Ineq Basic

Samuel de Araújo Brandão

December 20, 2025

This document collects my solutions to the OTIS problem sets from the **Ineq Basic** unit, written during my preparation for mathematical olympiads.

The solutions reflect my understanding and problem-solving approach at the time of writing. Some arguments were informed by discussions, official notes, or published sources; when so, attribution is provided (see section 3).

If you find errors or have suggestions, please contact me at samuelbaraujo19@gmail.com .

Contents

1 Practice Problems

11AM01 **(USAMO 2011 P1)** Let a, b, c be positive real numbers such that $a^2+b^2+c^2+(a+b+c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$

01IMO2 **(IMO 2001 P2)** Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

05IMO3 **(IMO 2005 P3)** Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{x^2+y^5+z^2} + \frac{z^5-z^2}{x^2+y^2+z^5} \geq 0.$$

11MOPR42 [9♣] **(MOP 2011 R4.2)** For positive real numbers a, b, c with $a+b+c=3$ prove that

$$\sum_{\text{cyc}} \sqrt{\frac{a^3+b^3}{a+b}} + 9\sqrt[3]{abc} \leq 12.$$

12IMO2 [5♣] **(IMO 2012 P2)** Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

03ELM04 [5♣] **(ELMO 2003 P4)** Let $x, y, z \geq 1$ be real numbers such that

$$\frac{1}{x^2-1} + \frac{1}{y^2-1} + \frac{1}{z^2-1} = 1.$$

Prove that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \leq 1.$$

04IMO4 [5♣] **(IMO 2004 P4)** Let $n \geq 3$ be an integer. Let t_1, t_2, ..., t_n be positive real numbers such that

$$n^2+1 > (t_1+t_2+\cdots+t_n)\left(\frac{1}{t_1}+\frac{1}{t_2}+\cdots+\frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

04SLA5 [9♣] **(Shortlist 2004 A5)** If a, b ,c are three positive real numbers such that $ab+bc+ca=1$, prove that

$$\sqrt[3]{\frac{1}{a}+6b} + \sqrt[3]{\frac{1}{b}+6c} + \sqrt[3]{\frac{1}{c}+6a} \leq \frac{1}{abc}.$$

98IRN [5♣] **(Iran 1998 P5)** When $x(\geq 1)$, $y(\geq 1)$, $z(\geq 1)$ satisfy $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$, prove in equality.

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$$

95IMO2 [5♣] **(IMO 1995 P2)** Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

12JMO3 [3♣] **(USAJMO 2012 P3)** Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+3b^3}{5a+b}+\frac{b^3+3c^3}{5b+c}+\frac{c^3+3a^3}{5c+a}\geq\frac{2}{3}(a^2+b^2+c^2)$$

98SLA3 [3♣] **(Shortlist 1998 A3)** Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)}\geq\frac{3}{4}.$$

00IMO2 [3♣] **(IMO 2000 P2)** Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1.$$

18RUS112 [3♣] **(All Russian Olympiad 2018 Grade 11 P2)** Let $n \geq 2$ and $x_1, x_2, \dots, x_n$ positive real numbers. Prove that

$$\frac{1+x_1^2}{1+x_1x_2}+\frac{1+x_2^2}{1+x_2x_3}+\cdots+\frac{1+x_n^2}{1+x_nx_1}\geq n.$$

2 Solutions

2.1 Lecture Problems

2.1.1 USAMO 2011 P1

Problem Statement

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3.$$

The key is to correctly homogenize the inequality as shown below

$$\sum_{\text{cyc}} \frac{2ab + 2}{(a + b)^2} \geq \sum_{\text{cyc}} \frac{2ab + ab + bc + ca + a^2 + b^2 + c^2}{(a + b)^2}$$

and observe that

$$\sum_{\text{cyc}} 3ab + bc + ca + a^2 + b^2 + c^2 = \sum_{\text{cyc}} (a + b)^2 + (c + a)(c + b).$$

Hence $\sum_{\text{cyc}} \frac{(c + a)(c + b)}{(a + b)^2} \geq 6$, by the AM-GM inequality.

2.1.2 IMO 2001 P2

Problem Statement

Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

By Hölder,

$$\left(\sum_{\text{cyc}} \frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum_{\text{cyc}} a\sqrt{a^2+8bc}\right) \geq (a+b+c)^2.$$

Hence, is it enough to prove that

$$(a+b+c)^2 \geq \sum_{\text{cyc}} a\sqrt{a^2+8bc} \iff 2(a^2b^2+b^2c^2+c^2a^2) \geq 2(a^2bc+ab^2c+abc^2),$$

which is clearly true by the Muirhead's inequality.

2.1.3 IMO 2005 P3

Problem Statement

Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

$$\sum_{\text{cyc}} \frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \sum_{\text{cyc}} \frac{x^5 - x^3yz}{x^5 + xyz(y^2 + z^2)}$$

By the Cauchy-Shwarz inequality,

$$\left(\sum_{\text{cyc}} \frac{x^6}{x^6 + x^2yz(y^2 + z^2)}\right)\left(\sum_{\text{cyc}} x^6 + x^2yz(y^2 + z^2)\right) \geq (x^3 + y^3 + z^3)^2,$$

$$\left(\sum_{\text{cyc}} \frac{x^4yz}{x^6 + x^2yz(y^2 + z^2)}\right)\left(\sum_{\text{cyc}} x^6 + x^2yz(y^2 + z^2)\right) \geq (x^2\sqrt{yz} + y^2\sqrt{xz} + z^2\sqrt{xy})^2.$$

Hence, it is enough to prove that $(x^3+y^3+z^3)^2 \geq \left(x^2\sqrt{yz} + y^2\sqrt{xz} + z^2\sqrt{xy}\right)^2$, which is clearly true by the Muirhead's inequality.

2.2 Mandatory

2.2.1 IMO 2012 P2

Problem Statement

Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

2.2.2 ELMO 2003 P4

Problem Statement

Let $x, y, z \geq 1$ be real numbers such that

$$\frac{1}{x^2-1} + \frac{1}{y^2-1} + \frac{1}{z^2-1} = 1.$$

Prove that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \leq 1.$$

2.2.3 IMO 2004 P4

Problem Statement

Let $n \geq 3$ be an integer. Let t_1, t_2, ..., t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

2.2.4 MOP 2011 R4.2

Problem Statement

For positive real numbers a, b, c with $a + b + c = 3$ prove that

$$\sum_{\text{cyc}} \sqrt{\frac{a^3 + b^3}{a + b}} + 9\sqrt[3]{abc} \leq 12.$$

2.3 Not Mandatory

2.3.1 Shortlist 2004 A5

Problem Statement

If a, b ,c are three positive real numbers such that $ab+bc+ca=1$, prove that

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a}\le\frac{1}{abc}.$$

First solution After homogenizing the inequality, we must prove that

$$\begin{aligned}
&\sum_{cyc}\sqrt[3]{\frac{(ab+bc+ca)^2}{a}+6b(ab+bc+ca)} &=\\
&\sum_{cyc}\sqrt[3]{\frac{(ab+bc+ca)(7ab+bc+ca)}{a}} &\le\\
&\frac{(ab+bc+ca)^2}{abc}.
\end{aligned}$$

By the Hölder's inequality,

$$\left(\sum_{cyc}\sqrt[3]{\frac{(ab+bc+ca)(7ab+bc+ca)}{a}}\right)^3\le\left(\sum_{cyc}ab+bc+ca\right)\left(\sum_{cyc}7ab+bc+ca\right)\left(\sum_{cyc}\frac{1}{a}\right).$$

Therefore, it suffices to prove the following.

$$\begin{aligned}
\left(\sum_{cyc}ab+bc+ca\right)\left(\sum_{cyc}7ab+bc+ca\right)\left(\sum_{cyc}\frac{1}{a}\right)\le\frac{(ab+bc+ca)^6}{(abc)^3}. &\iff\\
\frac{24(ab+bc+ca)^3}{abc}\le\frac{(ab+bc+ca)^6}{(abc)^3} &\iff\\
24(abc)^2\le(ab+bc+ca)^3 &\iff\\
2\sqrt[3]{3}(abc)^{\frac{2}{3}}\le ab+bc+ca.
\end{aligned}$$

By the AM-GM inequality, $ab+bc+ca\ge3(abc)^{\frac{2}{3}}$. Hence, we must prove that $3\ge2\sqrt[3]{3}$. Which is true, since $27>24$.

Consequently,

$$\frac{1}{abc}\ge\sum_{cyc}\sqrt[3]{\frac{1}{a}+6b}.$$

Second solution, found by DottedCalculator By the Hölder's inequality,

$$\left(\sum_{cyc}\left(\frac{1}{a}+6b\right)\right)\left(\sum_{cyc}1\right)^2\ge\left(\sum_{cyc}\sqrt[3]{\frac{1}{a}+6b}\right)^3.$$

Therefore, it suffices to prove that

$$\frac{1}{(abc)^3}\ge54(a+b+c)+9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\iff1\ge54(abc)^3(a+b+c)+9(abc)^2=$$
$$9(abc)^2(6abc(a+b+c)+1)$$

since $ab+bc+ca=1$.

By the AM-GM inequality, $\frac{1}{27} \geq (abc)^2$, and by the Titu's Lemma, $a^2b^2 + b^2c^2 + c^2a^2 \geq \frac{(ab+bc+ca)^2}{3} = \frac{1}{3}$. Hence, $9(abc)^2 \leq \frac{1}{3}$ and $6abc(a+b+c)+1 \leq 3$. It happens because

$$6abc(a+b+c)+1 = 3(ab+bc+ca)^2 - 3(a^2b^2+b^2c^2+c^2a^2)+1 = 4-3(a^2b^2+b^2c^2+c^2a^2).$$

Thus,

$$1 \geq 9(abc)^2(6abc(a+b+c)+1).$$

Consequently,

$$\frac{1}{abc} \geq \sum_{\text{cyc}} \sqrt[3]{\frac{1}{a}+6b}.$$

2.3.2 Iran 1998 P5

Problem Statement

When $x(\geq 1)$, $y(\geq 1)$, $z(\geq 1)$ satisfy $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$, prove in equality.

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{\text{cyc}} x\right)\left(\sum_{\text{cyc}} 1-\frac{1}{x}\right) \geq \left(\sum_{\text{cyc}} \sqrt{x-1}\right)^2.$$

However, since $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$,

$$\sum_{\text{cyc}}\left(1-\frac{1}{x}\right)=1.$$

Therefore,

$$\sum_{\text{cyc}} x \geq \left(\sum_{\text{cyc}} \sqrt{x-1}\right)^2 \implies \sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

2.3.3 IMO 1995 P2

Problem Statement

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3\left(b+c\right)}+\frac{1}{b^3\left(c+a\right)}+\frac{1}{c^3\left(a+b\right)}\geq\frac{3}{2}.$$

First solution By Cauchy-Schwarz,

$$\left(\sum_{\text{cyc}}\frac{1}{a^3(b+c)}\right)\left(\sum_{\text{cyc}}a(b+c)\right)\geq\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2=(ab+bc+ca)^2.$$

Hence, it is enough to prove that

$$(ab+bc+ca)^2\geq\frac{3}{2}\left(\sum_{\text{cyc}}a(b+c)\right)\iff(ab+bc+ca)^2\geq 3(ab+bc+ca)(abc)^{\frac{2}{3}},$$

which is true by AM-GM.

Second solution By Cauchy-Schwarz,

$$\left(\sum_{\text{cyc}}\frac{1}{a^3(b+c)}\right)\left(\sum_{\text{cyc}}a(b+c)\right)\geq\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2=(ab+bc+ca)^2,$$

and by Titu's lemma,

$$\sum_{\text{cyc}}\frac{(\frac{1}{a})^2}{a(b+c)}\geq\frac{ab+bc+ca}{2}\geq\frac{3(abc)^{\frac{2}{3}}}{2}=\frac{3}{2}$$

2.3.4 USAJMO 2012 P3

Problem Statement

Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+3b^3}{5a+b}+\frac{b^3+3c^3}{5b+c}+\frac{c^3+3a^3}{5c+a} \geq \frac{2}{3}(a^2+b^2+c^2)$$

By Titu's lemma we have

$$\sum_{\text{cyc}} \frac{a^4}{a(5a+b)} \geq \frac{a^2+b^2+c^2}{6} = x \quad \text{and} \quad 3\sum_{\text{cyc}} \frac{b^4}{b(5a+b)} \geq \frac{a^2+b^2+c^2}{2} = y$$

since $\sum_{\text{cyc}} 5ab+b^2 \leq \sum_{\text{cyc}} 5a^2+ab \leq 6(a^2+b^2+c^2)$. Hence $x+y=\dfrac{2}{3}(a^2+b^2+c^2)$.

2.3.5 Shortlist 1998 A3

Problem Statement

Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

First solution By Cauchy-Schwarz,

$$\begin{aligned}\left(\sum_{\text{cyc}} \frac{x^3}{(1+y)(1+z)}\right)\left(\sum_{\text{cyc}}(1+y)(1+z)\right) &\geq (x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}})^2 \\ &= x^3 + y^3 + z^3 + 2((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}).\end{aligned}$$

Therefore, it suffices to prove that

$$x^3 + y^3 + z^3 + 2((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}) \geq \frac{3}{4}(3 + 2(x+y+z) + xy + yz + zx) \iff$$
$$2\sum_{\text{sym}} x^3 + 4\sum_{\text{sym}}(xy)^{\frac{3}{2}} \geq \frac{3}{2}\sum_{\text{sym}} xyz + 3\sum_{\text{sym}}(x^5y^2z^2)^{\frac{1}{3}} + \frac{3}{2}\sum_{\text{cyc}}(x^4y^4z)^{\frac{1}{3}}$$

Second solution By Titu's lemma,

$$\sum_{\text{cyc}} \frac{x^4}{x(1+y)(1+z)} \geq \frac{(x^2+y^2+z^2)^2}{3xyz + 2(xy+yz+za) + x + y + z} \geq \frac{(x^2+y^2+z^2)^2}{4(x^2+y^2+z^2)} \geq \frac{3}{4}$$

2.3.6 IMO 2000 P2

Problem Statement

Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

First solution Since $abc = 1$, let $a = \frac{x}{y}$, $b = \frac{y}{z}$ and $c = \frac{z}{x}$. Therefore, it suffices to prove that

$$\prod_{\text{cyc}}\left(\frac{x}{y} - 1 + \frac{z}{y}\right) = \prod_{\text{cyc}}\left(\frac{x - y + z}{y}\right) \leq 1 \iff \prod_{\text{cyc}}(x - y + z) \leq xyz.$$

Let $u = x - y + z$,$v = y - z + x$ and $w = z - x + y$. Hence, it is enough to show that

$$uvw \leq \frac{(u + v)(v + w)(w + u)}{8},$$

which is true by AM-GM. Thus,

$$\prod_{\text{cyc}}\left(a - 1 + \frac{1}{b}\right) \leq 1.$$

Second solution Since $abc = 1$, let $a = \frac{x}{y}$, $b = \frac{y}{z}$ and $c = \frac{z}{x}$. Therefore, it suffices to prove that

$$\prod_{\text{cyc}}\left(\frac{x}{y} - 1 + \frac{z}{y}\right) = \prod_{\text{cyc}}\left(\frac{x - y + z}{y}\right) \leq 1 \iff \prod_{\text{cyc}}(x - y + z) \leq xyz.$$

Now, notice that

$$\prod_{\text{cyc}}(x - y + z) = \sum_{\text{sym}} x^2y - 2xyz - x^3 - y^3 - z^3.$$

Hence, it is enough to show that

$$\sum_{\text{sym}} x^2y - x^3 - y^3 - z^3 \leq 3xyz,$$

which is true by the Schur's inequality. Thus,

$$\prod_{\text{cyc}}\left(a - 1 + \frac{1}{b}\right) \leq 1.$$

2.3.7 All Russian Olympiad 2018 Grade 11 P2

Problem Statement

Let $n \geq 2$ and $x_1, x_2, \ldots, x_n$ positive real numbers. Prove that

$$\frac{1+x_1^2}{1+x_1x_2} + \frac{1+x_2^2}{1+x_2x_3} + \cdots + \frac{1+x_n^2}{1+x_nx_1} \geq n.$$

By the AM-GM inequality,

$$\sum_{\text{cyc}} \frac{1+x_i^2}{1+x_ix_{i+1}} \geq n\sqrt[n]{\prod_{\text{cyc}} \frac{1+x_i^2}{1+x_ix_{i+1}}}.$$

Therefore, it suffices to show that

$$\prod_{\text{cyc}} \frac{1+x_i^2}{1+x_ix_{i+1}} \geq 1 \iff \prod_{\text{cyc}}(1+x_i^2) \geq \prod_{\text{cyc}}(1+x_ix_{i+1}),$$

which is clearly true, since

$$(1+x_i^2)(1+x_{i+1}^2) \geq (1+x_ix_{i+1})^2$$

by the Cauchy-Schwarz inequality. Hence,

$$\prod_{\text{cyc}}(1+x_i^2) \geq \prod_{\text{cyc}}(1+x_1x_{i+1}).$$

Thus,

$$\sum_{\text{cyc}} \frac{1+x_i^2}{1+x_ix_{i+1}} \geq n.$$

3 References