Problem 1.

A line passing through the incenter I of the triangle ABC intersect its incircle at D and E and its circumcircle at F and G, in such a way that the point D lies between I and F. Prove that: $DF \cdot EG \geq r^2$.

Solution

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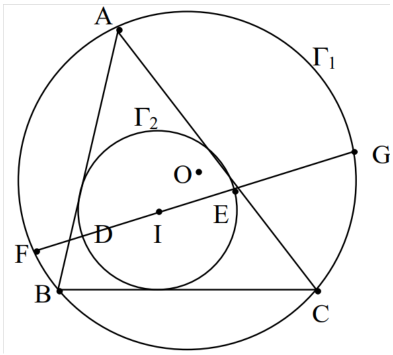

Figure 1: An illustration of the first problem. Source

$DF \cdot EG = (IF - ID)(IG - IE) = (IF - r)(IG - r)$
where r is the inradius. Hence
$IF \cdot IG - r(IF + IG) + r^2 = -Pot_{\Gamma_1} I - GFr + r^2$
$= R^2 - IO^2 - GFr + r^2$.
Since the distance IO between the incenter and circumcenter satisfies $IO = \sqrt{R^2 - 2Rr}$, it follows
$R^2 - IO^2 - GFr + r^2 = 2Rr - GFr + r^2 \Rightarrow 2Rr$
$- GFr + r^2 \geq r^2 \iff 2Rr \geq GFr$.
This inequality holds since GF is a chord of Γ_1.
$DF \cdot EG = r^2 \iff 2Rr = GFr$.

Problem 2.

Solution