USAJMO 2012

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This document contains solutions to the USAJMO 2012 problems, written by me during my preparation for the International Mathematical Olympiad.

The content reflects my own understanding and problem-solving process. Some solutions may have been inspired by the work of others or required external help, in which case proper attribution is given (see section 3).

If you notice any errors or have suggestions for improvement, I would greatly appreciate hearing from you at samuelbaraujo19@gmail.com.

Contents

1 Problems

1. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+3b^3}{5a+b}+\frac{b^3+3c^3}{5b+c}+\frac{c^3+3a^3}{5c+a}\geq\frac{2}{3}(a^2+b^2+c^2)$$

2 Solutions

2.1 Problem 2.

Problem Statement

Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+3b^3}{5a+b}+\frac{b^3+3c^3}{5b+c}+\frac{c^3+3a^3}{5c+a}\geq\frac{2}{3}(a^2+b^2+c^2)$$

By Titu's lemma we have

$$\sum_{\text{cyc}}\frac{a^4}{a(5a+b)}\geq\frac{a^2+b^2+c^2}{6}=x\quad\text{and}\quad 3\sum_{\text{cyc}}\frac{b^4}{b(5a+b)}\geq\frac{a^2+b^2+c^2}{2}=y$$

since $\sum_{\text{cyc}} 5ab+b^2\leq\sum_{\text{cyc}} 5a^2+ab\leq 6(a^2+b^2+c^2)$. Hence $x+y=\dfrac{2}{3}(a^2+b^2+c^2)$.

3 References