Solutions **IMO Shortlist 2004**

Samuel de Araújo Brandão

December 15, 2025

This document contains solutions to the **IMO Shortlist 2004** problems, written by me during my preparation for the International Mathematical Olympiad.

The content reflects my own understanding and problem-solving process. Some solutions may have been inspired by the work of others or required external help, in which case proper attribution is given (see section 3).

If you notice any errors or have suggestions for improvement, I would greatly appreciate hearing from you at samuelbaraujo19@gmail.com.

Contents

1 Problems

1. If a, b ,c are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

2 Solutions

2.0.1 A5

Problem Statement

If a, b ,c are three positive real numbers such that $ab+bc+ca=1$, prove that

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a}\leq\frac{1}{abc}.$$

First solution After homogenizing the inequality, we must prove that

$$\sum_{\text{cyc}}\sqrt[3]{\frac{(ab+bc+ca)^2}{a}+6b(ab+bc+ca)} =$$
$$\sum_{\text{cyc}}\sqrt[3]{\frac{(ab+bc+ca)(7ab+bc+ca)}{a}} \leq$$
$$\frac{(ab+bc+ca)^2}{abc}.$$

By the Hölder's inequality,

$$\left(\sum_{\text{cyc}}\sqrt[3]{\frac{(ab+bc+ca)(7ab+bc+ca)}{a}}\right)^3\leq\left(\sum_{\text{cyc}}ab+bc+ca\right)\left(\sum_{\text{cyc}}7ab+bc+ca\right)\left(\sum_{\text{cyc}}\frac{1}{a}\right).$$

Therefore, it suffices to prove the following.

$$\begin{aligned}\left(\sum_{\text{cyc}}ab+bc+ca\right)\left(\sum_{\text{cyc}}7ab+bc+ca\right)\left(\sum_{\text{cyc}}\frac{1}{a}\right)&\leq\frac{(ab+bc+ca)^6}{(abc)^3}. &&\iff\\ \frac{24(ab+bc+ca)^3}{abc}&\leq\frac{(ab+bc+ca)^6}{(abc)^3} &&\iff\\ 24(abc)^2&\leq(ab+bc+ca)^3 &&\iff\\ 2\sqrt[3]{3}(abc)^{\frac{2}{3}}&\leq ab+bc+ca.\end{aligned}$$

By the AM-GM inequality, $ab+bc+ca\geq 3(abc)^{\frac{2}{3}}$. Hence, we must prove that $3\geq 2\sqrt[3]{3}$. Which is true, since $27>24$.

Consequently,

$$\frac{1}{abc}\geq\sum_{\text{cyc}}\sqrt[3]{\frac{1}{a}+6b}.$$

Second solution, found by DottedCalculator By the Hölder's inequality,

$$\left(\sum_{\text{cyc}}\left(\frac{1}{a}+6b\right)\right)\left(\sum_{\text{cyc}}1\right)^2\geq\left(\sum_{\text{cyc}}\sqrt[3]{\frac{1}{a}+6b}\right)^3.$$

Therefore, it suffices to prove that

$$\frac{1}{(abc)^3}\geq 54(a+b+c)+9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\iff 1\geq 54(abc)^3(a+b+c)+9(abc)^2= 9(abc)^2(6abc(a+b+c)+1)$$

since $ab+bc+ca=1$.

By the AM-GM inequality, $\frac{1}{27} \geq (abc)^2$, and by the Titu's Lemma, $a^2b^2 + b^2c^2 + c^2a^2 \geq \frac{(ab+bc+ca)^2}{3} = \frac{1}{3}$. Hence, $9(abc)^2 \leq \frac{1}{3}$ and $6abc(a+b+c)+1 \leq 3$. It happens because

$$6abc(a+b+c)+1 = 3(ab+bc+ca)^2 - 3(a^2b^2+b^2c^2+c^2a^2)+1 = 4-3(a^2b^2+b^2c^2+c^2a^2).$$

Thus,

$$1 \geq 9(abc)^2(6abc(a+b+c)+1).$$

Consequently,

$$\frac{1}{abc} \geq \sum_{\text{cyc}} \sqrt[3]{\frac{1}{a} + 6b}.$$

3 References