

Ineq Basic

Samuel de Araújo Brandão

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This document collects my solutions to the OTIS problem sets from the **Ineq Basic** unit, written during my preparation for mathematical olympiads.

The solutions reflect my understanding and problem-solving approach at the time of writing. Some arguments were informed by discussions, official notes, or published sources; when so, attribution is provided (see [section 3](#)).

If you find errors or have suggestions, please contact me at samuelbaraujo19@gmail.com.

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1 Practice Problems

11AM01 (**USAMO 2011 P1**) Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$

01IM02 (**IMO 2001 P2**) Prove that for all positive real numbers a, b, c ,

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

05IM03 (**IMO 2005 P3**) Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

12IM02 (**IMO 2012 P2**) Let $n \geq 3$ be an integer, and let a_2, a_3, \dots, a_n be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

03ELM04 (**ELMO 2003 P4**) Let $x, y, z \geq 1$ be real numbers such that

$$\frac{1}{x^2 - 1} + \frac{1}{y^2 - 1} + \frac{1}{z^2 - 1} = 1.$$

Prove that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \leq 1.$$

04IM04 (**IMO 2004 P4**) Let $n \geq 3$ be an integer. Let t_1, t_2, \dots, t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n) \left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

11MOPR42 (**MOP 2011 R4.2**) For positive real numbers a, b, c with $a + b + c = 3$ prove that

$$\sum_{\text{cyc}} \sqrt{\frac{a^3 + b^3}{a + b}} + 9\sqrt[3]{abc} \leq 12.$$

04SLA5 (**Shortlist 2004 A5**) If a, b, c are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

98IRN (**Iran 1998 P5**) When $x (\geq 1), y (\geq 1), z (\geq 1)$ satisfy $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, prove in equality.

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

95IM02 (IMO 1995 P2) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

12JM03 (USAJMO 2012 P3) Let a, b, c be positive real numbers. Prove that

$$\frac{a^3 + 3b^3}{5a+b} + \frac{b^3 + 3c^3}{5b+c} + \frac{c^3 + 3a^3}{5c+a} \geq \frac{2}{3}(a^2 + b^2 + c^2)$$

98SLA3 (Shortlist 1998 A3) Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

2 Solutions

2.1 Lecture Problems

2.1.1 USAMO 2011 P1

Problem Statement

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3.$$

The key is to correctly homogenize the inequality as shown below

$$\sum_{\text{cyc}} \frac{2ab + 2}{(a + b)^2} \geq \sum_{\text{cyc}} \frac{2ab + ab + bc + ca + a^2 + b^2 + c^2}{(a + b)^2}$$

and observe that

$$\sum_{\text{cyc}} 3ab + bc + ca + a^2 + b^2 + c^2 = \sum_{\text{cyc}} (a + b)^2 + (c + a)(c + b).$$

Hence $\sum_{\text{cyc}} \frac{(c + a)(c + b)}{(a + b)^2} \geq 6$, by the AM-GM inequality.

2.1.2 IMO 2001 P2

Problem Statement

Prove that for all positive real numbers a, b, c ,

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

By Hölder,

$$\left(\sum_{\text{cyc}} \frac{a}{\sqrt{a^2 + 8bc}} \right) \left(\sum_{\text{cyc}} a\sqrt{a^2 + 8bc} \right) \geq (a + b + c)^2.$$

Hence, it is enough to prove that

$$(a + b + c)^2 \geq \sum_{\text{cyc}} a\sqrt{a^2 + 8bc} \iff 2(a^2b^2 + b^2c^2 + c^2a^2) \geq 2(a^2bc + ab^2c + abc^2),$$

which is clearly true by the Muirhead's inequality.

2.1.3 IMO 2005 P3

Problem Statement

Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

$$\sum_{\text{cyc}} \frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \sum_{\text{cyc}} \frac{x^5 - x^3yz}{x^5 + xyz(y^2 + z^2)}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left(\sum_{\text{cyc}} \frac{x^6}{x^6 + x^2yz(y^2 + z^2)} \right) \left(\sum_{\text{cyc}} x^6 + x^2yz(y^2 + z^2) \right) \geq (x^3 + y^3 + z^3)^2, \\ & \left(\sum_{\text{cyc}} \frac{x^4yz}{x^6 + x^2yz(y^2 + z^2)} \right) \left(\sum_{\text{cyc}} x^6 + x^2yz(y^2 + z^2) \right) \geq (x^2\sqrt{yz} + y^2\sqrt{xz} + z^2\sqrt{xy})^2. \end{aligned}$$

Hence, it is enough to prove that $(x^3 + y^3 + z^3)^2 \geq (x^2\sqrt{yz} + y^2\sqrt{xz} + z^2\sqrt{xy})^2$, which is clearly true by the Muirhead's inequality.

2.2 Mandatory

2.2.1 IMO 2012 P2

Problem Statement

Let $n \geq 3$ be an integer, and let a_2, a_3, \dots, a_n be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

2.2.2 ELMO 2003 P4

Problem Statement

Let $x, y, z \geq 1$ be real numbers such that

$$\frac{1}{x^2 - 1} + \frac{1}{y^2 - 1} + \frac{1}{z^2 - 1} = 1.$$

Prove that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \leq 1.$$

2.2.3 IMO 2004 P4

Problem Statement

Let $n \geq 3$ be an integer. Let t_1, t_2, \dots, t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \dots + t_n) \left(\frac{1}{t_1} + \frac{1}{t_2} + \dots + \frac{1}{t_n} \right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

2.2.4 MOP 2011 R4.2

Problem Statement

For positive real numbers a, b, c with $a + b + c = 3$ prove that

$$\sum_{\text{cyc}} \sqrt{\frac{a^3 + b^3}{a + b}} + 9\sqrt[3]{abc} \leq 12.$$

2.3 Not Mandatory

2.3.1 Shortlist 2004 A5

Problem Statement

If a, b, c are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

2.3.2 Iran 1998 P5

Problem Statement

When $x(\geq 1)$, $y(\geq 1)$, $z(\geq 1)$ satisfy $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, prove in equality.

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

2.3.3 IMO 1995 P2

Problem Statement

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

First solution By Cauchy-Schwarz,

$$\left(\sum_{\text{cyc}} \frac{1}{a^3(b+c)} \right) \left(\sum_{\text{cyc}} a(b+c) \right) \geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 = (ab+bc+ca)^2.$$

Hence, it is enough to prove that

$$(ab+bc+ca)^2 \geq \frac{3}{2} \left(\sum_{\text{cyc}} a(b+c) \right) \iff (ab+bc+ca)^2 \geq 3(ab+bc+ca)(abc)^{\frac{2}{3}},$$

which is true by AM-GM.

Second solution By Cauchy-Schwarz,

$$\left(\sum_{\text{cyc}} \frac{1}{a^3(b+c)} \right) \left(\sum_{\text{cyc}} a(b+c) \right) \geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 = (ab+bc+ca)^2,$$

and by Titu's lemma,

$$\sum_{\text{cyc}} \frac{\left(\frac{1}{a}\right)^2}{a(b+c)} \geq \frac{ab+bc+ca}{2} \geq \frac{3(abc)^{\frac{2}{3}}}{2} = \frac{3}{2}$$

2.3.4 USAJMO 2012 P3

Problem Statement

Let a, b, c be positive real numbers. Prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geq \frac{2}{3}(a^2 + b^2 + c^2)$$

By Titu's lemma we have

$$\sum_{\text{cyc}} \frac{a^4}{a(5a+b)} \geq \frac{a^2 + b^2 + c^2}{6} = x \quad \text{and} \quad 3 \sum_{\text{cyc}} \frac{b^4}{b(5a+b)} \geq \frac{a^2 + b^2 + c^2}{2} = y$$

since $\sum_{\text{cyc}} 5ab + b^2 \leq \sum_{\text{cyc}} 5a^2 + ab \leq 6(a^2 + b^2 + c^2)$. Hence $x + y = \frac{2}{3}(a^2 + b^2 + c^2)$.

2.3.5 Shortlist 1998 A3

Problem Statement

Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

First solution By Cauchy-Schwarz,

$$\begin{aligned} \left(\sum_{\text{cyc}} \frac{x^3}{(1+y)(1+z)} \right) \left(\sum_{\text{cyc}} (1+y)(1+z) \right) &\geq (x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}})^2 \\ &= x^3 + y^3 + z^3 + 2((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}). \end{aligned}$$

Therefore, it suffices to prove that

$$\begin{aligned} x^3 + y^3 + z^3 + 2((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}) &\geq \frac{3}{4}(3 + 2(x+y+z) + xy + yz + zx) \iff \\ 2 \sum_{\text{sym}} x^3 + 4 \sum_{\text{sym}} (x^{11}y^{11}z^5)^{\frac{1}{9}} &\geq \frac{3}{2} \sum_{\text{sym}} abc + \sum_{\text{sym}} (x^5y^2z^2)^{\frac{1}{3}} + \sum_{\text{cyc}} (x^4y^4z)^{\frac{1}{3}} \end{aligned}$$

which is true, since

- $\sum_{\text{sym}} x^3 \geq \sum_{\text{sym}} (x^5y^2z^2)^{\frac{1}{3}},$
- $\sum_{\text{sym}} x^3 \geq \sum_{\text{cyc}} (x^4y^4z)^{\frac{1}{3}},$
- $4 \sum_{\text{sym}} (x^{11}y^{11}z^5)^{\frac{1}{9}} \geq \frac{3}{2} \sum_{\text{sym}} abc,$

are all true by the Muirhead's inequality

Second solution By Titu's lemma,

$$\sum_{\text{cyc}} \frac{x^4}{x(1+y)(1+z)} \geq \frac{(x^2 + y^2 + z^2)^2}{3xyz + 2(xy + yz + za) + x + y + z} \geq \frac{(x^2 + y^2 + z^2)^2}{4(x^2 + y^2 + z^2)} \geq \frac{3}{4}$$

3 References