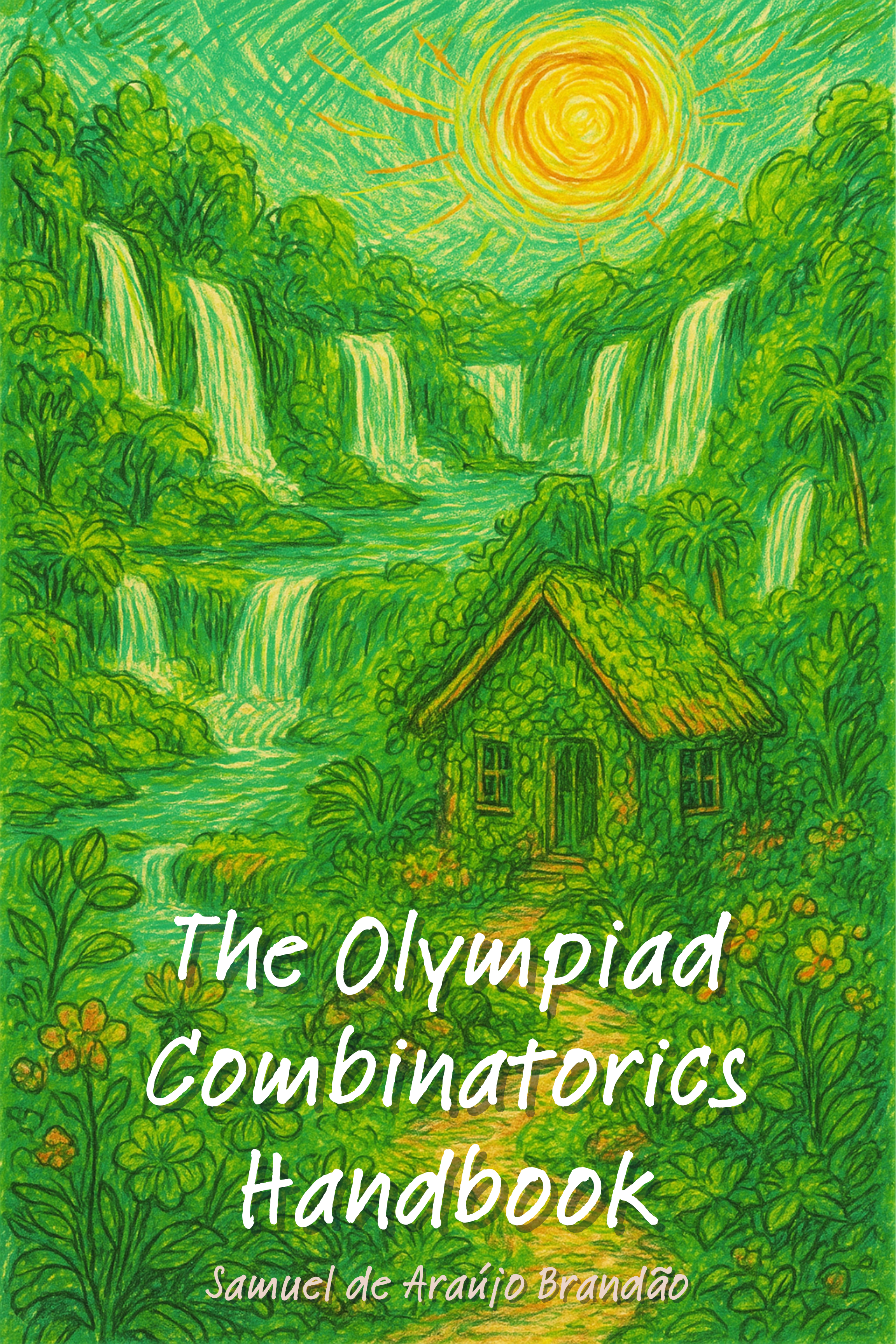

The Olympiad Combinatorics Handbook

Samuel de Araújo Brandão

The Olympiad Combinatorics Handbook

Samuel de Araújo Brandão

Preface

Acknowledgements

Introduction

Contents

Part I

Fundamentals

Chapter 1

Counting Basics

1.1 Permutations

A permutation can be defined as the arrangement of objects where the order matters.

Example 1.1 How many ways can one arrange 4 math books, 3 chemistry books, 2 physics books, and 1 biology book on a bookshelf so that all the math books are together, all the chemistry books are together, and all the physics books are together?

Solution. Consider the math books first. There are 4 possibilities for the first book, 3 possibilities for the second, 2 possibilities for the third and 1 for the fourth.

By the *Multiplication Principle*, there are $4 \times 3 \times 2 \times 1 = 4! = 24$ ways to arrange the math books. The symbol "!" denotes factorial, for integers $n \geq 1$, $n! = n \cdot (n-1) \cdot \ldots \cdot 1$

Similarly, there are 3! ways to arrange the chemistry books, 2! ways to arrange the physics books and only 1! way to arrange the biology book. Besides that, there are 4! ways to arrange the four subject blocks. The order matters, since it is a permutation.

Again, by the Multiplication Principle, one can say that there are

$$4! \cdot (4! \cdot 3! \cdot 2! \cdot 1!) = 6912$$

ways to arrange the books on the bookshelf.

Note 1.2 Do not worry, later problems and solutions are going to be harder than this one, it is only an introduction.

Note 1.3 $0! = 1$. This might be a surprise for you, but makes sense, since there is exactly one way to arrange 0 elements: the empty arrangement.

Example 1.4 From the five letters a,b,c,d,e, how many different 3-letter arrangements without repetition can be formed?

This time, we can't use 5! or 3! directly, but the idea is the same. For the first letter, there are 5 possibilities, 4 possibilities for the second and 3 possibilities for the third. In total, there are $5 \cdot 4 \cdot 3 = 60$. This count is the number of k-permutations, as you can see in Definition 1.1.

Definition 1.1: Number of Permutations

The number of different groups of k objects chosen from a total of n objects with regard to order is equal to

$$P(n, k) = \frac{n!}{(n-k)!}$$

When $n, k \in \mathbb{N}$ and $0 \leq k \leq n$.

Note 1.5 You might see the *number of permutations* denoted by ${}_nP_k$ instead of $P(n, k)$ either.

1.2 Combinations

A combination can be defined as the arrangement of objects where the order does not matter.

1.2.1 Binomial Coefficient

Definition 1.2: Binomial Coefficient

The number of different groups of k objects chosen from a total of n objects without regard to order is equal to

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

when $n \in \mathbb{N}, k \in \mathbb{Z}$ and $0 \leq k \leq n$. $\binom{n}{k} = 0 \iff k < 0$ or $k > n$ and $\binom{n}{0} = \binom{n}{n} = 1$.

We derive it from $P(n,k)$ by dividing by $k!$, since order is ignored. We have to count abc and bca, for example, a repetition and divide $P(n,k)$ by $k!$, as we can see in the following.

Example 1.6 How many ways are there to choose 2 letters from the set $\{a, b, c\}$?

Solution. Let's first demonstrate it using the Definition 1.1. $P(3,2) = 6$ gives ab, ac, ba, bc, ca, cb. Each subset is counted 2! times. Divide by 2!. Hence $\binom{3}{2} = 3$. That is exactly what goes behind Definition 1.4!

Example 1.7 How many ways are there to divide 9 people into one committee of 3, one committee of 4 and one committee of 2?

Solution. For the first committee, we decide between 9 people, so there are $\binom{9}{3} = 84$ possibilities. Once it is done, there are 6 people left for the second committee, so there are $\binom{6}{4} = 15$ possibilities. After that, the remainder must go in the third committee. In total, there are $\frac{9!}{3!4!2!} = 84 \cdot 15 = 1260$ ways to divide the 9 people.

This example illustrates what is called the *multinomial coefficient*:

1.2.2 Multinomial Coefficient

Definition 1.3: Multinomial Coefficient

For nonnegative integers $k_1, k_2, \ldots, k_m$ with $k_1 + k_2 + \cdots + k_m = n$, the multinomial coefficient is defined as

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!k_2!\cdots k_m!}.$$

As you can see, the multinomial coefficient is a generalization of the binomial coefficient, where $k_m = n - k$ and $m = 2$.

It would have been easier to solve the Example 1.7 using the multinomial coefficient, as you can see below.

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{9}{3, 4, 2} = \frac{9!}{3!4!2!} = 1260.$$

Let's finish this section with a really well known example.

Example 1.8 In how many ways can the letters of the word `MISSISSIPPI` be arranged?

Let's use the multinomial coefficient to solve this one. There are

- 1 letter `M`,
- 4 letters `I`,
- 4 letters `S`,
- 2 letters `P`.

Therefore,

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{11}{1, 4, 4, 2} = \frac{11!}{1!4!4!2!} = 34650$$

1.2.3 Symmetry

Definition 1.4: Binomial Coefficient

Choosing k objects is the same as rejecting $n-k$ objects:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Example 1.9 In tossing a fair coin 10 times, how many sequences contain exactly 8 heads?

Solution. To solve this one, you could either count the times that exactly 8 heads appear or count the times that exaclty 2 tails appear, as you can see:

$$\binom{10}{8} = \binom{10}{2} = 45.$$

1.3 Pascal's Identity

Theorem 1.1: Pascal's Identity

For all $n \geq 1$ and $1 \leq k \leq n-1$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. It is well known that

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!} \quad \text{and} \quad \binom{n-1}{k} = \frac{(n-1)!}{k!(n-1-k)}.$$

Therefore,

$$\begin{aligned}\frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-1-k)}&=\frac{(n-1)!k+(n-1)!(n-k)}{k!(n-k)!}\\&=\frac{(n-1)!(k+(n-k))}{k!(n-k)!}\\&=\frac{n!}{k!(n-k)!}.\end{aligned}$$ □

Example 1.10 A club has n members. In how many ways can you choose a committee of size k if you separate the cases:

- one special member (say Alice) is included,
- or Alice is not included?

Solution.

- If Alice is included: choose the other $k-1$ from $n-1$ members, resulting in $\binom{n-1}{k-1}$.
- If Alice is not included: choose the k from $n-1$ members, resulting in $\binom{n-1}{k}$.

So total:

$$\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}.$$

That is exactly Pascal's Identity in story form.

1.4 The Binomial Theorem

Theorem 1.2: Binomial Theorem

For any nonnegative integer n and $x,y \in \mathbb{R}$ or $\mathbb{C}$

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

1.5 Sum of Row

1.6 Alternating Sum

1.7 Vandermonde's Identity

Corollary 1.3: Vandermonde's Identity

1.8 Hockey-Stick Identity

Corollary 1.4: Hockey-Stick Identity

1.9 Grid Paths and Recursion

Corollary 1.5: Grid Paths Recursion

1.10 Pigeonhole Principle

Theorem 1.6: Pigeonhole Principle

If $n+1$ objects are placed into n boxes, then some box contains at least two objects.

Proof. Supose, for the sake of contradiction, that no box has more than one object. Then each of the n boxes contain at most one object. Hence, the total number of objects would be at most n.

But we assumed there were $n+1$ objects. Contradiction. Therefore, at least one box contains two or more objects. □

This is indeed a really simple concept, but at the same time, a tremendously important one.

Example 1.11 Choose 7 different numbers from $\{1, 2, \ldots, 12\}$. Prove that some pair sums to 13.

Solution. There are exactly 6 sums resulting in 13 using each number of the set $\{1, 2, \ldots, 12\}$ exactly once:

$$1 + 12, \quad 2 + 11, \quad 3 + 10, \quad 4 + 9, \quad 5 + 8, \quad 6 + 7.$$

Since we have to choose 7 numbers from the set, but there are 6 sums resulting in 13, by the Pigeonhole Principle, we'll need to choose two numbers from a single sum, and those two sum to 13.

Example 1.12 Place 5 points in the unit square. Show two are at distance less than or equal $\frac{\sqrt{2}}{2}$

Solution. Partition the unit square in four $\frac{1}{2} \cdot \frac{1}{2}$ subsquares. Since there are 5 points and 4 squares, by the Pigeonhole Principle, a square must contain 2 points. The greatest distance between two points in a subsquare is $\sqrt{\frac{1}{2}^2 + \frac{1}{2}^2} = \frac{\sqrt{2}}{2}$.

Example 1.13 In any set of $n + 1$ integers, chosen from $\{1, 2, \ldots, 2n\}$, prove that one number divides another.

Solution.

Lemma 1.7 Every positive integer a can be expressed as $2^b c$, where $b = \{x \geq 0 \mid x \in \mathbb{Z}^+\}$ and $c = 2y + 1$ for some positive integer y.

Proof. Let's split this proof.

- a is even: keep dividing a by 2 as long as the result is even. Eventually, you will stop, because dividing an integer by 2 cannot go on forever. When you stop, the number you reach must be odd. Call that number c.
- a is odd: $b = 0$, so $2^b = 1$ and $c = a$. □

Since there are n odd numbers inside $\{1, 2, \ldots, 2n\}$ and we chose $n+1$ numbers, by the Pigeonhole Principle there must be two numbers that have the same c in $2^b c$. Those two numbers are certainly divisible by each other.

Proof. Let's say that, WLOG, $b_2 < b_1 \implies \frac{2^{b_1} c}{2^{b_2} c} = 2^{b_1 - b_2}$, an integer number. $\square$

Example 1.14 Among 6 people, prove there exist 3 mutual acquaintances or 3 mutual strangers.

Solution. Let's fix one person, call them A. There are 5 people left. A has 5 possible relations: each of the 5 others is friend or stranger. By the Pigeonhole Principle, at least 3 of them must fall into the same category (since $5 = 2 \cdot 2 + 1$).

- A has 3 friends B, C, D: if any two of B, C and D are friends, say B and C, then we have a triangle of friends: A, B, C. If none of them are friends with each other, then B, C, D are pairwise strangers, so we have a triangle of strangers.

- A has 3 strangers E, D, F: if any two of E, F, and G are strangers, then with A we form a triangle of strangers. If they are all mutual friends, then E, D and F form a triangle of friends.

1.10.1 Generalized form

Theorem 1.8: Generalized Pigeonhole Principle

If n objects are distributed among m boxes, then some box contains at least

$$\left\lceil \frac{n}{m} \right\rceil$$

objects.

Proof. Suppose, for the sake of contradiction, that no box has more than $\lfloor \frac{n}{m} \rfloor$ objects. Then, each of the m boxes contain at most $\lceil \frac{n}{m} \rceil - 1 = \lfloor \frac{n}{m} \rfloor$. Hence, the total number of objects would be at most $m \cdot \lfloor \frac{n}{m} \rfloor$.

But we assumed there were n objects. Contradiction, since $\lfloor \frac{n}{m} \rfloor < \frac{n}{m}$. Therefore, at least one box contains at least $\lceil \frac{n}{m} \rceil$ objects. $\square$

1.11 Problems