

Ineq Basic

Samuel de Araújo Brandão

October 29, 2025

This document collects my solutions to the OTIS problem sets from the **Ineq Basic** unit, written during my preparation for mathematical olympiads.

The solutions reflect my understanding and problem-solving approach at the time of writing. Some arguments were informed by discussions, official notes, or published sources; when so, attribution is provided (see [section 3](#)).

If you find errors or have suggestions, please contact me at samuelbaraujo19@gmail.com.

Contents

1 Practice Problems	2
2 Solutions	4
2.1 Lecture Problems	4
2.1.1 USAMO 2011 P1	4
2.1.2 IMO 2001 P2	5
2.1.3 IMO 2005 P3	6
2.2 Mandatory	7
2.2.1 IMO 2012 P2	7
2.2.2 ELMO 2003 P4	8
2.2.3 IMO 2004 P4	9
2.2.4 MOP 2011 R4.2	10
2.3 Not Mandatory	11
2.3.1 Shortlist 2004 A5	11
2.3.2 Iran 1998 P5	12
2.3.3 IMO 1995 P2	13
2.3.4 USAJMO 2012 P3	14
2.3.5 Shortlist 1998 A3	15
3 References	16

1 Practice Problems

11AM01 (**USAMO 2011 P1**) Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$

01IM02 (**IMO 2001 P2**) Prove that for all positive real numbers a, b, c ,

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

05IM03 (**IMO 2005 P3**) Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

12IM02 (**IMO 2012 P2**) Let $n \geq 3$ be an integer, and let a_2, a_3, \dots, a_n be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

03ELM04 (**ELMO 2003 P4**) Let $x, y, z \geq 1$ be real numbers such that

$$\frac{1}{x^2 - 1} + \frac{1}{y^2 - 1} + \frac{1}{z^2 - 1} = 1.$$

Prove that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \leq 1.$$

04IM04 (**IMO 2004 P4**) Let $n \geq 3$ be an integer. Let t_1, t_2, \dots, t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n) \left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

11MOPR42 (**MOP 2011 R4.2**) For positive real numbers a, b, c with $a + b + c = 3$ prove that

$$\sum_{\text{cyc}} \sqrt{\frac{a^3 + b^3}{a + b}} + 9\sqrt[3]{abc} \leq 12.$$

04SLA5 (**Shortlist 2004 A5**) If a, b, c are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

98IRN (**Iran 1998 P5**) When $x (\geq 1), y (\geq 1), z (\geq 1)$ satisfy $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, prove in equality.

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

95IM02 (IMO 1995 P2) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

12JM03 (USAJMO 2012 P3) Let a, b, c be positive real numbers. Prove that

$$\frac{a^3 + 3b^3}{5a+b} + \frac{b^3 + 3c^3}{5b+c} + \frac{c^3 + 3a^3}{5c+a} \geq \frac{2}{3}(a^2 + b^2 + c^2)$$

98SLA3 (Shortlist 1998 A3) Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

2 Solutions

2.1 Lecture Problems

2.1.1 USAMO 2011 P1

Problem Statement

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3.$$

The key is to correctly homogenize the inequality as shown below

$$\sum_{\text{cyc}} \frac{2ab + 2}{(a + b)^2} \geq \sum_{\text{cyc}} \frac{2ab + ab + bc + ca + a^2 + b^2 + c^2}{(a + b)^2}$$

and observe that

$$\sum_{\text{cyc}} 3ab + bc + ca + a^2 + b^2 + c^2 = \sum_{\text{cyc}} (a + b)^2 + (c + a)(c + b).$$

Hence $\sum_{\text{cyc}} \frac{(c + a)(c + b)}{(a + b)^2} \geq 6$, by the AM-GM inequality.

2.1.2 IMO 2001 P2

Problem Statement

Prove that for all positive real numbers a, b, c ,

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

By Hölder,

$$\left(\sum_{\text{cyc}} \frac{a}{\sqrt{a^2 + 8bc}} \right) \left(\sum_{\text{cyc}} a\sqrt{a^2 + 8bc} \right) \geq (a + b + c)^2.$$

Hence, it is enough to prove that

$$(a + b + c)^2 \geq \sum_{\text{cyc}} a\sqrt{a^2 + 8bc} \iff 2(a^2b^2 + b^2c^2 + c^2a^2) \geq 2(a^2bc + ab^2c + abc^2),$$

which is clearly true by the Muirhead's inequality.

2.1.3 IMO 2005 P3

Problem Statement

Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

2.2 Mandatory

2.2.1 IMO 2012 P2

Problem Statement

Let $n \geq 3$ be an integer, and let a_2, a_3, \dots, a_n be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

2.2.2 ELMO 2003 P4

Problem Statement

Let $x, y, z \geq 1$ be real numbers such that

$$\frac{1}{x^2 - 1} + \frac{1}{y^2 - 1} + \frac{1}{z^2 - 1} = 1.$$

Prove that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \leq 1.$$

2.2.3 IMO 2004 P4

Problem Statement

Let $n \geq 3$ be an integer. Let t_1, t_2, \dots, t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \dots + t_n) \left(\frac{1}{t_1} + \frac{1}{t_2} + \dots + \frac{1}{t_n} \right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

2.2.4 MOP 2011 R4.2

Problem Statement

For positive real numbers a, b, c with $a + b + c = 3$ prove that

$$\sum_{\text{cyc}} \sqrt{\frac{a^3 + b^3}{a + b}} + 9\sqrt[3]{abc} \leq 12.$$

2.3 Not Mandatory

2.3.1 Shortlist 2004 A5

Problem Statement

If a, b, c are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

2.3.2 Iran 1998 P5

Problem Statement

When $x(\geq 1)$, $y(\geq 1)$, $z(\geq 1)$ satisfy $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, prove in equality.

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

2.3.3 IMO 1995 P2

Problem Statement

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

2.3.4 USAJMO 2012 P3

Problem Statement

Let a, b, c be positive real numbers. Prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geq \frac{2}{3}(a^2 + b^2 + c^2)$$

2.3.5 Shortlist 1998 A3

Problem Statement

Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

3 References