Solutions IMO Shortlist 1998

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This document contains solutions to the Solutions IMO Shortlist 1998 problems, written by me during my preparation for the International Mathematical Olympiad.

The content reflects my own understanding and problem-solving process. Some solutions may have been inspired by the work of others or required external help, in which case proper attribution is given (see section 3).

If you notice any errors or have suggestions for improvement, I would greatly appreciate hearing from you at samuelbaraujo19@gmail.com.

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1 Problems

1. Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

2 Solutions

2.1 Problem 2.

Problem Statement

Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

First solution By Cauchy-Schwarz,

$$\begin{aligned}\left(\sum_{\text{cyc}} \frac{x^3}{(1+y)(1+z)}\right)\left(\sum_{\text{cyc}}(1+y)(1+z)\right) &\geq (x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}})^2 \\ &= x^3 + y^3 + z^3 + 2((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}).\end{aligned}$$

Therefore, it suffices to prove that

$$x^3 + y^3 + z^3 + 2((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}) \geq \frac{3}{4}(3 + 2(x+y+z) + xy + yz + zx) \iff$$
$$2\sum_{\text{sym}} x^3 + 4\sum_{\text{sym}}(x^{11}y^{11}z^5)^{\frac{1}{9}} \geq \frac{3}{2}\sum_{\text{sym}} abc + \sum_{\text{sym}}(x^5y^2z^2)^{\frac{1}{3}} + \sum_{\text{cyc}}(x^4y^4z)^{\frac{1}{3}}$$

which is true, since

- $\sum_{\text{sym}} x^3 \geq \sum_{\text{sym}}(x^5y^2z^2)^{\frac{1}{3}}$,
- $\sum_{\text{sym}} x^3 \geq \sum_{\text{cyc}}(x^4y^4z)^{\frac{1}{3}}$,
- $4\sum_{\text{sym}}(x^{11}y^{11}z^5)^{\frac{1}{9}} \geq \frac{3}{2}\sum_{\text{sym}} abc$,

are all true by the Muirhead's inequality

Second solution By Titu's lemma,

$$\sum_{\text{cyc}} \frac{x^4}{x(1+y)(1+z)} \geq \frac{(x^2+y^2+z^2)^2}{3xyz + 2(xy+yz+za) + x+y+z} \geq \frac{(x^2+y^2+z^2)^2}{4(x^2+y^2+z^2)} \geq \frac{3}{4}$$

3 References