2} = 1$, $\sum_{\text{cyc}} \frac{ab+1}{(a+b)^2} \ge 3+3=6$. $\square$

3.3 Problem B.3 ✓

Problem statement

Let a, b, c, d be positive reals with $(a+c)(b+d) = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

By *Cauchy-Schwarz inequality*,

$$\left(\sum_{\text{cyc}} \frac{a^3}{b+c+d}\right)\left(\sum_{\text{cyc}} a(b+c+d)\right) \geq \left(\sum_{\text{cyc}} a^2\right)^2 \implies \frac{1}{3} \cdot \left(\sum_{\text{cyc}} a(b+c+d)\right) \leq \left(\sum_{\text{cyc}} a^2\right)^2,$$

and by *Muirhead's inequality*

$$3(a^2+b^2+c^2+d^2) \geq \sum_{\text{sym}} ab = \sum_{\text{cyc}} a(b+c+d),$$

because $(2,0,0,0) \succ (1,1,0,0)$.

Given that we want to show that $3(a^2+b^2+c^2+d^2)^2 \geq \sum_{\text{sym}} ab$, we must simply show that $a^2+b^2+c^2+d^2 \geq 1 = (a+b)(c+d)$, which is trivial by *AM-GM inequality*. $\square$

4 Additional Solutions

4.1 Problem C.1 ✗

Problem statement

Write a computer program to find the number of ordered pairs of prime numbers (p, q) such that when

$$N = p^2 + q^3$$

is written in decimal (without leading zeros), each digit from 0 to 9 appears exactly once. For example, $(109, 1163)$ is one such pair because $109^2 + 1163^3 = 1573049628$.

4.2 Problem C.2 ✗

Problem statement

Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f\Big(xf(x) + f(y)\Big) = f(x)^2 + y$$

holds for all real numbers x and y.

4.3 Problem C.3 ✗

Problem statement

Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

4.4 Problem C.4 ✗

Problem statement

Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

5 References

This document was made possible thanks to the help and inspiration of the following resource:

[1] NOIC - Núcleo Olímpico de Iniciação Científica *Potência de Ponto*, 2023. Available in: https://noic.com.br/wp-content/uploads/2023/10/potencia.pdf

[2] OBM - Olimpíada Brasileira de Matemática. *Potência de Ponto, Eixo Radical, Centro Radical e Aplicações*, 2017. Available in: https://www.obm.org.br/content/uploads/2017/01/eixos-2.pdf

[3] Evan Chen. *JMO 2012 Solution Notes*, 2025. Available in: https://web.evanchen.cc/exams/JMO-2012-notes.pdf

[4] Evan Chen. *USAMO 2011 Solution Notes*, 2025 Available in: https://web.evanchen.cc/exams/USAMO-2011-notes.pdf

[5] Evan Chen. *IMO 2009 Solution Notes*, 2025. Available in: https://web.evanchen.cc/exams/IMO-2009-notes.pdf