Orthogonal Equipartitions of 3-color Points in the Plane

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Abstract

Orthogonal equipartition is a problem of subdividing the plane to regions containing equal numbers of points using only vertical and horizontal lines. Previously, Sergey Bereg proved that an equipartition of a set of 2-color points into two regions needs at most 1 horizontal and 1 vertical segments. We will be interested in a set of 3-color points. We give an algorithm to find an equipartition into two regions with at most 3 vertical and 2 horizontal segments. We also find examples that cannot be equipartitioned with less than 5 segments.

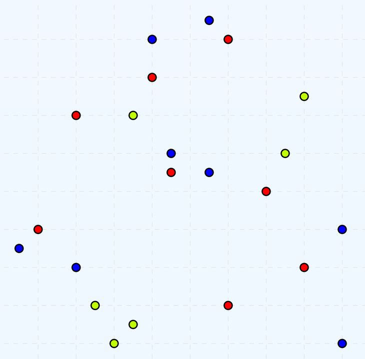
Keywords: Equitable subdivision, Orthogonal partition, Colored point sets

Objective

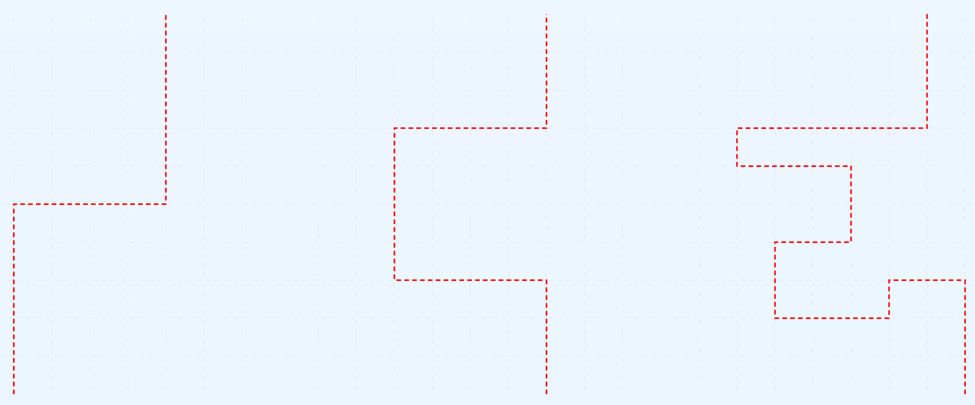
Minimizing the number of line segments of orthogonal equipartition of 3-color points in general position in the plane.

Definition

Let A be a set of points in the plane. The points in set A are lying in **general position** if no three points lie on the same line. In other words, any three points can make a triangle.



The **orthogonal partition** consists of vertical line segments and horizontal line segments such that the intersection of any two segment is either an empty set or a vertex of two segments.



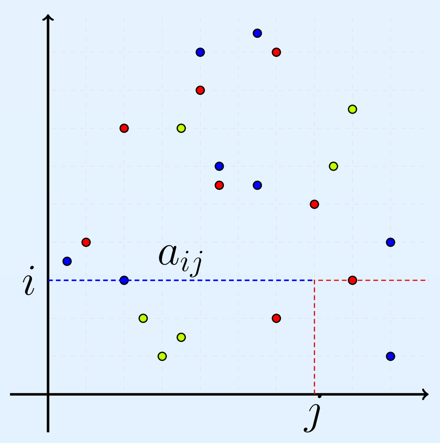
Let a and b be positive integers. A partition of the plane into two regions A and B is called **$(a, b)-$balanced with respect to a measure λ** if

$$\lambda(A) = \frac{a}{(a+b)} \text{ and } \lambda(B) = \frac{b}{(a+b)}$$

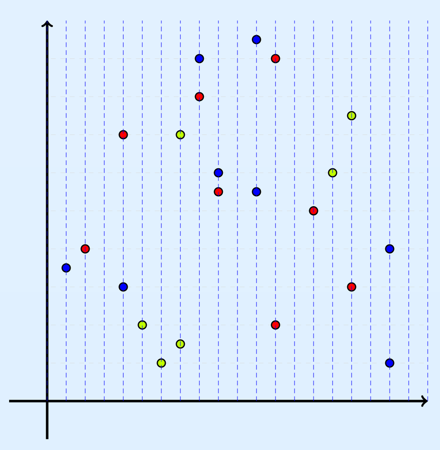
and called **$(a, b)-$balanced** if it is $(a, b)-$balanced with respect to both measure λ and μ. We say that $(a, b)-$balanced partition is **equitable** if $a = b$.

Notation

Let A is a set of $k-$color points in general position on lattice $\mathbb{Z}^2$ with length $n \times n$, where $n > 1$ and $k > 2$ be integers. Defined a_{ij} is a $k \times 1$-dimensional matrix of the number of points of each color over $y = i$ from $x = 0$ to $x = j$.

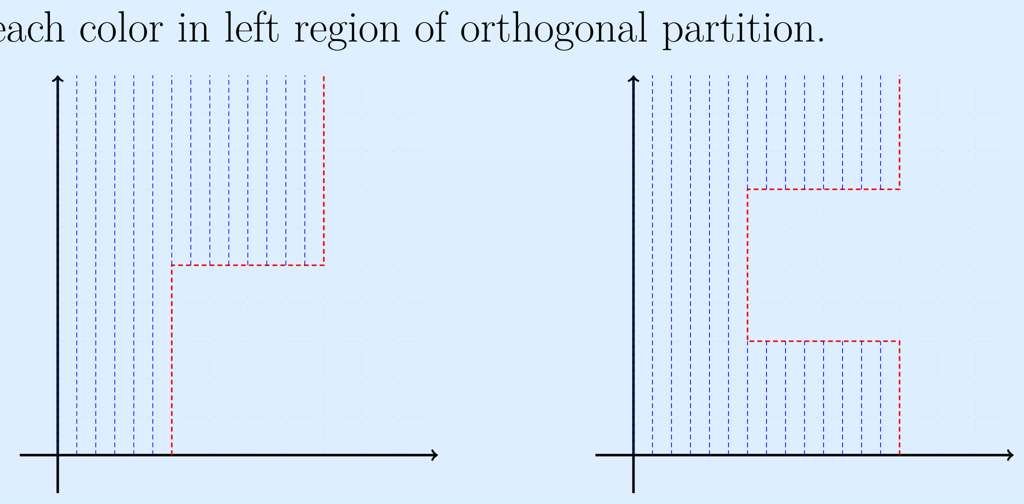


Defined S_j by $k \times 1$-dimensional matrix of the number of points of each color on the left side of a line $x = j$, that is $S_j = \sum_{i=1}^{n} a_{ij}$.



Note that S_n is $k \times 1$-dimensional matrix of the number of points of each color in the plane.

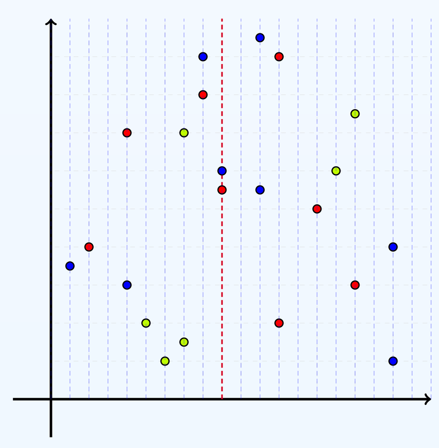
Defined V by the k-dimension vector of the number of points of each color in left region of orthogonal partition.



First Vertical Line

Find a smallest $j \in \{0, 1, \ldots, n\}$ satisfying

$$\left\lceil \frac{S_n}{2} \right\rceil \leq S_j$$

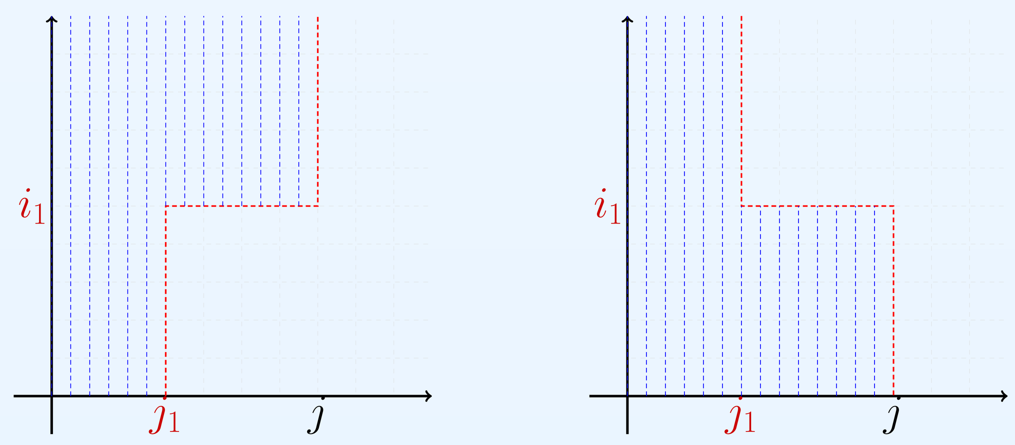


It can be written in the form

$$V = S_j = \sum_{i=0}^{n} a_{ij}.$$

3 Line Segments

Given index $0 < i_1 < n$ and $0 < j_1 < n$ into $y-$axis and $x-$axis of first vertical line, respectively. The orthogonal partition when we added index i_1, j_1 as shown below.



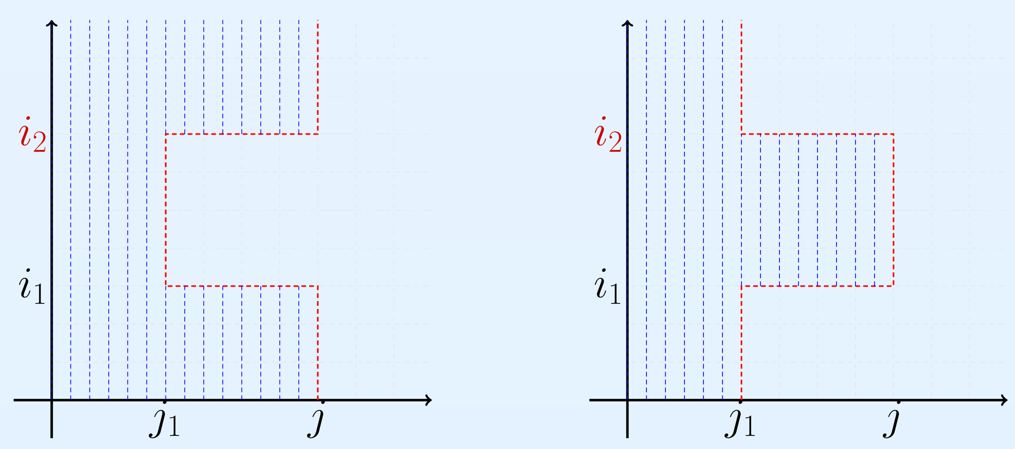
It can be written in the form

$$V = \sum_{i=0}^{i_1} a_{ij} + \sum_{i=i_1}^{n} a_{ij_1} \quad \text{or} \quad V = \sum_{i=0}^{i_1} a_{ij_1} + \sum_{i=i_1}^{n} a_{ij},$$

where $0 \leq j_1 \leq j$.

Some Cases of 5 Line Segments

Given an index $0 < i_2 < n$ into $y-$axis of three line segments. The orthogonal partition when we added index i_1, j_1 as shown below.



It can be written in the form

$$V = \sum_{i=0}^{i_1} a_{ij} + \sum_{i=i_1}^{i_2} a_{ij_1} + \sum_{i=i_2}^{n} a_{ij},$$

or

$$V = \sum_{i=0}^{i_1} a_{ij_1} + \sum_{i=i_1}^{i_2} a_{ij} + \sum_{i=i_2}^{n} a_{ij_1},$$

where $0 \leq j_1 \leq j$ and $0 < i_1 < i_2 < n$.

Verify a Solution

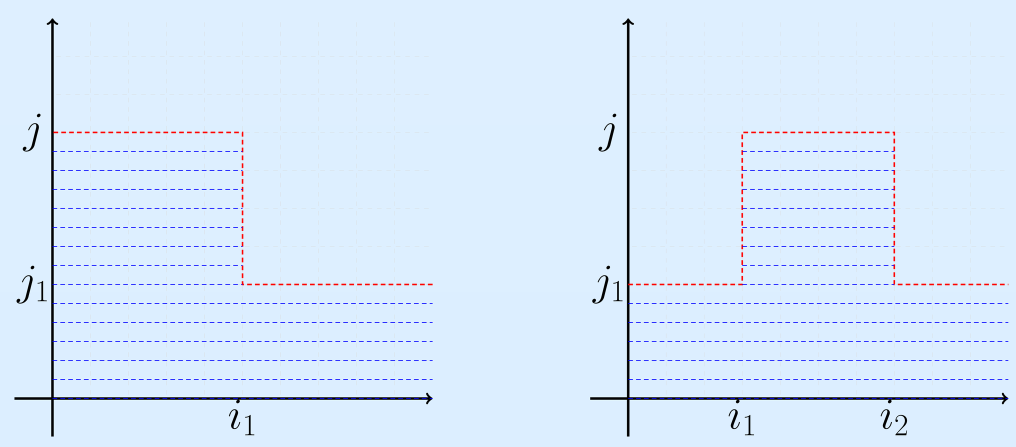
This orthogonal partition is a solution when

$$\left| \frac{S_n}{2} - V \right| \leq \frac{1}{2},$$

and it is a equitable when every element in matrix S_n are even.

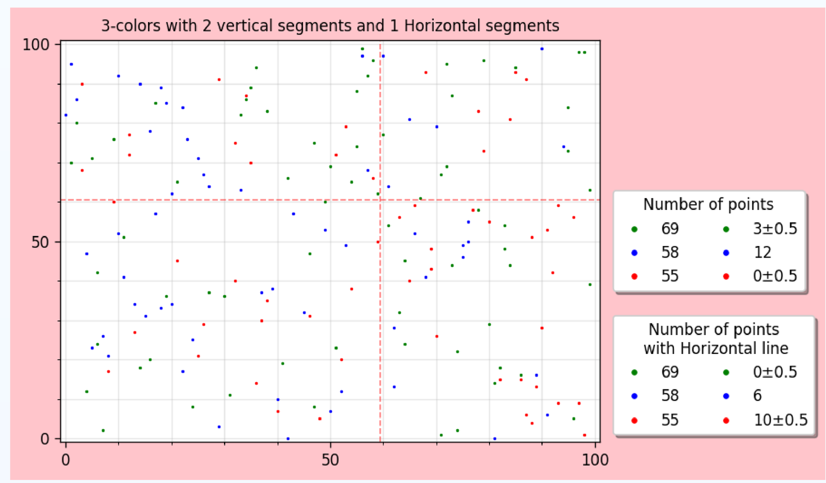
Rotational Coordinate

For the algorithm above, we can swap between the coordinate x, y and find orthogonal equipartitions with same iteration.

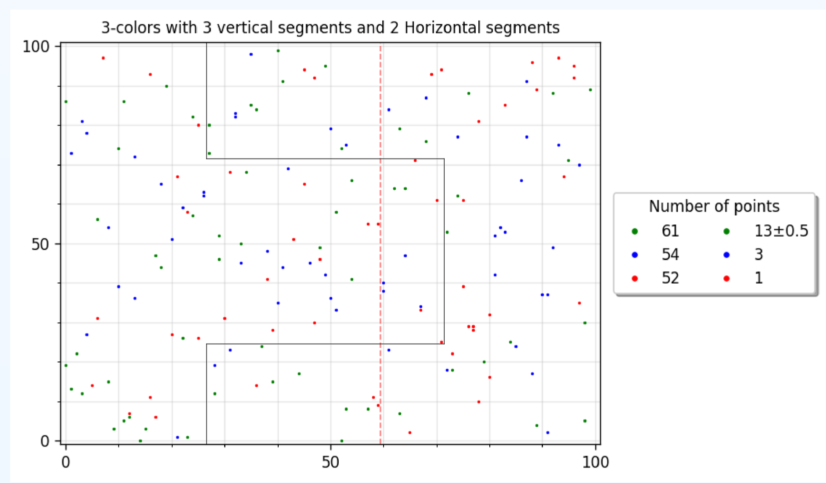


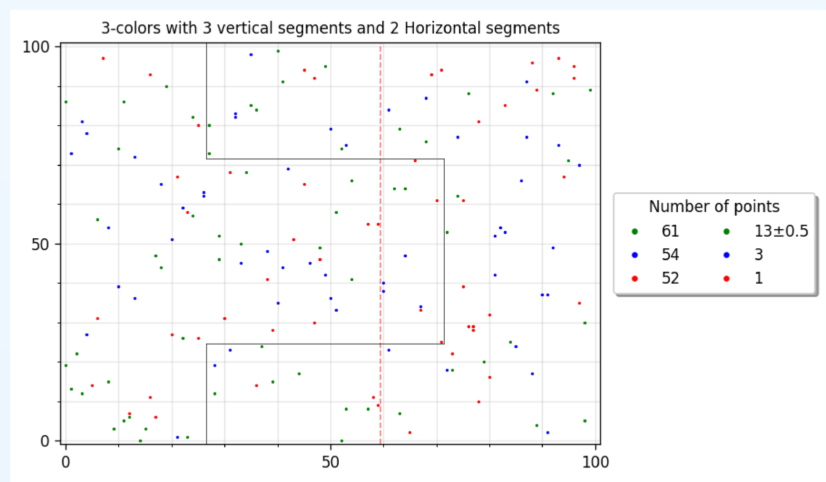
Result

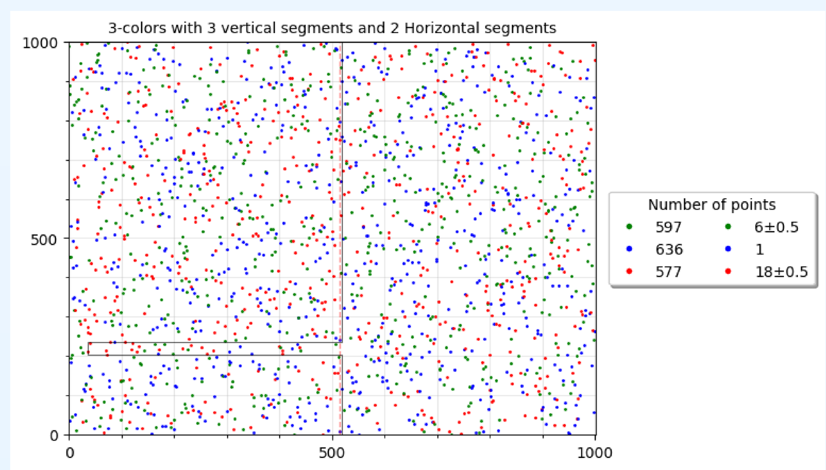
The counterexample of 3 line segments for 3-color points.



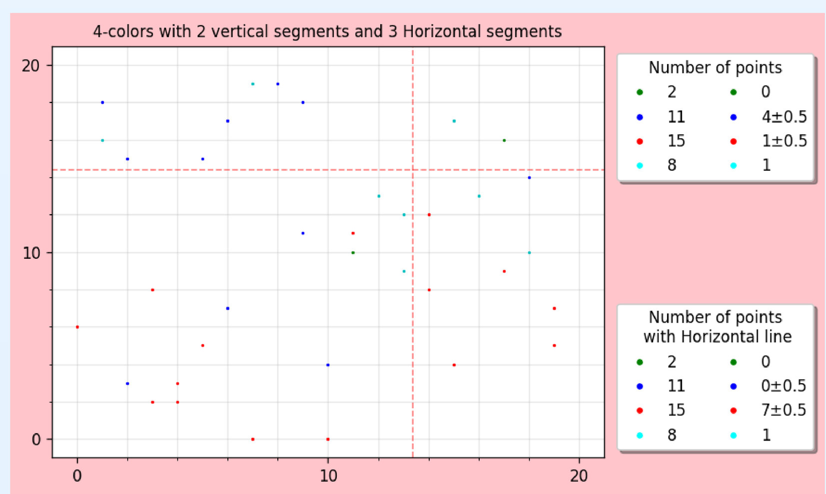
An example of 5 line segments for 3-color points.

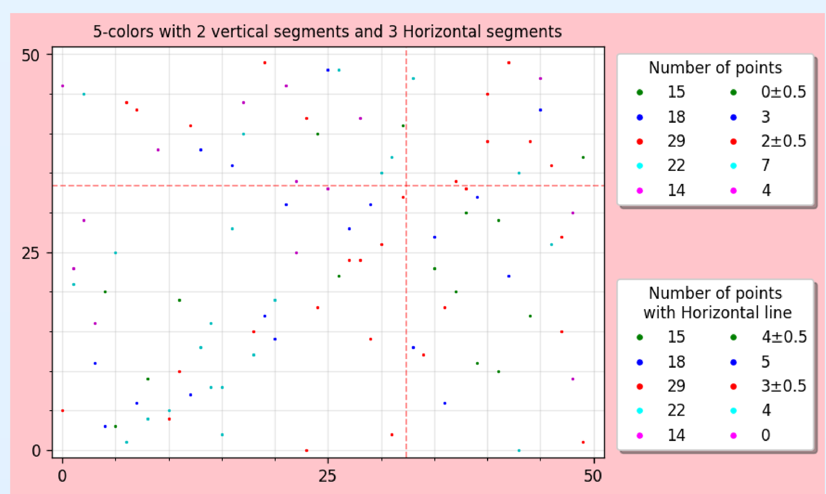






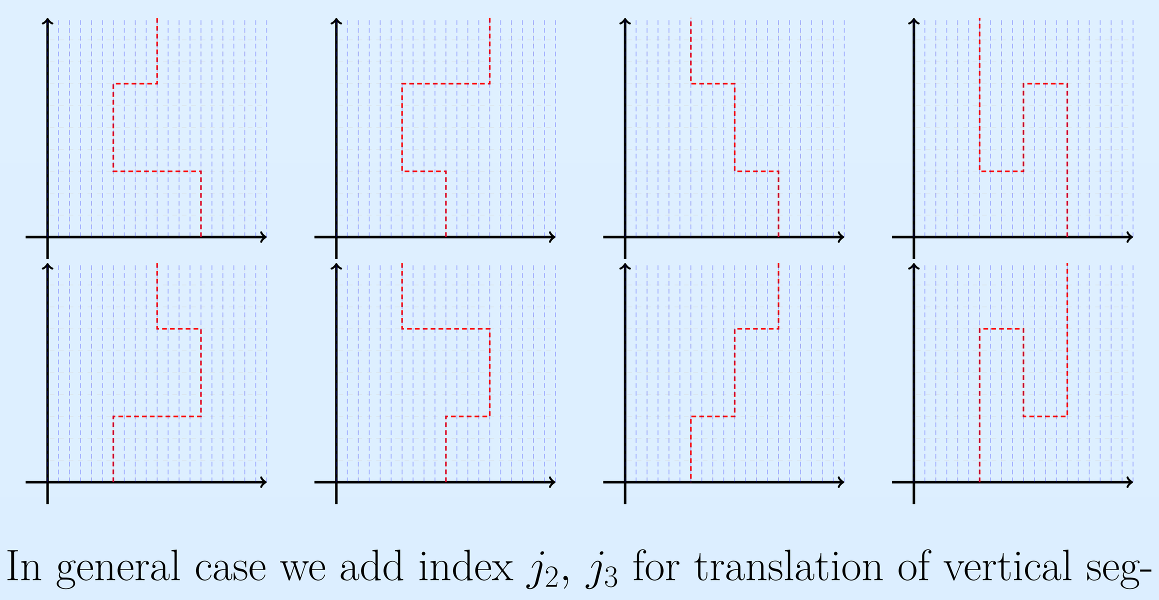
An counterexample of some cases of 5 line segments for 4, 5-color points, respectively.





Conclusion

In case 3-color points in general position have to use more than 3 line segments and it can find orthogonal equipartitions with 5 line segments for every sample from our generator. For k greater than 3, we have counterexample for some cases of 5 line segments. However, It maybe partitioning by general case of 5 line segments as shown below.



In general case we add index j_2, j_3 for translation of vertical segments in previous some cases of 5 segments. It can be written in the form

$$V = \sum_{i=0}^{i_1} a_{ij_1} + \sum_{i=i_1}^{i_2} a_{ij_2} + \sum_{i=i_2}^{n} a_{ij_3},$$

where $i_1 \leq i_2$, or

$$V = \sum_{i=0}^{i_1} a_{ij_1} - \sum_{i=i_1}^{i_2} a_{ij_2} + \sum_{i=i_2}^{n} a_{ij_3},$$

where $i_1 > i_2$.

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