Elem Geo

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This document collects my solutions to the OTIS problem sets from the **Elem Geo** unit, written during my preparation for mathematical olympiads.

The solutions reflect my understanding and problem-solving approach at the time of writing. Some arguments were informed by discussions, official notes, or published sources; when so, attribution is provided (see section 3).

If you find errors or have suggestions, please contact me at samuelbaraujo19@gmail.com .

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1 Practice Problems

ZABF24E3 **(Monge's Theorem)** Let ω_1, ω_2, ω_3 be three pairwise incongruent circles, and let X_{12}, X_{23}, X_{31} be the pairwise exsimilicenters. Show that X_{12}, X_{23}, X_{31} are collinear.

21USEMO4 **(USEMO 2021 P4)** Let ABC be a triangle with circumcircle ω, and let X be the reflection of A in B. Line CX meets ω again at D. Lines BD and AC meet at E, and lines AD and BC meet at F. Let M and N denote the midpoints of AB and AC. Can line EF share a point with the circumcircle of triangle AMN?

18JMO3 **(USAJMO 2018 P3)** Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ = FR$.

17TSTST5 **(USA TSTST 2017 P5)** Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

16TSTST2 **(USA TSTST 2016 P2)** Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.

22USEMO4 [9♣] **(USEMO 2022 P4)** Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points P, Q, R, S lie in the interiors of segments AB, BC, CD, DA, respectively, such that

$$\angle PDA = \angle PCB,\ \angle QAB = \angle QDC,\ \angle RBC = \angle RAD, \text{ and } \angle SCD = \angle SBA.$$

Let AQ intersect BS at X, and DQ intersect CS at Y. Prove that lines PR and XY are either parallel or coincide.

23TSTST1 [5♣] **(USA TSTST 2023 P1)** Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that $\angle RAS + \angle BAC = \angle BGC$.

16EGMO4 [5♣] **(EGMO 2016 P4)** Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

11IRNTST1 [5♣] **(Iran TST 2011 P1)** In acute triangle ABC angle B is greater than$C$. Let M is midpoint of BC. D and E are the feet of the altitude from C and B respectively. K and L are midpoint of ME and MD respectively. If KL intersect the line through A parallel to BC in T, prove that $TA = TM$.

06SLG2 [3♣] **(Shortlist 2006 G2)** Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points P, Q, B and C are concyclic.

17SLG3 [9♣] **(Shortlist 2017 G3)** Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

20ELMO4 [5♣] **(ELMO 2020 P4)** Let acute scalene triangle ABC have orthocenter H and altitude AD with D on side BC. Let M be the midpoint of side BC, and let D' be the reflection of D over M. Let P be a point on line $D'H$ such that lines AP and BC are parallel, and let the circumcircles of $\triangle AHP$ and $\triangle BHC$ meet again at $G \neq H$. Prove that $\angle MHG = 90^\circ$.

04IMO1 [3♣] **(IMO 2004 P1)** Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

23AMO1 [3♣] **(USAMO 2023 P1)** In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

99AMO6 [3♣] **(USAMO 1999 P6)** Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

2 Solutions

2.1 Lecture Notes

2.1.1 USEMO 2021 P4

Problem Statement

Let ABC be a triangle with circumcircle ω, and let X be the reflection of A in B. Line CX meets ω again at D. Lines BD and AC meet at E, and lines AD and BC meet at F. Let M and N denote the midpoints of AB and AC. Can line EF share a point with the circumcircle of triangle AMN?

2.1.2 USAJMO 2018 P3

Problem Statement

Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ = FR$.

2.1.3 USA TSTST 2017 P5

Problem Statement

Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

2.1.4 USA TSTST 2016 P2

Problem Statement

Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.

2.2 Mandatory

2.2.1 USEMO 2022 P4

Problem Statement

Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points P, Q, R, S lie in the interiors of segments AB, BC, CD, DA, respectively, such that

$$\angle PDA = \angle PCB,\ \angle QAB = \angle QDC,\ \angle RBC = \angle RAD,\ \text{and } \angle SCD = \angle SBA.$$

Let AQ intersect BS at X, and DQ intersect CS at Y. Prove that lines PR and XY are either parallel or coincide.

2.2.2 USA TSTST 2023 P1

Problem Statement

Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that $\angle RAS + \angle BAC = \angle BGC$.

2.2.3 EGMO 2016 P4

Problem Statement

Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

2.2.4 Iran TST 2011 P1

Problem Statement

In acute triangle ABC angle B is greater than$C$. Let M is midpoint of BC. D and E are the feet of the altitude from C and B respectively. K and L are midpoint of ME and MD respectively. If KL intersect the line through A parallel to BC in T, prove that $TA = TM$.

2.2.5 Shortlist 2006 G2

Problem Statement

Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points P, Q, B and C are concyclic.

2.3 Not mandatory

2.3.1 Shortlist 2017 G3

Problem Statement

Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

2.3.2 ELMO 2020 P4

Problem Statement

Let acute scalene triangle ABC have orthocenter H and altitude AD with D on side BC. Let M be the midpoint of side BC, and let D' be the reflection of D over M. Let P be a point on line $D'H$ such that lines AP and BC are parallel, and let the circumcircles of $\triangle AHP$ and $\triangle BHC$ meet again at $G \neq H$. Prove that $\angle MHG = 90°$.

2.3.3 IMO 2004 P1

Problem Statement

Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

2.3.4 USAMO 2023 P1

Problem Statement

In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

2.3.5 IMO 1997 P2

Problem Statement

It is known that $\angle BAC$ is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that $AU = TB + TC$.

3 References