

# Ineq Basic

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This document collects my solutions to the OTIS problem sets from the **Ineq Basic** unit, written during my preparation for mathematical olympiads.

The solutions reflect my understanding and problem-solving approach at the time of writing. Some arguments were informed by discussions, official notes, or published sources; when so, attribution is provided (see [section 3](#)).

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# 1 Practice Problems

**11AM01 (USAMO 2011 P1)** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$ . Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$

**01IM02 (IMO 2001 P2)** Prove that for all positive real numbers  $a, b, c$ ,

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

**05IM03 (IMO 2005 P3)** Let  $x, y, z$  be three positive reals such that  $xyz \geq 1$ . Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

**12IM02 (IMO 2012 P2)** Let  $n \geq 3$  be an integer, and let  $a_2, a_3, \dots, a_n$  be positive real numbers such that  $a_2 a_3 \cdots a_n = 1$ . Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

**03ELM04 (ELMO 2003 P4)** Let  $x, y, z \geq 1$  be real numbers such that

$$\frac{1}{x^2 - 1} + \frac{1}{y^2 - 1} + \frac{1}{z^2 - 1} = 1.$$

Prove that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \leq 1.$$

**04IM04 (IMO 2004 P4)** Let  $n \geq 3$  be an integer. Let  $t_1, t_2, \dots, t_n$  be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n) \left( \frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \right).$$

Show that  $t_i, t_j, t_k$  are side lengths of a triangle for all  $i, j, k$  with  $1 \leq i < j < k \leq n$ .

**11MOPR42 (MOP 2011 R4.2)** For positive real numbers  $a, b, c$  with  $a + b + c = 3$  prove that

$$\sum_{\text{cyc}} \sqrt{\frac{a^3 + b^3}{a + b}} + 9\sqrt[3]{abc} \leq 12.$$

**04SLA5 (Shortlist 2004 A5)** If  $a, b, c$  are three positive real numbers such that  $ab + bc + ca = 1$ , prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

**98IRN (Iran 1998 P5)** When  $x(\geq 1), y(\geq 1), z(\geq 1)$  satisfy  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ , prove in equality.

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

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**95IM02 (IMO 1995 P2)** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

**12JM03 (USAJMO 2012 P3)** Let  $a, b, c$  be positive real numbers. Prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geq \frac{2}{3}(a^2 + b^2 + c^2)$$

**98SLA3 (Shortlist 1998 A3)** Let  $x, y$  and  $z$  be positive real numbers such that  $xyz = 1$ . Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

**00IM02 (IMO 2000 P2)** Let  $a, b, c$  be positive real numbers so that  $abc = 1$ . Prove that

$$\left(a - 1 + \frac{1}{b}\right) \left(b - 1 + \frac{1}{c}\right) \left(c - 1 + \frac{1}{a}\right) \leq 1.$$

**18RUS112 (All Russian Olympiad 2018 Grade 11 P2)** Let  $n \geq 2$  and  $x_1, x_2, \dots, x_n$  positive real numbers. Prove that

$$\frac{1 + x_1^2}{1 + x_1 x_2} + \frac{1 + x_2^2}{1 + x_2 x_3} + \dots + \frac{1 + x_n^2}{1 + x_n x_1} \geq n.$$

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## 2 Solutions

### 2.1 Lecture Problems

#### 2.1.1 USAMO 2011 P1

##### Problem Statement

Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$ . Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$

The key is to correctly homogenize the inequality as shown below

$$\sum_{\text{cyc}} \frac{2ab+2}{(a+b)^2} \geq \sum_{\text{cyc}} \frac{2ab+ab+bc+ca+a^2+b^2+c^2}{(a+b)^2}$$

and observe that

$$\sum_{\text{cyc}} 3ab + bc + ca + a^2 + b^2 + c^2 = \sum_{\text{cyc}} (a+b)^2 + (c+a)(c+b).$$

Hence  $\sum_{\text{cyc}} \frac{(c+a)(c+b)}{(a+b)^2} \geq 6$ , by the AM-GM inequality.

### 2.1.2 IMO 2001 P2

#### Problem Statement

Prove that for all positive real numbers  $a, b, c$ ,

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

By Hölder,

$$\left( \sum_{\text{cyc}} \frac{a}{\sqrt{a^2 + 8bc}} \right) \left( \sum_{\text{cyc}} a\sqrt{a^2 + 8bc} \right) \geq (a + b + c)^2.$$

Hence, is it enough to prove that

$$(a + b + c)^2 \geq \sum_{\text{cyc}} a\sqrt{a^2 + 8bc} \iff 2(a^2b^2 + b^2c^2 + c^2a^2) \geq 2(a^2bc + ab^2c + abc^2),$$

which is clearly true by the Muirhead's inequality.

### 2.1.3 IMO 2005 P3

#### Problem Statement

Let  $x, y, z$  be three positive reals such that  $xyz \geq 1$ . Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

$$\sum_{\text{cyc}} \frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \sum_{\text{cyc}} \frac{x^5 - x^3yz}{x^5 + xyz(y^2 + z^2)}$$

By the Cauchy-Schwarz inequality,

$$\left( \sum_{\text{cyc}} \frac{x^6}{x^6 + x^2yz(y^2 + z^2)} \right) \left( \sum_{\text{cyc}} x^6 + x^2yz(y^2 + z^2) \right) \geq (x^3 + y^3 + z^3)^2,$$

$$\left( \sum_{\text{cyc}} \frac{x^4yz}{x^6 + x^2yz(y^2 + z^2)} \right) \left( \sum_{\text{cyc}} x^6 + x^2yz(y^2 + z^2) \right) \geq (x^2\sqrt{yz} + y^2\sqrt{xz} + z^2\sqrt{xy})^2.$$

Hence, it is enough to prove that  $(x^3 + y^3 + z^3)^2 \geq (x^2\sqrt{yz} + y^2\sqrt{xz} + z^2\sqrt{xy})^2$ , which is clearly true by the Muirhead's inequality.

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## 2.2 Mandatory

### 2.2.1 IMO 2012 P2

#### Problem Statement

Let  $n \geq 3$  be an integer, and let  $a_2, a_3, \dots, a_n$  be positive real numbers such that  $a_2 a_3 \cdots a_n = 1$ . Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

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### 2.2.2 ELMO 2003 P4

#### Problem Statement

Let  $x, y, z \geq 1$  be real numbers such that

$$\frac{1}{x^2 - 1} + \frac{1}{y^2 - 1} + \frac{1}{z^2 - 1} = 1.$$

Prove that

$$\frac{1}{x + 1} + \frac{1}{y + 1} + \frac{1}{z + 1} \leq 1.$$



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### 2.2.3 IMO 2004 P4

#### Problem Statement

Let  $n \geq 3$  be an integer. Let  $t_1, t_2, \dots, t_n$  be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \dots + t_n) \left( \frac{1}{t_1} + \frac{1}{t_2} + \dots + \frac{1}{t_n} \right).$$

Show that  $t_i, t_j, t_k$  are side lengths of a triangle for all  $i, j, k$  with  $1 \leq i < j < k \leq n$ .

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### 2.2.4 MOP 2011 R4.2

#### Problem Statement

For positive real numbers  $a, b, c$  with  $a + b + c = 3$  prove that

$$\sum_{\text{cyc}} \sqrt{\frac{a^3 + b^3}{a + b}} + 9\sqrt[3]{abc} \leq 12.$$

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## 2.3 Not Mandatory

### 2.3.1 Shortlist 2004 A5

#### Problem Statement

If  $a, b, c$  are three positive real numbers such that  $ab + bc + ca = 1$ , prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

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### 2.3.2 Iran 1998 P5

#### Problem Statement

When  $x(\geq 1)$ ,  $y(\geq 1)$ ,  $z(\geq 1)$  satisfy  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ , prove in equality.

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

### 2.3.3 IMO 1995 P2

#### Problem Statement

Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

**First solution** By Cauchy-Schwarz,

$$\left( \sum_{\text{cyc}} \frac{1}{a^3(b+c)} \right) \left( \sum_{\text{cyc}} a(b+c) \right) \geq \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 = (ab+bc+ca)^2.$$

Hence, it is enough to prove that

$$(ab+bc+ca)^2 \geq \frac{3}{2} \left( \sum_{\text{cyc}} a(b+c) \right) \iff (ab+bc+ca)^2 \geq 3(ab+bc+ca)(abc)^{\frac{2}{3}},$$

which is true by AM-GM.

**Second solution** By Cauchy-Schwarz,

$$\left( \sum_{\text{cyc}} \frac{1}{a^3(b+c)} \right) \left( \sum_{\text{cyc}} a(b+c) \right) \geq \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 = (ab+bc+ca)^2,$$

and by Titu's lemma,

$$\sum_{\text{cyc}} \frac{(\frac{1}{a})^2}{a(b+c)} \geq \frac{ab+bc+ca}{2} \geq \frac{3(abc)^{\frac{2}{3}}}{2} = \frac{3}{2}$$

### 2.3.4 USAJMO 2012 P3

#### Problem Statement

Let  $a, b, c$  be positive real numbers. Prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geq \frac{2}{3}(a^2 + b^2 + c^2)$$

By Titu's lemma we have

$$\sum_{\text{cyc}} \frac{a^4}{a(5a + b)} \geq \frac{a^2 + b^2 + c^2}{6} = x \quad \text{and} \quad 3 \sum_{\text{cyc}} \frac{b^4}{b(5a + b)} \geq \frac{a^2 + b^2 + c^2}{2} = y$$

since  $\sum_{\text{cyc}} 5ab + b^2 \leq \sum_{\text{cyc}} 5a^2 + ab \leq 6(a^2 + b^2 + c^2)$ . Hence  $x + y = \frac{2}{3}(a^2 + b^2 + c^2)$ .

### 2.3.5 Shortlist 1998 A3

#### Problem Statement

Let  $x, y$  and  $z$  be positive real numbers such that  $xyz = 1$ . Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

**First solution** By Cauchy-Schwarz,

$$\begin{aligned} \left( \sum_{\text{cyc}} \frac{x^3}{(1+y)(1+z)} \right) \left( \sum_{\text{cyc}} (1+y)(1+z) \right) &\geq (x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}})^2 \\ &= x^3 + y^3 + z^3 + 2((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}). \end{aligned}$$

Therefore, it suffices to prove that

$$\begin{aligned} x^3 + y^3 + z^3 + 2((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}) &\geq \frac{3}{4} (3 + 2(x+y+z) + xy + yz + zx) \iff \\ 2 \sum_{\text{sym}} x^3 + 4 \sum_{\text{sym}} (xy)^{\frac{3}{2}} &\geq \frac{3}{2} \sum_{\text{sym}} xyz + 3 \sum_{\text{sym}} (x^5 y^2 z^2)^{\frac{1}{3}} + \frac{3}{2} \sum_{\text{cyc}} (x^4 y^4 z)^{\frac{1}{3}} \end{aligned}$$

**Second solution** By Titu's lemma,

$$\sum_{\text{cyc}} \frac{x^4}{x(1+y)(1+z)} \geq \frac{(x^2 + y^2 + z^2)^2}{3xyz + 2(xy + yz + za) + x + y + z} \geq \frac{(x^2 + y^2 + z^2)^2}{4(x^2 + y^2 + z^2)} \geq \frac{3}{4}$$

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### 3 References