Introduction to Algorithms

6.046J/18.401J/SMA5503

Lecture 12

Prof. Erik Demaine

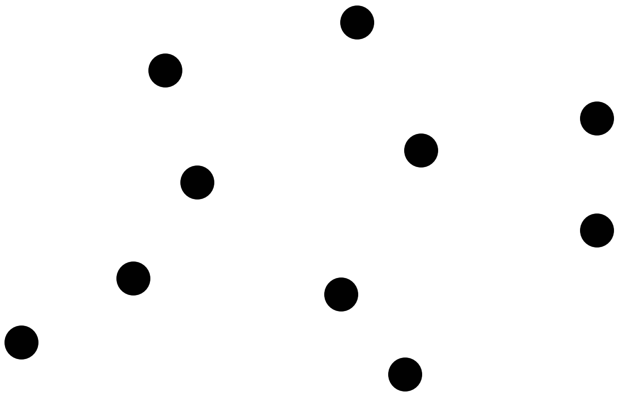
Computational geometry

Algorithms for solving "geometric problems" in 2D and higher.

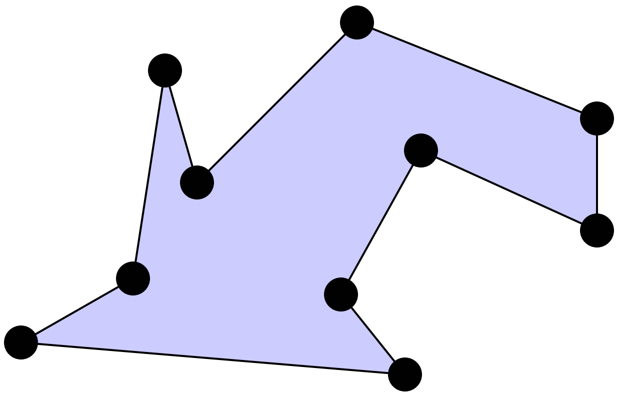
Fundamental objects:

point   line segment   line

Basic structures:



point set

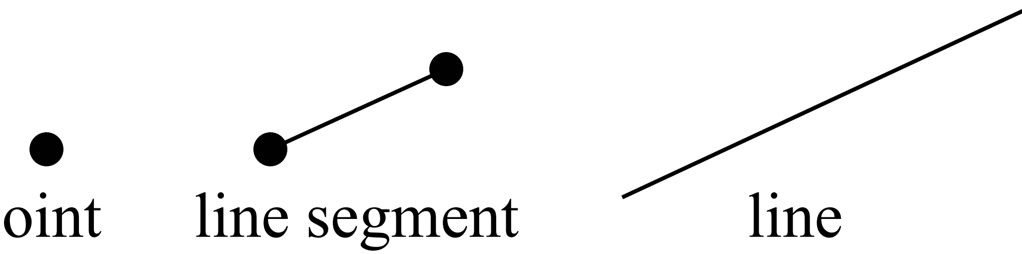


polygon

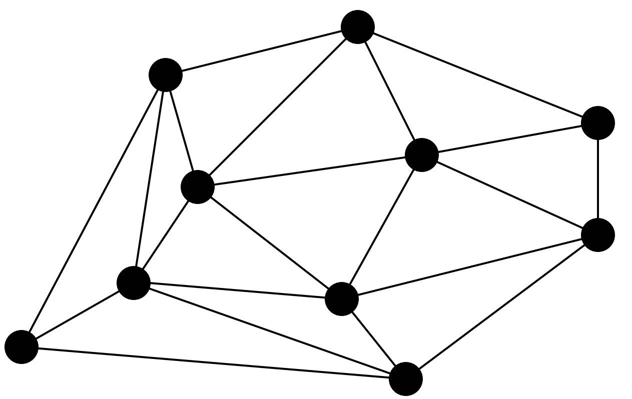
Computational geometry

Algorithms for solving "geometric problems" in 2D and higher.

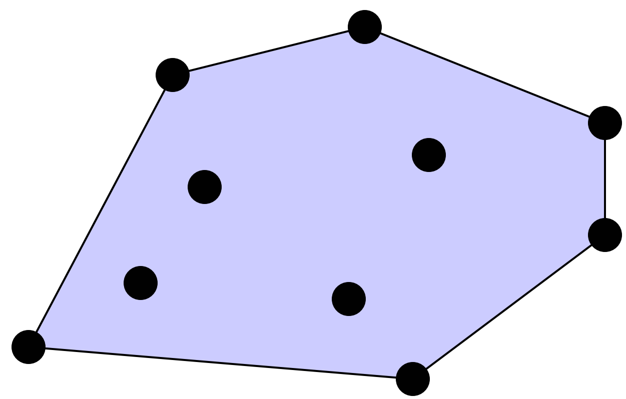
Fundamental objects:



Basic structures:



triangulation



convex hull

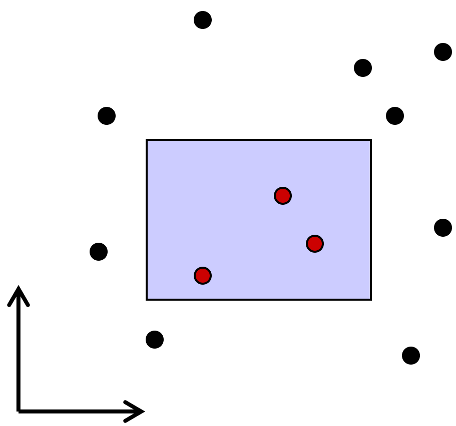
Orthogonal range searching

Input: n points in d dimensions

- E.g., representing a database of n records each with d numeric fields

Query: Axis-aligned ***box*** (in 2D, a rectangle)

- Report on the points inside the box:
  - Are there any points?
  - How many are there?
  - List the points.

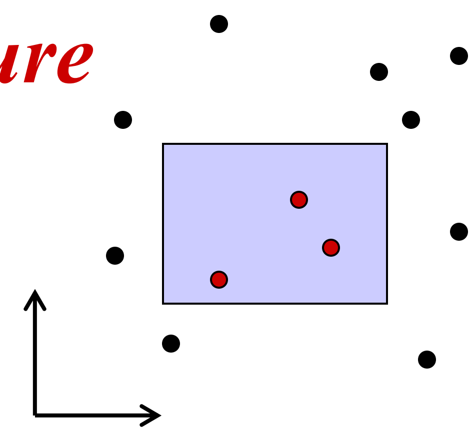


Orthogonal range searching

Input: n points in d dimensions

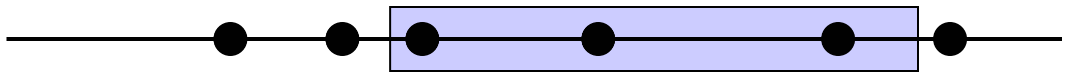
Query: Axis-aligned ***box*** (in 2D, a rectangle)
- Report on the points inside the box

Goal: Preprocess points into a data structure to support fast queries
- Primary goal: ***Static data structure***
- In 1D, we will also obtain a dynamic data structure supporting insert and delete



1D range searching

In 1D, the query is an interval:

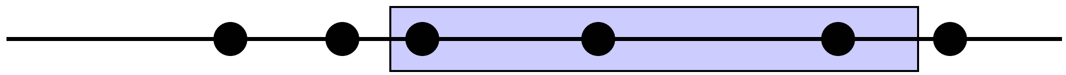


First solution using ideas we know:

- Interval trees
  - Represent each point x by the interval $[x, x]$.
  - Obtain a dynamic structure that can list k answers in a query in $O(k \lg n)$ time.

1D range searching

In 1D, the query is an interval:



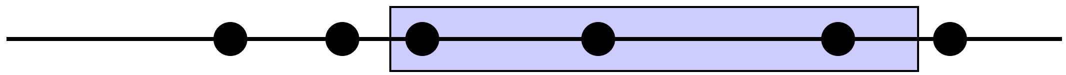
Second solution using ideas we know:

- Sort the points and store them in an array
  - Solve query by binary search on endpoints.
  - Obtain a static structure that can list k answers in a query in $O(k + \lg n)$ time.

Goal: Obtain a dynamic structure that can list k answers in a query in $O(k + \lg n)$ time.

1D range searching

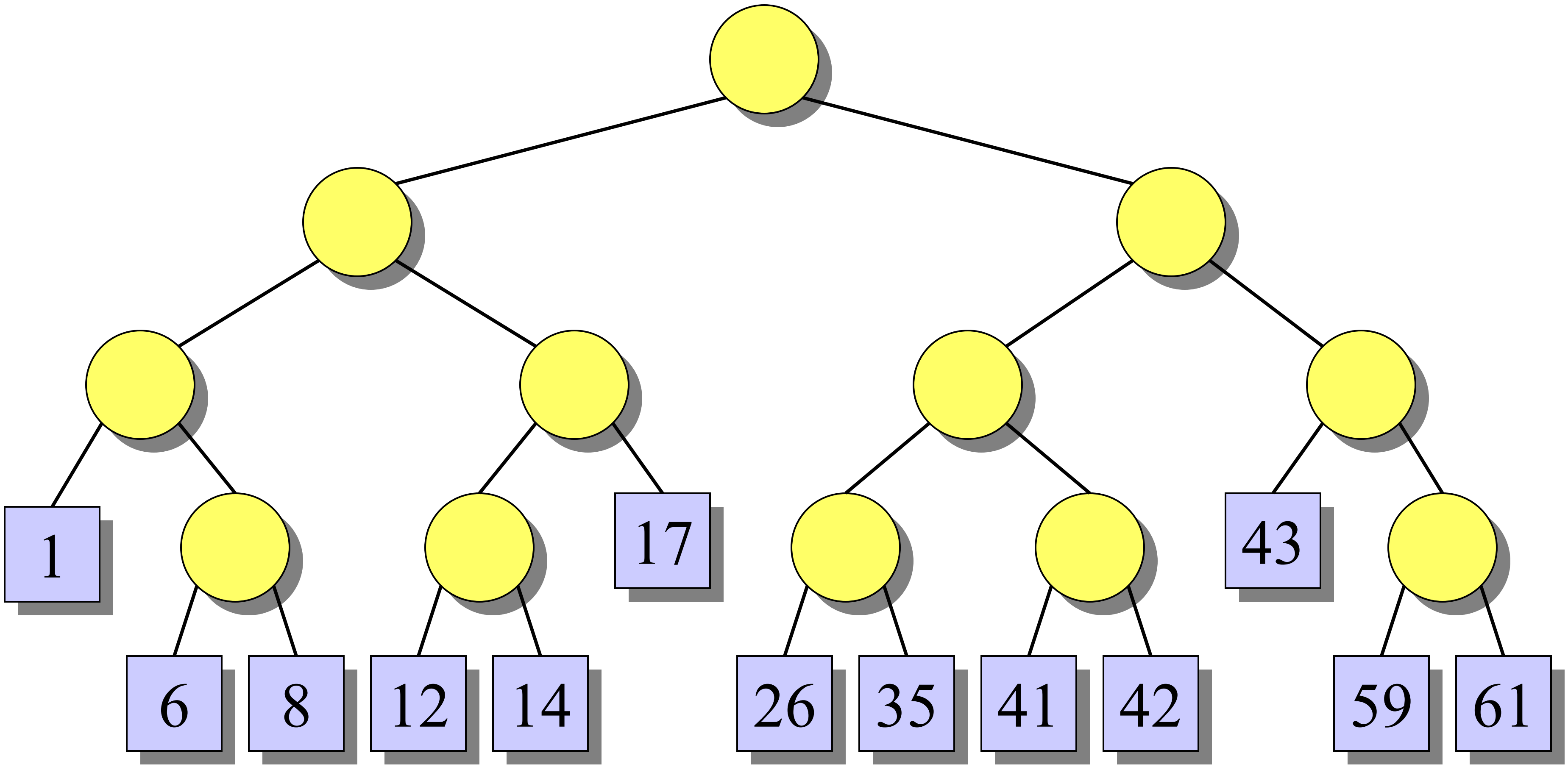
In 1D, the query is an interval:



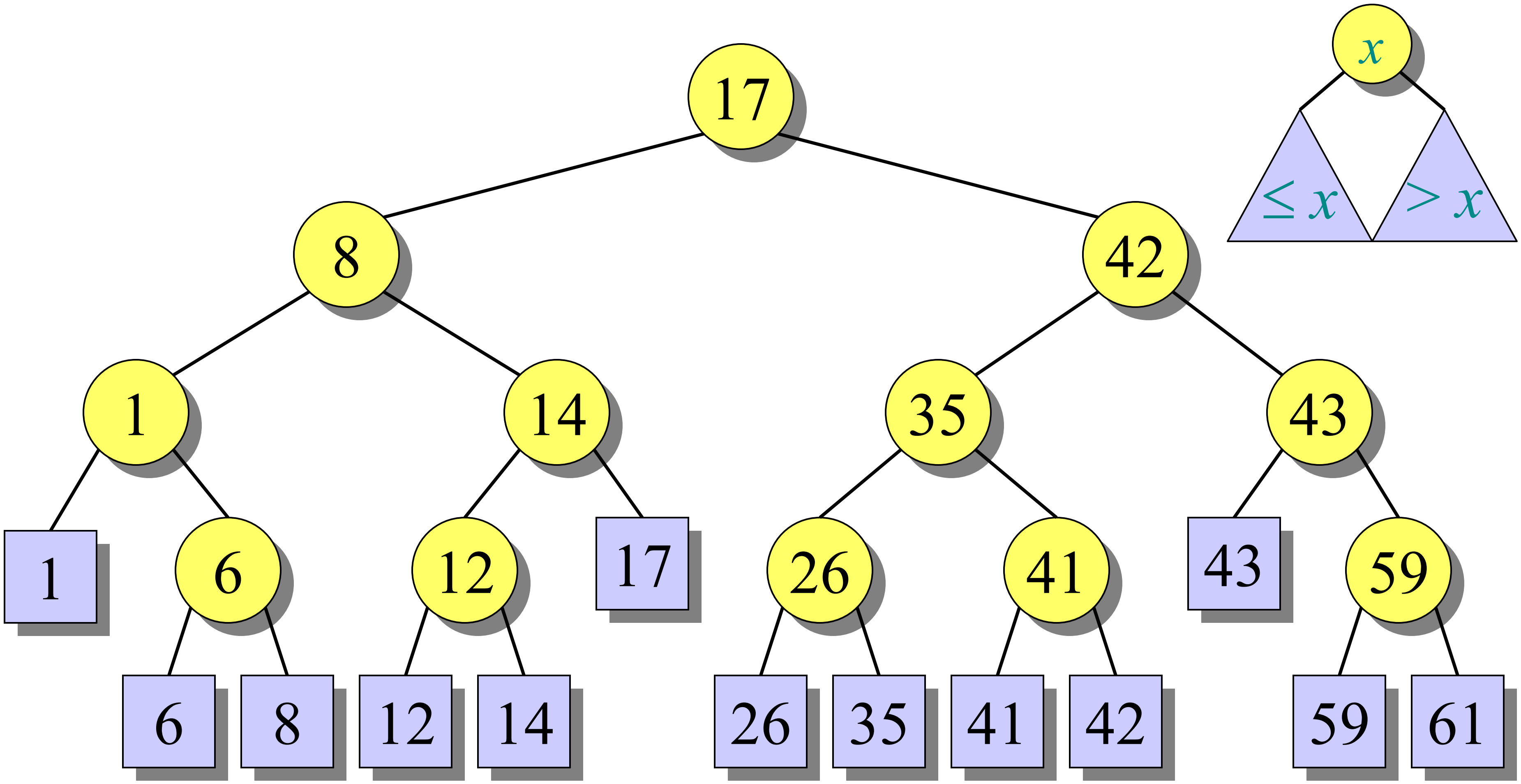
New solution that extends to higher dimensions:

- Balanced binary search tree
  - New organization principle:
    Store points in the *leaves* of the tree.
  - Internal nodes store copies of the leaves to satisfy binary search property:
    - Node x stores in $key[x]$ the maximum key of any leaf in the left subtree of x.

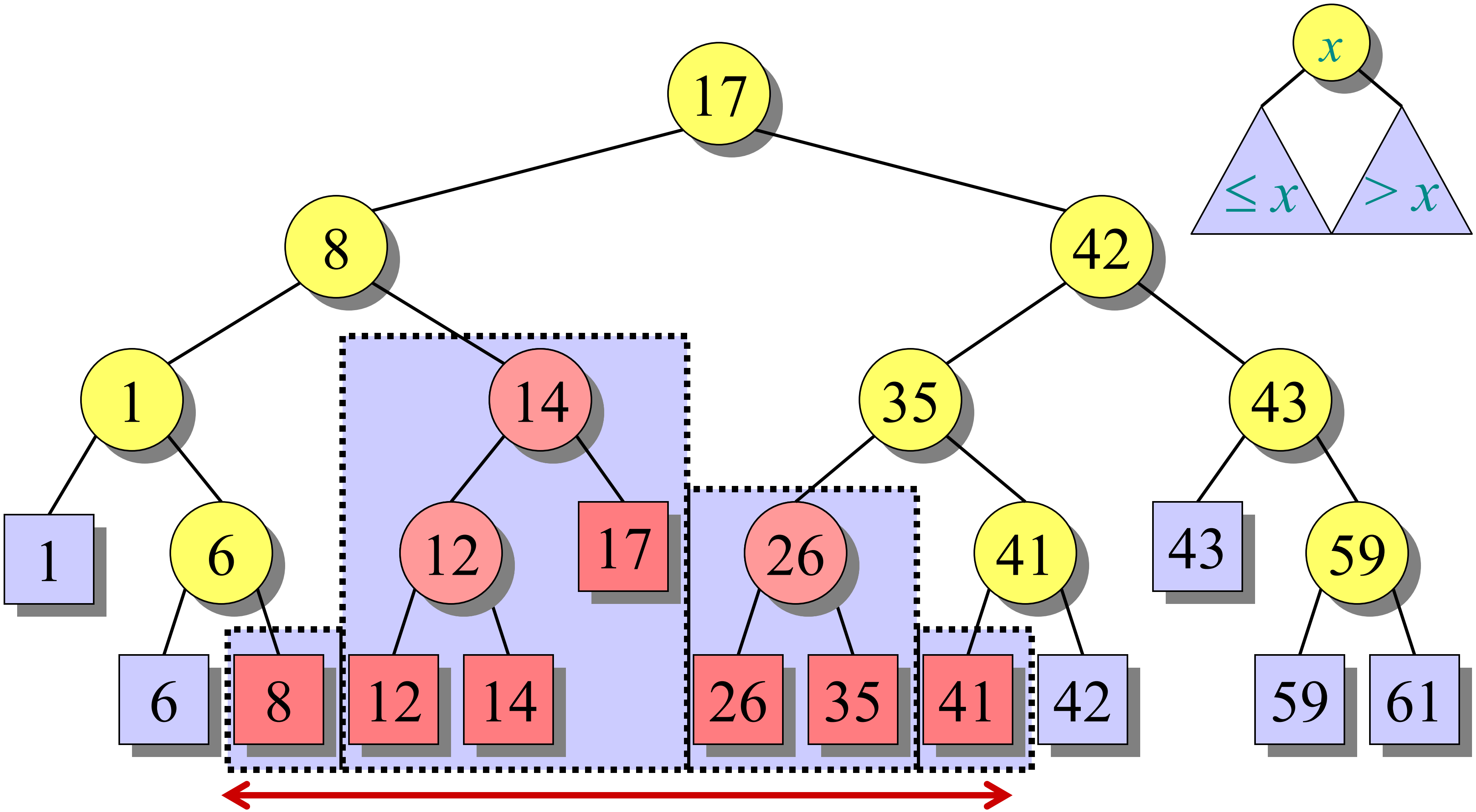
Example of a 1D range tree



Example of a 1D range tree

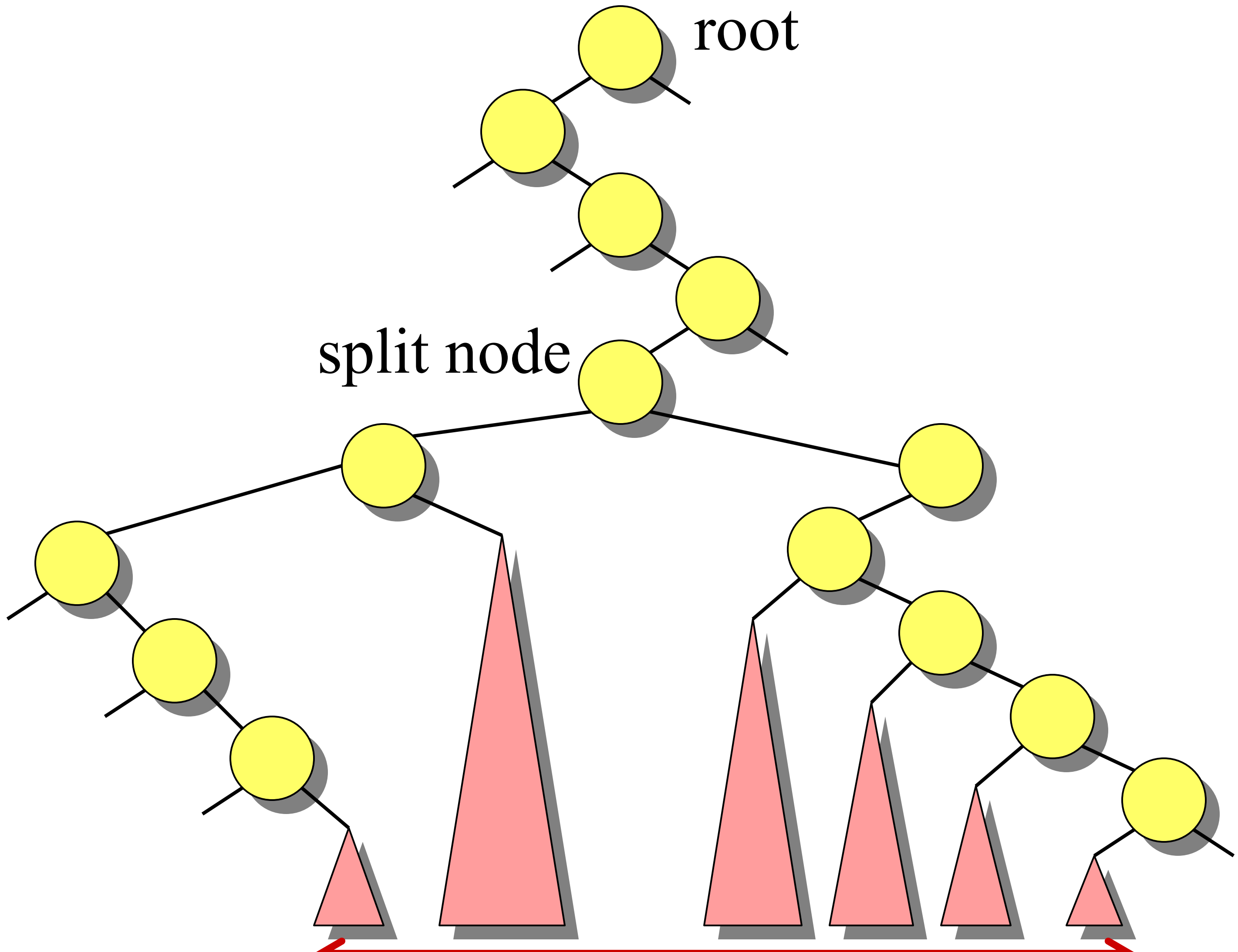


Example of a 1D range query



RANGE-QUERY([7, 41])

General 1D range query



Pseudocode, part 1: Find the split node

1D-RANGE-QUERY$(T, [x_1, x_2])$

 $w \leftarrow \mathrm{root}[T]$

 while w is not a leaf and $(x_2 \le key[w]$ or $key[w] < x_1)$

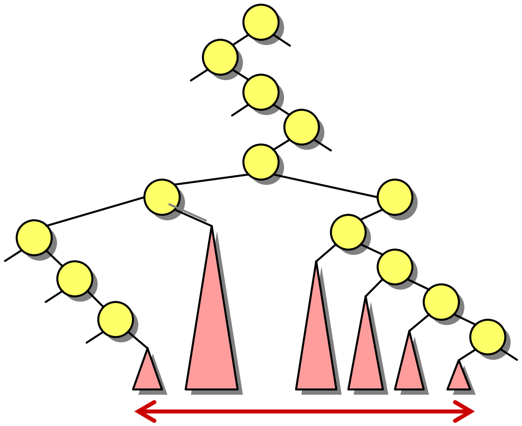
  do if $x_2 \le key[w]$

   then $w \leftarrow left[w]$

   else $w \leftarrow right[w]$

 ▷ w is now the split node

 [*traverse left and right from w and report relevant subtrees*]



Pseudocode, part 2: Traverse left and right from split node

1D-RANGE-QUERY$(T, [x_1, x_2])$

[find the split node]

▷ w is now the split node

if w is a leaf

then output the leaf w if $x_1 \le key[w] \le x_2$

else $v \leftarrow left[w]$ ▷ Left traversal

while v is not a leaf

do if $x_1 \le key[v]$

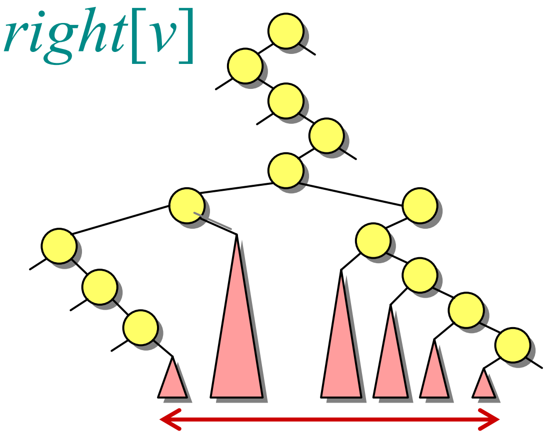
then output the subtree rooted at $right[v]$

$v \leftarrow left[v]$

else $v \leftarrow right[v]$

output the leaf v if $x_1 \le key[v] \le x_2$

[symmetrically for right traversal]

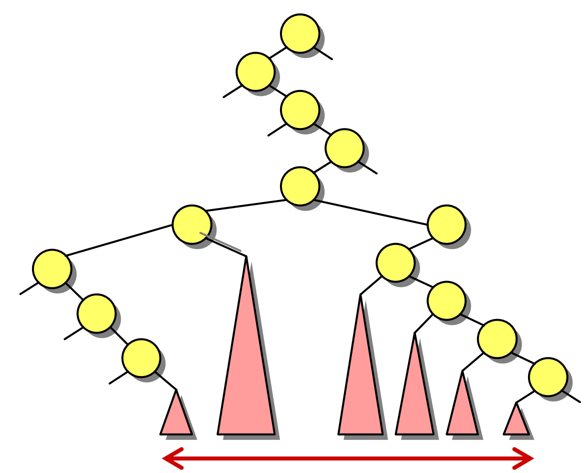


Analysis of 1D-RANGE-QUERY

Query time: Answer to range query represented by O(lg n) subtrees found in O(lg n) time.

Thus:

- Can test for points in interval in O(lg n) time.
- Can count points in interval in O(lg n) time if we augment the tree with subtree sizes.
- Can report the first k points in interval in O(k + lg n) time.



Space: O(n)

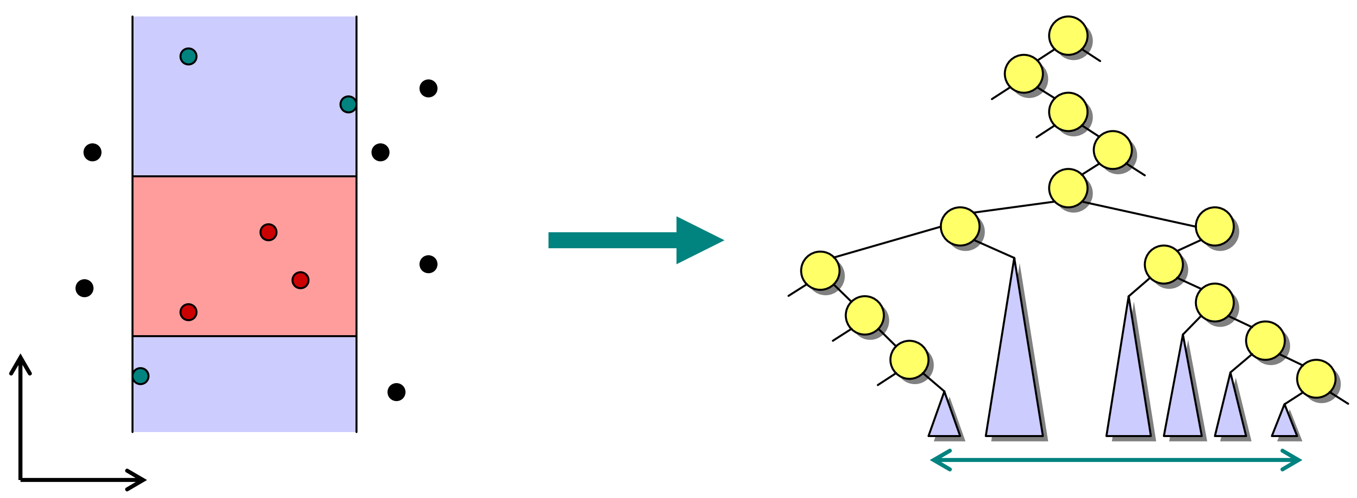
Preprocessing time: O(n lg n)

2D range trees

Store a ***primary*** 1D range tree for all the points based on x-coordinate.

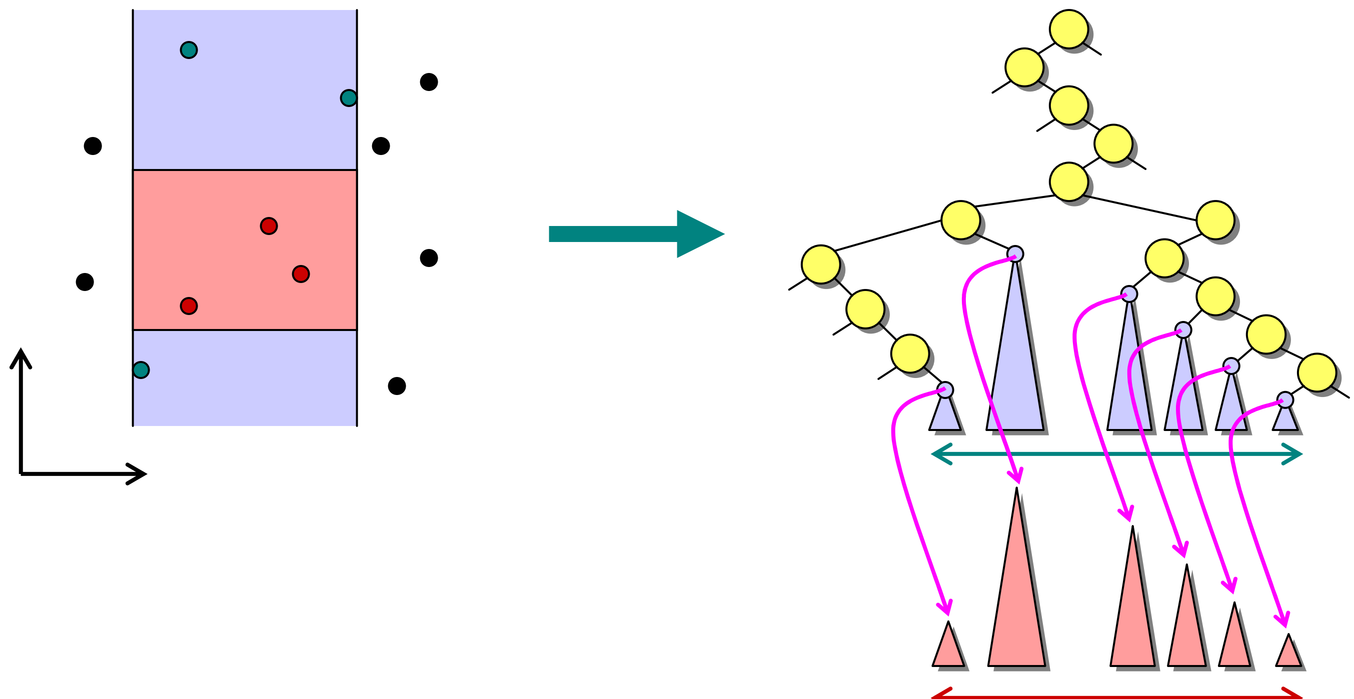
Thus in $O(\lg n)$ time we can find $O(\lg n)$ subtrees representing the points with proper x-coordinate.

How to restrict to points with proper y-coordinate?



2D range trees

Idea: In primary 1D range tree of x-coordinate, every node stores a ***secondary*** 1D range tree based on y-coordinate for all points in the subtree of the node. Recursively search within each.



Analysis of 2D range trees

Query time: In $O(\lg^2 n) = O((\lg n)^2)$ time, we can represent answer to range query by $O(\lg^2 n)$ subtrees. Total cost for reporting k points: $O(k + (\lg n)^2)$.

Space: The secondary trees at each level of the primary tree together store a copy of the points. Also, each point is present in each secondary tree along the path from the leaf to the root. Either way, we obtain that the space is $O(n \lg n)$.

Preprocessing time: $O(n \lg n)$

d-dimensional range trees

Each node of the secondary y-structure stores a tertiary z-structure representing the points in the subtree rooted at the node, etc.

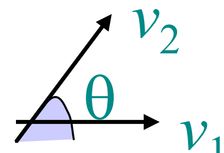
Query time: $O(k + \lg^d n)$ to report k points.
Space: $O(n \lg^{d-1} n)$
Preprocessing time: $O(n \lg^{d-1} n)$

Best data structure to date:
Query time: $O(k + \lg^{d-1} n)$ to report k points.
Space: $O(n (\lg n / \lg \lg n)^{d-1})$
Preprocessing time: $O(n \lg^{d-1} n)$

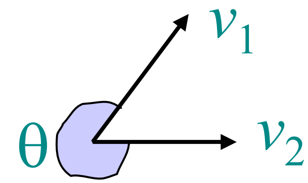
Primitive operations: Crossproduct

Given two vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$, is their counterclockwise angle θ

- ***convex*** ($< 180^o$),
- ***reflex*** ($> 180^o$), or
- borderline (0 or 180^o)?



convex



reflex

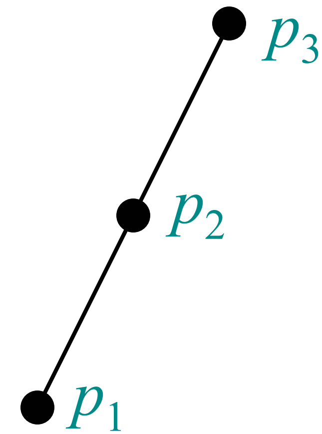
Crossproduct $v_1 \times v_2 = x_1\, y_2 - y_1\, x_2$
$= |v_1|\,|v_2| \sin\theta$.

Thus, $\text{sign}(v_1 \times v_2) = \text{sign}(\sin\theta)$ > 0 if θ convex,
< 0 if θ reflex,
$= 0$ if θ borderline.

Primitive operations: Orientation test

Given three points p_1, p_2, p_3 are they

- in ***clockwise (cw) order***,
- in ***counterclockwise (ccw) order***, or
- ***collinear***?



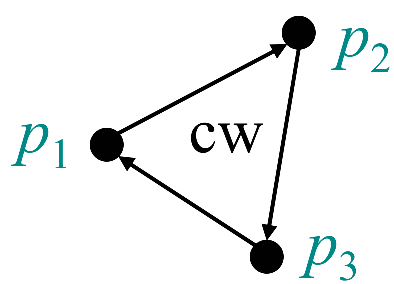
collinear

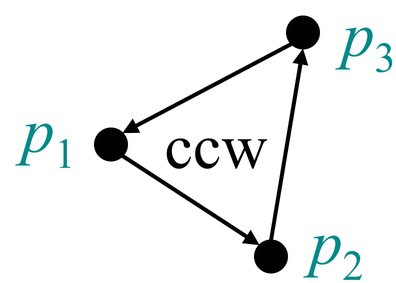
$$(p_2 - p_1) \times (p_3 - p_1)$$

> 0 if ccw

< 0 if cw

$= 0$ if collinear

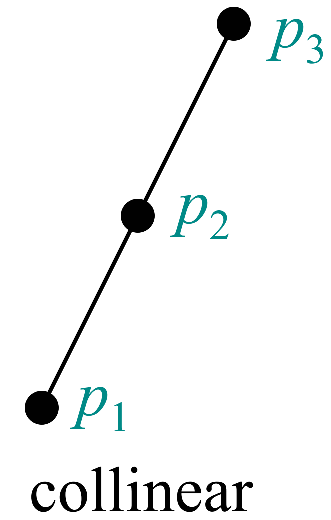




Primitive operations: Sidedness test

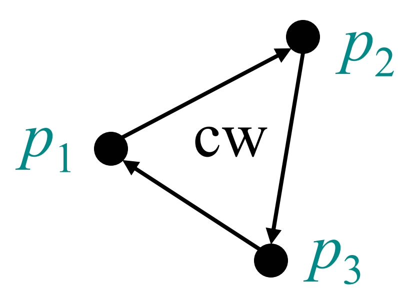
Given three points p_1, p_2, p_3 are they

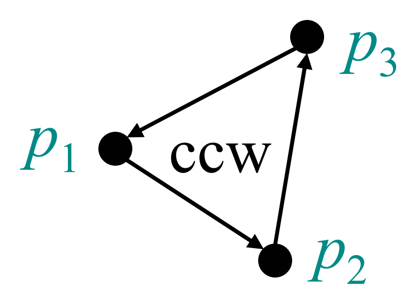
- in ***clockwise (cw) order***,
- in ***counterclockwise (ccw) order***, or
- ***collinear***?



Let L be the oriented line from p_1 to p_2.
Equivalently, is the point p_3

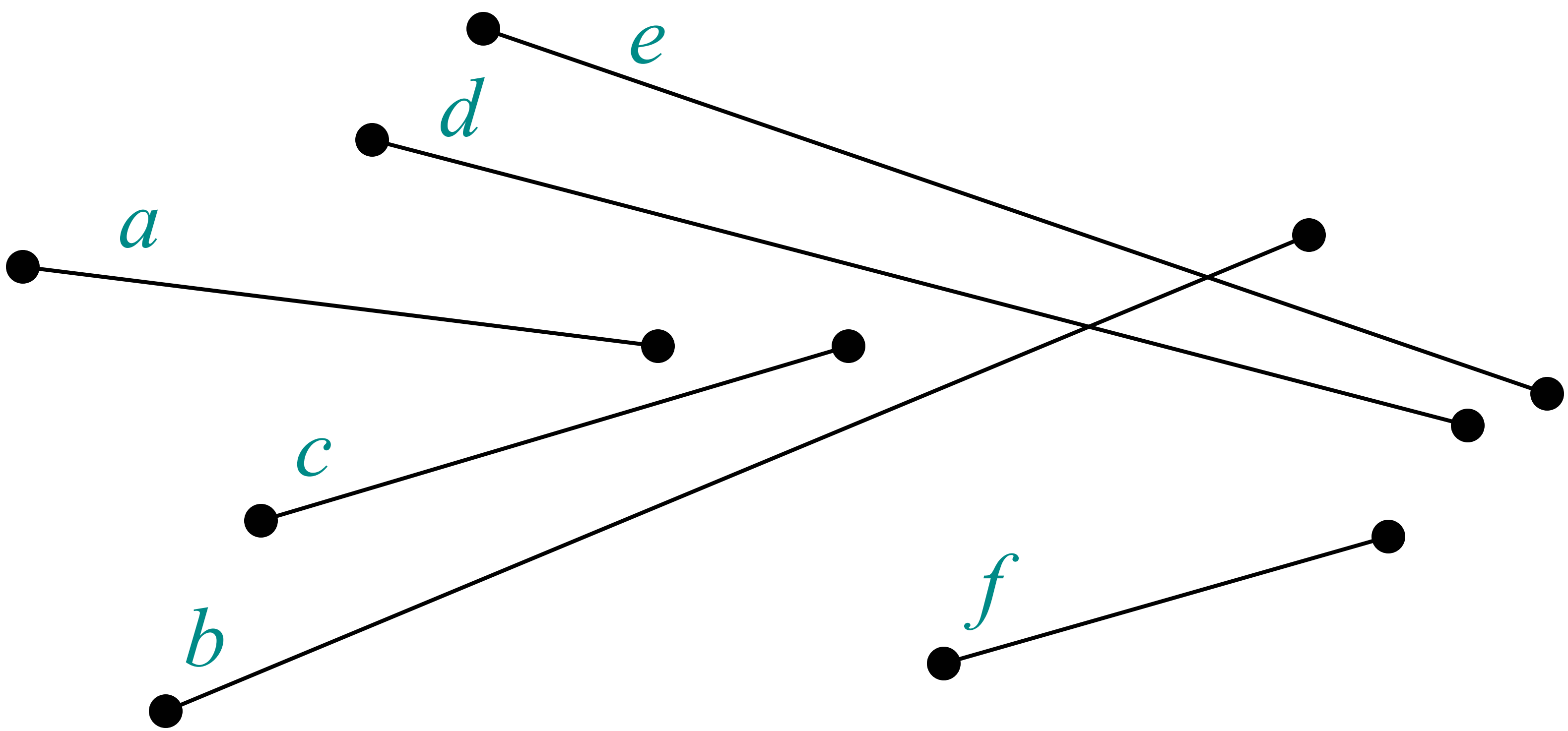
- ***right*** of L,
- ***left*** of L, or
- ***on*** L?





Line-segment intersection

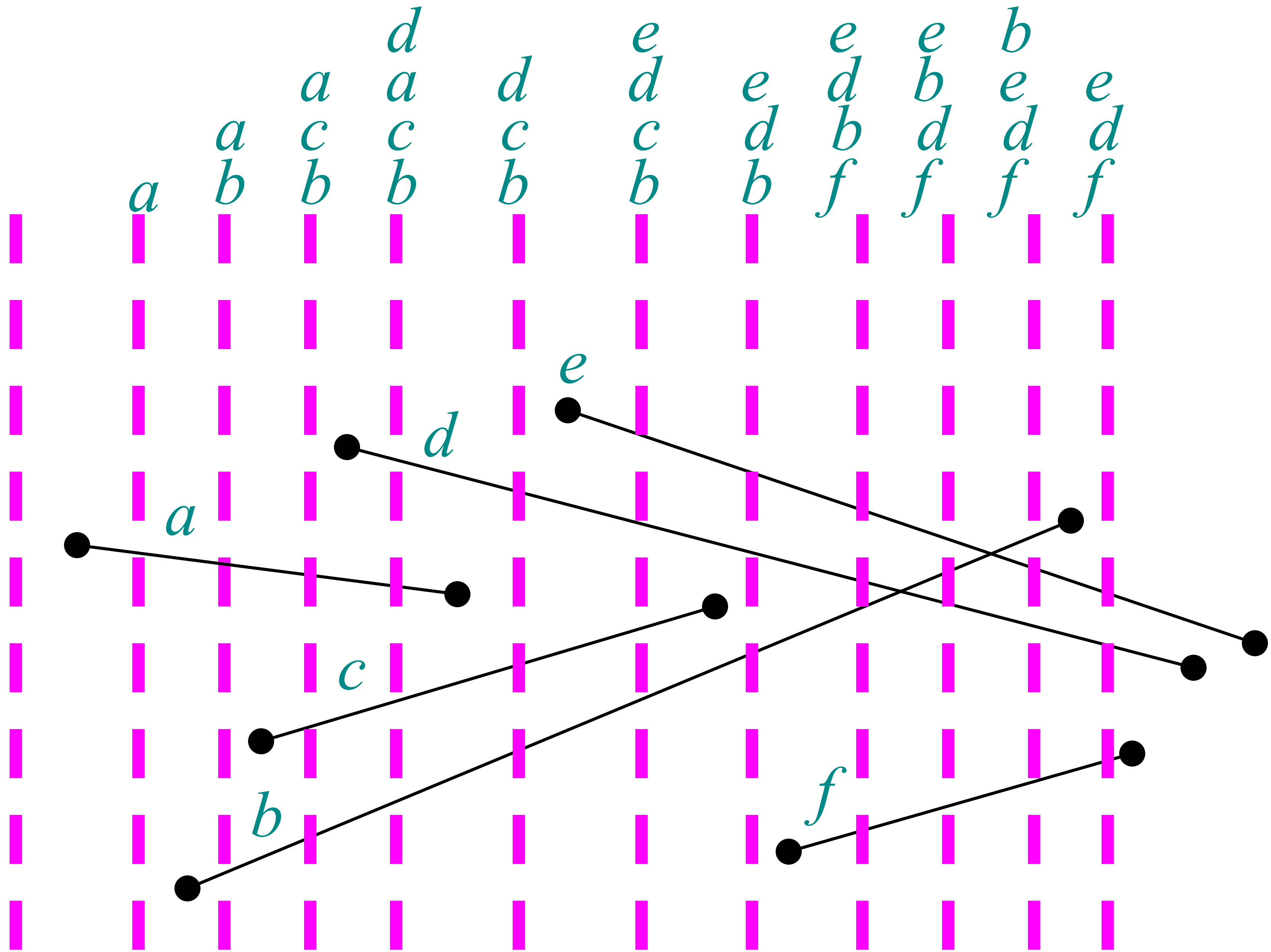
Given n line segments, does any pair intersect?
Obvious algorithm: $O(n^2)$.



Sweep-line algorithm

- Sweep a vertical line from left to right (conceptually replacing x-coordinate with time).
- Maintain dynamic set S of segments that intersect the sweep line, ordered (tentatively) by y-coordinate of intersection.
- Order changes when
  - new segment is encountered,
  - existing segment finishes, or
  - two segments cross
- Key ***event points*** are therefore segment endpoints.

$\left.\begin{array}{l}\text{new segment is encountered,}\\ \text{existing segment finishes, or}\end{array}\right\}$ segment endpoints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Sweep-line algorithm

Process event points in order by sorting segment endpoints by x-coordinate and looping through:

- For a left endpoint of segment s:
  - Add segment s to dynamic set S.
  - Check for intersection between s and its neighbors in S.
- For a right endpoint of segment s:
  - Remove segment s from dynamic set S.
  - Check for intersection between the neighbors of s in S.

Analysis

Use red-black tree to store dynamic set S.

Total running time: $O(n \lg n)$.

Correctness

Theorem: If there is an intersection, the algorithm finds it.

Proof: Let X be the leftmost intersection point.
Assume for simplicity that

- only two segments s_1, s_2 pass through X, and
- no two points have the same x-coordinate.

At some point before we reach X,
s_1 and s_2 become consecutive in the order of S.
Either initially consecutive when s_1 or s_2 inserted,
or became consecutive when another deleted. ◻