

# **Elem Geo**

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This document collects my solutions to the OTIS problem sets from the **Elem Geo** unit, written during my preparation for mathematical olympiads.

The solutions reflect my understanding and problem-solving approach at the time of writing. Some arguments were informed by discussions, official notes, or published sources; when so, attribution is provided (see [section 3](#)).

If you find errors or have suggestions, please contact me at [samuelbaraujo19@gmail.com](mailto:samuelbaraujo19@gmail.com).

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# 1 Practice Problems

21USEM04 (**USEMO 2021 P4**) Let  $ABC$  be a triangle with circumcircle  $\omega$ , and let  $X$  be the reflection of  $A$  in  $B$ . Line  $CX$  meets  $\omega$  again at  $D$ . Lines  $BD$  and  $AC$  meet at  $E$ , and lines  $AD$  and  $BC$  meet at  $F$ . Let  $M$  and  $N$  denote the midpoints of  $AB$  and  $AC$ . Can line  $EF$  share a point with the circumcircle of triangle  $AMN$ ?

18JM03 (**USAJMO 2018 P3**) Let  $ABCD$  be a quadrilateral inscribed in circle  $\omega$  with  $\overline{AC} \perp \overline{BD}$ . Let  $E$  and  $F$  be the reflections of  $D$  over lines  $BA$  and  $BC$ , respectively, and let  $P$  be the intersection of lines  $BD$  and  $EF$ . Suppose that the circumcircle of  $\triangle EPD$  meets  $\omega$  at  $D$  and  $Q$ , and the circumcircle of  $\triangle FPD$  meets  $\omega$  at  $D$  and  $R$ . Show that  $EQ = FR$ .

17TSTST5 (**USA TSTST 2017 P5**) Let  $ABC$  be a triangle with incenter  $I$ . Let  $D$  be a point on side  $BC$  and let  $\omega_B$  and  $\omega_C$  be the incircles of  $\triangle ABD$  and  $\triangle ACD$ , respectively. Suppose that  $\omega_B$  and  $\omega_C$  are tangent to segment  $BC$  at points  $E$  and  $F$ , respectively. Let  $P$  be the intersection of segment  $AD$  with the line joining the centers of  $\omega_B$  and  $\omega_C$ . Let  $X$  be the intersection point of lines  $BI$  and  $CP$  and let  $Y$  be the intersection point of lines  $CI$  and  $BP$ . Prove that lines  $EX$  and  $FY$  meet on the incircle of  $\triangle ABC$ .

16TSTST2 (**USA TSTST 2016 P2**) Let  $ABC$  be a scalene triangle with orthocenter  $H$  and circumcenter  $O$ . Denote by  $M, N$  the midpoints of  $\overline{AH}, \overline{BC}$ . Suppose the circle  $\gamma$  with diameter  $\overline{AH}$  meets the circumcircle of  $ABC$  at  $G \neq A$ , and meets line  $AN$  at a point  $Q \neq A$ . The tangent to  $\gamma$  at  $G$  meets line  $OM$  at  $P$ . Show that the circumcircles of  $\triangle GNQ$  and  $\triangle MBC$  intersect at a point  $T$  on  $\overline{PN}$ .

22USEM04 (**USEMO 2022 P4**) Let  $ABCD$  be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points  $P, Q, R, S$  lie in the interiors of segments  $AB, BC, CD, DA$ , respectively, such that

$$\angle PDA = \angle PCB, \angle QAB = \angle QDC, \angle RBC = \angle RAD, \text{ and } \angle SCD = \angle SBA.$$

Let  $AQ$  intersect  $BS$  at  $X$ , and  $DQ$  intersect  $CS$  at  $Y$ . Prove that lines  $PR$  and  $XY$  are either parallel or coincide.

23TSTST1 (**USA TSTST 2023 P1**) Let  $ABC$  be a triangle with centroid  $G$ . Points  $R$  and  $S$  are chosen on rays  $GB$  and  $GC$ , respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that  $\angle RAS + \angle BAC = \angle BGC$ .

16EGM04 (**EGMO 2016 P4**) Two circles  $\omega_1$  and  $\omega_2$ , of equal radius intersect at different points  $X_1$  and  $X_2$ . Consider a circle  $\omega$  externally tangent to  $\omega_1$  at  $T_1$  and internally tangent to  $\omega_2$  at point  $T_2$ . Prove that lines  $X_1T_1$  and  $X_2T_2$  intersect at a point lying on  $\omega$ .

11IRNTST1 (**Iran TST 2011 P1**) In acute triangle  $ABC$  angle  $B$  is greater than  $C$ . Let  $M$  is midpoint of  $BC$ .  $D$  and  $E$  are the feet of the altitude from  $C$  and  $B$  respectively.  $K$  and  $L$  are midpoint of  $ME$  and  $MD$  respectively. If  $KL$  intersect the line through  $A$  parallel to  $BC$  in  $T$ , prove that  $TA = TM$ .

06SLG2 (**Shortlist 2006 G2**) Let  $ABCD$  be a trapezoid with parallel sides  $AB > CD$ . Points  $K$  and  $L$  lie on the line segments  $AB$  and  $CD$ , respectively, so that  $AK/KB = DL/LC$ . Suppose that there are points  $P$  and  $Q$  on the line segment  $KL$  satisfying

$$\angle APB = \angle BCD \quad \text{and} \quad \angle CQD = \angle ABC.$$

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Prove that the points  $P$ ,  $Q$ ,  $B$  and  $C$  are concyclic.

- 17SLG3 (**Shortlist 2017 G3**) Let  $O$  be the circumcenter of an acute triangle  $ABC$ . Line  $OA$  intersects the altitudes of  $ABC$  through  $B$  and  $C$  at  $P$  and  $Q$ , respectively. The altitudes meet at  $H$ . Prove that the circumcenter of triangle  $PQH$  lies on a median of triangle  $ABC$ .

- 20ELM04 (**ELMO 2020 P4**) Let acute scalene triangle  $ABC$  have orthocenter  $H$  and altitude  $AD$  with  $D$  on side  $BC$ . Let  $M$  be the midpoint of side  $BC$ , and let  $D'$  be the reflection of  $D$  over  $M$ . Let  $P$  be a point on line  $D'H$  such that lines  $AP$  and  $BC$  are parallel, and let the circumcircles of  $\triangle AHP$  and  $\triangle BHC$  meet again at  $G \neq H$ . Prove that  $\angle MHG = 90^\circ$ .

- 04IM01 (**IMO 2004 P1**) Let  $ABC$  be an acute-angled triangle with  $AB \neq AC$ . The circle with diameter  $BC$  intersects the sides  $AB$  and  $AC$  at  $M$  and  $N$  respectively. Denote by  $O$  the midpoint of the side  $BC$ . The bisectors of the angles  $\angle BAC$  and  $\angle MON$  intersect at  $R$ . Prove that the circumcircles of the triangles  $BMR$  and  $CNR$  have a common point lying on the side  $BC$ .

- 23AM01 (**USAMO 2023 P1**) In an acute triangle  $ABC$ , let  $M$  be the midpoint of  $\overline{BC}$ . Let  $P$  be the foot of the perpendicular from  $C$  to  $AM$ . Suppose that the circumcircle of triangle  $ABP$  intersects line  $BC$  at two distinct points  $B$  and  $Q$ . Let  $N$  be the midpoint of  $\overline{AQ}$ . Prove that  $NB = NC$ .

- 97IM02 (**IMO 1997 P2**) It is known that  $\angle BAC$  is the smallest angle in the triangle  $ABC$ . The points  $B$  and  $C$  divide the circumcircle of the triangle into two arcs. Let  $U$  be an interior point of the arc between  $B$  and  $C$  which does not contain  $A$ . The perpendicular bisectors of  $AB$  and  $AC$  meet the line  $AU$  at  $V$  and  $W$ , respectively. The lines  $BV$  and  $CW$  meet at  $T$ .

Show that  $AU = TB + TC$ .

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## 2 Solutions

### 2.1 Lecture Notes

#### 2.1.1 USEMO 2021 P4

##### Problem Statement

Let  $ABC$  be a triangle with circumcircle  $\omega$ , and let  $X$  be the reflection of  $A$  in  $B$ . Line  $CX$  meets  $\omega$  again at  $D$ . Lines  $BD$  and  $AC$  meet at  $E$ , and lines  $AD$  and  $BC$  meet at  $F$ . Let  $M$  and  $N$  denote the midpoints of  $AB$  and  $AC$ . Can line  $EF$  share a point with the circumcircle of triangle  $AMN$ ?

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### 2.1.2 USAJMO 2018 P3

#### Problem Statement

Let  $ABCD$  be a quadrilateral inscribed in circle  $\omega$  with  $\overline{AC} \perp \overline{BD}$ . Let  $E$  and  $F$  be the reflections of  $D$  over lines  $BA$  and  $BC$ , respectively, and let  $P$  be the intersection of lines  $BD$  and  $EF$ . Suppose that the circumcircle of  $\triangle EPD$  meets  $\omega$  at  $D$  and  $Q$ , and the circumcircle of  $\triangle FPD$  meets  $\omega$  at  $D$  and  $R$ . Show that  $EQ = FR$ .

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### 2.1.3 USA TSTST 2017 P5

#### Problem Statement

Let  $ABC$  be a triangle with incenter  $I$ . Let  $D$  be a point on side  $BC$  and let  $\omega_B$  and  $\omega_C$  be the incircles of  $\triangle ABD$  and  $\triangle ACD$ , respectively. Suppose that  $\omega_B$  and  $\omega_C$  are tangent to segment  $BC$  at points  $E$  and  $F$ , respectively. Let  $P$  be the intersection of segment  $AD$  with the line joining the centers of  $\omega_B$  and  $\omega_C$ . Let  $X$  be the intersection point of lines  $BI$  and  $CP$  and let  $Y$  be the intersection point of lines  $CI$  and  $BP$ . Prove that lines  $EX$  and  $FY$  meet on the incircle of  $\triangle ABC$ .

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#### 2.1.4 USA TSTST 2016 P2

##### Problem Statement

Let  $ABC$  be a scalene triangle with orthocenter  $H$  and circumcenter  $O$ . Denote by  $M, N$  the midpoints of  $\overline{AH}, \overline{BC}$ . Suppose the circle  $\gamma$  with diameter  $\overline{AH}$  meets the circumcircle of  $ABC$  at  $G \neq A$ , and meets line  $AN$  at a point  $Q \neq A$ . The tangent to  $\gamma$  at  $G$  meets line  $OM$  at  $P$ . Show that the circumcircles of  $\triangle GNQ$  and  $\triangle MBC$  intersect at a point  $T$  on  $\overline{PN}$ .

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## 2.2 Mandatory

### 2.2.1 USEMO 2022 P4

#### Problem Statement

Let  $ABCD$  be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points  $P, Q, R, S$  lie in the interiors of segments  $AB, BC, CD, DA$ , respectively, such that

$$\angle PDA = \angle PCB, \angle QAB = \angle QDC, \angle RBC = \angle RAD, \text{ and } \angle SCD = \angle SBA.$$

Let  $AQ$  intersect  $BS$  at  $X$ , and  $DQ$  intersect  $CS$  at  $Y$ . Prove that lines  $PR$  and  $XY$  are either parallel or coincide.

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## 2.2.2 USA TSTST 2023 P1

### Problem Statement

Let  $ABC$  be a triangle with centroid  $G$ . Points  $R$  and  $S$  are chosen on rays  $GB$  and  $GC$ , respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that  $\angle RAS + \angle BAC = \angle BGC$ .

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### 2.2.3 EGMO 2016 P4

#### Problem Statement

Two circles  $\omega_1$  and  $\omega_2$ , of equal radius intersect at different points  $X_1$  and  $X_2$ . Consider a circle  $\omega$  externally tangent to  $\omega_1$  at  $T_1$  and internally tangent to  $\omega_2$  at point  $T_2$ . Prove that lines  $X_1T_1$  and  $X_2T_2$  intersect at a point lying on  $\omega$ .

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#### 2.2.4 Iran TST 2011 P1

##### Problem Statement

In acute triangle  $ABC$  angle  $B$  is greater than  $C$ . Let  $M$  is midpoint of  $BC$ .  $D$  and  $E$  are the feet of the altitude from  $C$  and  $B$  respectively.  $K$  and  $L$  are midpoint of  $ME$  and  $MD$  respectively. If  $KL$  intersect the line through  $A$  parallel to  $BC$  in  $T$ , prove that  $TA = TM$ .

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### 2.2.5 Shortlist 2006 G2

#### Problem Statement

Let  $ABCD$  be a trapezoid with parallel sides  $AB > CD$ . Points  $K$  and  $L$  lie on the line segments  $AB$  and  $CD$ , respectively, so that  $AK/KB = DL/LC$ . Suppose that there are points  $P$  and  $Q$  on the line segment  $KL$  satisfying

$$\angle APB = \angle BCD \quad \text{and} \quad \angle CQD = \angle ABC.$$

Prove that the points  $P$ ,  $Q$ ,  $B$  and  $C$  are concyclic.

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## 2.3 Not mandatory

### 2.3.1 Shortlist 2017 G3

#### Problem Statement

Let  $O$  be the circumcenter of an acute triangle  $ABC$ . Line  $OA$  intersects the altitudes of  $ABC$  through  $B$  and  $C$  at  $P$  and  $Q$ , respectively. The altitudes meet at  $H$ . Prove that the circumcenter of triangle  $PQH$  lies on a median of triangle  $ABC$ .

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### 2.3.2 ELMO 2020 P4

#### Problem Statement

Let acute scalene triangle  $ABC$  have orthocenter  $H$  and altitude  $AD$  with  $D$  on side  $BC$ . Let  $M$  be the midpoint of side  $BC$ , and let  $D'$  be the reflection of  $D$  over  $M$ . Let  $P$  be a point on line  $D'H$  such that lines  $AP$  and  $BC$  are parallel, and let the circumcircles of  $\triangle AHP$  and  $\triangle BHC$  meet again at  $G \neq H$ . Prove that  $\angle MHG = 90^\circ$ .

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### 2.3.3 IMO 2004 P1

#### Problem Statement

Let  $ABC$  be an acute-angled triangle with  $AB \neq AC$ . The circle with diameter  $BC$  intersects the sides  $AB$  and  $AC$  at  $M$  and  $N$  respectively. Denote by  $O$  the midpoint of the side  $BC$ . The bisectors of the angles  $\angle BAC$  and  $\angle MON$  intersect at  $R$ . Prove that the circumcircles of the triangles  $BMR$  and  $CNR$  have a common point lying on the side  $BC$ .

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### 2.3.4 USAMO 2023 P1

#### Problem Statement

In an acute triangle  $ABC$ , let  $M$  be the midpoint of  $\overline{BC}$ . Let  $P$  be the foot of the perpendicular from  $C$  to  $AM$ . Suppose that the circumcircle of triangle  $ABP$  intersects line  $BC$  at two distinct points  $B$  and  $Q$ . Let  $N$  be the midpoint of  $\overline{AQ}$ . Prove that  $NB = NC$ .

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### 2.3.5 IMO 1997 P2

#### Problem Statement

It is known that  $\angle BAC$  is the smallest angle in the triangle  $ABC$ . The points  $B$  and  $C$  divide the circumcircle of the triangle into two arcs. Let  $U$  be an interior point of the arc between  $B$  and  $C$  which does not contain  $A$ . The perpendicular bisectors of  $AB$  and  $AC$  meet the line  $AU$  at  $V$  and  $W$ , respectively. The lines  $BV$  and  $CW$  meet at  $T$ . Show that  $AU = TB + TC$ .

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### **3 References**