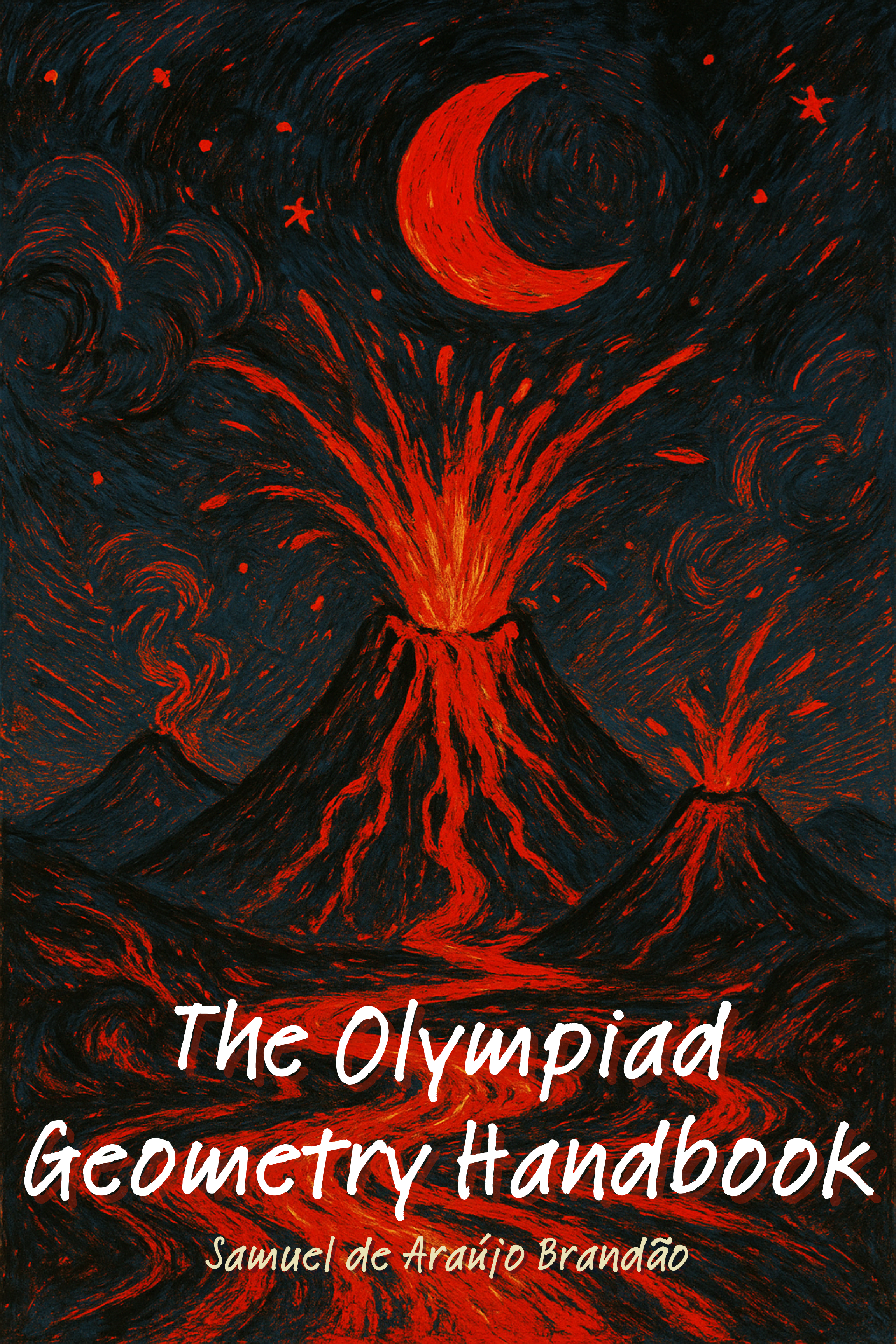

The Olympiad Geometry Handbook

Samuel de Araújo Brandão

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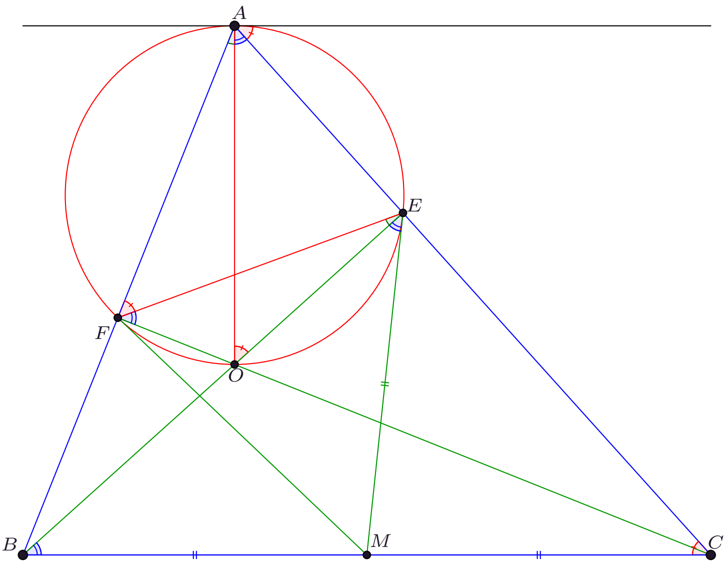
Contents

Part I

Fundamentals

Lemma 0.1: Three Tangents Lemma

Let ABC be an acute triangle. Let BE and CF be altitudes of $\triangle ABC$, and denote by M the midpoint of BC. Prove that ME, MF, and the line through A parallel to BC are all tangents to (AEF).



USAJMO 2011 P5

Problem Statement

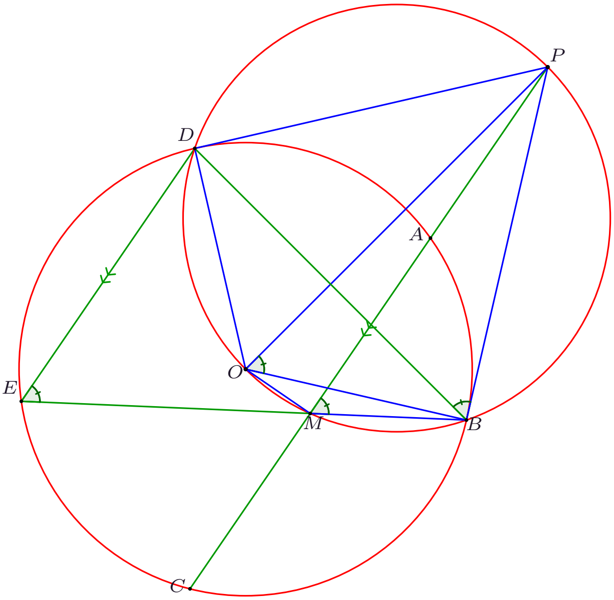
Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that

(i) lines PB and PD are tangent to ω,

(ii) P, A, C are collinear, and

(iii) $DE \parallel AC$.

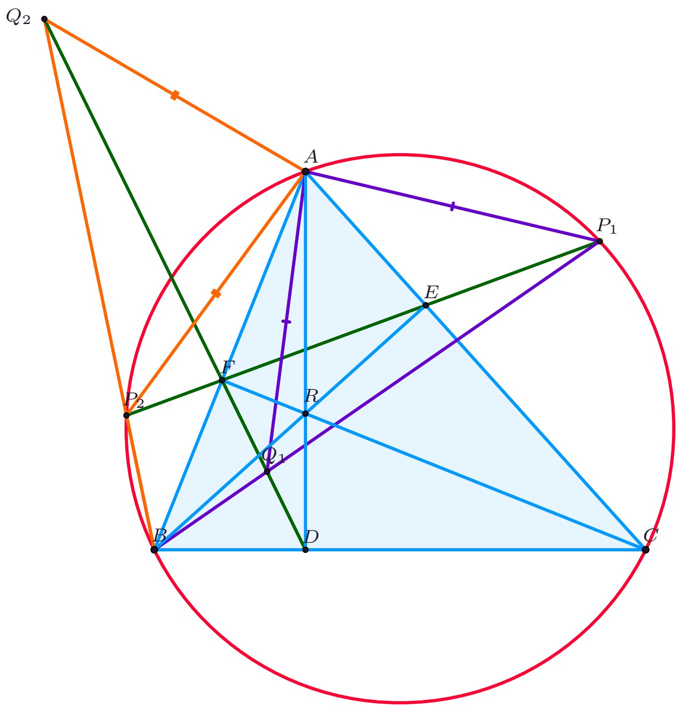
Prove that BE bisects AC.



Shortlist 2010 G1

Problem Statement

Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.



Let $\measuredangle$ denote directed angles mod $180°$. The line EF meets the circumcircle at two points. Directed angles allow us to treat both at once, so we fix one of them.

Our goal is to show $\measuredangle PQA = \measuredangle APQ$, i.e., that $\triangle APQ$ is isosceles. We will first prove that A, F, P, Q are concyclic.

Since $\triangle DEF$ is an orthic triangle, $\measuredangle CFA = \measuredangle ADC = 90°$. Therefore, $FACD$ is cyclic, culminating in $\measuredangle ACD = \measuredangle AFD = \measuredangle AFQ$. However, $APCB$ is cyclic, so $\measuredangle APB = \measuredangle ACB = \measuredangle ACD$. Thus, $\measuredangle APQ = \measuredangle AFQ$. Which means that $AFPQ$ is a cyclic quadrilateral.

$FECB$ is a cyclic quadrilateral either, by the same $FACD$'s reason. Therefore, putting together everything we've seen so far:

$$\measuredangle PQA = \measuredangle PFA = \measuredangle EFB = \measuredangle ECB = \measuredangle ACB = \measuredangle APB.$$

Hence, $AP = AQ$.